PART 1

Mathematics Scope

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An Intuitionistic Fuzzy Pseudo Enlarged Ideal of a BH-Algebra

Hussein Hadi Abbas[1] and Khoudair Obeas Kadem[2]

[1]*Department of Mathematics, Faculty of Education for Girls, University of Kufa, Iraq.*

[2]*Department of Mathematics, Faculty of Computer Science and Mathematics, University of Kufa, Iraq*

khudhayer1981@gmail.com

Abstract. In this Work the concepts of an intuitionistic fuzzy pseudo ideal of a pseudo BH-algebra are insert. several propositions and examples are scrupulous to study properties of this idea**.**

Keywords. BH-Algebra, Pseudo BH-Algebra, intuitionistic fuzzy pseudo ideal in pseudo BH-algebra, intuitionistic Enlarged ideal in pseudo BH-algebra.

1. Introduction:

The algebraic design named BCK-algebra & BCI- algebra a generality of BCK-algebra are come in by K. ISEKI and Y. IMAI in 1966[2]. In 1998 Y. B. Jun, et al show the idea of a BH-algebra [8]. furthermore, Y.B.Jun, et al introduce the idea of a pseudo BH-algebra in 2015[8]. In 2017, A.H. Nouri andH.H. Abbass thoughtful some kinds of ideals of pseudo BH-algebra [9].The most writer deem the year 1965 is the starting of a fuzzy logic when L. A. Zadeh knew a subset in fuzzy sets [1]. In 1991Xi. O thoughtful BCK-algebra a fuzzy sense [10]. Ever after then, the researchers have on a comprehensive scale. fuzzy Ideals about an Element of Pseudo BH-algebra defined by A. A. mutesher & H. H. Abbass[11]. H.H. Abbass & H.A. Dahham offer a fuzzy completely closed ideal of BH-algebra in 2012[5]. A fuzzy closed ideal relies on an element in BH-algebra thoughtful by H. M. A.Saeed & H. H. Abbass in 2011 [7], we intuitionistic fuzzy if pseudo ideal and pseudo enlarged ideal in a pseudo BH-algebra.

2. Preliminaries.

In this work, several basic connotations about a BH-algebra, ideal in BH-algebra, intuitionistic enlarged ideal in BH-algebra, pseudo ideal pseudo BH-algebra, intuitionistic fuzzy ideal in BH-algebra are given

2.1. **Definition**

A set X is not equl $\emptyset$ with a dual operation $(*)$ and aconstant 0 is named a **BH** – algebra if achieved : $\forall\ \mu , \lambda \in X$

- $\mu * \mu = 0$
- $\mu * \lambda = 0$ and $\lambda * \mu = 0 \Longrightarrow \mu = \lambda$
- $\mu * 0 = \mu$

2.2. **Definition**

Assume that $S \neq \emptyset$ is a subset of a BH-X is named a BH-subalgebra of X signify by BH-S if $\mu * \lambda \in S, \forall\ \mu , \lambda \in S$.

2.3. **Definition**

Assume that I$\neq \emptyset$ and a subset of a BH-X. Therefore I is named an ideal of X if that was achieved: $\forall$ μ , $\lambda \in$ X

- $0 \in$ I.
- $\mu * \lambda \in$ I and $\lambda \in$ I $\Longrightarrow \mu \in$ I.

2.4. **Definition**

Assume that I $\neq \emptyset$ and a subset of a P.BH-algebra X and there is no need an ideal of X, a subset J of X is named an Enlarged ideal of X related to I, and signify by E. I if that was achieved : for every μ , λ $\in$ X

- **I** is a subset of **J**
- $0 \in$ **J**
- $\boldsymbol{\mu * \lambda \in}$ **I and** $\boldsymbol{\lambda \in}$ **I** $\boldsymbol{\Longrightarrow \mu \in}$ **J.**

2.5. **Definition**

A pseudo **BH** indicates **P.BH** is a set X is not equl $\emptyset$ with a fixed 0 and dual operations $*, \#$ check the next conditions :

- $\mu * \mu = \mu \# \mu = 0, \forall \mu \in X$
- $\mu * \lambda = 0$ & $\lambda \# \mu = 0 \Longrightarrow \mu = \lambda, \forall \mu , \lambda \in X$
- $\mu * 0 = \mu \# 0 = \mu, \forall \mu \in X$.

2.6. **Definition**

Assume that S $\neq \emptyset$ is a subset of a P.BH-X is named a P.BH-subalgebra of X signify by P.BH-S if that was achieved : $\mu * \lambda$ and $\mu \# \lambda \in$ S, $\forall \mu , \lambda \in$ S.

2.7. **Definition**

Assume that I $\neq \emptyset$ and subset of a P.BH-X. Therefore I is named a pseudo ideal & signify by **P. I** of X if achieved : $\forall \mu , \lambda \in$ X

- $0 \in$ I.
- $\mu * \lambda, \mu \# \lambda \in$ I and $\lambda \in$ I $\Longrightarrow \mu \in$ I.

2.8. **Definition**

Assume that I$\neq \emptyset$ and subset of a P.BH-algebra X, a subset **J** of X is named a **pseudo Enlarged ideal** of X related to **I**, and signify by **P. E. I** if that was achieved : $\forall \mu , \lambda \in$ X

- **I** is a subset of **J**
- $0 \in$ **J**
- $\mu * \lambda \in$ **I,** $\mu \# \lambda \in$ **I** and $\lambda \in$ **I** $\Longrightarrow \mu \in$ **J**.

2.9. **Definition**

Assume X that is a non-empty set, fuzzy subset ω, σ in X are a formula from X into [0 , 1] of the real number.

2.10. **Definition**

Assume that A is an intuitionistic fuzzy set in X, shortened by **I. F. S** and the set U $(\omega , \alpha) = \{ \mu \in X : \omega_A(\mu) \geq \alpha \}$ is named upper α -level cut of A and L $(\sigma , \alpha) = \{ \mu \in X : \sigma_A(\mu) \leq \alpha \}$ is named lower α - level cut of A.

2.11. **Definition**

Assume A = $(\omega_A(\mu), \sigma_A(\mu))$ & $\mathcal{B} = (\omega_{\mathcal{B}}(\mu), \sigma_{\mathcal{B}}(\mu))$ are I. F. S in X : $\forall \mu \in X$

- $(A \cup \mathcal{B})(\mu) = \{< \mu, \max (\omega_A(\mu), \omega_{\mathcal{B}}(\mu)), \min (\sigma_A(\mu), \sigma_{\mathcal{B}}(\mu)) >| \mu \in X \}$
- $(A \cap \mathcal{B})(\mu) = \{< \mu, \min (\omega_A(\mu), \omega_{\mathcal{B}}(\mu)), \max (\sigma_A(\mu), \sigma_{\mathcal{B}}(\mu)) >| \mu \in X \}$

$A \cup \mathcal{B}$ & $A \cap \mathcal{B}$ are I. F. S in X , $\forall \mu \in X$ in broadly , if $\{ A_i , i \in \Omega\}$ be a chain of intuitionistic sets in X

$(\cap_{A_i}) (\mu) = (\inf \omega_{A_i}(\mu) , \sup \sigma_{A_i}(\mu))$

$(\cup_{A_i}) (\mu) = (\sup \omega_{A_i}(\mu) , \inf \sigma_{A_i}(\mu))$

Which are too I. F. S in X.

3. The Main Results

In the work, is defined the concepts of intuitionistic fuzzy pseudo enlarged ideal in **P.BH**-algebra. for our conversation, we will study the advantages of these concepts.

3.1. **Definition**

Assume A & B are two I. F. S of a BH-algebra X, so that $A \subseteq B$ then B is named **intuitionistic fuzzy enlarged ideal** of X related to A & signify by **I. F. E. I** if that was achieved :

- $\omega_B(0) \geq \omega_B(\mu)$ & $\sigma_B(0) \leq \sigma_B(\mu)$, $\forall \mu \in X$.
- $\omega_B(\mu) \geq \min \{ \omega_A(\mu * \lambda), \omega_A(\lambda) \}$, $\forall \mu , \lambda \in X$.
- $\sigma_B(\mu) \leq \max \{ \sigma_A(\mu * \lambda), \sigma_A(\lambda) \}$, $\forall \mu , \lambda \in X$.

3.2. **Example**

Assume that X = { 0, k, v, h } is a BH-algebra with the next cayley tables :

*	0	k	v	h

0	0	0	0	0
k	k	0	0	k
v	v	v	0	v
h	h	h	h	0

Define A = (ω_A(μ), σ_A(μ)), B = (ω_B(μ) , σ_B(μ)) are two I. F.S of X by

$$\omega_A(\mu) = \begin{cases} 0.5 & if \quad \mu = 0, h \\ 0.4 & if \quad \mu = k, v \end{cases}$$

$$\sigma_A(\mu) = \begin{cases} 0.2 & if \quad \mu = 0 \\ 0.4 & if \quad \mu = k, v, h \end{cases}$$

$$\omega_B(\mu) = \begin{cases} 0.6 & if \quad \mu = 0, k \\ 0.5 & if \quad \mu = v, h \end{cases}$$

$$\sigma_B(\mu) = \begin{cases} 0.2 & if \quad \mu = 0, v \\ 0.3 & if \quad \mu = k, h \end{cases}$$

Then B is an I. F. E. I of X related to A.

3.3. **Definition**

Assume A & B are an I. F. S of a BH-algebra X so that A ⊆ B then B is named **intuitionistic fuzzy pseudo enlarged ideal** of X related to A, signify by **I. F. P. E. I** if that was achieved : ∀ μ , λ ∈ X

- $\omega_B(0) \geq \omega_B(\mu)$ & $\sigma_B(0) \leq \sigma_B(\mu)$
- $\omega_B(\mu) \geq \inf \{ \omega_A(\mu * \lambda), \omega_A(\mu \# \lambda), \omega_A(\lambda) \}$
- $\sigma_B(\mu) \leq \sup \{ \sigma_A(\mu * \lambda), \sigma_A(\mu \# \lambda), \sigma_A(\lambda) \}$

3.4.**Example**

Assume that X = { 0, k, v, h }is a P.BH with the following cayley tables :

*	0	k	v	h
0	0	0	0	0
k	k	0	0	k
v	v	v	0	v
h	h	h	h	0

#	0	k	v	h
0	0	0	0	0
k	k	0	0	h
v	v	v	0	v
h	h	h	h	0

Define A = (ω_A(μ), σ_A(μ)), B = (ω_B(μ), σ_B(μ)) are two I. F. S of X by

$$\omega_A(\mu) = \begin{cases} 0.5 & if \quad \mu = 0, k \\ 0.4 & if \quad \mu = v, h \end{cases}$$

$$\sigma_A(\mu) = \begin{cases} 0.2 & if \quad \mu = 0 \\ 0.3 & if \quad \mu = k, v, h \end{cases}$$

$$\omega_B(\mu) = \begin{cases} 0.6 & if \quad \mu = 0 \\ 0.5 & if \quad \mu = k, v, h \end{cases}$$

$$\sigma_B(\mu) = \begin{cases} 0.1 & if \quad \mu = 0, h \\ 0.3 & if \quad \mu = k, v \end{cases}$$

Then B is an I. F. P. E. I of X related to A.

3.5.**Theorem**

Assume that { B_i | $i \in \Omega$ } is a family of I. F. P. E. I of a P.BH-algebra X related to A. Then $\cap_{i\in\Omega}$ Bi is an I. F. P. E. I of X related to A.

Proof :-

- Assume that μ ∈ X, $i \in \Omega$, $\omega_{Bi}(0) \geq \omega_{Bi}$(μ) ⇒ $inf_{i\in\Omega}\omega_{Bi}(0) \geq inf_{i\in\Omega}\omega_{Bi}$(μ) ⇒ $\omega_{\cap Bi}(0) \geq \omega_{\cap Bi}$(μ). Let μ ∈ X, $i \in \Omega$, $\sigma_{Bi}(0) \leq \sigma_{Bi}$(μ) ⇒ $sup_{i\in\Omega}\sigma_{Bi}(0) \leq sup_{i\in\Omega}\sigma_{Bi}$(μ) ⇒ $\sigma_{\cup Bi}(0) \leq \sigma_{\cup Bi}$(μ).
- Assume μ , λ ∈ X, $i \in \Omega$, $\omega_{\cap Bi}$(μ) = inf { ω_{Bi}(μ)}
  ≥ inf{ inf {ω_{Ai}(μ ∗ λ), ω_{Ai}(μ ⋕ λ), ω_{Ai}(λ) }
  [since B_i is an I. F.P. E. I of X related to A, ∀ $i \in \Omega$] ⇒
  $\omega_{\cap Bi}$(μ) ≥ inf { $\omega_{\cap Ai}$(μ ∗ λ), $\omega_{\cap Ai}$(μ ⋕ λ), $\omega_{\cap Ai}$(λ) }.
- Assume μ , λ ∈ X, $i \in \Omega$, $\sigma_{\cup Bi}$(μ) = sup{ σ_{Bi}(μ) }
  ≤ sup{sup{ σ_{Ai}(μ ∗ λ), σ_{Ai}(μ ⋕ λ), σ_{Ai}(λ) }
  [since B_i is an I. F. P. E. I of X related to A, ∀$i \in \Omega$] ⇒
  $\sigma_{\cap Bi}$(μ) ≤ sup{ $\sigma_{\cup Ai}$(μ ∗ λ), $\sigma_{\cup Ai}$(μ ⋕ λ), $\sigma_{\cup Ai}$(λ)}.
  Then $\cap_{i\in\Omega}$ Bi is an I. F. P. E. I of X related to A. ■

3.6.**Theorem**

Assume that X is a P.BH-algebra. A = (ω_A, σ_A) & B = (ω_B, σ_B) are two I. F. S of X, such that A ⊆ B then B is an I. F. P. E. I of X related to A ⇔ the set upper level **U**(ω_B,α_1) is P. E. I of X related to **U**(ω_A,α_1) or empty of X, ∀α_1 ∈ [0,1] and the set lower level **L** (σ_B,α_2) is P. E. I of X related to **L** (σ_A,α_2) or empty of X, ∀α_2 ∈ [0,1].

Proof:- Let B = (ω_B, σ_B) be an I. F. P. E. I of X related to A & **U** (ω_B,α_1) ≠ **L** (σ_B,α_2) ≠ ∅, for every α_1, α_2 ∈ [0,1]. Obviously 0 ∈ **U** (ω_B,α_1) ∩ **L** (σ_B,α_2) since $\omega_B(0) \geq \alpha_1$ & $\sigma_B(0) \leq \alpha_2$.

Assume μ , λ ∈ X such that μ ∗ λ, μ ⋕ λ ∈ **U**(ω_A,α_1) & λ ∈ **U**(ω_A,α_1)

Then $\omega_A(\mu * \lambda) \geq \alpha_1$, $\omega_A(\mu \# \lambda) \geq \alpha_1$, and $\omega_A(\lambda) \geq \alpha_1$.

Therefore, inf {$\omega_A(\mu * \lambda)$, $\omega_A(\mu \# \lambda)$, $\omega_A(\lambda)$ } ≥ α_1, but

$\omega_B(\mu) \geq$ inf {$\omega_A(\mu * \lambda)$, $\omega_A(\mu \# \lambda)$, $\omega_A(\lambda)$ } [since B is an I. F. P. E. I of X related to A] previously, $\omega_B(\mu) \geq \alpha_1 \Rightarrow \mu$ ∈ **U**(ω_B,α_1).

Of above **U**(ω_B,α_1) is P. E. I of X. Now assume μ , λ ∈ X such that

μ ∗ λ, μ ⋕ λ ∈ **L** (σ_A,α_2) & λ ∈ **L** (σ_A,α_2) then $\sigma_A(\mu * \lambda) \leq \alpha_2$, $\sigma_A(\mu \# \lambda) \leq \alpha_2$ and $\sigma_A(\lambda) \leq \alpha_2$, therefore, sup{$\sigma_A(\mu * \lambda)$, $\sigma_A(\mu \# \lambda)$, $\omega_A(\lambda)$ } ≤ α_2 but

$\sigma_B(\mu) \leq$ sup{$\sigma_A(\mu * \lambda)$, $\sigma_A(\mu \# \lambda)$, $\omega_A(\lambda)$ }[since B is an I. F. P. E. I of X related to A] previously, $\sigma_B(\mu) \leq \alpha_2 \Rightarrow \mu$ ∈ **U**(ω_B,α_2)

then **L** (σ_B,α_2) is an P. E. I of X. Conversely, assume that α_1, α_2 ∈ [0,1] and **U** (ω_B,α_1) **&** **L** (σ_B,α_2) are P. E. I of X related to **U**(ω_A,α_1)& **L** (σ_A,α_2) respectively, ∀ μ ∈ X.

Let $\omega_B(\mu) = \alpha_1$ & $\sigma_B(\mu) = \alpha_2$ then μ ∈ **U** (ω_B,α_1) ∩ **L** (σ_B,α_2) &**U** (ω_B,α_1) ≠ **L** (σ_B,α_2) ≠ ∅ since **U** (ω_B,α_1) **&** **L** (σ_B,α_2) are P. E. I of X then 0 ∈ **U** (ω_B,α_1) ∩ **L** (σ_B,α_2) Hence [$\omega_B(0) \geq \alpha_1 = \omega_B(\mu)$] & [$\sigma_B(0) \leq \alpha_2 = \sigma_B(\mu)$], ∀ μ ∈ X, we take the opposite. Let u, v ∈ X such that, $\omega_B(u) <$ inf {$\omega_A(u * v)$, $\omega_A(u \# v)$, $\omega_A(v)$ }, now let

$\alpha_3 = \frac{1}{2}$($\omega_B(u)$ + inf {$\omega_A(u * v)$, $\omega_A(u \# v)$, $\omega_A(v)$ }),

then $\omega_B(u) < \alpha_3 <$ inf {$\omega_A(u * v)$, $\omega_A(u \# v)$, $\omega_A(v)$ }.

Hence u ∉ **U** (ω_B,α_3), $u * v$, $u \# v$ ∈ **U** (ω_A,α_3) and v ∈ **U** (ω_A,α_3), then **U** (ω_B,α_3) is not P. E. I**.** And let k, h ∈ X such that

$\sigma_B(k) >$ sup {$\sigma_A(k * h)$, $\sigma_A(k \# h)$, $\sigma_A(h)$ },

now let $\alpha_4 = \frac{1}{2}$($\sigma_B(k)$ + sup {$\sigma_A(k * h)$, $\sigma_A(k \# h)$, $\sigma_A(h)$ })

then sup {$\sigma_A(k * h)$, $\sigma_A(k \# h)$, $\sigma_A(h)$ }$< \alpha_4 < \sigma_B(k)$
Hence $k * h$, $k \# h$ ∈ **L** (σ_A,α_4) and h ∈ **L** (σ_A,α_4), but k ∉ **L** (σ_B,α_4), then **L** (σ_B,α_4) is not P. E. I. This is impossible from the assumption, therefore, B=(ω_B, σ_B) is an I. F. P. E. I of X related to A. ■

3.7. **Remark**

Assume that A = (ω_A, σ_A) is an I. F.S of X then the mappings $\acute{A}$=($\acute{\omega}_A$, $\acute{\sigma}_A$) is define as follows $\acute{\omega}_A(\mu) = \omega_A(\mu) + 1 - \omega_A(0)$ and $\acute{\sigma}_A(\mu) = \sigma_A(\mu) - \sigma_A(0)$.

3.8. **Theorem**

Assume that X is a P.BH such that $\acute{B}$ is an I. F. S of X so that $\omega_B(0) = \omega_A(0)$ & $\sigma_B(0) = \sigma_A(0)$, then B is an I. F. P. E. I of X related to A $\Leftrightarrow \acute{B}$ is an I. F. P. E. I of X related to $\acute{A}$.
Proof :- Suppose B is an I. F. P. E. I of X related to A and $\mu \in X \Rightarrow$

$\omega_B(\mu) \geq \omega_A(\mu)$, $\omega_B(0) \geq \omega_A(0)$ & $\sigma_B(\mu) \leq \sigma_A(\mu)$, $\sigma_B(0) \leq \sigma_A(0)$

$\Rightarrow \acute{\omega}_B(\mu) = \omega_B(\mu) + 1 - \omega_B(0)$ & $\acute{\omega}_A(\mu) = \omega_A(\mu) + 1 - \omega_A(0) \Rightarrow$

$\omega_B(\mu) + 1 - \omega_B(0) \geq \omega_A(\mu) + 1 - \omega_A(0)$ [since B is an I. F. P. E. I of X related to A]. Then $[\acute{\omega}_B(\mu) \geq \acute{\omega}_A(\mu)]$ &

$\acute{\sigma}_B(\mu) = \sigma_B(\mu) - \sigma_B(0)$ & $\acute{\sigma}_A(\mu) = \sigma_A(\mu) - \sigma_A(0)$ then

$\sigma_B(\mu) - \sigma_B(0) \leq \sigma_A(\mu) - \sigma_A(0)$ [since B is an I. F. P. E. I of X related to A] $\Rightarrow [\acute{\sigma}_B(\mu) \leq \acute{\sigma}_A(\mu)]$

i. $\acute{\omega}_B(0) = \omega_B(0) + 1 - \omega_B(0) \Rightarrow \acute{\omega}_B(0) = 1 \Rightarrow [\acute{\omega}_B(0) \geq \acute{\omega}_B(\mu)]$ for every $\mu \in X$. And

$\acute{\sigma}_B(0) = \sigma_B(0) - \sigma_B(0) \Rightarrow \acute{\sigma}_B(0) = 0 \Rightarrow [\acute{\sigma}_B(0) \leq \acute{\sigma}_B(\mu)]$

ii. $\acute{\omega}_B(\mu) = \omega_B(\mu) + 1 - \omega_B(0)$

$$\begin{aligned}
&\geq \inf\{\omega_A(\mu * \lambda), \omega_A(\mu \# \lambda), \omega_A(\lambda)\} + 1 - \omega_A(0) \\
&\geq \inf\{\omega_A(\mu * \lambda) + 1 - \omega_A(0), \omega_A(\mu \# \lambda) + 1 - \omega_A(0), \omega_A(\lambda) + 1 - \omega_A(0)\} \\
&\geq \inf\{\acute{\omega}_A(\mu * \lambda), \acute{\omega}_A(\mu \# \lambda), \acute{\omega}_A(\lambda)\} \quad \Rightarrow
\end{aligned}$$

$\acute{\omega}_B(\mu) \geq \inf\{\acute{\omega}_A(\mu * \lambda), \acute{\omega}_A(\mu \# \lambda), \acute{\omega}_A(\lambda)\}$

iii. $\acute{\sigma}_B(\mu) = \sigma_B(\mu) - \sigma_B(0)$

$$\begin{aligned}
&\leq \sup\{\sigma_A(\mu * \lambda), \sigma_A(\mu \# \lambda), \sigma_A(\lambda)\} - \sigma_A(0) \\
&\leq \sup\{\sigma_A(\mu * \lambda) - \sigma_A(0), \sigma_A(\mu \# \lambda) - \sigma_A(0), \sigma_A(\lambda) - \sigma_A(0)\} \\
&\leq \sup\{\acute{\sigma}_A(\mu * \lambda), \acute{\sigma}_A(\mu \# \lambda), \acute{\sigma}_A(\lambda)\} \Rightarrow
\end{aligned}$$

$\acute{\sigma}_B(\mu) \leq \sup\{\acute{\sigma}_A(\mu * \lambda), \acute{\sigma}_A(\mu \# \lambda), \acute{\sigma}_A(\lambda)\}$

Thence $\acute{B}$ is an I. F. P. E. I of X related to $\acute{A}$. Conversely,

assume that $\acute{B}$ is an I. F. P. E. I of X related to $\acute{A}$ & $\mu \in X$

$\acute{\omega}_B(\mu) \geq \acute{\omega}_A(\mu) \Rightarrow \acute{\omega}_B(0) \geq \acute{\omega}_A(0) \Rightarrow$

$\omega_B(\mu) = \acute{\omega}_B(\mu) + 1 - \omega_B(0)$ & $\omega_A(\mu) = \acute{\omega}_A(\mu) + 1 - \omega_A(0)$

$\acute{\omega}_B(\mu) + 1 - \omega_B(0) \geq \acute{\omega}_A(\mu) + 1 - \omega_A(0)$

[since $\acute{B}$ is an I. F. P. E. I of X related to $\acute{A}$] therefore, $[\omega_B(\mu) \geq \omega_A(\mu)]$ &

$\acute{\sigma}_B(\mu) \leq \acute{\sigma}_A(\mu) \Rightarrow \acute{\sigma}_B(0) \leq \acute{\sigma}_A(0) \Rightarrow$

$\sigma_B(\mu) = \acute{\sigma}_B(\mu) - \sigma_B(0)$ & $\sigma_A(\mu) = \acute{\sigma}_A(\mu) - \sigma_A(0)$

$\acute{\sigma}_B(\mu) - \sigma_B(0) \leq \acute{\sigma}_A(\mu) - \sigma_A(0)$

[since $\acute{B}$ is an I. F. P. E. I of X related to $\acute{A}$] therefore, [$\sigma_B(\mu) \leq \sigma_A(\mu)$]. Now

i. $\omega_B(0) = \acute{\omega}_B(0) - 1 + \omega_B(0) \geq \acute{\omega}_B(\mu) - 1 + \omega_A(0) = \omega_B(\mu) \Rightarrow$

[$\omega_B(0) \geq \omega_B(\mu)$] for every $\mu \in X$ &

$\sigma_B(0) = \acute{\sigma}_B(0) + \sigma_B(0) \leq \acute{\sigma}_B(\mu) + \sigma_B(0) = \sigma_B(\mu) \Rightarrow$ [$\sigma_B(0) \leq \sigma_B(\mu)$] for every $\mu \in X$.

ii. $\omega_B(\mu) = \acute{\omega}_B(\mu) - 1 + \omega_B(0) \geq \inf\{\{\acute{\omega}_A(\mu * \lambda), \acute{\omega}_A(\mu \# \lambda), \acute{\omega}_A(\lambda)\} - 1 + \omega_A(0)\}$

$\geq \inf\{\acute{\omega}_A(\mu * \lambda) - 1 + \omega_A(0), \acute{\omega}_A(\mu \# \lambda) - 1 + \omega_A(0), \acute{\omega}_A(\lambda) - 1 + \omega_A(0)\}$

$\geq \inf\{\omega_A(\mu * \lambda), \omega_A(\mu \# \lambda), \omega_A(\lambda)\} \Rightarrow$

$\omega_B(\mu) \geq \inf\{\omega_A(\mu * \lambda), \omega_A(\mu \# \lambda), \omega_A(\lambda)\}$

iii. $\sigma_B(\mu) = \acute{\sigma}_B(\mu) + \sigma_B(0) \leq \sup\{\{\acute{\sigma}_A(\mu * \lambda), \acute{\sigma}_A(\mu \# \lambda), \acute{\sigma}_A(\lambda)\} + \sigma_A(0)\}$

$\leq \sup\{\acute{\sigma}_A(\mu * \lambda) + \sigma_A(0), \acute{\sigma}_A(\mu \# \lambda) + \sigma_A(0), \acute{\sigma}_A(\lambda) + \sigma_A(0)\}$

$\leq \sup\{\sigma_A(\mu * \lambda), \sigma_A(\mu \# \lambda), \sigma_A(\lambda)\} \Rightarrow$

$\sigma_B(\mu) \leq \sup\{\sigma_A(\mu * \lambda), \sigma_A(\mu \# \lambda), \sigma_A(\lambda)\}$

Then B is an I. F. P. E. I of X related to A.

4. Conclusion

In this work, the ideas (**I.P.I** & **I.P.E.I** & **I.F.P.I**)of a **P.BH-algebra** are offered. moreover, the consequences are studied in idiom of the relationship WITH an **I.P.E.I**, **I.F.P.I** & **I.F.P.E.I** of a **P,BH- algebra.**

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Laguerre and Touchard Polynomials for Linear Volterra Integral and Integro Differential Equations

Jalil T. Abdullah[1] **and Haleema S. Ali**[2]
[1]Wasit University/ Iraq
[2] Al Karkh University of Science / Iraq

jalil.talab@gmail.com

Abstract. In this paper, efficient numerical methods are given to solve linear Volterra integral (VI) equations and Volterra Integro differential (VID) equations of the first and second types with exponential, singular, regular and convolution kernels .These methods based on Laguerre polynomials (LPs) and Touchard polynomials (TPs) that convert these equations into a system of linear algebraic equations. The results are compared with one another method and with each other. The results show that these methods are applicable and efficient. In addition, the accuracy of solution is presented and also the calculations and Graphs are done by using matlab2018 program.

Keywords: Volterra integral and integro differential equation, Laguerre polynomials, Touchard polynomials, approximate numerical solutions

الخلاصة:

في هذه الورقة البحثية، تم اعطاء طرق عددية فعالة لحل معادلات فولتيرا التكاملية والتفاضلية التكاملية الخطية من النوع الاول ثانوية اسية، منفردة، منتظمة والالتفافية. هذه الطرق التي تستند على اساس متعددتي حدود لكويروتشارد تؤدي الى والثاني مع تحويل هذه المعادلات الى نظام المعادلات الجبرية الخطية . تمت مقارنة النتائج مع طريقة اخرى واحدة ومع بعضها البعض. وتبين النتائج ان هذه الطرق قابلة للتطبيق وفعالة. بالإضافة الى ذلك، تم تقديم دقة الحل وكذلك الحسابات والرسوم البيانية تمت باستخدام برنامج الماتلاب 2018.

الكلمات المفتاحية: معادلة فولتيرا التكاملية والتفاضلية التكاملية ، كثيرات حدود لكوير ، كثيرات حدود تشارد، الحلول العددية التقريبية.

1. Introduction:

The idea of this work is to illustrate the results of the solutions for linear Volterra integral (VI) equations and linear Volterra integro differential (VID) equations in two methods using the (LPs) and (TPs). Such equations are model of problems in many applications, like, heat conduction, dynamics of viscoelastic, electrodynamics [1]. The solutions of integral and integro differential equations have an essential role in several applied areas which include “mechanics, chemistry, physics, biology, astronomy and potential theory” [2].

The general formulas of the linear (VI) equations of the 2nd and 1st types [3, 4] respectively are defined by:

$$Q(\alpha) = w(\alpha) + \gamma \int_{a}^{\alpha} Y(\alpha,\tau)\, Q(\tau)d\tau \qquad b_1 \le \alpha \le b_2 \qquad \dots (1)$$

$$-w(\alpha) = \gamma \int_a^{\alpha} Y(\alpha, \tau)\, Q(\tau) d\tau \quad . \qquad b_1 \le \alpha \le b_2 \qquad \dots (1a)$$

Also the general formula of linear Abel's singular of the 1st type [4, 5 and 6] is defined as follows:

$$w(\alpha) = \gamma \int_a^{\alpha} \frac{1}{\sqrt{\alpha - \tau}}\, Q(\tau) d\tau \quad , \qquad b_1 \le \alpha \le b_2 \qquad \dots (2)$$

The general formula of the linear (VID) equation of the 1st order and 2nd type [4] is defined as follows:

$$Q'(\alpha) = w(\alpha) + \gamma \int_0^{\alpha} Y(\alpha, \tau)\, Q(\tau) d\tau, \quad , \quad b_1 \le \alpha \le b_2, \qquad \dots (3)$$

with initial condition $Q(0) = Q_0$, ... (3a)

where $Q'(\alpha) = \frac{dQ}{d\alpha}$, b_1, b_2 are constants, Q (α) is the unknown function that must be determined, γ is a known constant, it represents the physical meaning of the material, and Y (α, τ) is a kernel of the Integral equations (IEs), which is a known continuous or dis-continuous function holds characteristic or property of the material, w (α) is a known function represents the integration surface and $Q(0) = Q_0$ is a constant initial condition for eq. (3).

There are many approximate numerical methods used and developed by the scientific researchers to obtain the approximate numerical solutions for the (VI) equations and (VID) equations, mentioned as follows: [7] proposed numerical methods to solve weakly (VI) equations of the 1st type. [8] gave numerical method for the approximation of the (VI) equations with oscillatory Bessel kernels. [9] applied Chebyshev wavelet method to solve the (VI) equations with weakly singular of kernels. [10] used the standard Galerkin polynomial method to solve weakly singular kernels for the (VI) equations. [11] extended the single step pseudo spectral method to the multi step pseudo spectral method for the (VI) equations of 2nd type. [12] applied the Galerkin weight residual method and (LPs) as a trial function for solving the (VI) equations of the 1st, 2nd type with singular and regular kernels. [13] used the (LPs) for solving system of generalized Abel integral equations. [14] used iterative methods to solve the (VID) equations with singular kernel. [15] applied collocation method to solve the (VID) equations. [16] applied "Galerkin the weight residual method" with the (TPs) as a trial function to get numerical solutions to (IEs).

This article is arranged as follows: Laguerre polynomials, function of approximation using the (LPs), Touchard polynomials, function of approximation using the (TPs), solution the (VI) equation using the (LPs) method, accuracy of solutions, convergence rate, illustrative examples, tables and figures are provided, summary of conclusions and recommendations. Finally the references are mentioned.

2. Laguerre Polynomials [12 and 13]:

This section, begin with definition of the (LPs) which was studied in 1782 by Adrien-Marie Legendre. The (LPs) consisting of the polynomial sequence of binomial type, it's defined on [0, ∞) as follows:

$$V_k(\alpha) = \sum_{s=0}^{k} (-1)^s \frac{1}{s!} \binom{k}{s} \alpha^s = \sum_{s=0}^{k} \frac{(-1)^s}{(s!)^2 (k-s)!} \alpha^s \quad , k = 0, 1, 2, \dots n \text{ and } \alpha \in [0, \infty) \quad \cdots (4)$$

where k and s represent the degree and the index for the (LPs) respectively.

The first five polynomials of the (LPs) are given below:

$V_0(\alpha) = 1$

$V_1(\alpha) = 1 - \alpha$

$V_2(\alpha) = \frac{1}{2}(2 - 4\alpha + \alpha^2).$

$V_3(\alpha) = \frac{1}{6}(6 - 18\alpha + 9\alpha^2 - \alpha^3)$

$V_4(\alpha) = \frac{1}{24}(24 - 96\alpha + 72\alpha^2 - 16\alpha^3 + \alpha^4)$

3. Function of Approximation using the (LPs):

Suppose that the function Q_k (α) is approximated using the (LPs) as follows:

$$Q_k(\alpha) = \vartheta_0 V_0(\alpha) + \vartheta_1 V_1(\alpha) + \cdots + \vartheta_k V_k(\alpha) = \sum_{s=0}^{k} \vartheta_s V_s(\alpha) \quad 0 \le \alpha < \infty , \qquad \cdots (5)$$

for s≥0, the function $\{V_s(\alpha)\}_{s=0}^{k}$ denotes the Laguerre basis polynomials of kth degree, as defined in Eq. (4). ϑ_s $(s = 0,1,\dots,k)$ are the unknowns Laguerre coefficients that calculate later.

Writing Eq. (5) as a dot product:

$$Q_k(\alpha) = [V_0(\alpha) \ \ V_1(\alpha) \dots V_k(\alpha)] \cdot \begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ . \\ . \\ . \\ \vartheta_k \end{bmatrix}, \qquad \cdots (6)$$

Eq. (6) can be written in the following form:

$$Q_k(\alpha) = \begin{bmatrix} 1 & \alpha & \alpha^2 \dots \alpha^k \end{bmatrix} . \begin{bmatrix} \theta_{00} & \theta_{01} & \theta_{02} & \dots & \theta_{0k} \\ 0 & \theta_{11} & \theta_{12} & \cdots & \theta_{1k} \\ 0 & 0 & \theta_{22} & \cdots & \theta_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \theta_{kk} \end{bmatrix} . \begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ . \\ . \\ . \\ \vartheta_k \end{bmatrix}, \qquad \cdots (7)$$

where $\theta_{\rho\rho}$ $(\rho = 0,1,2,\dots,k)$ are known values of the power basis that are used to find the (LPs), also the square matrix is an upper triangular and non-singular. For example, if k= 1, and 2, the operational matrices are shown as in Eqs. (8) and (9) respectively:

$$Q_1(\alpha) = [1\ \ \alpha\ ].\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}.\begin{bmatrix}\vartheta_0\\ \vartheta_1\end{bmatrix} \qquad \ldots (8)$$

$$Q_2(\alpha) = [1\ \ \alpha\ \ \alpha^2].\begin{bmatrix}1 & 1 & 1\\ 0 & -1 & -2\\ 0 & 0 & 1/2\end{bmatrix}.\begin{bmatrix}\vartheta_0\\ \vartheta_1\\ \vartheta_2\end{bmatrix}. \qquad \ldots (9)$$

Since the derivative of Eq. (4) is:

$$V'_k(\alpha) = \frac{d}{d\alpha}\sum_{s=0}^{k}(-1)^s\frac{1}{s!}\binom{k}{s}\alpha^s = \sum_{s=1}^{k}\frac{(-1)^s}{(s!)^2(k-s)!}\, s\, \alpha^{s-1}\ , k = 1, 2, \ldots n, \text{and } \alpha \in [0, \infty\,) \quad \cdots (10\,)$$

so, the derivative of Eqs. (7), (8) and (9) is respectively:

$$Q'_k(\alpha) = [0\ \ 1\ \ \alpha \ldots k\alpha^{k-1}].\begin{bmatrix}\theta_{00} & \theta_{01} & \theta_{02} & \ldots\ \theta_{0k}\\ 0 & \theta_{11} & \theta_{12} & \cdots\ \theta_{1k}\\ 0 & 0 & \theta_{22} & \cdots \theta_{2k}\\ \vdots & \vdots & \vdots\ \ddots & \vdots\\ 0 & 0 & 0 & \cdots \theta_{kk}\end{bmatrix}.\begin{bmatrix}\vartheta_0\\ \vartheta_1\\ .\\ .\\ .\\ \vartheta_k\end{bmatrix}, \qquad \cdots (10a)$$

$$Q'_1(\alpha) = [0\ \ 1\ ].\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}.\begin{bmatrix}\vartheta_0\\ \vartheta_1\end{bmatrix} \qquad \ldots (10b)$$

$$Q'_2(\alpha) = [0\ \ 1\ \ 2\alpha].\begin{bmatrix}1 & 1 & 1\\ 0 & -1 & -2\\ 0 & 0 & 1/2\end{bmatrix}.\begin{bmatrix}\vartheta_0\\ \vartheta_1\\ \vartheta_2\end{bmatrix}. \qquad \ldots (10c)$$

4. Touchard Polynomials [16, 17, 18 and 19]:

(TPs) were first studied by the French mathematician Jacques Touchard 1885–1968, consisting of the polynomial sequence of binomial type, it's defined on [0, 1] as following:

$$O_k(\alpha) = \sum_{s=0}^{k} A(k,s)\alpha^s = \sum_{s=0}^{k}\binom{k}{s}\alpha^s\ , \quad \binom{k}{s} = \frac{k!}{s!\,(k-s)!}\ , \qquad \ldots (11)$$

where k and s represent the degree and the index for the (TPs) respectively.
The first five polynomials of the (TPs) are written below:

$O_0(\alpha) = 1$
$O_1(\alpha) = 1 + \alpha$
$O_2(\alpha) = 1 + 2\alpha + \alpha^2$
$O_3(\alpha) = 1 + 3\alpha + 3\alpha^2 + \alpha^3$
$O_4(\alpha) = 1 + 4\alpha + 6\alpha^2 + 4\alpha^3 + \alpha^4.$

5. Function of Approximation using the (TPs):

Suppose that the function $Q_k\,(\alpha)$ is approximated using the (TPs) as follows:

$$Q_k(\alpha) = \vartheta_0 O_0(\alpha) + \vartheta_1 O_1(\alpha) + \cdots + \vartheta_k O_k(\alpha) \ = \sum_{s=0}^{k}\vartheta_s O_s(\alpha)\,, 0 \le \alpha \le 1 \qquad \ldots (12)$$

for s≥0, the function $\{O_s(\alpha)\}_{s=0}^{k}$ denotes the Touchard basis polynomials of kth degree, as defined in Eq. (11). ϑ_s $(s = 0,1, \ldots, k)$ are the unknowns Touchard coefficients that determine later.

Writing Eq. (12) as a dot product:

$$Q_K(\alpha) = [O_0(\alpha) \;\; O_1(\alpha) \dots O_k(\alpha)] \cdot \begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ \cdot \\ \cdot \\ \cdot \\ \vartheta_k \end{bmatrix}, \qquad \dots (13)$$

Eq. (13) can be written as follows:

$$Q_k(\alpha) = \left[1 \;\; \alpha \;\; \alpha^2 \dots \alpha^k\right] \cdot \begin{bmatrix} \varepsilon_{00} & \varepsilon_{01} & \varepsilon_{02} & \dots & \varepsilon_{0k} \\ 0 & \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{1k} \\ 0 & 0 & \varepsilon_{22} & \cdots & \varepsilon_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \varepsilon_{kk} \end{bmatrix} \cdot \begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ \cdot \\ \cdot \\ \cdot \\ \vartheta_k \end{bmatrix}, \qquad \dots (14)$$

where $\varepsilon_{\rho\rho}$ (ρ= 0, 1, 2,…, k) are known constants of the power basis that are used to find the (TPs), also the square matrix is an upper triangular and non-singular. For instance, if k=2 and 3, the operational matrices are shown in Eqs. (15) and (16) respectively:

$$Q_2(\alpha) = [1 \;\; \alpha \;\; \alpha^2] \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ \vartheta_2 \end{bmatrix}, \qquad \dots (15)$$

$$Q_3(\alpha) = [1 \;\; \alpha \;\; \alpha^2 \; \alpha^3] \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \end{bmatrix}. \qquad \dots (16)$$

Since, the derivative of Eq. (11) is:

$$O'_k(\alpha) = \frac{d}{d\alpha} \sum_{s=0}^{k} A(k,s)\alpha^s = \sum_{s=1}^{k} \binom{k}{s} s\alpha^{s-1}, \text{where } \binom{k}{s} = \frac{k!}{s!\,(k-s)!}, \qquad \dots (17)$$

then, the derivative of Eqs. (14), (15) and (16) respectively is:

$$Q'_k(\alpha) = \left[0 \;\; 1 \;\; 2\alpha \dots k\alpha^{k-1}\right] \cdot \begin{bmatrix} \varepsilon_{00} & \varepsilon_{01} & \varepsilon_{02} & \dots & \varepsilon_{0k} \\ 0 & \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{1k} \\ 0 & 0 & \varepsilon_{22} & \cdots & \varepsilon_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \varepsilon_{kk} \end{bmatrix} \cdot \begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ \cdot \\ \cdot \\ \cdot \\ \vartheta_k \end{bmatrix}, \qquad \dots (17a)$$

$$Q'_2(\alpha) = [0 \;\; 1 \;\; 2\alpha] \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ \vartheta_2 \end{bmatrix}, \qquad \dots (17b)$$

$$Q'_3(\alpha) = [0 \;\; 1 \;\; 2\alpha \; 3\alpha^2] \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \end{bmatrix}. \qquad \dots (17c)$$

6. Solution the (VI) Equation of the 2^{nd} type using the (LPs):

In this section, the (LPs) is used to find the solutions for the (VI) equation. Since Eq. (1) is:

$$Q(\alpha) = w(\alpha) + \gamma \int_a^{\alpha} Y(\alpha,\tau)\, Q(\tau) d\tau\ , \qquad b_1 \le \alpha \le b_2 \qquad \dots (18)$$

by using Eq. (5), suppose that:

$$Q(\alpha) \cong Q_k(\alpha) = \sum_{s=0}^{k} \vartheta_s V_s(\alpha), \qquad \dots (19)$$

now, substituting Eq. (19) into Eq. (18), gives:

$$\sum_{s=0}^{K} \vartheta_s V_s(\alpha) = w(\alpha) + \gamma \int_a^{\alpha} Y(\alpha,\tau) \sum_{s=0}^{K} \vartheta_s V_s(\tau)\, d\tau\,, \qquad \dots (20)$$

By using Eq. (6), then Eq. (20) becomes:

$$[V_0(\alpha)\ \ V_1(\alpha) \dots V_k(\alpha)].\begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ . \\ . \\ . \\ \vartheta_k \end{bmatrix} = w(\alpha) + \gamma \int_a^{\alpha} Y(\gamma,\tau)[V_0(\tau)\ \ V_1(\tau) \dots V_k(\tau)].\begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ . \\ . \\ . \\ \vartheta_k \end{bmatrix} d\tau\ \ , \dots (21)$$

And by using Eq. (7), so, Eq. (21) is converted to:

$$\begin{aligned} &[1\ \ \alpha\ \ \alpha^2 \dots \alpha^k].\begin{bmatrix} \theta_{00} & \theta_{01} & \theta_{02} & \dots & \theta_{0k} \\ 0 & \theta_{11} & \theta_{12} & \cdots & \theta_{1k} \\ 0 & 0 & \theta_{22} & \cdots & \theta_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \theta_{kk} \end{bmatrix}.\begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ . \\ . \\ . \\ \vartheta_k \end{bmatrix} \\ &= w(\alpha) \\ &+ \gamma \int_a^{\alpha} Y(\alpha,\tau)\, [1\ \ \tau\ \ \tau^2 \dots \tau^k].\begin{bmatrix} \theta_{00} & \theta_{01} & \theta_{02} & \dots & \theta_{0k} \\ 0 & \theta_{11} & \theta_{12} & \cdots & \theta_{1k} \\ 0 & 0 & \theta_{22} & \cdots & \theta_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \theta_{kk} \end{bmatrix}.\begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ . \\ . \\ . \\ \vartheta_k \end{bmatrix} d\tau \dots (22) \end{aligned}$$

Now, after simplifying Eq. (22), the unknown Laguerre coefficients ($\vartheta_0\ ,\ \vartheta_1\ , \dots, \vartheta_k$) are obtained by selecting points α_β ($\beta = 0, 1, \dots, k$) in the interval [b_1,b_2]. Consequently, Eq. (22) converts to a system of (k+1) linear algebraic equations in (k+1) unknown coefficients, this system can be solved using "Gauss elimination method" to obtain these coefficients, which have the unique solutions. These coefficients are substituted into Eq. (5), to get the approximate numerical solution for Eq. (1).

The same procedure can be applied to Eqs. (1a) and (2) when using the (TPs).

7. Solution the (VID) Equation of the 1st order and 2nd type using the (LPs):

In this section, the (TPs) is used to find the solutions for the (VID) equation. Since Eq. (3) is:

$$Q'(\alpha) = w(\alpha) + \gamma \int_0^{\alpha} Y(\alpha,\tau)\, Q(\tau) d\tau, \quad , \ \ b_1 \le \alpha \le b_2, \qquad \dots (23)$$

$$Q(0) = Q_0, \qquad \dots (23a)$$

by using Eqs.(7) and (10a), suppose that:

$$Q(\alpha) \cong Q_k(\alpha) = \left[1 \ \alpha \ \alpha^2 \dots \alpha^k\right].\begin{bmatrix} \theta_{00} & \theta_{01} & \theta_{02} & \dots \theta_{0k} \\ 0 & \theta_{11} & \theta_{12} & \cdots \theta_{1k} \\ 0 & 0 & \theta_{22} & \cdots \theta_{2k} \\ \vdots & \vdots & \vdots \ddots & \vdots \\ 0 & 0 & 0 & \cdots \theta_{kk} \end{bmatrix}.\begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ . \\ . \\ . \\ \vartheta_k \end{bmatrix}, \qquad \cdots (24)$$

$$Q'(\alpha) \cong Q'_k(\alpha) = \left[0 \ 1 \ \alpha \dots k\alpha^{k-1}\right].\begin{bmatrix} \theta_{00} & \theta_{01} & \theta_{02} & \dots \theta_{0k} \\ 0 & \theta_{11} & \theta_{12} & \cdots \theta_{1k} \\ 0 & 0 & \theta_{22} & \cdots \theta_{2k} \\ \vdots & \vdots & \vdots \ddots & \vdots \\ 0 & 0 & 0 & \cdots \theta_{kk} \end{bmatrix}.\begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ . \\ . \\ . \\ \vartheta_k \end{bmatrix}, \qquad \cdots (25)$$

now, by substituting Eqs. (24) and (25) into Eq. (23), gives:

$$\left[1 \ \alpha \ \alpha^2 \dots \alpha^k\right].\begin{bmatrix} \theta_{00} & \theta_{01} & \theta_{02} & \dots \theta_{0k} \\ 0 & \theta_{11} & \theta_{12} & \cdots \theta_{1k} \\ 0 & 0 & \theta_{22} & \cdots \theta_{2k} \\ \vdots & \vdots & \vdots \ddots & \vdots \\ 0 & 0 & 0 & \cdots \theta_{kk} \end{bmatrix}.\begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ . \\ . \\ . \\ \vartheta_k \end{bmatrix}$$
$$= w(\alpha)$$
$$+ \gamma \int_a^\alpha Y(\alpha,\tau)\left[0 \ 1 \ \tau \dots k\tau^{k-1}\right].\begin{bmatrix} \theta_{00} & \theta_{01} & \theta_{02} & \dots \theta_{0k} \\ 0 & \theta_{11} & \theta_{12} & \cdots \theta_{1k} \\ 0 & 0 & \theta_{22} & \cdots \theta_{2k} \\ \vdots & \vdots & \vdots \ddots & \vdots \\ 0 & 0 & 0 & \cdots \theta_{kk} \end{bmatrix}\begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ . \\ . \\ . \\ \vartheta_k \end{bmatrix} d\tau \ \dots (26)$$

So, after simplifying Eq. (26), the unknown Touchard coefficients (ϑ_0 , ϑ_1 , ..., ϑ_k) are obtained by selecting points α_β ($\beta = 0, 1, \dots, k$) in the interval [b_1,b_2], with the initial condition Eq. (23a). Therefore, Eq. (26) converts to a system of (k+1) linear algebraic equations in (k+1) unknown coefficients, this system can be solved using “Gauss elimination method” to obtain theses coefficients, which have unique solutions. These coefficients are substituted into Eq. (5), to get the approximate numerical solution for Eq. (3).
The same procedure can be applied when using the (TPs).

8. Accuracy of Solutions:
In this section, the accuracy of the proposed methods is tested.

8.1: For the (VI) equation:

Since Eq. (20) has the following formula:

$$\sum_{s=0}^{k} \vartheta_s V_s(\alpha) = w(\alpha) + \gamma \int_a^\alpha Y(\alpha,\tau) \sum_{s=0}^{k} \vartheta_s V_s(\tau)\, d\tau , \qquad \dots (27)$$

Since Eq. (5) has the following form:

$$Q_k(\alpha) = \sum_{s=0}^{k} \vartheta_s V_s(\alpha),$$

And the unknown Laguerre coefficients $(\vartheta_0, \vartheta_1, \dots, \vartheta_k)$ were determined by using Eq. (22). Also, by using Eq. (19), we have:

$$Q(\alpha) \cong Q_k(\alpha) = \sum_{s=0}^{k} \vartheta_s V_s(\alpha), \qquad \dots (28)$$

then, Eq. (28) is the unique approximate solution for Eq. (27), and it's substituted into Eq. (27). Now, suppose that $\alpha = \alpha_\theta \in [0, 1]$, $\theta = 0, 1, 2, \dots, k$, and then, the error function:

$$AR\,(\alpha_\theta) = \left| \sum_{s=0}^{k} \vartheta_s\, V_s(\alpha_\theta) - w(\alpha_\theta) - \gamma \int_a^{\alpha} Y(\alpha_\theta, \tau) \sum_{s=0}^{k} \vartheta_s\, V_s(\tau_\theta) d\tau \right| \cong 0\,, \text{then}$$

$AR\,(\alpha_\theta) \leq \epsilon$, for each α_θ in [0, 1] and $\epsilon > 0$.

Then, the difference for error function $AR\,(\alpha_\theta)$ at each point α_θ will be smaller than any positive integer $\epsilon > 0$. Thus, the error function $AR\,(\alpha)$ can be estimated using the relation:

$$AR_k\,(\alpha) = \sum_{s=0}^{k} \vartheta_s\, V_s(\alpha) - w(\alpha) - \gamma \int_a^{\alpha} Y(\alpha, \tau) \sum_{s=0}^{k} \vartheta_s\, V_s(\tau) d\tau,$$

then, $AR_k\,(\alpha) \leq \epsilon$.

This procedure is suitable for Eqs. (1a) and (2). Also this procedure can be applied using the (TPs).

8.2 For the (VID) equation:

Since Eq. (3) with initial condition is:

$$Q'(\alpha) = w(\alpha) = \gamma \int_0^{\alpha} Y(\alpha, \tau)\, Q(\tau) d\tau, \quad , \; b_1 \leq \alpha \leq b_2, \qquad \dots (29)$$

$$Q(0) = Q_0,$$

since Eq. (5) has the following form:

$$Q_k(\alpha) = \sum_{s=0}^{k} \vartheta_s V_s(\alpha),$$

and the unknown Laguerre coefficients $(\vartheta_0, \vartheta_1, \dots, \vartheta_k)$ were determined by using Eq. (26). Also, by using Eq. (19), we have:

$$Q(\alpha) \cong Q_k(\alpha) = \sum_{s=0}^{k} \vartheta_s V_s(\alpha), \qquad \dots (30)$$

is the approximate numerical solution for Eq. (29) also, Eq. (30) and its derivative is substituted into Eq. (29). Now, suppose that $\alpha = \alpha_\theta \in [0, 1]$, $\theta = 0, 1, 2, \dots, k$, and then, the error function:

$$AR\,(\alpha_\theta) = \left| \left(\sum_{s=0}^{k} \vartheta_s V_s(\alpha_\theta) \right)' - w(\alpha_\theta) - \gamma \int_a^{\alpha} Y(\alpha_\theta, \tau_\theta) \sum_{s=0}^{k} \vartheta_s\, V_s(\tau_\theta) d\tau_\theta \right| \cong 0\,, \text{then}$$

$AR\,(\alpha_\theta) \leq \epsilon$, for each α_θ in [0, 1] and $\epsilon > 0$.

Then, the difference for error function $AR\,(\alpha_\theta)$ at each point α_θ will be smaller than any positive integer $\epsilon > 0$.

Thus, the error function $AR\,(\alpha_\theta)$ can be estimated using the relation:

$$AR_k(\alpha) = \left(\sum_{s=0}^{k} \vartheta_s V_s(\alpha_\theta)\right)' - w(\alpha) - \gamma \int_a^\alpha Y(\alpha,\tau) \sum_{s=0}^{k} \vartheta_s V_s(\tau)d\tau,$$

then, $AR_k(\alpha) \le \epsilon$.
Note: This procedure can be applied using the (TPs) for Eq. (3).

9. Convergence Rate:
In this section, the error function can be defined by the following relation [20]:

$$\|AR_k(\alpha)\| = \left(\int_0^1 AR_k^2(\alpha)d\alpha\right)^{1/2} \cong \left(\frac{1}{k}\sum_{s=0}^{k} ER_s^2(\alpha_s)\right)^{1/2},$$

where $\|AR_k(\alpha)\|$ is an arbitrary vector norm of error function,
$AR_k(\alpha) = Q(\alpha) - Q_k(\alpha)$, where $Q(\alpha)$ and $Q_k(\alpha)$, are the exact and approximate numerical solutions respectively.

10. Illustrative Examples:
In this section, the (LPs) and (TPs) are used to solve linear (VI) and (VID) equations. These two polynomials have been applied to six numerical examples, and the convergence of solutions using the error function is given.

Example 1: Solve the linear (VI) equation of 1st type with the exponential kernel [20]:

$$\int_0^\alpha e^{(\alpha-\tau)} Q(\tau)\, d\tau = \sin(\alpha), \quad \alpha \in [0,1],$$

where $Q(\alpha) = \cos(\alpha) - \sin(\alpha)$ is the exact solution.

For k = 2, 3, 4, 5 and 6, the approximate results using:
1. The (LPs) are:
$Q_2(\alpha) = -0.8489V_0(\alpha) + 2.6884V_1(\alpha) - 0.8392V_2(\alpha)$.
$Q_3(\alpha) = 0.1621V_0(\alpha) - 0.4997V_1(\alpha) + 2.5137V_2(\alpha) - 1.1761V_3(\alpha)$.
$Q_4(\alpha) = 0.7515V_0(\alpha) - 2.9774V_1(\alpha) + 6.4215V_2(\alpha) - 3.9166V_3(\alpha) + 0.7210V_4(\alpha)$.
$Q_5(\alpha) = -0.2178V_0(\alpha) + 2.1162V_1(\alpha) - 4.2887V_2(\alpha) + 7.3473V_3 - 5.2041V_4 + 1.2472V_5$.
$Q_6(\alpha) = -0.6581\, V_0 + 4.8926V_1 - 11.5853\, V_2 + 17.5776V_3 - 13.2746V_4 + 4.6438V_5 - 0.5958V_6$
2. The (TPs) are:
$Q_2(\alpha) = 1.5906\, O_0(\alpha) - 0.1707\, O_1(\alpha) - 0.4196 O_2(\alpha)$.
$Q_3(\alpha) = 1.2960\, O_0(\alpha) + 0.6034\, O_1(\alpha) - 1.0954\, O_2(\alpha) + 0.1960\, O_3(\alpha)$.
$Q_4(\alpha) = 1.3568\, O_0(\alpha) + 0.3984\, O_1(\alpha) - 0.8371 O_2(\alpha) + 0.0519\, O_3(\alpha) + 0.0300\, O_4(\alpha)$.
$Q_5(\alpha) = 1.3872\, O_0(\alpha) + 0.2747\, O_1(\alpha) - 0.6368\, O_2(\alpha) - 0.1097\, O_3 + 0.0950\, O_4 - 0.0104 O_5$.
$Q_6 = 1.3835 O_0 + 0.2923 O_1 - 0.6713 O_2 - 0.0738 O_3 + 0.0741 O_4 - 0.0039 O_5 - 8.2750E-4\, O_6$.

The solutions were approximated in five different degrees. The comparison of error functions of the proposed methods and those in [20] is shown in Table 1, showing the (LPs) and the (TPs) methods having a higher accuracy than in [20] with the same degrees, and that both proposed methods having the same accuracy.
Figure 1 shows the comparison of result for k=6 with exact solution. They seem to be identical.

Table1. Comparison of the Error Function of Example1.

k	$\|\|AR_k\|\|$		
	Method in [20]	(LPs) Method	(TPs) Method
2	5.06401E−02	1.1940E−01	1.1940E−01
3	2.07936E−03	6.7190E−03	6.7191E−03
4	6.14967E−04	1.2897E−03	1.2897E−03
5	1.42477E−04	3.3898E−05	3.3775E−05
6	5.41139E−05	3.7027E−06	3.5964E−06

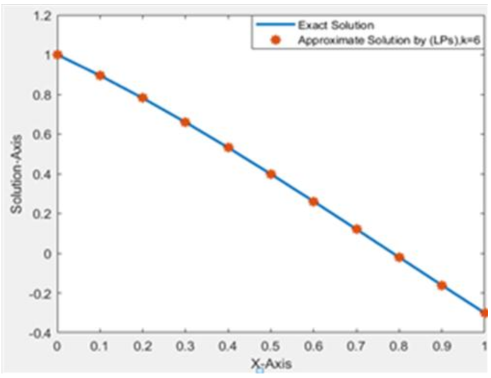

Figure 1(a). The (LPs) of Example1 for k=6

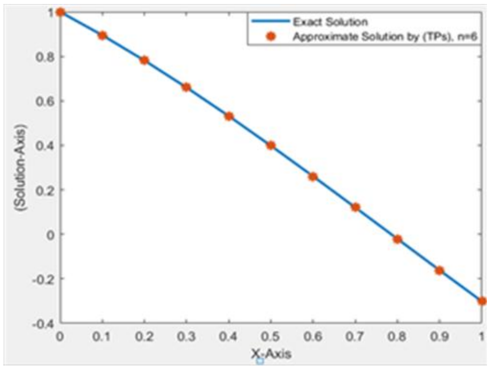

Figure 1(b). The (TPs) of Example1 for k=6

Example 2: Solve the Abel's (IEs) (linear (VI) equation of 1st type with singular kernel) [20]:

$$\int_0^{\alpha} \frac{1}{\sqrt{\alpha-\tau}}\, Q(\tau)d\tau = \frac{2\sqrt{\alpha}}{105}(105 - 56\alpha^2 + 48\alpha^3), \quad \alpha \in [0,1],$$

where $Q(\alpha) = \alpha^3 - \alpha^2 + 1$ is the exact solution.

For k = 2, 3 and 4, the approximate results using:

1. The (LPs) are:

$Q_2(\alpha) = -0.0441V_0(\alpha) + 2.0183V_1(\alpha) - 0.9714V_2(\alpha).$

$Q_3(\alpha) = 5V_0(\alpha) - 14V_1(\alpha) + 16V_2(\alpha) - 6V_3(\alpha).$

$Q_4(\alpha) = 5V_0(\alpha) - 14V_1(\alpha) + 16V_2(\alpha) - 6V_3(\alpha) + 4.5324E - 12V_4(\alpha).$

2. The (TPs) are:

$Q_2(\alpha) = \ 0.5925\, O_0(\alpha) + 0.8960\, O_1(\alpha) - 0.4857 O_2(\alpha).$

$Q_3(\alpha) = \ -O_0(\alpha) + \ 5\, O_1(\alpha) - 4\, O_2(\alpha) + O_3(\alpha).$

$Q_4(\alpha) = -O_0(\alpha) + 5\, O_1(\alpha) - 4\, O_2(\alpha) + O_3(\alpha) + 1.8885E - 13\, O_4(\alpha).$

The solutions were approximated in three different degrees. The comparison of error functions of the proposed methods and those in [20] is shown in Table 2, showing the (LPs) and (TPs) methods having

a higher accuracy than in [20] with the same degrees, and that both proposed methods having the same accuracy.
Figure 2 shows the comparison of result for k=4 with exact solution. They seem to be identical.

Table2. Comparison of the Error Function of Example 2.

k	$\|AR_k\|$		
	Method of [20]	(LPs) Method	(TPs) Method
2	6.39819E−02	5.1892E−01	5.1892E−01
3	2.42366E−02	4.2274E−07	5.4209E−07
4	3.26226E−03	3.6611E−07	4.6947E−07

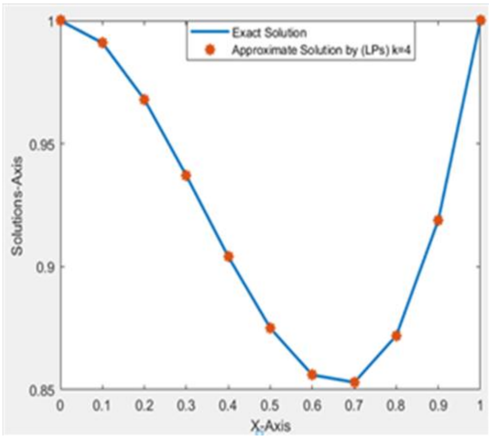

Figure 2(a). The (LPs) of Example 2

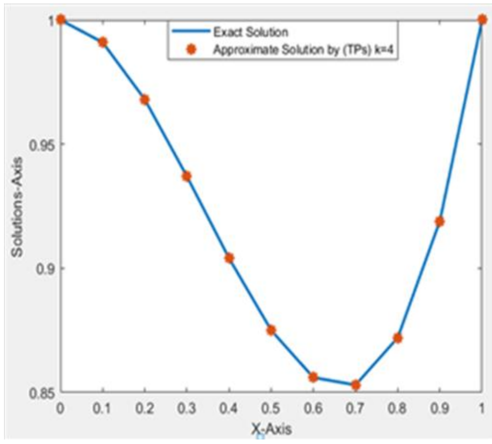

Figure 2(b). The (TPs) of Example 2 for

Example 3: Solve the linear (VI) equation of 2nd type with the regular kernel [4]:

$$Q(\alpha) = \alpha + \alpha^4 + \frac{1}{2}\alpha^2 + \frac{1}{5}\alpha^5 - \int_0^{\alpha} Q(\tau)\, d\tau \quad , \quad \alpha \in [0,1],$$

where the exact solution is $Q(\alpha) = \alpha + \alpha^4$.
For k = 2, 3, 4, 5 and 6, the approximate results using:
1. The (LPs) are:
$Q_2(\alpha) = 1.4402\ V_0(\alpha) - 1.9327\ V_1(\alpha) + 0.4959\ V_2(\alpha).$
$Q_3(\alpha) = 6.3321\ V_0(\alpha) - 17.5993\ V_1(\alpha) + 17.2449\ V_2(\alpha) - 5.9799\ V_3(\alpha).$
$Q_4(\alpha) = 25\ V_0(\alpha) - 97\ V_1(\alpha) + 144\ V_2(\alpha) - 96V_3(\alpha) + 24\ V_4(\alpha).$
$Q_5(\alpha) = 25V_0(\alpha) - 97V_1(\alpha) + 144V_2(\alpha) - 96V_3(\alpha) + 24V_4(\alpha) - 4.2929E - 11V_5(\alpha).$
$Q_6(\alpha) = 25V_0 - 97V_1 + 144V_2 - 96V_3 + 24V_4 + 4.4567E - 09V_5 - 7.9628E - 10V_6$
2. The (TPs) are:
$Q_2(\alpha) = -0.68955\ O_0(\alpha) + 0.44500\ O_1(\alpha) + 0.24796\ O_2(\alpha).$

$Q_3(\alpha) = -2.3957\, O_0(\alpha) + 4.7342\, O_1(\alpha) - 3.3374\, O_2(\alpha) + 0.99666\, O_3(\alpha).$
$Q_4(\alpha) = -2.8232E-13\, O_0(\alpha) - 3\, O_1(\alpha) + 6\, O_2(\alpha) - 4\, O_3(\alpha) + O_4(\alpha).$
$Q_5(\alpha) = -1.5673E-12\, O_0(\alpha) - 3\, O_1(\alpha) + 6\, O_2(\alpha) - 4\, O_3(\alpha) + O_4(\alpha) + 3.5774E-13\, O_5(\alpha).$
$Q_6(\alpha) = -7.9198E-12\, O_0 - 3\, O_1 + 6\, O_2 - 4\, O_3 + O_4 + 9.3103E-12\, O_5 - 1.1059E-12\, O_6$

The solutions were approximated in five different degrees. The comparison of error functions of the (LPs) method and those in the (TPs) method is shown in Table 3, showing the (TPs) method having a higher accuracy than in the (LPs) method with the same degrees. Figure 3 shows the comparison of result for k=6 with exact solution. They seem to be identical.

Table 3. Comparison of the Error Function of the (LPs) and (TPs) of Example 3.

k	$\|AR_k\|$	
	(LPs) Method	(TPs) Method
2	7.1586E−01	7.1586E−01
3	2.0767E−01	2.0767E−01
4	3.3525E−06	1.2191E−06
5	2.9986E−06	1.0904E−06
6	2.7373E−06	9.9539E−07

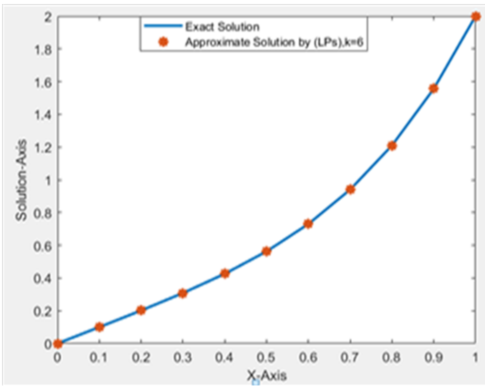

Figure 3(a). The (LPs) of Example 4 for k=6

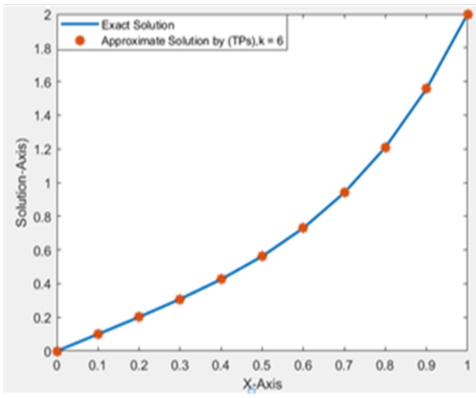

Figure 3(b). The (TPs) of Example 4 for k=6

Example 4: Solve the linear (VI) equation of 2nd type with the convolution kernel [4]:

$$\mathbf{Q(\alpha) = \alpha + \int_0^{\alpha} (\tau - \alpha)Q(\tau)d\tau\,, \alpha \in [0, 1],}$$

where $\mathbf{Q(\alpha) = \sin(\alpha)}$ is the exact solution.
For k= 2, 3, 4 and 5, the approximate results using:

1. The (LPs) are:
$\mathbf{Q_2(\alpha) = 0.8187\, V_0(\alpha) - 0.6212\, V_1(\alpha) - 0.1985\, V_2(\alpha).}$

$$\mathbf{Q_3(\alpha) = \ 0.0277V_0(\alpha) + 1.9126\,V_1(\alpha)\ -2.9081V_2(\alpha) + 0.9677\,V_3(\alpha).}$$
$$\mathbf{Q_4(\alpha) = 0.2570V_0(\alpha) + 0.9372V_1(\alpha) - 1.3506V_2(\alpha) - 0.1387V_3(\alpha) + 0.2950V_4(\alpha)}$$
$$\mathbf{Q_5(\alpha) = 0.9497V_0(\alpha) - 2.7377V_1(\alpha) + 6.4523V_2(\alpha) - 8.4278V_3(\alpha) + 4.7009V_4 - 0.9374V_5.}$$

2. The (TPs) are:

$$\mathbf{Q_2(\alpha) = -1.1184\,O_0(\alpha) + 1.2167\,O_1(\alpha) - 0.0993\,O_2(\alpha).}$$
$$\mathbf{Q_3(\alpha) = \ -0.8416\,O_0(\alpha) + \ 0.5215\,O_1(\alpha) + 0.4814\,O_2(\alpha)\ - 0.1613\,O_3(\alpha).}$$
$$\mathbf{Q_4(\alpha) = -0.8121O_0(\alpha) + \ 0.4262O_1(\alpha) + \ 0.5963O_2(\alpha) - 0.2227O_3(\alpha) + 0.0123\,O_4(\alpha)}$$
$$\mathbf{Q_5(\alpha) = -0.8402O_0(\alpha) + \ 0.5358O_1(\alpha) + 0.4262O_2(\alpha) - 0.0911O_3 - 0.0385O_4 + 0.0078O_5}$$

The solutions were approximated in five different degrees. The comparison of error functions of the (LPs) method and those in the (TPs) method is shown in Table 4, showing the (LPs) method having a higher accuracy than in the (TPs) method with the same degrees. Figure 4 shows the comparison of results for k=4 and 5 with exact solution. They seem to be identical.

Table4. Comparison of the Error Function of the (LPs) and (TPs) of Example 4.

k	$\|\|AR_k\|\|$	
	(LPs) Method	(TPs) Method
2	7.0865E−02	7.0865E−02
3	3.2692E−03	3.2693E−03
4	6.3587E−04	6.3589E−04
5	1.6865E−05	1.6908E−05

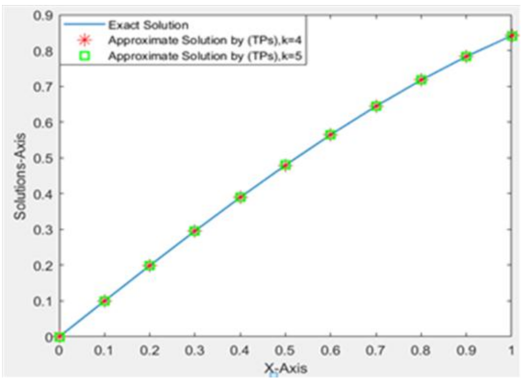

Figure 4(b). The (TPs) of Example 4

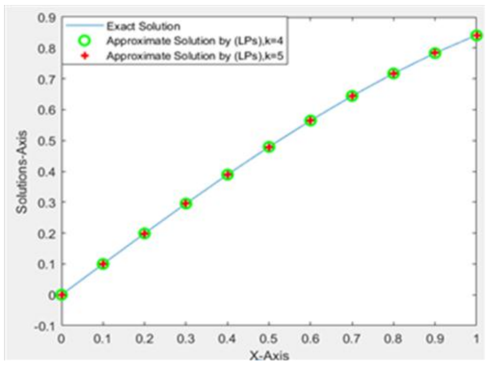

Figure 4(a). The (LPs) of Example 4 for k=4 and 5

Example 5: Solve the (VID) equation of the 2nd type with constant kernel [4]:

$$Q'(\alpha) = 6 - 3\alpha^2 + \int_0^{\alpha} Q(\tau)\,d\tau,\ \ Q(0) = 0\,,$$

where $Q(\alpha) = 6\alpha$ is the exact solution.

For k= 2, 3 and 4, the same exact solution is obtained, so, using the (LPs), we have:

$$Q_2(\alpha) = 6V_0(\alpha) - 6V_1(\alpha) = Q(\alpha) = 6\alpha$$

$$Q_3(\alpha) = 6V_0(\alpha) - 6V_1(\alpha) = Q(\alpha) = 6\alpha$$

$$Q_4(\alpha) = 6V_0(\alpha) - 6V_1(\alpha) = Q(\alpha) = 6\alpha$$

Also using the (TPs), we have:

$$Q_2(\alpha) = -6\,O_0(\alpha) + 6\,O_1(\alpha) = Q(\alpha) = 6\alpha$$

$$Q_3(\alpha) = -6\,O_0(\alpha) + 6\,O_1(\alpha) = Q(\alpha) = 6\alpha$$

$$Q_4(\alpha) = -6\,O_0(\alpha) + 6\,O_1(\alpha) = Q(\alpha) = 6\alpha$$

The solutions were approximated in three different degrees and the exact solution was obtained the same and this shows that the error function is zero in this case. Figure 5 displays the comparison of results for k=2, 3 and 4 with exact solution. They seem to be identical.

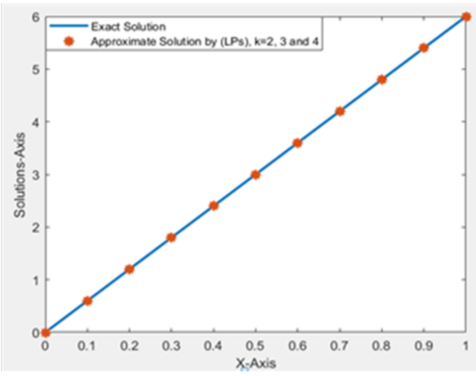

Figure 5(a). The (LPs) of Example 5

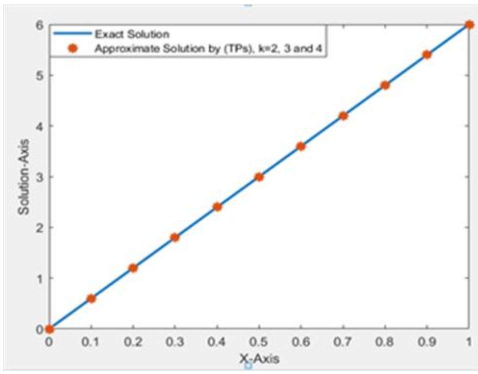

Figure 5(b). The (TPs) of Example 5

Example 6: Solve the generalized Abel's integro differential equation of the 2nd type [2]:

$$Q'(\alpha) = -Q(\alpha) - \alpha + 0.2\int_0^{\alpha} \frac{Q'(\tau) + 1}{\sqrt{(\alpha - \tau)}}\,d\tau\,, \quad 0 < \alpha \leq 1, Q(0) = 1,$$

where $Q(\alpha) = 1 - \alpha$, is the exact solution.

For k= 2, 3 and 4, the same exact solution is obtained, then, using the (LPs) and (TPs), the results are respectively:

$Q_2(\alpha) = Q_3(\alpha) = Q_4(\alpha) = Q(\alpha) = 1 - \alpha$ and $Q_2(\alpha) = Q_3(\alpha) = Q_4(\alpha) = Q(\alpha) = 1 - \alpha$.

The solutions were approximated in three different degrees and the exact solution was obtained the same and this shows that the error function is zero in this case. Figure 6 displays the comparison of results for k=2, 3 and 4 with exact solution. They seem to be identical.

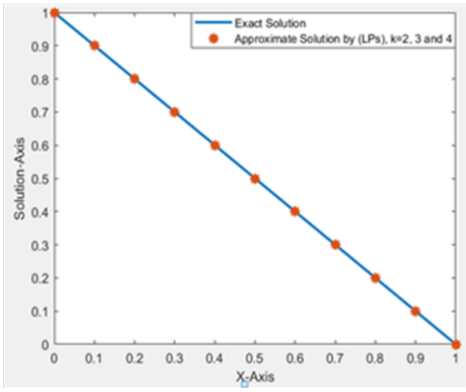

Figure 6(a). The (LPs) of Example 6

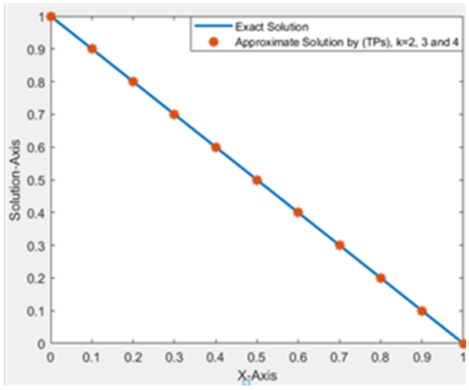

Figure 6(b). The (TPs) of Example 6

11. Conclusions and Recommendations:

In this work, two effective approximate numerical methods base on the (LPs) and (TPs) have been used to get approximate numerical solutions for four examples of linear (VI) equation and two examples of the linear (VID) equation. The error function of these methods were established and appeared its accuracy.The results of both proposed methods in Tables 1 and 2 were better than in [20]. The results of error function for example 3 in Table 3 were decreasing with increased polynomials degrees, also, the results in Table 4 have shown that the (LPs) method is better than the (TPs) method. In examples ***5***and ***6***, the approximate solutions were exactly the same as exact solution, so, the error functions were zero in these cases for both proposed methods. In general, all results indicate that the errors function decreasing with increasing the degree of polynomials as shown in the relevant Tables and Figures. Therefore, the methods used in this article can be applied to other types of integral equations, like, nonlinear integral and integro differential equations.

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Domination Polynomial of the Composition of Complete Graph and Star Graph

Ihsan A. Alwan
Ihsan.ab92@gmail.com

Ahmed A. Omran
pure.ahmed.omran@uobabylon.edu.iq
Department of Mathematics, College of Education for Pure Science,
University of Babylon, Babylon, Iraq

Abstract. Graph domination by vertices is finding a subset D from the vertex set $V(G)$, "in a graph G such that D is a dominating set if every vertex in set $V - D$ is adjacent to at least one vertex in set D". In this paper, $\mathcal{D}(\mathrm{G}, \mathrm{i})$ when G is a composition of complete graph $\mathrm{K_r}$ and star graph S_m, is constructed where "$\mathrm{D}(\mathrm{G}, \mathrm{i})$, is the family of all dominating sets of a graph G with cardinality i and $d(G,i) = |\mathcal{D}(G,i)|$". A recursive formula for $d(K_r[S_m], i)$ is obtained. The domination polynomial of graph $K_r[S_m]$ is determined by using this recursive formula.

Keywords: Domination number, Domination polynomial, Dominating set, composition.

1 Introduction

Assume that $G = (V, E)$ with n vertices is a simple graph. The set $N(v) = \{u \in V | uv \in E\}$ and the set $N[v] = N(v) \cup \{v\}$ are the open and closed neighborhood of $v, v \in G$ respectively [1-5]. A degree for every vertex $v \in V$, is the number of edges incident with v or equivalently, $deg(v) = |N(v)|$. "The minimum degree and the maximum degree of vertices of G are $\delta(G)$ and $\Delta(G)$, respectively" [6-11].

In a graph G, the set $D \subseteq V$ is a "dominating set" if every vertex $v \in V$ is either an element of D or is adjacent to an element of D. The minimum cardinality of a dominating set in a graph G is the "domination number $\gamma(G)$". Any dominating set with cardinality equal to $\gamma(G)$ is called $\gamma - set$. For a detailed treatment of this parameter, see some types of domination by vertices [12-21]. In a graph G an i-subset is a subset of $V(G)$ with cardinality equal to "i". "The family of all dominating sets of G which are i-subsets is $D(G,i)$ where, $d(G,i) = |\mathcal{D}(G,i)|$". The polynomial $D(G,x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G,i)\, x^i$ "is defined as domination polynomial of G" [4- 6]. "The composition of two graphs G_1 and G_2, is the graph $G_1[G_2]$ where, the vertex set of the graph $G_1[G_2]$ is $V(G_1) \times V(G_2)$ such that the vertex (r, x) is adjacent to vertex (t, y) if and only if r is adjacent to t in the graph G_1 or $r = t$ and x is adjacent to y in the graph G_2". [21], and [22].

Theorem 1.1. [21] Suppose K and L be two graphs with at least two vertices. If $\gamma(L) = 1$, then $\gamma(K[L]) = \gamma(K)$.

Theorem 1.2. [21] The following some properties of a composition held for all graph M, L and K.

i. $M[L] \neq L[M]$.
ii. $(M[L])[K] \cong M\big([L[K]]\big)$.
iii. $K_1[M] \cong M$.
iv. $M[K_1] \cong M$.
v. $(M \cup L)[K] = M[K] \cup L[K]$.

"Lemma 1.3. [4] The following properties held for all graph G of order n.

i. If G is connected, then $d(G,n)=1$ and $d(G,n-1)=n$.
ii. $d(G,i)=0$ if and only if $i<\gamma(G)$ or $i>n$.
iii. $D(G,x)$ has no constant term.
iv. $D(G,x)$ is a strictly increasing function in $[0,\infty)$.
v. Let G be a graph and H be any induced subgraph of G. Then
vi. $deg(D(G,x))\geq deg(D(H,x))$."

2 Dominating sets for composition of complete graph and star graph

Suppose K_r be a complete graph of order r and, suppose S_m be a star graph of order m, $m\geq 3$. The composition of complete graph and star graph is $K_r[S_m]$ with $n=rm$ vertices. Let $\mathcal{D}(K_r[S_m],i)$, be the family of dominating sets of $K_r[S_m]$ of cardinality "i".

Theorem 2.1. [23] the following properties held for each star graph S_m with order m, $\forall m\geq 3$

$d(S_m,\ i)=\binom{m-1}{i-1}\ \forall i<m-1,\ m\geq 3.$

$D(S_m,\ x)=\sum_{i=1}^{n}\binom{m-1}{i-1}x^i+x^{m-1}$

Theorem 2.2. [23] Let $K_{a_j}=K_{a_1,a_2,a_3,\cdots,a_r}$ be complete r-partite graph with order $n=a_1+a_2+a_3+\cdots+a_r$, the following properties held $\forall\ 2\leq a_1\leq a_2\leq a_3\leq\cdots\leq a_r$

i. $d\left(K_{a_j},i\right)=\binom{n}{i}-\binom{a_1}{i}-\binom{a_2}{i}-\cdots-\binom{a_r}{i}\ \ \forall\ i<a_1\leq a_2\leq\cdots\leq a_r$
ii. $d\left(K_{a_j},i\right)=\binom{n}{i}\ \ \forall\ i\geq a_1\geq a_2\geq\cdots\geq a_r$
iii. $D\left(K_{a_j},x\right)=\sum_{i=2}^{n}\binom{n}{i}x^i-\sum_{i=2}^{a_1-1}\binom{a_1}{i}x^i-\sum_{i=2}^{a_2-1}\binom{a_2}{i}x^i-\cdots-\sum_{i=2}^{a_r-1}\binom{a_r}{i}x^i$

Theorem 2.3. Let the graph $K_r[S_m]$ be a composition of complete graph and star graph with order $n=rm$, $m\geq 3$, then

i. $d(K_r[S_m],i)=\binom{n}{i}-r\binom{m-1}{i}\quad\forall\ 1\leq i<m-1$
ii. $d(K_r[S_m],i)=\binom{n}{i}\quad\forall\ m-1\leq i\leq n$

Proof:-

i. The number of subsets with cardinality i of $K_r[S_m]$ is $\binom{n}{i}$. Let $v\in S_m$ be the center vertex of S_m, the vertices of star except v forms be the dominating set of S_m. Since every vertex of K_r composition with the vertices of star except v then there exist $r\binom{m-1}{i}\ \forall\ i<m-1$ of subsets which are not dominating sets of $K_r[S_m]$, then
$d(K_r[S_m],i)=\binom{n}{i}-r\binom{m-1}{i}\ \ \forall\ \ 1\leq i\leq m-1.$
ii. By (i) and since every subset with cardinality $i,\forall i\geq m-1$ is the dominating set of $K_r[S_m]$ then $\binom{n}{i}$ is the number dominating sets with cardinality $i,\forall\ i\geq m-1$ then $d(K_r[S_m],i)=\binom{n}{i}\ \forall\ m-1\leq i\leq n$. □

Theorem 2.4. The graph $K_r[S_m]$ be a composition of complete graph and star graph with order $n=rm$, $m\geq 3$, then

i. $d(K_r[S_m],i)=d(S_n,i)+d(S_{n-1},i)+d(S_{n-2},i)+\cdots+d(S_{n-r+1},i)+d\left(K_{a_j},i\right)$

$\forall\, a_j = \frac{rm-r}{r} = m-1,\ j = 1,2,3,\dots,r,\ \ \forall\ i < n-r$

ii. $d(K_r[S_m], i) = d(S_n, i) + d(S_{n-1}, i) + d(S_{n-2}, i) + \cdots + d(S_{n-r+1}, i) + d\left(K_{a_j}, i\right) - 1$

$\forall\, a_j = \frac{rm-r}{r} = m-1,\ j = 1,2,3,\dots,r,\ \ \forall\ n-r \le i \le n-1$

Proof:-

i. let $u_1, u_2, \dots, u_r \in S_n$ such that u_1 is the center vertex of S_n. It is obvious that S_n be a spanning subgraph of $K_r[S_m]$, $S_n \setminus u_1$ be a subgraph of $K_r[S_m]$ such that u_2 is the center vertex of subgraph starS_{n-1}, and so on ... That means in general $S_n \setminus \{u_1, \dots, u_{r-1}\}$ be a subgraph of $K_r[S_m]$ such that u_r is the center vertex of S_{n-r+1} and since $K_r[S_m] \setminus \{u_1, u_2, \dots, u_r\}$ is a complete r-partite subgraph of $K_r[S_m]$, then $S_n \cup S_{n-1} \cup S_{n-2} \cup \dots \cup S_{n-r+1} \cup K_{m-1,\dots,m-1} = K_r[S_m]$. The $d(S_j, i)$ is the number of dominating sets with cardinality i of S_j,j$= n-r+1, n-r+2, \dots, n$. The number $d(K_{m-1,\dots,m-1}, i)$ represent the number of dominating sets with cardinality i of $(K_{m-1,\dots,m-1})$, and $d(K_r[S_m], i)$ is the number of dominating sets with cardinality i of $K_r[S_m]$. Since $d(K_r[S_m], n) = 1$ by lemma (1.3), $d(S_{n-r+1}, n-r) \cap d(K_{m-1,\dots,m-1}, n-r) = 1$, and
$d(S_{n-r+1}, n-r+1) \cap d(S_{n-r+2}, n-r+1) = 1 \dots d(S_{n-1}, n-1) \cap d(S_n, n-1) = 1$,
then $d(K_r[S_m], i) = d(S_n, i) + d(S_{n-1}, i) + d(S_{n-2}, i) + \cdots + d(S_{n-r+1}, i) + d\left(K_{a_j}, i\right)$
$\forall\, a_j = \frac{rm-r}{r} = m-1,\ j = 1,2,3,\dots,r,\ \ \forall\ i < n-r$

ii. By (1) $d(S_{n-r+1}, n-r) \cap d(K_{m-1,\dots,m-1}, n-r) = 1$ and $d(S_{n-r+1}, n-r+1) \cap d(S_{n-r+2}, n-r+1) = 1 \dots d(S_{n-1}, n-1) \cap d(S_n, n-1) = 1$, then $d(K_r[S_m], i) = d(S_n, i) + d(S_{n-1}, i) + d(S_{n-2}, i) + \cdots + d(S_{n-r+1}, i) + d\left(K_{a_j}, i\right) - 1$, $\forall\, a_j = \frac{rm-r}{r} = m-1,\ j = 1,2,3,\dots,r, \forall\, n-r \le i \le n-1$. □

Example 2.5. Let $r = 2,\ m \ge 3$ then $n \ge 6$ and $1 \le i \le n$.
By using Theorems 2.4 and 2.5 we calculate the coefficients of $D(K_r[S_m], x)$ for $6 \le n \le 14$ in Table1. Let $d(K_r[S_m], i) = |D(K_r[S_m], i)|$. There are interesting relationships between the numbers of $d(K_r[S_m], i)(1 \le i \le n)$ in the next table.

		i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
n	r	m														
6	2	3	2	15	20	15	6	1								
8	2	4	2	22	56	70	56	28	8	1						
10	2	5	2	33	112	210	252	210	120	45	10	1				
12	2	6	2	46	200	485	792	924	792	495	220	66	12	1		
14	2	7	2	61	324	971	1990	3003	3432	3003	2002	1001	364	91	14	1

Table 1: $|\mathcal{D}(G, i)|$ of $K_r[S_m]$ with cardinality i

Corollary 2.6. The following properties held for coefficients of $D(K_r[S_m], x)\ with\ order\ n = rm,\ \forall m \ge 3$

i. $\gamma\,(K_r[S_m]) = 1$
ii. $d(K_r\,[S_m\,], 1) = r$
iii. $\delta(K_r[S_m]) = n - m + 1$
iv. $\Delta(K_r[S_m]) = n - 1$

Proof

i. since $\gamma(S_m) = 1$, and by definition of the composition of complete graph and star graph, then $\gamma(K_r[S_m]) = 1$.
ii. let v be the center vertex of S_m and since K_r of order r, and by (i), then $d(K_r\,[S_m\,], 1) = r$.
iii. Let S_m be a star graph contain $m - 1$ end vertices, v be any end vertex of S_m and u be any vertex of K_r. So by definition of composition, (u, v) is not adjacent to $(m - 1)$ vertices of $K_r\,[S_m\,]$ but adjacent to other vertices of $K_r\,[S_m\,]$, therefore, $\delta(K_r[S_m]) = n - (m - 1) = n - m + 1$.
iv. Let $v \in S_m$, and v is the center vertex of S_m, and let $u_j \in K_r,\ j = 1,2, \dots, r$, then by definition of composition $\left(u_j, v\right)$ is adjacent to all vertices of $K_r[S_m]$, then $\Delta(K_r[S_m]) = n - 1$. □

Proposition 2.7. The following properties held for all $D(K_r[S_m], x), \forall\, m \geq 3$

i. $D(K_r[S_m], x) = \sum_{i=1}^{m-2} \binom{n}{i} x^i - \sum_{i=1}^{m-2} r \binom{m-1}{i} x^i + \sum_{m-1}^{n} \binom{n}{i} x^i$
ii. $D(K_r[S_m], x) = \sum_{i=1}^{n} d(S_n, i)\, x^i + \sum_{i=1}^{n} d(S_{n-1}, i)\, x^i + \cdots + \sum_{i=1}^{n} d(S_{n-r+1} i)\, x^i + \sum_{i=1}^{n} d\left(K_{a_j}, i\right) x^i - \sum_{n-r}^{n-1} x^i$

Proof :-

i. According to definition of domination polynomial and Theorem 2.4, then

$$D(K_r[S_m], x) = \sum_{i=1}^{n} d(K_r[S_m], i)\, x^i = \sum_{i=1}^{n} \left[\binom{n}{i} - r\binom{m-1}{i} + \binom{n}{i}\right] x^i$$

$$= \sum_{i=1}^{m-2} \binom{n}{i} x^i - \sum_{i=1}^{m-2} r \binom{m-1}{i} x^i + \sum_{m-1}^{n} \binom{n}{i} x^i.$$

ii. By using definition of domination polynomial and according to Theorem 2.5, then

$$D(K_r[S_m], x) = \sum_{i=1}^{n} d(K_r[S_m], i)\, x^i = \sum_{i=1}^{n} \left[d(S_n, i) + d(S_{n-1}, i) + d(S_{n-2}, i) + \cdots + d(S_{n-r+1}, i) + d\left(K_{a_j}, i\right) - 1\right] x^i$$

$$= \sum_{i=1}^{n} d(S_n, i)\, x^i + \sum_{i=1}^{n} d(S_{n-1}, i)\, x^i + \cdots + \sum_{i=1}^{n} d(S_{n-r+1}, i)\, x^i + \sum_{i=1}^{n} d\left(K_{a_j}, i\right) x^i - \sum_{n-r}^{n-1} x^i \quad \forall\, a_j = m - 1, \quad j = 1,2,3, \dots, r.$$ □

Example 2.8. Let the graph $K_2[S_4]$ with order 8, be the composition of K_2 and S_4, then by Proposition (2.7) we have

$D(K_2[S_4], x) = \sum_{i=1}^{2}\binom{8}{i}x^i - \sum_{i=1}^{2}2\binom{3}{i}x^i + \sum_{3}^{8}\binom{8}{i}x^i$

$= [8x + 28x^2] - [6x + 6x^2] + [56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8]$

$= 2x + 22x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$. (see figure 2.1)

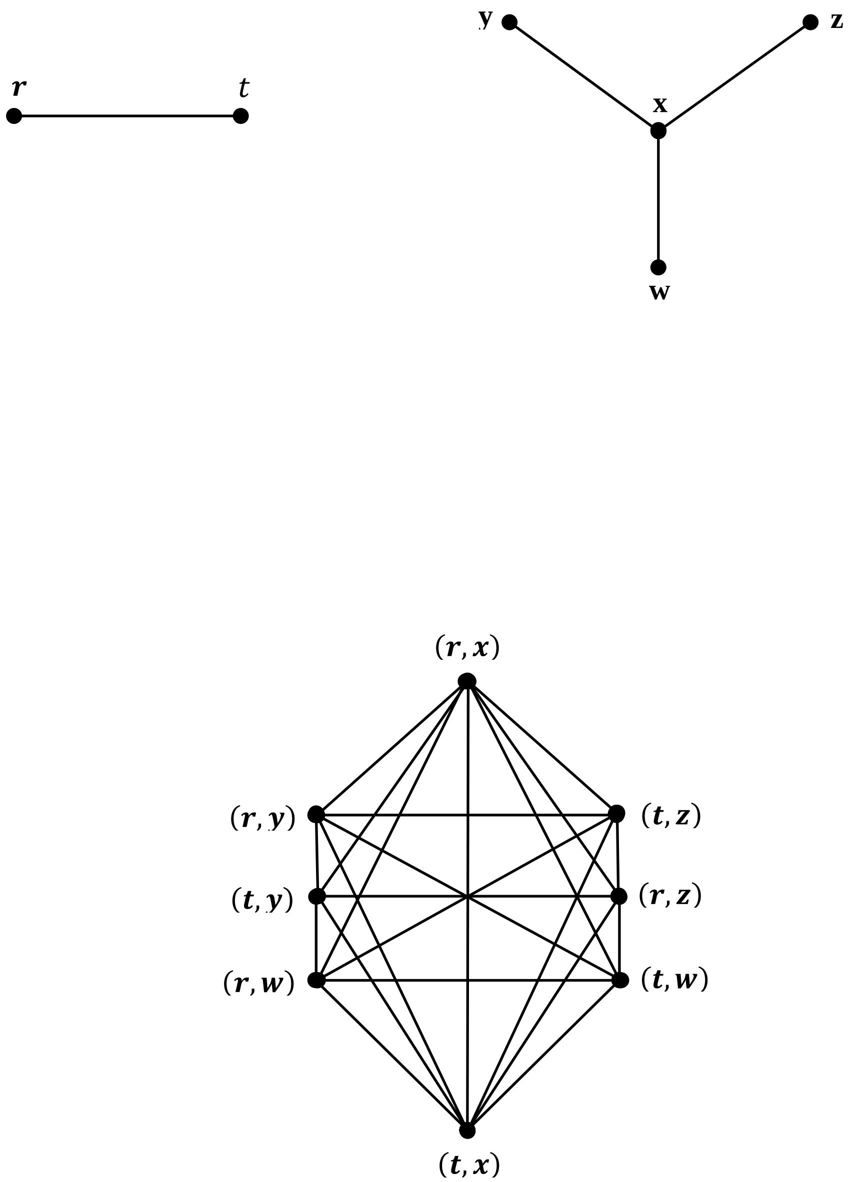

Figure 2.1: $K_2[S_4]$

Conclusion

In this paper, we studied the dominating sets for composition of complete graph and star graph and find the formula of domination polynomial of the composition of a complete graph and star graph.

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Unit Regular Clean Rings

[1]Israa Th. Younis [2] Prof .Dr. Nazar H. Shuker

[1]Department of Mathematics, college of computer science and Mathematics, University of Mosul.

[2] Department of Mathematics, college of computer science and Mathematics, University of Mosul.

israthanon8080@gmail.com

Abstract. A ring R is called unit regular clean, if every element is the sum of an idempotent and a unit regular elements. In this paper we introduce the notion of unit regular clean ring. we investigate some of it's basic properties and it's relation with clean ring.

Keyword: Clean ring ,unit regular ring ,unit regular element ,r-clean ring

1- Introduction:

Throughout this paper,R is an associative ring with identity. $U(R)$, $Ur(R)$ and $Id(R)$ are respectively, the set of units, unit regular and idempotent elements. $J(R)$ is the Jacobson radical of R

An element x of a ring R is said to be clean if $x = e + u$ for some $u \in U(R)$ and $e \in Id(R)$. A ring R is called clean if each of its element is clean. Clean ring, was firstly presented by, Nicholson [7]. Many researchers worked on this subject and investigated properties of clean rings, see for example [2, 5, 8, 10]. In 1936 Von Newmann defined that: an element $r \in R$ is called regular if $r = ryr$ for some $y \in R$. The ring R is said to be regular if each of its element is regular, some of the properties of regular rings have been studied in [6]. A ring R is called abelian if every idempotent in R is central [3].

A ring R is said to be unit regular if for each $a \in R$, there exists a unit $u \in R$ such that $a = aua$. Camillo and Yu [5, Theorem 5] proved that: "every unit regular ring is clean".

In [7], Nicholson and Varadrajan proved that the converse is not necessarily be true. In [1] Ashrafi and Nasibi defined that a ring R is said to be r-clean if every element of it can be written as the sum of idempotent and regular elements.

We say that an element x of a ring R is a unit regular clean (briefly, ur- clean) if $x = e + a$ where $a \in (R)$ and $e \in Id(R)$.

A ring R is said to be ur- clean if each of its element is ur- clean.

Clearly unit regular rings and clean rings are ur-clean. we also provide an example of ur- clean ring which is not clean. In this work we give some properties of ur- clean rings and its relation with clean ring.

2- Unit regular clean ring

In this section we introduce the notion of unit regular clean ring, we give some of it's properties and provide some examples.

Definition 2.1 An element x of a ring R is unit regular clean, (briefly, ur- clean) if $x = r + e$ where $r \in ur(R)$ and $e \in Id(R)$. A ring R is ur- clean if each of its elements is ur-clean.

Clearly, unit regular rings and clean rings are ur-clean. but the converse is not necessarily be true. as the following example shows.

Example 2.2 The ring of integers, Modulo 4, $Z4$ is not unit regular because 2 is not unit regular in $Z4$. However it is easy to check that $Z4$ is ur- clean. In general ur- clean is not necessarily be clean see [11, Theorem 4.1].

Next ,we shall give part of basic properties of ur-clean rings.

Proposition 2.3: If R is a ring, then $x \in R$ is ur-clean element if and only if $(1 - x)$ is ur-clean element.

Proof: Let x is ur-clean element then $x = e + a$ where, $e \in Id(R)$ and $a \in Ur(R)$, then

$1 - \mathrm{x} = (1 - \mathrm{e}) + (-a)$, but $(1 - \mathrm{e})$ is idempotent since

$(1 - \mathrm{e})2 = 1 - 2\mathrm{e} + \mathrm{e}2 = 1 - 2\mathrm{e} + \mathrm{e} = 1 - \mathrm{e}$ .Clearly $(-\mathrm{a}) \in \mathrm{Ur}(\mathrm{R})$ since [$\mathrm{a} = e.u$, a is unit regular then $-\mathrm{a} = \mathrm{e}(-\mathrm{u})$ is a unit regular]

Hence $1 - x$ is ur-clean element.

Conversely: let $(1 - \mathrm{x})$ is ur-clean element then

$1 - \mathrm{x} = \mathrm{e} + \mathrm{a}$ where $\mathrm{e} \in \mathrm{Id}(\mathrm{R})$ and $\mathrm{a} \in Ur$ (R)

$-x = e + a - 1 =$ then, $x = (1 - e) - a = (1 - e) + (-a)$

$(1 - \mathrm{e})$ is an idempotent and $-\mathrm{a}$ is unit regular which implies that x is ur-clean element.■

Note that, for any ring R, and any ideal I of R , if $\mathrm{R/I}$ is ur-clean then R is not necessarily to be ur-clean as the following examples shows.

Example 2.4:

1- If P is prime number then $\mathrm{Z/p} \cong \mathrm{Zp}$ is ur-clean, but the ring Z is not clean.

2- The ring of integers modulo 12, $Z12$. Let I = {0, 3, 6, 9} be an ideal of $Z12$. Now $Z12/I$ is ur-clean since $Z12/I$ is a field; but $Z12$ is not ur-clean ring.

Following [9], idempotent can be lifted modulo, as one sided ideal I of a ring R.if for $x \in R$ with $x - x2 \in \mathrm{I}$, there exists an idempotent $e \in R$ such that $\mathrm{e} - \mathrm{x} \in \mathrm{I}$.

The following result, gives a sufficient condition for R to be ur-clean

Theorem 2.5: Let $\mathrm{I} \subseteq \mathrm{J(R)}$ be any ideal, of a ring R then R is ur-clean if and only if the quotient ring $\mathrm{R/I}$ is ur-clean and idempotent lift modulo I.

Proof: Let $x + I \in R/I, x \in R$ such that $x = e + a$ where e is an idempotent and a is unit regular element.

Now, $x + I = e + a + I = (e + I) + (a + I)$.

Clearly, $(\mathrm{e} + \mathrm{I})$ is an idempotent element of R/I and

$(a + I) = (aua + I) = (a + I)(u + I)(a + I)$.

So $(\mathrm{a} + \mathrm{I})$ is unit regular then R/I is ur-clean ring.

Conversely: Suppose that the quotient ring $\mathrm{R/I}$ is ur- clean and idempotent lift modulo I and let r be any element in R. since $\mathrm{R/I}$ is ur-clean we can Write

$\mathrm{r} + \mathrm{I} = \mathrm{x} + \mathrm{e} + \mathrm{I}$ for some unit regular $x + I$, and idempotent lift modulo I, we assume e is an idempotent of the ring R, since $r - e + I = x + I$ is unit regular element of $\mathrm{R/I}$. So $r - e$ is unit regular of R , it follows that r may be written as the sum of idempotent and unit regular of R by writing , $\mathrm{r} = (\mathrm{r} - \mathrm{e}) + \mathrm{e}$, This proves the sufficiency.■

Theorem 2.6: If R is abelian ur-clean ring and $2 \in u(R)$, then every element of R can be written as a sum of idempotent and two units.

Proof: Let $\mathrm{x} \in \mathrm{R}$, then $\mathrm{x} = \mathrm{e} + a$, where $\mathrm{e} \in \mathrm{Id}\ (\mathrm{R}), a = aua$, since ua is idempotent say e`, then $\mathrm{a} = \mathrm{u}.\mathrm{e}`$

Let $\mathrm{v} = 2\mathrm{e}` - 1$, clearly $\mathrm{v}2 = 1$ So $2\mathrm{e}` = \mathrm{v} + 1$, Since $2 \in \mathrm{U(R)}$ then
$\mathrm{e}` = 2 - 1\,\mathrm{v} + 2 - 1$ .So $\mathrm{a} = \mathrm{u}.\mathrm{e}` = \mathrm{u}\,(2 - 1\,\mathrm{v} + 2 - 1)$

$= \mathrm{u}\,2 - 1\,\mathrm{v} + \mathrm{u}2 - 1 = \mathrm{u}1 + \mathrm{u}2$

Hence $x = \mathrm{e}` + \mathrm{u}1 + \mathrm{u}2$.■

In [4] Camillo and Khurana gave the following result.

If a is unit regular element then $a = e + u$ and $a\,\mathrm{R} \cap \mathrm{e}\,\mathrm{R} = 0$.

Theorem 2.7: Let R be abelian ur-clean ring ,for any $x \in R$ there exists an idempotent e, such that ex is idempotent.

Proof: Since x is ur- clean, then $x = e_1 + a$ where e_1 is idempotent and a is unit regular then $a = e + u$ and $a R \cap e R = 0$

Since $ae = ea \in aR \cap eR$, then $ae = 0$. So $ex = ee_1 + ea$, then $ex = e.e_1$, since e and e_1 are central idempotents, then ee_1 is idempotent. ■

In [1] Ashrafi proved that "if R be an abelian r-clean ring, then eRe is also r-clean ring". We do like wise of ur- clean ring.

Theorem 2.8: Let R be an abelian ur-clean ring then eRe is also ur- clean ring.

Proof: Let $a \in eRe \subseteq R$, then $a = e_1 + r$ and $e_1 r = re_1$ where e_1 is idempotent and $r \in Ur(R)$ where $R$is ur-clean.

Since $a \in eRe$, then $a = ee_1e + er'e$, it follows that $a = e_1 e + r'e$ we want to show that re is unit regular and $e_1 e$ is an idempotent.

for this consider $(e_1e)^2 = (e_1e) . (e_1e) = e_1 (ee_1)e = e_1 (e_1e) e$

$$= e_1e_1) (ee) = (e_1^2 \, e2) = e_1e$$

Therefore e_1e is idempotent.

Now consider $eue \in eRe$

$(re) (eue) (re) = (re) (eue)(er) = (re) u (er)$

$$= (er) u (er) = e (rur) e = ere \in eRe$$

Then re is unit regular, implies that eRe is ur-clean ring. ■

Theorem 2.9: Let R be a ring with every $a \in R$ there is $b \in R$, such that $a + b \in J(R)$ and $a.b = a$, Then R is ur-clean

Proof: Let $a \in R$, then there is $b \in R$ such that $a + b \in J(R)$ and $a.b = a$

Then $a + b - 1 \in U(R)$. Let $a + b - 1 = u$. Now $au = a (a + b - 1) = a^2 + ab - a = a^2$

So $au = a^2$, and hence $a = a^2u - 1$

Therefore a is unit regular. If we write $a = 0 + a$, then R is ur-clean.■

Theorem 2.10: Let R be a ring with every a in R there is b in R such that $a+\ b$ is unit and $a.b=\ 0$, Then R is reduced ur-clean ring.

Proof: Let $a \in R$,then there exists $b \in R$ such that $a+b$ is unit and $a.b\ =o$

Now, if we set $a\ +\ b\ =\ v$ then $av\ =\ a(a+b)=\ a2+ab\ =a2$. Clearly

R is reduced ring, if $a2\ =\ o$, then $av=\ 0$ implies that, $a\ =\ o$.

So $a\ =\ a2\ v-1$ this implies $a\ (1-\ av-1)\ =\ 0$. Hence $1-av-1\ \in$ r (*a*) = ℓ (*a*).

Then $(1-av-1)\ a\ =\ o$. Hence $a\ =\ av-1\ a$, so it unit regular.

If we set $a\ =\ 0\ +\ a$ then a is ur −clean.■

3- The relation between ur-clean and clean rings

In this section we give the relationship between ur-clean and clean rings. Clearly every clean ring is ur-clean ring since unit is unit regular, but the converse is not necessarily be true.

Theorem 3.1: Let R be an abelian ring, then R is ur-clean if and only if R is clean.

Proof: One direction is trivial.

Conversely: let R be ur-clean ring and $x\ \in R$, then $x\ =\ e\ +\ r$ where $e\ \in\ Id\ (R)$ and $r\ \in\ Ur\ (R)$. So there is $u \in R$ such that $rur\ =\ r$

Clearly $e` = r.u$ and $u.r$ are idempotents and

$(re`\ +\ (1-e`)(ue`\ +\ (1-e`))=1$, also since R is abelian we have

$(ue`+(1-e`))(re`+(1-e`))=1$ then

$(re`+(1-e`))$ is unit and hence $e`u+(1-e`)$ is unit

-$(e`u+(1-e`))$ is a unit ,since $1-e$ is idempotent

So, $-r\ =\ (1-e`)+\ (-(e`u+\ (1-e`))$ is clean that is x is clean.■

Theorem 3.2: Let R be abelian ur-clean ring such that each pair of distinct idempotents in R are orthogonal then R is clean.

Proof: Since every abelian regular ring is clean then for each $\in R$, x can be written as $x=e_1\ +\ e_2\ +a$ where $e_1,e_2\ \in\ Id\ (R)$ and $a\ \in ur(R)$

Now since e_1, e_2 are orthogonal then $e = e_1 + e_2 \in Id(R)$ and hence $x = e + a$ which shows that R is clean.■

Theorem 3.3: If R is a directly finite ur-clean ring, and 0 and 1, are the only idempotents in R, then R is clean.

Proof: Since R is ur-clean ring, each $x \in R$ can be written as $x = r + e$, where r is a unit regular element and e is an idempotent element of R.

If $r = 0$, then

$x = e = (2e - 1) + (1 - e)$. Also, since $(2e - 1)$ is a unit of R and $(1 - e)$ is an idempotent element of R, so x is a clean. Hence R is clean.

If $r \neq 0$, then there exists $u \in R$ such that $rur = r$. Thus ru an idempotent element of R .

So by hypothesis, $ru = 0$ or $ru = 1$.

Now if $ru = 0$, then $r = rur = 0$, which is contradiction. Therefore

$ru = 1$ and since R is directly finite so $ur = ru = 1$.

Thus, r is a unit of R. So x is clean element, and hence R is clean ring.■

Theorem 3.4: Let R be abelian ring and for every $a \in R$, there exists $b \in R$ such that $a + b \in Ur(R)$ and $a.b = 0$, then there is e in R such that ae is clean element.

Proof: Let $a \in R$, and $a + b \in Ur(R)$ then $a + b = e_1.u$ where e is idempotent and u is unit. Now

$a(a + b) = ae_1.u$ so, $a^2 = ae_1u$ and hence
$ae_1 = a^2 u - 1$, so $ae_1 = (ae_1)^2.u - 1$ clearly $ae_1e_2 + (e_2 - 1)$ is unit since

$(ae_1e_2 +)(e_2 - 1))\ (e_2u - 1 + (e_2 - 1)) = 1$ where $e_2 = ae_1u - 1$

So $ae_1e_2 + e_2^{-1} = v$ and hence $ae_1e_2 = 1 - e_2 + v$ but e_1e_2 is idempotent say e so $ae = (1 - e_2) + v$

This means that ae is clean element and hence it is ur-clean.■

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Using Modified Conjugate Gradient Method to Improve SCA

Ayad Hamad Khalaf[1] , Prof. Dr. Ban Ahmed Mitras[2]

[1] Student, department of mathematics, college of computer sciences & mathematics, Mosul University. Email: ayadlohebe@Gmail.com

[2] Prof. Dr. department of mathematics, college of computer sciences & mathematics, Mosul University. Email: dr.banah.mitras@Gmail.com

Abstract: This research is to improve Sine - Cosine Algorithm (SCA) that is like any other intelligent techniques that encounter some problem such as slow convergence and the dropping in local solution. To overcome these problems. SCA has been developed and improved through three directions, First: Hybrid of SCA with Modified conjugate gradient method (MCG) that has improved through that derivation of parameter of new conjugate factor (β^{new}) and attest its characteristic such as descent and global to construct improve algorithm called SCA-MCG. The second direction was a hybrid of SCA with classic optimization methods such as conjugate gradient (CG) algorithm to construct SCA-CG, , and parallel Tangent (PT) algorithm to construct SCA-PT. Third combining both previous methods, using the Hybrid value with SCA to construct SCA-CG-PT Algorithm of high quality accounts in all directions mentioned above. To improve the initial population which randomly generated by using excellent characteristics of MCG-CG-PT as well as using this improvement as initial population for SCA. Numerical results have proved the efficiency of improved Algorithm and the results was excellent if we compared with SCA. In addition, we got optimum global values for most functions by achieving functions minimum.

*Keyword***:** SCA algorithm, meta-heuristic algorithms, conjugate gradient and PARTAN methods

1. Introduction:

Optimization refers to the finding optimum values of the given system facts, for all possible values. In mathematics, it means to find minimum or maximum value of a function contains a certain number of variants. It can found in all fields of study that seek to develop basic optimization techniques so it is of a high importance for most researchers in their works. Optimization started in 1960 through many directions and methods through 2 main parts of algorithms, the first is Deterministic, and the other is Stochastic. Most of classical algorithms are deterministic, such as CG, PARTAN, QN, and others. Most of them based on slope or what called derivation, (derivation base algorithm). The second part of algorithm, Stochastic that is divided into Heuristic and meta-heuristic. It is important to mention that, recent trends of study refer to the lack of certain definition of these Heuristic and Meta Heuristic. (Glover 1986).

SCA is a Heuristic and inspired by sine and cosine functions. It suggested by (Seyedali Marjalili 2016) to solve optimization and apply it to improve airplanes' performance [9]. Many improvement and modifications as well as Hybrid suggested. In 2017, (simye ,Busra,Pakite) present a study about Constructed optimization problems using SCA[6] . Same year witnessed presenting another study of a Hybrid of SCA to solve global optimization problems by (R.M,rizk Allah) [13] , In 2018 Zhiliujun , Chiver, et al) present a study about Modifuing SCA based on search of circular uninvited adjunct. In the same year, (Ramzy Ali , Dunisis) present a study about Chaotic SCA[12] . Finally, in 2019 (yasmin, R.sindhu et. al.) present a study of SCA Hybrid using biogeography for problems of choosing merit [14]. In the same year (mouhoub, Mohamed, et al.) present a study about improving SCA to choose merit in sorting texts [2]. Two researchers (Lalit, Kusum) present a research paper about choosing merit [8]. (Chandrasekaran), also, present a study to improve SCA on the problem of sending Dynamic economy [3]. (Gholizadeh1, S. & Sojoudizadeh) present a study to modified sine cosine algorithm for Sizing Optimization of Truss Structures [7]. With an explanation of CG method to show its characteristic. Since it is one of the classical methods and use it in generating initial community used with SCA, using its characteristics to get optimum and global solution. PT and its uses with SCA had referred to also. After checking results, we made third modification by combining CG and PT with SCA. The numerical results were better when applied on special functions. Finally, new conjugate factor had derived, and then its globalization and slope was a tested.

It show efficiency when used in Hybrid SCA plus combining suggested and Classical and Heuristic to produce improved and Hybrid algorithm of high Characteristic tested on a set of special functions . The problem of the research focused on finding global optimum solutions for optimization problems to get rid of slow convergence, and fall in local solutions.

The study aims at presenting improved algorithm that hybrid of sine-cosine algorithm SCA with a set of classical algorithms named as SCA-MCG, SCA-CG, SCA-PT and SCA-CG-PT.

2. Conjugate Gradient Method:

In unconstrained optimization, we minimize an objective function that depends on real variables with no restrictions on the values of these variables. The unconstrained optimization problem is:

$$Min \quad f(x) : x \in R^n, \tag{1}$$

Where $f : R^n \to R$ is a continuously differentiable function, bounded from down. A nonlinear conjugate gradient method generates a sequence $\{x_k\}, k$ is integer number, $k \geq 0$. Starting from an initial point x_0, the value of x_k calculate by the following equation:

$$x_{k+1} = x_k + \lambda_k d_k , \tag{2}$$

Where the positive step size $\lambda_k > 0$ obtained by a line search and the directions d_k generated as:

$$d_{k+1} = -g_{k+1} + \beta_k d_k , \tag{3}$$

Where $d_0 = -g_0$, the value of β_k is determine according to the algorithm of Conjugate Gradient (CG), and its known as a conjugate gradient parameter, $s_k = x_{k+1} - x_k$ and $g_k = \nabla f(x_k) = f'(x_k)$, consider $\|.\|$ is the Euclidean norm, and $y_k = g_{k+1} - g_k$. The termination conditions for the conjugate gradient line search often based on some version of the Wolfe conditions . The standard Wolfe conditions [4] :

$$f(x_k + \lambda_k d_k) - f(x_k) \le \rho \lambda_k g_k^T d_k \, , \tag{4}$$

$$g(x_k + \lambda_k d_k)^T d_k \ge \sigma g_k^T d_k \, , \tag{5}$$

Where d_k is a descent search direction and $0 < \rho \le \sigma < 1$, where β_k defined by one of the following formulas:

$$\beta_k^{(HS)} = \frac{y_k^T g_{k+1}}{y_k^T d_k}; \quad \beta_k^{(FR)} = \frac{g_{k+1}^T g_{k+1}}{g_k^T g_k} \; ; \quad \beta_k^{(PRP)} = \frac{y_k^T g_{k+1}}{g_k^T g_k} \tag{6}$$

$$\beta_k^{(CD)} = -\frac{g_{k+1}^T g_{k+1}}{g_k^T d_k} \; ; \; \beta_k^{(LS)} = -\frac{y_k^T g_{k+1}}{g_k^T d_k} \quad ; \quad \beta_k^{(DY)} = \frac{g_{k+1}^T g_{k+1}}{y_k^T s_k} \tag{7}$$

Al-Bayati and Al-Assady In (Al-Bayati and Al-Assady, 1986) proposed three forms for the scalar β_k defined by [1]:

$$\beta_k^{AB1} = \frac{\|y_k\|^2}{\|g_k\|^2} \quad ; \; \beta_k^{AB2} = -\frac{\|y_k\|^2}{d_k^T g_k} \; ; \quad \beta_k^{AB3} = \frac{\|y_k\|^2}{d_k^T y_k} \tag{8}$$

3. Extension Dai and Yuan Method:

Yabe and Sakaiwa in 2005 extended the Dai and Yuan method as [4]:

$$\beta_k = \frac{\|g_{k+1}\|^2}{\tau_{k+1}} \tag{9}$$

Where τ_{k+1} be a positive parameter.

By setting $\tau_k = d_k^T y_k$ formula (9) reduce to this DY method as:

$$\beta_k^{DY} = \frac{\|g_{k+1}\|^2}{d_k^T y_k} \tag{10}$$

4. Proposed A New Conjugancy Coefficient:

We have the quasi-Newton condition

$$y_k = G_k s_k \tag{11}$$

Where $G_k = \dfrac{\partial^2 f}{\partial x_k^2}$ is the Hessian Matrix

We multiply both sides of equation (11) by s_k and we get

$$[y_k = G_k s_k] * s_k \tag{12}$$

$$\Rightarrow y_k^T s_k = G s_k^T s_k \tag{13}$$

$$G = \frac{y_k^T s_k}{\|s_k\|^2} . I_{nxn} \tag{14}$$

Where I is the identity matrix

Let $d_{k+1}^N = -G_k^{-1} g_{k+1}$ (15)

Eq. (15) is the Newton direction. From eq.(15) and (15) we get:

$$d_{k+1}^N = -\frac{y_k^T s_k}{\|s_k\|^2} g_{k+1} \tag{16}$$

Multiply both sides of equation (16) by y_k^T and we get

$$y_k^T d_{k+1}^N = -\left[\frac{y_k^T s_k}{\|s_k\|^2}\right] y_k^T g_{k+1} \tag{17}$$

$$\Rightarrow y_k^T d_{k+1}^{CG} = -y_k^T g_{k+1} + \beta_k d_k^T y_k \tag{18}$$

From (17) and (18) we have:

$$- y_k^T g_{k+1} + \beta_k d_k^T y_k = -\left[\frac{y_k^T s_k}{\|s_k\|^2}\right] y_k^T g_{k+1} \tag{19}$$

We assume that $\beta_k = \beta_k^{(DY)} = \dfrac{g_{k+1}^T g_{k+1}}{y_k^T d_k}$

Then we have

$$-y_k^T g_{k+1} + \frac{\|g_{k+1}\|^2}{d_k^T y_k} d_k^T y_k = -\left[\frac{y_k^T s_k}{\|s_k\|^2}\right] y_k^T g_{k+1} \tag{20}$$

From eq. (9) we get:

$$-y_k^T g_{k+1} + \beta_k \tau_k = -\left[\frac{y_k^T s_k}{\|s_k\|^2}\right] y_k^T g_{k+1} \tag{21}$$

Then, we have

$$\beta_k = \frac{-\left[\frac{y_k^T s_k}{\|s_k\|^2}\right] y_k^T g_{k+1} + y_k^T g_{k+1}}{\tau_k} \tag{22}$$

Since $\tau_{k+1} > 0$ then from [5], we have: $\tau_k = \lambda = \left[\frac{\|s_k\|^2}{2(f_k - f_{k+1}) + 2g_{k+1}^T s_k}\right]$ then:

$$\beta_k = \left(-\left[\frac{y_k^T s_k}{\|s_k\|^2}\right] y_k^T g_{k+1} + y_k^T g_{k+1}\right) \div \frac{\|s_k\|^2}{2(f_k - f_{k+1}) + 2g_{k+1}^T s_k} \tag{23}$$

$$\beta_k = \left(-\left[\frac{y_k^T s_k}{\|s_k\|^2}\right] y_k^T g_{k+1} + y_k^T g_{k+1}\right) \times \frac{\left(2(f_k - f_{k+1}) + 2g_{k+1}^T s_k\right)}{\|s_k\|^2} \tag{24}$$

$$\beta_k = \left(1 - \left[\frac{y_k^T s_k}{\|s_k\|^2}\right]\right) \frac{y_k^T g_{k+1}.\left(2(f_k - f_{k+1}) + 2g_{k+1}^T s_k\right)}{\|s_k\|^2} \tag{25}$$

Let $A = f_k - f_{k+1}$ then:

$$\beta_k = \left(1 - \left[\frac{y_k^T s_k}{\|s_k\|^2}\right]\right) \frac{y_k^T g_{k+1}.\left(2A + 2g_{k+1}^T s_k\right)}{\|s_k\|^2} \tag{26}$$

Or

$$\beta_k = \frac{1}{\|s_k\|^2}\left(\left[1-\frac{y_k^T s_k}{\|s_k\|^2}\right] y_k^T g_{k+1}\left(2A+2g_{k+1}^T s_k\right)\right) \tag{27}$$

4.1 Outlines of the Proposed Algorithm:

Step (1):*The initial step*: We select starting point $x_0 \in R^n$, and we select the accuracy solution $\varepsilon > 0$ is a small positive real number and we find $d_k = -g_k$, $\lambda_0 = Min\,ary(g_0)$, and we set $k = 0$.

Step (2): *The convergence test*: If $\|g_k\| \le \varepsilon$ then stop and set the optimal solution is x_k. Else, go to step (3).

Step (3): *The line search*: We compute the value of λ_k by Cubic method and that satisfy the Wolfe conditions in Eqs. (4),(5) and go to step(4).

Step (4): *Update the variables*: $x_{k+1} = x_k + \lambda_k d_k$ and compute $f(x_{k+1}), g_{k+1}$ and $s_k = x_{k+1} - x_k$, $y_k = g_{k+1} - g_k$.

Step (5): *Check:* if $\|g_{k+1}\| \le \varepsilon$ then stop. Else continue.

Step (6): *The search direction*: We compute the scalar $\beta_k^{(New)}$ by use the equation (27) and set $k = k+1$, and go to step (4).

5. The Convergence Analysis:

Theoretical Properties for the New CG-Method.

In this section, we focus on the convergence behavior on the β_k^{New} method with exact line searches. Hence, we make the following basic assumptions on the objective function.

Assumption 1:

f Is bounded below in the level set $L_{x_0} = \{x \in R^n \mid f(x) \le f(x_0)\}$; in some neighborhood U of the level set L_{x_0}, f is continuously differentiable and its gradient ∇f is Lipchitz continuous in the level set L_{x_0}, namely, there exists a constant L> 0 such that:

$\|\nabla f(x) - \nabla f(y)\| \le L\|x - y\|$ for all x,y $\in$ L_{x_0} (28)

5.1 Sufficient Descent Property:

We will show that in this section the proposed algorithm that defined in the equations (27) and (3) satisfy the sufficient descent property that satisfy the convergence property.

Theorem 1:

The search direction d_k that generated by the proposed algorithm of modified CG satisfy the descent property for all k, when the step size λ_k satisfied the Wolfe conditions (4),(5) .

Proof: we will use the indication to prove the descent property, for $k = 0$, $d_0 = -g_0 \Rightarrow d_0^T g_0 = -\|g_0\| < 0$, then we proved that the theorem is true for $k = 0$,we assume that $\|s_k\| \le \eta;\ \|g_{k+1}\| \le \Gamma\ and\ \|g_k\| \le \eta 1$ and assume that the theorem is true for any k i.e. $d_k^T g_k < 0$ or $s_k^T g_k < 0$ since $s_k = \lambda_k d_k$, now we will prove that the theorem is true for $k+1$ then:

$$d_{k+1} = -g_{k+1} + \beta_k^{(New)} d_k \tag{29}$$

$$\beta_k^{new} = \left(1 - \left[\frac{y_k^T s_k}{\|s_k\|^2}\right]\right)\frac{y_k^T g_{k+1}.\left(2A + 2g_{k+1}^T s_k\right)}{\|s_k\|^2} \tag{30}$$

i.e. $$d_{k+1} = -g_{k+1} + \left(1 - \left[\frac{y_k^T s_k}{\|s_k\|^2}\right]\right)\frac{y_k^T g_{k+1}.\left(2A + 2g_{k+1}^T s_k\right)}{\|s_k\|^2} d_k \tag{31}$$

Multiply both sides of the equation (31) by g_{k+1}^T we get:

$$g_{k+1}^T d_{k+1} = -\|g_{k+1}\|^2 + \left(1 - \left[\frac{y_k^T s_k}{\|s_k\|^2}\right]\right)\frac{y_k^T g_{k+1}.\left(2A + 2g_{k+1}^T s_k\right)}{\|s_k\|^2} .g_{k+1}^T d_k \tag{32}$$

$$\frac{g_{k+1}^T d_{k+1} + \|g_{k+1}\|^2}{\|g_{k+1}\|^2} \le \left(1 - \left[\frac{\|y_k\|\|s_k\|}{\|s_k\|^2}\right]\right) \frac{y_k^T g_{k+1}.(2A + 2g_{k+1}^T s_k)}{\|s_k\|^2} . \frac{g_{k+1}^T d_k}{\|g_{k+1}\|^2}$$

(33)

$$\frac{g_{k+1}^T d_{k+1} + \|g_{k+1}\|^2}{\|g_{k+1}\|^2} \le \left(y_k^T g_{k+1}\right) \frac{\left(2A + 2g_{k+1}^T s_k\right)}{\|s_k\|^2} . \frac{g_{k+1}^T d_k}{\|g_{k+1}\|^2}$$

(34)

$$\frac{g_{k+1}^T d_{k+1} + \|g_{k+1}\|^2}{\|g_{k+1}\|^2} \le \left(\|y_k\|\|g_{k+1}\|\right) \frac{\left(2A + 2\|g_{k+1}\|\|s_k\|\right)}{\|s_k\|^2} . \frac{g_{k+1}^T d_k}{\|g_{k+1}\|^2} \tag{35}$$

$$\frac{g_{k+1}^T d_{k+1} + \|g_{k+1}\|^2}{\|g_{k+1}\|^2} \le \left(2A + 2\|g_{k+1}\|\|s_k\|\right) . \frac{g_{k+1}^T d_k}{\|g_{k+1}\|\|s_k\|} \tag{36}$$

Using strong Wolfe condition

$$\frac{g_{k+1}^T d_{k+1} + \|g_{k+1}\|^2}{\|g_{k+1}\|^2} \le 2A \frac{g_{k+1}^T d_k}{\|g_{k+1}\|\|s_k\|} + 2\|g_{k+1}\|\|s_k\| \frac{-\rho g_k^T d_k}{\|g_{k+1}\|\|s_k\|} \tag{37}$$

$$\frac{g_{k+1}^T d_{k+1} + \|g_{k+1}\|^2}{\|g_{k+1}\|^2} \le 2A \frac{g_{k+1}^T d_k}{\|g_{k+1}\|\|s_k\|} \tag{38}$$

Using S=λd

$$\frac{g_{k+1}^T d_{k+1} + \|g_{k+1}\|^2}{\|g_{k+1}\|^2} \le \frac{2A}{\lambda} \frac{g_{k+1}^T d_k}{\|g_{k+1}\|\|d_k\|} \le 1 \tag{39}$$

Where $0 < \lambda < 1$

$$\frac{\|g_{k+1}\|^2}{g_{k+1}^T d_{k+1} + \|g_{k+1}\|^2} \ge \frac{\lambda}{2A} \frac{\|g_{k+1}\|\|d_k\|}{g_{k+1}^T d_k} = \delta > 1 \tag{40}$$

$$\frac{g_{k+1}^T d_{k+1} + \|g_{k+1}\|^2}{\|g_{k+1}\|^2} \le \frac{1}{\delta} \tag{41}$$

$$g_{k+1}^T d_{k+1} + \|g_{k+1}\|^2 \le \frac{1}{\delta}\|g_{k+1}\|^2 \tag{42}$$

$$g_{k+1}^T d_{k+1} \le -(1-\frac{1}{\delta})\|g_{k+1}\|^2 \tag{43}$$

Let $c = (1-\frac{1}{\delta})$ (44)

Then $g_{k+1}^T d_{k+1} \le -c\|g_{k+1}\|^2$ (45)

For some positive constant c>0. This condition has often used to analyze the global convergence of conjugate gradient methods with inexact line search.

5.2 Global Convergence Property:

The conclusion of the following lemma used to prove the global convergence of nonlinear conjugate gradient methods, under the general Wolfe line search.

Lemma 1:

Suppose assumptions (1) (i) and (ii) hold and consider any conjugate gradient method (27) and (3), where d_k is a descent direction and λ_k is obtained by the strong Wolfe line search. If

$$\sum_{k\ge 1}^{\alpha} \frac{1}{\|d_k\|^2} = \alpha \tag{46}$$

Then $\liminf_{k\to\infty} \|g_k\| = 0$ (47)

For uniformly convex functions that satisfy the above assumptions, we can prove that the norm of d_{k+1} given by (27) is bounded above. Assume that the function f is a uniformly convex function, i.e. there exists a constant $\mu \ge 0$ such that for all $x, y \in S$,

$$(g(x)-g(y))^T (x-y) \ge \mu\|x-y\|^2, \tag{48}$$

Using lemma 1 the following result can be proved.

Theorem 2:

Suppose that the assumptions (i) and (ii) hold. Consider the algorithm (3), (27). If $\|s_k\|$ tends to zero, and there exists nonnegative constants $\eta 1$ and $\eta 2$ such that:

$$\|g_k\|^2 \geq \eta 1 \|s_k\|^2, \qquad \|g_{k+1}\|^2 \geq \eta 2 \|s_k\| \tag{49}$$

and f is a uniformly convex function, then.

$$\liminf_{k\to\infty} \|g_k\| = 0 \tag{50}$$

Proof: From eq. (27) We have:

$$\beta_k^{new} = \left(1 - \left[\frac{y_k^T s_k}{\|s_k\|^2}\right]\right) \frac{y_k^T g_{k+1}.\left(2A + 2g_{k+1}^T s_k\right)}{\|s_k\|^2} \tag{51}$$

$$\left|\beta_k^{new}\right| = \left|\left(1 - \left[\frac{y_k^T s_k}{\|s_k\|^2}\right]\right) \frac{y_k^T g_{k+1}.\left(2A + 2g_{k+1}^T s_k\right)}{\|s_k\|^2}\right|$$

(52)

$$\leq \left(1 - \left[\frac{\|y_k\|\|s_k\|}{\|s_k\|^2}\right]\right) \frac{\|y_k\|\|g_{k+1}\|\left(2A + 2\|g_{k+1}\|\|s_k\|\right)}{\|s_k\|^2} \tag{53}$$

$$\leq \left(\|y_k\|\|g_{k+1}\|\right) \frac{\left(2A + 2\|g_{k+1}\|\|s_k\|\right)}{\|s_k\|^2} \tag{54}$$

But $\|y_k\| \leq L\|s_k\|$,. Then

$$\leq \left(L\|s_k\|\|g_{k+1}\|\right) \frac{\left(2A + 2\|g_{k+1}\|\|s_k\|\right)}{\|s_k\|^2} \tag{55}$$

$$\leq \left(L\|g_{k+1}\|\right) \frac{\left(2A + 2\|g_{k+1}\|\|s_k\|\right)}{\|s_k\|} \tag{56}$$

$$\leq \left(L\Gamma\right) \frac{\left(2A + 2\Gamma\eta\right)}{\|s_k\|} \tag{57}$$

Hence,

$$\|d_{k+1}\| \leq \|g_{k+1}\| + \left|\beta_k^N\right| \|s_k\| \tag{58}$$

$$\|d_{k+1}\| \leq \gamma + (L\Gamma)\frac{(2A + 2\Gamma\eta)}{\|s_k\|}\|s_k\| = \gamma + (2AL\Gamma + 2\Gamma^2 L\eta)$$

(59)

$$\sum_{k\geq 1} \frac{1}{\|d_{k+1}\|^2} = \infty \qquad (60)$$

$$\frac{1}{\|\gamma + (2AL\Gamma + 2\Gamma^2 L\eta)\|^2} \sum_{k\geq 1} 1 = \infty$$

(61)

6. Parallel Tangent Method:

The name of parallel tangent (PARTAN) has no significance as far as the mechanics of the search procedure are concerned; however, the name has an interesting geometrical origin, which shown in the two-dimensional case of Fig. 1. [10].

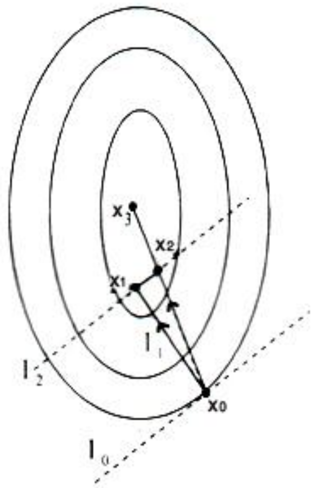

Figure 1. Locus of the search for a quadratic function.

The strong point common to all PARTAN methods, is that the acceleration step from x_0 through x_2 to x_3 is taken through the two points x_0 and x_2 at which the two parallel lines L_0 and L_2 are tangent to the equi-magnitude contours. To see this consider any two lines in the $x_1 x_2$ plane which are parallel and which intersect a straight ravine of $f(x_1, x_2)$ (Fig. (2)). Observed that the point of tangency defines a line, which parallels the ravine. Hence, by searching along the parallel ravine-line, we effectively follow the ridge. The gradient descent searches are used to find $x_1, x_2, x_4, x_6, \ldots$ and acceleration steps are used to locate $x_3, x_5, x_7, x_9, \ldots$. With PARTAN, the acceleration steps conducted through the following pairs of points:

$$(x_0, x_2), (x_1, x_4), (x_3, x_6), \ldots, (x_{2k-3}, x_{2k}), \ldots$$

The locus of the gradient-PARTAN search would look as depicted in Fig. (3) below.

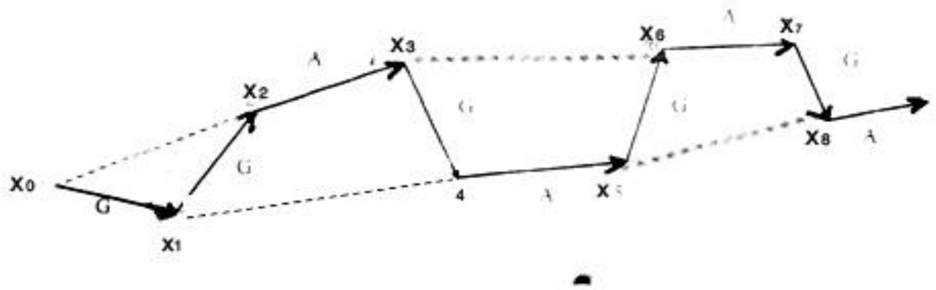

Figure 2. The path taken by the gradient-PARTAN.

6.1. A General Outlines of the PARTAN Algorithm:

Starting procedure: For the first step,

Let, $d_0=-g_0$ and $x_1 = x_0 + \lambda_0 d_0$, Next, choose $d_2=-g_2$

Then, the fourth point is generated by moving in direction that is collinear with (x_3-x_1) so that $d_3= -(x_3-x_1)$

This referred to as an acceleration step. Continuing the procedure:

After determining x_4, the procedure continued by successively alternating gradient and acceleration steps. Thus

$$d_i=-g_i \qquad \text{for } i=0,2,\ldots,2n-2 \tag{62}$$

$$d_i=-(x_i-x_{i-2}) \qquad \text{for } i=3,5,\ldots,2n-1 \tag{63}$$

This method will reach the minimum of an n dimensional quadratic surface in no more than $2n$ steps[11].

7. Sine - Cosine Algorithm (SCA):

Mathematical sine - cosine algorithm is one of the meta-heuristic suggest by (Seyedali Marjalili 2016) which depends in general on sine and cosine functions that starts with improving a set of arbitrary solutions , then we estimate these solutions repeatedly using objective function which improved by a set of rules representing the essence of improving technique . Since techniques based on community aim at optimization for improving problems, there no guarantee to find solution in one term. With existence of a sufficient number of arbitrary solutions and improving steps (repetition), there is high probability to get optimum solutions and global values. SCA method based on finding and improving solutions, changing

$$X_k^{t+1} = X_k^t + r_1 \times \sin(r_2) \times \left| r_3 P_k^t - X_k^t \right| \tag{64}$$

$$X_k^{t+1} = X_k^t + r_1 \times \cos(r_2) \times \left| r_3 P_k^t - X_{kk}^t \right| \quad (65)$$

Where sine and cosine, are the well know Mathematic functions. X_i^t is the present solution position in dimension i-th, With repetition t-th . r1 , r2 , r3 , are arbitrary numbers , as well as | | absolute value , r4 is an arbitrary number with in period [0,1] . [9]

If we combined eqs.(64) and (65), we get the following:

$$X_k^{t+1} = \begin{cases} X_k^t + r_1 \times \sin(r_2) \times \left| r_3 P_k^t - X_k^t \right|, & r_4 < 0.5 \\ X_k^t + r_1 \times \cos(r_2) \times \left| r_3 P_k^t - X_{kk}^t \right| & r_4 \geq 0.5 \end{cases} \quad (66)$$

Where r_4 is a random number

The range of sine and cosine in Eqs. (64) to (66) changed using:

$$r_1 = a - t\frac{a}{T} \quad . \quad (67)$$

Where t is current iteration; T maximum number of iterations and a is a constant.

7.1. Outlines of SCA :

Step (1): Select arbitrary initial community (search agents) solutions X.

Step (2): Calculate cost function for each search agents.

Step (3): Return best solution.

Step (4): Select best search agent according to cost function.

Step (5): Update r_1,r_2, r_3 and r_4 .

Step (6): Update search agent position using the equation (64).

Step (7): While t < max no. iterations, go to step 2.

Step (8): Return best solution you got according to its degree to get global solution [9].

8. Modified Conjugate Gradient method (MCG)

It is a Hybrid method, where a conjugate factor that derived and used in Modifying conjugate Gradient algorithm CG, PT in addition to SCA method to produce on improved algorithm of high efficiency:

Below Improved method.

Step (1): Preparing and generating the initial community.

Step (2): improving Initial community by MCG, CG and PT.

Step (3): Calculating suitability function of improved community.

Step (4): Calculating the best position of all search agents to produce Improved new generation.

Step (5): Updating position of each search agents using the SCA algorithm

Step (6): SCA works by using certain repetitions until to reach optimum value or achieve the stop condition when it finish repetition case.

Step (7): To get either minimum value of the function or about or to get the global value.

The following figure represent the new SCA-MCG algorithm

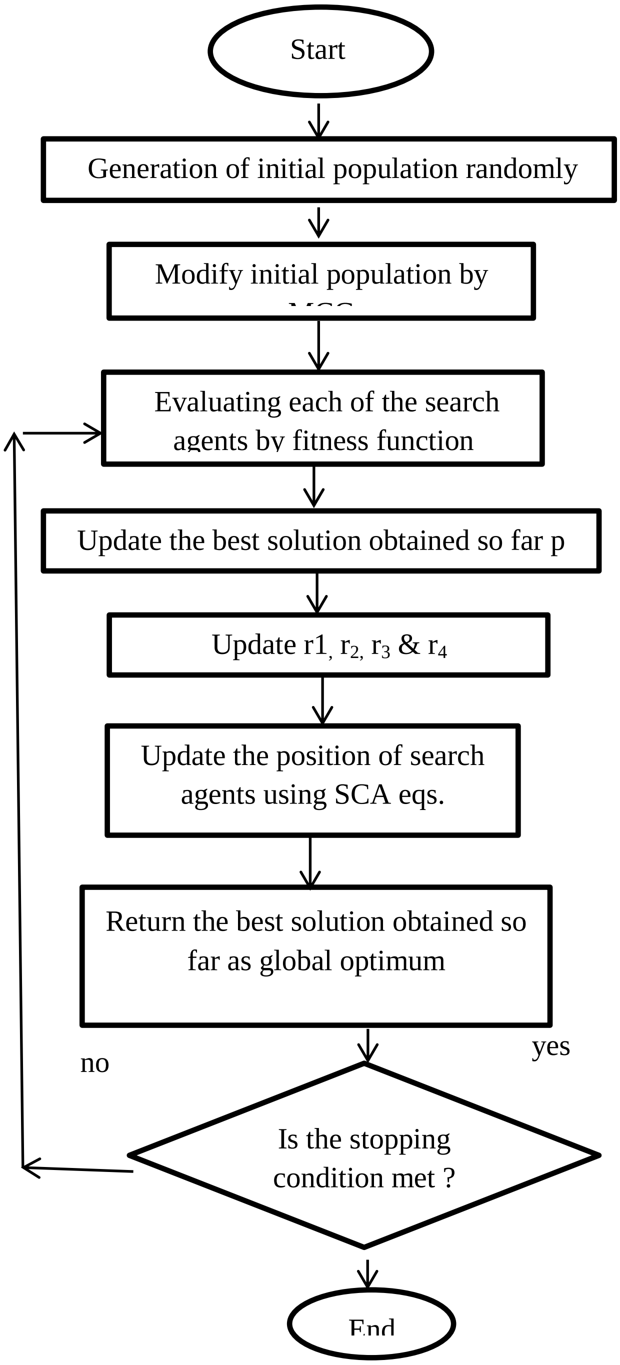

Figure (3): The Proposed SCA-MCG algorithm

9. Practical part:

To evaluate action and probability of the suggested algorithm in solving optimization problems and getting best results. It applied on a set of standard functions mentioned in table (1), to compare with SCA itself. This table includes test functions, functions extends, minimums and maximums, as well as its (F_{min}).

Table l. Test Function

Function	Dim	Range	F_{min}
$F_1(x) = \sum_{i=1}^{n} x_i^2$	30	[-100,100]	0
$F_2(x) = \sum_{i=1}^{n}(\sum_{j-1}^{n} x_i)^2$	30	[-100,100]	0
$F_3(x) = \sum_{i=1}^{n} ix_i^4 + \text{random}[0,1)$	30	[-1.28,1.28]	0
$F_4(x) = \sum_{i=1}^{n}[x_i^2 - 10\cos(2\pi x_i) + 10]$	30	[-5.12,5.12]	0
$F_5(x) = \sum_{i=1}^{n} x_i^2 - \prod_{i=1}^{n}\cos\left(\frac{x_i}{\sqrt{i}}\right) + 1$	30	[-600,600]	0
$F_6 = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 + x_1x_2 - 4x_2^2 + 4x_2^4$	2	[-5,5]	-1.031
$F_7(x) = \sum_{i=1}^{n} -x_i \sin(\sqrt{\lvert x_i \rvert})$	30	[-500,500]	-418.9
$F_8(x) = \sum_{i=1}^{n} \lvert x_i \rvert + \prod_{i=1}^{n} \lvert x_i \rvert$	30	[-10,10]	0
$F_9(x) = max_i \{\lvert x_i \rvert . 1 \le i \le n\}$	30	[-100,100]	0
$F_{10}(x) = -20exp\left(-0.2\sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}\right) - exp(\frac{1}{n}\sum_{i=1}^{n} cos(2\pi x_i) + 20 + e$	30	[-32,32]	0

In Tables (2-4) Functions mentioned below have been applied on all mentioned algorithm. Results show the difference SCA and its improving methods, we notice that we got global values in most function which refers that improving methods for SCA, were of high efficiency, and In Tables (2-4) we notice the efficiency of the algorithms that are directly proportional to the increase in the number of search elements. The more the number of search elements increases, the better the numerical results. Note that function F_{10} gives constant results for all methods used and for all the number of different elements.

Table 2. compare SCA with all other Proposed Hybrid methods at No. of element =10 and iteration=500

Function	SCA	SCA-CG	SCA-PT	SCA-CG-PT	SCA-MCG
F1	2.00E-30	1.46E-226	1.27E-227	0	0
F2	2.13E-26	6.41E-210	1.64E-222	0	0
F3	4.17E-78	0	0	0	0
F4	0	0	0	0	0
F5	0.0026593	0	0	0	0
F6	-1.0316	-1	-1	0	0
F7	-1257.2739	-6.70E-18	-4.08E-22	-6.05E-173	0
F8	7.04E-20	1.23E-117	1.22E-116	2.27E-215	0
F9	3.71E-13	6.54E-111	2.09E-109	5.50E-211	0
F10	8.8818E-16	8.8818E-16	8.8818E-16	8.8818E-16	8.8818E-16

Table 3. compare SCA with all other Proposed Hybrid methods at No. of element =30 and iteration=500

Function	SCA	SCA-CG	SCA-PT	SCA-CG-PT	SCA-MCG
F1	1.26E-49	2.70E-239	3.68E-238	0	0
F2	1.72E-34	3.48E-225	8.68E-227	0	0
F3	1.47E-80	0	0	0	0
F4	0	0	0	0	0
F5	0.030856	0	0	0	0
F6	-1.0316	-1	-1	0	0
F7	-1357.09	-418.983	-418.983	-1.32E-124	0
F8	1.92E-23	3.06E-120	1.21E-122	1.86E-220	0
F9	3.66E-18	2.08E-115	2.70E-113	7.51E-215	0
F10	8.8818E-16	8.8818E-16	8.8818E-16	8.8818E-16	8.8818E-16

Table 4. compare SCA with all other Proposed Hybrid methods at No. of element =50 and iteration=500

Function	SCA	SCA-CG	SCA-PT	SCA-CG-PT	SCA-MCG
F1	3.58E-47	4.04E-242	4.04E-242	0	0
F2	9.74E-40	6.08E-229	6.08E-229	0	0
F3	1.01E-88	0	0	0	0
F4	0	0	0	0	0
F5	0	0	0	0	0
F6	-1.0316	-1	-1	0	0
F7	-1401.8469	-4.19E+02	-4.19E+02	-1.18E-108	0
F8	1.15E-26	1.10E-124	1.10E-124	6.63E-225	0
F9	2.48E-20	7.79E-119	7.79E-119	3.23E-218	0
F10	8.8818E-16	8.8818E-16	8.8818E-16	8.8818E-16	8.8818E-16

10. Conclusions:

The process of improving and hybrid of Heuristic SCA with the suggested method of MCG and other classical ones such as CG and PT leads to increase the convergence speed and avoid falling in local solutions. It also assists in improving the resulted solution kind through the increase of detecting algorithm efficiencies. Where the results show the possibility of the improved algorithm to solve different optimization problems, after comparison, results were excellent, where global optimum value had reached in most test function. This shown in the numerical results of this study.

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Solving Max-Cut Optimization Problem

Basim Rabaa Jumaa
College of Education for Pure Sciences
University of Babylon
basim4345@gmail.com (07712276044)

Dr Ahmed Sabah Al-Jilawi
College of Education for Pure Sciences
University of Babylon
aljelawy2000@yahoo.com (07719409850)

Abstract : The goal of this paper is to find a better method that converges faster of Max-Cut problem. One strategy is to the comparison between Bundle Method and the Augmented Lagrangian method. We have also developed the theoretical convergence properties of these methods.

Keywords: Max-Cut, Augmented Lagrangian Method, Bundle Methods and Constrained Optimization problems.

1. INTRODUCTION

Optimization is a primary mathematical method aimed at finding the value of variables that provide the minimum value for a mathematical function. Optimization algorithms are a basic and efficient technique in mathematical programming, to arrive at a solution, generally with the help of a computer. Optimization algorithms start with a first estimate of the value of the variables and by an iterative technique generates a sequence of get better estimates, or iterates, until an optimal solution is reached.

A great algorithm should be accurate, fast, efficient and robust. A good approximation of an optimal solution should be generated.

We here present a short overview of multiplier methods [**1**]. The beginning in the field of multiplier methods begins with Joh (1943). Kuhn and Tucker (1951) are eminent scientists who have conducted extensive research in the field of multiplier methods. Its results on the necessary conditions and adequate conditions are important in this field. Arrow and Hurwicz (1956) introduced the Lagrangian function [**2**]. King (1966) [**3**] developed the augmented Lagrangian algorithms.

Hestenes and Powell [**4**] showed that the algorithm is locally convergent if the second-order sufficient conditions are satisfied. Miele et al. (1971) [**5**] and Rockafellar (1970) [**6**] introduced an augmented Lagrangian method for inequality constrained convex programming.

This method has been studied by Rockafellar in several papers [**7**]. The augmented Lagrangian method has got a powerful theoretical tool for convex programming. Arrow, Gould, and Howe (1971) [**8**] studied Rockafellar's augmented Lagrangian method and Pierre (1971) [**9**] introduced a special augmented Lagrangian method with local convergence properties. This method was also studied by Lowe (1974) [**10**], Bertsekas (1982) [**11**], Humes (2000), R. A. Polyak (2001), R. A. Castillo (2003), J. M. Martinez (2006), S. Leyffer (2007) and H. Z. Luo et al. (2011).

Recently, many researchers have been interested in Lagrange's enhanced methods, such as Leyffer (2016) [**12**], Kanzow et al. (2018) [**13**] and Lourenço (2018) [**14**]. The benefit of the augmented Lagrangian method is that it is robust, and we do not need a feasible beginning point. The augmented Lagrangian method has been used to solve optimization problems with both equality and inequality constraints [**11**].

Also, the Bundle method was independently created by Claude Lemarechal [**15**] and Philip Wolfe [**16**] in (1975). Since then a large number of variants of bundle methods have been developed, such as proximal bundle (1990) [**17**], trust region bundle (2001) [**18**]. Bundle methods are at the moment the most efficient and promising methods for smooth optimization and they have been successfully used in many practical applications, for example, in engineering, economics, mechanics and optimal control.(2002) [**19**]. The convergence of the minimization algorithm was studied and compare them with different versions of the bundle methods using the results of numerical experiments (2013) [**20**]. Bundle methods have been extensively studied to solve convex and nonconvex optimization problems (2015) [**21**]. The a simple version of the bundle method with linear programming is suggested. (2019) [**22**].

2. MAX-CUT Problem

The maximum cut (MAX-CUT) problem is an fascinating area of combinatorial optimization and has several applications in different fields, for instance, physics, computer science, and mathematics. This problem is NP-hard [**23**] The abbreviation NP denoting for non-deterministic polynomial time, which means NP-hard is a difficult problem that can not be solved accurately. Several papers have studied the MAX-CUT problem. This line of research was started by Goemans-Williamson (1995) [**24**] with their approximation for the MAX-CUT problem based on semidefinite programming relaxation. Poljak showed that linear programming techniques cannot accomplish a better approximation solution [**25**], which is why semidefinite programming has attracted great importance and research activity. (For more details see [**26**]).

Suppose $G = (V, E)$ A nondirected graph is with the vertex set V and edge set E, and suppose $w_{ij} = w_{ji}$ be edge weight $ij \in E$ and $w_{ij} = 0$ if $ij \notin E$. The adjacency matrix of G is given by $A = [a_{ij}]$ such that $a_{ij} = w_{ij}$.

We can write the Mac-Cut problem model as

$$(MC) \quad \begin{cases} \text{maximize} & \sum_{i<j} w_{ij} \left(\frac{1-x_i x_j}{2}\right) \\ \text{suchthat} & x \in \{-1,1\}^n, \end{cases}$$

the objective function $\sum_{i,j \in E} w_{ij} \left(\frac{1-x_i x_j}{2}\right)$ = number of cut edges. For example, if

$$x = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \in \{-1,1\}^n.$$

Then

$$\begin{cases} x_1x_2 = (-1)(1) = -1 \\ x_2x_3 = (1)(1) = 1 \\ x_1x_5 = (-1)(-1) = 1 \\ x_4x_5 = (1)(-1) = -1 \end{cases} \rightarrow \begin{cases} \text{edge}(1,2) \ \text{ is cut} \\ \text{edge}(2,3) \ \text{ is not cut} \\ \text{edge}(1,5) \ \text{ is not cut} \\ \text{edge}(4,5) \ \text{ is cut} \end{cases}$$

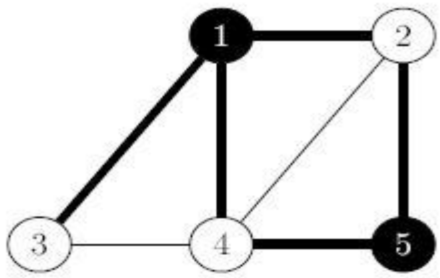

Figure 1: Example MAX- CUT .

3. The bundle and Augmented Lagrangian Methods

In the optimization problem, we wish to minimize or maximize some function subject to some constraint. The general problem of optimization given by [**27, 28**]:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{suchthat} & x \in X. \end{array} \tag{1}$$

The function f is defined from a convex set $X \subseteq \mathbb{R}^n$ into $\mathbb{R}$. A point $x^* \in X$ is a local solution of problem (1) if there exists a neighborhood $B(x^*, k)$ where $f(x^*) \leq f(x)$ for every $x \in B(x^*, k) \cap X = \{x \in X \mid \ \| x - x^* \| \leq k\}$.

3.1 Optimality Conditions for Unconstrained Optimization

In this section ,We consider the problem of unconstrained optimization. If $X = \mathbb{R}^n$, i.e., minimize f sans constraints [**27, 28**], it can be expressed by:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \qquad f(x). \tag{2}$$

• If f is continuously differentiable, then a necessary condition for $x^* \in \mathbb{R}^n$ is a solution of problem (2)

$$\nabla f(x^*) = 0.$$

• If f is twice continuously differentiable, then a necessary condition for $x^* \in \mathbb{R}^n$ is a solution of problem (2)

$$\nabla f(x^*) = 0, \quad \nabla^2 f(x^*) \succeq 0.$$

• The sufficient conditions for $x^* \in \mathbb{R}^n$ is a local solution of problem (3)

$$\nabla f(x^*) = 0, \quad \nabla^2 f(x^*) \succ 0.$$

Theorem 3.1(First-Order Necessary Condition (FONC)) [**28**]

Suppose f: $\mathbb{R}^n \to \mathbb{R}$ be differentiable. If x^* is a local minimizer of f, then $\nabla f(x^*) = 0$.

Proof: Define $h: \mathbb{R} \to \mathbb{R}$ as $h(\rho) = f(x^* + \rho w)$ for some $w \in \mathbb{R}^n$, then

$h'(\rho) = w^T \nabla f(x^* + \rho w)$. If $\rho = 0$, than $h'(0) = w^T \nabla f(x^*)$. By definition,

$$h'(0) = \lim_{\rho \to 0} \frac{f(x^* + \rho w) - f(x^*)}{\rho}.$$

Since x^* is a local minimizer, there exists $k > 0$, where $f(x^* + \rho w) \geq f(x^*)$ for every $0 < \rho \leq k$, thus we get $w^T \nabla f(x^*) \geq 0$. Since w it is arbitrary, we can substitute w by $-w$, and thus $-w^T \nabla f(x^*) \geq 0$. So, $w^T \nabla f(x^*) = 0$, for every $w \in \mathbb{R}^n$. Thus, $\nabla f(x^*) = 0$. □

Theorem 3.2 (Second-Order Necessary Condition (SONC)) [28]

Suppose f: $\mathbb{R}^n \to \mathbb{R}$ be twice differentiable. If x^* is a local minimizer of f, then $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive semidefinite.

Proof: Suppose $w \in \mathbb{R}^n$. We want to prove that $w^T \nabla^2 f(x^*) w \geq 0$.

By using Taylor expansion of f at x^*, we get

$$f(x^* + \rho w) = f(x^*) + \rho w^T \nabla f(x^*) + \frac{\rho^2}{2} w^T \nabla^2 f(x^*) w + o(\rho^2).$$

Since $\nabla f(x^*) = 0$ by (FONC) theorem, we have

$$f(x^* + \rho w) = f(x^*) + \frac{\rho^2}{2} w^T \nabla^2 f(x^*) w + o(\rho^2).$$

Divide the sides on ρ^2, we have

$$\frac{f(x^*+\rho w) - f(x^*)}{\rho^2} = \frac{1}{2} w^T \nabla^2 f(x^*) w + \frac{o(\rho^2)}{\rho^2}.$$

We take the limit to both sides, and use the fact of that x^* is a local minimizer, we get

$$0 \leq \lim_{\rho \to 0} \frac{f(x^*+\rho w) - f(x^*)}{\rho^2} = \lim_{\rho \to 0} \{\frac{1}{2} w^T \nabla^2 f(x^*) w + \frac{o(\rho^2)}{\rho^2}\}.$$

Since

$$\lim_{\rho \to 0} \frac{o(\rho^2)}{\rho^2} = 0,$$

we conclude that $w^T \nabla^2 f(x^*) w \geq 0$. So , $\nabla^2 f(x^*)$ is positive semidefinite. □

Theorem 3.3 (Second-Order Sufficient Condition (SOSC)) [28]

Suppose f: $\mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. If $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive definite, then x^* is a strict local minimizer.

3.2. The Augmented Lagrangian Method

This method started to be used in the 1970s. Initially, it was called the multipliers method. Now, this method is called the augmented Lagrangian method. The goal of this method is to solve constrained optimization problems. This is done by substitute a constrained problem with a series of unconstrained problems [4]. The augmented Lagrangian method is analogous to the bundle method since in both of them a bundle term is added to the objective. The difference in the augmented Lagrangian method is the Lagrange multiplier term is added to it [**27**].

The augmented Lagrangian method was introduced by Hestenes [**4**]. To introduce the augmented Lagrangian method, we change the constraint $h_i(x) = 0$ to the constraint $h_i(x) + \beta\alpha = 0$. therefore, we get the problem

$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subjectto} & h_i(x) + \beta\alpha = 0, & i = 1, \dots, E, \\ & x \in X. & \end{array} \tag{3}$$

We apply the bundle method to the problem (3), can get the augmented Lagrangian function as follows. We start with the bundle problem for the problem (3)

$$\underset{x}{\text{argmin}}\ f(x) + \frac{1}{2\beta}(h(x) + \beta\alpha)^T(h(x) + \beta\alpha),$$

this expands to

$$\underset{x}{\text{argmin}}\ f(x) + \frac{1}{2\beta}(h(x)^T h(x) + 2\beta\alpha^T h(x) + \beta^2\alpha^T\alpha),$$

after simplification we get

$$\underset{x}{\text{argmin}}\ f(x) + \alpha^T h(x) + \frac{1}{2\beta} \parallel h(x) \parallel^2.$$

Thus, the augmented Lagrangian function is

$$\mathcal{F}(x, \alpha, \beta) = f(x) + \alpha^T h(x) + \frac{1}{2\beta} \parallel h(x) \parallel^2.$$

We apply the equality augmented Lagrangian and the bundle methods to the linear programming (LP) problem. The initial results will provide an idea to work in semidefinite programming. The bundle method and augmented Lagrangian methods can be used for equality

The generally LP problem is given as

$$(LP) \quad \begin{cases} \text{minimize} & \langle c, x \rangle \\ \text{subjectto} & \langle a_i, x \rangle = b_i, \quad i = 1, \dots, m \\ & x \geq 0, \quad x \in \mathbb{R}^n. \end{cases}$$

where $a_i \in \mathbb{R}^n$, for $i = 1, \dots, m$, the numbers $b_i \in \mathbb{R}$, for $i = 1, \dots, m$, and the vector $c \in \mathbb{R}^n$, $x_i \geq 0$, for $i = 1, \dots, n$. The Lagrangian is given by

$$\mathcal{L}(x, y) = \langle c, x \rangle + \langle y, b_i - \langle a_i, x \rangle \rangle$$

The general form for the augmented Lagrangian is

$$\mathcal{L}_\alpha(x, y) = f(x) + \langle y, b_i - \langle a_i, x \rangle \rangle + \frac{1}{2\alpha} \parallel b_i - \langle a_i x \rangle \parallel^2.$$

Consider the primal (P) and the dual (D) standard linear programming problem:

$$(P) \quad \begin{cases} \text{maximize} & \langle c, x \rangle \\ \text{subjectto} & Ax = b \\ & x \geq 0 \end{cases}$$

and

$$(D) \quad \begin{cases} \text{minimize} & \langle b, y \rangle \\ \text{subjectto} & A^T y \geq c \end{cases}$$

Example 3.1 Consider the following simple linear programming problem:

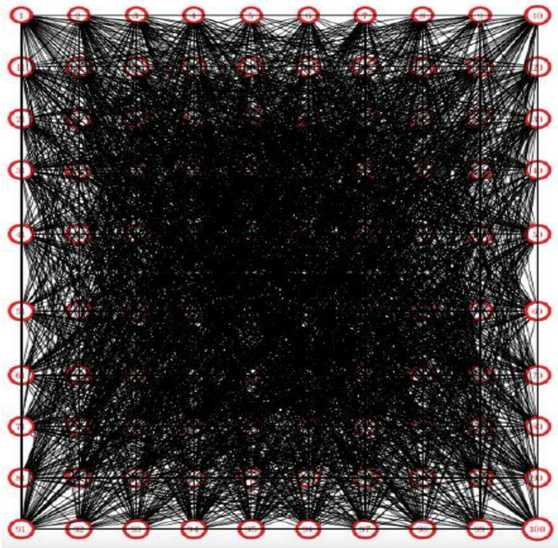

Figure 2: Biq Mac library (100 nodes and 2475 edges)

$$(P) \quad \begin{cases} \text{maximize} & x_1 + 2x_2 + 4x_3 \\ \text{subjectto} & x_1 + x_2 + x_3 = 9 \\ & x \geq 0. \end{cases} \tag{4}$$

The dual of problem (4) is given by

$$(D) \quad \begin{cases} \text{minimize} & 9y_1 \\ \text{subjectto} & y_1 \geq 1 \\ & y_1 \geq 2 \\ & y_1 \geq 4 \end{cases} \tag{5}$$

The optimal solution of problem (4) is $x_1 = x_2 = 0,\ x_3 = 9$ and the optimal value is $x_1 + 2x_2 + 4x_3 = 36$; an optimal solution of problem (5) is $y_1 = 4$, and the optimal value is $9y_1 = 36$.

4. Algorithms and Numerical Computation

In this section, We discuss the numerical results of the algorithms by using Julia Language (JuliaBox). The numerical results were generated using the augmented Lagrangian method, which was Validated with the bundle method. This test was done on a specific graph that was imported from the Biq Mac library [**29**] in Figure 2. This graph consists of 100 nodes connected with 2475 edges. The figure shows that the exact MAX-CUT solution equals 1430

4.1. Augmented Lagrangian Methods

The multiplier method is to update the Lagrange Multiplier estimate [**29**] α and sometimes the bundle parameter β in all iteration. The method of the multiplier is summarized in the Algorithm [1].

Algorithm [1] : The Augmented Lagrangian Methods

---

1. Choose x^0, and $\beta^0 > 0$, choose α^0.
2. Find x^{k+1} such that

$$x^{k+1} = \underset{x}{\operatorname{argmin}} \ \mathcal{F}(x, \alpha^k, \beta^k).$$

3. Update β^k and α^k.
4. Set $k = k + 1$ and repeat.

4.2. Bundle Methods [30]

We define another method that can be considered as a stabilization of the plane's cutting method. We start by adding an additional point called the center,y^k, to the bundle of information. We will continue to use the same linear model for our function f, but it is no longer a solution **LP** on each iteration. Instead, we will compute the next iterate of the Algorithm [2].

Algorithm [2] : Bundle Method

---

1. Let $\delta > 0, m \in (0; 1), y^0, x^0 = y^0$, and $k = 0$. Compute $f(y^0)$
2. Compute the next iterate

$$x^{k+1} = \underset{y \in \mathbb{R}^n}{\operatorname{argmin}} f_k(x) + \frac{\alpha}{2} \parallel x - y^k \parallel^2$$

3. Define $\delta_k := f(y^k) - [f_k(x^{k+1}) + \frac{\alpha}{2} \parallel x^{k+1} - y^k \parallel^2] \geq 0$
4. If $\delta_k < 0$ Stop
5. Compute $f(x^{k+1})$
6. Update the model

$$f_{k+1}(x) := max\{f_k(x), f(x^{k+1}) + \langle s^{k+1}, x - x^{k+1} \rangle\}$$

7. Set $k = k + 1$ and go to **Step 2**

4.3. Numerical Resultes

In this section, we will review our results and we are assessing the performance of the development algorithm proposed. The figures in this section illustrate the number of function calls of the approaches being used to solve the max-cut problems. Different sizes of cases were tested, results were extracted and shown in this section.

In Figure [3] and Figure[4], it is obvious that the Augmented Lagrangian Method provides a more rapid convergence. The bundle methods converge after 4s, while the augmented Lagrangian methods required only 3s for the convergence.

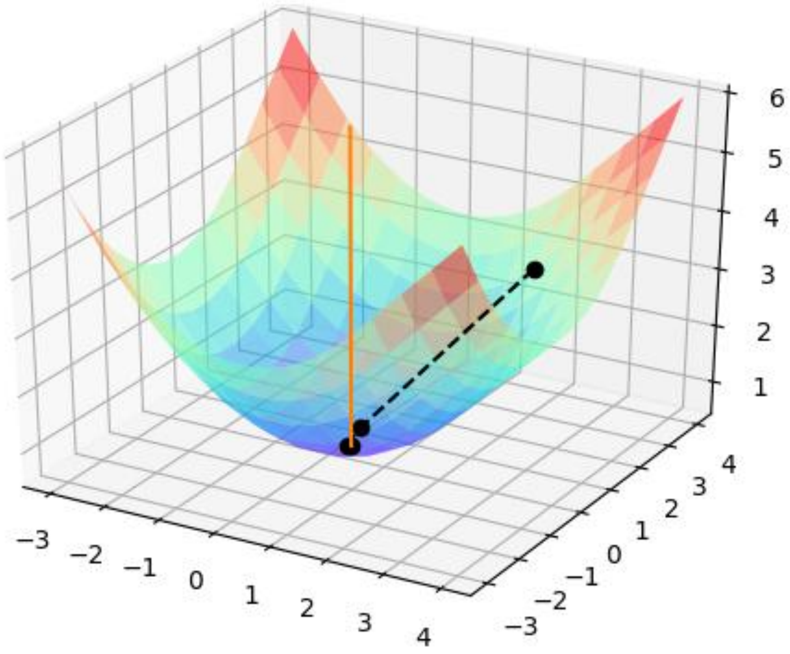

Figure 3: Augmented Lagrangian

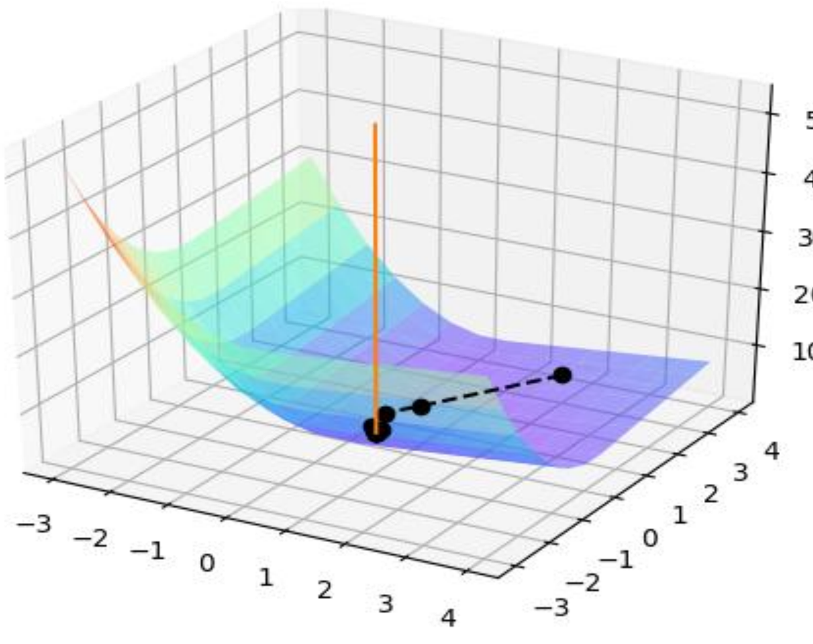

Figure 4: The Bundle

CPU time may change between runs due to the use of other software on the same computer. Accordingly, it was more accurate to plot the number of function calls rather than the CPU time, Which cannot be affected by any other program that is run at the same time by using our program, it is evident that the augmented Lagrangian method performed faster and required fewer function calls, while the bundle method required more function calls.

4.4. L-BFGS and BFGS methods [**31**]

In the section, we present methods solve of minimizing the problem optimization [L-BFGS, BFGS and CQ methods]

- Limited-memory BFGS (L-BFGS) is an optimization algorithm in the quasi-Newton method family that is using a limited amount of computer memory to approximate the Broyden – Fletcher – Goldfarb – Shanno algorithm (BFGS).
- It is a common algorithm for parameter estimation in machine learning. The target problem for the algorithm is to minimize over unconstrained values for the real vector.
- The algorithm L-BFGS solves the problem of minimizing an objective, given its gradient, by Iteratively measure approximations of the Hessian inverse matrix.
- The conjugate gradient (CQ) method Is an algorithm for the numerical solution of specific linear equation systems, namely those whose matrix is symmetric positive-definite.

Table [1] and Table [2] reports the CPU time and number of function calls for several graphs in the Biq Mac library that have 100 nodes and 2475 edges [**29**].

Augmented Lagrangian Method										
	Aug. (LBFGS) m=1		Aug. (LBFGS) m=2		Aug. (LBFGS) m=10		Aug. (BFGS)		Aug. (CQ)	
Problem	time	fcalls	time	fcalls	time	fcalls	time	fcalls	time	fcalls
g05_100.0	1.22	112	1.31	120	1.38	118	1.88	161	0.75	80
g05_100.1	1.19	118	1.22	108	1.25	102	1.39	119	0.73	82
g05_100.2	1.24	115	1.31	125	1.34	118	1.60	158	0.65	81
g05_100.3	1.37	130	1.27	122	1.70	139	2.15	181	0.87	100
g05_100.4	1.80	163	1.68	150	1.63	152	1.72	158	0.88	100
g05_100.5	1.44	120	1.50	130	1.47	122	1.71	155	0.80	90
g05_100.6	1.30	110	1.30	109	1.49	117	1.35	116	0.74	86
g05_100.7	1.25	106	1.20	108	1.26	108	1.20	100	0.77	79
g05_100.8	1.43	119	1.45	119	1.50	126	1.46	126	0.88	90
g05_100.9	1.46	102	1.30	96	1.36	100	2.01	91	1.12	72

Table 1: CPU time and function calls number of iterations for augmented Lagrangian method

Bundle Method										
	Bundle. (LBFGS) m=1		Bundle. (LBFGS) m=2		Bundle. (LBFGS) m=10		Bundle. (BFGS)		Bundle. (CQ)	
Problem	time	fcalls	time	fcalls	time	fcalls	time	fcalls	time	fcalls
g05_100.0	2.65	240	2.50	233	2.46	234	3.21	302	1.44	161
g05_100.1	2.00	204	2.01	202	2.40	203	3.31	255	1.05	133
g05_100.2	2.30	250	2.52	260	2.82	261	3.18	355	1.58	187
g05_100.3	2.85	295	2.65	270	2.79	270	5.44	444	1.65	218
g05_100.4	3.15	338	3.00	329	3.40	322	5.27	515	2.00	234
g05_100.5	2.50	266	2.68	282	2.58	267	4.02	416	1.70	192
g05_100.6	3.00	234	2.88	240	2.37	242	3.22	340	1.47	160
g05_100.7	2.26	238	2.10	212	2.21	222	2.72	281	1.18	150
g05_100.8	2.61	267	2.26	234	2.55	239	3.81	355	1.81	186
g05_100.9	3.00	229	2.82	218	2.75	216	3.77	315	1.82	149

Table 2: CPU time and function calls number of iterations for bundle method

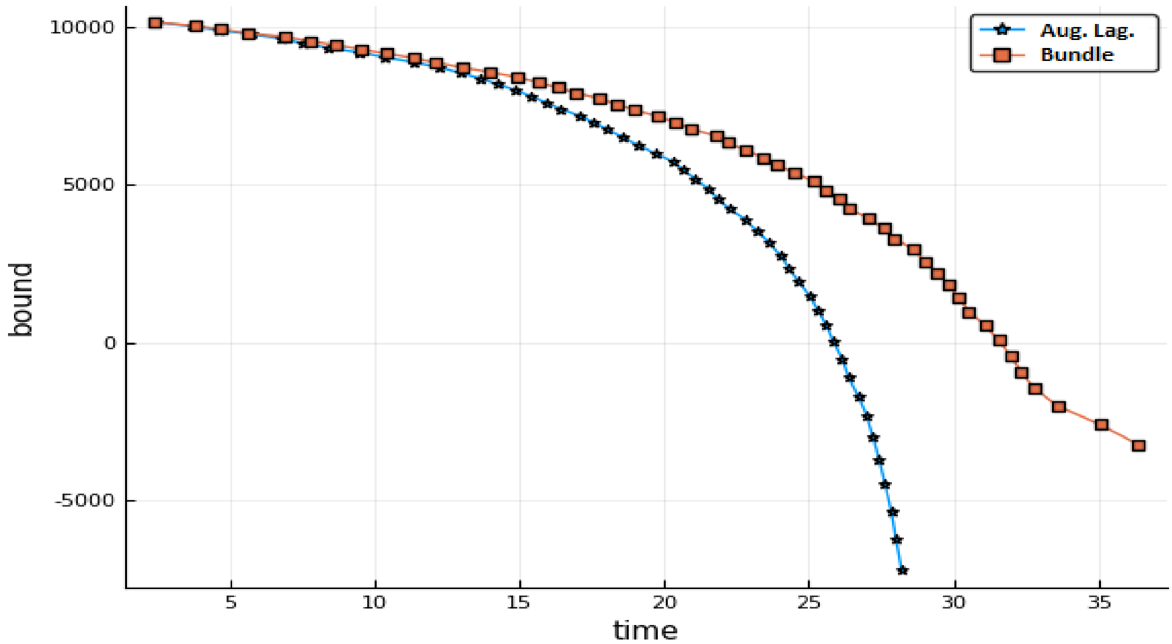

Figure 5: Bounds vs CPU time for augmented Lagrangian and bundle methods.

In figure [5] we plotted the bounds against the CPU time to compare the performance of the augmented Lagrangian methods and the bundle methods. This test was performed on a specific graph imported from the Biq Mac library [**29**]. It is evident that the augmented Lagrangian method performed faster and required 28 function calls to converge, while the bundle method required more than 37 function calls.

5. The Theoretical Convergence Properties of the bundle and Augmented Lagrangian Methods

In this section, we discuss the current major convergence theorems. We start with the convergence of the bundle and augmented Lagrangian methods. Recall that

$$P_q(x,\beta) = f(x) + \frac{1}{2\beta} \parallel h(x) \parallel^2,$$

and, the augmented Lagrangian function is

$$\mathcal{F}(x,\alpha,\beta) = f(x) + \alpha^T h(x) + \frac{1}{2\beta} \parallel h(x) \parallel^2.$$

suppose us define $D(x)$ to be $E \times n$ Jacobian of $h(x)$, such that

$$h(x) = [h_1(x), \dots, h_E(x)]^T.$$

Hence

$$D(x)^T = [\nabla h_1(x), \dots, \nabla h_E(x)].$$

Theorem 5.1 (Convergence of augmented Lagrangian method) [**32**]

Suppose f and h be twice continuously differentiable functions. suppose

$$y^k = \alpha^k + \frac{h(x^k)}{\beta^k},$$

and

$$\parallel \nabla_x \mathcal{F}(x^k, \alpha^k, \beta^k) \parallel \leq \epsilon^k,$$

where $\epsilon^k \to 0$ as $k \to \infty$. If x^k converges to x^*, where $\nabla h_i(x^*)$, $i = 1, \dots, E$, are linearly independent, then $y^k \to y^*$ with y^* satisfying $\nabla f(x^*) = D(x^*)^T y^*$. If additionally, either $\beta^k \to 0$ with bounded α^k or $\alpha^k \to y^*$ with bounded β^k, then x^* satisfies the (FONC) and y^* is the vector of Lagrange multipliers.

Proof: Since the augmented Lagrangian function is

$$\mathcal{L}(x,\beta,\alpha) = f(x) + \beta^T h(x) + \frac{1}{2\alpha} h(x)^T g(x),$$

we have that

$$\nabla_x \mathcal{L}(x, \beta, \alpha) = \nabla f(x) + D(x)^T \beta + \frac{1}{\alpha} D(x)^T h(x),$$

It can be rewritten

$$\nabla_x \mathcal{L}(x, \beta, \alpha) = \nabla f(x) + D(x)^T \left(\beta + \frac{h(x)}{\alpha}\right).$$

So , we have that

$$\| \nabla_x \mathcal{L}(x^k, \beta^k, \alpha^k) \| = \| \nabla f(x^k) + D(x^k)^T y^k \| \leq \epsilon^k,$$

by assumption $y^k \to y^*$, where $y^* = -(D(x^*)^+)^T \nabla f(x^*)$, and that

$\nabla f(x^*) + D(x^*)^T y^* = 0$. Now, by definition of y^k,

$$\| h(x^k) \| = \alpha^k \| \beta^k - y^k \| \leq \alpha^k \| y^k - y^* \| + \alpha^k \| \beta^k - y^* \|.$$

By assumption, $\alpha^k \to 0$ with bounded β^k or $\beta^k \to y^*$ with bounded α^k, so we have that $h(x^k) \to 0$ in either case. Since x^k converges to x^* and h is continuous,

$h(x^*) = 0$. Thus (x^*, y^*) satisfies the (FONC). □

The Bundle Method [**33**]

The aim is to provide the rate convergence of the bundle method for solving convex optimization problems of a form below

$$\min_{x \in \mathbb{R}^n} F(x) \tag{6}$$

and $F: \mathbb{R}^n \longrightarrow R$ is a convex function.

The bundle method is linked to the basic idea of the proximal point method,who uses the Moreau-Yosida regularization for $F(.)$,

$$F_\rho(y) = \min_x \{F(x) + \frac{\rho}{2} \| x - y \|^2\}, \ \rho > 0,$$

to build the proximal step for (6) ,

$$prox_F(y) = \operatorname*{argmin}_x \{F(x) + \frac{\rho}{2} \| x - y \|^2\}$$

The proximal point method apply the iteration $x^{k+1} = prox_F(x^k)$,

$k = 1,2, \ldots$ and is converging to a minimum of $F(.)$, if a minimum exists [**34**]. The basic idea of the bundle method is to replace the problem (1) with a series of approximate problems of the following form:

$$\min_{x} \; F^k(x) + \frac{\rho}{2} \parallel x - y \parallel^2$$

Here k = 1,2, . . . is the iteration number, x^k is the best approximation to the solution, and $F^k(.)$ is a piecewise linear convex lower approximation of the function $F(.)$.Two versions of the method differ in the way it constructs this approximation.

Theorem 5.2 (Convergence of bundle method): [**33**]

Let Argmin $F \neq \emptyset$ and $\varepsilon = 0$. Then a point $x^* \in ArgminF$ exists such that :

$$\lim_{k\to\infty} x^k = \lim_{k\to\infty} z^k = x^*$$

Proof: The proof of this result (in slightly different versions) it can be found in many references, such as [[**34**] Thm.4.9],[[**35**] Thm.XV.3.2.4], [[**36**] Thm.7.16]

6. Conclusions

The aim of the research has been achieved, and the following points have been clarified :

1- We test comparison between two methods the bundle method and Lagrangian augmentation method.

2- We prove the properties of theoretical convergence and we studied algorithms.

3- The graphs available in the Big Mac library were used to evaluate the method. These drawings included different features with a large number of edges and nodes

4- The results showed that the augmented Lagrangian method reached the goal in fewer the number of function calls of the bundle method and also was timed faster in CPU time.

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Qualitative Analysis and Traveling wave Solutions for the Nonlinear Convection Equations with Absorption

Bashayir N. Abed[1] **Salam J. Majeed**[2] **Habeeb A. Aal-Rkhais**[3]

[123]Department of Mathematics, College of Computer Science and Mathematics,

University of Thi-Qar, Iraq

ABSTRACT: We discuss qualitative behavior of the solutions for the nonlinear parabolic equation which modeling nonlinear convection equation with absorption. This model represents the movement of growing population that is ruled by convection process. In this paper, we concentrate on proving the existence of traveling wave solutions for the nonlinear convection-reaction equations. In addition, we consider the model when the speed of advective wave may breakdown and the problem has a shock wave solution. The mathematical interesting of the waves comes from the behaviors of singular differential equation and discussing the stability of the solution.

Keywords: traveling-waves, convection-reaction process, characteristic methods, stability.

1. Introduction

The traveling waves have played a very important role in many nonlinear parabolic equations modeling reaction-diffusion-convection processes. In this paper we are interested in solutions of nonlinear advection equation model

$$\frac{\partial \varphi}{\partial t} = a\frac{\partial \varphi^{\gamma}}{\partial x} + b\varphi^{\beta}(1-\varphi) \qquad (1)$$

where $\beta > 0, \gamma > 1, \gamma + \beta > 2$; a and b are positive constants and $\varphi = \varphi(x;t)$ is a nonnegative unknown density (concentration), with space x and time t. The study and application of this model clearly appeared in many areas of science such as biological and physical models including shock waves and traveling waves of oscillatory chemical reactions. Existence and uniqueness of the local solution and traveling waves for reaction-diffusion-convection equation are introduced in [1, 2, 3, 7, 8, 12]. Several models of partial differential equations are represented as pattern formations, critical patch sizes, traveling waves, ecological invasion and many others in [5, 9, 10].Combining population growth dynamics with models of movement has ecological interest. the Fisher model is one of the classical model of ecology that represents dispersion and population growth see [6]. In addition, we consider the standard nonlinear reaction-diffusion-convection equation in one dimension. Generally, this equation can show shock wave solutions [4, 5, 11].

In this paper, we considerate on the problem of a nonlinear advection-absorption process which has traveling wave solutions for special situations in one dimensional space. Particularly, we consider the population at a particular position which grows according to the diffusion process that is very weak with respect to the advection effects. It is interesting that we discuss the solutions in the form $\varphi = \varphi(x \pm \lambda t)$ where $\lambda > 0$ represents the speed of the wave and it travels without changing shape.

2. Traveling Wave Solutions

In this section, we find travelling-wave solutions for the equation (1), and give the asymptotic behavior of these solutions of (1) and a description of a nonlinear convection process with a logistic population growth. The second part of (1) represents the nonlinear absorption term. the solution represents population density which is changed per unit time. In the spatially homogeneous status, the steady states of equation (1) for $b > 0$, $\varphi = 0$ and $\varphi = 1$ which are unstable and stable respectively. Before we discuss the existence of solutions, it is appropriate to change the variable $\varphi = v^{1/(\gamma-1)}$ in the equation (1) and it becomes

$$\frac{\partial v}{\partial t} - a\gamma v\frac{\partial v}{\partial x} = b(\gamma-1)v^{\alpha}\left(1 - v^{\frac{1}{\gamma-1}}\right) \qquad (2)$$

where $\alpha = (\gamma + \beta - 2)/\gamma - 1$.

Theorem 1. If $\lambda > 0$, $\gamma > 1$, $\beta > 0$, $\gamma + \beta > 2$; then the traveling wave solution $v(x;t) = f(\xi)$, $\xi = x + \lambda t$ of (2) is satisfied for $0 \le f \le 1$, with the boundary conditions $\lim_{\xi\to-\infty} f(\xi) = 0$ and $\lim_{\xi\to+\infty} f(\xi) = 1$.

Proof. Let us use rescaling technique to equation (2) by writing new variables as $\tau = bt$ and $y = (b/a\gamma)x$.Then equation (2) becomes

$$\frac{\partial v}{\partial t} - v\frac{\partial v}{\partial x} = (\gamma-1)v^{\alpha}\left(1 - v^{\frac{1}{\gamma-1}}\right), \qquad (3)$$

where $\gamma > 1$. We consider nonnegative solution to (3) for $v \le 1$ because the uniformly steady states of the solutions are only $v = 0$ and $v = 1$. We can formulate the traveling wave solution as

$$v(x;t) = f(\xi), \qquad \xi = x + \lambda t \qquad (4)$$

where $\lambda > 0$ is the wave speed. Then the wave fronts of the solutions move to the left in the ξ-plane. We substitute the function (4) in the equation (3), then $f(\xi)$ satisfies

$$\frac{\partial f}{\partial \xi} = (\gamma - 1)(\lambda - f)^{-1} f^{\alpha}\left(1 - f^{\frac{1}{\gamma-1}}\right) \qquad (5)$$

where differentiation is satisfied according to the variable ξ. A singularity of the solution happens at $f(\xi) = \lambda$. We can get the wave front solution $f(\xi)$ to have limiting values. The problem is to govern the traveling wave solution with respect to λ where the solution of (5) is nonnegative and exists. It satisfies $f'(\xi) > 0$ and,

$$\lim_{\xi\to-\infty} f(\xi) = 0 \text{ and } \lim_{\xi\to+\infty} f(\xi) = 1\,, \qquad (6)$$

which are steady states and also $f(\xi)$ can be monotonically increasing. Where equation (5) has steady states at $f(\xi) = 0$ and $f(\xi) = 1$ and stability of them relies too much on value of λ. Linearity of the equation (5) displays that the solution $f(\xi) = 0$ is unstable for $\lambda > 0$, and $f(\xi) = 1$ is stable for $\lambda > 0$. Also, it is generally unstable for $0 < \lambda < 1$. If $\lambda = 1$, we can reduce equation (5) into $f'(\xi) = f(\xi)$ provided that$f(\xi) \neq 1$. Definitely,$f(\xi) = 1$ is a singularity of (5), and $f(\xi)$ is exponentially increasing. ■

Next, we introduce in particular case the traveling wave of (5) with $\gamma = 2$ and $\alpha = \beta$, for $\lambda > 1$. Let us consider the equation (2) which becomes the following equation

$$\frac{\partial v}{\partial t} - 2av\frac{\partial v}{\partial x} = bv^{\beta}(1 - v), \qquad (7)$$

and after rescaling equation (7) by assuming $\tau = bt$ and $y = (b/2a)x$, we get

$$\frac{\partial v}{\partial t} - v\frac{\partial v}{\partial x} = v^{\beta}(1 - v). \qquad (8)$$

Then the similar way in Theorem 1, the traveling wave solution $f(\xi)$ satisfies

$$\frac{\partial f}{\partial \xi} = f^{\beta}(\lambda - f)^{-1}(1 - f) \qquad (9)$$

Then for $0 < f(\xi) < 1$, we have three cases to get the solution of the ODEs. First, if $\beta = 1/2$, then the solution of (9) is

$$ln\left(\frac{1 + \sqrt{f}}{1 - \sqrt{f}}\right)^{\lambda-1} = \xi - 2\sqrt{f} + 2C_1. \qquad (10)$$

If the parameter $\beta = 1$, then the solution of (9) is in the following form

$$ln\frac{f^{\lambda}}{(1-f)^{\lambda-1}} = \xi + C_2. \tag{11}$$

Finally, let us choose that $\beta = 2$, then the solution of (9) has the following form

$$ln\left(\frac{f}{1-f}\right)^{\lambda-1} = \frac{\lambda}{f} + (x - C_3), \tag{12}$$

where $C_i, i = 1,2,3$; are constants of integration. The solutions (10)-(12) of the equation (9) for $\beta = 1/2, 1, 2$; respectively are satisfied with the initial condition $f(0) = 1/2$, for all $\lambda > 0$ with the boundary conditions (6) at $-\infty$ for any constants $C_1 = (1 + (\lambda - 1)\ln 3)/2, C_2 = -\ln 2$, $C_3 = (2\lambda + 2(\lambda - 1)\ln 2)$. Also, the boundary conditions (6) are satisfied at $+\infty$ for $\lambda > 1$ but they are not satisfied for $\lambda < 1$. The solution f is exponentially increasing and satisfied travelling wave solutions for $\lambda = 1$. Because the traveling wave solutions are invariant, the equation (9) is unchanged if $\xi \to \xi + c$, where c is any constant. Let us take $\xi = 0$ to be the origin point so the behavior of solutions is invariant to any shifting from the origin.

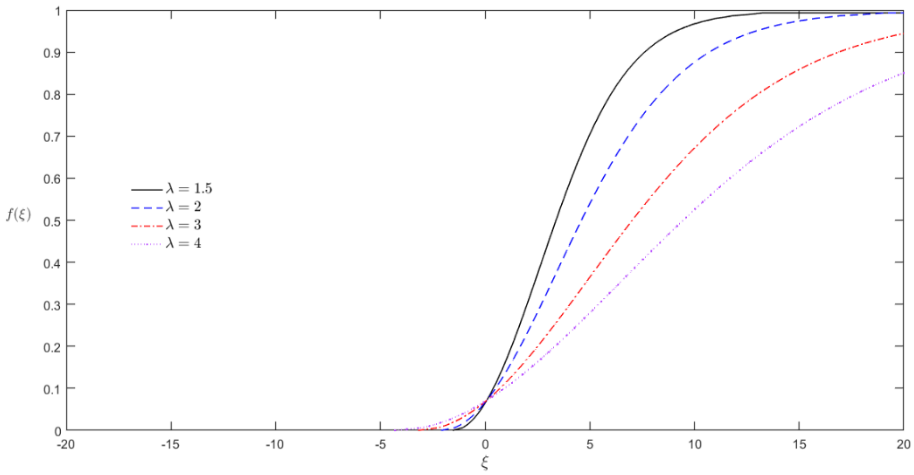


Fig.1: Traveling wave solution $f(\xi)$ where $\beta = 1/2, \gamma = 2, \alpha = 1/2$

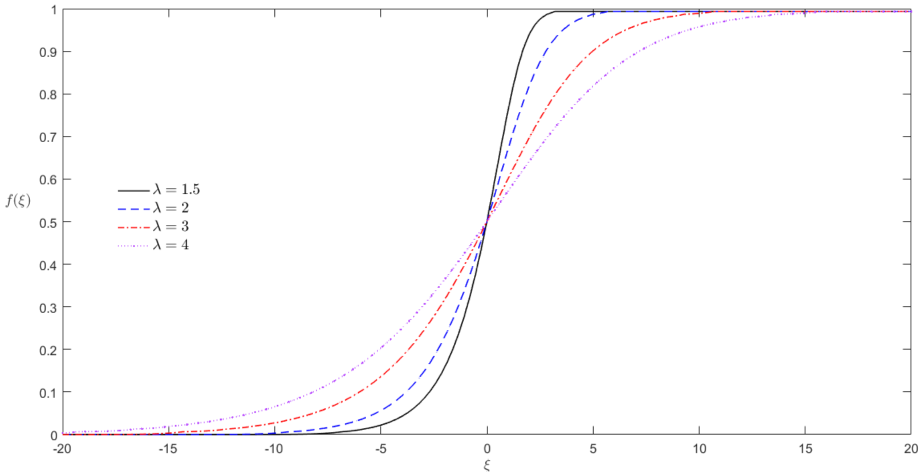


Fig.2: Traveling wave solution $f(\xi)$ where $\beta = 1, \gamma = 2, \alpha = 1$

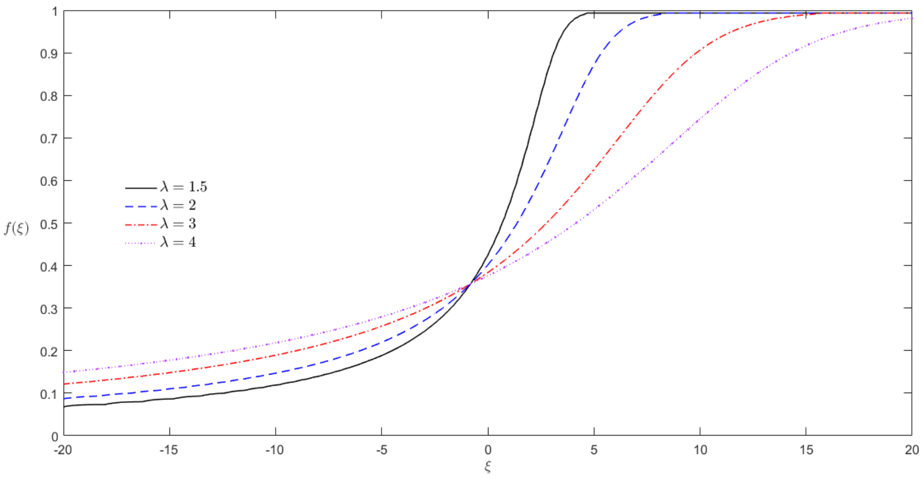


Fig.3: Traveling wave solution $f(\xi)$ where $\beta = 2, \gamma = 2, \alpha = 2$.

Therefore, traveling wave solution $f = f(\xi)$ of (9) with $\beta = 1/2, 1, 2$ are shown in Figs.1-3 ; respectively. They are matching to values $\lambda = 1.5, \lambda = 2, \lambda = 3$ and $\lambda = 4$. We observe that the derivative of the solution at $\xi = 0$ explains the steepness of the traveling waves is decreasing but the wave speed is increasing.

3. Methods of Characteristics

Let us consider in this section the stability of the traveling wave solution. If we impose a small perturbation on the wavefront at initial time such as $t = 0$, then it decays away. Also, the behavior of the initial conditions effects on the speed of propagation of the wave. Development of traveling wave solutions the partial differential equation (3) with the initial condition $v(x, 0) = v_0(x)$ are satisfied. Now, we use the characteristic methods to solve the initial value problem of characteristic equations

$$\frac{dx}{d\tau} = -v, \quad \frac{dt}{d\tau} = 1, \quad \frac{dv}{d\tau} = v^{\alpha}(1 - v).$$

With the initial conditions that can be parameterized in the following forms

$$x(s, 0) = s, \quad t(s, 0) = 0, \qquad v(s, 0) = v_0(s)$$

Integrating the equation for t yields $t = \tau$. For v, after substituting t for τ, we consider particular case when $\alpha = 1,\ \beta = 1,\ \gamma > 1$; explicit solution

$$v(s, t) = e^{t} v_0(s)\ [\,(e^{t} - 1)v_0(s) + 1]^{-1} \qquad (13)$$

Also, we obtain the characteristic curves as follow

$$x = s - \ln\big((e^{t} - 1)v_0 + 1\big). \qquad (14)$$

On the other hand, if we suppose that such $\alpha = 2,\ \beta = \gamma,\ \gamma > 1$; we get implicit solution which has a complicated form and is not easy to consider its characteristic curves and behavior. The solution of equation (13) evolves along the characteristic curve (14) at $(s, 0),\ s \in R$. We can assume initial guess of initial conditions

$$v_0(x) = 1 \text{ if } x \le a \text{ and } v_0(x) = 0 \text{ if } x \ge b,$$

where $a < b$ and $v_0(x)$ is continuous in $a \le x \le b$. Let us begin with the above initial condition, and because the slope equals to the origin point (zero) for all x when $v_0(x) = 0$ for $x \le a$, the characteristic curves will intersect. Also, the derivative of $v_0(x)$ is nonnegative and will move up to be shocks. Depending on the nature of traveling wave and the observation of the above initial data, we should consider the initial condition with the following inequality

$$0 \le v_0'(x) \le v_0(x) \le 1, \quad \forall x \in R,$$

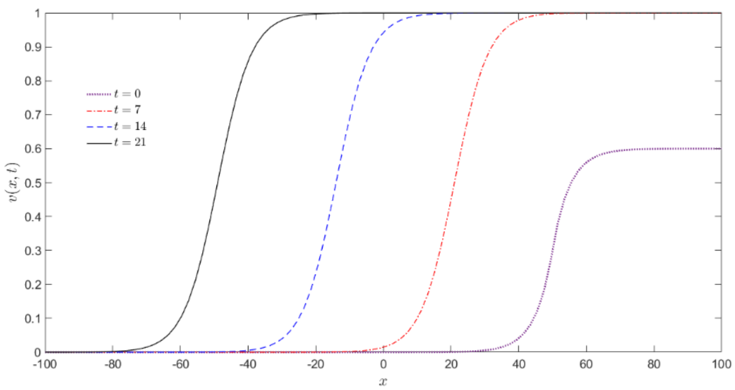
should be satisfied. This restriction is significant because if $v_0'(x) > v_0(x)$, then $\partial v / \partial x$ may blow up at some $t > 0$.

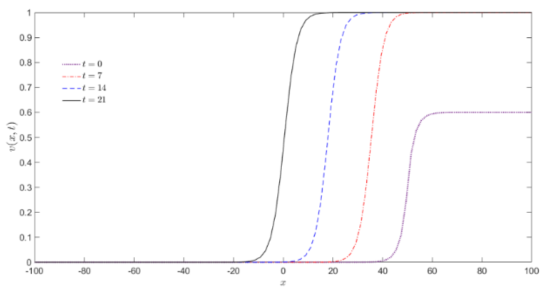
Therefore, we observe that if initial conditions are smooth, then the curves may steepen into shocks-like solutions. Thus, from the above analysis, we shall assume the form of the initial data as

$$v(x,0) = \begin{cases} Ce^{\mu(x-50)}, & x \leq 50, \\ C(2 - e^{-\mu(x-50)}), & x > 50, \end{cases} \tag{15}$$

with the nonnegative constant C and $C \leq 0.5$ and $0 < \mu < 1$ we consider the traveling wave in the form (4) with the initial condition (15) and boundary conditions (6) . Then , for $\mu > 1$, the derivative of the solution $\partial v/\partial x$ with the initial condition (15) for $x \leq x_0$ will be blows up for some $t > 0$. Also for $\mu = 1$, then $\partial v/\partial x$ is unbounded and the solution v does not represent the traveling wave. We observe numerically that the traveling wave solutions for equation (3) with the initial condition (15) for $0 < \mu < 1$ are satisfied with the wave speed λ depends on the value of μ and is inversely proportional to μ .

In Fig.4, numerical development of the traveling wave solutions is shown for $\mu = 0.2$,0.4 and 0.8 with wave speeds $\lambda = 1.5$, 2 , 3 and 4. For more motivation , the wave speed depending on the parameter μ has a fundamental analysis in [10].





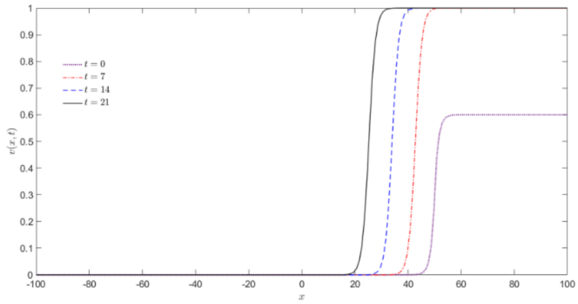


Fig.4: Development of solutions of (3) starting from initial condition (16) with

$C = 0.3$. Top:$\mu = 0.2$, middle: $\mu = 0.4$, bottom: $\mu = 0.8$

4. Stability of Traveling Wave Solutions

In this section , we try to investigate the stability of traveling wave solutions in particular cases where $\gamma = 2$, $\alpha = \beta$ and for $\lambda > 1$. Let us write the equation (8) by assuming $v(x,t) = \psi(y,t)$ where $y = x + \lambda t$, and we get

$$\frac{\partial \psi}{\partial t} + (\lambda - \psi)\frac{\partial \psi}{\partial y} = \psi^{\beta}(1 - \psi). \qquad (16)$$

Suppose that f(y) is a traveling wave solution of the equation (9) which is defined for $\lambda > 0$. Let us consider the equation (14) that has a solution in the form

$$\psi(y,t) = f(y) + P(y,t) \qquad (17)$$

where $P(y,t)$ is a small perturbations of f(y). Thus, for some $x_0 \in IR$, we suppose that $P(y,t) = 0$ for $y < x_0$, which means that the perturbation can be vanished on the interfaces of the waves. Let us substitute the form (17) in the equation(16), then we obtain a partial differential equation of the perturbation $P(y,t)$ as

$$\frac{\partial P}{\partial t} + (\lambda - f(y))\frac{\partial P}{\partial y} + (2f(y) - 1 - f'(y))P - P\frac{\partial P}{\partial y} + P^2 = 0, \qquad (18)$$

By varnishing the last two terms since P is too small and $\frac{dP}{dy}$ is very small at low density (if is advection). Also if $\beta = 2$, we use the similar calculation thus (18) becomes

$$\frac{\partial P}{\partial t} + (\lambda - f(y))\frac{\partial P}{\partial y} + (2f(y) - 1 - f'(y))P = 0. \qquad (19)$$

Since $\lim_{t\to\infty} P(y,t) = 0$ for any fixed y. We shall apply the same technique that introduced in [10], for $\lambda > 1$ to investigate the stability of the traveling wave solution $f(y)$ of (19) to small perturbations $P(y, t)$.

5. Conclusion

Existence and uniqueness of the solutions for the nonlinear parabolic equation which modeling nonlinear convection equation with absorption have introduced in several studies in [1, 2, 7, 8, 12]. Proving the existence of traveling wave solutions for the nonlinear convection-reaction equations in some cases was discussed. Also, Shock wave solutions happens in some restrictions of the parameters where the speed of traveling wave may breakdown. Qualitative techniques displayed the traveling wave depends on the behavior of the initial conditions particularly at the edges of the waves. The equation(1) with $\gamma = \beta = 1$ has speed of the traveling wave that depends on the initial conditions at infinity. We satisfy that traveling wave solution which has a compact support cannot grow from the initial data.

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Comparison between Confine MO-Connectedness and Connectedness, Confine MO-Countability and Countability

Hasanain Al-Abbasi[1] and Luay A A Al-Swidi[2]

University of Babylon, College of Education for Pure Sciences, Department of Mathematics[1]

University of Babylon, College of Education for Pure Sciences, Department of Mathematics[2]

Tel: +964 7811242210, E-mail: hasanainalabbasi@gmail.com[1]

Tel: +964 7812914256, E-mail: drluayha11@yahoo.com[2]

Abstract. The main goal of this work is to introduce a comparison between connected and confine MO-connected space, locally connected and confine MO-locally connected space, extremally disconnected and confine MO-extremally disconnected space, confine MO-first countable space and first countable space, confine MO-second countable space and second countable space. New properties and relations are introduced.

Keyword. confine MO-topology, connected space**,** locally connected, extremally disconnected, first countable space, second countable space.

1. Introduction

There are numerous different types of function spaces, and there are several various topologies that can be configured on a given collection of functions [2]. A function space is an precious example of a topological space. There are numerous researchers studied various kinds of function spaces via placing various topologies on the collection of functions [11]. In 1945 Ralph Fox defined the compact-open topology by using a collection of continuous functions between two topological spaces [12]. In 1981 Panos Lambrions introduced the bounded-open topology [9]. In 1996 Kathryn Porter introduced the regular open-open topology by using a collection of continuous maps between two topological spaces [7]. In 2016 R. Saadati introduced the quasicompact-open topology by using continuous real-valued maps on space C(X) [10]. In 2017 Sanjay Mishra and others introduced the generalized pre-open compact topology by using collection of real-valued continuous maps respecting a Tychonoff space [13]. In the paper that was accepted for publication at the International Scientific Conference of the University of Babylon (ISCUB-2019) we presented a new kind of topology on function spaces was the confine measurable open topology (confine MO-topology) which was defined as follows: Let $\left(\text{Ҳ}, \sum_{\mathcal{T}_{\text{Ҳ}}}\right)$ and $\left(\text{Џ}, \sum_{\mathcal{T}_{\text{Џ}}}\right)$ be two Borel measurable spaces, a function $\text{Ƒ}: \text{Ҳ} \longrightarrow \sum_{\mathcal{T}_{\text{Џ}}}$ is said to be set-valued Borel function (şʙ-function), a collection of şʙ-functions denoted by Ş. The pair $(\mathfrak{F}, \text{Ş})$ is said to be şʙ-function space (Ϫ-space) where $\mathfrak{F} = (\text{Ҳ} \times P(\text{Џ}))$. So that $\text{Ƒ} \in \text{Ş}$ is said to be measurable şʙ-function (ᴍşʙ-function) if, $\text{Ƒ}^{-1}(\text{Ʊ}) \in \sum_{\mathcal{T}_{\text{Ҳ}}}$, for every open subset Ʊ of $\sum_{\mathcal{T}_{\text{Џ}}}$. A collection of ᴍşʙ-functions denoted by Ɱ. The pair $(\mathfrak{F}, \text{Ɱ})$ is said to be measurable şʙ-functions space ($\mathfrak{P}$-space).

Let $\text{ß}(\text{ƀ}, \text{Ʊ}) = \{\text{Ƒ} \in \text{Ɱ}: \text{Ƒ}(\text{ƀ}) \subseteq \text{Ʊ}, \text{for fixed } \text{ƀ} \in \text{Ҳ}, \{\text{ƀ}\} \in \sum_{\mathcal{T}_{\text{Ҳ}}} \text{ and } \text{Ʊ} \in \mathcal{T}_{\text{Џ}}\}$ then $\acute{\text{S}}_{\text{Ɱ}_{\text{ƀ}}} = \{\text{ß}(\text{ƀ}, \text{Ʊ}), \text{for fixed } \text{ƀ} \in \text{Ҳ}, \{\text{ƀ}\} \in \sum_{\mathcal{T}_{\text{Ҳ}}}$ and Ʊ is an open set of Џ$\}$ is a subbase in Ɱ and the union of finite interaction of $\acute{\text{S}}_{\text{Ɱ}_{\text{ƀ}}}$ is a topology on Ɱ is called the confine MO-topology of Ɱ denoted by $\mathcal{T}_{\text{Ɱ}_{\text{ƀ}}}$. The pair $(\text{Ɱ}, \mathcal{T}_{\text{Ɱ}_{\text{ƀ}}})$ is called the confine MO-topological space.

The main goal of this study is to provide a comparison between the topological space and the confine measurable open topology (confine MO-topology), which this comparison includes a many properties of topological space such as connected space, locally connected, first countable and second countable space, thus we obtain new relationships between various types of topological spaces.

2. Preliminaries

Definition 2.1 [3]. Let $(\mathfrak{F}, \text{M̧})$ be a $\mathfrak{P}$-space and $\text{x̧} \in \text{X̧}$ then $(\mathfrak{F}, \text{M̧})$ is said to be

- **a)** First scarce $\mathfrak{P}$-space at x̧, if $\forall\ \text{ų} \in \text{Ų}\ \exists\ \text{Ƒ} \in \text{M̧}$ such that $\text{Ƒ}(\text{x̧}) = \{\text{ų}\}$.
- **b)** Second scarce $\mathfrak{P}$-space at x̧, if $\forall\ \text{Ƒ} \in \text{M̧},\ \exists\ \text{ų} \in \text{Ų}$ such that $\text{Ƒ}(\text{x̧}) = \{\text{ų}\}$.
- **c)** Principle scarce $\mathfrak{P}$-space at x̧, if $\forall\ \text{ų}_1, \text{ų}_2 \in \text{Ų}, \text{ų}_1 \neq \text{ų}_2$ there exist $\text{Ƒ}_1, \text{Ƒ}_2 \in \text{M̧}$ such that $\text{ų}_1 \in \text{Ƒ}_1(\text{x̧}), \text{ų}_2 \in \text{Ƒ}_2(\text{x̧})$ and $\text{Ƒ}_1 \neq \text{Ƒ}_2$.

Definition 2.2 [3]. Let $(\text{M̧}, \mathcal{T}_{\text{M̧}_\mathfrak{b}})$ be a confine MO-topological space and $ß(\mathfrak{b}, \text{Ʋ}) \subseteq \text{M̧}$ then:

- **a)** $ß(\mathfrak{b}, \text{Ʋ})$ is said to be exact set if $\cup_{\text{Ƒ} \in ß(\mathfrak{b}, \text{Ʋ})} \text{Ƒ}(\mathfrak{b}) = \text{Ʋ}$. $(\text{M̧}, \mathcal{T}_{\text{M̧}_\mathfrak{b}})$ is said to be exact space, if $\forall\ ß(\mathfrak{b}, \text{Ʋ}) \subseteq \text{M̧}$, $ß(\mathfrak{b}, \text{Ʋ})$ is an exact set.
- **b)** $ß(\mathfrak{b}, \text{Ʋ})$ is said to be plenary set if $\left(ß(\mathfrak{b}, \text{Ʋ})\right)^c = ß(\mathfrak{b}, \text{Ʋ}^c)$. $(\text{M̧}, \mathcal{T}_{\text{M̧}_\mathfrak{b}})$ is said to be plenary space if $\forall\ ß(\mathfrak{b}, \text{Ʋ}) \subseteq \text{M̧}$, $ß(\mathfrak{b}, \text{Ʋ})$ is a plenary set.

Definition 2.3 [3]. Let $(\text{M̧}, \mathcal{T}_{\text{M̧}_\mathfrak{b}})$ be a confine MO-topological space, then $(\text{M̧}, \mathcal{T}_{\text{M̧}_\mathfrak{b}})$ is said to be

- **a)** United space if $\forall\ ß(\mathfrak{b}, \text{Ʋ}_1), ß(\mathfrak{b}, \text{Ʋ}_2) \subseteq \text{M̧},\ ß(\mathfrak{b}, \text{Ʋ}_1) \cup ß(\mathfrak{b}, \text{Ʋ}_2) = ß(\mathfrak{b}, \text{Ʋ}_1 \cup \text{Ʋ}_2)$.
- **b)** limpid space if $\forall\ ß(\mathfrak{b}, \text{Ʋ}_1), ß(\mathfrak{b}, \text{Ʋ}_1) \subseteq \text{M̧}$ such that $ß(\mathfrak{b}, \text{Ʋ}_1) \cap ß(\mathfrak{b}, \text{Ʋ}_1) = \Phi \Longrightarrow \text{Ʋ}_1 \cap \text{Ʋ}_2 = \Phi$.

Definition 2.4 [3]. Let $(\text{M̧}, \mathcal{T}_{\text{M̧}_\mathfrak{b}})$ be an exact united plenary limpid space then $(\text{M̧}, \mathcal{T}_{\text{M̧}_\mathfrak{b}})$ is said to caliper topological space.

Definition 2.5 [3]. Let $(\text{M̧}, \mathcal{T}_{\text{M̧}_\mathfrak{b}})$ be a confine MO-topological space. The union of all confine MO-open subsets of $ß(\mathfrak{b}, \mathcal{A})$ is called the confine MO-interior of $ß(\mathfrak{b}, \mathcal{A})$ and is denoted by $\left(ß(\mathfrak{b}, \mathcal{A})\right)^\circ$. The confine MO-interior of $ß(\mathfrak{b}, \mathcal{A})$ is a confine MO-open subset of M̧.

Definition 2.6 [3]. Let $(\text{M̧}, \mathcal{T}_{\text{M̧}_\mathfrak{b}})$ be a confine MO-topological space and $ß(\mathfrak{b}, \mathcal{A}) \subset \text{M̧}$. The intersection of all confine MO-closed supersets of $ß(\mathfrak{b}, \mathcal{A})$ is called the confine MO-closure of $ß(\mathfrak{b}, \mathcal{A})$ and is denoted by $\overline{ß(\mathfrak{b}, \mathcal{A})}$. The confine MO-closure of $ß(\mathfrak{b}, \mathcal{A})$ is a confine MO-closed subset of M̧.

Definition 2.7 [3]. Let $(\text{M̧}, \mathcal{T}_{\text{M̧}_\mathfrak{b}})$ be a confine MO-topological space and $\text{Ƒ} \in \text{M̧}$. A confine MO-neighbourhood of Ƒ is a subset $ß(\mathfrak{b}, \text{Ɲ})$ of M̧ such that there exists a confine MO-open set $ß(\mathfrak{b}, \text{Ʋ}) \subset \text{M̧}$ such that $\text{Ƒ} \in ß(\mathfrak{b}, \text{Ʋ}) \subset ß(\mathfrak{b}, \text{Ɲ})$. The set of all confine MO-neighbourhood of Ƒ is denoted $ß(\mathfrak{b}, \text{Ɲ})(\text{Ƒ})$.

Definition 2.8 [3]. Let $(\text{M̧}, \mathcal{T}_{\text{M̧}_\mathfrak{b}})$ be a confine MO-topological space and $ß(\mathfrak{b}, \mathcal{A}) \subset \text{M̧}$. A confine MO-neighbourhood of $ß(\mathfrak{b}, \mathcal{A})$ is a subset $ß(\mathfrak{b}, \text{Ɲ})$ of M̧ such that there exists a confine MO-open set $ß(\mathfrak{b}, \text{Ʋ}) \subset \text{M̧}$ such that $ß(\mathfrak{b}, \mathcal{A}) \subset ß(\mathfrak{b}, \text{Ʋ}) \subset ß(\mathfrak{b}, \text{Ɲ})$. The set of all confine MO-neighbourhood of $ß(\mathfrak{b}, \mathcal{A})$ is denoted $ß(\mathfrak{b}, \text{Ɲ})(ß(\mathfrak{b}, \mathcal{A}))$.

Definition 2.9 [3]. Let $(\text{M̧}, \mathcal{T}_{\text{M̧}_\mathfrak{b}})$ be a confine MO-topological space. A collection **B** of subsets of M̧ is said to form a confine MO-base for $\mathcal{T}_{\text{M̧}_\mathfrak{b}}$ iff $\mathbf{B} \subset \mathcal{T}_{\text{M̧}_\mathfrak{b}}$ and if for each point $\text{Ƒ} \in \text{M̧}$ and each confine MO-neighbourhood $ß(\mathfrak{b}, \text{Ɲ})$ of $ß(\mathfrak{b}, \mathcal{B})$ there exist $ß(\mathfrak{b}, \mathcal{B}) \in \mathbf{B}$ such that $\text{Ƒ} \in ß(\mathfrak{b}, \mathcal{B}) \subset ß(\mathfrak{b}, \text{Ɲ})$.

Theorem 2.10 [3]. Let $(\text{M̧}, \mathcal{T}_{\text{M̧}_\mathfrak{b}})$ be an exact united space then:

a) If ß(ƀ, Ɲ) is a confine MO-neighbourhood of ß(ƀ, 𝒜) then Ɲ is a neighbourhood of 𝒜.
b) If Ɲ is a neighbourhood of 𝒜 then ß(ƀ, Ɲ) is a confine MO-neighbourhood of ß(ƀ, 𝒜).

Proof:

a) Let (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be an exact united space and ß(ƀ, Ɲ) be a confine MO-neighbourhood of ß(ƀ, 𝒜) then there is a confine MO-open set ß(ƀ, Ʊ) of Ɱ such that ß(ƀ, 𝒜) ⊆ ß(ƀ, Ʊ) ⊆ ß(ƀ, Ɲ) since (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) is an exact space then by lemma 2.12 (a) we have 𝒜 ⊆ Ʊ ⊆ Ɲ and since (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) is an exact united space then Ʊ ∈ $\mathcal{T}_{Ụ}$ hence Ɲ is a neighbourhood of 𝒜 .
b) Let Ɲ be a neighbourhood of 𝒜 then there is an open set Ʊ of Ụ such that 𝒜 ⊆ Ʊ ⊆ Ɲ then by lemma 2.11 (c) we have ß(ƀ, 𝒜) ⊆ ß(ƀ, Ʊ) ⊆ ß(ƀ, Ɲ) such that ß(ƀ, Ʊ) is a confine MO-open set hence ß(ƀ, Ɲ) is a confine MO-neighbourhood of ß(ƀ, 𝒜).

Lemma 2.11 [4]. Let (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be a confine MO-Topological space then:

a) ß(ƀ, {Ụ}) = Ɱ.
b) If $Ʊ_1, Ʊ_2$ ⊆ Ụ, $Ʊ_1 \cap Ʊ_2 = \Phi$ then ß(ƀ, $Ʊ_1$) ∩ ß(ƀ, $Ʊ_2$) = Φ.
c) If $Ʊ_1, Ʊ_2$ ⊆ Ụ, $Ʊ_1 \subseteq Ʊ_2$ then ß(ƀ, $Ʊ_1$) ⊆ ß(ƀ, $Ʊ_2$).

Lemma 2.12 [4]. Let (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be an exact space and ß(ƀ, $Ʊ_1$), ß(ƀ, $Ʊ_2$) are subsets of Ɱ such that:

a) ß(ƀ, $Ʊ_1$) ⊆ ß(ƀ, $Ʊ_2$) then $Ʊ_1 \subseteq Ʊ_2$.
b) ß(ƀ, $Ʊ_1$) = ß(ƀ, $Ʊ_2$) then $Ʊ_1 = Ʊ_2$.

Theorem 2.13 [4]. Let (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be an exact united space then ß(ƀ, Ʊ) be an open set of Ɱ iff Ʊ be an open set of Ụ.

Proposition 2.14 [4]. Let (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be a confine ᴍşʙ-function topological space and ß(ƀ, Ʊ) is a subset of Ɱ then ß(ƀ, $Ʊ^{\circ}$) ⊆ $\big($ß(ƀ, Ʊ)$\big)^{\circ}$.

Proposition 2.15 [4]. Let (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be an exact united space and ß(ƀ, Ʊ) is a subset of Ɱ then $\big($ß(ƀ, Ʊ)$\big)^{\circ}$ ⊆ ß(ƀ, $Ʊ^{\circ}$).

Theorem 2.16 [4]. Let (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be a plenary space, if ß(ƀ, Ʊ) is a subset of Ɱ then $\overline{\text{ß(ƀ, Ʊ)}}$ ⊆ ß(ƀ, $\overline{Ʊ}$).

Proposition 2.17 [4]. Let (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be an exact united limpid space, ß(ƀ, Ʊ) ⊆ Ɱ then ß(ƀ, $\overline{Ʊ}$) ⊆ $\overline{\text{ß(ƀ, Ʊ)}}$.

Result 2.18 [4]. Let (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be a caliper space, ß(ƀ, Ʊ) ⊆ Ɱ then ß(ƀ, $\overline{Ʊ}$) = $\overline{\text{ß(ƀ, Ʊ)}}$.

Proposition 2.19 [4]. Let (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be a plenary space and Ʊ be a closed subset of Ụ then ß(ƀ, Ʊ) is a closed subset of Ɱ.

Proposition 2.20 [4]. Let (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be an exact united plenary space and ß(ƀ, Ʊ) is a closed subset of Ɱ then Ʊ be a closed subset of Ụ.

3. Confine MO-connectedness and connectedness.

Definition 3.1. Let (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be a confine MO-topological space and ß(ƀ, 𝒜) ⊂ Ɱ then ß(ƀ, 𝒜) is said to be confine MO-connected set iff cannot be represented as the union of two disjoint non-empty confine MO-open subsets in ß(ƀ, 𝒜).

Definition 3.2. Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be a confine MO-topological space then Ɱ is said to be a confine MO-connected space iff cannot be represented as the union of two disjoint non-empty confine MO-open subsets.

Theorem 3.3. Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be a confine MO-topological space then Ɱ is a confine MO-connected space iff the only subsets of Ɱ which are both confine MO-open and confine MO-closed are Ɱ and the empty set.

Definition 3.4. The maximal confine MO-connected subsets of a confine MO-topological space $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ are called the confine MO-connected components of Ɱ.

Definition 3.5. Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be a confine MO-topological space and $Ƒ \in Ɱ$ then Ɱ is said to be a confine MO-locally connected at a point Ƒ if every confine MO-neighbourhood of Ƒ contains a confine MO-connected open neighbourhood. A space Ɱ is said to be a confine MO-locally connected iff Ɱ is a confine MO-locally connected at each of its points.

Definition 3.6. Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be a confine MO-topological space and $Ƒ \in Ɱ$ then Ɱ is said to be a confine MO-weakly locally connected at a point Ƒ if every confine MO-neighbourhood $ß(ƀ, Ɲ)$ of Ƒ contains a confine MO-connected set $ß(ƀ, \mathcal{A})$ such that $Ƒ \in ß(ƀ, \mathcal{A})^{\circ} \subset ß(ƀ, \mathcal{A}) \subset ß(ƀ, Ɲ)$. A space Ɱ is said to be a confine MO-weakly locally connected iff Ɱ is a confine MO-weakly locally connected at each of its points.

Definition 3.7. A confine MO-topological space $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ is said to be a confine MO-extremally disconnected space iff the confine MO-closure of every confine MO-open set is a confine MO-open set.

Lemma 3.8. Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be an united space and $ß(ƀ, Ʊ)$ be a confine MO-connected set in Ɱ then Ʊ is a connected set in Ų.

Proof: Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be an united space and $ß(ƀ, Ʊ)$ be a confine MO-connected set in Ɱ. Suppose that Ʊ is a disconnected set in Ų then there exist two disjoint open sets $Ʊ_1, Ʊ_2$ such that $Ʊ = Ʊ_1 \cup Ʊ_2$ thus $ß(ƀ, Ʊ) = ß(ƀ, Ʊ_1 \cup Ʊ_2)$ since $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be an united space then $ß(ƀ, Ʊ) = ß(ƀ, Ʊ_1 \cup Ʊ_2) = ß(ƀ, Ʊ_1) \cup ß(ƀ, Ʊ_2)$ but $Ʊ_1, Ʊ_2$ are two disjoint open sets in Ų therefore $ß(ƀ, Ʊ_1), ß(ƀ, Ʊ_2)$ are two disjoint open sets in Ɱ thus $ß(ƀ, Ʊ)$ is a disconnected set in Ɱ this contradiction hence Ʊ is a connected set in Ų.

Lemma 3.9. Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be an exact united limpid space and Ʊ is a connected set in Ų then $ß(ƀ, Ʊ)$ is a confine MO-connected set in Ɱ.

Proof: Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be an exact united limpid space and Ʊ be a connected set in Ų. Suppose that $ß(ƀ, Ʊ)$ is a disconnected set in Ɱ then there exist two disjoint open sets $ß(ƀ, Ʊ_1), ß(ƀ, Ʊ_2)$ such that $ß(ƀ, Ʊ) = ß(ƀ, Ʊ_1) \cup ß(ƀ, Ʊ_2)$ since $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ is an united space then $ß(ƀ, Ʊ) = ß(ƀ, Ʊ_1) \cup ß(ƀ, Ʊ_2) = ß(ƀ, Ʊ_1 \cup Ʊ_2)$ since $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ is an exact space then $Ʊ = Ʊ_1 \cup Ʊ_2$ since $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ is an exact united limpid space then we have $Ʊ_1, Ʊ_2$ are two disjoint open sets in Ų thus Ʊ is a disconnected set in Ų this contradiction hence $ß(ƀ, Ʊ)$ is a confine MO-connected set in Ɱ.

Theorem 3.10. Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be an united confine MO-connected space then $(Ų, \mathcal{T}_{Ų})$ is a connected space.

Proof: Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be an united confine MO-connected space. Suppose that $(Ų, \mathcal{T}_{Ų})$ is a disconnected space then there exist two disjoint open sets $Ʊ_1, Ʊ_2$ such that $Ų = Ʊ_1 \cup Ʊ_2$ thus $Ɱ = ß(ƀ, Ų) =$

$ß(ƅ, Ʊ_1 \cup Ʊ_2)$ since $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ be an united space then $Ɱ = ß(ƅ, Ų) = ß(ƅ, Ʊ_1 \cup Ʊ_2) = ß(ƅ, Ʊ_1) \cup ß(ƅ, Ʊ_2)$ but $Ʊ_1, Ʊ_2$ are two disjoint open sets in $Ų$ therefore $ß(ƅ, Ʊ_1), ß(ƅ, Ʊ_2)$ are two disjoint open sets in $Ɱ$ thus $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ is a disconnected space this contradiction hence $(Ų, \mathcal{T}_{Ų})$ is a connected space.

Theorem 3.11. Let $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ be an exact united limpid space and $(Ų, \mathcal{T}_{Ų})$ be a connected then $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ is a confine MO-connected space.

Proof: Let $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ be an exact united limpid space and $(Ų, \mathcal{T}_{Ų})$ be a connected space. Suppose that $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ is a disconnected space then there exist two disjoint open sets $ß(ƅ, Ʊ_1), ß(ƅ, Ʊ_2)$ such that $Ɱ = ß(ƅ, Ʊ_1) \cup ß(ƅ, Ʊ_2)$ since $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ is an united space then $Ɱ = ß(ƅ, Ʊ_1) \cup ß(ƅ, Ʊ_2) = ß(ƅ, Ʊ_1 \cup Ʊ_2)$ since $Ɱ = ß(ƅ, Ų)$ and $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ is an exact space then $Ų = Ʊ_1 \cup Ʊ_2$ since $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ is an exact united limpid space then we have $Ʊ_1, Ʊ_2$ are two disjoint open sets in $Ų$ thus $(Ų, \mathcal{T}_{Ų})$ is a disconnected space this contradiction hence $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ is a confine MO-connected space.

Theorem 3.12. Let $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ be an exact united limpid space then $ß(ƅ, \mathcal{C})$ be a confine MO-component set in $Ɱ$ iff $\mathcal{C}$ is a component set in $Ų$.

Proof: Let $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ be an exact united limpid space and $ß(ƅ, \mathcal{C})$ be a confine MO-component set in $Ɱ$ then $\mathcal{C}$ is a connected set in $Ų$. Suppose that $\mathcal{C}^*$ is a connected set in $Ų$ such that $\mathcal{C} \subseteq \mathcal{C}^*$ thus $ß(ƅ, \mathcal{C}) \subseteq ß(ƅ, \mathcal{C}^*)$ since $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ be an exact united limpid space then $ß(ƅ, \mathcal{C}^*)$ is a connected set in $Ɱ$ but $ß(ƅ, \mathcal{C})$ be a confine MO-component set in $Ɱ$ this contradiction hence $\mathcal{C}$ is a component set in $Ų$.

Now let $\mathcal{C}$ be a component set in $Ų$ then $ß(ƅ, \mathcal{C})$ is a confine MO-connected set in $Ɱ$. Suppose that $ß(ƅ, \mathcal{C}^*)$ is a confine MO-component set in $Ɱ$ such that $ß(ƅ, \mathcal{C}) \subseteq ß(ƅ, \mathcal{C}^*)$ since $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ be an exact space then $\mathcal{C} \subseteq \mathcal{C}^*$ since $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ be an exact united space then $\mathcal{C}^*$ is a confine MO-connected set in $Ų$ but $\mathcal{C}$ be a component set in $Ų$ this contradiction hence $ß(ƅ, \mathcal{C})$ is a component set in $Ɱ$.

Theorem 3.13. Let $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ be a confine MO-locally connected space. If $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ is an exact united space and $(\mathfrak{F}, Ɱ)$ is a principle scarce $\mathfrak{P}$-space at $ƅ$ then $(Ų, \mathcal{T}_{Ų})$ is a locally connected space.

Proof: Let $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ be a confine MO-locally connected space and $(\mathfrak{F}, Ɱ)$ be a principle scarce $\mathfrak{P}$-space at $ƅ$. Suppose that $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ is an exact united space and $ų \in Ų$, $Ʊ$ is an open set of $Ų$ such that $ų \in Ʊ$ then there exist $Ƒ \in Ɱ$ such that $Ƒ(ƅ) = \{ų\}$ thus $Ƒ(ƅ) \subseteq Ʊ$ so that $Ƒ \in ß(ƅ, Ʊ)$ such that $ß(ƅ, Ʊ)$ is an open set in $Ɱ$ then there exist a confine MO-connected open set $ß(ƅ, Ʋ)$ such that $Ƒ \in ß(ƅ, Ʋ) \subseteq ß(ƅ, Ʊ)$ since $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ be an exact space then $Ƒ(ƅ) = \{ų\} \subseteq Ʋ \subseteq Ʊ$ so that $ų \in Ʋ \subseteq Ʊ$ but $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ is an exact united space then $Ʋ$ is a connected open set hence $(Ų, \mathcal{T}_{Ų})$ is a locally connected space.

Theorem 3.14. Let $(Ų, \mathcal{T}_{Ų})$ be a locally connected. If $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ is an exact united limpid space and $(\mathfrak{F}, Ɱ)$ is a second scarce $\mathfrak{P}$-space at $ƅ$ then $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ is a confine MO-locally connected space.

Proof: Let $(Ų, \mathcal{T}_{Ų})$ be a locally connected space and $(\mathfrak{F}, Ɱ)$ be second scarce $\mathfrak{P}$-space at $ƅ$. Suppose that $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ is an exact united limpid space and $Ƒ \in Ɱ$, $ß(ƅ, Ʊ)$ is an open set such that $Ƒ \in ß(ƅ, Ʊ)$ then $Ƒ(ƅ) = \{ų\} \subseteq Ʊ$ thus $ų \in Ʊ$ since $Ʊ$ is an open set of $Ų$ and $(Ų, \mathcal{T}_{Ų})$ is a locally connected space then there exist a connected open set $Ʋ$ such that $ų \in Ʋ \subseteq Ʊ$ thus $Ƒ(ƅ) = \{ų\} \subseteq Ʋ \subseteq Ʊ$ so that $Ƒ \in ß(ƅ, Ʋ) \subseteq ß(ƅ, Ʊ)$ but $(\mathfrak{F}, Ɱ)$ is an exact united limpid space and $Ʋ$ is a connected open then $ß(ƅ, Ʋ)$ confine MO-connected open in $Ɱ$ hence $(Ɱ, \mathcal{T}_{Ɱ_ƅ})$ is a confine MO-locally connected space.

Theorem 3.15. Let (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be a confine MO-weakly locally connected space. If (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) is an exact united space and (𝔉, Ɱ) is a principle scarce 𝔓-space at ƀ then (Ų, $\mathcal{T}_{Ų}$) is a weakly locally connected space.

Proof: Let (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be a confine MO-weakly locally connected space and (𝔉, Ɱ) be a principle scarce 𝔓-space at ƀ**.** Suppose that (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) is an exact united space and ų ∈ Ų, Ɲ is a neighbourhood of ų then there exist Ƒ ∈ Ɱ such that Ƒ(ƀ) = {ų} thus Ƒ(ƀ) ⊆ Ɲ then Ƒ ∈ ß(ƀ, Ɲ) such that ß(ƀ, Ɲ) is a confine MO-neighbourhood of Ƒ then there exist a confine MO-connected set ß(ƀ, Ѵ) such that Ƒ ∈ (ß(ƀ, Ѵ))° ⊆ ß(ƀ, Ѵ) ⊆ ß(ƀ, Ɲ) since (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be an exact united space then (ß(ƀ, Ѵ))° ⊆ ß(ƀ, Ѵ°) ⊆ ß(ƀ, Ѵ) so that (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be an exact space then we have Ƒ(ƀ) = {ų} ⊆ Ѵ° ⊆ Ѵ ⊆ Ɲ thus ų ∈ Ѵ° ⊆ Ѵ ⊆ Ɲ but (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) is an exact united space then Ѵ is a connected set hence (Ų, $\mathcal{T}_{Ų}$) is a weakly locally connected space.

Theorem 3.16. Let (Ų, $\mathcal{T}_{Ų}$) be a weakly locally connected space. If (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) is an exact united limpid space and (𝔉, Ɱ) is a second scarce 𝔓-space at ƀ then (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) is a confine MO-weakly connected space.

Proof: Let (Ų, $\mathcal{T}_{Ų}$) be a weakly locally connected space and (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be an exact united limpid space. Suppose that (𝔉, Ɱ) is a second scarce 𝔓-space at ƀ and Ƒ ∈ Ɱ, ß(ƀ, Ɲ) is a confine MO-neighbourhood of Ƒ then Ƒ(ƀ) = {ų} ⊆ Ɲ thus ų ∈ Ɲ since (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) is an exact united space then Ɲ is a neighbourhood of ų since Ų is a weakly locally connected space then there exist a connected set Ѵ such that ų ∈ Ѵ° ⊆ Ѵ ⊆ Ɲ thus Ƒ(ƀ) = {ų} ⊆ Ѵ° ⊆ Ѵ ⊆ Ɲ so that Ƒ ∈ ß(ƀ, Ѵ°) ⊆ ß(ƀ, Ѵ) ⊆ ß(ƀ, Ɲ) since ß(ƀ, Ѵ°) ⊆ (ß(ƀ, Ѵ))° then Ƒ ∈ (ß(ƀ, Ѵ))° ⊆ ß(ƀ, Ѵ) ⊆ ß(ƀ, Ɲ) but (𝔉, Ɱ) is an exact united limpid space and Ѵ is a connected set then ß(ƀ, Ѵ) is a confine MO-connected set in Ɱ hence (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) is a confine MO-weakly locally connected space.

Theorem 3.17. Let (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be a caliper then is a confine MO-extremally disconnected space iff (Ų, $\mathcal{T}_{Ų}$) is an extremally disconnected space.

Proof: Let (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be a caliper confine MO-extremally disconnected space and Ʋ be an open sets in Ų then ß(ƀ, Ʋ) is a confine MO-open set in Ɱ therefore $\overline{ß(ƀ, Ʋ)}$ is a confine MO-open set in Ɱ since (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) is a caliper MO-space then ß(ƀ, $\overline{Ʋ}$) = $\overline{ß(ƀ, Ʋ)}$ thus ß(ƀ, $\overline{Ʋ}$) is a confine MO-open set in Ɱ since (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) is a caliper space then $\overline{Ʋ}$ is an open set in Ų hence (Ų, $\mathcal{T}_{Ų}$) is an extremally disconnected space.

Now let (Ų, $\mathcal{T}_{Ų}$) be an extremally disconnected space and ß(ƀ, Ʋ) is a confine MO-open set in Ɱ then Ʋ is an open sets in Ų therefore $\overline{Ʋ}$ is an open set in Ų thus $\overline{ß(ƀ, Ʋ)}$ is a confine MO-open set in Ɱ since (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) is a caliper MO-space then ß(ƀ, $\overline{Ʋ}$) = $\overline{ß(ƀ, Ʋ)}$ thus $\overline{ß(ƀ, Ʋ)}$ is a confine MO-open set in Ɱ hence (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) be a confine MO-extremally disconnected.

4. Confine MO-Countability and Countability.

Definition 4.1. A confine MO-topological space (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) is said to be a confine MO-first countable space if each point has a countable confine MO-neighboruhood basis.

Definition 4.2. A confine MO-topological space (Ɱ, $\mathcal{T}_{Ɱ_ƀ}$) is said to be a confine MO-second countable space if $\mathcal{T}_{Ɱ_ƀ}$ has a countable confine MO-basis.

Definition 4.3. A confine MO-topological space $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ is said to be a confine MO-separable space if it contains a countable confine MO-dense subset.

Definition 4.5. Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be a confine MO-topological space and $ß(ƀ, \mathcal{A}) \subset Ɱ$ then $ß(ƀ, \mathcal{A})$ is said to be confine MO-dense in $Ɱ$ iff $\overline{ß(ƀ, \mathcal{A})} = Ɱ$.

Lemma 4.6. Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be a caliper confine MO-topological space and $ß(ƀ, Ʊ)$ be a confine MO-dense set in $Ɱ$ then $Ʊ$ is a dense set in $Ų$.

Proof: Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be a caliper confine MO-topological space and $ß(ƀ, Ʊ)$ be a confine MO-dense set then $\overline{ß(ƀ, Ʊ)} = Ɱ$ since $Ɱ = ß(ƀ, Ų)$ then $\overline{ß(ƀ, Ʊ)} = ß(ƀ, Ų)$ so that by result 2.15 we have $ß(ƀ, \overline{Ʊ}) = \overline{ß(ƀ, Ʊ)} = ß(ƀ, Ų)$ implies that $ß(ƀ, \overline{Ʊ}) = ß(ƀ, Ų)$ thus $\overline{Ʊ} = Ų$, hence $Ʊ$ is a dense set in $Ų$.

Lemma 4.7. Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be an exact united limpid space and $Ʊ$ be a dense set in $Ų$ then $ß(ƀ, Ʊ)$ is a confine MO-dense set in $Ɱ$.

Proof: Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be an exact united limpid space and $Ʊ$ be a dense set in $Ų$ then $\overline{Ʊ} = Ų$ thus $ß(ƀ, \overline{Ʊ}) = ß(ƀ, Ų)$ so that $ß(ƀ, Ų) = ß(ƀ, \overline{Ʊ}) \subseteq \overline{ß(ƀ, Ʊ)}$ implies that $ß(ƀ, Ų) \subseteq \overline{ß(ƀ, Ʊ)}$ but $\overline{ß(ƀ, Ʊ)} \subseteq ß(ƀ, Ų)$ therefore $\overline{ß(ƀ, Ʊ)} = ß(ƀ, Ų) = Ɱ$ thus $\overline{ß(ƀ, Ʊ)} = Ɱ$, hence $ß(ƀ, Ʊ)$ is a confine MO-dense set in $Ɱ$.

Theorem 4.8. Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be a countable caliper confine MO-topological space and $(Ų, \mathcal{T}_Ų)$ is a countable topological space then $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ is a confine MO-separable space iff $(Ų, \mathcal{T}_Ų)$ is a separable space.

Theorem 4.9. Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be an exact united confine MO-first countable space and $(\mathfrak{F}, Ɱ)$ be a first scarce $\mathfrak{P}$-space at $ƀ$ then $(Ų, \mathcal{T}_Ų)$ is a first countable space.

Proof: Let $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be an exact united confine MO-first countable space and $(\mathfrak{F}, Ɱ)$ be a first scarce $\mathfrak{P}$-space at $ƀ$. Suppose that $ų \in Ų$ then there exist $Ƒ \in Ɱ$ such that $Ƒ(ƀ) = \{ų\}$ so that $Ƒ$ has a countable confine MO-neighbourhood basis $\boldsymbol{N}_Ƒ$. Consider $\boldsymbol{N}_ų = \{Ʋ : ß(ƀ, Ʋ) \in \boldsymbol{N}_Ƒ\}$ since $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ is an exact united space then $\boldsymbol{N}_ų$ is a countable collection of confine MO-neighbourhood of $ų$. Now we confirm $\boldsymbol{N}_ų$ is a basis of $ų$. Let $Ɲ$ be a neighbourhood of $ų$ then $ß(ƀ, Ɲ)$ is a confine MO-neighbourhood of $Ƒ$ thus there exist $ß(ƀ, Ʋ) \in \boldsymbol{N}_Ƒ$ such that $Ƒ \in ß(ƀ, Ʋ) \subseteq ß(ƀ, Ɲ)$ so that $Ƒ(ƀ) = \{ų\} \subseteq Ʋ \subseteq Ɲ$ thus $ų \in Ʋ \subseteq Ɲ$ but $Ʋ \in \boldsymbol{N}_ų$ therefore $\boldsymbol{N}_ų$ is a basis of $ų$ hence $(Ų, \mathcal{T}_Ų)$ is a first countable space.

Theorem 4.10. Let $(Ų, \mathcal{T}_Ų)$ be a first countable space. If $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ is an exact united and $(\mathfrak{F}, Ɱ)$ is a second scarce $\mathfrak{P}$-space at $ƀ$ then $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ is a confine MO-first countable space.

Proof: Let $(Ų, \mathcal{T}_Ų)$ be a first countable space and $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ be an exact united. Suppose that $(\mathfrak{F}, Ɱ)$ is a second scarce $\mathfrak{P}$-space at $ƀ$ and $Ƒ \in Ɱ$ then there exist $ų \in Ų$ such that $Ƒ(ƀ) = \{ų\}$ so that $ų$ has a countable neighbourhood basis $\boldsymbol{N}_ų$. Consider $\boldsymbol{N}_Ƒ = \{ß(ƀ, Ʋ) : Ʋ \in \boldsymbol{N}_ų\}$ then $\boldsymbol{N}_Ƒ$ is a countable collection of confine MO-neighbourhood of $Ƒ$. Now we confirm $\boldsymbol{N}_Ƒ$ is a confine MO-basis of $Ƒ$. Let $ß(ƀ, Ɲ)$ be a confine MO-neighbourhood of $Ƒ$ since $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ is an exact united space then $Ɲ$ is a neighbourhood of $ų$ thus there exist $Ʋ \in \boldsymbol{N}_ų$ such that $ų \in Ʋ \subseteq Ɲ$ so that $Ƒ(ƀ) = \{ų\} \subseteq Ʋ \subseteq Ɲ$ thus $Ƒ \in ß(ƀ, Ʋ) \subseteq ß(ƀ, Ɲ)$ but $ß(ƀ, Ʋ) \in \boldsymbol{N}_Ƒ$ therefore $\boldsymbol{N}_Ƒ$ is a confine MO-basis of $Ƒ$ hence $(Ɱ, \mathcal{T}_{Ɱ_ƀ})$ is a confine MO-first countable space.

Theorem 4.11. Let $(Ɱ, \mathcal{T}_{Ɱ_{𝔟}})$ be an exact united confine MO-second countable space and $(𝔉, Ɱ)$ be a first scarce 𝔓-space at 𝔟 then $(Ų, \mathcal{T}_{Ų})$ is a second countable space.

Proof: Let $(Ɱ, \mathcal{T}_{Ɱ_{𝔟}})$ be a confine MO-second countable space such that $\boldsymbol{B}_{Ɱ}$ is a countable basis of $(Ɱ, \mathcal{T}_{Ɱ_{𝔟}})$. Suppose that $(𝔉, Ɱ)$ is a first scarce 𝔓-space at 𝔟 and consider $\boldsymbol{B}_{Ų} = \{Ꝟ: Ꝟ$ is a subset of Ų such that $ß(𝔟, Ꝟ) \in \boldsymbol{B}_{Ɱ}\}$ since $(Ɱ, \mathcal{T}_{Ɱ_{𝔟}})$ is an exact united space then $\boldsymbol{B}_{Ų}$ is a countable collection of open subsets of Ų now we confirm $\boldsymbol{B}_{Ų}$ is a basis of $(Ų, \mathcal{T}_{Ų})$. Let $Ʊ \in \mathcal{T}_{Ų}$ and $ų \in Ʊ$ since $(𝔉, Ɱ)$ is a first scarce 𝔓-space at 𝔟 then there exist $Ƒ \in Ɱ$ such that $Ƒ(𝔟) = \{ų\}$ thus $Ƒ(𝔟) \subseteq Ʊ$ implies that $Ƒ \in ß(𝔟, Ʊ)$ since $ß(𝔟, Ʊ)$ is a confine MO-open sub set of Ɱ then there exist $ß(𝔟, Ꝟ) \in \boldsymbol{B}_{Ɱ}$ such that $Ƒ \in ß(𝔟, Ꝟ) \subseteq ß(𝔟, Ʊ)$ so that $Ƒ(𝔟) = \{ų\} \subseteq Ꝟ \subseteq Ʊ$ thus $ų \in Ꝟ \subseteq Ʊ$ but $Ꝟ \in \boldsymbol{B}_{Ų}$ therefore $\boldsymbol{B}_{Ų}$ is a basis of $(Ų, \mathcal{T}_{Ų})$ hence $(Ų, \mathcal{T}_{Ų})$ is a second countable space.

Theorem 4.12. Let $(Ų, \mathcal{T}_{Ų})$ be a second countable space and $(𝔉, Ɱ)$ be a second scarce 𝔓-space at 𝔟 then $(Ɱ, \mathcal{T}_{Ɱ_{𝔟}})$ is a confine MO-second countable space.

Proof: Let $(Ų, \mathcal{T}_{Ų})$ be a second countable space such that $\boldsymbol{B}_{Ų}$ is a countable basis of $(Ų, \mathcal{T}_{Ų})$. Suppose that $(𝔉, Ɱ)$ is a second scarce 𝔓-space at 𝔟 and consider $\boldsymbol{B}_{Ɱ} = \{ß(𝔟, Ꝟ): ß(𝔟, Ꝟ)$ is a confine MO-open set of Ɱ such that $Ꝟ \in \boldsymbol{B}_{Ų}\}$ then $\boldsymbol{B}_{Ɱ}$ is a countable collection of confine MO-open subsets of Ɱ now we confirm $\boldsymbol{B}_{Ɱ}$ is a confine MO-basis of $(Ɱ, \mathcal{T}_{Ɱ_{𝔟}})$. Let $ß(𝔟, Ʊ) \in \mathcal{T}_{Ɱ_{𝔟}}$ and $Ƒ \in ß(𝔟, Ʊ)$ then $Ƒ(𝔟) \subseteq Ʊ$ since $(𝔉, Ɱ)$ is a second scarce 𝔓-space at 𝔟 then there exist $ų \in Ų$ such that $Ƒ(𝔟) = \{ų\}$ thus $\{ų\} \subseteq Ʊ$ implies that $ų \in Ʊ$ since Ʊ is an open sub set of Ų then there exist $Ꝟ \in \mathbf{B}_{Ų}$ such that $ų \in Ꝟ \subseteq Ʊ$ so that $Ƒ(𝔟) = \{ų\} \subseteq Ꝟ \subseteq Ʊ$ thus $Ƒ \in ß(𝔟, Ꝟ) \subseteq ß(𝔟, Ʊ)$ but $ß(𝔟, Ꝟ) \in \mathbf{B}_{Ɱ}$ therefore $\mathbf{B}_{Ɱ}$ is a confine MO-basis of $(Ɱ, \mathcal{T}_{Ɱ_{𝔟}})$ hence $(Ɱ, \mathcal{T}_{Ɱ_{𝔟}})$ is a confine MO-second countable space.

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The Split Anti Fuzzy Domination in Anti Fuzzy Graphs

H J Yousif [1] and A A Omran[2]

[1,2]Department of Mathematics, College of Education for Pure Science, University of Babylon, Iraq.

Email:* hayderjasim36@gmail.com and Pure.Ahmed.Omran@uobabylon.edu.iq

Abstract. We will discuss the concept of a split anti-fuzzy dominating set (SAFD) in anti fuzzy graph (GAF) and investigate the relationship of γ_{sAf} (G_{AF})(split anti fuzzy domination number) with other known parameters of anti-fuzzy graph. Some bounds and interesting results for this parameter are obtained. The split anti-fuzzy domination on some standard anti-fuzzy graph has been discussed with some suitable graphs.

Keywords: anti fuzzy graph (G_{AF}), Anti fuzzy dominating set (AFD) and Split anti fuzzy Domination number.

1. Introduction

The fuzzy set introduced by L.A. Zadeh [1] to explain vagueness mathematically and tried to resolve problems by giving a particular grade of membership to every member of a given set, which laid the basis of set theory. In (1975) the fuzzy Graph introduced by A. Rosenfeld [2]. The basic idea of fuzzy graph introduced by Kauffmann [3], and fuzzy relation represents the relationship between the objects of the given set. Domination in fuzzy graphs has been introduced by A.Somasundaram and S. Somasundaram [4] and they defined by effective edge. Domination in fuzzy graphs by strong edge it was discussed by A. Nagoorgani and V. T. Chandrasekaran [5] Anti fuzzy structures on graphs has been introduced by Muhammad Akram [6] and discussed the concepts of self-centroid anti fuzzy graphs and constant anti fuzzy graphs and other concepts. on anti fuzzy graph and domination on anti fuzzy graph has been introduced by R. Muthuraj and A. Sasireka [7, 8] Antipodal anti fuzzy graph has been discussed by Seethalakshmi, R.B. Gnanajothi [9]. Split domination in Fuzzy graph has been introduced by Q. M. Mahioub and N.D Soner [10]. The Strong Split Domination Number of Fuzzy Graphs introduced by C.Y.Ponnappan, P.Surulinathan and S. Basheer Ahamed [11]. In this paper, we introduce the concept of Split anti fuzzy domination on Anti Fuzzy Graph. Some theorems are discussed and suitable examples are given.

2. Basic Definitions:

2.1. Definition [6]: Let $\eta: V\rightarrow [0, 1]$ and $\rho: V\times V \rightarrow [0, 1]$, then $G_{AF} = (\eta, \rho)$ is known as anti fuzzy Graph if $\rho (u_1, u_2) \geq \eta (u_1) \vee \eta (u_2)$ $\forall$ $u_1, u_2 \in V$ and is denoted by $G_{AF} = (\eta, \rho)$ and $\vee$: refer to maximum.

2.2. Definition [6]: $G_A{}^* = (\eta^*, \rho^*)$ is known as underlying crisp graph of $G_{AF} = (\eta, \rho)$
Where $\eta^* = \{w \in V / \eta (w) > 0\}$ and $\rho^* = \{(u, w) \in V \times V / \rho (u, w) > 0\}$.

Note: ρ is taken into account as reflexive and symmetric. For each example, η is selected suitably. i.e., only undirected G_{AF} are studied.

2.3. Definition [7]: The size Ş and order Ƥ of $G_{AF} = (\eta, \rho)$ are defined to be Ş $=\sum_{uv\in E} \rho(u, v)$

And $\dot{P} = \sum_{v \in V} \eta(v)$, Denoted by S (G_{AF}) and O (G_{AF}) respectively.

2.4. Definition [8]: G_{AF} is complete if ρ (u, w) = max [η (u), η (w), ∀ u, w ∈ η *] and it is denote by K_η

2.5. Definition [9]: The complement of G_{AF} = (η, ρ) is an anti-fuzzy graph such that: $\eta = \bar{\eta}$ and $\overline{\rho(x, y)}$= 1- ρ (u, w) + max [η (u), η (w)] for all ρ (u, w) ∈ E.

2.6. Definition [8]: The effective edge e = (u, w) in G_{AF} is defined as if ρ (u, w) = max [η (u), η (w)].

2.7. Definition [8]: Let w be a vertex in $G_{AF,}$ N (w) = {u: (w, u) is an effective edge} is known as The Neighbourhood of w and N[w] ∪{w} is known as the closed neighbourhood of w.

2.8. Definition [6]: The G_{AF} = (η, ρ) is connected if there exist a fuzzy path between any two vertices of G_{AF}.

2.9. Definition [12]: The G_{AF} = (η, ρ) is a strong anti fuzzy graph if ρ (u, w) =max [η (u), η (w)], ∀ ρ (u, w) ∈ ρ *.

2.10. Definition [12]: The v-nodal in G_{AF} is defined as every vertex has equal fuzzy values. i.e η (x) = k, ∀ x ∈V (G_{AF}).

2.11. Definition [12]: The e-nodal in G_{AF} is defined as every edge has an equal fuzzy values. i.e. ρ (x, y) = k ∀ (x, y) ∈E (G_{AF}).

2.12. Definition [12]: The uninodal in G_{AF} is defined as for every vertices and edges in G_{AF} have the equal Fuzzy values i.e. η (x) = k = ρ (x, y).

2.13. Definition [13]: Let A ⊆V(G_{AF}) is known as an anti-fuzzy vertex cover of G_{AF} if for each effective
Edge e = (u, w), at least (one) of u ,w is in A. The maximum anti-fuzzy cardinality of anti-fuzzy vertex cover is known as anti-fuzzy vertex covering number of G_{AF} and is represented by α_0 (G_{AF}).

Note: If e = (v, w) is an effective edge in an anti fuzzy graph G_{AF}, then we say that v and e cover each other.

2.14. Definition: A vertex w is known as an isolated vertex if ρ (w, u) > η (w) ∨ η (u) ∀ u ∈ V-{w}.

2.15. Definition: Let S ⊑ V(G_{AF}) is known as the independent anti-fuzzy set if

$$\begin{cases} \rho\,(w,u) = 0 & \forall\, u\,, w \in\; S \text{ such that } \rho\,(w,u) \notin E\,(GAF) \\ \rho\,(w,u) > \eta\,(w) \vee \eta\,(u)\; \forall\, u\,, w \in\; S \text{ such that } \rho\,(w,u) \in\; E(GAF) \end{cases}$$

2.16. Definition: An independent anti – fuzzy set S of G_{AF} is called the maximal independent anti fuzzy set if there is no independent anti- fuzzy set S* of G_{AF} such that |S* | > | S |.

2.17. Definition: The maximum fuzzy cardinality over all maximal independent anti fuzzy set of G_{AF} is known as the independence number of G_{AF} and is denoted by β_0 (G_{AF}).

2.18. Definition: Two vertices u_1 and u_2 of G_{AF} dominate each other if ρ (u_1, u_2) = max [η (u_1), η (u_2)].

2.19. Definition: A vertex subset $\mathcal{D}$ of V(G_{AF})is known as anti-fuzzy dominating (AFD) set of G_{AF} if for each vertex $u_1 \in V - \mathcal{D}$ there exists a vertex $u_2 \in \mathcal{D}$ such that u_2 dominates u_1. The AFD set $\mathcal{D}$ of G_{AF} is called minimal AFD set of G_{AF} if no proper subset $\mathcal{D}$ * of $\mathcal{D}$ is AFD of G_{AF}.

2.20. Definition: The maximum fuzzy cardinality among all minimal AFD set of G_{AF} is called the anti fuzzy domination number and is denoted by γ_{Af} (G_{AF}).

3. Split anti fuzzy Domination of G_{AF}.

In this section the SAFD set and split anti fuzzy domination number on G_{AF} are defined, uninodal anti fuzzy graph is discussed, and these concepts are studied on some kinds of simple G_{AF}.

3.1. Definition: AFD set $\mathcal{D}$ of G_{AF} is known as SAFD set of G_{AF} if the induced anti fuzzy subgraph

< V – $\mathcal{D}$ > is disconnected.

3.2. Definition: The SAFD set $\mathcal{D}$ of G_{AF} is known as minimal SAFD set of G_{AF} if no proper subset $\mathcal{D}$* of $\mathcal{D}$ is SAFD set of G_{AF}.

3.3. Definition: The maximum fuzzy cardinality among all minimal SAFD set of G_{AF} is known as the split anti fuzzy domination number of G_{AF} and is denoted by γ_{sAf} (G_{AF}).

3.1. Example: Consider G_{AF} in Figure1.

Such that V = {v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}} and I? (u, v) = η (u) ∨ η (v) ∀(u, v) ∈ E (G_{AF})

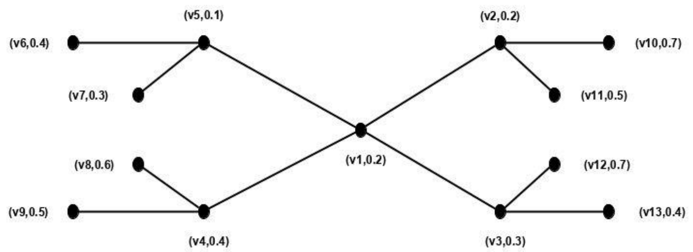

Figure.1

We see that the vertex subset $\mathcal{D}_1$ = {v_2, v_3, v_4, v_5}, $\mathcal{D}_2$ = {v_1, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}},

$\mathcal{D}_3$ = {v_4, v_5, v_{10}, v_{11}, v_{12}, v_{13}} and $\mathcal{D}_4$ = {v_2, v_3, v_6, v_7, v_8, v_9} are minimal SAFD Set of G_{AF} and hence, γ_{sAf} (G_{AF}) = max {$|\mathcal{D}_1|$, $|\mathcal{D}_2|$, $|\mathcal{D}_3|$, $|\mathcal{D}_4|$} = max {1, 4.3, 2.8, 2.3} = 4.3

Observation 3.1: A minimal SAFD set of GAF with $|\mathcal{D}|$ = γ_{sAf} (G_{AF}) is denoted by γ_{sAf}- set of G_{AF}.

3.1. Preposition: Let anti fuzzy graph G_{AF} = K_η then SAFD set does not exist.

3.2. Preposition: Let G_{AF} = $K_{1,\eta}$ a star anti fuzzy graph then γ_{sAf} ($K_{1,\eta}$) = η (v), v is a root vertex.

3.3. Preposition: Let G_{AF} = $K_{\eta 1,\eta 2}$ be a complete anti fuzzy bipartite graph where $|V_1|$= m and $|V_2|$=n where m=$\sum \eta(v)$, v ∈ V_1 and n=$\sum \eta(v)$, v ∈ V_2 then γ_{sAf} ($K_{\eta 1,\eta 2}$) = max {m, n}.

3.1. Theorem: Let $\mathcal{D}$ be a (SAFD) set of G_{AF} be a minimal SAFD set of G_{AF} if and only if for every vertex $u_2 \in \mathcal{D}$ one of the next situations holds:

(a) There exists a vertex $u_1 \in V - \mathcal{D}$ such that N (u_1) ∩ $\mathcal{D}$ = {u_2};

(b) u_2 is an isolated in $\mathcal{D}$;

(c) < (V − $\mathcal{D}$) ∪{u_2} > is connected.

Proof: Consider $\mathcal{D}$ is a minimal SAFD of GAF and $u_2 \in \mathcal{D}$ such that u_2 does not satisfy any one of the three situations, Then by (a) and (b) $\mathcal{D}^*$ = $\mathcal{D}$ − {u_2} is AFD set of GAF and by condition (c) < V – $\mathcal{D}^*$ > is disconnected. This implies that $\mathcal{D}^*$ is a minimal SAFD set of G_{AF}; this is a contradiction with minimalist $\mathcal{D}$. Therefore, for every vertex $u_2 \in \mathcal{D}$ satisfies one of the above conditions.

Conversely, assume that for every vertex $u_2 \in \mathcal{D}$ one of the above situations holds. Further,

if $\mathcal{D}$ is not minimal, then there exists a vertex $u_2 \in \mathcal{D}$ such that $\mathcal{D}$ − {u_2} is SAFD set of G_{AF} and there exists a vertex $u_1 \in \mathcal{D}$ − {u_2} such that u_1 dominates u_2. That is $u_1 \in N(u_2)$. Therefore, u_2 does not satisfy the conditions (b) and (c), thus it must satisfy the condition (a). Then there exists $u_1 \in V - \mathcal{D}$ such that N (u_1) ∩ $\mathcal{D}$ = {u_2}. Since $\mathcal{D}$ − {u_2} is a SAFD set of G_{AF}, then there exists h ∈ $\mathcal{D}$ − {u_2} such that h ∈ N(u_1). Therefore, h ∈ N (u_1) ∩ $\mathcal{D}$, h ≠u_2, is a contradiction with N (u_1) ∩ $\mathcal{D}$ = {u_2}. Clearly, $\mathcal{D}$ is a minimal SFD set for G_{AF} □

3.2. Theorem: The AFD set $\mathcal{D}$ of GAF is a (SAFD) set of G_{AF} if and only if there exist u_1, $u_2 \in V - \mathcal{D}$ such that every u_1-u_2 path contains a vertex of $\mathcal{D}$.

Proof: Suppose that $\mathcal{D}$ is a minimal SAFD set of G_{AF}, then < V − $\mathcal{D}$> is disconnected, take u_1, u_2 ∈ V − $\mathcal{D}$ such that every u_1-u_2 path-joining u_1 and u_2 must contain a vertex of $\mathcal{D}$.

Conversely, assume that u_1, $u_2 \in V - \mathcal{D}$ such that every u_1-u_2 path contains a vertex of $\mathcal{D}$. Let $\mathcal{D}$ be an AFD set of G_{AF}, < V − $\mathcal{D}$ > either connected or disconnected. < V − $\mathcal{D}$ > is connected, then for any two vertices u_1, $u_2 \in V - \mathcal{D}$ there is a u_1-u_2 path joining u_1 and u_2 in < V − $\mathcal{D}$ > which does not contain a vertex of $\mathcal{D}$, this impossible with our assumption. Therefore, $\mathcal{D}$ is a SAFD set of G_{AF}. □

3.4. Preposition: Let G_{AF} = (η, ρ) be a strong anti fuzzy graph and $\mathcal{D}$ be a γ_{sAf} (G_{AF}) – set of G_{AF}, Then V − $\mathcal{D}$ is AFD set of G_{AF}.

Proof: Assume that $\mathcal{D}$ is a minimal SAFD set of G_{AF}. If V − $\mathcal{D}$ is not AFD set of G_{AF}, then there exists w ∈ $\mathcal{D}$ which does not dominate any vertex of V − $\mathcal{D}$. Thus $\mathcal{D}^*$= $\mathcal{D}$ − {w} is a SADF set of G_{AF}, this is a contradiction, therefore V − $\mathcal{D}$ is AFD set of G_{AF}. □

3.5. Preposition: For any strong anti fuzzy graph G_{AF}= (η, ρ),

$$\gamma_{Af}\ (GAF)\ +\gamma_{sAf}\ (GAF)\ \leq\ Ƥ.$$

Proof: Let $\mathcal{D}$ be a γ_{sAf}– set of G_{AF}, thus from Preposition 3.3, V − $\mathcal{D}$ is AFD set of G_{AF}. Therefore

γ_{Af} (G_{AF}) ≤ |V − $\mathcal{D}$ | = Ƥ − γ_{sAf} (G_{AF}). Hence γ_{Af} (G_{AF}) + γ_{sAf} (G_{AF}) ≤ Ƥ. □

3.6. Preposition: Let $\mathcal{D}$ be a γ_{sAf}– set of G_{AF} = (η, ρ). If < $\mathcal{D}$ > is disconnected anti fuzzy subgraph of G_{AF}, then γ_{sAf} (G_{AF}) ≤ Ƥ /2.

Proof: Let $\mathcal{D}$ be a γ_{sAf}– set of G_{AF}, thus V − $\mathcal{D}$ is AFD set of G_{AF}, since < $\mathcal{D}$ > is disconnected, then V − $\mathcal{D}$ is a SAFD set of G_{AF}. Therefore, γ_{sAf} (G_{AF}) ≤ |V − $\mathcal{D}$ | = Ƥ − γ_{sAf} (G_{AF}). Hence γ_{sAf} (G_{AF}) ≤ Ƥ/2. □

3.7. Preposition: For any anti fuzzy graph G_{AF}= (η, ρ), γ_{Af} (G_{AF}) ≤ γ_{sAf} (G_{AF});

Proof: from definitions of γ_{Af} (G_{AF}) and γ_{sAf} (G_{AF}). □

3.8. Preposition: V − A is a SAFD set of strong anti fuzzy graph G_{AF} = (η, ρ) If A is maximal Independent anti fuzzy set of G_{AF}.

Proof: Since A is maximal independent anti fuzzy set of strong anti-fuzzy graph G_{AF}, then V − A is AFD set of G_{AF}. Further < A > =< V − (V − A) > is disconnected. This implies V − A is a SAFD set. □

3.3. Theorem: A set S_i ⊑V (G_{AF}) is independent anti fuzzy set of GAF if and only if V(G_{AF})-S_i is an anti-vertex covering of G_{AF}.

Proof: Let S_i be an independent anti-fuzzy set of G_{AF}. By the definition of independent anti fuzzy set, there exist no effective edge between any two vertices in S_i, thus no edges of G_{AF} has at least one end in S_i Then V(G_{AF}) - S_i contains at least one end for every edge, Hence ,V(G_{AF})-S_i is an anti-vertex covering of G_{AF}. And similarly if S_C is anti-vertex covering then it is clear that V(G_{AF})-S_C is independent anti-fuzzy Set. □

3.4. Theorem: If G_{AF} is an anti-fuzzy graph, then Ƥ ≤ α_0+ β_0, where α_o, β_o are anti-fuzzy covering number and independence number respectively.

Proof: Let G_{AF} be an anti-fuzzy graph. Let S_i be a maximal anti independent set and S_C be an anti-vertex covering of G_{AF}. By theorem3.3, we get V(G_{AF})-S_C is an anti-independent set of G_{AF}.

Hence |V- S_C| ≤ |S_i | ⇒ Ƥ - α_o ≤ β_o ⇒ Ƥ ≤ α_0+ β_0. □

3.5. Theorem: Let G_{AF} = (η, ρ) be a uninodal anti-fuzzy graph then γ_{sAf} (G_{AF}) ≤ α_0(G_{AF}), where α_0 (G_{AF}) is a vertex covering number of G_{AF}.

Proof: Let A be a maximal independent anti-fuzzy set of G_{AF}, then it is contains at least two vertices and for each vertex u ∈ A there exists w ∈ V − A such that I? (u, w) = η(u) ∨ η(w). Thus V − A is a SAFD set of G_{AF}. Hence γ_{sAf} (G_{AF}) ≤ |V − A | = Ƥ − β_0 (G_{AF}) = α_0(G_{AF}). □

3.6. Theorem: Let G_{AF}= (η, ρ) be any anti fuzzy graph with end-vertex, γ_{Af} (G_{AF}) = γ_{sAf}(G_{AF}). Furthermore, there exists a SAFD set of G_{AF} containing all vertices adjacent to anti fuzzy end-vertices.

Proof: Suppose that $\mathcal{D}$ is AFD set of G_{AF} and v be an end vertex of G_{AF}, then there exists a cut vertex u adjacent to v and I? (u, v) = η(u) ∨ η(v). Assume that u ∈ $\mathcal{D}$, then $\mathcal{D}$ is a SAFD set of G_{AF}, if u ∈ V − $\mathcal{D}$ then v ∈ $\mathcal{D}$ Hence $\mathcal{D}$ − {v} ∪{u} is SAFD set. Repeating this process for all such cut-vertices adjacent to end-Vertices, we obtain a SAFD set of G_{AF} containing all cut-vertices adjacent to end-vertices of G_{AF}. □

3.7. Theorem: Let G_{AF}= (η, ρ) be any anti fuzzy graph, then γ_{sAf} (G_{AF}) = t, t ∈ [0, 1],

t = η (w), w ∈ V (G_{AF}) if and only if GAF has only one cut vertex w ∈ V(G_{AF}) which has n − 1

neighbors of vertices.

Proof: Assume that $\mathcal{D}$ = {w} is a γ_{sAf} −set of G_{AF}, thus < V − {w}> is disconnected. Hence v is a cut vertex of G_{AF}, so N (w) = {V − {w}} then w has n − 1 neighbors in G_{AF}. Assume that there exists another cut vertex say u in G_{AF} which has n − 1 neighbors in G_{AF}, then u is adjacent to all remaining vertices of G_{AF}. In this case < V − {w} > is connected, this is a contradiction. Then w is only the cut vertex of G_{AF} has n − 1 neighbors in G_{AF}.

Conversely, assume that w is only one cut vertex of G_{AF} has n − 1 neighbors in G_{AF}, then w is adjacent to all vertices of G_{AF}. Hence there exists u∈ V − {w}, u ≠ w which it is not adjacent with other vertex of V − {w}, the < V − {w} > is disconnected. Thus = {w} is SAFD set of G_{AF} and hence γ_{sAf} (G_{AF}) = t, t = η (w). □

3.8. Theorem: Every SAFD set of G_{AF} = (η, ρ) is a split dominating set in crisp graph G_A* = (η ∗, I? *).

Proof: Let $\mathcal{D}$ be a SAFD set of G_{AF} = (η, ρ) then for each vertex u ∈V- $\mathcal{D}$ there exist w ∈ $\mathcal{D}$ such that I? (u, w) = η (u) ∨ η (w) > 0, and < V- $\mathcal{D}$ > is disconnected. Thus I? (u ,w) ∈ μ*, hence each vertex in V- $\mathcal{D}$ is dominated by at least one vertex in $\mathcal{D}$ and < V- $\mathcal{D}$ > is disconnected , thus $\mathcal{D}$ is a split dominating set in G_A*= (η ∗, I? *).□

Note: The convers theorem 3.8 is not true.

3.8.1. Example: Let G_A*= (η ∗, I? *) and G_{AF} = (η, I?), be a crisp graph of GA and anti fuzzy graph are considered in figure (2) and figure (3) respectively.

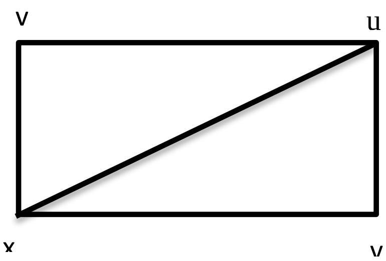

Figure.2 crisp graph (G_A*)

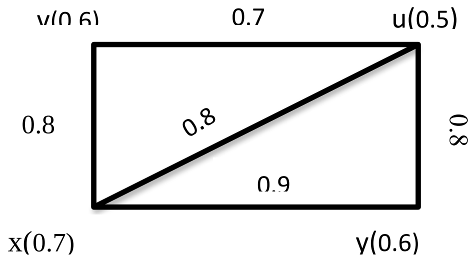

Figure.3 anti fuzzy graph (G_{AF})

We see that the split dominating set in crisp graph $G_A^* = (\eta *, \boldsymbol{\rho} *)$, $\boldsymbol{\mathcal{D}} = \{x, u\}$ which is not a split anti fuzzy dominating set in anti fuzzy graph $G_{AF} = (\sigma, \mu)$.

4. Conclusion

In this work, we studied (SAFD) set and a split anti fuzzy domination number of an anti-fuzzy graph (G_{AF}). For some standard an anti-fuzzy graphs, we found the exact value of γ_{sAf} (G_{AF}). In addition, we got some relationships between split anti-fuzzy domination number and for some parameters.

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On Certain Types of Topological Spaces Associated with Digraphs

Khalid Shea Khairallah Al' Dzhabri[1] and Mohammad Falih Hani Al'murshidy[2]

[1]University of Al-Qadisiyah, College of Education, Department of mathematics
[2]Ministry of Education, Babylon Education Directorate

[1]khalid.aljabrimath@qu.edu.iq, [2] mohfalih17@yahoo.com

Abstract. In this work, we constructed a new types of topological structures by associated with digraphs called $\mathrm{DG}_{\mathrm{E}}^{\sigma}-$ topological space and $\mathrm{DG}_{\mathrm{E}}^{\mu}-$ topological space by induced two alternate definitions $\mathrm{DG}^{\sigma}-$ open set and $\mathrm{DG}^{\mu}-$ open set respectivally. Investigatedsome properties ofthe topologies determined by a digraph with respect to each of these alternated definitions.

1. Introduction

Graph theory is important mathematical tool in many subjects ply an important role in discrete mathematic. There is a closed connection between topologies and digraphs. In 1967, J.N. Evans, et al. [3] There was found a correlation between the set of all topologies and the set of all transitive digraphs. In 1968, T.N. Bhargve and T.J. Ahllborn [7] studied and dissect the topological spaces with digraphs explain that every digraph $D = (V, E)$ is offset by topology $(\mathrm{V}, \mathcal{T}_{\mathrm{E}})$ where $\mathcal{T}_{\mathrm{E}}$={U: U ∈ 2^{V}, Uopen};. In 2013, A. H.Mahdi and S.N.Al-khafaji [1], construccted a topollogy on finiite undirectd graphs and a topology on subgraphs on the set of edges and discussed the connectedness of each of the graph and the topological spacee that induces by that finite undirectd graph . In 2015, Khalid Al'Dzhabri in [6] find the correspondnce between the finite topology and the graph of finite reflexive –transitive relations. In 2018, K. A. Abdu and A. Kilicman[4] by using the set of edges of any digraph studied associated of applyiing the topology on digrphs called compatible edge topology and incompatible edge topology. In 2020, Khalid Al'Dzhabri, A. Hamza Mahdi and Y. Saheb Eissa [5] constructed each digraph to topology and studied new operators called DG −operators. In our work, we constructed a new types of topological structures by associated with digraphs called $\mathrm{DG}_{\mathrm{E}}^{\sigma}-$ topological space and$\mathrm{DG}_{\mathrm{E}}^{\mu}-$ topological space by induced two alternate definitions $DG^{\sigma}-$ open set and$DG^{\mu}-$ open set respectivally.

2. Preliminaries

In this part, we recall that some definitions and facts and update another definition by using our new concepts .

Definition2.1[2]: A digraph (directed graph)is a setV of vertices and a set Eof order pairs of vertices such that $\emptyset \subseteq \mathrm{E} \subseteq \mathrm{V} \times \mathrm{V}$ and denoted by $\mathrm{D} = (\mathrm{V}, \mathrm{E})$ or simply by $\mathrm{D}(\mathrm{V})$ if the set Eis fixed .

Definition2.2[2]:Let $\acute{\mathrm{V}} \subseteq \mathrm{V}$,the digrph $\mathrm{D} = (\acute{\mathrm{V}}, \mathrm{E} \cap \acute{\mathrm{V}} \times \acute{\mathrm{V}})$ denoted simply by $\mathrm{D}(\acute{\mathrm{V}})$,is a subdigrph of the digraph $\mathrm{D} = (\mathrm{V}, \mathrm{E})$.

Definintion2.3[2]:An element of E is called an arc or (directed edge) of the digraph $\mathrm{D} = (\mathrm{V}, \mathrm{E})$and is denoted by uv ∈ E ;and said to be an arc from u to v.

Definition2.4 [2]:A directed path (dipath) of length L from u_i to u_j is an ordered $(L+1)$ $-$tuple of vertices of $D=(V,E)$, $u_i, u_{k_1}, u_{k_2}, u_{k_3}, \dots, u_{k_{(L-1)}}, u_j$ in which L is a positive integer and $\{u_i u_{k1}, u_{k1} u_{k2}, u_{k2} u_{k3}, u_{k(L-1)} u_j\}$ is a subset of the arc set E of $D=(V,E)$ .The vertex u_i is called the initial vertex ,the vertices $u_{k_1}, u_{k_2}, \dots, u_{k_{(L-1)}}$ is called intermediate vertices ,and u_j is called the terminal vertex of the digraph .

Definition2.5[2]:An directed edge from u_i tou_i is called a loop at u_i and denoted by $u_i u_i \in E$.

Definition3.6:If there exists a dipath from u_i to u_j in $D=(V,E)$,we say that u_i indegree to u_j or u_j outdegree from u_i and denoted by $\psi(i,j)$ .The ordered pair (u_i,u_j) is called an indegrees pair . Ifu_i is not indegree to u_j,we write $\widetilde{\psi}(i,j)$.

Definition2.7:If both $\psi(i,j)$ and $\psi(j,i)$ that is if u_i is indegree to u_j and u_j indegree to u_i we say that u_i andu_j are symmetrically indegrable and denoted by $\psi^*(i,j)$

Remarrk2.8:We note that the relation ψ^* is an equivalence relation on a set V in $D=(V,E)$.

Definition3.9[2]:Let $D=(V,E)$ be a digraph .Then $D=(V,E)$ is called a transitive digraph if $uv \in E$ and $vw \in E$ implies that $uw \in E$.

Now by using $\psi(i,j)$ in the definition 3.6 we give the following definitions.

Definition2.10:Let $D=(V,E)$ be a digraph . Then D is called

i) $\psi-$ strongly connected , if $\psi^*(i,j)$, for every u_i and u_j in V.

ii) ψ $-$unilaterally connected , if $\psi(i,j)$ or $\psi(j,i)$ for every u_i and u_j in V.

iii)ψ $-$weakly connected , if $D=(V, E\cup E^c)$ is ψ $-$strongly connected where $E^c=\{vu: uv \in E\}$.

iv) ψ $-$disconnected if $D=(V,E)$ is not even ψ $-$weakly connected.

Remark 2.11: 1) Adigraph $D=(V,E)$ is called be of type:

i)ψ_4, if $D=(V,E)$ is ψ $-$strongly connected .

ii) ψ_3, if $D=(V,E)$ is ψ $-$unilaterally connected but notψ $-$strongly connected.

iii) ψ_2, if $D=(V,E)$ isψ $-$weakly connected but notψ $-$unilaterally connected.

iv) ψ_1, if $D=(V,E)$ is ψ $-$disconnected.

2) A digraph $D=(V,E)$ of type $\psi_i$is said to be in the connectedness state ψ_i, for $i=1,2,3$ or 4.

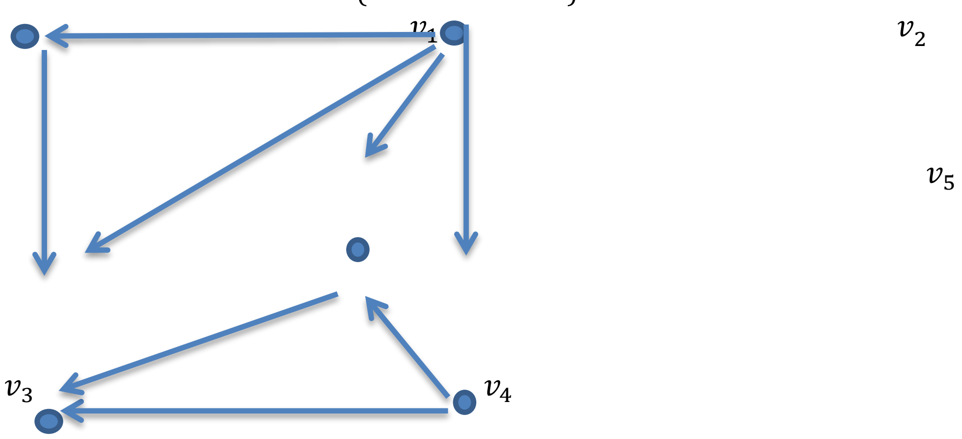
3. On DG $-$ open set.

This section, introduced by Khalid Al'Dzhabri, Abd Alhamza Mahdi and Yousif Saheb [5]by constructed a topology which associated with digraph called DG$-$ topological space induced by DG $-$open set.

Definition 3.1[5]: Let $D=(V,E)$ be a digraph a subset A of V is called DG $-$ open set if for $u_i \in A$ and an directed edge$u_j u_i \in E$, then $u_j \in A$.

Remark 3.2 [5]:From the definition above the topology associated with the digraph $D=(V,E)$ denoted by τ_{DG} and $\tau_{DG}=\{A: A is DG - open set\}$.And (V,τ_{DG}) is called DG $-$ topological space .

Example3.3: consider the digraph $D=(V,E)$where $V=\{v_1, v_2, v_3, v_4, v_5\}$



And the topology corresponding to the above digraph $\tau_{DG} = \{\emptyset, V, \{v_2\}, \{v_1, v_2\}, \{v_2, v_4\}, \{v_1, v_2, v_4\}, \{v_2, v_4, v_5\}, \{v_1, v_2, v_4, v_5\}\}$

Theorem3.4 [5]:Let $D = (V, E)$ be a digraph then (V, τ_{DG}) is topology on a set V associated with the digraph $D = (V, E)$.

Proof:

[O1] Clearly $\emptyset$ and $V \in \tau_{DG}$

[O2] Let $\{U_\alpha\}$ be a collection of subsets of V in τ_{DG} ,let $u \in \cup_\alpha U_\alpha$ and $uv \in E$ then $\exists U_{\alpha_0} \in U_\alpha$ for some α with $uv \in E$ implies that $v \in U$, so $\cup_\alpha U_\alpha \in \tau_{DG}$.

[O3] Let $U_i \in \tau_{DG}, \forall i = 1,2,3 \dots, n$. Now let $u \in \cap_{i=1}^{n} U_i$ and $vu \in E$,then $u \in U_i$ for all i and $v \in U_i$ and therefore a family $\cap_{i=1}^{n} U_i \in \tau_{DG}$ .Hence τ_{DG} is topology on V .

Definition3.5: Let(V, τ_{DG})be a $DG-$ topological space , then (V, τ_{DG}) is called a $DG -$ topologically connected if V can not be expressed as union of two disjoint non empty a $DG -$ open set and other wise (V, τ_{DG}) is called a $DG -$disconnected space.

Theorem3.6: The digraph $D = (V, E)$ is $\psi -$ weakly connected iff (V, τ_{DG}) is a $DG -$ topologically connected .

Proof: Suppose that (V, τ_{DG}) is a $DG -$ connected space , then V cannot be expressed as the union of two disjoint $DG -$ open sets and this iff every non empty proper subset of V is not a $DG -$ open or is not $DG -$ closed . Equivalently , by definition 3.1 , for each proper subset say A of V , there exists an directed edge from A^c to A or there exists an directeed edge from A to A^c in the digraph $D(V)$, that is in $D = (V, E \cup E^c)$, where $E^c = \{uv: vu \in E)$ there exists an directed edgefrom A^c to A and there exists an directed edgefrom A to A^c for each proper subset A of V . Hence by definition 3.1the only $DG-$ open set in $D = (V, E \cup E^c)$are$\emptyset$and V . Thus$D = (V, E \cup E^c)$ is $\psi -$strongly connected and hence $D = (V, E)$ is $\psi -$ weakly connected .

4. On DG^σ −open set.

In this section, we constructed a topology associated with digraph called DG_E^σ −topological space which induced by DG^σ −open set.

Definition 4.1:Let $D = (V, E)$ be a digrph, subset A of V is called DG^σ −opeen set if for $u_i \in A$ and $u_j \in A^c$implies that $u_i u_j \notin E$. In other words, a subst A of V is DG^σ −opeen set, if there does not exist an directed edge in $D = (V, E)$ from A to A^c.

Theorem 4.2:Let $D = (V, E)$ be a digraph then:

a subset A of a digraph $D = (V, E)$ is DG^σ −opeen set, iff A is DG^σ −closed.

A subset A of a digraph$D = (V, E)$ is DG^σ −open set, iff A is $DG^\sigma -$ open of a digraph $D = (V, E^c)$, where$E^c = \{u_j u_i \in E: u_i u_j \in E\}$.

Proof: we can compare between the definition 4.1 and the definition 3.1 then we note that $(E^c)^c = E$ and that the digraph $D = (V, E^c)$ obtained from the digraph $D = (V, E)$ by reversing the direction of each and every directed edge of $D = (V, E)$ and this operation does not alter the connectedness state ψ_i, for i = 1,2,3 or 4. of a digraph.

Theorem 4.3:Each digraph$D = (V, E)$ determines, with respect to DG^σ −open sets, a unique $DG_E^\sigma -$ topological space $(V, \tau_{DG_E^\sigma})$ and this topological space is identical to the $DG_{E^c}^\sigma -$ topological space $(V, \tau_{DG_{E^c}^\sigma})$ determined with respect to DG^σ −open set by digraph $D = (V, E^c)$, where $E^c = \{u_j u_i \in E: u_i u_j \in E\}$.

Proof: Let A be an arbitrary subset of V. By theorem 4.2 a subst A of a digraph $D = (V, E)$ is DG^σ −opeen set, iff A is $DG^\sigma -$ open of a digraph $D = (V, E^c)$, where$E^c = \{u_j u_i \in E: u_i u_j \in E\}$.And thus $\tau_{DG_E^\sigma} = \{A: A \subseteq V$ of $D = (V, E)$,is $DG^\sigma -$ open set$\}$ and this $DG_E^\sigma -$ topology is identical to the $DG_{E^c}^\sigma -$ topology such that $\tau_{DG_{E^c}^\sigma} = \{A: A \subseteq V$ of $D = (V, E^c)$, is $DG^\sigma -$ open set$\}$ and hence by theorem 3.4, $D = (V, E^c)$ determines a unique $DG_{E^c}^\sigma -$ topological space $(V, \tau_{DG_{E^c}^\sigma})$ which is identical to the $DG_E^\sigma -$ topological space $(V, \tau_{DG_E^\sigma})$.

Definition 4.4: Let$(V, \tau_{DG_E^\sigma})$be a $DG_E^\sigma -$ topological space , then $(V, \tau_{DG_E^\sigma})$. is called a $DG_E^\sigma -$ topologically connected if V can not be expressed as uniion of two disjoint non empty a $DG^\sigma -$ opeen set and other wise $(V, \tau_{DG_E^\sigma})$ is called a $DG_E^\sigma -$disconnected space.

Theorem 4.5:The digraph $D = (V, E)$ is $\psi -$ weakly connected iff $(V, \tau_{DG_E^\sigma})$ is a $DG_E^\sigma -$ topologically connected .

Proof:$D = (V, E)$ is of the same connectedness states ψ_i, for $i = 1,2,3$ or 4. of a digrph $D = (V, E^c)$, $E^c = \{u_j u_i \in E: u_i u_j \in E\}$. In particular, $D = (V, E)$ is $\psi -$ weakly connected iff $D = (V, E^c)$ is $\psi -$ weakly connected and from theorem 4.3 $(V, \tau_{DG_E^\sigma})$ is identical to the $(V, \tau_{DG_{E^c}^\sigma})$ and similarly from theorem 3.6 $D = (V, E)$ is $\psi -$ weakly connected iff $(V, \tau_{DG_E^\sigma})$ is a $DG_E^\sigma -$ topologically connected with respect to $DG^\sigma -$ open set.

5. On DG^μ –open set

In this section, we constructed a topology associated with digraph called $DG_E^\mu -$topological space which induced by $DG^\mu -$open set.

Definition 5.1:Let $D = (V, E)$ be a digraph a subset A of V is called $DG^\mu -$open set if for $u_i \in A^c$ and $u_j \in A$ implies that $u_i u_j \in E$. In other words, a subset A of V is $DG^\mu -$open set, if there exist an directed edge in $D = (V, E)$ fromA^c toA .

Theorem 5.2: A set $A \subseteq V$ of digraph $D = (V, E)$ is $DG^\mu -$open set iff a set $A \subseteq V$ of digraph $D = (V, E^*)$ is $DG^\mu -$open set, where $E^* = V \times V \backslash E$.

Proof: we can compare between the definition 5.1 and the definition 3.1 we note that $(E^*)^* = E$ and the digraph $D = (V, E^*)$ may be determine from the digraph $D = (V, E)$ be including in $D = (V, E^*)$ iff the directed edges which do not appear in the digraph $D = (V, E)$ and this operation does in some cases, change the connectedness states ψ_i. For example the digraph $D = (V, V \times V)$ is type of ψ_4 but the digraph $D = (V, \emptyset)$ is type of ψ_1.

Theorem 5.3:Each digraph$D = (V, E)$ determines, with respect to $DG^\mu -$open sets, a unique $DG_E^\mu -$ topological space $(V, \tau_{DG_E^\mu})$ and this topological space is identical to the $DG_{E^*}^\mu -$ topological space $(V, \tau_{DG_{E^*}^\mu})$ determined with respect to $DG^\mu -$open set by digraph $D = (V, E^*)$, where $E^* = V \times V \backslash E$.

Proof: Let A be an arbitrary subset of V. By theorem 5.2 a subset A of a digraph $D = (V, E)$ is $DG^\mu -$open set, iff A is $DG^\mu -$ open of a digraph $D = (V, E^*)$, where$E^* = V \times V \backslash E$.And thus $\tau_{DG_E^\mu} = \{A: A \subseteq V$ of $D = (V, E)$, is $DG^\mu -$ open set$\}$ and this $DG_E^\mu -$ topology is identical to the $DG_{E^*}^\mu -$ topology such that $\tau_{DG_{E^*}^\mu} = \{A: A \subseteq V$ of $D = (V, E^*)$, is $DG^\mu -$ open set$\}$ and hence by theorem 3.4 $D = (V, E^*)$ determines a unique $DG_{E^*}^\mu -$ topological space $(V, \tau_{DG_{E^*}^\mu})$ which is identical to the $DG_E^\mu -$ topological space $(V, \tau_{DG_E^\mu})$.

Definition 5.4:Let$(V, \tau_{DG_E^\mu})$be a $DG_E^\mu -$ topological space , then $(V, \tau_{DG_E^\mu})$. is called a $DG_E^\mu -$ topologically connected if V can not be expressed as union of two disjooint non empty a $DG^\mu -$ open set and other wise $(V, \tau_{DG_E^\mu})$ is called a $DG_E^\mu -$disconnected space.

Remark 5.4:In general, the "connectedness classification" of a digraph $D = (V, E)$ is not consistent with $DG_E^\mu -$ topologically connected of the $DG_E^\mu -$ topological space$\left(V, \tau_{DG_E^\mu}\right)$. For example : let $V = \{u_1, u_2\}$. The digrph $D = (V, E^*)$ is type of ψ_1 , but the $DG_{E^*}^\mu -$ topological space$(V, \tau_{DG_{E^*}^\mu})$ determine with respect $DG^\mu -$ open sets, by $D = (V, E^*)$ is an indiscrete space and hence is $DG_E^\mu -$ topologically connected.

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Matroidal Structure Based On Soft-Sets

Kholod M. Hassan and Saied A. Johnny

AL-Diwaniyah Education Directorate, AL-Diwaniyah, Iraq

Email: kholodmohamed@qu.edu.iq
Email: saiedjhonnykh@gmail.com

Abstract: A matroidal structure that generalizes the properties of independence. Relevant applications are found in graph theory and linear algebra. This paper will focus on the definitions of a matroid in terms of generalization for a crisp set called soft-set and soft-point, also we give some results related to this concept. A soft-matroid is defined and examples of soft-systems which form are given. The novel concept of independent soft-set is introduced. The notion maximal of independent soft-sets and minimal dependent soft-sets, with examples from linear algebra and soft-graph theory, are illustrated. Finally, we investigate some fundamental properties of soft-matroid.

1. Introduction:

Soft-set theory was initiated by the Russian researcher Molodtsov in 1999, [1] as a new mathematical tool for many mathematical theories. Up to now, the algebraic structure of the soft-sets has been investigated by some authors, (see [2], [3], [4], [5] and [6]). Maji et al. [7], presented applications of soft-sets in some decision making problems. Ali et el. [8], proposed several new operations in soft-set theory. Moreover, many works have been devoted to application of soft-sets in various algebraic structures. In 2002, Maji et al they introduced the definition of processes on soft-sets and their properties. After that Sujoy et al [9], specified the definition of soft-set which is called a soft-point a generalization of a crisp-point.

A matroid is a structure that generalizes the properties of independence, [10]. Relevant applications are found in graph theory and linear algebra. There are several but equivalent ways to define a matroid, each related to the concept of independence. Various basic examples of matroids are presented and basic concepts are clarified in the context of these examples, [11] and [12].

In recent years, a graph theory is one of the branches of mathematics, which aims to describe phenomena and concepts of an ambiguous, vague, undefined and imprecise meaning. Since the graph theory has a rich potential, researches on a graph theory and its applications in various fields are progressing rapidly.

There exists several different approaches for studying the generalization of the matroidal structure and one of the most important was presented by [13], [14] and [15] such as fuzzy-graph theory, and soft- graph theory which are all generalizations of the graph theory.

The purpose of this paper is to make contribution for investigating on soft-graph theory and we focus on a soft-point of a soft-set and give some new properties within this concept. Moreover, our study in this paper focuses on the analytical part of some aspects of the soft-graph theory.

2. Background Material:

We first review some elementary concepts of soft-sets and soft-graphs that are necessary for this paper.

2.1. Definition [1 , 9]:

(i) A pair (F, A) is said to be a soft-set over a universal set $\mathfrak{U}$, where $A \subseteq \mathfrak{E}$ for a set of parameters and F is a set-valued mapping $\mathrm{F} : A \to \wp(\mathfrak{U})$.

It is apparent that a soft set $\mathrm{F}_A = (\mathrm{F}, A)$ over a universe $\mathfrak{U}$ can be viewed as a parameterized family of subsets of $\mathfrak{E}$.

(ii) A soft set F_A is said to be a soft-point and its denoted by: $\mathfrak{p}_e^x = \{(e, \mathrm{F}(e))\}$, if exactly one $e \in A$, $\mathrm{F}(e) = \{x\}$ for some $x \in \mathfrak{U}$ and $\mathrm{F}(e') = \emptyset$ for all $e' \in A \backslash \{e\}$. i.e. the fact that $\mathfrak{p}_e^x$ a soft-point of F_A and will be denoted by $\mathfrak{p}_e^x \tilde{\in} \mathrm{F}_A$, if $x \in \mathrm{F}(e)$.

2.2. Definition [2 , 9]:

(i) Let F_A and H_B be two soft-sets. Then F_A is said to be a sub-soft-set of H_B, denoted by $\mathrm{F}_A \tilde{\subseteq} H_B$, if:

- $A \subseteq B$;
- $\mathrm{F}(e) \subseteq H(e)$ for all $e \in A$.

(ii) Two soft-sets F_A and H_B are equal, if $\mathrm{F}_A \tilde{\subseteq} H_B$ and $H_B \tilde{\subseteq} \mathrm{F}_A$.

(iii) Two soft-point $\mathfrak{p}_e^x$ and $\mathfrak{p}_{e'}^y$ are equal, if $x = y$ and $e = e'$.

2.3. Definition [2]:

(i) A soft-set F_A is said to be null soft-set and denoted by $\emptyset_A$, if for all $e \in A$, implies that $\mathrm{F}(e) = \emptyset$.

(ii) A soft-set F_A is said to be an absolute soft-set and denoted by $\mathfrak{U}_A$, if for all $e \in A$, implies that $\mathrm{F}(e) = \mathfrak{U}$.

(iii) A complement of a soft-set F_A, denoted F_A^c and defined by:

, $\mathrm{F}^c(e) = \mathfrak{U} \backslash \mathrm{F}(e)$ for all $e \in A$. $\mathrm{F}^c : A \to \wp(\mathfrak{U})$

2.4. Remark [9]:

(i) $\emptyset_A^c = \mathfrak{U}_A$ and $\mathfrak{U}_A^c = \emptyset_A$.

(ii) If $\mathfrak{p}_e^x \tilde{\in} \mathrm{F}_A$, $\mathfrak{p}_e^x \tilde{\notin} \mathrm{F}_A^c$, i.e. $x \notin \mathrm{F}^c(e)$.

2.5. Definition [9]:

A soft-set F_A is said to be a finite, if $\mathrm{F}(e)$ is finite for all $e \in A$.

2.6. Definition [2 , 3]:

(i) The intersection (union) of two soft-sets F_A and G_B is the soft-set H_C, which is defined by:

$H_A = \mathrm{F}_A \tilde{\cap} G_B$ $(H_A = \mathrm{F}_A \tilde{\cup} G_B)$, where $C = A \cap B$ $(C = A \cup B)$ and for all $e \in C$, written

as $H(e) = \mathrm{F}(e) \cap H(e)$ $\left(H(e) = \begin{cases} \mathrm{F}(e) & ; \quad e \in A \setminus B \\ H(e) & ; \quad e \in B \setminus A \\ \mathrm{F}(e) \cup H(e) & ; \quad e \in A \cap B \end{cases}\right)$.

It is clear that every soft-set can be expressed as a union of all soft-points belong to it.

(ii) The difference of two soft-sets F_A and G_A is the soft-set H_A, which is defined by:

$H_A = \mathrm{F}_A \tilde{\setminus} G_A$, and for all $e \in A$, write $H(e) = \mathrm{F}(e) \setminus H(e)$.

2.7. Definition [13]:

Let a pair $\mathbb{G} = (V, E)$ be a crisp graph and A any non-empty set. Let R be a subset of $A \times V$ be an arbitrary relation from A to V. A mapping (or set-valued mapping) from A to V, written as: $\mathrm{F} : A \to \wp(V)$ can be defined as $\mathrm{F}(e) = \{v \in V : eRv\}$ and a mapping $H : A \to \wp(E)$, can be defined as $H(e) = \{xv \in E : \{x, v\} \subseteq \mathrm{F}(e)\}$. A pair (F, A) is a soft-set over V and (H, A) is a soft-set over E.

2.8. Definition [13]: (Soft-graph)

A 4-tiple $\mathbb{G}^* = (\mathbb{G}, \mathrm{F}_A, H_A, A)$ is said to be a soft-graph, if it satisfies the following conditions:

(g_1) $\mathbb{G}$ is a graph;

(g_2) A is a non-empty set of parameters;

(g_3) F_A is a soft-set over V;

(g_4) H_A is a soft-set over E;

(g_5) $(\mathrm{F}(e), H(e))$ is a sub-graph of $\mathbb{G}$ for all $e \in A$.

3. Main results:

In this section, our definition of soft-matroid as introduced. We prove some systems of $\mathbb{G}^*$ are equivalent to the soft-matroid. In our use of the terms independent soft-set, dependent soft-sets, bases of soft-matriod and circuit of a soft-matroid. Finally, give some examples and results related to these concepts.

The number of any soft-points in finite soft-set F_A is said to be the cardinal number and its denoted as $|\mathrm{F}_A|$.

3.1. Definition**:** A soft-matroid $\widetilde{\mathcal{M}}$ is a structure or an ordered pair $(\mathrm{F}_A, \mathcal{G})$ of a soft-graph, which consisting of a finite soft-set F_A and a collection $\mathcal{G}$ of a sub-soft-sets of F_A satisfying the following three conditions:

(μ_1) $\emptyset_A \in \mathcal{G}$.

(μ_2) If $G_A \in \mathcal{G}$ and $G'_A \,\widetilde{\subseteq}\, G_A$, then $G'_A \in \mathcal{G}$.

(μ_3) If G_A , $H_A \in \mathcal{G}$ with $|G_A| < |H_A|$, then there exists $\mathfrak{p}_e^x$ of $H_A \,\widetilde{\setminus}\, G_A$ such that $G_A \widetilde{\cup} \mathfrak{p}_e^x \in \mathcal{G}$.

If $\widetilde{\mathcal{M}}$ is the soft-matroid $(F_A, \mathcal{G})$, then $\widetilde{\mathcal{M}}$ is called soft-matroid on F_A .

3.2. Examples:

(i) Let $\mathfrak{U} = A = \{1, 2\}$, $\mathrm{F}_A = \{\mathfrak{p}_1^1 , \mathfrak{p}_2^1 , \mathfrak{p}_1^2, \mathfrak{p}_2^2\}$ and $\mathcal{G} = \{\emptyset_A\}$. Then $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$ is a soft-matroid.

(ii) Let $\mathfrak{U} = A = \{1\}$, $F_A = \{\mathfrak{p}_1^1\}$ and $\mathcal{G} = \{\emptyset_A , F_A\}$. Then $\widetilde{\mathcal{M}} = (F_A, \mathcal{G})$ is a soft-matroid.

(iii) Let $\mathfrak{U} = A = \{1 , 2\}$, $\mathrm{F}_A = \{\mathfrak{p}_1^1 , \mathfrak{p}_2^1 , \mathfrak{p}_1^2 , \mathfrak{p}_2^2\}$ and $\mathcal{G} = \{\emptyset_A , G_A , H_A\}$, with $G_A = \{\mathfrak{p}_1^1 , \mathfrak{p}_2^1\}$ and $H_A = \{\mathfrak{p}_1^2 , \mathfrak{p}_2^2\}$. Then $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$ is not soft-matroid.

(iv) Let $\mathfrak{U} = \{1, 2, 3\}$, $A = \{1, 2\}$, $\mathrm{F}_A = \{\mathfrak{p}_1^1, \mathfrak{p}_2^1, \mathfrak{p}_3^1 , \mathfrak{p}_1^2, \mathfrak{p}_2^2, \mathfrak{p}_3^2\ \mathfrak{p}_1^3, \mathfrak{p}_2^3, \mathfrak{p}_3^3\ \}$ with $G_A = \{\mathfrak{p}_1^1, \mathfrak{p}_2^1\}$, $H_A = \{\mathfrak{p}_1^1 , \mathfrak{p}_2^1 , \mathfrak{p}_1^2 , \mathfrak{p}_2^2\}$, $K_A = \{\mathfrak{p}_1^2 , \mathfrak{p}_2^2\}$ and $\mathcal{G} = \{\emptyset_A , G_A , H_A\}$. Then $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$ is not soft-matroid.

3.3. Theorem**:** A structure $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$ is a soft-matroid if and only if satisfies the following conditions:

(i) $\mathcal{G} \neq \emptyset$.

(ii) If $G_A \in \mathcal{G}$ and $G'_A \,\widetilde{\subseteq}\, G_A$, then $G'_A \in \mathcal{G}$.

(iii) If $G_A, H_A \in \mathcal{G}$ and $|H_A| = |G_A| + 1$, then there exists $\mathfrak{p}_e^x$ of $H_A \,\widetilde{\setminus}\, G_A$ such that $G_A \widetilde{\cup} \mathfrak{p}_e^x \in \mathcal{G}$.

Proof: Suppose first that $\widetilde{\mathcal{M}}$ is a soft-matroid. It is enough to prove that (iii), since $|\mathrm{G_A}| < |\mathrm{H_A}|$, we must have from (μ_3) of the Definition (3.1), there exists at least one soft-point $\mathfrak{p}_e^x$ of $\mathrm{H_A} \,\widetilde{\setminus}\, \mathrm{G_A}$ such that $\mathrm{G_A} \widetilde{\cup} \mathfrak{p}_e^x \in \mathcal{G}$.

Conversely, to show that the three soft-matroid conditions, from (i) of a hypothesis above, we have $\emptyset_\mathrm{A} \in \mathcal{G}$. It is clear that (ii) equivalent to (μ_2). Also, from (iii), we have $|\mathrm{H_A}| = |\mathrm{G_A}| + 1$ and hence, there exists $\mathfrak{p}_e^x$ of $\mathrm{H_A} \,\widetilde{\setminus}\, \mathrm{G_A}$ such that $\mathrm{G_A} \widetilde{\cup} \mathfrak{p}_e^x \in \mathcal{G}$.

This implies that $\widetilde{\mathcal{M}}$ is a soft-matroid.

3.4. Definition: Let $\widetilde{\mathcal{M}}_1 = (\mathrm{F}_{1_A}, \mathcal{G}_1)$ and $\widetilde{\mathcal{M}}_2 = (\mathrm{F}_{2_A}, \mathcal{G}_2)$ be two soft-matroids on a disjoint soft-sets F_{1_A} and F_{2_A} respectively. Let $\mathrm{F}_A = \mathrm{F}_{1_A} \widetilde{\cup} \mathrm{F}_{2_A}$ and $\mathcal{G} = \{G_{1_A} \widetilde{\cup} G_{2_A} : G_{1_A} \in \mathcal{G}_1 ; G_{2_A} \in \mathcal{G}_2\}$. Then $\widetilde{\mathcal{M}}$ is a soft-matroid on F_A. This soft-matroid $\widetilde{\mathcal{M}}$ is the direct sum $\widetilde{\mathcal{M}}_1 \oplus \widetilde{\mathcal{M}}_2$ of $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$.

3.5. Remark**:** Given two soft-matroids $\widetilde{\mathcal{M}}_1 = (\mathrm{F}_A, \mathcal{G}_1)$ and $\widetilde{\mathcal{M}}_2 = (\mathrm{F}_A, \mathcal{G}_2)$, we are interested in their intersection, which is defined by:

$$\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_1 \widetilde{\cap} \widetilde{\mathcal{M}}_2 = (\mathrm{F}_A, \mathcal{G}_1 \widetilde{\cap} \mathcal{G}_2) .$$

In general, $\widetilde{\mathcal{M}}_1 \widetilde{\cap} \widetilde{\mathcal{M}}_2$ is not a soft-matroid.

3.6. Definition**:** Let $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$ be a soft-matroid. The soft-subset of F_A that is in $\mathcal{G}$ is said to be the independent soft-set. A sub-soft-set of F_A that is not in $\mathcal{G}$ is said to be a dependent soft-set.

3.7. Example**:** Let $\mathfrak{U} = \{1, 2, 3,4\}$, $A = \{1\}$ $\mathrm{F}_A = \{\mathfrak{p}_1^1, \mathfrak{p}_1^2, \mathfrak{p}_1^3, \mathfrak{p}_1^4\}$ and $\mathcal{G} = \{G_A \widetilde{\subsetneq} \mathrm{F}_A : |G_A| \leq 2\}$. Then $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$ is a soft-matroid and its clear the form of independent soft-sets. To know the sub-soft-set of F_A that is not in $\mathcal{G}$ (dependent soft-sets), we give the following formula $\{G_A \widetilde{\subseteq} \mathrm{F}_A : |G_A| > 2\}$.

3.8. Definition**:** Let $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$ be a soft-matroid. The maximal independent soft-set is an independent soft-set that is not a sub-soft-set of any other. Also, The minimal dependent soft-set is an dependent soft-set which has no proper sub-soft-set.

3.9. Example**:** For example (3.7) data.

The collection of all maximal independent soft-sets in $\widetilde{\mathcal{M}}$ given by $\{G_A \widetilde{\subsetneq} \mathrm{F}_A : |G_A| = 2\}$ and the collection of all minimal dependent soft-sets in $\widetilde{\mathcal{M}}$ given by $\{G_A \widetilde{\subsetneq} \mathrm{F}_A : |G_A| = 3\}$.

3.10. Theorem**:** A structure $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$ is soft-matroid if and only if satisfies the following conditions:

(i) $\mathcal{G} \neq \emptyset$.

(ii) If $G_A \in \mathcal{G}$ and $G'_A \widetilde{\subseteq} G_A$, then $G'_A \in \mathcal{G}$.

(iii) If $\mathrm{F}'_A \widetilde{\subseteq} \mathrm{F}_A$ with G_A and G'_A are maximal members in $\{H_A : H_A \in \mathcal{G}$ and $H_A \widetilde{\subseteq} \mathrm{F}'_A\}$, then $|G_A| = |G'_A|$.

Proof: Clear.

3.11. Definition**:** Let $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$ be a soft-matroid. A collection $\mathfrak{B}$ of sub-soft-set of F_A such that for all G_A , $G'_A \in \mathfrak{B}$:

$(\mathscr{b}_1)$ $|G_A| = |G'_A|$.

$(\mathscr{b}_2)$ For all $\mathfrak{p}_e^x \widetilde{\in} G_A \widetilde{\setminus} G'_A$, there exists $\mathfrak{p}_{e'}^y \widetilde{\in} G'_A \widetilde{\setminus} G_A$ such that $(G_A \widetilde{\setminus} \mathfrak{p}_e^x) \widetilde{\cup} \mathfrak{p}_{e'}^y \in \mathfrak{B}$.

$(\mathscr{b}_3)$ For all $\mathfrak{p}_e^x \widetilde{\in} G_A \widetilde{\setminus} G'_A$, there exists $\mathfrak{p}_{e'}^y \widetilde{\in} G'_A \widetilde{\setminus} G_A$ such that $(G'_A \widetilde{\setminus} \mathfrak{p}_{e'}^y) \widetilde{\cup} \mathfrak{p}_e^x \in \mathfrak{B}$.

Then $\mathfrak{B}$ is said to be a collection of bases for $\widetilde{\mathcal{M}}$ and it is denoted by $\mathfrak{B}(\widetilde{\mathcal{M}})$.

3.12. Example:

(i) Let $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$ be a soft-matroid. A collection of maximal independent soft-sets in $\mathcal{G}$ is a collection of bases for $\widetilde{\mathcal{M}}$.

(ii) For example (3.7) data. Let $\mathcal{G} = \wp(\mathrm{F}_A)$ with $\mathfrak{B}(\widetilde{\mathcal{M}}) = \{G_A, G'_A\}$ and $G_A = \{\mathfrak{p}_1^1, \mathfrak{p}_1^2\}$ and $G_A = \{\mathfrak{p}_1^3, \mathfrak{p}_1^4\}$. Then $\mathfrak{B}(\widetilde{\mathcal{M}})$ is not collection of bases for $\widetilde{\mathcal{M}}$.

3.13. Theorem**:** Let $\mathfrak{B}$ be a collection of a sub-soft-set of F_A. Then $\mathfrak{B}$ is the collection of bases for a soft-matroid $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$ if and only if it satisfies the following conditions:

(i) $\mathfrak{B} \neq \emptyset$.

(ii) If $G_A, G'_A \in \mathfrak{B}$ and $\mathfrak{p}_e^x \tilde{\in} G_A \tilde{\setminus} G'_A$, then there exists $\mathfrak{p}_{e'}^y \tilde{\in} G'_A \tilde{\setminus} G_A$ such that $(G'_A \tilde{\setminus} \mathfrak{p}_{e'}^y) \tilde{\cup} \mathfrak{p}_e^x \in \mathfrak{B}$.

Proof: Assume $\mathfrak{B}$ is the collection of bases for $\widetilde{\mathcal{M}}$. The first direction is clear.

Conversely, suppose that the above conditions are met. Suffice it to prove the members of $\mathfrak{B}$ are equal-cardinal.

Now, if for all $G_A, G'_A \in \mathfrak{B}$, with $|G_A| < |G'_A|$. From Definition (3.11), we must have G_A and G'_A are both in $\mathcal{G}$. Also, from Definition (3.1. $\mathfrak{l}_3$), implies that there exists $\mathfrak{p}_{e'}^y \tilde{\in} G'_A \tilde{\setminus} G_A$ with $G_A \tilde{\cup} \mathfrak{p}_{e'}^y \in \mathcal{G}$. This contradicts the maximality of G_A. Hence $|G_A| \geq |G'_A|$ and similarly, $|G'_A| \geq |G_A|$.

3.14. Theorem**:** Let F_A be a soft-set and $\mathfrak{B}$ be a collection of a sub-soft-set of F_A satisfying (i) and (ii) conditions in Theorem (3.13). Let $\mathcal{G}$ be a collection of a sub-soft-set of F_A that are contained in some members of $\mathfrak{B}$. Then $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$ is a soft-matroid having $\mathfrak{B}$ as it is a collection of bases.

Proof: Since $\mathfrak{B} \neq \emptyset$ and $\mathcal{G} \subseteq \mathfrak{B}$, implies that $\mathcal{G}$ satisfies ($\mathfrak{l}_1$). Moreover, if $G_A \in \mathcal{G}$, then $G_A \subseteq H_A$ for some $H_A \in \mathfrak{B}$. Thus if $G'_A \tilde{\subseteq} G_A$, then $G'_A \tilde{\subseteq} H_A$. So $G'_A \in \mathcal{G}$. i.e. $\mathcal{G}$ satisfies ($\mathfrak{l}_2$).
Finally, to show that $\mathcal{G}$ satisfies ($\mathfrak{l}_3$). Assume that ($\mathfrak{l}_3$) false for $\mathcal{G}$. Then there exist $G_A, H_A \in \mathcal{G}$ with $|G_A| < |H_A|$ such that, for all $\mathfrak{p}_e^x \tilde{\in} H_A \tilde{\setminus} G_A$, the soft-set $G_A \tilde{\cup} \mathfrak{p}_e^x \notin \mathcal{G}$. From our hypotheses above $\mathcal{G} \subseteq \mathfrak{B}$, there exist $G'_A, H'_A \in \mathcal{G}$ such that $G_A \tilde{\subseteq} G'_A$ and $H_A \tilde{\subseteq} H'_A$.
Assume that such a soft set H'_A is chosen, so that $|H'_A \tilde{\setminus} (G'_A \tilde{\cup} H_A)|$ is a minimal. By the choice of G_A and H_A ;

$$H_A \tilde{\setminus} G'_A = H_A \tilde{\setminus} G_A \qquad (1)$$

Now, suppose that $H'_A \tilde{\setminus} (G'_A \tilde{\cup} H_A) \neq \emptyset_A$.
Let $\mathfrak{p}_e^x \tilde{\in} H'_A \tilde{\setminus} (G'_A \tilde{\cup} H_A)$. Then from (iii) of our hypothesis, there exists $\mathfrak{p}_{e'}^y \tilde{\in} G'_A \tilde{\setminus} H'_A$ such that $(H'_A \tilde{\setminus} \mathfrak{p}_e^x) \tilde{\cup} \mathfrak{p}_{e'}^y \in \mathfrak{B}$. But then $|[(H'_A \tilde{\setminus} \mathfrak{p}_e^x) \tilde{\cup} \mathfrak{p}_{e'}^y] \tilde{\setminus} (G'_A \tilde{\cup} H_A)| < |H'_A \tilde{\setminus} (G'_A \tilde{\cup} H_A)|$, which is a contradiction with H'_A. Hence $H'_A \tilde{\setminus} (G'_A \tilde{\cup} H_A) = \emptyset_A$ and so $H'_A \tilde{\setminus} G'_A = H_A \tilde{\setminus} G'_A$. Thus by (1), we must have:

$$H'_A \tilde{\setminus} G'_A = H_A \tilde{\setminus} G_A \qquad (2)$$

Next, we show that $G'_A \tilde{\setminus} (G_A \tilde{\cup} H'_A) = \emptyset_A$.
If not, then there exists $\mathfrak{p}_e^x \tilde{\in} G'_A \tilde{\setminus} (G_A \tilde{\cup} H'_A)$ and $\mathfrak{p}_{e'}^y \tilde{\in} H'_A \tilde{\setminus} G'_A$, so that $(G'_A \tilde{\setminus} \mathfrak{p}_e^x) \tilde{\cup} \mathfrak{p}_{e'}^y \in \mathfrak{B}$. Now, $G_A \tilde{\cup} \mathfrak{p}_{e'}^y \tilde{\subseteq} [(G'_A \tilde{\setminus} \mathfrak{p}_e^x) \tilde{\cup} \mathfrak{p}_{e'}^y]$, so $G_A \tilde{\cup} \mathfrak{p}_{e'}^y \in \mathcal{G}$. Since $\mathfrak{p}_{e'}^y \tilde{\in} H'_A \tilde{\setminus} G'_A$, it follows by (2) that $\mathfrak{p}_{e'}^y \in H_A \tilde{\setminus} G_A$ and so we have a contradiction to our assumption that ($\mathfrak{l}_3$) false.

We conclude that $G'_A \tilde{\setminus} (G_A \tilde{\cup} H'_A) = \emptyset_A$. Hence $G'_A \tilde{\setminus} H'_A = G_A \tilde{\setminus} H'_A$. But;

$$G_A \tilde{\setminus} H'_A \tilde{\subseteq} G_A \tilde{\setminus} H_A \qquad (3)$$

By (ii) from hypothesis above, we have $|G'_A \tilde{\setminus} H'_A| = |H'_A \tilde{\setminus} G'_A|$. Therefore by (2) and (3), implies that $|G_A \tilde{\setminus} H_A| \geq |H_A \tilde{\setminus} G_A|$, so that $|G_A| \geq |H_A|$. This contradiction completes the proof $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$ is a soft-matroid.

3.15. Definition**:** A minimal dependent soft-set in an arbitrary soft-matroid $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$ is said to be a circuit of $\widetilde{\mathcal{M}}$ and we shall denote the collection of circuits of $\widetilde{\mathcal{M}}$ by $\mathcal{C}(\widetilde{\mathcal{M}})$.

3.16. Example**:** A collection of minimal dependent sub-soft-sets in $\{G_A \tilde{\subseteq} \mathrm{F}_A : |G_A| > 2\}$ of an example (3.7) is a circuits of $\widetilde{\mathcal{M}}$.

3.17. Theorem**:** A collection $\mathfrak{C}$ of a sub-soft-sets of F_A satisfying:

(i) $\emptyset_A \notin \mathfrak{C}$.

(ii) If G_A and H_A are distinct members of $\mathfrak{C}$ with $\mathfrak{p}_e^x \tilde{\in} G_A \tilde{\cap} H_A$, then there exists $K_A \in \mathfrak{C}$ such that $K_A \tilde{\subseteq} \left(G_A \tilde{\cup} H_A\right) \tilde{\setminus} \mathfrak{p}_e^x$.

Let $\mathcal{G}$ be a collection of sub-soft-sets of F_A that contain no member of $\mathfrak{C}$. Then $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$ is a soft-matroid having $\mathfrak{C}$ as its collection of circuits.

Proof: We shall first show that $\mathcal{G}$ satisfies $\mathfrak{r}_1$, $\mathfrak{r}_2$ and $\mathfrak{r}_3$. It is clear that $\emptyset_A$ does not contain in $\mathfrak{C}$. So $\emptyset_A \in \mathcal{G}$ and $\mathfrak{r}_1$ holds. If $\mathcal{G}$ contains no members of $\mathfrak{C}$ and $G_A \in \mathcal{G}$ and $G'_A \tilde{\subseteq} G_A$, then $G'_A \notin \mathfrak{C}$. Thus $\mathfrak{r}_2$ holds.

To prove $\mathfrak{r}_3$, suppose that G'_A, $H'_A \in \mathcal{G}$ with $|G'_A| < |H'_A|$. Assume that $\mathfrak{r}_3$ fails for all G'_A and H'_A. Now, $\mathcal{G}$ has a member that is a sub-soft-set of $G'_A \tilde{\cup} H'_A$ with $|G'_A \tilde{\cup} H'_A| > |G'_A|$. Choose such a sub-soft-set W'_A for which $|G'_A \tilde{\setminus} W'_A|$ is minimal. As $\mathfrak{r}_3$ fails, $G'_A \tilde{\setminus} W'_A \neq \emptyset_A$, so we can choose $\mathfrak{p}_e^x$ from $G'_A \tilde{\setminus} W'_A$. Now, for each $\mathfrak{p}_{e'}^y$ of $W'_A \tilde{\setminus} G'_A$.

Then $\left(W'_A \tilde{\cup} \mathfrak{p}_e^x\right) \tilde{\setminus} \mathfrak{p}_{e'}^y \tilde{\subseteq} G'_A \tilde{\cup} H'_A$ and $|G'_A \tilde{\setminus} (\left(W'_A \tilde{\cup} \mathfrak{p}_e^x\right) \tilde{\setminus} \mathfrak{p}_{e'}^y)| < |G'_A \tilde{\setminus} W'_A|$. Therefore $\left(W'_A \tilde{\cup} \mathfrak{p}_e^x\right) \tilde{\setminus} \mathfrak{p}_{e'}^y \notin \mathcal{G}$, so $\left(W'_A \tilde{\cup} \mathfrak{p}_e^x\right) \tilde{\setminus} \mathfrak{p}_{e'}^y$ contains a member W_A^y of $\mathcal{C}$. Evidently, $\mathfrak{p}_{e'}^y$ is not in W_A^y. Moreover, $\mathfrak{p}_e^x \tilde{\in} W_A^y$ otherwise $W_A^y \tilde{\subseteq} W'_A$ contradicting the fact that $W'_A \in \mathcal{G}$. Let $\mathfrak{p}_{e''}^z \tilde{\in} W'_A \tilde{\setminus} G'_A$. If $W_A^z \tilde{\cap} \left(W'_A \tilde{\setminus} G'_A\right) = \emptyset_A$, where W_A^z is a member of $\mathfrak{C}$. Then $W_A^z \tilde{\subseteq} ((G'_A \tilde{\cap} H'_A) \tilde{\cup} \mathfrak{p}_e^x) \tilde{\setminus} \mathfrak{p}_{e''}^z \tilde{\subseteq} G'_A$; a contradiction.

Therefore, there exists $\mathfrak{p}_{e''}^w \tilde{\in} W_A^z \tilde{\cap} \left(W'_A \tilde{\setminus} G'_A\right)$. Now, $\mathfrak{p}_e^x \tilde{\in} W_A^z \tilde{\cap} W_A^w$, so (ii), implies that there exists a member of W''_A of $\mathcal{C}$ such that $W''_A \tilde{\subseteq} (W_A^z \tilde{\cup} W_A^w) \tilde{\setminus} \mathfrak{p}_e^x$. But, both W_A^z and W_A^w are sub- soft-sets of $G'_A \tilde{\cup} \mathfrak{p}_e^x$ and hence $W''_A \tilde{\subseteq} G'_A$, a contradiction. We conclude that $\mathfrak{r}_3$ holds. Thus $\widetilde{\mathcal{M}}$ is a soft-matroid.

Now, to prove that $\mathfrak{C}$ is a collection of circuits of $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$, we note that the following statements are equivalent:

(c_1) C_A is a circuit of $\widetilde{\mathcal{M}}$.

(c_2) $C_A \notin \mathcal{G}$ and $C_A \tilde{\setminus} \mathfrak{p}_e^x \in \mathcal{G}$ for all $\mathfrak{p}_e^x \tilde{\in} C_A$.

(c_3) C_A has a member C'_A of $\mathcal{C}$ as a sub-soft-set, but $C'_A \tilde{\subsetneq} C_A$.

(c_4) $C_A \in \mathfrak{C}$.

2.18. Corollary**:** Let $\mathfrak{C}$ be a collection of a soft-subsets of F_A. Then $\mathfrak{C}$ is the collection of circuits of a soft-matroid $\widetilde{\mathcal{M}} = (\mathrm{F}_A, \mathcal{G})$ if and only if $\mathfrak{C}$ satisfyies the following conditions:

(c'_1) $\emptyset_A \notin \mathfrak{C}$.

(c'_2) If G_A and H_A are members of $\mathfrak{C}$ such that $G_A \tilde{\subseteq} H_A$, then $G_A = H_A$.

(c'_3) If G_A and H_A are distinct members of $\mathfrak{C}$ with $\mathfrak{p}_e^x \tilde{\in} G_A \tilde{\cap} H_A$, then there exists $W_A \in \mathfrak{C}$ such that $W_A \tilde{\subseteq} \left(G_A \tilde{\cup} H_A\right) \tilde{\setminus} \mathfrak{p}_e^x$.

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On spectral asymptotic for the second-derivative operators

Ehsan Hameed[1] and Yahea Saleem[2]

[1] Department of Mathematics, College of Computer Science and Mathematics, University of Thi-Qar. Iraq.

[2] Department of Mathematics, College of Education for Pure Science, University of Basrah, Iraq.

ehsan.m@utq.edu.iq

Abstract. In this work we focus on spectral asymptotic for the second derivative operators. Here we study Schrödinger operator with zero-range potentials, because this operator has great importance for understanding the solvable problems in quantum mechanics and atomic physics. It appears in different models such as the mathematical physics, applied mathematics and theoretical physics. We have two objectives in this work. We first demonstrated that this operator has a continuous spectrum contains an infinite number of bands separated by gaps. We then explained that the bands to gaps ratio tends to zero under certain conditions.

1. Introduction

3. The differential operators are ubiquitous in many natural systems, ranging from quantum to atomic physics applications. These applications are used to give rise a solvable model of complicated physical phenomena [1,2,5]. Because the method of solid-state physics reproduces the geometry of the problem extremely well, therefore, there is a particular interest in the applications of these models. Kroing and Penney [10] were the first who described this model by the Hamiltonian operator

4. $$\mathbf{H} = -\frac{d^2}{dx^2} + \sum_{n\in Z} \alpha_n \, \delta(x-n),$$

5. where δ is the Dirac delta function and α_n are the actual coupling constants that describes each point interactions. They also explained the spectrum of permissible energy values which consists of continuous region separated by finite intervals. Further, this operator is used to solve the complicated physical phenomena. The point interactions found in many different models by considering boundary conditions at the individual points. The generalized point interaction in one dimension with boundary conditions

6. $$\begin{pmatrix} \psi(0^+) \\ \psi(0^+) \end{pmatrix} = e^{i\theta} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \psi(0^-) \\ \psi(0^-) \end{pmatrix},$$

7. is studied in [12, 13]. He also discussed the existence and the physical properties of the one-dimensional δ′-interaction Hamiltonian. Bloch theorem is used to explain that any such operator coincides with some self-adjoint extension of the unperturbed second-derivative operator restricted to the set of functions vanishing in a neighbourhood of the origin [7]. Moreover, the connected extensions of the Schrödinger operator are studied and described by the boundary conditions at the origin in [8],

8. $$\begin{pmatrix} \psi(0^+) \\ \psi(0^+) \end{pmatrix} = e^{i\theta} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \psi(0^-) \\ \psi(0^-) \end{pmatrix},$$

9. where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}$ are real, and $\boldsymbol{\alpha\delta - \beta\gamma = 1},\ \boldsymbol{0 \le \theta < 2\pi}$. The spectrum of the generalized Kroing-Penney model has infinitely many gaps and the behaviour depend substantially on the parameters of generalized point interaction [6]. Moreover, the spectral asymptotic for operators with partial derivatives have been the subject of extensive research for over a century. Therefore, it drew the attention of many remarkable mathematicians and physicists. The mathematical framework used to describe this spectral asymptotic was based on the Bloch theorem. In our work we used the transfer matrix to describe this behaviour.

10. The main result of this paper is contained in three Propositions which describe the asymptotic behaviour of the operator $\boldsymbol{\mathcal{L}}$ corresponding to the values of three independent real parameters. We show that the spectrum of this operator is absolutely continuous and fills in an infinite number of bands separated by gaps.

11. Let us give a brief outline of the contents of the paper: In section 2, we define the second-derivative operator and discuss the classes of unitary of equivalent of this operator. We also derive the reduction relation in Proposition 2.1. Then, we study the transfer matrix to obtain the dispersion relation which uses to calculate the spectral bands. In section 3, we investigate the spectral asymptotic by three Propositions (3.1), (3.2) and (3.3).

2. Preliminaries

12. At the beginning let us briefly recall the definition of the second-derivative operator $\boldsymbol{\mathcal{L}}$. We consider here the operator $\boldsymbol{\mathcal{L} \equiv \mathcal{L}(\mathcal{A}, \theta)}$ where $\boldsymbol{\mathcal{A}} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ \boldsymbol{\gamma} & \boldsymbol{\delta} \end{pmatrix} \in \boldsymbol{SL(2, R)}$ such that $\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma}, \boldsymbol{\delta} \in \mathbb{R}$ and $\boldsymbol{0 \le \theta < 2\pi}$, acting in the Hilbert space $\boldsymbol{L_2}(\mathbb{R})$ defined on the functions from $\boldsymbol{W_2^2}\{\mathbb{R}\backslash \{\boldsymbol{n}\}_{\boldsymbol{n}\in\mathbb{Z}}$ (Sobolev space) satisfying the boundary conditions,

13.
$$\begin{pmatrix} \boldsymbol{u_R(n)} \\ \boldsymbol{u'_L(n)} \end{pmatrix} = \boldsymbol{e^{i\theta}\mathcal{A}} \begin{pmatrix} \boldsymbol{u_R(n)} \\ \boldsymbol{u'_L(n)} \end{pmatrix}, \quad \boldsymbol{n} \in \mathbb{Z}. \tag{2.1}$$

14. In addition, this coincides with a self-adjoint operator extension of the operator $\boldsymbol{\mathcal{L} = -d^2/dx^2}$ limited to all functions from $\boldsymbol{W_2^2}(\mathbb{R})$, disappearance in a neighbourhood of the points $\boldsymbol{x = n}$ [9].

15. Now, in order to illustrate the spectral asymptotic of the second derivative operator, we first are going to describe the classes of unitary equivalent operators of this operator. There are three independent real parameters to describe these classes which are $\boldsymbol{t = \alpha + \delta},\ \boldsymbol{\beta}$ **and** $\boldsymbol{\gamma}$. The following proposition explains the relationship between these parameters to each other, as well as determining the values of these parameters to calculate the spectral asymptotic of the second derivative operator.

16. Proposition 2.1. If $\boldsymbol{t}, \boldsymbol{\beta}$ **and** $\boldsymbol{\gamma}$ be three independent real parameters describing the operator $\boldsymbol{\mathcal{L}}$ such that $\boldsymbol{t = \alpha + \delta}$, then $\boldsymbol{t} \ \ge \boldsymbol{2\sqrt{\beta\gamma + 1}}$.

Proof. Since $\boldsymbol{t = \alpha + \delta}$, then multiplication this equation by $\boldsymbol{\alpha}$ we get:

$$\boldsymbol{\alpha t = \alpha^2 + \alpha\delta}.$$

But
$$\boldsymbol{\alpha\delta - \beta\gamma = 1},$$

thus
$$\boldsymbol{\alpha\delta = 1 + \beta\gamma}.$$

Implies that $\qquad \alpha t - \alpha^2 = 1 + \beta\gamma,$

then

$$\alpha = \frac{t \mp \sqrt{t^2 - 4(\beta\gamma + 1)}}{2}.$$

By the same way we get

$$\delta = \frac{t \mp \sqrt{t^2 - 4(\beta\gamma + 1)}}{2},$$

therefore $\qquad \alpha + \delta = t \mp \sqrt{t^2 - 4(\beta\gamma + 1)}\,.$

Since $\qquad t^2 - 4(\beta\gamma + 1) \quad \geq \; 0,$

implies that $\qquad t^2 \; \geq 4(\beta\gamma + 1).$

Then $\qquad t \; \geq 2\sqrt{\beta\gamma + 1}\,. \qquad (2.2)$

■

Now, we are going to study the transfer matrix for the purpose of describing the second derivative operator spectrum. Subsequently, this matrix is given by [3, 4]

$$\mathcal{T}_\lambda = \begin{pmatrix} \cos\kappa & \frac{1}{\kappa}\sin\kappa \\ -\kappa\sin\kappa & \cos\kappa \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha\cos\kappa + \frac{\gamma}{\kappa}\sin\kappa & \beta\cos\kappa + \frac{\delta}{\kappa}\sin\kappa \\ -\alpha\kappa\sin\kappa + \gamma\cos\kappa & -\beta\kappa\sin\kappa + \delta\cos\kappa \end{pmatrix} \qquad (2.3)$$

where $\kappa = \sqrt{\lambda}$. And since $\det\mathcal{T}_\lambda = 1$, therefore, the specific determinant of this matrix is given by

$$\det(\mathcal{T}_\lambda - \lambda I) = \lambda^2 - \lambda \mathrm{Tr}\mathcal{T}_\lambda + 1.$$

Furthermore, the operator's spectrum coincides with the set of λ where the spectrum of this operator is calculated as zeros of the following inequality [11],

$$|\mathrm{Tr}\mathcal{T}_\lambda| \leq 2.$$

Thus

$$\left|(\alpha + \delta)\cos\kappa + \left(\frac{\gamma}{\kappa} - \beta\kappa\right)\sin\kappa\right| \leq 2.$$

Let us now define the function g by

$$g(\kappa) = t\cos\kappa \; + \; \left(\frac{\gamma}{\kappa} - \beta\kappa\right)\sin\kappa\,. \qquad (2.4)$$

Consequently, we can be determined the operator's spectrum by solving the following equation

$$|g(\kappa)| \leq 2 \qquad (2.5)$$

This equation is called the dispersion relation which used to obtain the spectral bands in the following section.

3. Spectral asymptotic for the periodic operator

In this section, we study the spectral asymptotic for the second derivative operator $\boldsymbol{\mathcal{L}}$. There are infinite numbers of bands in this operator, which has a continuous spectrum (i.e. consist of all eigenvalues such that the resolvent of operator $\boldsymbol{\mathcal{L}}$ exists and defined on a set which is dense in $\boldsymbol{L_2}(\mathbb{R})$) and it is tending to ∞. The following three Propositions give an explicit description of the spectral asymptotic corresponding to the parameters of this operator.

Proposition 3.1. Assume that $\boldsymbol{\beta}$ **and** $\boldsymbol{\gamma}$ are arbitrary satisfying the equation (2.2). If $\boldsymbol{\beta \neq 0}$, then there are infinite numbers of bands $\Delta_{\boldsymbol{n}} = [\boldsymbol{A_n^2}, \boldsymbol{B_n^2}]$ of the operator $\boldsymbol{\mathcal{L}}$, which has a continuous spectrum and located in the intervals $[(\boldsymbol{\pi n} - \boldsymbol{\pi/2})^2, (\boldsymbol{\pi n} + \boldsymbol{\pi/2})^2]$ for large values of n. And their edges are asymptotically which are given by

$$\boldsymbol{A_n = \pi n + \frac{t-2}{\beta\pi}\frac{1}{n} + \left(-\frac{1}{3\beta^3\pi^3}t^3 + \frac{1-\beta}{\beta^3\pi^3}t^2 + \frac{\gamma+4}{\beta^2\pi^3}t - \frac{4}{3\beta^3\pi^3} - \frac{4+2\gamma}{\beta^2\pi^3}\right)\frac{1}{n^3} + O\left(\frac{1}{n^5}\right), \quad as\; n \to \infty,}$$

(3.1)

$$\boldsymbol{B_n = \pi n + \frac{t+2}{\beta\pi}\frac{1}{n} + \left(-\frac{1}{3\beta^3\pi^3}t^3 - \frac{1+\beta}{\beta^3\pi^3}t^2 + \frac{\gamma-4}{\beta^2\pi^3}t + \frac{4}{3\beta^3\pi^3} + \frac{2\gamma-4}{\beta^2\pi^3}\right)\frac{1}{n^3} + O\left(\frac{1}{n^5}\right), \quad as\; n \to \infty.}$$

In addition, the length and the midpoint of the band are asymptotically which given by:

$$\boldsymbol{|\Delta_n| = \frac{8}{|\beta|} + \frac{4}{\pi^2}\left(-\frac{1}{|\beta|\beta^2}t^2 - \frac{2}{|\beta|\beta}t + \frac{4}{3\beta^3} + \frac{2\gamma}{|\beta|\beta}\right)\frac{1}{n^2} + O\left(\frac{1}{n^4}\right), \quad as\; n \to \infty, \quad (3.4)}$$

and

$$\boldsymbol{M_n = \pi^2 n^2 + \frac{2t}{\beta} + \frac{1}{\pi^2}\left(-\frac{2}{3\beta^3}t^3 - \frac{1}{\beta^2}t^2 + \frac{2\gamma}{\beta^2}t - \frac{4}{\beta^2}\right)\frac{1}{n^2} + O\left(\frac{1}{n^4}\right), \quad as\; n \to \infty, \quad (3.5)}$$

respectively.

Proof. At first, let us to prove that there is only one band $\Delta_{\boldsymbol{n}}$ of continuous spectrum in each interval $\mathbf{I}_{\boldsymbol{n}}$ for the large enough values of $\boldsymbol{\kappa}$.

Now, by the equation (2.4) we get

$$\boldsymbol{g(\pi n + \pi/2) = t\cos(\pi n + \pi/2) + (\tfrac{\gamma}{\pi n+\pi/2} - \beta(\pi n + \pi/2)\sin(\pi n + \pi/2)}$$

$$= (-1)^{n+1}\beta\pi n + O(1) \ \ as \quad n \to \infty.$$

This equation determines the values of the end points of each interval $\mathbf{I}_n$. Since it has alternating signs, and when n is sufficiently large, thus $|g(\pi n + \pi/2)| > 2$. Consequently, that means there is one spectral band when the interval is considered.

Let $g'(\kappa) = 0$ we get:

$$0 = g'(\kappa) = -\left(t + \frac{\gamma}{\kappa^2} + \beta\right)\sin\kappa + \left(\frac{\gamma}{\kappa} - \beta\kappa\right)\cos\kappa,$$

implies that

$$\tan\kappa = \kappa(\gamma - \beta\kappa^2)/(\kappa^2 (t+\beta) + \gamma. \tag{3.6}$$

This function is rational and by the comparison test it tends to $\pm\infty$ as $\kappa \to \infty$.

Note that

1- if $t + \beta = 0, \gamma \neq 0$, then $(\kappa(\gamma - \beta\kappa^2))/((t+\beta)\kappa^2 + \gamma) = \kappa - \beta/\gamma\ \kappa^3.$

2- if $t + \beta \neq 0$, γ arbitrary, then $\frac{\kappa(\gamma - \beta\kappa^2)}{(t+\beta)\kappa^2 + \gamma} = -\beta/(t+\beta)\ \kappa + (\gamma(t+2\beta))/((t+\beta)^2)\ 1/\kappa + O(1/\kappa^2).$

3- if $t + \beta = 0$, $\gamma = 0$, then the relation (3.6) takes the form:

$$(t + \gamma/\kappa^2 + \beta)\sin\kappa = (\gamma/\kappa - \beta\kappa)\cos\kappa,$$

implies that $-\beta\kappa\cos\kappa = 0.$

But $\cos\kappa = 0$ when $\kappa = n\pi + \pi/2$, hence, there is one extreme point in each interval I_n for the function g when $n \to \infty$. Consequently, because the function g is continuous and monotonically between these points, then for n is sufficiently large, there is only one band where $|g(\kappa)| \leq 2$ in each interval I_n.

In order to calculate the end points of each band Δ_n, let us to solve the equation $|g(\kappa)| = 2$ [11]. Consider the first case $\beta > o$, then the left and right end points of the intervals Δ_n satisfy the following equations

$$t\cos A_n + ({}^{\gamma}/_{A_n} - \beta A_n)\sin A_n = (-1)^n 2, \tag{3.7}$$

$$t\cos B_n + ({}^{\gamma}/_{B_n} - \beta B_n)\sin B_n = (-1)^n 2, \tag{3.8}$$

respectively.

On the other hand, due to the points A_n and B_n are closed to πn for large n, then let us to use the following representation of the asymptotic

$$A_n = \pi n + \frac{a}{n} + \frac{a'}{n^3} + O\left(\frac{1}{n^5}\right), \qquad B_n = \pi n + \frac{b}{n} + \frac{b'}{n^3} + O\left(\frac{1}{n^5}\right)\ as\ n \to \infty.$$

Substituting these representations into (3.7) and (3.8), we get:

$$A_n = \pi n + \frac{1}{\pi}\left[\frac{t}{\beta} - \frac{2}{|\beta|}\right]\frac{1}{n} + [-\frac{t^3}{3\beta^3\pi^3} - \left(1 - \frac{1}{|\beta|}\right)\frac{t^2}{\beta^2\pi^3} + +(\frac{\gamma}{\beta^2\pi^3} + \frac{\gamma}{\beta^2\pi^3})t - \frac{4}{3|\beta|^3\pi^3}$$
$$- \frac{2}{\beta^3\pi^3}(2\beta - \gamma|\beta|)]\frac{1}{n^3} + O\left(\frac{1}{n^5}\right), as\ n \to \infty$$

$$B_n = \pi n + \frac{1}{\pi}[\frac{t}{\beta} + \frac{2}{|\beta|}]\frac{1}{n}\ + [-\frac{t^3}{3\beta^3\pi^3} - (1 + \frac{1}{|\beta|})\frac{t^2}{\beta^2\pi^3} + (\frac{\gamma}{\beta^2\pi^3} - \frac{4|\beta|}{\beta^3\pi^3})t + \frac{4}{3|\beta|^3\pi^3}$$
$$- \frac{2}{\beta^3\pi^3}(2\beta - \gamma|\beta|)]\frac{1}{n^3} + O(\frac{1}{n^5}) \quad as \quad n \to \infty.$$

In the similar way we can be analysed of the case when $\beta < 0$, which leads to formula (3.1). Finally, the $|\Delta_n|$ and M_n of the band are given by

$$|\Delta_n| = B_n^2 - A_n^2 = \frac{8}{|\beta|} + \frac{4}{\pi^2}\left(-\frac{1}{|\beta|\beta^2}t^2 - \frac{2}{|\beta|\beta}t + \frac{4}{3|\beta|^3} + \frac{2\gamma}{|\beta|\beta}\right)\frac{1}{n^2} + O\left(\frac{1}{n^4}\right), \qquad as\ n \to \infty,$$

And

$$M_n = \frac{A_n^2 + B_n^2}{2} = \pi^2 n^2 + \frac{2t}{\beta} + \frac{1}{\pi^2}\left(-\frac{2}{3\beta^3}t^3 - \frac{1}{\beta^2}t^2 + \frac{2\gamma}{\beta^2}t - \frac{4}{\beta^2}\right)\frac{1}{n^2}\ + O\left(\frac{1}{n^4}\right), as\ n \to \infty,$$

respectively. ∎

Additionally, the length of the gaps $\mathcal{G}_n$ is calculated as the following

$$|\mathcal{G}_n| = A_{n+1}^2 - B_n^2 = \pi^2(2n+1) - \frac{8}{\beta} + O\left(\frac{1}{n^2}\right).$$

Implies that

$$\frac{|\Delta_n|}{|\mathcal{G}_n|} = \frac{4}{\pi^2|\gamma|n} + O\left(\frac{1}{n^2}\right), \qquad \text{as } n \to \infty. \tag{3.9}$$

As a result, we conclude that the bands to gaps ratio tends to zero at high energies.

Proposition 3.2. Assume that $\beta = 0$, $t > 2$, and γ is an arbitrary, then there are infinite numbers of bands $\Delta_n = [A_n^2, B_n^2]$ of the operator $\mathcal{L}$, which has a continuous spectrum and located in the intervals $\mathrm{I}_n = [\pi^2 n^2\ ,\ \pi^2(n+1)^2]$ for large values of n. And their edges are asymptotically which are given by

$$A_n = \pi n + \cos^{-1}\frac{2}{t} + \frac{\gamma}{\pi t n} + O\left(\frac{1}{n^2}\right), \qquad as\ n \to \infty,$$

(3.10)

$$B_n = \pi(n+1) - \cos^{-1}\frac{2}{t} + \frac{\gamma}{\pi t n} + O\left(\frac{1}{n^2}\right), \quad as\ n \to \infty.$$

In addition, the length and the midpoint of the band are asymptotically which given by:

$$|\Delta_n| = 2\pi\left(\pi - 2\cos^{-1}\frac{2}{t}\right)n + \left(\pi^2 - 2\pi\cos^{-1}\frac{2}{t}\right) + O\left(\frac{1}{n}\right),\ as\ n \to \infty, \tag{3.11}$$

and

$$M_n = \pi^2\left(n+\frac{1}{2}\right)^2 + \left(\cos^{-1}\frac{2}{t} - \frac{\pi}{2}\right)^2 + \frac{2\gamma}{t} + O\left(\frac{1}{n}\right), as\ n \to \infty, \tag{3.12}$$

respectively.

Proof. At first, let us to prove that there is only one band Δ_n of continuous spectrum in each interval I_n for the large enough values of κ.

Now, since $\beta = 0$, and by the equation (2.4) we get

$$g(\kappa) = t\cos\kappa + \frac{\gamma}{\kappa}\sin\kappa\,, \tag{3.13}$$

and

$$g(n\pi) = t\cos n\pi + \frac{\gamma}{n\pi}\sin n\pi = (-1)^n t.$$

Since the function $g(\kappa)$ is continuous and $g(n\pi)$ has alternating signs, moreover, when n is sufficiently large, $|g(n\pi)| > 2$, then we conclude that there is only one spectral band in each interval.

The zeroes of $g'(\kappa)$ we get

$$0 = g'(\kappa) = -t\sin\kappa + \frac{\gamma}{\kappa}\cos\kappa - \frac{\gamma}{\kappa^2}\sin\kappa.$$

Impels that the equation for extreme points is given by

$$\tan\kappa = \frac{\gamma\kappa}{\kappa^2 t + \gamma},$$

and because this function is decreasing if κ is sufficiently large, then there is only one solution in each interval.

Note that if $\gamma = 0$, then $g(\kappa) = t\cos\kappa$. Also, since $g(\kappa) = (-1)^n t$, $t = 2$, then $t\cos\kappa = \mp 2$.

Consequently,

$$\kappa = \mp\cos^{-1}\frac{2}{t} + n\pi.$$

Hence, there is one extreme point in each interval I_n for the function g when $n \to \infty$. Consequently, because the function g is continuous and monotonically between these points, therefore, for n is sufficiently large, there is only one band where $|g(\kappa)| \leq 2$ in each interval I_n. Now, when $t > 2$ then $\cos^{-1}\frac{2}{t}$ satisfies

$$0 < \cos^{-1}\frac{2}{t} < \pi/2.$$

On the other hand, due to the A_n and B_n points are closed to $\pi n + \cos^{-1}\frac{2}{t}$ and $\pi(n+1) - \cos^{-1}\frac{2}{t}$ respectively, then let us to use the following representation of the asymptotic

$$A_n = n\pi + \cos^{-1}\frac{2}{t} + a_n \quad , \quad B_n = (n+1)\pi - \cos^{-1}\frac{2}{t} + b_n,$$

where a_n, b_n are real constant.

The equation for the left end point,

$$(-1)^n 2 = (-1)^n t[(\frac{2}{t} \cos a_n - \sin(\cos^{-1}\frac{2}{t}) \sin a_n \;] + \frac{\gamma}{n\pi + \cos^{-1}(2/t)}$$
$$[(\sin(\cos^{-1}\frac{2}{t}) \cos a_n) + \frac{2}{t} \sin a_n)].$$

By using the perturbation theory to keep the first terms, we get

$$a_n = \frac{\gamma}{\pi t n} + O\left(\frac{1}{n^2}\right), \quad as\ n \to \infty,$$

thus

$$A_n = n\pi + \cos^{-1}\frac{2}{t} + \frac{\gamma}{\pi t n} + O\left(\frac{1}{n^2}\right) \quad as\ n \to \infty.$$

By the same way we can prove the representation for B_n, i.e.

$$B_n = (n+1)\pi - \cos^{-1}\frac{2}{t} + \frac{\gamma}{\pi t n} + O\left(\frac{1}{n^2}\right), \quad as\ n \to \infty.$$

Furthermore,

$$|\Delta_n| = 2\pi\left(\pi - 2\cos^{-1}\frac{2}{t}\right)n + \left(\pi^2 - 2\pi\cos^{-1}\frac{2}{t}\right) + O\left(\frac{1}{n}\right), \quad as\ n \to \infty,$$

and

$$M_n = \pi^2(n + \frac{1}{2})^2 + (\cos^{-1}\frac{2}{t} - \frac{\pi}{2})^2 + \frac{2\gamma}{t} + O(\frac{1}{n}), \quad as\ n \to \infty,$$

respectively. ∎

In addition, the length of the gaps $\mathcal{G}_n$ is calculated as the following

$$|\mathcal{G}_n| = 4\pi\cos^{-1}\frac{2}{t}(n+1) + O\left(\frac{1}{n}\right) \text{ as } n \to \infty.$$

Implies that

$$\frac{|\Delta_n|}{|\mathcal{G}_n|} = \frac{\pi/2 - 2\cos^{-1}\frac{2}{t}}{\cos^{-1}\frac{2}{t}} + O\left(\frac{1}{n}\right) \text{ as } n \to \infty. \tag{3.14}$$

As a result, we conclude that the bands to gaps ratio tends to the finite non-zero limit depending on the parameter t only at high energies.

Proposition 3.3. Assume that $\beta = 0$, $t = 2$, and $\gamma \neq 0$; then there are infinite numbers of bands $\Delta_n = [A_n^2, B_n^2]$ of the operator $\mathcal{L}$, which has a continuous spectrum and located in the intervals $\mathrm{I}_n = [\pi^2 n^2 , \pi^2(n+1)^2]$. And their edges are asymptotically which given by

if $\gamma > 0$, then $A_n = \pi n + \frac{\gamma}{n\pi} + O\left(\frac{1}{n^2}\right)$, $\quad B_n = \pi(n+1),\ as\ n \to \infty,$ (3.15)

if $\gamma < 0$, then $A_n = \pi n,\ B_n = \pi(n+1) - \frac{|\gamma|}{\pi n} + O\left(\frac{1}{n}\right), as\ n \to \infty$ (3.16)

In addition, the length and the midpoint of the band are asymptotically which given by:

$$|\Delta_n| = 2\pi^2 n + (\pi^2 - 2|\gamma|) + O\left(\frac{1}{n}\right),\quad as\ n \to \infty, \tag{3.17}$$

and

$$M_n = \pi^2 n^2 + \pi^2 n + \frac{\pi^2}{2} + \gamma + O\left(\frac{1}{n}\right)\ ,\ as\ n \to \infty, \tag{3.18}$$

respectively.

Proof. By using the similar way which used in the previous two propositions we can prove this proposition. ∎

Furthermore, the length of the gaps $\mathcal{G}_n$ is calculated as the following

$$|\mathcal{G}_n| = 2|\gamma| + O\left(\frac{1}{n}\right) \text{ as } n \to \infty.$$

Implies that

$$\frac{|\Delta_n|}{|\mathcal{G}_n|} = \frac{\pi^2}{|\gamma|}n + O\left(\frac{1}{n}\right) \text{ as } n \to \infty. \tag{3.19}$$

As a result, we conclude that the bands to gaps ratio tends to infinity at high energies.

4. Conclusions

As mentioned in the introduction, the goal of this study was to describe a spectral asymptotic of the second derivative operator corresponding to the values of three independent real parameters. We first used the transfer matrix method to obtain the dispersion relation which allowed to describe the spectrum of this operator. Then, we observed there are three different spectral asymptotics for this operator depending on independent parameters which are described in three propositions. More importantly, we proved analytically that there are infinite numbers of bands of this operator $\mathcal{L}$ filled with a pure absolutely continuous spectrum. Furthermore, we proved analytically that the bands to gaps ratio tends to zero at particular case when $\beta \neq 0$.

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Locally Finite Associative Algebras and Their Lie Subalgebras

HASAN M. SHLAKA

Department of Mathematics, Faculty of Computer Science and Mathematics, University of Kufa, Iraq

Email address: hasan.shlaka@uokufa.edu.iq

Abstract. An infinite dimensional associative algebra $\mathcal{A}$ over a field $\mathbb{F}$ is called locally finite associative algebra if every finite set of elements is contained in a finite dimensional subalgebra of $\mathcal{A}$. Given any associative algebra $\mathcal{A}$ over field $\mathbb{F}$ of any characteristic. Consider a new multiplication on $\mathcal{A}$ called the Lie multiplication which defined by $[a,b]=ab-ba$ for all $a,b\in\mathcal{A}$, where ab is the associative multiplication in $\mathcal{A}$. Then $L=\mathcal{A}^{(-)}$ together with the Lie multiplication form a Lie subalgebra of $\mathcal{A}$. It is natural to expect that the structures of L and $\mathcal{A}$ are connected closely. In this paper, we study and discuss the structure of infinite dimensional locally finite Lie and associative algebras. The relation between them, their ideals and their inner ideals is considered. A brief discussion of the simple associative algebras and simple Lie algebras is also be provided.

2. Introduction

Throughout this paper, unless otherwise stated, $\mathbb{F}$ is an algebraically closed field of characteristic positive characteristic p, $\mathcal{A}$ is an infinite dimensional locally finite associative algebra over $\mathbb{F}$ and L is an infinite dimensional locally finite Lie algebra over $\mathbb{F}$.

In 2004, Bahturin, Baranov and Zalesski [1] studied simple locally finite Lie subalgebra of the locally finite associative ones. A locally finite (Lie or Associative) algebra $\mathcal{A}$ is an algebra in which for every finite set of elements of $\mathcal{A}$ is contained in a finite dimensional subalgebra P of $\mathcal{A}$. The Lie structure of associative rings or algebras were investigated by the American Mathematician Herstein in 1954 (see [20] and [21]) after defining a new multiplication called the Lie Multiplication by

$$[x,y]:=xy-yx \quad \text{for all} \quad x,y\in\mathcal{A}, \qquad (1.1)$$

where xy is the usual associative multiplication in the simple associative ring $\mathcal{A}$ over its centre $Z(\mathcal{A})$. Then $\mathcal{A}^{(-)}$ together with the multiplication in (1.1) form a Lie algebra over $Z(\mathcal{A})$. We denote by $\mathcal{A}^{(1)}=[\mathcal{A},\mathcal{A}]$ to be the Lie subalgebra of $\mathcal{A}^{(-)}$ together with the multiplication defined in (1.1). Moreover, if an involution $*$ is defined on A, then for any subalgebra $\mathcal{U}$ of $\mathcal{A}$

$$\text{skew}(\mathcal{U}):=\{a\in\mathcal{U}: a^*=-a\} \qquad (1.2)$$

form a Lie algebra with the Lie multiplication that defined as (1.1). Recall that an involution $*:\mathcal{A}\to\mathcal{A}$ is an anti-automorphism, defined by $*(a)=a^*$, satisfy the following conditions $*(a+b)=a^*+b^*$, $*(ab)=b^*a^*$ and $*(*((a))=a$ for all $a,b\in A$. Involutions of the first kind only is considered in this paper, that is, involutions with the following property: $*(\alpha a)=\alpha a^*$.

Baxter [11] Focused on the study of the Lie algebras come from simple associative rings with involution in 1958 and Ericson [17] studied the Lie subalgebras of prime rings with involutions in 1972. A revision to Herstein's Lie theory was giving by Martindale [22] 1986. All of these studies focused on the structure of the Lie ideals and Lie subalgebras that obtained from simple associative rings or algebras. Recall that a subspace I of L is called a *subalgebra* of L if $I^{(1)}\subseteq I$ and an *ideal* if $[I,L]\subseteq I$. Although simple Lie algebras have no ideals except themselves and the trivial ones, it has been proved in [12] that all simple Lie algebras of classical type have non-zero inner ideals.

In 1976, the American mathematician Georgia Benkart introduced the notion *inner ideals of Lie algebras. An inner ideal* is a vector subspace B of L which satisfies the property $[B,[B,L]] \subseteq B$. By the definition of the Lie ideals, one can see that every ideal is an inner ideal. However, Inner ideals are more difficult to be studied as some of them are even not Lie subalgebras. Benkart showed that the structure of the Lie inner ideals are similar to the structure of the ad-nilpotent elements of Lie algebras [13]. Therefore, inner ideals are important in classifying Lie algebras because by using certain restriction on the ad-nilpotent elements one can distinguish the simple Lie algebras of classical type and of the non-classical ones in the case when $p > 2$. In several papers (See for example [14], [15] [18] and [19]) Fernández López et al generalized Benkart's theory over inner ideals.

In this paper, we discuss the structure of the infinite dimensional simple locally finite algebras. We start Section 2 with some preliminaries. Section 3 stats some facts about the plain, diagonal and non-diagonal modules of finite dimensional Lie algebra and Section 4 consists of the infinite dimensional case where the some types of local systems of locally finite algebras (associative or Lie) are considered. Section 5 is the completion of Section 3 where the infinite dimensional cases of plain diagonal and non-diagonal Lie algebras are highlighted. In Section 6 we investigate the structure of (involution) simple and associative algebras. The main results of this paper are found in Sections 7 and 8, where the simple locally finite Lie algebras of simple and involution simple associative algebras are considered.

3. Preliminaries

A *perfect Lie algebra* is a Lie algebra L with the property $L^{(1)} = L$ and a *perfect associative algebra* is an associative algebra $\mathcal{A}$ such that $\mathcal{A}^2 = \mathcal{A}$ [4].

Definition 2.1. [1] A *locally finite (associative, Lie,…etc) algebra* is an algebra (associative, Lie,…etc) $\mathcal{A}$ over a field $\mathbb{F}$ in which for every finite set of elements in $\mathcal{A}$ we can find a finite dimensional subalgebra of $\mathcal{A}$ that contained it.

Recall that a set Γ is said to be a *directed partially ordered set* if there is an ordering relation $\leq$ defined on Γ such that for each $\alpha, \beta \in \Gamma$, there is $\gamma \in \Gamma$ such that $\alpha, \beta \leq \gamma$ [2].

Remark 2.2. Suppose that for each $\alpha, \beta \in \Gamma$ with $\mathcal{A}_\alpha \subseteq \mathcal{A}_\beta$ we set $\alpha \leq \beta$. Then for each $\alpha, \beta \in \Gamma$, there is $\gamma \in \Gamma$ such that $\alpha, \beta \leq \gamma$, so Γ is a directed partially ordered set. Thus, $\lim_{\rightarrow} \mathcal{A}_\alpha$ is the direct limits of an infinite chain of algebras ($\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_i \subset \mathcal{A}_{i+1}$ ...). Therefore, $\mathcal{A}$ is the *inductive limit* $\mathcal{A} = \lim_{\rightarrow} \mathcal{A}_\alpha$ of the algebras $\mathcal{A}_\alpha$.

We denote by $\mathcal{M}_n(\mathbb{F})$ the vector space of all $n \times n$-matrices together with the matrix multiplication defined on it.

Remark 2.3. Every $\mathcal{M}_n(\mathbb{F})$ can be generalized to be an $(n+1) \times (n+1)$-matrix $\mathcal{M}_{n+1}(\mathbb{F})$ by putting $\mathcal{M}_n(\mathbb{F})$ in the left upper hand corner and bordering the last column and row by 0's.

Example 2.4. As an example of locally finite associative algebra is the algebra $\mathcal{M}_\infty(\mathbb{F})$ of infinite matrices with finite numbers of non-zero entries, that is,

$$\mathcal{M}_\infty(\mathbb{F}) = \bigcup_{n=1}^{\infty} \mathcal{M}_n(\mathbb{F}). \qquad (2.1)$$

By using the Lie multiplication in (1.1) on $\mathcal{M}_n(\mathbb{F})$, we obtain a Lie algebra called the *general linear Lie algebra* $\mathfrak{gl}_n(\mathbb{F}) = \mathcal{M}_n(\mathbb{F})^{(-)}$. There are three simple Lie subalgebras of $\mathfrak{gl}_n(\mathbb{F})$. These are *the* *special linear* $\mathfrak{sl}_n(\mathbb{F})$, *the Orthogonal* $\mathfrak{so}_n(\mathbb{F})$ *and the Symplectic* $\mathfrak{sp}_{2n}(\mathbb{F})$ *Lie algebras* are subalgebras of $\mathfrak{gl}_n(\mathbb{F})$ which are defind, respectively, by

$$\mathfrak{sl}_n(\mathbb{F}) = [\mathfrak{gl}_n(\mathbb{F}), \mathfrak{gl}_n(\mathbb{F})] = \{X \in \mathfrak{gl}_n(\mathbb{F}): tr(X) = 0\}; \quad (2.2)$$

$$\mathfrak{so}_n(\mathbb{F}) = \{X \in \mathfrak{gl}_n(\mathbb{F}): X^t = -X\}; \quad (2.3)$$

$$\mathfrak{sp}_{2n}(\mathbb{F}) = \{X \in \mathfrak{gl}_n(\mathbb{F}): X^\tau = -X\}, \quad (2.4)$$

where $tr(X)$ is the trace of the matrix X, X^t is the matrix transpose of X and X^τ is the symplectic transpose of a matrix X defined by $X^\tau = -JX^tJ$ with $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ (I_n is the identity $n \times n$-matrix).

Remark 2.5. 1) It follows from [7] that t in (2.3) and τ in (2.4) are involutions on $\mathcal{M}_n(\mathbb{F})$.
2) $\mathfrak{gl}_n(\mathbb{F})$, $\mathfrak{sl}_n(\mathbb{F})$, $\mathfrak{so}_n(\mathbb{F})$ and $\mathfrak{sp}_{2n}$ are the *Lie algebras of classical type*.
3) $\mathfrak{sl}_n(\mathbb{F})$, $\mathfrak{so}_n(\mathbb{F})$ and $\mathfrak{sp}_{2n}$ are called *the simple Lie algebras of classical type.*

The simple Lie algebras of classical type in Remark 2.5(2) can be constructed from a vector space V as follows: Consider the vector subspace $\mathfrak{gl}(V)$ of $End(V)$ together with the Lie multiplication defined in (1.1). Then we get the *general* $\mathfrak{gl}(V)$ and *the special* $\mathfrak{sl}(V)$ *linear Lie algebras*, where $\mathfrak{sl}(V)$ is a subalgebra of $\mathfrak{gl}(V)$ defined by $\mathfrak{sl}(V) = [\mathfrak{gl}(V), \mathfrak{gl}(V)]$.

If there is (skew)symmetric bilinear form (ϑ) ψ on V, then we get *the Orthogonal* $\mathfrak{so}(V,\psi)$ *or the Symplectic* $\mathfrak{sp}(V,\vartheta)$ *Lie algebras, respectively.* To simplify notations, we denote by $\mathfrak{so}(V)$ and $\mathfrak{sp}(V)$ to be the *Orthogonal* and the *Symplectic* Lie algebras, respectively.

Lemma 2.6. [7] *Let V, V_1 and V_2 be vector spaces over $\mathbb{F}$. Suppose that each of them is of dimension n and $p = 0$.*

1. *If $*$ is an involution on the algebra $End(V) \cong \mathcal{M}_n(\mathbb{F})$, then there is a basis of V such that $*$ is expressed as $X \longmapsto X^t$ or $X \longmapsto X^\tau$ for each $X \in End(V)$. In particular, $skew\big(End(V)\big) \cong \mathfrak{so}_n(\mathbb{F})$ or $\mathfrak{sp}_n(\mathbb{F})$.*
2. *Let $*$ be an involution defined on the algebra $End(V_1) \oplus End(V_2)$ such that $End(V_1)^* = End(V_2)$. Then there are bases of V_1 and V_2 such that $*$ is expressed as $(X_1, X_2) \longmapsto (X_2^t, X_1^t)$ for each $X_i \in End(V_i) \cong \mathcal{M}_n(\mathbb{F})$. In particular,*
$$skew\big(End(V_1) \oplus End(V_2)\big) = \{\, (X, X^t) \mid X \in \mathcal{M}_n(\mathbb{F}) \,\} \cong \mathfrak{gl}_n(\mathbb{F})$$
.

Example 2.8. [3] Consider the locally finite associative algebra $\mathcal{M}_\infty(\mathbb{F})$ in Example 2.4. We construct three locally finite Lie subalgebras of $\mathcal{M}_\infty(\mathbb{F})$. Those are the *stable special linear* $\mathfrak{sl}_\infty(\mathbb{F})$, *stable Symplectic* $\mathfrak{sp}_\infty(\mathbb{F})$ *and stable Orthogonal* $\mathfrak{so}_\infty(\mathbb{F})$ Lie subalgebras of $\mathcal{M}_\infty(\mathbb{F})$ that defined to be the union (or the direct limit) of the natural embeddings, respectively,

$$\mathfrak{sl}_2(\mathbb{F}) \to \mathfrak{sl}_3(\mathbb{F}) \to \cdots \to \mathfrak{sl}_n(\mathbb{F}) \to \cdots;$$
$$\mathfrak{sp}_2(\mathbb{F}) \to \mathfrak{sp}_4(\mathbb{F}) \to \cdots \to \mathfrak{sp}_{2n}(\mathbb{F}) \to \cdots;$$
$$\mathfrak{so}_2(\mathbb{F}) \to \mathfrak{so}_3(\mathbb{F}) \to \cdots \to \mathfrak{so}_n(\mathbb{F}) \to \cdots.$$

Definition 2.9. [2] A locally finite (associative or Lie) algebra $\mathcal{A}$ over a field $\mathbb{F}$ is said to be *locally semi(simple)* in the case when for every finite set of elements S of $\mathcal{A}$ we can find a finite dimensional (semi)simple subalgebra of $\mathcal{A}$ which contains S.

Example 2.10. Let $\mathcal{A}$ be a simple locally finite associative algebra over $\mathbb{F}$. Then for every finite set of elements S of $\mathcal{A}$, there is a finite dimensional simple subalgebra $\mathcal{A}_\alpha$ (for $\alpha = 1,2,\ldots$) of $\mathcal{A}$ that contains S, so there is a chain

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \cdots$$

of simple subalgebras of $\mathcal{A}$ such that $\mathcal{A} = \bigcup_{\alpha=1}^{\infty} \mathcal{A}_\alpha$. Moreover, we can identify each $\mathcal{A}_\alpha$ with $\mathcal{M}_{n_\alpha}(\mathbb{F})$ (for all $\alpha = 1,2,\ldots$), where n_α is an integer number (because $\mathbb{F}$ is algebraically closed). Note that each embedding $\mathcal{A}_\alpha \subseteq \mathcal{A}_{\alpha+1}$ is written as follows:

$$X \mapsto \text{diag}(X, \dots, X, 0, \dots, 0), \qquad X \in \mathcal{M}_{n_\alpha}(\mathbb{F}).$$

4. Plain, diagonal and non-diagonal modules of finite dimensional Lie algebras.

Suppose that L is perfect. Then there is a Levi (maximal semisimple) subalgebra Q of L such that $L = Q \oplus \mathcal{R}$, where $\mathcal{R}$ is a solvable radical of L (Levi-Malcev Theorem). As $\mathcal{R}$ is an ideal of L, we have $L/\mathcal{R} \cong Q$. Let V be a simple L-module. Since $\mathcal{A}$ is perfect, $Rad(L)$ annihilates V, so $QVQ = V$ (because V is simple). Let $Q_1, \dots, Q_k$ be the simple ideals of Q such that $Q = Q_1 \oplus \dots \oplus Q_k$. Then V is a completely reducible Q-module and $V = V_1 \oplus \dots \oplus V_k$, where V_i is a simple Q_i-module.

Remark 3.1. 1) If $Q_i \cong \mathfrak{sl}(V_i), \mathfrak{so}(V_i), \mathfrak{sp}(V_i)$ for each $1 \le i \le k$, then every natural Q_i-module V_i is an L-module.
2) Suppose that $L \subseteq L'$ is a perfect Lie algebra. If W is an L'-module, then $W\downarrow_L$ denotes the *restriction* of W *to* L.

Definition 3.2. Suppose that L is perfect and finite dimensional. Let V be an L-module.
1. Suppose that $Q_i \cong \mathfrak{sl}(V_i)$ for each $1 \le i \le k$. Then V is said to be a *plain L-module* if each V_i is a natural L-module.
2. Suppose that L' is a perfect Lie algebra such that L' is finite dimensional. Let $V_1', \dots, V_k'$ be natural L'-modules. An embedding $L \subseteq L'$ is called a *plain embedding* if $(V_1' \oplus \dots \oplus V_k')\downarrow_L$ is a plain L-module.

Example 3.3. Suppose that $L = \mathfrak{sl}_n(\mathbb{F})$ and $L' = \mathfrak{sl}_m(\mathbb{F})$ for some positive integers n and m with $n < m$. Let V and T be the natural and the trivial 1-dimensional L-modules, respectively. Then
1. The embedding $L \subseteq L'$ is called a *natural embedding* if for every L'-module V' we have,
$$V'\downarrow_L = V \oplus T \oplus \dots \oplus T.$$
2. The embedding $L \subseteq L'$ is a plain embedding if the L-module V' is *plain*, that is,
$$V'\downarrow_L = \underbrace{V \oplus \dots \oplus V}_{\ell} \oplus \underbrace{T \oplus \dots \oplus T}_{r} \quad \text{for some positive integers } \ell \text{ and } r.$$

Example 3.4. The embedding $\mathfrak{sl}(V) \subseteq \mathfrak{sl}(W)$ is called a *plain embedding* if we can find a basis of W such that $X \to diag\left(\underbrace{X, \dots, X}_{\ell}, \underbrace{0, \dots, 0}_{z}\right)$, (for all $X \in \mathfrak{sl}(V)$)
where the integers ℓ and z do not depend on X and $z + \ell \dim V = \dim W$.

Definition 3.5. Suppose that L is perfect and finite dimensional. Let V be an L-module.
1. Suppose that $Q_i \cong \mathfrak{sl}(V_i), \mathfrak{so}(V_i), \mathfrak{sp}(V_i)$ for each $1 \le i \le k$. Then V is said to be a *diagonal L-module* in the case when each V_i is either a natural or a dual to natural L-module. Otherwise, V is said to be a *non-diagonal L-module*.
2. Suppose that L' is a perfect Lie algebra such that L' is finite dimensional. Suppose that $V_1', \dots, V_k'$ are natural L'-modules. An embedding $L \subseteq L'$ is called *diagonal embedding* if $(V_1' \oplus \dots \oplus V_k')\downarrow_L$ is diagonal.

Example 3.6. [5] Let L and L' be classical simple Lie algebras (See Remark 2.5(3)) over $\mathbb{F}$. Suppose that V, V^* and T be a natural, a dual and a trivial 1-dimensional L-modules, respectively. Let V' be an L'-module. The embedding $L \subseteq L'$ is diagonal if
$$V'\downarrow_L = \underbrace{V \oplus \dots \oplus V}_{\ell} \oplus \underbrace{V^* \oplus \dots \oplus V^*}_{z} \oplus \underbrace{T \oplus \dots \oplus T}_{r} \quad \text{for some positive integers } \ell, z \text{ and } r.$$

Example 3.7. The embedding $\mathfrak{sl}(V) \subseteq \mathfrak{sl}(W)$ is called a *diagonal embedding* if we can find a basis of W such that

$$X \to diag\left(\underbrace{X, \dots, X}_{\ell}, \underbrace{-X^t, \dots, -X^t}_{r}, \underbrace{0, \dots, 0}_{z}\right), \qquad (X \in \mathfrak{sl}(V))$$

where $z + (\ell + r)\dim V = \dim W$.

Proposition 3.8. [9] *Let L_1 be a simple Lie algebra of rank greater than* 10. *Suppose that $L_1 \subseteq L_2 \subseteq L_3$, where L_i are all perfect and finite dimensional Lie algebras. Suppose that $p = 0$ and $L_2 \subseteq L_3$ is a non-diagonal embedding. If $W_{\downarrow_{L_1}}$ is non-trivial for every L_2-module W, then there is a natural L_3-module V such that $V_{\downarrow_{L_1}}$ is a non-diagonal L_1-module.*

5. Local Systems of Locally Finite Algebras

Definition 4.1. [4] Suppose that $\mathcal{A}$ is a locally finite algebra.

1. A *local system* of $\mathcal{A}$ is a set $\{\mathcal{A}_\alpha\}_{\alpha\in\Gamma}$ of finite dimensional subalgebras of $\mathcal{A}$ satisfying the following conditions:
   i. $\mathcal{A} = \cup_{\alpha\in\Gamma} \mathcal{A}_\alpha$.
   ii. There exist $\gamma \in \Gamma$ for each pair $\alpha, \beta \in \Gamma$ such that $\mathcal{A}_\alpha, \mathcal{A}_\beta \subseteq \mathcal{A}_\gamma$.
2. A local system $\{\mathcal{A}_\alpha\}_{\alpha\in\Gamma}$ of $\mathcal{A}$ is called *perfect* in the case when $\mathcal{A}_\alpha$ are perfect algebras.
3. A local system $\{\mathcal{A}_\alpha\}_{\alpha\in\Gamma}$ of $\mathcal{A}$ is called *conical* if it is perfect and if Γ has a minimal element 1 satisfying the following conditions:
   i. $\mathcal{A}_1 \subseteq \mathcal{A}_\alpha$ for all $\alpha \in \Gamma$;
   ii. $\mathcal{A}_1$ is simple;
   iii. If V is a natural $\mathcal{A}_\alpha$-module, then the restriction $V_{\downarrow_{\mathcal{A}_1}}$ to $\mathcal{A}_1$ contains a proper composition factor.

Remark 4.2. [4] Definition 4.1(3.iii.) implies that the rank of every simple ideal S of any Levi (maximal semisimple) subalgebra of $\mathcal{A}_\alpha$ (for every $\alpha \in \Gamma$) is greater than or equal to the rank of $\mathcal{A}_1$.

Lemma 4.3. *Suppose that $\mathcal{A}$ is a simple locally finite associative (or Lie) algebra. Suppose that $p = 0$. The following holds:*

1. [2] *If $\{\mathcal{A}_\alpha\}_{\alpha\in\Gamma}$ is local system, then there exists $\beta \in \Gamma$ for each $\alpha \in \Gamma$ such that $\mathcal{A}_\alpha \subset \mathcal{A}_\beta$ and $\mathcal{A}_\alpha \cap Rad\mathcal{A}_\beta = 0$.*
2. [9] *$\mathcal{A}$ Possesses a perfect local system.*
3. [9] *If $\{\mathcal{A}_\alpha\}_{\alpha\in\Gamma}$ is a local system of perfect algebras of $\mathcal{A}$, then there exists $\alpha' \in \Gamma$ for each $\alpha \in \Gamma$ such that $RadL_\beta \cap L_\alpha = 0$ for all $\beta \geq \alpha'$.*

Theorem 4.4. [1] *Suppose that $p = 0$ and $\mathcal{A}$ is locally finite. Then*

1. *If $\mathcal{A}$ is simple with involution $*$, then $\mathcal{A}$ contains a local system which is conical of arbitrary large rank;*
2. *If $\mathcal{A}$ is simple, then $\mathcal{A}$ contains a local system which is conical of arbitrary large rank.*

Proposition 4.5. *Suppose that $\mathcal{A}$ is simple and $\{\mathcal{A}_\alpha\}_{\alpha\in\Gamma}$ is a local system of $\mathcal{A}$. Let $\{I_\alpha\}_{\alpha\in\Gamma}$ be a system of ideals such that I_α is an ideal of $\mathcal{A}_\alpha$ for each $\alpha \in \Gamma$. Then either $\bigcap_{\alpha\in\Gamma} I_\alpha = 0$ or for each $k \in \Gamma$ there is $\beta_k \in \Gamma$ with $\mathcal{A}_k \subseteq I_{\beta_k}$.*

Proof. Put $I = \bigcap_{\alpha\in\Gamma} I_\alpha$. Suppose that $I \neq 0$. Let

$$I'_\alpha = \cap \{ X_\alpha \mid X_\alpha \text{ is an ideal of } \mathcal{A}_\alpha \text{ with } X_\alpha \supset I \}.$$

Then I'_α is an ideal of $\mathcal{A}_\alpha$ with $I'_\alpha \subseteq I_\alpha$ for each $\alpha \in \Gamma$. Let X_α be a member in I'_α. Then X_α be an ideal of $\mathcal{A}_\alpha$ with $I \subseteq X_\alpha$. Note that for any $\mathcal{A}_\eta \in \{\mathcal{A}_\alpha\}_{\alpha\in\Gamma}$ with $\mathcal{A}_\eta \subset \mathcal{A}_\alpha$ we have $X_\alpha \cap \mathcal{A}_\eta$ is an ideal of $\mathcal{A}_\eta$ with $I \subseteq X_\alpha \cap \mathcal{A}_\eta$, so $I'_\eta \subseteq I'_\alpha$. Hence, $I' = \bigcup_{\alpha\in\Gamma} I'_\alpha$ is an ideal of $\mathcal{A}$ with $I \subseteq I'$. The simplicity of $\mathcal{A}$ implies that $\mathcal{A} = I' = \bigcup_{\alpha\in\Gamma} I'_\alpha$. Thus, $\{I'_\alpha\}_{\alpha\in\Gamma}$ is a local system of $\mathcal{A}$. Let $k \in \Gamma$. Since $\mathcal{A}_k$ is a finite dimensional, there exists $\alpha_k \in \Gamma$ such that $\mathcal{A}_k \subseteq I'_{\alpha_k}$, but $I'_\alpha \subseteq I_\alpha$. Therefore, for each $k \in \Gamma$, there is $\alpha_k \in \Gamma$ such that $\mathcal{A}_k \subseteq I_{\alpha_k}$, as required. ■

6. Plain, diagonal and non-diagonal locally finite Lie algebras

Let $L \subseteq L'$ be perfect Lie algebras. If L and L' are finite dimensional and $V'_1, \dots, V'_k$ are natural L'-modules, then an embedding $L \subseteq L'$ is called a *plain* (*resp. diagonal*) *embedding* if $(V'_1 \oplus \dots \oplus V'_k)\downarrow_L$ is a plain (resp. diagonal) L-module.

Definition 5.1. [5] Suppose that L is simple. Then a *plain* (resp. *diagonal*) *local system* of L is a perfect local system $\{L_\alpha\}_{\alpha\in\Gamma}$ such that the embedding $L_\alpha \subseteq L_\beta$ is plain (resp. diagonal) for all $\alpha \leq \beta$.

Example 5.2. If L is simple and $p = 0$. Then by Lemma 4.3(2), L has a perfect local system, say $\{L_\alpha\}_{\alpha\in\Gamma}$. For each $\alpha \in \Gamma$, we denote by Q_α is a Levi subalgebra of L_α and $\{Q^1_\alpha, Q^2_\alpha, \dots, Q^{n_\alpha}_\alpha\}$ is the set of the simple ideals of Q_α, so

$$Q_\alpha = Q^1_\alpha \oplus \dots \oplus Q^{n_\alpha}_\alpha.$$

Let V^k_α be the standard Q^k_α-module. As L_α is perfect, for each k there is a unique indecomposable L_α-module $\mathcal{V}^k_\alpha$ in which the restriction $\mathcal{V}^k_\alpha\downarrow_{Q^k_\alpha}$ is isomorphic to V^k_α.

An embedding $L_\alpha \subset L_\beta$ for $\alpha < \beta$ is a diagonal embedding if

$$\mathcal{V}^k_\beta\downarrow_{L_\alpha} = \left\{\mathcal{V}^1_\alpha, \dots, \mathcal{V}^{n_\alpha}_\alpha, \mathcal{V}^{1^*}_\alpha, \dots, \mathcal{V}^{n^*_\alpha}_\alpha, T_\alpha\right\}, \qquad 1 \leq k \leq n_\beta,$$

where T_α is a trivial and one dimensional L_α-module and $\mathcal{V}^{i^*}_\alpha$ is the dual to $\mathcal{V}^i_\alpha$.

Remark 5.3 [4] Suppose that $\{L_\alpha\}_{\alpha\in\Gamma}$ is a conical system of L. Then all simple components of L_α (for each $\alpha \in \Gamma$) are of classical type if the rank of L_1 is greater than or equal to 9.

Definition 5.4. [5] Suppose that L is simple, then L is said to be *plain* (resp. *diagonal*) if L has a plain (resp. diagonal) local system.

Example 5.5. Consider the zero trace $n \times n$-matrices $X \in \mathcal{M}_n(\mathbb{F})$. Then

1. $\mathfrak{sl}_\infty(\mathbb{F})$ and $\mathfrak{sl}_{2^\infty}(\mathbb{F})$ can be defined to be the limit of the sequence of the natural embeddings:

   $$\varphi_1 : \mathfrak{sl}_n(\mathbb{F}) \to \mathfrak{sl}_{n+1}(\mathbb{F})$$

   and

   $$\varphi_2 : \mathfrak{sl}_{2^n}(\mathbb{F}) \to \mathfrak{sl}_{2^{n+1}}(\mathbb{F}),$$

   where φ_1 and φ_2 are defined as follows: $\varphi_1(X) = \mathrm{diag}(X, 0)$ and $\varphi_2(X) = \mathrm{diag}(X, X)$, respectively. Then $\mathfrak{sl}_\infty(\mathbb{F})$ and $\mathfrak{sl}_{2^\infty}(\mathbb{F})$ are both simple of diagonal type.

2. A generalization of $\mathfrak{sl}_{2^\infty}(\mathbb{F})$ can be done as follows: Consider the sequence $\mathcal{N} = (\ell_1, \ell_2, \ell_3, \dots)$ of the positive integers ℓ_i. Let $q_n = \ell_1\ell_2\ell_3 \dots$. Then $\mathfrak{sl}_{\mathcal{N}}(\mathbb{F})$ is defined to be the limit of the sequence of diagonal matrix embedding.

   $$\varphi_q : \mathfrak{sl}_{q^n}(\mathbb{F}) \to \mathfrak{sl}_{q^{n+1}},$$

where φ_q is defined as $\varphi_q(X) = \mathrm{diag}(X, X, \dots, X)$ (ℓ_{n+1} copies).

Definition 5.6. A subspace B of L is called an inner ideal of L if $[B,[B,L]] \subseteq B$.

Theorem 5.7. *Suppose that $p = 0$. The following holds:*
1. [4] *There exists a simple of diagonal type locally finite algebra that is not locally semisimple.*
2. *Suppose that L is simple over $\mathbb{F}$. Then*
   i. [4] *L is semisimple as Lie algebra and locally perfect as well.*
   ii. [9] *L contains a non-trivial inner ideal if it is of diagonal type and vice versa.*

Definition 5.8. [5] Suppose that L is simple. Then L is called *non-diagonal* if there is no diagonal local system of L.

Recall that the map $ad_x: L \to \mathfrak{gl}(L)$, $ad_x(y) = [x,y]$ for all $y \in L$, is linear. The *adjoint homomorphism* $ad: L \to \mathfrak{gl}(L)$, is a linear map defined by $x \longmapsto ad_x$ for all $x \in L$ [16].

Example 5.9. Consider the Lie algebra $\mathfrak{sl}_{ad}(\mathbb{F})$ which is defined to be the limit of the sequence of embeddings

$$\mathfrak{sl}_2(\mathbb{F}) \to \mathfrak{sl}(\mathfrak{sl}_2(\mathbb{F})) \cong \mathfrak{sl}_3(\mathbb{F}) \to \mathfrak{sl}(\mathfrak{sl}_3(\mathbb{F})) \cong \mathfrak{sl}_8(\mathbb{F}) \to \cdots$$

where all embeddings are induced by the adjoint map $x \mapsto adx$. Then $\mathfrak{sl}_{ad}(\mathbb{F})$ is a simple of non-diagonal type (see [4 Corollary 2.11] for the proof).

Theorem 5.10. [9] *Suppose that L is simple of non-diagonal type and $p = 0$. The following hold*
1. *If $\{L_\alpha\}_{\alpha\in\Gamma}$ is a conical system of L of rank > 10, then for every $\alpha \in \Gamma$, there is $\beta \geq \alpha$ such that $L_\beta \subseteq L_\gamma$ is non-diagonal embedding for all $\gamma \geq \beta'$.*
2. *L has no non-zero proper inner ideals.*

7. Simple and simple with Involution associative algebras.

Recall Wedderburn theorem (see [6, Theorem 1]) that if $\mathcal{A}$ is finite dimensional, then $\mathcal{A}$ can be written as $\mathcal{A} = S \oplus Rad(\mathcal{A})$, where S is semisimple subalgebra of $\mathcal{A}$ and a $Rad(\mathcal{A})$ is a nilpotent ideal (the radical) of $\mathcal{A}$.

Lemma 6.1. Suppose that $\mathcal{A}$ is semisimple and finite dimensional. If $p = 0$, then $[\mathcal{A},\mathcal{A}]$ is a semisimple finite dimensional Lie algebra over $\mathbb{F}$.

Proof. Consider the set of the simple ideals $\{S_1, \dots, S_k\}$ of $\mathcal{A}$, so

$$\mathcal{A} = S_1 \oplus \dots \oplus S_k.$$

Then for each $1 \leq i \leq k$, we have $S_i \cong \mathcal{M}_{n_i}(\mathbb{F})$ for some integer n_i, so $[S_i, S_i] \cong \mathfrak{sl}_{n_i}(\mathbb{F})$ (see (2.2)). Thus,

$$[\mathcal{A},\mathcal{A}] = [S_1,S_1] \oplus \dots \oplus [S_k,S_k] \cong \mathfrak{sl}_{n_1}(\mathbb{F}) \oplus \dots \oplus \mathfrak{sl}_{n_k}(\mathbb{F}).$$

Therefore, $[\mathcal{A},\mathcal{A}]$ is a semisimple and finite dimensional, as required. ■

Definition 6.2. [1] An associative algebra $\mathcal{A}$ is said to be an *involution simple associative algebra* if the only $*$-invariant ideals of $\mathcal{A}$ are $\{0\}$ and $\mathcal{A}$.

We have the following result. See [1, Proposition 2.8] for the proof.

Proposition 6.3. *Every involution simple algebra $\mathcal{A}$ over $\mathbb{F}$ is either simple with involution $*$ or the $\mathcal{A} = \mathcal{U} \oplus \mathcal{U}^*$, where $\mathcal{U}$ is simple ideal.*

We will need the following well-known result. See for example [7].

Lemma 6.4. *Let $\mathcal{A}$ be semisimple and finite dimensional with involution $*$. Suppose that $p = 0$. Then $[skew(\mathcal{A}), skew(\mathcal{A})]$ is semisimple Lie algebra.*

Proof. Let $S_1, \dots, S_k$ be the the involution simple ideals of $\mathcal{A}$, so $\mathcal{A} = S_1 \oplus \dots \oplus S_k$. Then by Proposition 6.3, for each $1 \le i \le k$, we have S_i is either simple with involution $*$ or $S_i = \mathcal{U}_i \oplus \mathcal{U}_i^*$ for some simple ideals $\mathcal{U}_i$ and $\mathcal{U}_i^*$ of S_i. Thus, by using Lemma 2.6, we get that

$$skew(S_i) \cong \begin{cases} \mathfrak{so}_{n_i}(\mathbb{F}), \mathfrak{sp}_{n_i}(\mathbb{F}) & \text{if } S_i \text{ is simple with involution} \\ \mathfrak{gl}_{n_i}(\mathbb{F}) & \text{if } S_i = \mathcal{U}_i \oplus \mathcal{U}_i^*. \end{cases}$$

Thus, $[skewS_i, skewS_i] \cong \mathfrak{sl}_{n_i}(\mathbb{F}), \mathfrak{so}_{n_i}(\mathbb{F}), \mathfrak{sp}_{n_i}(\mathbb{F})$ for each $i = 1, \dots, k$. Therefore,

$$[skew(\mathcal{A}), skew(\mathcal{A})] = [skew(\mathrm{S}_1), skew(\mathrm{S}_1)] \oplus \dots \oplus [skew(\mathrm{S}_k), skew(\mathrm{S}_k)],$$

is semisimple and finite dimensional.

Definition 6.5. A system $\{\mathcal{A}_\alpha\}_{\alpha\in\Gamma}$ is called a $*$-*invariant system* if for each $\mathcal{A}_\alpha \in \{A_\alpha\}_{\alpha\in\Gamma}$ we have $a_\alpha^* \in \mathcal{A}_\alpha$ for all $a_\alpha \in \mathcal{A}_\alpha$.

We have the following lemma (See [1]).

Lemma 6.6. *Let $\mathcal{A}$ be locally finite with involution $*$. Then $\mathcal{A}$ contains a $*$-invarint system.*

Proof. Consider the local system $\{\mathcal{A}_\alpha\}_{\alpha\in\Gamma}$ of $\mathcal{A}$. Then for each $\alpha \in \Gamma$, consider the subalgebra $\hat{\mathcal{A}}_\alpha$ of $\mathcal{A}$ that generated by $\mathcal{A}_\alpha + \mathcal{A}_\alpha^*$. Since $a_\alpha^* \in \hat{\mathcal{A}}$ for all $a_\alpha \in \mathcal{A}_\alpha$, we get that $\hat{\mathcal{A}}_\alpha$ is a $*$-invariant subalgebra of $\mathcal{A}$. Thus, $\{\hat{\mathcal{A}}_\alpha\}_{\alpha\in\Gamma}$ is a $*$-invariant local system of $\mathcal{A}$.

Proposition 6.7. *If $\boldsymbol{\mathcal{A}}$ is simple with involution and $\boldsymbol{p = 0}$.*

1. [1] *$\boldsymbol{\mathcal{A}}$ have a $*$-invariant conical system of large rank.*
2. [9] *If $\{\boldsymbol{\mathcal{A}_\alpha}\}_{\boldsymbol{\alpha\in\Gamma}}$ is a $*$-invariant conical system of $\boldsymbol{\mathcal{A}}$, then for every $\boldsymbol{\alpha \in \Gamma}$ there exists $\boldsymbol{\alpha' \in \Gamma}$ satisfying that for all $*$-invariant maximal ideals $\boldsymbol{I}$ of $\boldsymbol{A_\beta}$ ($\boldsymbol{\beta \ge \alpha'}$) we have $\boldsymbol{A_\alpha \cap I = 0}$.*

8. Locally finite Lie algebras of simple associative algebras

Definition 7.1. [1] 1) An associative algebra $\mathcal{A}$ is said to be an *envelope* of a Lie algera L if

i. L is a subalgebra of $\mathcal{A}$.
ii. L generates $\mathcal{A}$.

2) An envelope $\mathcal{A}$ of L is said to be a $\mathfrak{B}$-envelope of L if $L = [\mathcal{A}, \mathcal{A}]$.

In what follow, $U(L)$ is denoted to be the *universal enveloping algebra* of L and $A(L)$ the *augmented ideal* of $U(L)$ which defined to be the ideal of $U(L)$ of codimension 1. Recall that *universal enveloping algebra* $U(L)$ of L is an infinite-dimensional associative algebra [16]. If $\mathcal{A}$ is a $\mathfrak{B}$-enveloping of L, then $\mathcal{A}$ can be considered as the augmented ideal $\mathcal{A}(L)$ of $U(L)$ [1]. Therefore, and there is a $1-1$ correspondence between $\mathcal{A}$ and $H_{\mathcal{A}}$ with the following property $H_{\mathcal{A}} \cap L = 0$, $A(L)/H_{\mathcal{A}} \cong \mathcal{A}$.

Remark 7.2. We say that $\mathcal{A} \leq \mathcal{C}$ if and only if $H_{\mathcal{A}} \supseteq H_{\mathcal{C}}$.

Theorem 7.3. [1] *If L is simple plain and $p = 0$, then L generates two $\mathfrak{B}$-envelopes associative algebras $\mathcal{A}_+$ and $\mathcal{A}_-$ such that:*

1. *The radical $Rad(\mathcal{A}_\pm)$ annihilates $\mathcal{A}_\pm$.*
2. $\mathcal{A}_\pm/Rad(\mathcal{A}_\pm)$ is a simple $\mathfrak{B}$-envelope of L.
3. If $\mathcal{A}$ is a $\mathfrak{B}$-envelope of L, then $\mathcal{A}_+/Rad(\mathcal{A}_+) \leq \mathcal{A} \leq \mathcal{A}_+$ or $\mathcal{A}_-/Rad(\mathcal{A}_-) \leq \mathcal{A} \leq \mathcal{A}_-$.
4. The inverse of the mapping in (v) is defined by $\mathcal{A} \to [\mathcal{A},\mathcal{A}]$.

Recall that a subspace B of L is called an *inner ideal* of L if $[B,[B,L]] \subseteq B$ (see Definition 5.6). B is called *abelian* in the case when $[B,B] = 0$. An inner ideal of the Lie algebra $\mathcal{A}^{(-)}$ is called *Jordan-Lie* in the case when $B^2 = 0$ [10].

Theorem 7.5. *Let $\mathcal{A}$ be simple and $p = 0$.*

1. [1] *$[\mathcal{A},\mathcal{A}]$ is a simple and plain. Moreover, $\mathcal{A}$ is $\mathfrak{B}$-envelope of $[\mathcal{A},\mathcal{A}]$.*
2. *$[\mathcal{A},\mathcal{A}]$ contains a proper inner ideal.*
3. [9] *If B is an inner ideal of $[\mathcal{A},\mathcal{A}]$, then B is Jordan-Lie.*
4. *If B is inner ideal of $[\mathcal{A},\mathcal{A}]$, then B is abelian.*

Proof. Part (*1.*) is proved in [1]. For the proof see [1, Theorem 2.12].

2. By (*1*), $\mathcal{A}^{(1)}$ is a simple and diagonal, so by Theorem 5.7(2.ii), L contains a non-trivial inner ideal, as required.

3. This is proved in [9]. For the proof see [9, Corollary 4.14].

4. Let B be an inner ideal of $[\mathcal{A},\mathcal{A}]$. By using (3.), we get that $[B,B] \subseteq B^2 = 0$. ■

Definition 7.6. [9] An inner ideal B of $\mathcal{A}^{(-)}$ (or $[\mathcal{A},\mathcal{A}]$) is called *regular* if B is Jordan-Lie and $B\mathcal{A}B \subseteq B$.

Lemma 7.7. [10] *If $p \neq 2,3$, then an inner ideal B of $[\mathcal{A},\mathcal{A}]$ is regular if and only if there is right $\mathfrak{R}$ and left $\mathfrak{L}$ ideals of $\mathcal{A}$ with $\mathfrak{L}\mathfrak{R}=0$ such that*

$$\mathfrak{R}\mathfrak{L} \subseteq B \subseteq \mathfrak{R} \cap \mathfrak{L} \cap [\mathcal{A},\mathcal{A}].$$

We will need the following proposition. It represents a special case of [10, Proposition 6.20].

Proposition 7.8. [10] *If $p \neq 2,3$, then Jordan-Lie inner ideals of $\mathcal{A}^{(-)}$ and of $[\mathcal{A},\mathcal{A}]$ are regular.*

Theorem 7.9. *Let $\mathcal{A}$ be simple and locally semisimple. If $p = 0$, then.*

1. *$[\mathcal{A},\mathcal{A}]$ is locally semisimple as Lie algebra.*
2. *Consider the system $\{\mathcal{A}_\alpha\}_{\alpha\in\Gamma}$ of $\mathcal{A}$. If B is inner ideal of $[\mathcal{A},\mathcal{A}]$. Then $B \cap [\mathcal{A}_\alpha,\mathcal{A}_\alpha]$ is an inner ideal of $[\mathcal{A}_\alpha,\mathcal{A}_\alpha]$.*
3. *Every inner ideal of $\mathcal{A}^{(1)}$ is regular.*
4. *Every proper inner ideal of $\mathcal{A}^{(1)}$ can be written as $\mathfrak{R}\mathfrak{L}$ for some right $\mathfrak{R}$ and left $\mathfrak{L}$ ideals of $\mathcal{A}$ with $\mathfrak{L}\mathfrak{R}=0$.*

Proof. 1. Consider the semisimple system $\{\mathcal{A}_\alpha\}_{\alpha\in\Gamma}$. Then by Lemma 6.1, $[\mathcal{A}_\alpha,\mathcal{A}_\alpha]$ is a semisimple finite dimensional Lie algebra over $\mathbb{F}$. Therefore, $\{[\mathcal{A}_\alpha,\mathcal{A}_\alpha]\}_{\alpha\in\Gamma}$ is semisimple system of $\mathcal{A}^{(1)}$, so $\mathcal{A}^{(1)}$ is locally semisimple Lie algebra, as required.

2. This is obvious as $B \cap [\mathcal{A}_\alpha,\mathcal{A}_\alpha] \subseteq [\mathcal{A}_\alpha,\mathcal{A}_\alpha]$.

3. Let B be an inner ideal of $[\mathcal{A},\mathcal{A}]$. Then B is Jordan-Lie (Theorem 7.5(3)), that is, $B^2=0$, so we only need to prove that $B\mathcal{A}B \subseteq B$. Let $x \in B\mathcal{A}B$. Let $\{[S_\alpha,S_\alpha]\}_{\alpha\in\Gamma}$ be a semisimple local system of $[\mathcal{A},\mathcal{A}]$, where $\{S_\alpha\}_{\alpha\in\Gamma}$ is a semisimple local system of $\mathcal{A}$. By (2), $B_\alpha = B \cap [S_\alpha,S_\alpha]$ for all $\alpha \in \Gamma$. Then B_α is Jordan-Lie of $[S_\alpha,S_\alpha]$, so there exists $\beta \in \Gamma$ such that $x \in B_\beta S_\beta B_\beta$. Since $[S_\beta,S_\beta]$ is semisimple, By Proposition 7.8, B_β is a regular inner ideal of $[S_\beta,S_\beta]$, so $B_\beta S_\beta B_\beta \subseteq B_\beta$. Thus, $x \in B_\beta \subseteq B$. Therefore, B is regular.

4. This follows from (3) and Lemma 7.7. ■

9. Locally finite Lie algebras of involution simple associative algebras

Definition 8.1. An associative algebra $\mathcal{A}$ with an involution is called $\mathfrak{B}^*$-envelope if $\mathcal{A}$ is an envelope of L and $L = K^{(1)}$, where $K = \mathrm{skew}(\mathcal{A})$.

Theorem 8.2. [1] *If $p = 0$, then L generates a unique $\mathfrak{B}^*$-envelope associative algebras $\mathcal{A}$ such that*

1. *The Jacobson radical $Rad(\mathcal{A})$ annihilates $\mathcal{A}$.*
2. *$\mathcal{A}/Rad(\mathcal{A})$ is a simple $\mathfrak{B}^*$-envelope of L.*
3. *If $\mathcal{A}$ is a $\mathfrak{B}^*$-envelope of L, then either $\mathcal{A}/Rad(\mathcal{A}) \leq \mathcal{A} \leq \mathcal{A}$.*
4. *The mapping $L \mapsto \mathcal{A}/Rad(\mathcal{A})$ is a $1-1$ correspondence between L and the set of all involution*

simple infinite dimensional locally finite associative algebras.

5. *The inverse of the linear transformation in (iv) is defined to be $\mathcal{A} \longmapsto [skew\mathcal{A}, skew\mathcal{A}]$.*

Theorem 8.3. [1] *Let $p = 0$. Then $L = [K, K]$ is simple and diagonal. Moreover, $\mathcal{A}$ is $\mathfrak{B}^*$-envelope of $[K, K]$.*

An inner ideal of $K = skew(\mathcal{A})$ or $[K, K]$ is said to be *Jordan-Lie* if $B^2 = 0$ [23].

Definition 8.4. [9] An inner ideal B of $K = skew(\mathcal{A})$ (or $[K, K]$) is said to be a $*$-*regular* if B is Jordan-Lie and $skew(B\mathcal{A}B) \subseteq B$.

Lemma 8.5. [10] *Suppose that $K = skew(\mathcal{A})$ and $p = 0$. An inner ideal B of $K^{(1)}$ is $*$-regular if and only if there exists left ideal $\mathfrak{L}$ of $\mathcal{A}$ satisfying $\mathfrak{L}\mathfrak{L}^* = 0$ such that $\mathfrak{L}\hat{}* \ \mathfrak{L} \subseteq B \subseteq \mathfrak{L}\hat{} * \cap \mathfrak{L} \cap K^{(1)}$.*

Theorem 8.6. [9] *If $p = 0$ and $\mathcal{A}$ is locally $*$-semisimple, then the following hold.*

1. *$[K, K]$ is locally semisimple.*
2. *Suppose that $[K, K]$ is non-isomorphic to $\mathfrak{so}_\infty(\mathbb{F})$, then*
   i. If B is inner ideal of $K^{(1)}$, then B is $*$-regular;

   ii. If B is inner ideal of $K^{(1)}$ can be written in form $\mathfrak{L}^*\mathfrak{L}$ for some left $\mathfrak{L}$ ideal of $\mathcal{A}$ with $\mathfrak{L}\mathfrak{L}^* = 0$.

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The role of media coverage on the dynamical behavior of smoking model with and without spatial diffusion

Ahmed Ali Mohsen[1] and Raid Kamel Naji[2]

[1]Department of Mathematics, Ibn-Al-Haitham College of Education, Baghdad University. E-mail: aamuhseen@gmail.com;

[2]Department of mathematics, College of science, University of Baghdad.

rknaji@scbaghdad.edu.iq

Abstract: The spread of epidemic diseases still a major threat to the life of communities. Therefore, with the great development of the technology , the spread of diseases can be reduced by using media coverage awareness. In this paper a smoking model incorporating media coverage for warranting the population is proposed and studied. The dynamics of the model is investigated in two different cases: nonexistence and existence of diffusion. The existence, positivity and bounded-ness of solutions are investigated. The local and global stability by the help of Lyapunov function of all possible equilibrium points are investigated. Moreover, numerical simulations are carried out to validate the analytical results and specify the effect of varying the parameters.

Keywords: Smoking model, media, diffusion, stability.

1. Introduction

The smoke from the Cigarette is a very complex chemical mixture that is dangerous to human health and all the elements of the environment. It contains more than 3,800 toxic chemicals, the most important of which is the carbon monoxide (Co), which is one of the poisonous and dangerous gases on human life, ammonia (NH_3), Hydrogen sulfide (H_2S), formaldehyde (HCHO), Acetaldehyde (CH_3CHO), hydrogen cyanide (HCN), in addition to a large number of acids including: Carbonic acid (H_2CO_3), nitric acid (HNO_3), acetic acid (CH_3COOH) and formic acid (HCOOH), see [1].

Cigarette smoke also carries a huge range of organic compounds, which have proved dangerous, classified globally as highly dangerous. These substances include benzopyrene, which works to destroy the mucous membranes of the respiratory tract of smokers, and also destroys the airways of smokers. In one of the statistics from 2013, the number of premature deaths due to smoking to 5950 deaths, as well as 200,000 cases of hospitalization. And there are many diseases caused by smoking such as 44% Cancer, 30% Circulatory diseases, 25% Respiratory diseases and other [2-3]. All these reasons have invited many authors to understand and study the smoking epidemic for example: In [4], Castillo-Garsow et al suggested the tobacco model with recovery. Lahrouz, et al [5] proposed and studied mathematical model of smoking. Al-Shareef and Batarfi studied the effect of chain, mild and passive smoke see [6]. In [7], Sharomi and Gumel provided a rigorous mathematical study for assessing the dynamics of smoking and their impact on public health in a community. Zaman, studied the smoking dynamics with control strategy, he discussed qualitative behavior of tobacco model [8, 9]. Erturk and Momani [10] proposed analytic method for approximating a giving up smoking model. Zainab et al [11] studied global dynamics of a mathematical model on Smoking. Moreover many researchers proposed and studied models showed how the media effect of the spread of the diseases for example: Misra et al [12] studied the effects of awareness programs by media on the spread of infectious diseases. Smith et al [13] investigated the impact of media coverage on the influenza disease. Cui and Zhu [14] studied the impact of media on control of infectious disease. On the other hand, it is well known that location play a critical role in disease dynamics see for example [15-19]. In this work, we proposed and studied a mathematical model describing the effect of awareness through media program on the spread of smoking. Further, the effect of location on outbreak the smoking in the population is also considered through studying the model with reaction diffusion. Finally, local as well as global stability analysis of the proposed model are also investigated.

2. Construction of the model

The mathematical model offer us more understand about spread the infection disease, we know that the disease is transmitted by direct contact between healthy individuals with infected individuals. In fact, outbreak the smoking is very similar to the spread of epidemic and hence some populations start smoking due to contact with smokers. Consider a population of size N at time t. It is assumed that, the population divided into four classes: the 1st class consisting of individuals who do not smoke tobacco and maybe become smokers in future (potential smokers) and the size of individuals at time t for this class denoted by $P(t)$; 2nd class involving the smoker individuals and denoted their size at time t by $S(t)$; $Q(t)$ represents the size of individuals at time t in the 3rd class that contains individuals who temporarily quit smoking; $R(t)$ stands for the size of individuals at time t in the 4th class, which contains the recovery from smoking. On the other hand, the efficiency of awareness by media coverage to reducing the number of smokers (or smoking prevention) at time t will be denoted by $M(t)$. Accordingly, the dynamics of smoking model with the effect of awareness by media coverage to outbreak the smoking can be describe by the following system of nonlinear ODEs.

$$\begin{aligned}
\dot{P} &= \psi - \beta PS - \mu P - \gamma PM \\
\dot{S} &= \beta PS + \sigma\gamma PM - \mu S - \gamma SM + \varepsilon\delta Q \\
\dot{Q} &= e\gamma SM - \mu Q - \delta Q \\
\dot{R} &= \gamma(1-\sigma)PM + \gamma(1-e)SM - \mu R + \delta(1-\varepsilon)Q \\
\dot{M} &= \alpha(S+P) - \theta M
\end{aligned} \tag{1}$$

As the fourth equation is a linear differential equation with respect to variable $R(t)$, which is not appear in the other equations of system (1), hence system (1) can be reduced to the following system:

$$\begin{aligned}
\dot{P} &= \psi - \beta PS - \mu P - \gamma PM \\
\dot{S} &= \beta PS + \sigma\gamma PM - \mu S - \gamma SM + \varepsilon\delta Q \\
\dot{Q} &= e\gamma SM - \mu Q - \delta Q \\
\dot{M} &= \alpha(S+P) - \theta M
\end{aligned} \tag{2}$$

with initial condition $P(0) > 0$, $S(0) \geq 0$, $Q(0) \geq 0$ and $M(0) > 0$. Therefore, by solving system (2) and substituting the solution, say (P^*, S^*, Q^*, M^*), of it in the fourth equation of system (1) and solving the obtained linear differential equation we get for $t \to \infty$ that:

$$R = \frac{\gamma[(1-\sigma)P^* + (1-e)S^*]M^* + \delta(1-\varepsilon)Q^*}{\mu} \tag{3}$$

Moreover, all the parameters are assumed to be nonnegative with, $\psi > 0$ represents the recruitment of potential smokers population, $\mu > 0$ represents the natural death rate of the human populations. The parameter $\beta > 0$ is the contact rate between potential smokers and smokers. On other hand, the awareness level through media coverage that reached to the individuals is denoted by $\gamma > 0$, however portion of individuals who received awareness transfers to smoker class and temporarily quit smoking class with rates $(0 \leq \sigma \leq 1)$ and $(0 \leq e \leq 1)$ respectively. The parameter $\delta > 0$ represents the rate of losing the temporary quitters smoking individuals, in fact fraction of them with rate $(0 \leq \varepsilon \leq 1)$ transfers to smoker's class while the rest of individuals will transfer to recovery from smoking class. The parameter $\alpha > 0$ represents media campaigns rate performed by both smokers and nonsmokers, however the rate of disappearance of media coverage represented by $\theta > 0$. Keeping the above description of variables and parameters, it is easy to proof that system (1), and hence system (2), is defined on the following positively invariant set:

$$\Gamma = \left\{(P, S, Q, R, M) \in \mathbb{R}_+^5 : 0 \leq N \leq \frac{\psi}{\mu}, 0 \leq M \leq \frac{\alpha\psi}{\theta\mu}\right\}$$

where $N = P + S + Q + R$.

3. The existence of equilibrium points of system (2)

In this section, the existence conditions of all possible equilibrium points are determine. It is easy to shows that system (2) has three equilibrium points. The points and their existence conditions can be described as following:

- In the absence of smokers, that is $S = 0$. Then, system (2) has a unique positive equilibrium point in the interior of positive quadrant of PM −plane, namely smoking free equilibrium point (*SFEP*), which denoted by $E_0 = (P_0, 0, 0, M_0)$ where

$$P_0 = \frac{2\psi\theta}{\mu\theta + \sqrt{(\mu\theta)^2 + 4\alpha\gamma\psi\theta}}$$
$$M_0 = \frac{-\mu\theta + \sqrt{(\mu\theta)^2 + 4\alpha\gamma\psi\theta}}{2\gamma\theta} \tag{4a}$$

  provided that the following condition holds

$$\sigma = 0 \tag{4b}$$

- In the absence of temporarily quit smokers ($Q = 0$). Hence, system (2) has an equilibrium point in the interior of positive octant of PSM −space, namely free temporarily quit smoking equilibrium point (*FTQSEP*), which denoted by $E_1 = (P_1, S_1, 0, M_1)$ where:

$$M_1 = \frac{\alpha(S_1+P_1)}{\theta} \tag{5a}$$

while (P_1, S_1) is a positive root to the following two isoclines:

$$f(P,S) = \theta\psi - (\theta\beta + \alpha\gamma)SP - \theta\mu P - \alpha\gamma P^2 = 0 \tag{5b}$$

$$g(P,S) = (\beta\theta + \alpha\gamma(\sigma - 1))SP + \alpha\sigma\gamma P^2 - \mu\theta S - \alpha\gamma S^2 = 0 \tag{5c}$$

Clearly, as $S \to 0$, the two isoclines reduced to:

$$\theta\psi - \theta\mu P - \alpha\gamma P^2 = 0 \tag{5d}$$
$$\alpha\gamma\sigma P^2 = 0 \tag{5e}$$

Obviously, Eq. (5d) has a unique intersection positive point with P −axis that given by

$$p = \frac{-\theta\mu}{2\alpha\gamma} + \frac{1}{2\alpha\gamma}\sqrt{(\theta\mu)^2 + 4\alpha\gamma\theta\psi} \tag{6}$$

while, Eq. (5e) has zero root on the P −axis.

Therefore, straightforward computation shows that the two isoclines (5b) and (5c) have a unique intersection positive point (P_1, S_1) provided that:

$$\begin{aligned} \frac{dS}{dP} &= -\frac{\partial f/\partial P}{\partial f/\partial S} < 0 \\ \frac{dS}{dP} &= -\frac{\partial g/\partial P}{\partial g/\partial S} > 0 \end{aligned} \tag{7a}$$

Consequently, in addition to condition (7a), the following condition guarantees the existence of *FTQSEP*.

$$e = 0 \tag{7b}$$

- The coexistence equilibrium point or endemic equilibrium point (*EEP*), which denoted by $E_2 = (P_2, S_2, Q_2, M_2)$ where

$$M_2 = \frac{\alpha(S_2+P_2)}{\theta};\ Q_2 = \frac{e\,\gamma\,\alpha S_2(S_2+P_2)}{\theta(\mu+\delta)} \tag{8}$$

while (P_2, S_2) represents a positive intersection point of the two isoclines $f(S,P) = 0$, which is given by Eq. (5b), while the other isocline is given by

$$\begin{aligned} g_1(S,P) = {} & [[\beta\theta + \alpha\gamma(\sigma - 1)]SP + \alpha\sigma\gamma P^2 - \mu\theta S - \alpha\gamma S^2](\mu + \delta) \\ & +\alpha\gamma\varepsilon\delta e S(S + P) = 0 \end{aligned} \tag{9a}$$

Clearly, as $S \to 0$, the last two isocline reduced to the same polynomial equation given in Eq. (5d) and (5e). Hence they have the same nonnegative roots fall on the P −axis. Accordingly, (P_2, S_2) exists uniquely in the interior of positive quadrant PS −plane provided that

$$\begin{aligned} \frac{dS}{dP} &= -\frac{\partial f/\partial P}{\partial f/\partial S} < 0 \\ \frac{dS}{dP} &= -\frac{\partial g_1/\partial P}{\partial g_1/\partial S} > 0 \end{aligned} \tag{9b}$$

Hence the EEP exists uniquely in the $Int\ \mathbb{R}_+^4$ provided that in addition to condition (9b) the following condition holds

$$e > 0 \tag{9c}$$

Note that, the *EEP* and *FTQSEP* are coinciding in the interior of positive octant of PSM −space under the condition (7b).

4. Stability analysis of system (2)

In this section, the stability analysis of all equilibrium points of system (2) is studied. The Jacobian matrix of system (2) at (P, S, Q, M) can be written in the following form.

$$J(P, S, Q, M) = \left(a_{ij}\right)_{4\times4} \tag{10}$$

where

$$a_{11} = -\beta S - \mu - \gamma M,\ a_{12} = -\beta P,\ a_{13} = 0,\ a_{14} = -\gamma P$$

$$a_{21} = \beta S + \sigma\gamma M,\ a_{22} = \beta P - \mu - \gamma M,\ a_{23} = \varepsilon\delta,\ a_{24} = \sigma\gamma P - \gamma S$$

$$a_{31} = 0,\ a_{32} = e\gamma M,\ a_{33} = -\mu - \delta,\ a_{34} = e\gamma S$$

$$a_{41} = \alpha,\ a_{42} = \alpha,\ a_{43} = 0,\ a_{44} = -\theta$$

Consequently, the local stability of *SFEP* is investigated in the following theorem.

Theorem 1: The *SFEP* of system (2) is locally asymptotically stable (LAS) if the following sufficient conditions hold

$$\beta P_0 < \mu + \gamma M_0 \tag{11a}$$

$$(\mu + \delta)\beta P_0 + e\gamma\varepsilon\delta M_0 < (\mu + \delta)(\mu + \gamma M_0) \tag{11b}$$

Proof: The Jacobian matrix of system (2) at E_0 can be written:

$$J(E_0) = \begin{pmatrix} -(\mu + \gamma M_0) & -\beta P_0 & 0 & -\gamma P_0 \\ 0 & \beta P_0 - (\mu + \gamma M_0) & \varepsilon\delta & 0 \\ 0 & e\gamma M_0 & -(\mu + \delta) & 0 \\ \alpha & \alpha & 0 & -\theta \end{pmatrix} = \left(b_{ij}\right)_{4\times4} \tag{12}$$

Hence, the characteristic equation can be written as

$$\lambda^4 + B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4 = 0 \tag{13}$$

Such that

$$B_1 = -[b_{11} + b_{22} + b_{33} + b_{44}]$$

$$B_2 = b_{11}(b_{22} + b_{33}) + b_{11}b_{44} - b_{14}b_{41} + b_{22}b_{33} - b_{23}b_{32} + b_{44}(b_{22} + b_{33})$$

$$B_3 = -[(b_{11} + b_{44})(b_{22}b_{33} - b_{23}b_{32}) + (b_{22} + b_{33})(b_{11}b_{44} - b_{14}b_{41})]$$

$$B_4 = (b_{22}b_{33} - b_{23}b_{32})(b_{11}b_{44} - b_{14}b_{41})$$

while by using some algebraic computation we obtain that

$$\begin{aligned} B_1B_2 - B_3 = & -(b_{11} + b_{44})(b_{22} + b_{33})(b_{11} + b_{22}) \\ & -(b_{11} + b_{44})(b_{22} + b_{33})(b_{33} + b_{44}) \\ & -(b_{11} + b_{44})(b_{11}b_{44} - b_{14}b_{41}) - (b_{22} + b_{33})(b_{22}b_{33} - b_{23}b_{32}) \\ & -(b_{22} + b_{33})(b_{22}b_{33} - b_{23}b_{32}) \end{aligned}$$

However $\Delta = B_3(B_1B_2 - B_3) - {B_1}^2B_4$ can be written as:

$$\begin{aligned} \Delta = & -B_1(b_{11} + b_{44})^2(b_{22} + b_{33})(b_{22}b_{33} - b_{23}b_{32}) \\ & -B_1(b_{11} + b_{44})(b_{22} + b_{33})^2(b_{11}b_{44} - b_{14}b_{41}) \\ & +(b_{11} + b_{44})(b_{22} + b_{33})[(b_{22}b_{33} - b_{23}b_{32}) - (b_{11}b_{44} - b_{14}b_{41})]^2 \end{aligned}$$

Note that, according to the Routh-Hurwitz criterion, all the eigenvalues of $J(E_0)$ have negative real parts and then the *SFEP* of system (2) is locally asymptotically stable provided that $B_i > 0$ for $i = 1,2,3,4$; $B_1B_2 - B_3 > 0$ and $\Delta > 0$.

It is easy to verify that condition (11a) guarantees that the element b_{22} is negative and condition (11b) guarantees that the term $b_{22}b_{33} - b_{23}b_{32} > 0$. Hence due to the sign of matrix elements and the sufficient conditions (11a) and (11b) all the Routh-Hurwitz conditions are satisfied. Therefore, the proof is complete.

Theorem 2: The *FTQSEP* of system (2) is LAS if the following sufficient conditions hold

$$\beta P_1 < (\mu + \gamma M_1) \tag{14a}$$

$$\sigma P_1 < S_1 \tag{14b}$$

$$\alpha\gamma S_1 < \alpha\sigma\gamma P_1 + \theta\beta S_1 + \theta\sigma\gamma M_1 \tag{14c}$$

$$\alpha\sigma\gamma P_1[\beta P_1 + \gamma M_1] < 2\theta(\beta S_1 + \mu + \gamma M_1)[(\mu + \gamma M_1) - \beta P_1] \tag{14d}$$

Proof: The Jacobian matrix of system (2) at E_1 can be written:

$$J(E_1) = (c_{ij})_{4\times 4} = \begin{pmatrix} -(\beta S_1 + \mu + \gamma M_1) & -\beta P_1 & 0 & -\gamma P_1 \\ \beta S_1 + \sigma\gamma M_1 & \beta P_1 - (\mu + \gamma M_1) & \varepsilon\delta & \sigma\gamma P_1 - \gamma S_1 \\ 0 & 0 & -(\mu + \delta) & 0 \\ \alpha & \alpha & 0 & -\theta \end{pmatrix} \tag{15}$$

Hence, the characteristic equation can be written as

$$(c_{33} - \lambda)(\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3) = 0 \tag{16}$$

where the eigenvalue in the $Q -$ direction is given by $\lambda_Q = -(\mu + \delta) < 0$, while

$$C_1 = -[c_{11} + c_{22} + c_{44}]$$

$$C_2 = c_{11}c_{22} - c_{12}c_{21} + c_{11}c_{44} - c_{14}c_{41} + c_{22}c_{44} - c_{24}c_{42}$$

$$C_3 = -[c_{11}(c_{22}c_{44} - c_{24}c_{42}) + c_{12}(c_{24}c_{41} - c_{21}c_{44}) + c_{14}(c_{21}c_{42} - c_{22}c_{41})]$$

with

$$\begin{aligned} C_1C_2 - C_3 = &-(c_{11} + c_{22})[c_{11}c_{22} - c_{12}c_{21}] \\ &-(c_{11} + c_{44})[c_{11}c_{44} - c_{14}c_{41}] \\ &-(c_{22} + c_{44})[c_{22}c_{44} - c_{24}c_{42}] \\ &-2c_{11}c_{22}c_{44} + c_{12}c_{24}c_{41} + c_{14}c_{21}c_{42} \end{aligned}$$

Note that, according to the Routh-Hurwitz criterion, all the eigenvalues of $J(E_1)$ have negative real parts and then the *FTQSEP* of system (2) is locally asymptotically stable provided that $C_i > 0$ for $i = 1,3$ and $C_1C_2 - C_3 > 0$.

It is easy to verify that condition (14a) guarantees that the element c_{22} is negative and condition (14b) guarantees that the element c_{24} is negative, while condition (14c) guarantees that the term $c_{24}c_{41} - c_{21}c_{44} > 0$. On the other hand condition (14d) ensure that $-2c_{11}c_{22}c_{44} + c_{12}c_{24}c_{41} + c_{14}c_{21}c_{42} > 0$. Hence due to the sign of matrix elements and the sufficient conditions (14a)-(14d) all the Routh-Hurwitz conditions are satisfied. Therefore, the proof is complete.

Theorem 3: The *EEP* of system (2) is LAS if the following sufficient conditions hold

$$\beta P_2 < \mu + \gamma M_2 \tag{17a}$$

$$\sigma P_2 < S_2 \tag{17b}$$

$$\beta(\mu + \delta)P_2 + \varepsilon\delta e\gamma M_2 < (\mu + \gamma M_2)(\mu + \delta) \tag{17c}$$

$$P_2(\beta S_2 + \sigma\gamma M_2) + \varepsilon e\delta S_2 < \gamma P_2(\beta S_2 + \mu + \gamma M_2 + \theta) \tag{17d}$$

Proof: The Jacobian matrix of system (2) at E_2 is written as

$$J(E_2) = \left(z_{ij}\right)_{4\times 4} \tag{18}$$

where $z_{ij} = a_{ij}(P_2, S_2, Q_2, M_2)$, $\forall i, j = 1,2,3,4$. Hence, the characteristic equation can be written as

$$\lambda^4 + Z_1\lambda^3 + Z_2\lambda^2 + Z_3\lambda + Z_4 = 0 \tag{19}$$

where

$$Z_1 = -[z_{11} + z_{22} + z_{33} + z_{44}]$$

$$\begin{aligned} Z_2 = &\, z_{11}z_{22} - z_{12}z_{21} + z_{11}z_{33} + z_{11}z_{44} - z_{14}z_{41} + z_{22}z_{33} - z_{23}z_{32} \\ &+ z_{22}z_{44} - z_{24}z_{42} + z_{33}z_{44} \end{aligned}$$

$$\begin{aligned} Z_3 = -[&(z_{11} + z_{44})(z_{22}z_{33} - z_{23}z_{32}) + (z_{22} + z_{33})(z_{11}z_{44} - z_{14}z_{41}) \\ &- z_{12}z_{21}(z_{33} + z_{44}) - z_{33}z_{24}z_{42} - z_{24}(z_{11}z_{42} - z_{12}z_{41}) \\ &+ z_{42}(z_{14}z_{21} - z_{23}z_{34})] \end{aligned}$$

$$Z_4 = (z_{11}z_{44} - z_{14}z_{41})(z_{22}z_{33} - z_{23}z_{32}) + (z_{11}z_{42} - z_{12}z_{41})(z_{23}z_{34} - z_{24}z_{33}) \\ -z_{21}z_{33}(z_{12}z_{44} - z_{14}z_{42})$$

Moreover, we have that

$$Z_1Z_2 - Z_3 = -(z_{11} + z_{22})(z_{11}z_{22} - z_{12}z_{21}) - z_{11}z_{33}(z_{11} + 2z_{22} + z_{33}) \\ -z_{44}(z_{11} + z_{22} + z_{33} + z_{44})(z_{11} + z_{22} + z_{33}) + z_{12}z_{24}z_{41} \\ -(z_{22} + z_{33})(z_{22}z_{33} - z_{23}z_{32}) + z_{14}z_{41}(z_{11} + z_{44}) \\ +z_{24}z_{41}(z_{22} + z_{44}) + z_{42}(z_{14}z_{21} - z_{23}z_{34})$$

and $\Delta = Z_3(Z_1Z_2 - Z_3) - {Z_1}^2 Z_4$ can be written as:

$\Delta = X_1(X_2 + X_1) + X_3$

where

$$X_1 = (z_{11} + z_{44})(z_{22}z_{33} - z_{23}z_{32}) + (z_{22} + z_{33})(z_{11}z_{44} - z_{14}z_{41}) \\ -z_{12}z_{21}(z_{33} + z_{44}) + z_{24}z_{42}(z_{11} + z_{33}) + z_{12}z_{24}z_{41} + z_{42}(z_{14}z_{21} - z_{23}z_{34})$$

$$X_2 = (z_{11} + z_{22} + z_{33} + z_{44})[z_{11}(z_{22} + z_{33} + z_{44}) - z_{12}z_{21} - z_{14}z_{41} \\ +z_{44}(z_{22} + z_{33}) - z_{24}z_{42} + z_{22}z_{33} - z_{23}z_{32}]$$

$$X_3 = (z_{11} + z_{22} + z_{33} + z_{44})^2[(z_{11}z_{44} - z_{14}z_{41})(z_{22}z_{33} - z_{23}z_{32}) \\ +(z_{11}z_{42} - z_{12}z_{41})(z_{23}z_{34} - z_{33}z_{24}) - z_{33}z_{21}(z_{44}z_{12} - z_{14}z_{42})]$$

Note that, according to the Routh-Hurwitz criterion, all the eigenvalues of $J(E_2)$ have negative real parts and then the *EEP* of system (2) is locally asymptotically stable provided that $Z_i > 0$ for $i = 1,2,3,4$; $Z_1Z_2 - Z_3 > 0$ and $\Delta > 0$.

It is easy to verify that condition (17a) and (17b) guarantees that the elements z_{22} and z_{24} are negative respectively and condition (17c) guarantees that the term $z_{22}z_{33} - z_{23}z_{32} > 0$. While, the term $z_{14}z_{41}(z_{11} + z_{44}) - z_{14}z_{21} + z_{23}z_{34} > 0$, if the condition (17d) holds. Hence due to the sign of matrix elements and the sufficient conditions (17a) and (17d) all the Routh-Hurwitz conditions are satisfied. Therefore, the proof is complete.

It is well known that, for each equilibrium point there is a specific basin of attraction and the point will be a globally asymptotically stable if and only if their basin of attraction is the total domain. Therefore, in the following theorems, the basin of attraction or the global stability conditions of each point is determined.

Theorem 4: Assume that the *SFEP* is LAS. Then it has a basin of attraction that satisfies the following conditions

$$\left(\frac{\alpha P - \gamma P_0 M}{PM}\right)^2 < 4\left(\frac{\mu + \gamma M}{P}\right)\left(\frac{\theta}{M}\right) \qquad (20a)$$

$$(\alpha + \beta P_0) < \mu \qquad (20b)$$

Proof: Consider the following positive definite Lyapunov function, which is defined for all $P > 0$ and $M > 0$ in the domain of system (2).

$$V_1 = \left(P - P_0 - P_0 \ln \frac{P}{P_0}\right) + S + Q + \left(M - M_0 - M_0 \ln \frac{M}{M_0}\right)$$

Clearly, by differentiating V_1 with respect to t along the solution curve of system (2), it's obtaining that:

$$\begin{aligned} V_1{}' = & -\left(\frac{\mu+\gamma M}{P}\right)(P - P_0)^2 + \left(\frac{\alpha P-\gamma P_0 M}{PM}\right)(P - P_0)(M - M_0) \\ & -\frac{\theta}{M}(M - M_0)^2 - [\mu + (1-\varepsilon)\delta]Q - \gamma(1-e)\gamma SM \\ & -[\mu - (\alpha + \beta P_0)]S - \frac{\alpha M_0}{M} S \end{aligned}$$

Therefore by using the above conditions, it's observed that

$$\begin{aligned} V_1{}' < & -\left[\sqrt{\frac{\mu+\gamma M}{P}}\,(P - P_0) - \sqrt{\frac{\theta}{M}}\,(M - M_0)\right]^2 - [\mu + (1-\varepsilon)\delta]Q - \gamma(1-e)\gamma SM \\ & -[\mu - (\alpha + \beta P_0)]S - \frac{\alpha M_0}{M} S \end{aligned}$$

Obviously, $V_1{}' = 0$ at $E_0 = (P_0, 0, 0, M_0)$, moreover $V_1{}' < 0$ otherwise. Hence $V_1{}'$ is negative definite and then the solution starting from any initial point satisfy the above conditions will approaches asymptotically to *SFEP*. Hence the proof is complete.

Theorem 5: Assume that the *FTQSEP* is LAS. Then it has a basin of attraction that satisfies the following conditions

$$\beta P_1 < \gamma M_1 + \mu \tag{21a}$$

$$\left(\frac{\sigma\gamma M}{S}\right)^2 < \left(\frac{\beta S_1+\mu P+\gamma M}{P}\right)\left(\frac{\gamma M_1+\mu-\beta P_1}{S}\right) \tag{21b}$$

$$\left(\frac{\alpha}{M} - \frac{\gamma P_1}{P}\right)^2 < \left(\frac{\beta S_1+\mu P+\gamma M}{P}\right)\left(\frac{\theta}{M}\right) \tag{21c}$$

$$\left(\frac{\alpha}{M} - \frac{\sigma\gamma P_1}{S} - \gamma\right)^2 < \left(\frac{\gamma M_1+\mu-\beta P_1}{S}\right)\left(\frac{\theta}{M}\right) \tag{21d}$$

Proof: Consider the following positive definite Lyapunov function, which is defined for all $P > 0, S > 0$ and $M > 0$ in the domain of system (2).

$$V_2 = \left(P - P_1 - P_1 \ln \frac{P}{P_1}\right) + \left(S - S_1 - S_1 \ln \frac{S}{S_1}\right) + Q + \left(M - M_1 - M_1 \ln \frac{M}{M_1}\right)$$

Clearly, by differentiating V_2 with respect to t along the solution curve of system (2), it's obtaining that:

$$V_2' = -\left(\frac{\beta S_1+\mu P+\gamma M}{2P}\right)(P-P_1)^2 - \left(\frac{\sigma\gamma M}{S}\right)(P-P_1)(S-S_1)$$
$$-\left(\frac{\gamma M_1+\mu-\beta P_1}{2S}\right)(S-S_1)^2 - \left(\frac{\beta S_1+\mu P+\gamma M}{2P}\right)(P-P_1)^2$$
$$+\left(\frac{\alpha}{M}-\frac{\gamma P_1}{P}\right)(P-P_1)(M-M_1) - \frac{\theta}{2M}(M-M_1)^2$$
$$-\left(\frac{\gamma M_1+\mu-\beta P_1}{2S}\right)(S-S_1)^2 + \left(\frac{\alpha}{M}-\frac{\gamma P_1}{S}-\gamma\right)(S-S_1)(M-M_1)$$
$$-\frac{\theta}{2M}(M-M_1)^2 - (\mu+(1-\varepsilon)\delta)Q - \frac{\varepsilon\delta S_1 Q}{S}$$

Therefore by using the above conditions, it's observed that

$$V_2' < -\left[\sqrt{\frac{x_{11}}{2}}\,(P-P_1) - \sqrt{\frac{x_{22}}{2}}\,(S-S_1)\right]^2 - \left[\sqrt{\frac{x_{11}}{2}}\,(P-P_1) - \sqrt{\frac{x_{44}}{2}}\,(M-M_1)\right]^2$$
$$-\left[\sqrt{\frac{x_{22}}{2}}\,(S-S_1) - \sqrt{\frac{x_{44}}{2}}\,(M-M_1)\right]^2 - (\mu+(1-\varepsilon)\delta)Q - \frac{\varepsilon\delta S_1 Q}{S}$$

where $x_{11} = \left(\frac{\beta S_1+\mu P+\gamma M}{P}\right)$; $x_{22} = \left(\frac{\gamma M_1+\mu-\beta P_1}{S}\right)$; $x_{44} = \frac{\theta}{2M}$.

Obviously, $V_2' = 0$ at $E_1 = (P_1, S_1, 0, M_1)$, moreover $V_2' < 0$ otherwise. Hence V_2' is negative definite and then the solution starting from any initial point satisfy the above conditions will approaches asymptotically to *FTQSEP*. Hence the proof is complete.

Furthermore, in the following theorem the conditions that specify the basin of attraction of *EEP* are established.

Theorem 6: Assume that the *EEP* is LAS. Then it has a basin of attraction that satisfies the following conditions

$$\beta P_2 < \mu + \gamma M \tag{22a}$$

$$(\sigma\gamma M + \beta S - \beta P_2)^2 < \frac{2}{3}(\beta S + \beta P_2 + \mu)(\gamma M + \mu - \beta P_2) \tag{22b}$$

$$(\alpha - \gamma P_2)^2 < \frac{2}{3}\theta(\beta S + \beta P_2 + \mu) \tag{22c}$$

$$(\varepsilon\delta + e\gamma M)^2 < \frac{2}{3}(\mu + \delta)(\gamma M + \mu - \beta P_2) \tag{22d}$$

$$(\alpha + \sigma\gamma P_2 - \gamma S_2)^2 < \frac{4}{9}\theta(\gamma M + \mu - \beta P_2) \tag{22e}$$

$$(e\gamma S_2)^2 < \frac{2}{3}\theta(\mu + \delta) \tag{22f}$$

Proof: Consider the following positive definite Lyapunov function

$$V_3 = \frac{(P-P_2)^2}{2} + \frac{(S-S_2)^2}{2} + \frac{(Q-Q_2)^2}{2} + \frac{(M-M_2)^2}{2}$$

Hence, by differentiating V_3 with respect to t along the solution curve of system (2), we get that

$$\begin{aligned} V_3{}' = & -\tfrac{(\beta S+\beta P_2+\mu)}{2}(P-P_2)^2 + (\sigma\gamma\, M + \beta S - \beta P_2)(P-P_2)(S-S_2) \\ & -\tfrac{(\gamma\, M+\mu-\beta P_2)}{3}(S-S_2)^2 - \tfrac{(\beta S+\beta P_2+\mu)}{2}(P-P_2)^2 \\ & +(\alpha-\gamma\, P_2)(P-P_2)(M-M_2) - \tfrac{\theta}{3}(M-M_2)^2 \\ & -\tfrac{(\gamma\, M+\mu-\beta P_2)}{3}(S-S_2)^2 + (\varepsilon\delta + e\gamma\, M)(S-S_2)(Q-Q_2) \\ & -\tfrac{(\mu+\delta)}{2}(Q-Q_2)^2 - \tfrac{(\gamma\, M+\mu-\beta P_2)}{3}(S-S_2)^2 \\ & +(\alpha+\sigma\gamma\, P_2 - \gamma\, S_2)(S-S_2)(M-M_2) - \tfrac{\theta}{3}(M-M_2)^2 \\ & -\tfrac{(\mu+\delta)}{2}(Q-Q_2)^2 + e\gamma\, S_2(Q-Q_2)(M-M_2) - \tfrac{\theta}{3}(M-M_2)^2 \end{aligned}$$

Therefore by using the above conditions, it's observed that

$$\begin{aligned} V_3{}' = & -\left[\sqrt{\tfrac{q_{11}}{2}}(P-P_2) - \sqrt{\tfrac{q_{22}}{3}}(S-S_2)\right]^2 - \left[\sqrt{\tfrac{q_{11}}{2}}(P-P_2) - \sqrt{\tfrac{q_{44}}{3}}(M-M_2)\right]^2 \\ & -\left[\sqrt{\tfrac{q_{22}}{3}}(S-S_2) - \sqrt{\tfrac{q_{33}}{2}}(Q-Q_2)\right]^2 - \left[\sqrt{\tfrac{q_{22}}{3}}(S-S_2) - \sqrt{\tfrac{q_{44}}{3}}(M-M_2)\right]^2 \\ & -\left[\sqrt{\tfrac{q_{33}}{2}}(Q-Q_2) - \sqrt{\tfrac{q_{44}}{3}}(M-M_2)\right]^2 \end{aligned}$$

here $q_{11} = \beta S + \beta P_2 + \mu;\; q_{22} = \gamma\, M + \mu - \beta P_2;\; q_{33} = \mu + \delta;\; q_{44} = \theta.$

Obviously, $V_3{}' = 0$ at $E_2 = (P_2, S_2, Q_2, M_2)$, moreover $V_3{}' < 0$ otherwise. Hence $V_3{}'$ is negative definite and then the solution starting from any initial point satisfy the above conditions will approaches asymptotically to *EEP*. Hence the proof is complete.

5. Smoking model with diffusion

Obviously, system (1) does not consider the structure of smokers spreading and hence it is not suitable to understand the transmission of smoking in case of moving the individuals. Therefore, it is important to consider the diffusion terms in the model structure in order to investigate whether and how spatial heterogeneity can affect the smoking transmission dynamics. Consequently, the smoking model with diffusion is considered in this section, which is extended to the smoking model given in Eq. (1). Let Ω is a bounded domain in $\mathbb{R}_+^5$ with smooth boundary $\partial\Omega$ and η is the outward unit normal vector on the boundary, then the smoking model with diffusion can be written as:

$$\frac{\partial P}{\partial t} = \psi - \beta PS - \mu P - \gamma PM + D_1 \Delta P \qquad (23a)$$

$$\frac{\partial S}{\partial t} = \beta PS + \sigma\gamma PM - \mu S - \gamma SM + \varepsilon\delta Q + D_2\Delta S \tag{23b}$$

$$\frac{\partial Q}{\partial t} = e\gamma SM - \mu Q - \delta Q + D_3\Delta Q \tag{23c}$$

$$\frac{\partial R}{\partial t} = \gamma(1-\sigma)PM + \gamma(1-e)SM - \mu R + \delta(1-\varepsilon)Q + D_4\Delta R \tag{23d}$$

$$\frac{\partial M}{\partial t} = \alpha(S+P) - \theta M + D_5\Delta M \tag{23e}$$

with homogeneous Neumaun boundary condition

$$\frac{\partial P}{\partial \eta} = \frac{\partial S}{\partial \eta} = \frac{\partial Q}{\partial \eta} = \frac{\partial R}{\partial \eta} = \frac{\partial M}{\partial \eta} = 0, \quad x \in \partial\Omega, \ t > 0 \tag{24}$$

and initial conditions

$$\begin{aligned} &P(x,0) = P_0(x) \geq 0, \ S(x,0) = S_0(x) \geq 0, \ Q(x,0) = Q_0(x) \geq 0 \\ &R(x,0) = R_0(x) \geq 0, \ M(x,0) = M_0(x) \geq 0 \quad x \in \overline{\Omega} \end{aligned} \tag{25}$$

where $P(x,t), S(x,t), Q(x,t), R(x,t)$ and $M(x,t)$, denoted the numbers of potential smokers, smokers, temporary quit smoking, recovery and media at location *x* and time *t*. All parameters in system (23) have same meaning as those in system (1). However, the parameters $D_i \geq 0, i = 1,2,3,4,5$ are the diffusion coefficients of population respectively; while, Δ is Laplacian operator.

Similarly as in system (1), we can reduce system (23), by removing Eq. (23d) (recovery equation) from it, since the other equations in this system are independent of the recovery equation and hence system (23) becomes

$$\begin{aligned} \frac{\partial P}{\partial t} &= \psi - \beta PS - \mu P - \gamma PM + D_1\Delta P \\ \frac{\partial S}{\partial t} &= \beta PS + \sigma\gamma PM - \mu S - \gamma SM + \varepsilon\delta Q + D_2\Delta S \\ \frac{\partial Q}{\partial t} &= e\gamma SM - \mu Q - \delta Q + D_3\Delta Q \\ \frac{\partial M}{\partial t} &= \alpha(S+P) - \theta M + D_5\Delta M \end{aligned} \tag{26}$$

So that R can be determined from

$$R(x,t) = N - \left[P(x,t) + S(x,t) + Q(x,t)\right], \quad x \in \Omega, \ t > 0 \tag{27}$$

As the initial values are positive and the growth functions in the interaction functions of system (26) are assumed to be sufficiently smooth in $\mathbb{R}_+^4$ then standard partial differential equations theory shows that the solution of (26) is unique and continuous for all the positive time in Ω. Furthermore, we recall the positivity lemma in order to using it to proof the positivity and the uniformly bounded of the solution of (26).

Lemma 7 [17]: Suppose $K \in C(\overline{\Omega}\times[0,\tau]) \cap C^{2,1}(\Omega\times(0,\tau])$ and satisfies

$$K_t - D\Delta K \geq c(z,t)K\,,\ z\in\Omega,\ 0<t\leq\tau,$$

$$\frac{\partial K}{\partial \eta} \geq 0,\ z\in\partial\Omega,\ 0<t\leq\tau, \tag{28}$$

$$K(z,0)\geq 0,\quad z\in\overline{\Omega}$$

where $c(z,t)\in C(\overline{\Omega}\times[0,\tau])$. Then $K(z,t)\geq 0$ on $\overline{\Omega}\times[0,\tau]$. Moreover, $K(z,t)>0$ or $K\equiv 0$ in $\Omega\times[0,\tau]$.

Hence, according to lemma (7), we have the following theorem.

Theorem 8: Any solution of system (26) with a positive initial condition is positive.

Proof: Assume that (*P*, *S*, *Q*, *M*) be a solution of system (26) in $\Omega\times[0,T_{\max})$. Then for any τ with $0<\tau<T_{\max}$, we get from 1st equation of system (26) that:

$$P_t - D_1\Delta P \geq -(\beta S+\mu+\gamma M)P\,,\quad x\in\Omega, 0<t\leq\tau$$

Since $-(\beta S+\mu+\gamma M)$ is bounded due to the boundedness of the population in $\Omega\times[0,\tau]$, then by using the lemma (7) we obtain $P>0$ in $\Omega\times(0,\tau]$. By the same way we have $S>0$ in $\Omega\times(0,\tau]$ since that

$$S_t - D_2\Delta S \geq -(\mu+\gamma M)S\,,\quad x\in\Omega,\ 0<t\leq\tau\,,$$

Similarly, we have $Q>0$, due to the following

$$Q_t - D_3\Delta Q \geq -(\mu+\delta)Q\,,\ x\in\Omega,\ 0<t\leq\tau\,,$$

Again we applied the same lemma on last equation of system (26) we obtain that

$$M_t - D_5\Delta M \geq \theta M\,,\quad x\in\Omega,\ 0<t\leq\tau\,,$$

Hence, $M>0$. Now, since τ is arbitrary in $(0,T_{max})$, we obtain that $P>0$, $S>0$, $Q>0$ and $M>0$ in $\Omega\times[0,T_{\max})$.

Now, we show the bounded-ness of solution of system (26) and investigate that in following theorem

Theorem 8: Let $(P,S,Q) \in [C(\bar{\Omega} \times [0,T_{max})) \cap C^{2,1}(\Omega \times (0,T_{max}))]^3$ be the solution of system (26) with non-negative non-trivial initial values. Then $T = \infty$ and $P(x,t)+S(x,t)+Q(x,t) \le \max\{N, \|P_0(x)+S_0(x)+Q_0(x)\|_\infty\}$, where $N = \frac{\psi}{\mu}$.

Proof: We show that $P(x,t), S(x,t)$ and $Q(x,t)$ are bounded by $\Omega \times [0,T_{\max})$. Since

$$0 < P(x,0)+S(x,0)+Q(x,0) \le \|P_0(x)+S_0(x)+Q_0(x)\|_\infty$$

and

$$(P+S+Q)_t - D\Delta(P+S+Q) \le \psi - \mu(P+S+Q)$$

with $D = \max\{D_1, D_2, D_3\}$, then for $t \in [0,\infty)$, we have that

$$P(x,t)+S(x,t)+Q(x,t) \le \left[\frac{\psi}{\mu} + \left(\|P_0(x)+S_0(x)+Q_0(x)\|_\infty - \frac{\psi}{\mu}\right)e^{-\mu t}\right]$$

is the solution of the inequalities

$$\frac{dL(t)}{dt} = \psi - \mu L(t);\ L(0) = \|P_0(x)+S_0(x)+Q_0(x)\|_\infty$$

Such that, $L = (P+S+Q)$, hence, we have

$$0 < L(t) \le \max\left\{\frac{\psi}{\mu}, \|P_0(x)+S_0(x)+Q_0(x)\|\right\}, \text{ for } t \in [0,\infty) \text{ and thus,}$$

$$P(x,t)+S(x,t)+Q(x,t) \le L(x) \le \max\left\{\frac{\psi}{\mu}, \|P_0(x)+S_0(x)+Q_0(x)\|\right\}$$

As well, by the same way we have shown that the media equation is bounded by $\Omega \times [0,T_{\max})$. Such that, $M(x,0) \le \|M_0(x)\|_\infty$, then

$$M_t - D_5\Delta M \le \alpha(P+S) - \theta M$$

We have

$$M(x,t) \le \frac{\alpha(P+S)}{\theta} + \left(\|M_0(X)\|_\infty - \frac{\alpha(P+S)}{\theta}\right)e^{-\mu t}$$

If $t \to \infty$, we get

$$M(x,t) \le \max\left\{\frac{\alpha\psi}{\theta\mu}, \|M_0(X)\|_\infty\right\}$$

Thus the proof is complete.

6. Stability analysis of system (26)

In this section, the local and global stabilities of the equilibrium points of diffusion system (26) are discussed. It is easy to verify that the equilibrium points of diffusion system (26) and those of system (2) are the same. Then the stability analysis for each of them can be study as in the following theorems

Theorem 9: The *SFEP* of diffusion system (26) is LAS if the following sufficient conditions hold

$$\beta P_0 < \mu + \gamma M_0 + kD_2 \quad (29a)$$

$$(\mu + \delta + kD_3)\beta P_0 + e\gamma\varepsilon\delta M_0 < (\mu + \delta + kD_3)(\mu + \gamma M_0 + kD_2) \quad (29b)$$

Proof: The Jacobian matrix of system (26) at the *SFEP* is given by

$$J(E_0) = \begin{pmatrix} b_{11} - kD_1 & b_{12} & 0 & b_{14} \\ 0 & b_{22} - kD_2 & b_{23} & 0 \\ 0 & b_{32} & b_{33} - kD_3 & 0 \\ b_{41} & b_{42} & 0 & b_{44} - kD_5 \end{pmatrix} \quad (30a)$$

where $b_{ij}; i,j = 1,2,3,4$ are given by Eq. (12). Then the characteristic equation can be written as

$$\lambda^4 + \tilde{B}_1\lambda^3 + \tilde{B}_2\lambda^2 + \tilde{B}_3\lambda + \tilde{B}_4 = 0 \quad (30b)$$

Such that

$$\tilde{B}_1 = -[\tilde{b}_{11} + \tilde{b}_{22} + \tilde{b}_{33} + \tilde{b}_{44}]$$

$$\tilde{B}_2 = \tilde{b}_{11}(\tilde{b}_{22} + \tilde{b}_{33}) + \tilde{b}_{11}\tilde{b}_{44} - b_{14}b_{41} + \tilde{b}_{22}\tilde{b}_{33} - b_{23}b_{32} + \tilde{b}_{44}(\tilde{b}_{22} + \tilde{b}_{33})$$

$$\tilde{B}_3 = -[(\tilde{b}_{11} + \tilde{b}_{44})(\tilde{b}_{22}\tilde{b}_{33} - b_{23}b_{32}) + (\tilde{b}_{22} + \tilde{b}_{33})(\tilde{b}_{11}\tilde{b}_{44} - b_{14}b_{41})]$$

$$\tilde{B}_4 = (\tilde{b}_{22}\tilde{b}_{33} - b_{23}b_{32})(\tilde{b}_{11}\tilde{b}_{44} - b_{14}b_{41})$$

As well

$$\tilde{B}_1\tilde{B}_2-\tilde{B}_3=-\left(\tilde{b}_{11}+\tilde{b}_{44}\right)\left(\tilde{b}_{22}+\tilde{b}_{33}\right)\left(\tilde{b}_{11}+\tilde{b}_{22}\right)$$
$$-\left(\tilde{b}_{11}+\tilde{b}_{44}\right)\left(\tilde{b}_{22}+\tilde{b}_{33}\right)\left(\tilde{b}_{33}+\tilde{b}_{44}\right)$$
$$-\left(\tilde{b}_{11}+\tilde{b}_{44}\right)\left(\tilde{b}_{11}\tilde{b}_{44}-b_{14}b_{41}\right)$$
$$-\left(\tilde{b}_{22}+\tilde{b}_{33}\right)\left(\tilde{b}_{22}\tilde{b}_{33}-b_{23}b_{32}\right)$$

while $\Delta=\tilde{B}_3(\tilde{B}_1\tilde{B}_2-\tilde{B}_3)-\tilde{B}_1{}^2\tilde{B}_4$ can be written as

$$\Delta=-\tilde{B}_1\left(\tilde{b}_{11}+\tilde{b}_{44}\right)^2\left(\tilde{b}_{22}+\tilde{b}_{33}\right)\left(\tilde{b}_{22}\tilde{b}_{33}-b_{23}b_{32}\right)$$
$$-\tilde{B}_1\left(\tilde{b}_{11}+\tilde{b}_{44}\right)\left(\tilde{b}_{22}+\tilde{b}_{33}\right)^2\left(\tilde{b}_{11}\tilde{b}_{44}-b_{14}b_{41}\right)$$
$$+(\tilde{b}_{11}+\tilde{b}_{44})(\tilde{b}_{22}+\tilde{b}_{33})\left[\left(\tilde{b}_{22}\tilde{b}_{33}-b_{23}b_{32}\right)-(\tilde{b}_{11}\tilde{b}_{44}-b_{14}b_{41})\right]^2$$

where

$$\tilde{b}_{11}=-(\mu+\gamma M_0+kD_1)\ ;\quad \tilde{b}_{22}=(\beta P_0-\mu-\gamma M_0-kD_2)$$

$$\tilde{b}_{33}=-(\mu+\delta+kD_3)\ ;\quad \tilde{b}_{44}=-(\theta+kD_5)$$

Note that, all the Routh-Hurwitz conditions that guarantee the LAS of the *SFEP* of system (26) are satisfied provided that the conditions (29a)-(29b) hold.

Theorem 10: The *FTQSEP* of diffusion system (26) is LAS if in addition to condition (14b) the following sufficient conditions hold

$$\beta P_1<(\mu+\gamma M_1+kD_2) \tag{31a}$$

$$\alpha\gamma S_1<\alpha\sigma\gamma P_1+(\theta+kD_5)\beta S_1+(\theta+kD_5)\sigma\gamma M_1 \tag{31b}$$

$$\alpha\sigma\gamma P_1[\beta P_1+\gamma M_1]<2(\theta+kD_5)(\beta S_1+\mu+\gamma M_1+kD_1)\times[(\mu+\gamma M_1+kD_2)-\beta P_1] \tag{31c}$$

Proof: The Jacobian matrix of system (26) at *FTQSEP* can be written:

$$J(E_1)=\begin{pmatrix} c_{11}-kD_1 & c_{12} & 0 & c_{14} \\ c_{21} & c_{22}-kD_2 & c_{23} & c_{24} \\ 0 & 0 & c_{33}-kD_3 & 0 \\ c_{41} & c_{42} & 0 & c_{44}-kD_5 \end{pmatrix} \tag{32a}$$

where $c_{ij}; i,j=1,2,3,4$ are given in Eq. (15). Hence, the characteristic equation can be written as

$$(\hat{c}_{33}-\lambda)(\lambda^3+\hat{C}_1\lambda^2+\hat{C}_2\lambda+\hat{C}_4)=0 \tag{32b}$$

here the eigenvalue in the Q −direction is given by $\lambda_Q=-(\mu+\delta+kD_3)<0$, while the other three eigenvalues are the roots of the third degree polynomial, where

$$\hat{C}_1=-[\hat{c}_{11}+\hat{c}_{22}+\hat{c}_{44}]$$

$$\hat{C}_2=\hat{c}_{11}\hat{c}_{22}-c_{12}c_{21}+\hat{c}_{11}\hat{c}_{44}-c_{14}c_{41}+\hat{c}_{22}\hat{c}_{44}-c_{24}c_{42}$$

$$\hat{C}_3 = -[\hat{c}_{11}(\hat{c}_{22}\hat{c}_{44} - c_{24}c_{42}) + c_{12}(c_{24}c_{41} - c_{21}\hat{c}_{44}) + c_{14}(c_{21}c_{42} - \hat{c}_{22}c_{41})]$$

with

$$\hat{C}_1\hat{C}_2 - \hat{C}_3 = -(\hat{c}_{11} + \hat{c}_{22})[\hat{c}_{11}\hat{c}_{22} - c_{12}c_{21}] - (\hat{c}_{11} + \hat{c}_{44})[\hat{c}_{11}\hat{c}_{44} - c_{14}c_{41}]$$
$$-(\hat{c}_{22} + \hat{c}_{44})[\hat{c}_{22}\hat{c}_{44} - c_{24}c_{42}] - 2\hat{c}_{11}\hat{c}_{22}\hat{c}_{44} + c_{12}c_{24}c_{41} + c_{14}c_{21}c_{42}$$

here

$$\hat{c}_{11} = -(\beta S_1 + \mu + \gamma M_1 + kD_1) \ ; \quad \hat{c}_{22} = (\beta P_1 - \mu - \gamma M_1 - kD_2)$$

$$\hat{c}_{33} = -(\mu + \delta + kD_3) \ ; \quad \hat{c}_{44} = -(\theta + kD_5)$$

Note that, it is easy to verify that all the Routh-Hurwitz conditions that guarantee the LAS of the *FTQSEP* of system (26) are satisfied provided that the conditions (31a)-(31c) and (14b) hold.

Theorem 11: The *EEP* of diffusion system (26) is LAS if in addition to condition (17b) the following sufficient conditions hold

$$\beta P_2 < \mu + \gamma M_2 + kD_2 \tag{33a}$$

$$\beta(\mu + \delta + kD_3)P_2 + \varepsilon\delta e\gamma M_2 < (\mu + \gamma M_2 + kD_2)(\mu + \delta + kD_3) \tag{33b}$$

$$P_2(\beta S_2 + \sigma\gamma M_2) + \varepsilon e\delta S_2 < \gamma P_2(\beta S_2 + \mu + \gamma M_2 + kD_1 + \theta + kD_5) \tag{33c}$$

Proof: The Jacobian matrix of system (26) at *EEP* can be written:

$$J(E_2) = \begin{pmatrix} z_{11} - kD_1 & z_{12} & 0 & z_{14} \\ z_{21} & z_{22} - kD_2 & z_{23} & z_{24} \\ 0 & z_{32} & z_{33} - kD_3 & z_{34} \\ z_{41} & z_{42} & 0 & z_{44} - kD_5 \end{pmatrix} \tag{34a}$$

where $z_{ij}; i,j = 1,2,3,4$ are given in Eq. (18).So the characteristic equation can be written as

$$\lambda^4 + \hat{Z}_1\lambda^3 + \hat{Z}_2\lambda^2 + \hat{Z}_3\lambda + \hat{Z}_4 = 0 \tag{34b}$$

where

$$\hat{Z}_1 = -[\hat{z}_{11} + \hat{z}_{22} + \hat{z}_{33} + \hat{z}_{44}]$$

$$\hat{Z}_2 = \hat{z}_{11}\hat{z}_{22} - z_{12}z_{21} + \hat{z}_{11}\hat{z}_{33} + \hat{z}_{11}\hat{z}_{44} - z_{14}z_{41} + \hat{z}_{22}\hat{z}_{33} - z_{23}z_{32}$$
$$+\hat{z}_{22}\hat{z}_{44} - z_{24}z_{42} + \hat{z}_{33}\hat{z}_{44}$$

$$\hat{Z}_3 = -[(\hat{z}_{11} + \hat{z}_{44})(\hat{z}_{22}\hat{z}_{33} - z_{23}z_{32}) + (\hat{z}_{22} + \hat{z}_{33})(\hat{z}_{11}\hat{z}_{44} - z_{14}z_{41})$$
$$-z_{12}z_{21}(\hat{z}_{33} + \hat{z}_{44}) - \hat{z}_{33}z_{24}z_{42} - z_{24}(\hat{z}_{11}z_{42} - z_{12}z_{41})$$
$$+z_{42}(z_{14}z_{21} - z_{23}z_{34})]$$

$$\hat{Z}_4 = (\hat{z}_{11}\hat{z}_{44} - z_{14}z_{41})(\hat{z}_{22}\hat{z}_{33} - z_{23}z_{32}) + (\hat{z}_{11}z_{42} - z_{12}z_{41})(z_{23}z_{34} - z_{24}\hat{z}_{33})$$
$$-z_{21}\hat{z}_{33}(z_{12}\hat{z}_{44} - z_{14}z_{42})$$

Moreover, we have that

$$\hat{Z}_1\hat{Z}_2 - \hat{Z}_3 = -(\hat{z}_{11} + \hat{z}_{22})(\hat{z}_{11}\hat{z}_{22} - z_{12}z_{21}) - \hat{z}_{11}\hat{z}_{33}(\hat{z}_{11} + 2\hat{z}_{22} + \hat{z}_{33})$$
$$-\hat{z}_{44}(\hat{z}_{11} + \hat{z}_{22} + \hat{z}_{33} + \hat{z}_{44})(\hat{z}_{11} + \hat{z}_{22} + \hat{z}_{33}) + z_{12}z_{24}z_{41}$$
$$-(\hat{z}_{22} + \hat{z}_{33})(\hat{z}_{22}\hat{z}_{33} - z_{23}z_{32}) + z_{14}z_{41}(\hat{z}_{11} + \hat{z}_{44})$$
$$+z_{24}z_{41}(\hat{z}_{22} + \hat{z}_{44}) + z_{42}(z_{14}z_{21} - z_{23}z_{34})$$

and $\Delta = \hat{Z}_3(\hat{Z}_1\hat{Z}_2 - \hat{Z}_3) - \hat{Z}_1{}^2\hat{Z}_4$ can be written as:

$$\Delta = \hat{X}_1(\hat{X}_2 + \hat{X}_1) + \hat{X}_3$$

here

$$\hat{X}_1 = (\hat{z}_{11} + \hat{z}_{44})(\hat{z}_{22}\hat{z}_{33} - z_{23}z_{32}) + (\hat{z}_{22} + \hat{z}_{33})(\hat{z}_{11}\hat{z}_{44} - z_{14}z_{41})$$
$$-z_{12}z_{21}(\hat{z}_{33} + \hat{z}_{44}) + z_{24}z_{42}(\hat{z}_{11} + \hat{z}_{33}) + z_{12}z_{24}z_{41} + z_{42}(z_{14}z_{21} - z_{23}z_{34})$$

$$\hat{X}_2 = (\hat{z}_{11} + \hat{z}_{22} + \hat{z}_{33} + \hat{z}_{44})[\hat{z}_{11}(\hat{z}_{22} + \hat{z}_{33} + \hat{z}_{44}) - z_{12}z_{21} - z_{14}z_{41}$$
$$+\hat{z}_{44}(\hat{z}_{22} + \hat{z}_{33}) - z_{24}z_{42} + \hat{z}_{22}\hat{z}_{33} - z_{23}z_{32}]$$

$$\hat{X}_3 = (\hat{z}_{11} + \hat{z}_{22} + \hat{z}_{33} + \hat{z}_{44})^2[(\hat{z}_{11}\hat{z}_{44} - z_{14}z_{41})(\hat{z}_{22}\hat{z}_{33} - z_{23}z_{32})$$
$$+(\hat{z}_{11}z_{42} - z_{12}z_{41})(z_{23}z_{34} - \hat{z}_{33}z_{24}) - \hat{z}_{33}z_{21}(\hat{z}_{44}z_{12} - z_{14}z_{42})]$$

Such that

$$\hat{z}_{11} = -(\beta S_2 + \mu + \gamma M_2 + kD_1) \quad ; \quad \hat{z}_{22} = (\beta P_2 - \mu - \gamma M_2 - kD_2)$$

$$\hat{z}_{33} = -(\mu + \delta + kD_3) \quad ; \quad \hat{z}_{44} = -(\theta + kD_5)$$

Again by using Routh-Hurwitz criterion, we get that the *EEP* is LAS if the sufficient conditions (33a)-(33c) with (17b) hold.

Note that, according to the above theorems it's clear that, the equilibrium points of diffusion system (26) are always LAS if they are stable in system (2), that is mean without diffusion, but the converse is not necessarily true.

Next, in following theorems the globally asymptotically stability (GAS) of diffusion system (26) at *SFEP*, *FTQSEP* and *EEP* is carried out using the method described in [19].

Theorem 12: Assume that the *SFEP* of the diffusion system (26) is LAS, then it is GAS if the conditions (20a)-(20b) hold

Proof: Consider the following candidate Lyapunov function with $u(x,t)$ is a solution of diffusion system (26)

$$W_1 = \int_{\Omega} V\big(u(x,t)\big)dx \tag{35}$$

where $V(u)$ is a continuously differentiable function defined on some $\mathbb{R}_+^4$. Then the time derivative of W_1 along the positive solution of system (26) is written as

$$\frac{dW_1}{dt} = \int_{\Omega} \nabla V(u)\cdot(f(u) + D\Delta u)dx$$

where $f(u)$ is the vector field that given in right hand side of system (26) without diffusion, while $D\Delta u$ is the diffusion term with $D = (D_1, D_2, D_3, D_5)$ and $D_i \geq 0$. Therefore, we obtain that

$$\frac{dW_1}{dt} = \int_{\Omega} \nabla V(u)\cdot f(u)\,dx + \int_{\Omega} \nabla V(u)\cdot D\Delta u\,dx$$

which gives

$$\frac{dW_1}{dt} = \int_{\Omega} \nabla V(u)\cdot f(u)\,dx + \sum_{i=1}^{4} D_i \int_{\Omega} \frac{\partial V}{\partial u_i}\,\Delta u_i\,dx \tag{36}$$

Assume that, the integrand of the first term in Eq. (36) is already calculated as that for the system (2) given by theorem (4). However, the second term is simplified by using Green's formula, and we obtain

$$\int_{\Omega} \frac{\partial V}{\partial u_i}\,\Delta u_i\,dx = \int_{\partial\Omega} \frac{\partial V}{\partial u_i}\,\frac{\partial u_i}{\partial \eta}\,d\upsilon - \int_{\Omega} \nabla u_i\,.\nabla\left(\frac{\partial V}{\partial u_i}\right)dx \tag{37}$$

Since $\frac{\partial u}{\partial \eta} = 0$ on $\partial\Omega$. Therefore, Eq. (37) becomes

$$\int_{\Omega} \frac{\partial V}{\partial u_i}\,\Delta u_i\,dx = -\int_{\Omega} \nabla u_i\,.\nabla\left(\frac{\partial V}{\partial u_i}\right)dx \tag{38}$$

Accordingly, by using Eq. (38) in Eq. (36), it's obtain that

$$\frac{dW_1}{dt} = \int_{\Omega} \nabla V(u)\cdot f(u)\,dx - \sum_{i=1}^{4} D_i \int_{\Omega} \nabla u_i\,.\nabla\left(\frac{\partial V}{\partial u_i}\right)dx \tag{39}$$

Therefore, in order to construct the function V we should have

$$D_i \int_{\Omega} \nabla u_i\,.\nabla\left(\frac{\partial V}{\partial u_i}\right)dx \geq 0\text{, for all } i = 1,2,3,4. \tag{40}$$

Now by using the function $V \equiv V_1$, that given in theorem (4)

$$V = \left(P - P_0 - P_0 \ln \frac{P}{P_0}\right) + S + Q + \left(M - M_0 - M_0 \ln \frac{M}{M_0}\right)$$

Hence, in this case we have that

$$\int_\Omega \nabla u_i . \nabla\left(\frac{\partial V}{\partial u_i}\right) dx = \int_\Omega \left[P_0 \frac{|\nabla P|^2}{P^2} + M_0 \frac{|\nabla M|^2}{M^2}\right] \geq 0$$

Consequently, we obtain that

$$\begin{aligned}\frac{dW_1}{dt} = &-\left(\frac{\mu+\gamma M}{P}\right)(P - P_0)^2 + \left(\frac{\alpha P - \gamma P_0 M}{PM}\right)(P - P_0)(M - M_0) \\ &-\frac{\theta}{M}(M - M_0)^2 - [\mu + (1-\varepsilon)\delta]Q - \gamma(1-e)\gamma SM \\ &-[\mu - (\alpha + \beta P_0)]S - \frac{\alpha M_0}{M} S - D \int_\Omega \left[P_0 \frac{|\nabla P|^2}{P^2} + M_0 \frac{|\nabla M|^2}{M^2}\right] dx\end{aligned}$$

where $D = \min\{D_1, D_5\}$. Therefore by using the conditions (20a)-(20b), it's observed that

$$\begin{aligned}\frac{dW_1}{dt} < &-\left[\sqrt{\frac{\mu+\gamma M}{P}}\,(P - P_0) - \sqrt{\frac{\theta}{M}}\,(M - M_0)\right]^2 - [\mu + (1-\varepsilon)\delta]Q \\ &-\gamma(1-e)\gamma SM - [\mu - (\alpha + \beta P_0)]S - \frac{\alpha M_0}{M} S \\ &-D \int_\Omega \left[P_0 \frac{|\nabla P|^2}{P^2} + M_0 \frac{|\nabla M|^2}{M^2}\right] dx\end{aligned}$$

Obviously, $W_1{}' = 0$ at $E_0 = (P_0, 0,0, M_0)$, moreover $W_1{}' < 0$ otherwise. Hence $W_1{}'$ is negative definite and then the solution starting from any initial point satisfy the above conditions will approaches asymptotically to *SFEP*. Hence the proof is complete.

Theorem 13: Assume that the *FTQSEP* of the diffusion system (26) is LAS, then it is GAS if the conditions (21a)-(21d) hold

Proof: Similarly as in proof of theorem (12), we consider the following candidate Lyapunov function with $u(x,t)$ be a solution of diffusion system (26).

$$W_2 = \int_\Omega V_2\big(u(x,t)\big)dx \qquad (41)$$

with the function V_2 that given in theorem (5). Therefore, direct computation gives that

$$W_2{}' = -\left(\frac{\beta S_1+\mu P+\gamma M}{2P}\right)(P-P_1)^2 - \left(\frac{\sigma\gamma M}{S}\right)(P-P_1)(S-S_1)$$
$$-\left(\frac{\gamma M_1+\mu-\beta P_1}{2S}\right)(S-S_1)^2 - \left(\frac{\beta S_1+\mu P+\gamma M}{2P}\right)(P-P_1)^2$$
$$+\left(\frac{\alpha}{M}-\frac{\gamma P_1}{P}\right)(P-P_1)(M-M_1) - \frac{\theta}{2M}(M-M_1)^2$$
$$-\left(\frac{\gamma M_1+\mu-\beta P_1}{2S}\right)(S-S_1)^2 + \left(\frac{\alpha}{M}-\frac{\gamma P_1}{S}-\gamma\right)(S-S_1)(M-M_1)$$
$$-\frac{\theta}{2M}(M-M_1)^2 - (\mu+(1-\varepsilon)\delta)Q - \frac{\varepsilon\delta S_1 Q}{S}$$
$$-D\int_\Omega\left[P_1\frac{|\nabla P|^2}{P^2}+S_1\frac{|\nabla S|^2}{S^2}+M_1\frac{|\nabla M|^2}{M^2}\right]dx$$

where $D=\min\{D_1,D_2,D_5\}$. Therefore by using the conditions (21a)-(21d), it's observed that

$$W_2{}' < -\left[\sqrt{\frac{x_{11}}{2}}\,(P-P_1)-\sqrt{\frac{x_{22}}{2}}\,(S-S_1)\right]^2 - \left[\sqrt{\frac{x_{11}}{2}}\,(P-P_1)-\sqrt{\frac{x_{44}}{2}}\,(M-M_1)\right]^2$$
$$-\left[\sqrt{\frac{x_{22}}{2}}\,(S-S_1)-\sqrt{\frac{x_{44}}{2}}\,(M-M_1)\right]^2 - (\mu+(1-\varepsilon)\delta)Q - \frac{\varepsilon\delta S_1 Q}{S}$$
$$-D\int_\Omega\left[P_1\frac{|\nabla P|^2}{P^2}+S_1\frac{|\nabla S|^2}{S^2}+M_1\frac{|\nabla M|^2}{M^2}\right]dx$$

where x_{11}, x_{22} and x_{44} are given theorem (5). Obviously, $W_2{}'=0$ at $E_1=(P_1,S_1,0,M_1)$, moreover $W_2{}'<0$ otherwise. Hence $W_2{}'$ is negative definite and then the solution starting from any initial point satisfy the above conditions will approaches asymptotically to *FTQSEP*. Hence the proof is complete.

Theorem 14: Assume that the *EEP* of the diffusion system (26) is LAS, then it is GAS if the conditions (22a)-(22f) hold

Proof: Consider the following candidate Lyapunov function with $u(x,t)$ be a solution of diffusion system (26).

$$W_3=\int_\Omega V_3\big(u(x,t)\big)dx \tag{42}$$

with the function V_3 that given in theorem (6). Therefore, direct computation gives that

$$
\begin{aligned}
W_3{}' = & -\frac{(\beta S+\beta P_2+\mu)}{2}(P-P_2)^2+(\sigma\gamma\, M+\beta S-\beta P_2)(P-P_2)(S-S_2) \\
& -\frac{(\gamma\, M+\mu-\beta P_2)}{3}(S-S_2)^2-\frac{(\beta S+\beta P_2+\mu)}{2}(P-P_2)^2 \\
& +(\alpha-\gamma\, P_2)(P-P_2)(M-M_2)-\frac{\theta}{3}(M-M_2)^2 \\
& -\frac{(\gamma\, M+\mu-\beta P_2)}{3}(S-S_2)^2+(\varepsilon\delta+e\gamma\, M)(S-S_2)(Q-Q_2) \\
& -\frac{(\mu+\delta)}{2}(Q-Q_2)^2-\frac{(\gamma\, M+\mu-\beta P_2)}{3}(S-S_2)^2 \\
& +(\alpha+\sigma\gamma\, P_2-\gamma\, S_2)(S-S_2)(M-M_2)-\frac{\theta}{3}(M-M_2)^2 \\
& -\frac{(\mu+\delta)}{2}(Q-Q_2)^2+e\gamma\, S_2(Q-Q_2)(M-M_2)-\frac{\theta}{3}(M-M_2)^2 \\
& -D\int_\Omega\left[P_2\frac{|\nabla P|^2}{P^2}+S_2\frac{|\nabla S|^2}{S^2}+Q_2\frac{|\nabla Q|^2}{Q^2}+M_2\frac{|\nabla M|^2}{M^2}\right]dx
\end{aligned}
$$

where $D=\min\{D_1,D_2,D_3,D_5\}$. Therefore by using the conditions (22a)-(22f), it's observed that

$$
\begin{aligned}
W_3{}' = & -\left[\sqrt{\tfrac{q_{11}}{2}}(P-P_2)-\sqrt{\tfrac{q_{22}}{3}}(S-S_2)\right]^2-\left[\sqrt{\tfrac{q_{11}}{2}}(P-P_2)-\sqrt{\tfrac{q_{44}}{3}}(M-M_2)\right]^2 \\
& -\left[\sqrt{\tfrac{q_{22}}{3}}(S-S_2)-\sqrt{\tfrac{q_{33}}{2}}(Q-Q_2)\right]^2-\left[\sqrt{\tfrac{q_{22}}{3}}(S-S_2)-\sqrt{\tfrac{q_{44}}{3}}(M-M_2)\right]^2 \\
& -\left[\sqrt{\tfrac{q_{33}}{2}}(Q-Q_2)-\sqrt{\tfrac{q_{44}}{3}}(M-M_2)\right]^2 \\
& -D\int_\Omega\left[P_2\tfrac{|\nabla P|^2}{P^2}+S_2\tfrac{|\nabla S|^2}{S^2}+Q_2\tfrac{|\nabla Q|^2}{Q^2}+M_2\tfrac{|\nabla M|^2}{M^2}\right]dx
\end{aligned}
$$

here q_{11}; q_{22}; q_{33}; and q_{44} are given in theorem (6). Obviously, $W_3{}'=0$ at $E_2=(P_2,S_2,Q_2,M_2)$, moreover $W_3{}'<0$ otherwise. Hence $W_3{}'$ is negative definite and then the solution starting from any initial point satisfy the above conditions will approaches asymptotically to *EEP*. Hence the proof is complete.

8. Numerical simulation of systems (1)

In a bid to check our computation, some numerical simulations are carried out. The objective is to understand the global dynamics if system (1) and then study the effects of varying the parameters values. For the following set of hypothetical values of the parameters with different initial conditions the dynamical behavior of system (1) is investigated using the following sets of initial conditions (0.7,0.9,0.6,0.5,0.5), (1,2,3,1,4) and (3,0.5,5,3,1) respectively. The obtained trajectories are drawn in Fig . (1) below.

$$
\begin{gathered}
\psi=3\,,\ \beta=0.03\,,\ \mu=0.1\,,\ \gamma=0.1\,,\ \sigma=0\,,\ \varepsilon=0.03\,,\ e=0.1 \\
\delta=0.1\,,\ \alpha=0.05\,,\ \theta=0.02
\end{gathered}
\tag{43}
$$

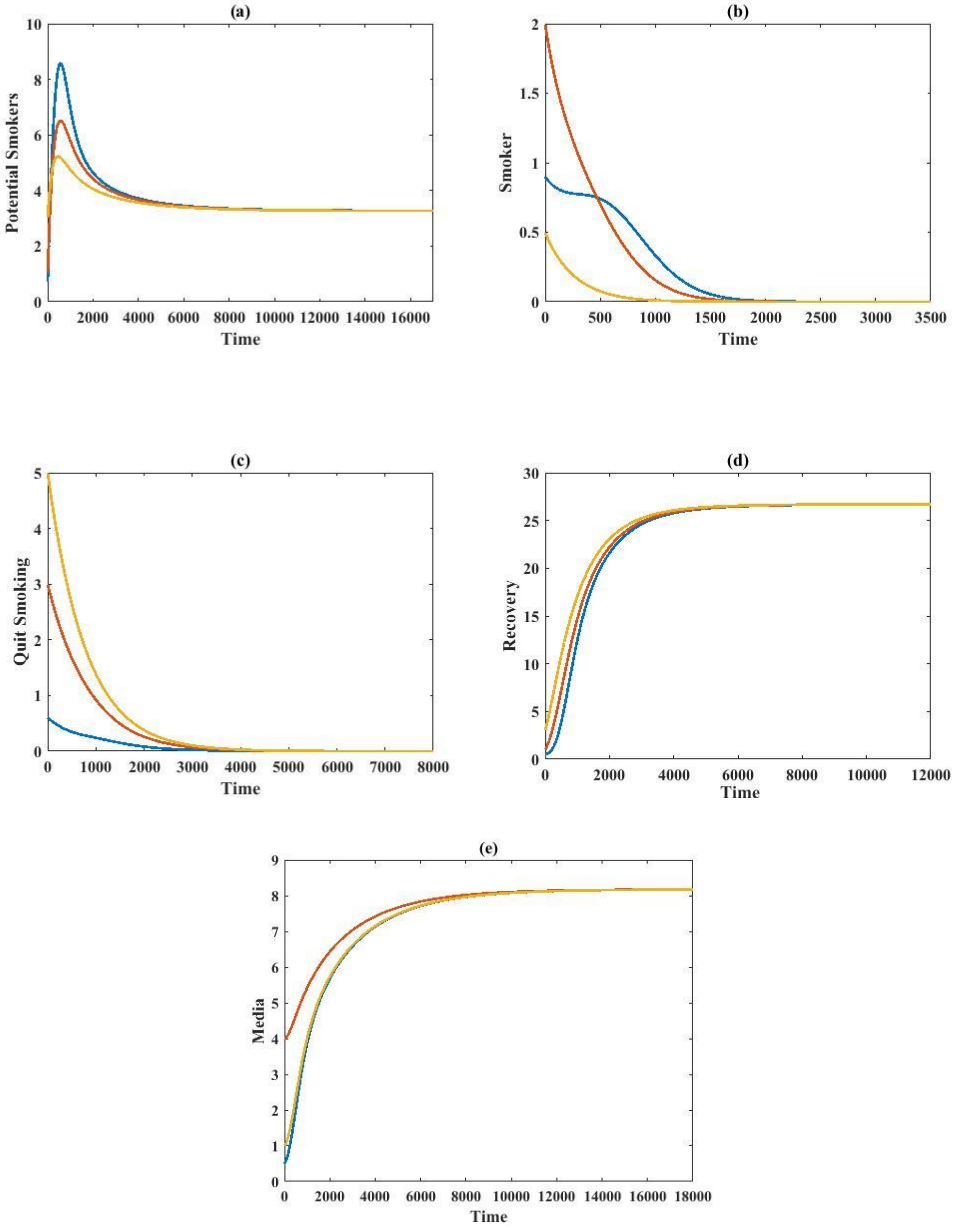

Fig. 1: *The trajectory of system (1) approaches asymptotically to a globally stable SFEP given by* $E_0 = (3.2, 0, 0, 26.7, 8.1)$. *(a) Trajectory of P(t), (b) Trajectory of S(t), (c) Trajectory of Q(t), (d) Trajectory of R(t), (e) Trajectory of M(t).*

Clearly, as shown in Fig. (1), system (1) has a globally asymptotically stable *SFEP* for the data (43). Now, for the following set of hypothetical parameters values with the same initial sets of values used in Fig. (1), the trajectories of system (1) are drawn in Fig. (2) below.

$$\psi = 3,\ \beta = 0.3,\ \mu = 0.1,\ \gamma = 0.1,\ \sigma = 0.1,\ \varepsilon = 0.03,\ e = 0.1 \\ \delta = 0.1,\ \alpha = 0.05,\ \theta = 0.02 \tag{44}$$

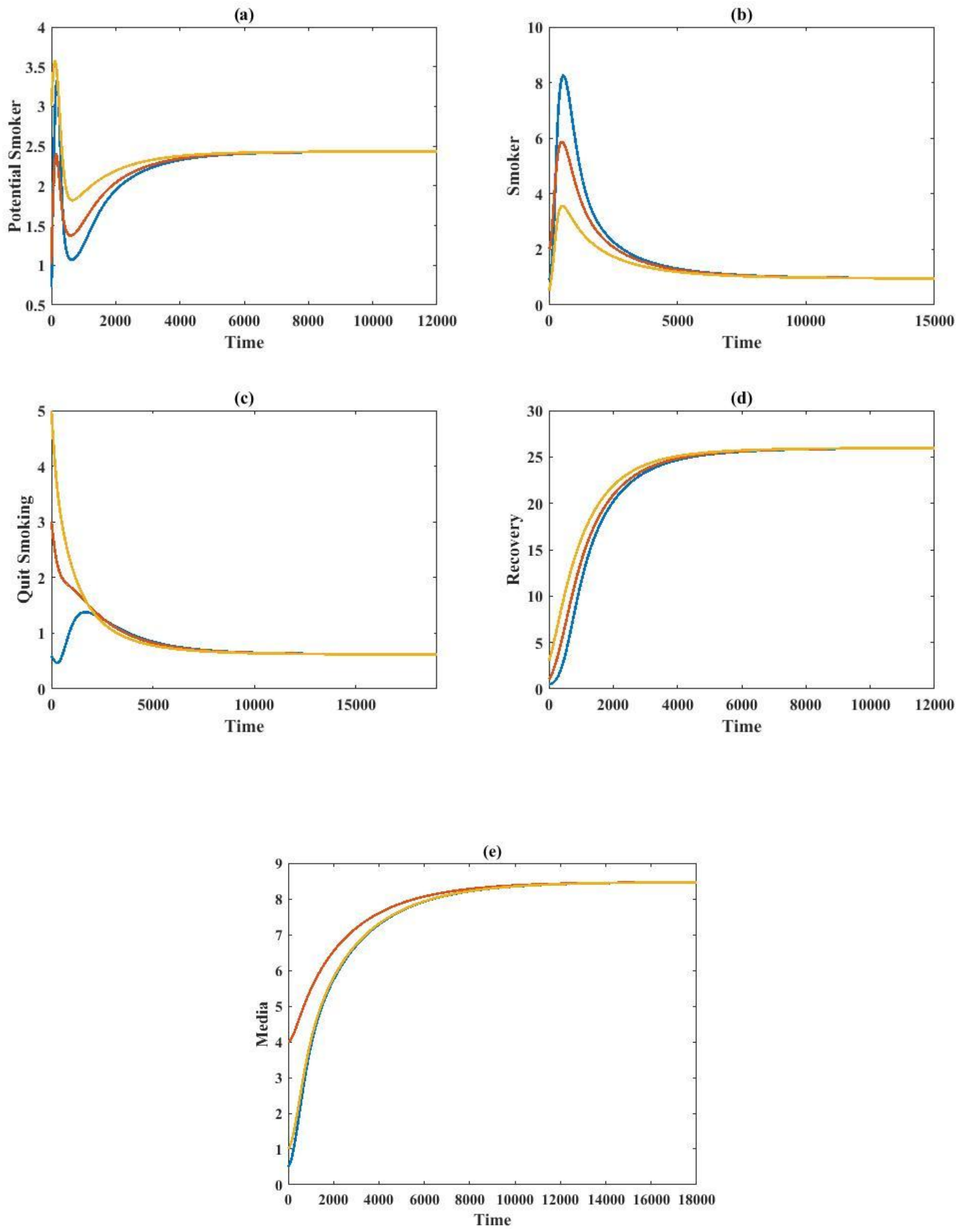

Fig. 2: *The trajectory of system (1) approaches asymptotically to a globally stable EEP given by* $E_2 = (2.4, 0.95, 0.6, 25.9, 8.4)$. *(a) Trajectory of P(t), (b) Trajectory of S(t), (c) Trajectory of Q(t), (d) Trajectory of R(t), (e) Trajectory of M(t).*

Now, we used the same set of hypothetical parameters values in Eq. (44) with $e = 0$, and the same initial sets of values used in Fig. (1), then system (1) has a globally asymptotically stable *FTQSEP*, hence the trajectories of system (1) are drawn in Fig. (3) below.

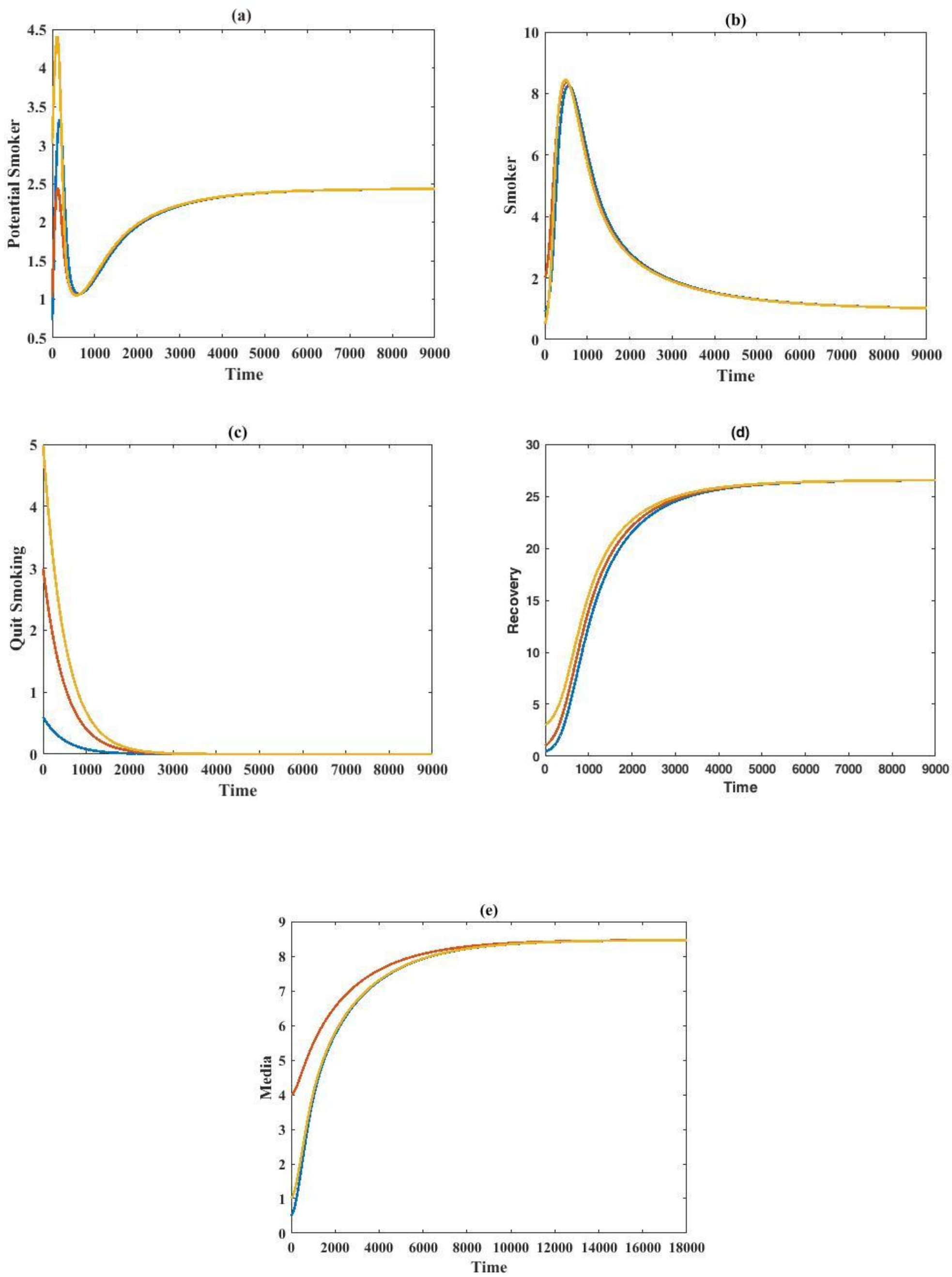

Fig. 3: *The trajectory of system (1) approaches asymptotically to a globally stable FTQSEP given by* $E_1 = (2.4, 0.95, 0, 26.6, 8.4)$. *(a) Trajectory of P(t), (b) Trajectory of S(t), (c) Trajectory of Q(t), (d) Trajectory of R(t), (e) Trajectory of M(t).*

Now, for the data set (44) with different values of contact rate β given by the parameters values $\beta = 0.3, 0.5, 0.0001$ respectively, system (1) is solved numerically and the obtained trajectories are drawn in Fig. (4) below.

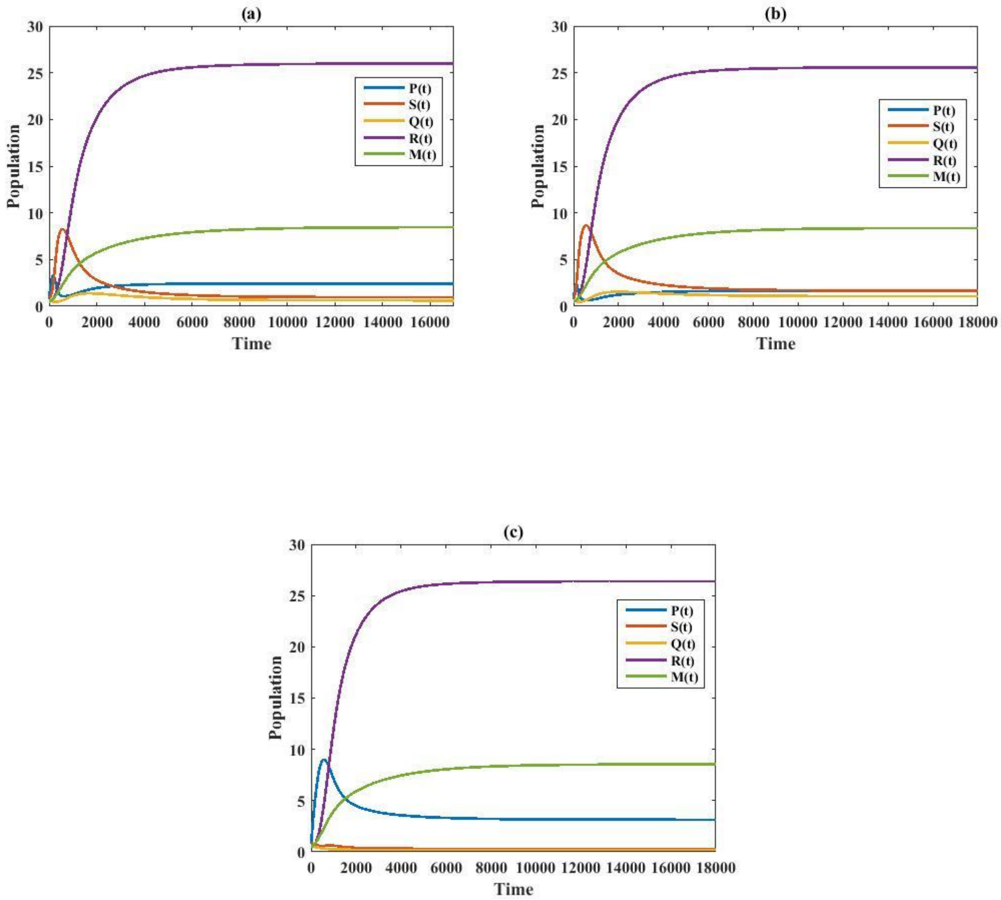

Fig. (4): *Time series of the trajectory of system (1) for the data (44) with different values of contact rate. (a) Trajectory of system (1) for* $\beta = 0.3$*, (b) Trajectory of system (1) for* $\beta = 0.5$*, (c) Trajectory of system (1) for* $\beta = 0.0001$*.*

According to Fig. (4), as the contact rate between the potential smoker individuals and smoker individuals increases, then the trajectory of system (1) approaches asymptotically to the (*EEP*) point as shown in the typical figure given by Fig. (4). In fact as β increases, it is observed that the populations of smoker, quit smoker and media coverage increase while the populations of potential smokers and recovered decrease. On the other hand, as the contact rate β decreases then the trajectory of system (1) still approaches asymptotically to the (*EEP*) but with opposite size of populations.

Now, for the data (44) with awareness level given by $\gamma = 0.2$ and different values of response to media coverage from the potential smoker individuals such that $1-\sigma = 0.99999, 0.8, 0$ respectively, system (1) is solved numerically and the obtained trajectories are drawn in Fig. (5) below.

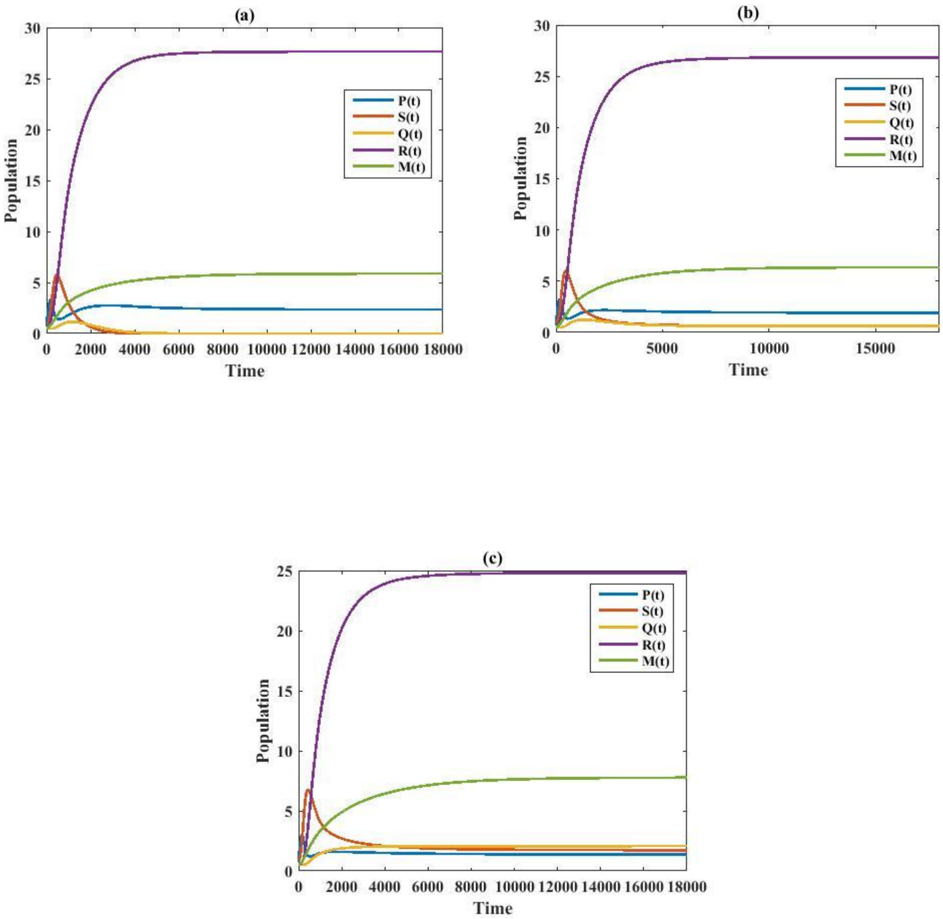

Fig. (5): *Time series of the trajectory of system (1) for the data (44) with* $\gamma = 0.2$ *and different values of response rate to the media coverage. (a) Trajectory of system (1) for* $1-\sigma = 0.99999$*, (b) Trajectory of system (1) for* $1-\sigma = 0.8$*, (c) Trajectory of system (1) for* $1-\sigma = 0$.

Clearly, as shown in Fig. (5), increase the efficiency rate of the media coverage makes the trajectory of system (1) approaches asymptotically to the (*SFEP*) gradually and vice versa.

Similarly, for the data (44) with awareness level given by $\gamma = 0.2$ and different values of response to media coverage from the smoker individuals such that $1-e = 1, 0.5, 0$ respectively, system (1) is solved numerically and the obtained trajectories are drawn in Fig. (6) below.

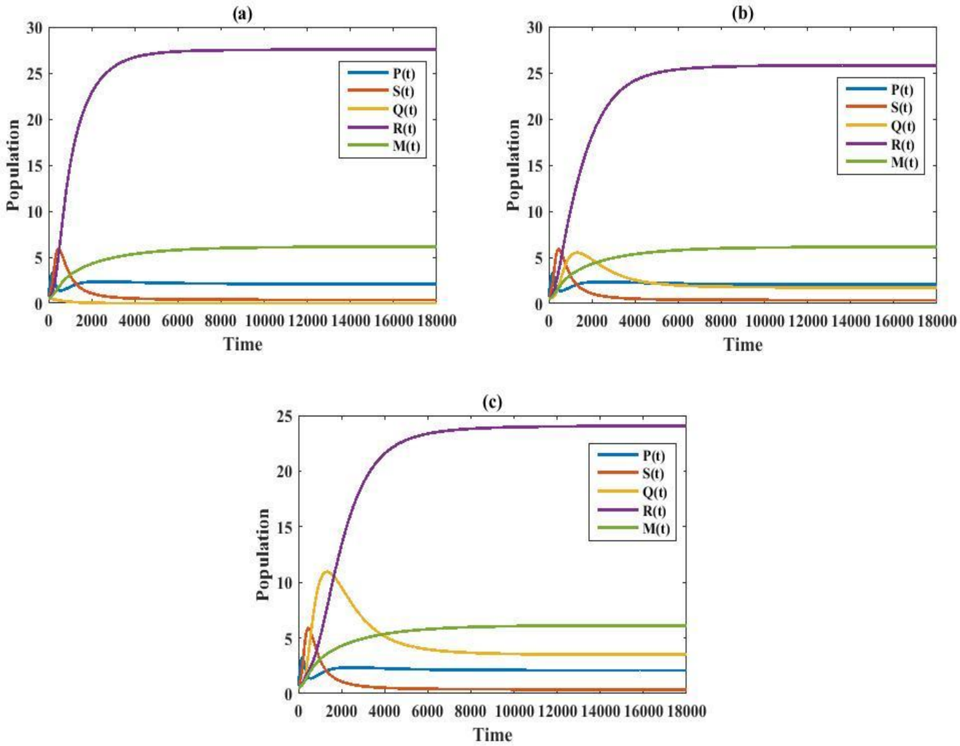

Fig. (6): *Time series of the trajectory of system (1) for the data (44) with* $\gamma = 0.2$ *and different values of response rate of smoker individuals to the media coverage. (a) Trajectory of system (1) for* $1 - e = 1$, *(b) Trajectory of system (1) for* $1 - e = 0.5$, *(c) Trajectory of system (1) for* $1 - e = 0$.

Clearly, as shown in Fig. (6), increase the efficiency rate of the media coverage on the smoker individuals makes the trajectory of system (1) approaches asymptotically to the (*FTQSEP*) gradually and vice versa.

9. Discussion

In this paper, a mathematical model has been studied and analyzed to study the effect of a warning by media on the dynamical behavior of smoking epidemic model. The existence and the stability analysis of all possible equilibrium points are studied analytically as well as numerically. Finally according to the numerically simulation the following results are obtained:

1. As the contact rate between the individuals of potential smokers and smokers increase the trajectory of system (1) approaches asymptotically to the (*EEP*).
2. As the response to the media coverage from the potential smokers increases then the trajectory of system (1) approaches asymptotically to the (*SFEP*). Otherwise the trajectory still approaches asymptotically to (*EEP*).
3. As the response to the media coverage from the smokers increases then the trajectory of system (1) approaches asymptotically to the (*SFEP*). Otherwise the trajectory still approaches asymptotically to (*EEP*).

4. The stability of the smoking system in presence of diffusion follows if the smoking system without diffusion is stable, but the converse is not necessarily true.

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Trigonometric Neural Networks L_p, p<1 Approximation

Eman Samir Bhaya, and Zaineb Hussain Abd Al-sadaa
University of Babylon, College of education for pure sciences, Mathematics Department
e-mail: emanbhaya@itnet.uobabylon.edu.iq

Abstract. Many researchers studied the approximation by neural networks approximation. However only using first or second modulus, that is with low speed approaching zero. Here we define a neural network. Then we use it to approximate functions from L_p_quasi normed spaces we prove upper and lower bounds trigonometric neural networks estimations using the modulus of smoothness of order k.

المستخلص. درس العديد من الباحثين التقريب باستخدام الشبكات العصبية، لكن باستخدام مقياس النعومة من الدرجة الأولى او الثانية و للدوال المستمرة فقط، و بالتالي سيكون الاقتراب الى الصفر بطيئ جدا. في هذا البحث قمنا بتعريف عائلة من الشبكات العصبية و استخدمناها لتقريب الدوال المثلثي في الفضاءات L_p عندما 0<p<1 . برهنا نظريات مباشرة و عكسية حول التقريب المثلثي باستخدام الشبكات العصبية و بدلالة مقياس النعومة من الرتبة k مما يجعل الاقتراب سريعا جدا نحو الصفر و ليس للدوال المستمرة فقط و انما لجميع الدوال في الفضاءات L_p عندما 0<p<1
.

1. Introduction

In the recent years, the approximation using neural networks have many good applications. Many results on the density of the FNNs on the space of continuous functions or on the space of integrable functions are introduced see for example [7], [11], [10], [18], [6]and [4].In these references we can read the result that for any continuous function f of multivariable defined on a compact subset of $\mathbb{R}^n$ we can find a FNN of one hidden layer as best approximation for f of the form

$$N(x) = \sum_{i=1}^{d} c_i \sigma_i(\sum_{j=1}^{d} w_{ij} x_j + \theta_i), x \in R^d, d \geq 1, \tag{1.1}$$

where $i = 1,2 \ldots . d$, θ_i is a real threshold, $w_i = (w_{i1}, w_{i2}, \ldots, w_{is})^T \in R^d$ is the weight that connect the neuron of index i of the hidden layer and the neuron that we input it., c_i is a real constant that connect the weight and the neuron that it output. and $\sigma_i(.)$ is the activation function of the neural network. In the above formula the d is very important: it draw the topology of the hidden layer of the neural network. In many of the approximation studied of the neural network is very difficult to specify the number d, and it is sufficient to say it is existing and large. [8]

We can see many kinds of forward neural networks; all these kinds are different. They are same by: Its input nodes and the links connecting them. We input these nodes then we make processing on them to get the outputs.

The approximation by neural network attracted attentions, especially in the recent years. See for example [1], [13], [16], [14], [5], [3] and [17]. In all studies above the authors study the degree of best neural approximation using modules of smoothness of order 1. The estimation in the above references cannot characterize the ability of the neural network in general. So in this section we will study the order of essential approximation on a special class of neural using trigonometric hidden layer in terms of the k^{th} order modulus of smoothness. We shall use upper and lower bound estimation of neural approximation. After upper and lower bounded, estimation we can write the order of essential neural approximation. We want to mention that we will use the multivariate function for approximation, and using k^{th} order modulus of smoothness for measuring the approximation order. We clear that there is a relationship between the speed of approximation and the number of hidden units.

2. Some Definitions and Notation

If N is the naturals, and R is the reals. Let N_o be the naturals with the zero number and 0 is the zero vector,$1_i = (0,0,\dots,1^{ith},0,\dots,0) \in N_0^d$. Let $|r| = \sum_{i=1}^{d} |r_i|$ for $r = (r_1, r_2, \dots, r_d) \in N_0^d$, $\|t\| = (\sum_{i=1}^{d} t_i{}^2)^{\frac{1}{2}}$ for $t = (t_1, t_2, \dots, t_d) \in R^d$, and $rt = \sum_{i=1}^{d} r_i t_i$.

Let $f \in L_{2\pi}^p, 0 < p < 1$. Write $C_{2\pi}$ the space of the continuous functions with2π periodic with respect to the variable in R^d. If $f \in L_{2\pi}^p, 0 < p < 1$, its quasi norm.

Define the symmetric difference of degree *r* for the function f as

$$\Delta_h^{(r)} f(x) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i f(x + \left(\frac{r}{2} - i\right) h).$$

Using $\Delta_h^r f(.)$, we define the modulus of smoothness of order *r* as:

$$\omega_r(f,t)_p = sup_{0<\|h\|\leq t} \left\| \Delta_h^{(r)} f(.) \right\|_p . \tag{2.1}$$

where

$$\|f\|_p = (\int_{-\pi}^{\pi} |f(x)|^p)^{1/p}.$$

We say that the function f belongs to Lipishtz space of order greater than $r, r \in N$, . Write $f \in Lip(\alpha)_r$, if $\omega_r(f,t)_p = O(t^\alpha)$, with an $\alpha \in (0, r]$.

Modulus of smoothness is a measurement of smoothness. The modulus of smoothness of many variables is an improvement of the modulus of smoothness of one variable. Let us list some properties of the modulus of smoothness. For $f \in L_{2\pi}^{p}, 0 < p < 1$, we have

(1)$\lim_{\delta \to 0} \omega_r(f, \delta, J)_p = 0$.

(2)$\omega_r(f, \delta, J)_r$ is nondecreasing function of δ.

(3)$\omega_r(f, \delta, J)_p \leq c\lambda^r \omega_r(f, \delta, J)_p$, for $\lambda \geq 1$.

We denote by $f * g(x) = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} f(t)g(x-t)dt$ the convolution of f and g, and by $f^\wedge(r) = < f, e^{-irt} >$ the Fourier transformation of function$f$, where$< f, g > = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} f(t)g(t)dt$ is inner product of f and g. The definition of the r-th K-functional of $f \in L_{2\pi}^{p}, 0 < p < 1$, and $\delta > 0, r \in N$, it mean

$$K_r(f, \delta^r)_p = inf_{D^\beta g \in L_{2\pi}^{p}} \{ \| f - g \|_p + \delta^r sup_{|\beta| = r} \| D^\beta g \|_p \}, \quad (2.2)$$

where $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_d, \beta = (\beta_1, \beta_2, \ldots, \beta_d) \in N_o^d$, and $D^\beta = \frac{\partial |\beta|}{\partial {x_1}^{\beta_1} \ldots \partial {x_d}^{\beta_d}}$,

is the operator of derivative. The K-functional operator was defined by K-Peetre in [15]. Then it developed by Johmen and Scherer in [12] and in [9] by Ditzian and Totik. The K-functional operator used to measure the distance between the neural liner space and the approximations space. One of the famous results for the K-functionless is it's equivalence with the modulus of smoothness define in (2.2), it mean there are constants C_1 and C_2, satisfy

$$C_1 \omega_r(f, \delta)_p \leq K_r(f, \delta^r)_p \leq C_2 \omega_r(f, \delta)_p. \quad (2.3)$$

Now let us introduce some notations from [16]. We have $\lambda \epsilon \mathbb{N}$ and $f_i \epsilon L_{2\pi}^{p}, o < p < 1$.

$p = (p_1, p_2, \ldots, p_d) \in N_o^d, q = (q_1, q_2, \ldots, q_d) \in N_0^d$

$$B_\lambda = (\frac{2}{\lambda + 2})^d, b_{\lambda, r} = \prod_{i=1}^{d} \sin \frac{r_i + 1}{\lambda + 2} \pi.$$

In our article we will use the notation $c(v_1, v_2)$ to denote such absolute crostatas which are may differ on different occurrences even in the same line, and depending on v_1 and v_2.

3. The Main Results

This section consists of the main results of this article.

Theorem.3.1. For $f_i \in L^p_{2\pi}$, $0 \leq p \leq 1$, we have

$$\left\| EN_\lambda(f_i) - f_i \right\|_p \leq c(p,k) W_k(f_i, \frac{1}{\lambda+2})_p$$

Proof.

Suppose $r = (r_1, r_2, \ldots, r_d) \in N_0^d$, λ is a natural number.

The Fejer – korovkin kernel k_λ of dimension d is defined by

$K\lambda(t) = B\lambda \left| \sum_{0 \leq ri \leq \lambda} b_{\lambda,r} e^{irt} \right|^2$, where

$b_{\lambda,r} = \prod_{i=1}^{d} \sin \frac{ri+1}{\lambda+2}\pi$, $B_\lambda = (\sum_{0 \leq ri \leq \lambda} (b_{\lambda,} r)^2)^{-1}$.

Then

$B_\lambda = (\sum_{o \leq ri \leq \lambda} (\prod_{i=1}^{d} \sin \frac{ri+1}{\lambda+2})^2)^{-1} = (\frac{2}{\lambda+2})^{-1} = (\frac{2}{\lambda+2})^d$,

and

$K_\lambda(t) = B_\lambda \left| \sum_{0 \leq ri \leq \lambda} b_\lambda, re^{irt} \right|^2 = 1 + 2B_\lambda \sum_{p \neq q \in N_o^d}^{0 \leq p_u, q_v \leq \lambda} b_{\lambda,p}, b_{\lambda,q} \cos(p-q)t.$

We define the operator

$EN_\lambda(f_i) = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} (\sum_{j=0}^{\frac{r}{2}-1} f_i\big(x + \big(\frac{r}{2} - j\big)t\big) K_\lambda(t) dt\big) + \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} (\sum_{j=\frac{r}{2}+1}^{r} f_i\big(x + \big(\frac{r}{2} - j\big)t\big) K_\lambda(t) dt + \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} C_r (-1)^{\frac{r}{2}} f_i(x) K_\lambda(t) dt,$

where $c_r = \frac{r! + ((\frac{r}{2})!)^2}{(\frac{r}{2}!)^2}$.

$K_\lambda^\wedge(0) = 1$, $K_\lambda^\wedge(r) = B_\lambda \sum_{p-q=r, p,q \in N_0^d}^{0 \leq p_u, q_v \leq \lambda} b_{\lambda,p}\ b_{\lambda,q}$,

and $K_\lambda^\wedge(1_i) = \cos \frac{\pi}{\lambda+2}$ [16].

Using (1), (2), and (3) to get

$$\left\| EN_\lambda(f_i) - f_i \right\|$$

$$= \| \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}(\sum_{j=0}^{\frac{r}{2}-1} f_i\left(x+\left(\frac{r}{2}-j\right)t)K_\lambda(t)dt\right)$$

$$+\frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}(\sum_{j=\frac{r}{2}+1}^{r} f_i\left(x+\left(\frac{r}{2}-j\right)t\right)K_\lambda(t)dt$$

$$+\frac{1}{(2\pi)^d}\int_{-\pi}^{\pi} C_r(-1)^{\frac{r}{2}} f_i(x)K_\lambda(t)dt \|_p$$

$$= \| \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}(\sum_{j=0}^{\frac{r}{2}-i} f_i\left(x+\left(\frac{r}{2}-j\right)t\right)K_\lambda(t)dt + \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}(\sum_{j=\frac{r}{2}+1}^{r} f_i\left(x+\left(\frac{r}{2}-j\right)t\right)K_\lambda(t)dt + \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi} c_r(-1)^{\frac{r}{2}}\ f_i(x)K_\lambda(t)dt - f_i \|_p$$

$$\leq \frac{c(p)}{(2\pi)^d}\int_{-\pi}^{\pi} K_\lambda(t)\omega_k(f_i, \|t\|)_p$$

$$\leq \frac{c(p)}{(2\pi)^d}\int_{-\pi}^{\pi} K_\lambda(t)\omega_k(f_i, \frac{\delta}{\delta}\|t\|)_p$$

$$\leq \frac{c(p)}{(2\pi)^d}\ \omega_k(f_i,\delta)_p \int_{-\pi}^{\pi} K_\lambda(t)(\delta^{-1}\|t\|)^k\ dt$$

$$\leq c(p)\omega_k(f_i,\delta)_p\ (\frac{1}{\delta^k}\left(\frac{1}{(2\pi)^d}\sum_{i=1}^{d}\int_{-\pi}^{\pi} {t_j}^{2k}\ K_\lambda(t)dt)^{\frac{1}{2}}\right).$$

Since

$t \leq \pi \sin\frac{t}{2}$, $0 \leq t \leq \pi$, $\pi \sin\frac{t}{2} \leq t$, $-\pi \leq t \leq 0$,

so

$t^{2k} \leq \pi^{2k}\left(\sin\frac{t}{2}\right)^{2k}$, $-\pi \leq t \leq \pi$.

Therefore, $t^{2k} \leq \pi^{2k}(\sin\frac{\pi}{2})^2$.

Consequently,

$$\frac{1}{(2\pi)^d}\int_{-\pi}^{\pi} {t_j}^{2k}\ K_\lambda(t)dt \leq \frac{1}{(2\pi)^d}\pi^{2k}\int_{-\pi}^{\pi}(\sin t_j)^{2k}\ K_\lambda(t)dt$$

$$= \pi^k(1-\cos\frac{\pi}{\lambda+2})^k .$$

If we take $\delta = \frac{1}{\lambda+2}$,

and since (1-cos $\frac{\pi}{\lambda+2}$) is bounded set so

$$\sum_{i=1}^{d}(1-\cos\frac{\pi}{\lambda+2})^k = c(\sum_{i=1}^{d}(1-\cos\frac{\pi}{\lambda+2})^k$$

$$\leq c(\frac{\pi}{\lambda+2})^{2k}$$, c is a positive constant.

Thus take $\delta = \frac{1}{\lambda+2}$

$$\left\| EN_\lambda(f_i) - f_i \right\|_p \leq c(p)(\frac{1}{(\frac{1}{\lambda+2})^k}\ ((c(\frac{\pi}{\lambda+2})^{2k}))^{1/2}\omega_k(f_i,\frac{1}{\lambda+2})_p$$

$$\leq c(p)((\lambda+2)^k\left(\frac{c\pi^k}{(\lambda+2)^k}\right)\omega_k(f_i,\frac{1}{\lambda+2}))_p$$

$$\leq c(p)(\pi^k)^k\ \omega_k(f_i,\frac{1}{\lambda+2})_p$$

$$\leq \mathrm{c(p,k)\omega_k(f_i,}\frac{1}{\lambda+2})_\mathrm{p}$$

Lemma .3.2. [8]. For $f_i \in L^p_{2\pi}, 0 < p < 1, we\ have$

$\lim_{\lambda\to\infty} \left\| EN_\lambda[f_i] - f_i \right\|_p = \lim_{\lambda\to\infty} \frac{1}{\lambda^2}\sum_{k=1}^{\lambda} K \left\| EN_k[f_i] - [f_i] \right\|_p$.

Theorem .3.3. [2]. If $f_i \in L^p_{2\pi}, 0 < p < 1, n \in \mathbb{N},\ \lambda\epsilon\mathbb{N}$, then

$$\omega_k\left(f_i,\frac{1}{\lambda+2}\right) \leq c(p)\sum_{\lambda=1}^{n} \left\| EN_\lambda(f_i) - (f_i) \right\|_p$$

Corollary.3.4. for $f_i \in L^p_{2\pi}, 0 < p < 1$, we get

$$c(p)\omega_k(f_i,\frac{1}{\lambda+2})_p \leq \left\| EN_\lambda[f_i] - [f_i] \right\|_p \leq c(p,k)\omega_k(f_i,\frac{1}{\lambda+2})_p$$

Proof.

By lemma (3.2)

$\lim_{\lambda\to\infty} \left\| EN_\lambda(f_i) - f_i \right\|_p \leq \frac{1}{\lambda^2}\sum_{k=1}^{\lambda} k \left\| EN_\lambda(f_i) - f_i \right\|_p$,

and by using theorem (3.1)

$\left\| EN_{\lambda}(f_i) - f_i \right\|_p \leq c(p,k)\omega_k\left(f_i, \frac{1}{\lambda+2}\right).$

Therefore, we get

$$c(p)\omega_k(f_i, \frac{1}{\lambda+2})_p \leq \left\| EN_{\lambda}[f_i] - [f_i] \right\|_p \leq c(p,k)\omega_k(f_i, \frac{1}{\lambda+2})_p$$

Theorem.3.5. If $f_i \in L^p_{2\pi}, 0 < p < 1$, then $\left\| EN_{\lambda}[f_i] - f_i \right\|_p = O(\lambda^{-\alpha}), 0 < \alpha < k-1$, if and only if $f_i \in Lip(\alpha)_k$

Proof.

Let $f_i \in Lip(\alpha)_k \, where \; 0 < \alpha < k-1$ we must prove that

$\left\| EN_{\lambda}[f_i] - f_i \right\|_p = O(\lambda^{-\alpha}).$

Since $f_i \in Lip(\alpha)_k$, then$\omega_k(f,\lambda)_p = O(\lambda^{-\alpha}).$ (3.5.1)

Using theorem.3.1 and (3.5.1) we get

$\left\| \mathrm{E}N_{\lambda}[f_i] - f_i \right\|_p \leq c(p)O(\lambda^{-\alpha}).$

Then

$$\left\| EN_{\lambda}[f_i] \right\|_p = O(\lambda^{-\alpha}).$$

Let $f_i \in L^p_{2\pi}, 0 < p < 1$, then $\left\| \mathrm{E}N_{\lambda}[f_i] - f_i \right\|_p = O(\lambda^{-\alpha}).$

We must prove that$f_i \in Lip(\alpha)_k$.

Now, $\left\| \mathrm{E}N_{\lambda}[f_i] - f_i \right\|_p = O(\lambda^{-\alpha})$

$$\leq c(\lambda^{-\alpha}),$$

and since $\left\| \mathrm{E}N_{\lambda}[f_i] - f_i \right\|_p \leq c(p,k)\omega_k(f_i,\lambda)$. Then

$$c(p)\omega_k(f_i,\lambda) = c(\lambda^{-\alpha})$$

$$\omega_k(f_i,\lambda) = O(\lambda^{-\alpha}).$$

Therefore, using definition of Lipschitian function we get $f_i \in Lip(\alpha)_k$

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Impress of rotation and an inclined MHD on waveform motion of the non-Newtonian fluid through porous canal

T Sh Alshareef

"Department of mathematics, College of education for pure science \Ibn AlHaitham\, University of Baghdad, Iraq"

"tamara.shihab8282@gmail.com"

Abstract: Waveform flow of non-Newtonian fluid through a porous medium of the non-symmetric sloping canal under the effect of rotation and magnetic force, which has applied by the inclined way, have studied analytically and computed numerically. Slip boundary conditions on velocity distribution and stream function are used. We have taken the influence of heat and mass transfer in the consideration in our study. We carried out the mathematical model by using the presumption of low Reynolds number and small wave number. The resulting equations of motion, which are representing by the velocity profile and stream function distribution, solved by using the method of a domain decomposition analysis and we obtained the exact solutions of velocity, temperature, and concentration. The expressions of velocity, temperature, and concentration of the particles of the fluid have obtained and examined graphically by utilizing the soft wave of the Mathematica program. The efforts of various variables on mathematical modeling of motion and energy are discussed in detail. We found that.

Keywords: rotation effect, non-Newtonian fluid, porous medium, magnetic force, waveform transport.

1- Introduction

Motion through the porous area takes place in the filtering of fluids and leakage of water in the beds of rivers. The moment under the ground, oils and water are some important examples of flows through a porous medium. An oil barrage mostly includes the formation of sediments such as sandstone and limestone in which the oil is entrapped. Another example of motion through a porous medium is the leakage under a dam, which is very important. There are examples of nature's porous medium such as rye bread and beach sand. The transport through porous media discussed by (Sceidgger, 1963). The waveform motion of Newtonian fluid in a vertical asymmetric porous channel is studied by (Srinivas S and Gayathri R, 2009). The peristaltic transport of Jeffrey fluid under the effort of a magnetic field in an asymmetric porous canal is studied by (kothandapani & Srnivas, 2008). The impact of porous medium and magnetic force on the waveform flow of Jeffrey fluid is studied by (Mahmood, Afifi, & Al. Isede, 2011). The influence of the thickness of the porous material on the waveform pumping of Jeffrey fluid when the tube wall is provided with non-erodible lining is made by (Rathod & Channakote, 2011).

The MHD flow of a fluid in a channel with elastic, rhythmically contraction walls is of interest in connection with a certain problem of the movement of conductive physiological fluids, e.g., the blood and with the need for theoretical research on the operation of a peristaltic MHD
Compressor. The effect of a moving magnetic field on blood flow was studied by (Stud V K, Sephone G S and Mishra R K G, 1977). And they observed that the effects of a suitable moving magnetic field accelerate the speed of blood. The blood as an electrically conducting fluid that constitutes a suspension of red cells in the plasma is considered by (Srivastava L M and Srivastava V P, 1984). The MHD flow of a conducting couple stress fluid in a slit channel with rhythmically contracting walls is analyzed by (Mekheimer Kh S, 2008). The MHD peristaltic motion of a Sisko fluid in an asymmetric channel is studied by (Wang Y, Hayat T, Ali N and oberlack M, 2008). The peristaltic transport of a Jeffrey fluid under the effect of the magnetic field in an under the effect of a magnetic field in a Symmetric channel with flexible rigid walls are examined by (Kothandapani M and Srinivas S, 2008).

The effects of an endoscope and magnetic field on the peristalsis involving Jeffrey fluid has investigated by (Hayat T, Ahmed N and Ali N, 2008). Given these facts, it will be interesting to study the peristaltic flow of conducting Jeffrey fluid flow in a channel bounded by permeable walls.
Waveform transport with heat and mass transfer has many applications in biomedical sciences and industry such as conduction in tissues, heat convection due to blood flow from the pores of tissues and radiation between environment and its surface, food processing and vasodilation. The processes of oxygenation and hemodialysis have also visualized by considering peristaltic flows with heat transfer. There is a certain role of mass transfer in all these processes. The mass transfer also occurs in many industrial processes like membrane separation process, reverse osmosis, distillation process, combustion process and diffusion of chemical impurities. The effect of heat transfer on the peristaltic flow of an electrically conducting fluid in a porous space is studied by (Hayat T, Qurashi M U and Hussain Q, 2009). The influence of heat transfer and slip-on peristaltic transport is analyzed by (Hayat T, Hina S and Hendi A A, 2012). Heat transfer analysis of peristaltic flow in a curved channel is analyzed by (Ali N, Sajid M, Javed T and Abbas Z, 2010).
It is also of interest to remember that non-slip boundary conditions are unsuitable for must non-Newtonian fluids because they display microscopically the slip condition of the walls. The fluids that displaying the boundary slip condition give applications in technology such that the polishing of artificial heart, there are many studies, which are, using this condition, see (Abdulhadi A M and Al-Hadad A H, 2015), (Chaube M K, Pandey S K and Tripathi D, 2010) & (Ali N, Wang Y, Hayat T and oberlack M, 2009). Recently, magnetic field and rotation effects on the peristaltic transport of Jeffrey fluid in an asymmetric channel studied by (Abd-Alla, A M. and Abo-Dahab, S M, 2015). The effect of the rotation on wave motion through the cylindrical bore in a micropolar porous medium is discussed by (Mahmoud S R, Abd-alla A M and El-Sheikh M A, 2011). The effects of rotation and MHD on the nonlinear peristaltic flow of Jeffrey fluid in an asymmetric channel through a porous medium has discussed by (Abdulhadi A M and Al-Hadad A H, 2016).
Now in this paper, we discuss the waveform motion of the non-Newtonian fluid through a porous medium of non- symmetric sloping canal under the effect of rotation and inclined magnetic field in two-dimensional channels. We studied the problem under the slip boundary conditions on the velocity distribution and stream function profile, in addition to the impact of heat and mass transfer in the channel. The governing equations are modeling and then solved analytically by using a domain decomposition method and we obtained the exact solutions of the velocity, temperature, and concentration distribution by using the approximations of long wavelength and low Reynolds number. We studied the effects of various parameters on the above distributions by displaying some graphs, which have shown by using the por rogram of Mathematica software.

2- Problem's Mathematical Pattern

Through our work, we have considered the waveform flow of non-Newtonian fluid through a porous medium of two-dimensional with non-symmetric and non-uniform inclined channel under the effect of rotation parameter of the channel and combined influence of inclined magnetic field as well as heat/mass transfer. We suppose that there is infinite number of waves, which are transporting with speed c_1 along the non-regular walls. We have chosen a system of rectangular coordinates for this channel with $\overline{X_1}$ along the direction of wave's propagation and parallel to the cort line and the axis $\overline{Y_1}$ is transverse to it. The mathematical model for the channel's walls can described by:

$$\left.\begin{aligned}\overline{G_{11}} &= -\overline{A} - \overline{B_1}, \text{ left wall,...}\\ \overline{G_{12}} &= -\overline{A} + \overline{B_2}, \text{ right wall,...}\end{aligned}\right\}, \qquad ...(2.1)$$

$$\left.\begin{aligned}&\overline{A} = e + \overline{m_1}\,\overline{X_1}\\ \text{where } &\overline{B_1} = e_1 \sin[\frac{2\pi}{\zeta}(\overline{X_1} - c_1\bar{t}) + \phi_1]\\ &\overline{B_2} = e_2 \sin[\frac{2\pi}{\zeta}(\overline{X_1} - c_1\bar{t})]\end{aligned}\right\} \quad ...(2.2)$$

Such that ζ is the wave's length, (2e) is the width of the channel at the inlet $(\overline{m_1} << 1)$ which is the non –uniform parameter, (e_1, e_2) are the wave's amplitudes, ϕ_1 is the phase difference of the waves which changes in the rate about $\phi_1 \in [0, \pi]$ in which if $\phi_1 = 0$ corresp onds to symmetric channel and the waves are out the phase and if $\phi_1 = \pi$ represent to the waves in the phase. Moreover the parameters e_1, e_2, e and ϕ_1 achieved the following condition:

$$e_1^{\,2} + e_2^{\,2} + 2e_1e_2\cos\phi_1 \le (2e)^2 \quad ...(2.3)$$

Also, it is worth noting through our study, we suppose the magnetic Reynolds number is small and hence the induced magnetic field is cancel.

3- Basic Equation

The system that governing the equations of motion and energy can give in the following formula:

$$\frac{\partial\overline{W_1}}{\partial\overline{X_1}} + \frac{\partial\overline{W_2}}{\partial\overline{Y_1}} = 0 \quad ...(3.1)$$

$$\rho_1(\frac{\partial\overline{W_1}}{\partial\bar{t}} + \overline{W_1}\frac{\partial\overline{W_1}}{\partial\overline{X_1}} + \overline{W_2}\frac{\partial\overline{W_1}}{\partial\overline{Y_1}}) - \rho_1\overline{\Omega}(\overline{\Omega}\overline{W_1} + 2\frac{\partial\overline{W_2}}{\partial\overline{Y_1}}) = \frac{\partial\overline{P}}{\partial\overline{X_1}} + \frac{\partial}{\partial\overline{X_1}}(\bar{\tau}_{\overline{X_1}\overline{X_1}}) + \frac{\partial}{\partial\overline{Y_1}}(\bar{\tau}_{\overline{X_1}\overline{Y_1}})$$

$$-\sigma B_0^2\cos\beta(\overline{W_1}\cos\beta - \overline{W_2}\sin\beta) - \frac{N_0}{k_1}\overline{W_1} + \rho_1 g\sin\alpha_1. \quad ...(3.2)$$

$$\rho_1(\frac{\partial\overline{W_2}}{\partial\bar{t}} + \overline{W_1}\frac{\partial\overline{W_2}}{\partial\overline{X_1}} + \overline{W_2}\frac{\partial\overline{W_2}}{\partial\overline{Y_1}}) - \rho_1\overline{\Omega}(\overline{\Omega}\overline{W_2} - 2\frac{\partial\overline{W_1}}{\partial\overline{Y_1}}) = -\frac{\partial\overline{P_1}}{\partial\overline{Y_1}} + \frac{\partial}{\partial\overline{X_1}}(\bar{\tau}_{\overline{X_1}\overline{Y_1}}) + \frac{\partial}{\partial\overline{Y_1}}(\bar{\tau}_{\overline{Y_1}\overline{Y_1}})$$

$$+\sigma B_0^2\sin\beta(\overline{W_1}\cos\beta - \overline{W_2}\sin\beta) - \frac{N_0}{k_1}\overline{W_2} - \rho_1 g Cos\alpha_1. \quad ...(3.3)$$

$$\rho_1\zeta_1(\overline{W_1}\frac{\partial\overline{F}}{\partial\overline{X_1}} + \overline{W_2}\frac{\partial\overline{F}}{\partial\overline{Y_1}}) = k_2(\frac{\partial^2\overline{F}}{\partial\overline{X_1}^2} + \frac{\partial^2\overline{F}}{\partial\overline{Y_1}^2}) + N_0(2(\frac{\partial\overline{W_1}}{\partial\overline{X_1}})^2 + 2(\frac{\partial\overline{W_2}}{\partial\overline{Y_1}})^2 + (\frac{\partial\overline{W_1}}{\partial\overline{Y_1}} + \frac{\partial\overline{W_2}}{\partial\overline{X_1}})^2) + \frac{g_m k_F}{k_s}(\frac{\partial^2\overline{f}}{\partial\overline{X_1}^2}$$

$$+\frac{\partial^2\overline{f}}{\partial\overline{Y_1}^2}) \quad ...(3.4)$$

$$(\overline{W_1}\frac{\partial\overline{f}}{\partial\overline{X_1}} + \overline{W_2}\frac{\partial\overline{f}}{\partial\overline{Y_1}}) = g_m(\frac{\partial^2\overline{f}}{\partial\overline{X_1}^2} + \frac{\partial^2\overline{f}}{\partial\overline{Y_1}^2}) + \frac{g_m k_F}{\overline{F}_m}(\frac{\partial^2\overline{F}}{\partial\overline{X_1}^2} + \frac{\partial^2\overline{F}}{\partial\overline{Y_1}^2}) \quad ...(3.5)$$

Where (ρ_1) is the fluid's density, $\overline{\Omega} = \overline{\Omega}\kappa$, κ is the unit vector parallel to $\bar{z}_1 -$axis, $\overline{\Omega}$ is the rotation parameter, $\overline{W} = [\overline{W_1}(\overline{X_1}, \overline{Y_1}), \overline{W_2}(\overline{X_1}, \overline{Y_1}), 0]$ is the vector of velocity in two-dimensional coordinates $(\overline{X_1}, \overline{Y_1})$, ρ_1 is the fluid's pressure, $\bar{\tau}$ is the flow's fluid time, σ is the fluid's electrical conductivity, B_0 is the strength of the applied magnetics force. The absence of an electrical field characterized by the Lorentz force $(\overline{J} \times \overline{B})$, which takes the following formula:

$$\overline{J}\times\overline{B}=-\sigma B_0^2\cos\beta(\overline{W_1}\cos\beta-\overline{W_2}\sin\beta)e_i\ ++\sigma B_0^2\sin\beta(\overline{W_1}\cos\beta-\overline{W_2}\sin\beta)e_i \quad ...(3.6)$$

Where (e_i, e_j) are the unit vectors, $\overline{J}$ is the induced current density. We observed that the effect of the magnetic field appears on the flow of $\overline{X_1Y_1}-\text{direction}$ due to the inclination angle β of magnetic field. Also, we have α_1 referred to inclination angle of the channel, g is the acceleration due to gravity, No is the fluid's viscosity, k_1 is the porosity parameter of the canal, ζ_1 is the specific heat at constant pressure, $\overline{F}$ is the fluid's temperature, $\overline{f}$ is the fluid's concentration, k_2 is the fluid's thermal conductivity, g_m is the coefficient of mass diffusivity, k_s is the concentration susceptibility, k_F is the thermal diffusion ratio and $\overline{F}_m$ is the fluid's mean temperature.

The constituent equations for non-Newtonian incompressible fluid which characterized by rate type fluid can be shown as the form:

$$\overline{S}=-\overline{\rho_1}\overline{I}+\overline{\tau} \quad ...(3.7)$$

Where $\overline{S}$ is the Cauchy stress tensor, $\overline{I}$ is the identity tensor and $\overline{\tau}$ is the extra stress for the fluid which is formed as [18]:

$$\overline{\tau}=\frac{N_0}{1+\lambda_1}(\dot{\overline{r}}+\zeta_2\ddot{\overline{r}}) \quad ...(3.8)$$

Where the ratio of repose to obstruction times is λ_1, ζ_2 is the obstruction time, $\dot{\overline{r}}$ is the rate of shear, such that:

$$\dot{\overline{r}}=(\nabla\overline{W})+(\nabla\overline{W})^{\mathrm{T}} \quad ...(3.9)$$

$$\ddot{\overline{r}}=[\frac{\partial}{\partial\overline{t}}+\overline{W_1}\frac{\partial}{\partial\overline{X_1}}+\overline{W_2}\frac{\partial}{\partial\overline{Y_1}}]\dot{\overline{r}} \quad ...(3.10)$$

Now, if we substitute (3.10) into (3.8), we obtain:

$$\overline{\tau}=\frac{N_0}{1+\lambda_1}((\frac{\partial}{\partial\overline{t}}+\overline{W_1}\frac{\partial}{\partial\overline{X_1}}+\overline{W_2}\frac{\partial}{\partial\overline{Y_1}}))\dot{\overline{r}} \quad ...(3.11)$$

Then the components of stress have given by:

$$\overline{\tau}_{\overline{X_1}\overline{X_1}}=\frac{N_0}{1+\lambda_1}(\dot{\overline{r}}_{\overline{X_1}\overline{X_1}}+\zeta_2\ddot{\overline{r}}_{\overline{X_1}\overline{X_1}})$$

$$=\frac{2N_0}{1+\lambda_1}[\frac{\partial\overline{W_1}}{\partial\overline{X_1}}+\zeta_2(\frac{\partial^2\overline{W_1}}{\partial\overline{t}\partial\overline{X_1}}+\overline{W_1}\frac{\partial^2\overline{W_1}}{\partial\overline{X_1}^2}+\overline{W_2}\frac{\partial^2\overline{W_1}}{\partial\overline{Y_1}\partial\overline{X_1}})] \quad ...(3.12)$$

$$\overline{\tau}_{\overline{X_1}\overline{Y_1}}=\frac{N_0}{1+\lambda_1}(\dot{\overline{r}}_{\overline{X_1}\overline{X_1}}+\zeta_2\ddot{\overline{r}}_{\overline{X_1}\overline{Y_1}})$$

$$=\frac{N_0}{1+\lambda_1}[(\frac{\partial\overline{W_1}}{\partial\overline{Y_1}}+\frac{\partial\overline{W_2}}{\partial\overline{X_1}})+\zeta_2[(\frac{\partial^2\overline{W_1}}{\partial\overline{t}\partial\overline{Y_1}}+\frac{\partial^2\overline{W_2}}{\partial\overline{t}\partial\overline{X_1}})+\overline{W_1}(\frac{\partial^2\overline{W_1}}{\partial\overline{X_1}\partial\overline{Y_1}}+\frac{\partial^2\overline{W_2}}{\partial\overline{X_1}})+\overline{W_2}(\frac{\partial^2\overline{W_1}}{\partial\overline{Y_1}^2}+\frac{\partial^2\overline{W_2}}{\partial\overline{Y_1}\partial\overline{X_1}})]] \quad ..(3.13)$$

$$\overline{\tau}_{\overline{Y_1}\overline{Y_1}}=\frac{N_0}{1+\lambda_1}(\dot{\overline{r}}_{\overline{Y_1}\overline{Y_1}}+\zeta_2\ddot{\overline{r}}_{\overline{Y_1}\overline{Y_1}})$$

$$=\frac{2N_0}{1+\lambda_1}[\frac{\partial\overline{W_2}}{\partial\overline{Y_1}}+\zeta_2(\frac{\partial^2\overline{W_2}}{\partial\overline{t}\partial\overline{Y_1}}+\overline{W_1}\frac{\partial^2\overline{W_1}}{\partial\overline{X_1}\partial\overline{Y_1}}+\overline{W_2}\frac{\partial^2\overline{W_2}}{\partial\overline{Y_1}^2})] \quad ...(3.14)$$

Now, if we introduce the following non-dimensional parameters into Eq. (1-14) we obtain:

$$x=\frac{\overline{X_1}}{\zeta},\ y=\frac{\overline{Y_1}}{e},\ t=\frac{c_1\bar{t}}{\zeta},u=\frac{\overline{W_1}}{c_1},\ v=\frac{\overline{W_2}}{c_1\sigma_1},\delta_1=\frac{e}{\zeta},\ g_1=\frac{\overline{G_1}}{e},\ g_2=\frac{\overline{G_2}}{e},\theta=\frac{\overline{F}-F_0}{F_1-F_0},\varphi=\frac{\overline{f}-f_0}{f_1-f_0}$$

$$\mathrm{Re}n=\frac{\rho_1 c_1 e}{N_0},\mathrm{Pr}n=\frac{N_0\zeta_1}{k_2},M=\sqrt{\frac{\sigma}{N_0}}eB_0,\ a=\frac{e_1}{e},\ b=\frac{e_2}{e},\ Sci=\frac{N_0}{\rho_1 g_m},Sor=\frac{\rho_1 g_m k_f(F_1-F_0)}{\overline{F_m}N_0(f_1-f_0)},$$

$$Da=\frac{k_1}{e^2},\ \mathrm{Ec}=\frac{c_1^{\,2}}{\zeta_1(F_1-F_0)},\ D_f=\frac{g_m k_f(f_1-f_0)}{\zeta_1 N_0 k_s(F_1-F_0)},\ \tau=\frac{e}{N_0 c_1}\bar{\tau},\ P_1=\frac{e^2\overline{P_1}}{c_1\zeta N_0},\ \mathrm{Fr}=\frac{c_1^{\,2}}{ge},$$

$$Brn=\mathrm{Pr}n.Ec,\mathrm{A}=\frac{\overline{A}}{e},\mathrm{B}_1=\frac{\overline{B_1}}{e},\mathrm{B}_2=\frac{\overline{B_2}}{e},\mathrm{m}_1=\frac{\overline{\mathrm{m}_1\zeta}}{e} \qquad ...(3.15)$$

where δ_1 is the wavenumber, $\mathrm{Pr}n$ is the Prandtl number, Ec is the Eckert number, $\mathrm{Re}n$ is the Reynolds number, Brn is the Brinkman number, M is Hartmann number, $a\,\&\,\mathrm{b}$ are the amplitudes of the wave, Sci is the Schmidt number, Sor is the soret number, Da is the porous medium parameter, $\theta\,\&\,\varphi$ are the non-dimensional of temperature and concentration respectively, $\overline{P_1}$ is the pressure of the fluid, $F_0\,\&\,F_1$ are the temperature of the fluid at upper and lower side of the walls, $c_0\,\&\,c_1$ are the concentration of the fluid at upper and lower part of the walls $g_1\,\&\,g_2$ are non-dimensional of upper and lowers walls of the channel, D_f is the Dufour number, Fr is the Froude number.

So, Equations (3.1)-(3.14) will become:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad ...(3.16)$$

$$\mathrm{Re}n\delta_1\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial P_1}{\partial x}+\delta_1\frac{\partial}{\partial x}\tau_{xx}+\frac{\partial}{\partial y}\tau_{xy}+\frac{\rho_1\overline{\Omega}^2e^2}{N_0}u+2\delta_1^2\,\mathrm{Re}n\overline{\Omega}\frac{\partial v}{\partial t}$$

$$-N_1^2u+\frac{\mathrm{Re}n}{Fr}\sin\alpha_1 \qquad ...(3.17)$$

$$\mathrm{Re}n\delta_1^3\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial P_1}{\partial y}+\delta_1^2\frac{\partial}{\partial x}\tau_{xy}+\delta_1\frac{\partial}{\partial y}\tau_{yy}+\delta_1^2\frac{\rho_1\overline{\Omega}^2e^2}{N_0}v-2\delta_1^2\,\mathrm{Re}n\overline{\Omega}\frac{\partial u}{\partial t}+$$

$$\delta_1M^2\sin\beta\cos\beta u-\delta_1^2N_2^2v-\delta_1\frac{\mathrm{Re}n}{Fr}\cos\alpha_1 \qquad ...(3.18)$$

$$\mathrm{Re}n\delta_1\,\mathrm{Pr}n\left(u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}\right)=\delta_1^2\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+Brn[2\delta_1^2(\frac{\partial u}{\partial x})^2+2\delta_1^2(\frac{\partial v}{\partial y})^2+(\frac{\partial u}{\partial y}+\delta_1^2\frac{\partial v}{\partial x})^2]$$

$$+\mathrm{Prn\ Df}\left(\delta_1^2\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right) \qquad ...(3.19)$$

$$\mathrm{Re}n\delta_1Sci\left(u\frac{\partial\varphi}{\partial x}+v\frac{\partial\varphi}{\partial y}\right)=\left(\delta_1^2\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)+Sci\,Sor\left(\frac{\partial^2\theta}{\partial y^2}+\delta_1^2\frac{\partial^2\theta}{\partial x^2}\right) \qquad ...(3.20)$$

In which $N_1=\sqrt{M^2\cos^2\beta+\frac{1}{Da}},\ N_2=\sqrt{M^2\sin^2\beta+\frac{1}{Da}}$ **and the components of shear stress are:**

$$\tau_{xx}=\frac{2\delta_1}{1+\lambda_1}[1+\frac{\zeta_2c_1\delta_1}{e}(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y})]\frac{\partial u}{\partial x} \qquad ...(3.21)$$

$$\tau_{xy}=\frac{1}{1+\lambda_1}[1+\frac{\zeta_2c_1\delta_1}{e}(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y})](\frac{\partial u}{\partial y}+\delta^2\frac{\partial v}{\partial x}) \qquad ...(3.22)$$

$$\tau_{yy}=\frac{-2\delta_1}{1+\lambda_1}[1+\frac{\zeta_2 c_1\delta_1}{e}(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y})]\frac{\partial v}{\partial y} \quad ...(3.23)$$

Now, by using the approximations of small wavenumber δ_1 and it's orders and the low value of Reynolds number (Ren), Eq. (3.16)-(3.22) will be in the following form:

$$\frac{\partial P_1}{\partial x}=\frac{\partial}{\partial y}\tau_{xy}+\frac{\rho_1\overline{\Omega}^2 e^2}{N_0}u-N_1^2 u+\frac{\mathrm{Re}\,n}{Fr}\mathrm{si} \quad ...(3.24)$$

$$\frac{\partial P}{\partial y}=0 \quad ...(3.25)$$

$$0=\frac{\partial^2\theta}{\partial y^2}+Brn(\frac{\partial u}{\partial y})^2+\mathrm{Pr}\,nD_f\frac{\partial^2\varphi}{\partial y^2} \quad ...(3.26)$$

$$0=\frac{\partial^2\varphi}{\partial y^2}+SciSor\frac{\partial^2\theta}{\partial y^2} \quad ...(3.27)$$

$$\tau_{xx}=0\ ,\tau_{xy}=\frac{1}{1+\lambda_1}\frac{\partial u}{\partial y},\quad \tau_{yy}=0 \quad ...(3.28)$$

Now, if we introduce the stream function $\varphi(x,y)$ in Eq. (3.24) by taking the formula $u=\frac{\partial\psi}{\partial y}$,

$v=-\frac{\partial\psi}{\partial x}$, **we get** **… (3.29)**

$$\frac{\partial P}{\partial x}=\frac{1}{1+\lambda_1}\frac{\partial^3\psi}{\partial y^3}-(N_1^2-\frac{\rho_1\overline{\Omega}^2 e^2}{N_0})\frac{\partial\psi}{\partial y}+\frac{\mathrm{Re}\,n}{Fr}\sin\alpha_1 \quad ...(3.30)$$

From Eq. (3.25) we deduce that the pressure ρ_1 of the fluid doesn't depend on y, so if we derive Eq. (3.30) with respect to y we obtain:

$$0=\frac{1}{1+\lambda_1}\frac{\partial^4\psi}{\partial y^4}+(\frac{\rho_1\overline{\Omega}^2 e^2}{N_0}-N_1^2)\frac{\partial^2\psi}{\partial y^2} \quad ...(3.31)$$

The boundary conditions, which are used through this study, can represent in the following:

$$\psi=\frac{\iota}{2},\ \frac{\partial\psi}{\partial y}+\beta_1\frac{\partial^2\psi}{\partial y^2}=-1 \ \text{ at } \ y=g_2$$

$$\psi=-\frac{\iota}{2},\ \frac{\partial\psi}{\partial y}-\beta_1\frac{\partial^2\psi}{\partial y^2}=-1 \ \text{ at } \ y=g_1 \quad ...(3.32)$$

$$\theta=0,\varphi=0 \text{ at } y=g_2$$

$$\theta=1,\varphi=1 \text{ at } y=g_1 \quad ...(3.33)$$

In which, $g_2=A+B_2, g_1=A-B_1, A=1+m_1x, B_1=a\sin[2\pi(x-t)+\phi_1], B_2=b\sin[2\pi(x-t)]$ ...(3-34)

4. Problem's solution

By using the method of "A domain decomposition", the Eq. (3.31) can be written as:

$$\frac{\partial^4\psi}{\partial y^4}=S^2\frac{\partial^2\psi}{\partial y^2} \quad ...(4.1)$$

In which $S^2=(1+\lambda_1)(N_1^2-\frac{\rho_1\overline{\Omega}^2 e^2}{N_0})$**, an operator ($\bar{\iota}$) can write Eq. (4.1) as:**

$$\bar{\iota}\psi=S^2(\psi_m)_{yy} \quad ...(4.2)$$

In which $\bar{\iota} = \frac{\partial^4}{\partial y^4}$ **is a fourth-order difference operators** $(\bar{\iota})^{-1}$ **is a fourth-fold integration operator defined by:**

$$(\bar{\iota})^{-1} = \int_0^y\int_0^y\int_0^y\int_0^y (.)dydydydy \qquad ...(4.3)$$

if we are operating with $(\bar{\iota})^{-1}$ **on Eq. (4.2), we have :**

$$\psi = c_{11} + c_{12}y + c_{13}\frac{y^2}{2!} + c_{14}\frac{y^3}{3!} + S^2(\bar{\iota})^{-1}(\psi_m)_{yy} \qquad ...(4.4)$$

In which the function $c_{ij}\,(i=1, j=1,2,3,4)$ **can be obtained by using the boundary conditions, Eq. (3.32)**

The standard of A domain decomposition method, are get:

$$\psi = \sum_{m=0}^{\infty}\psi_m \qquad ...(4.5)$$

Where the components $(\psi_m), m \geq 0$ **, will be located frequently. The following repeated relation is got from Eq. (4.4):**

$$\psi_0 = c_{11} + c_{12}y + c_{13}\frac{y^2}{2!} + c_{14}\frac{y^3}{3!} + ... \qquad ...(4.6)$$

$$\psi_{m+1} = S^2(\bar{\iota})^{-1}(\psi_m)_{yy}, \mathrm{m} \geq 0$$

Hence, we have:

$$\psi_1 = \frac{1}{S^2}c_{13}\frac{(Sy)^4}{4!} + \frac{1}{S^3}c_{14}\frac{(Sy)^5}{5!} + ... \qquad ...(4.7)$$

$$\psi_2 = \frac{1}{S^2}c_{13}\frac{(Sy)^6}{6!} + \frac{1}{S^3}c_{14}\frac{(Sy)^7}{7!} + ... \qquad ...(4.8)$$

$$\vdots$$

$$\psi_m = \frac{1}{S^2}c_{13}\frac{(Sy)^{2m+2}}{(2\mathrm{m}+2)!} + \frac{1}{S^3}c_{14}\frac{(Sy)^{2m+3}}{(2\mathrm{m}+3)!}, \mathrm{m} \geq 0 \qquad ...(4.9)$$

Thus from Eq. (4.5), the formula for ψ **is given as:**

$$\psi = c_{11} + c_{12}y + \frac{1}{S^2}c_{13}(\cosh Sy - 1) + \frac{1}{S^3}c_{14}(\sinh Sy - Sy) \qquad ...(4.10)$$

$$u = (-c_4 + c_2 s^2 + c_4 Cosh[sy] + c_3 sSinh[sy])/s^2 \qquad ...(4\text{-}11)$$

The expression of temperature and concentration distributions as follow:

$$\theta = \frac{Brc_3{}^2y^2}{4(-1+W1)} - \frac{Brc_4{}^2y^2}{4s^2(-1+W1)} + a_1 + ya_2 + \frac{Brc_4{}^2Cosh[2sy]}{8s^4(-1+W1)} + \frac{Brc_3{}^2Cosh[2sy]}{8s^2(-1+W1)} + \frac{Brc_3c_4Sinh[2sy]}{4s^3(-1+W1)} \qquad ...(4\text{-}12)$$

$$\varphi = -((Brc3^2ScSry^2)/(4(-1+W1))) + (Brc_4{}^2ScSry^2)/(4s^2(-1+W1)) + b_1 + yb_2 - (Brc_4{}^2ScSrCosh[2sy])/(8s^4(-1+W1)) - (Brc_3{}^2ScSrCosh[2sy])(8s^2(-1+W1)) - (Brc_3c_4ScSrSinh[2sy])/(4s^3(-1+W1)); \qquad ...(4\text{-}13)$$

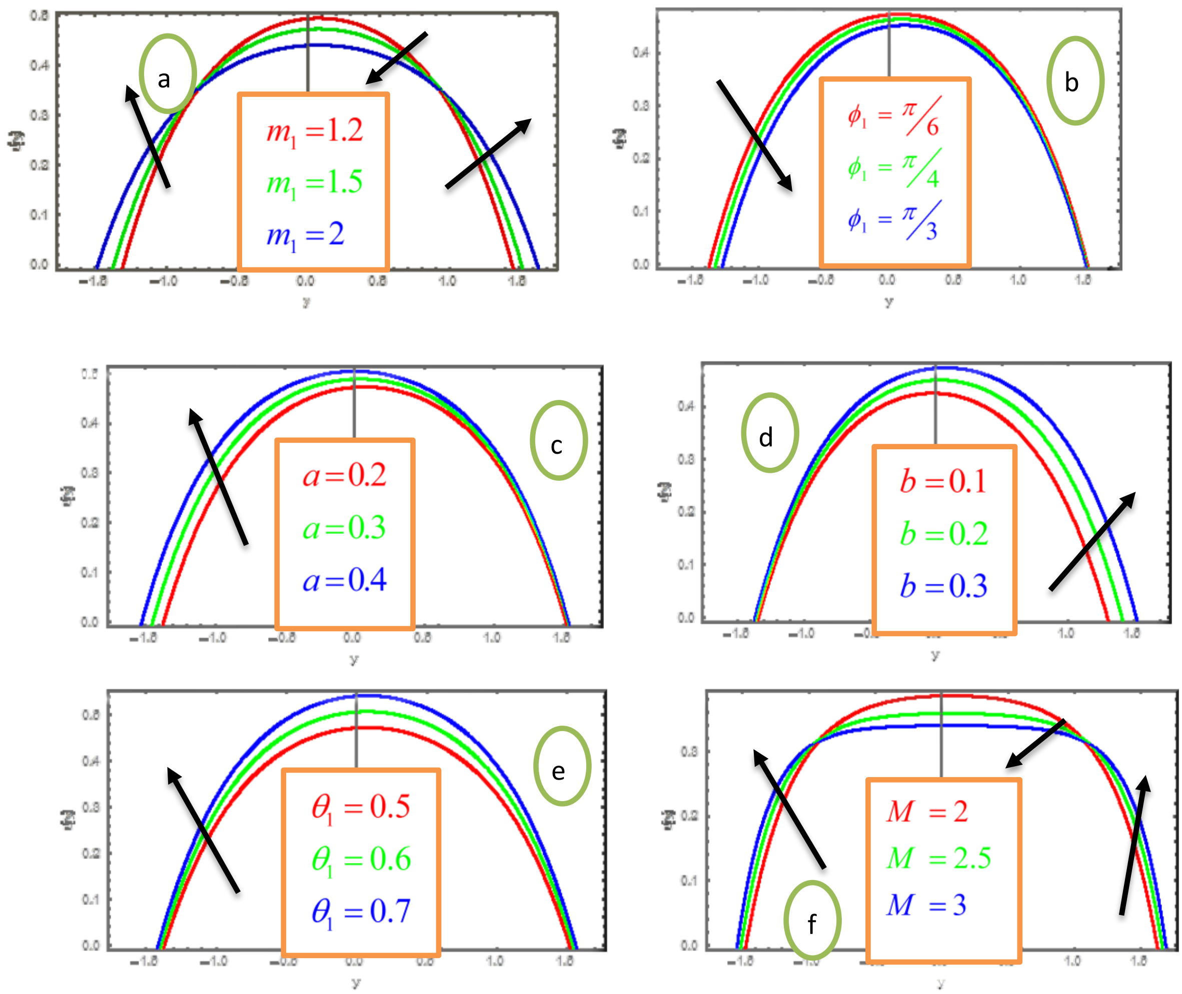
5 – Discussion of the problem's results

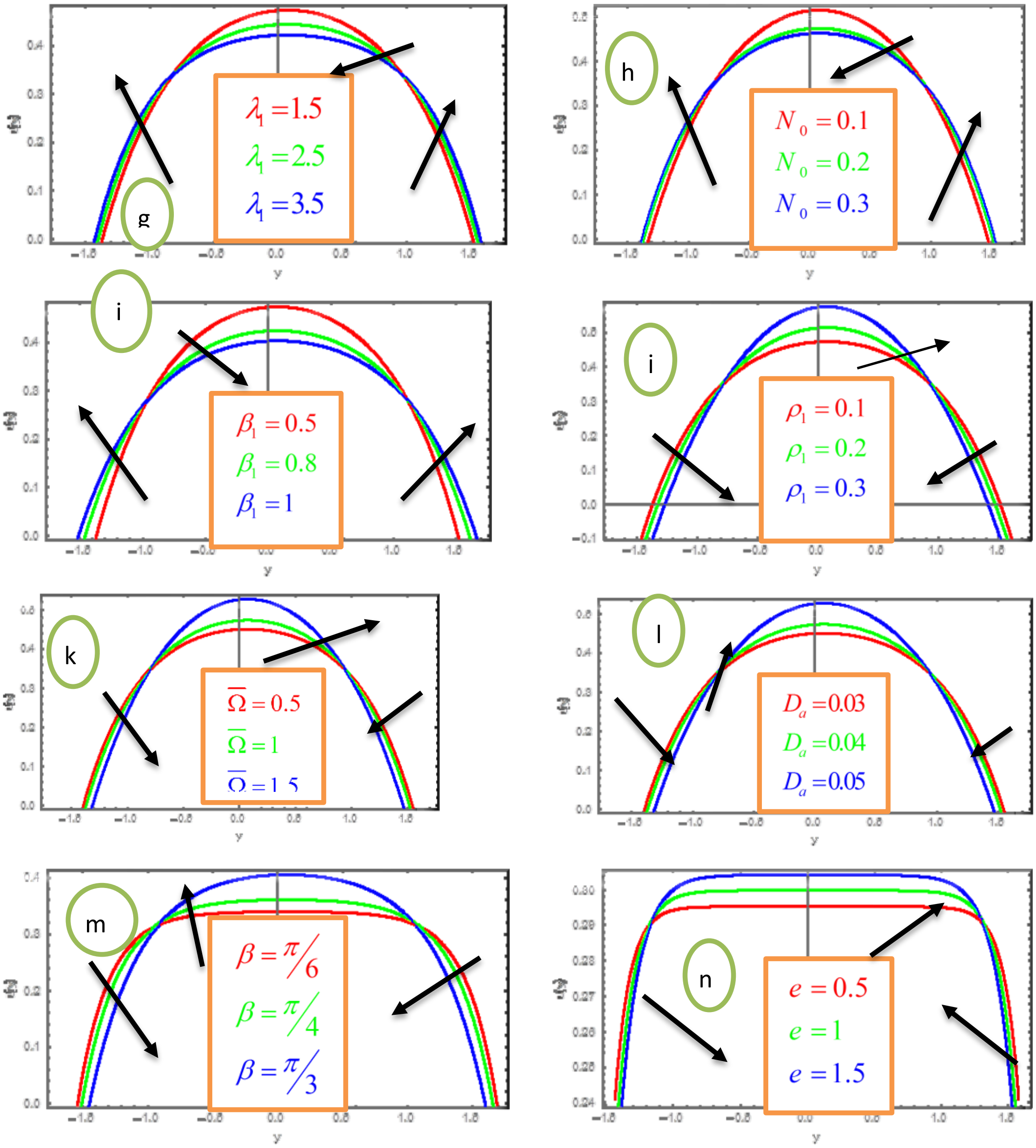
5.1 Velocity's distribution

By Equation (4-11), we can realize that velocity's profile is a function of y.

In this section, we have displayed the results of the problem and have discussed for different physical parameters of interest. Figure (1) have used to show the distribution of axial velocity

for various of the channel m_1, the phase difference of the channel (ϕ_1), the amplitudes of the channel's waves $(a\,\&\,b)$, Hartmann number (M), the fluid's material parameter (λ_1), fluid's density (ρ_1), fluid's viscosity (N_0), rotation parameter $(\overline{\Omega})$, channel's width $(2e)$, inclination angle of magnetic field (β), volume flow rate (θ_1) and the slip parameter (β_1). In figure (1-a), we observed that an increase in (m_1) leads to an increase in flow of fluid on the walls of the channel and decrease in the cort of channel at $y \in (-0.7, 0.9)$. Figure (1-b), shows the impact of (ϕ_1) on the velocity profile, it have found that the magnitude of velocity reduces at all the channel and especially at the lower wall of the channel. Figure (1-c,d,e) displays the effects of parameters $(a, b\,\&\,\theta_1)$ on the velocity, it have noted that their behavior on velocity is opposite to phase difference's behavior on it. The efforts of M, λ_1, N_0 and β_1 on the velocity distribution have sketched on the figures(1-f,g,h,i) and we noticed that the rising values of the last parameters results an increase to amount of flow on the sides of the channel and decreasing in the center.





Fig(1-a,b,…,n): Effect of parameters on velocity profile

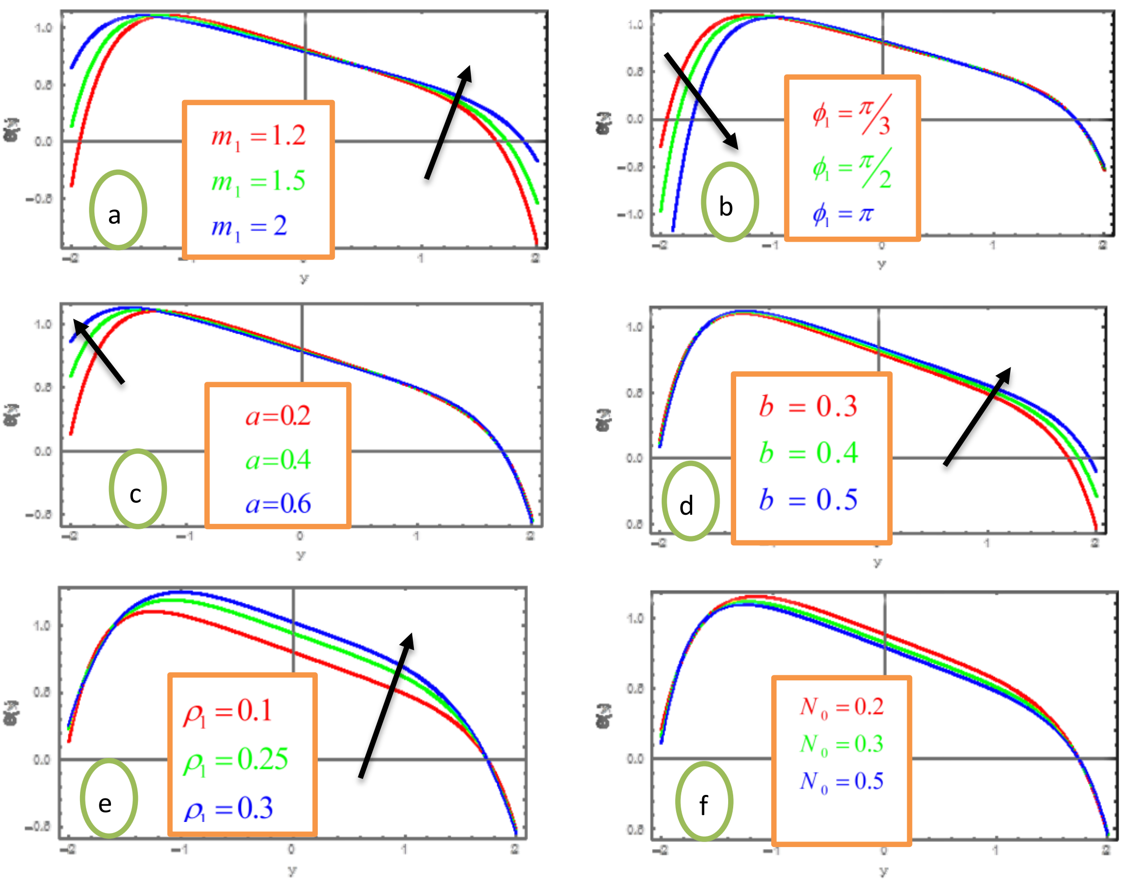
$m_1 = 1.5, t = 0.01, \phi_1 = \pi/6, = 0.2, b = 0.3, M = 1, \lambda_1 = 1.5, \rho_1 = 0.1,$

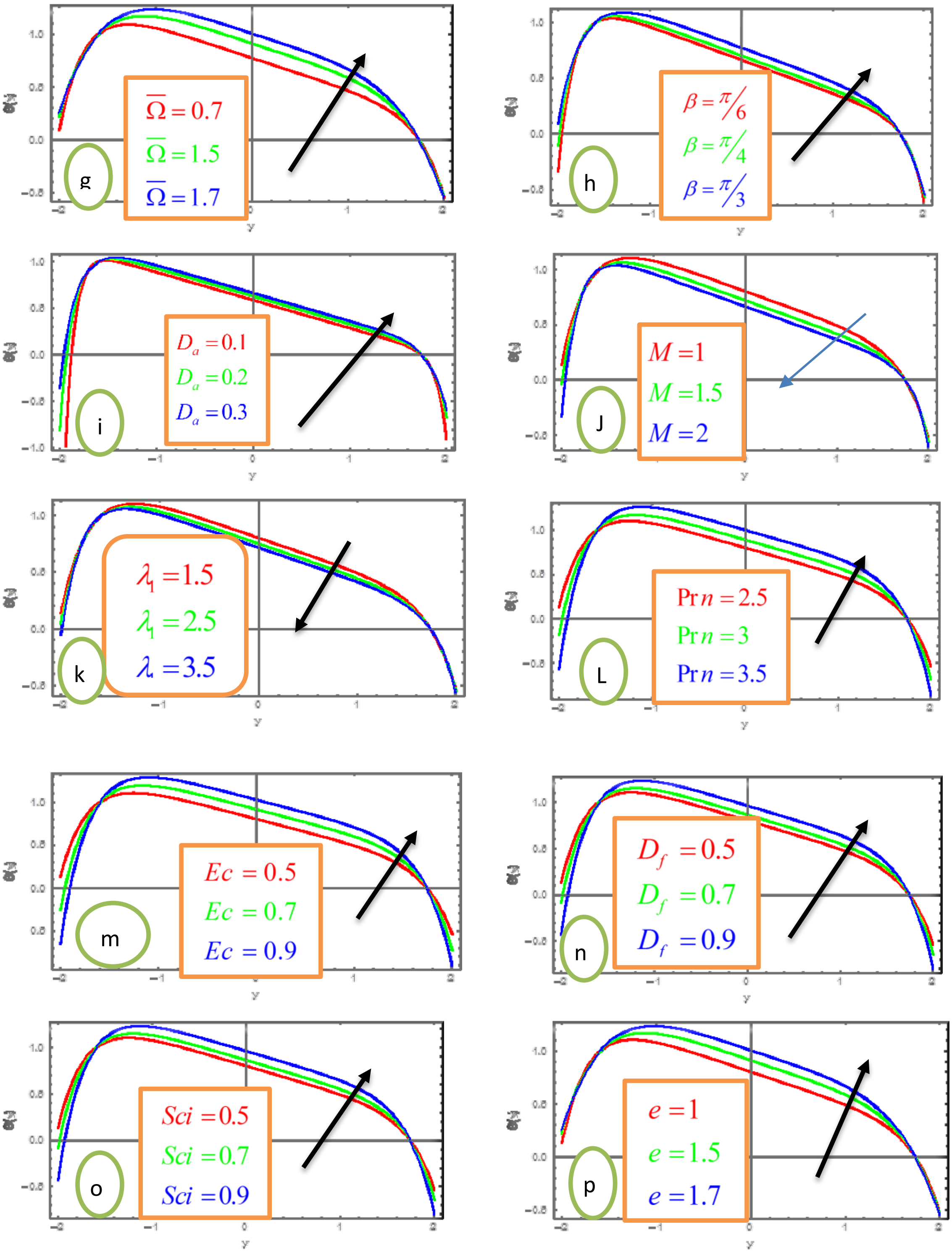
$N_0 = 0.2, \bar{\Omega} = 1, e = 1, \beta \to \pi/6, Da = 1, \theta_1 = 0.5, \beta_1 = 0.5, x = 0.3$

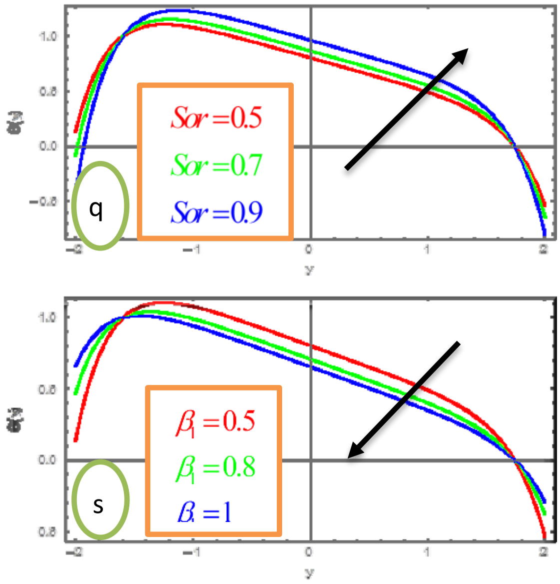
5.2 Temperature's characteristics

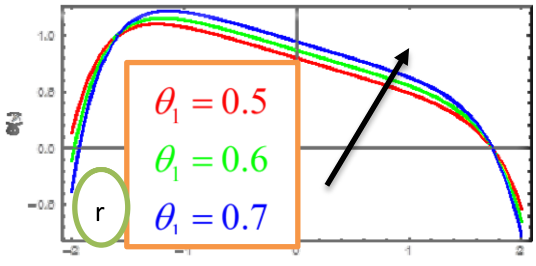
By Eq. (4-12), we can see that the distribution of fluid's temperature is a function of y. figure (2) have designed to show the changes of temperature distribution for various values of $m_1, \phi_1, a, b, M, \lambda_1, \rho_1, N_0, \bar{\Omega}, e, \beta, n, P, Ec, D_f, Sc, D_a, Sr, \theta_1, \beta_1$**.figure (2-a) have drawn to explain the effect of non-uniform parameter of channel** m_1 **on fluid's temperature, we have seen that the temperature increase on the walls of channel but it decreases in the part of center of channel when** $y \in (-1.2, 0.2)$**. The impact of phase difference** ϕ_1 **on the distribution of temperature, it observed that an increase in this parameter leads to reduce in the heat of fluid**

on the lower part of channel but there is a slight increase in the cort of channel when $y \in (-0.8, 0.02)$ a of wave's amplnd we can see that in figure (2-b). opposite effectiveness can see in figure (2-c)for the influence of wave's amplitude a on the heat of fluids and we see that it's temperature is low in the area when $y \in (-1.2, 0.3)$. Figure (2-d,e,f,g,h,i) are displayed the efforts of wave's amplitude b , fluid's density ρ_1, fluid's viscosity N_0, rotation parameter $\overline{\Omega}$, inclination angle of magnetic force β and Darcy number D_a on heats distribution, we have noted that the temperature of fluid increases in the all parts of channel with an increase of these parameters. adverse effective can notice in figure (2-j,k) for the actions of Hartmann number M and fluid's material parameter λ_1. Figure(2-l,m,n,o,p,q,r) is sketched to show the impress of Prandtl number $\Pr n$, Eckert number Ec, Dufour number D_f, Schmidt number Sci ,half width e, soret number Sor and volume flow rate θ_1,on the fluid's heat, we viewed that with an excess values of previous variables, the temperature of the fluid will raise at the middle of the channel but it goes down little at the endings of walls. In figure (2-s) , the effectuation of slip parameter β_1 on the fluid's temperature is offered, we have seen that this parameter show up cross attitude for the prandtl number's manner $\Pr n$ on the heat of fluid.




g
$\bar{\Omega} = 0.7$
$\bar{\Omega} = 1.5$
$\bar{\Omega} = 1.7$
h
$\beta = \pi/6$
$\beta = \pi/4$
$\beta = \pi/3$
i
$D_a = 0.1$
$D_a = 0.2$
$D_a = 0.3$
J
$M = 1$
$M = 1.5$
$M = 2$
k
$\lambda_1 = 1.5$
$\lambda_1 = 2.5$
$\lambda_1 = 3.5$
L
$\Pr n = 2.5$
$\Pr n = 3$
$\Pr n = 3.5$
m
$Ec = 0.5$
$Ec = 0.7$
$Ec = 0.9$
n
$D_f = 0.5$
$D_f = 0.7$
$D_f = 0.9$
o
$Sci = 0.5$
$Sci = 0.7$
$Sci = 0.9$
p
$e = 1$
$e = 1.5$
$e = 1.7$



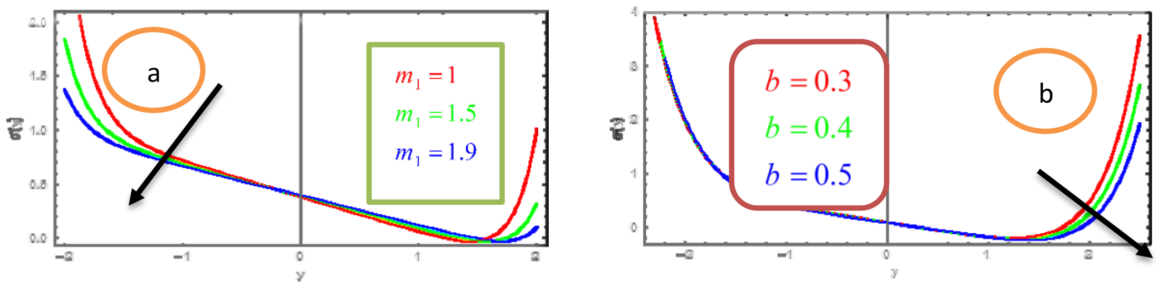


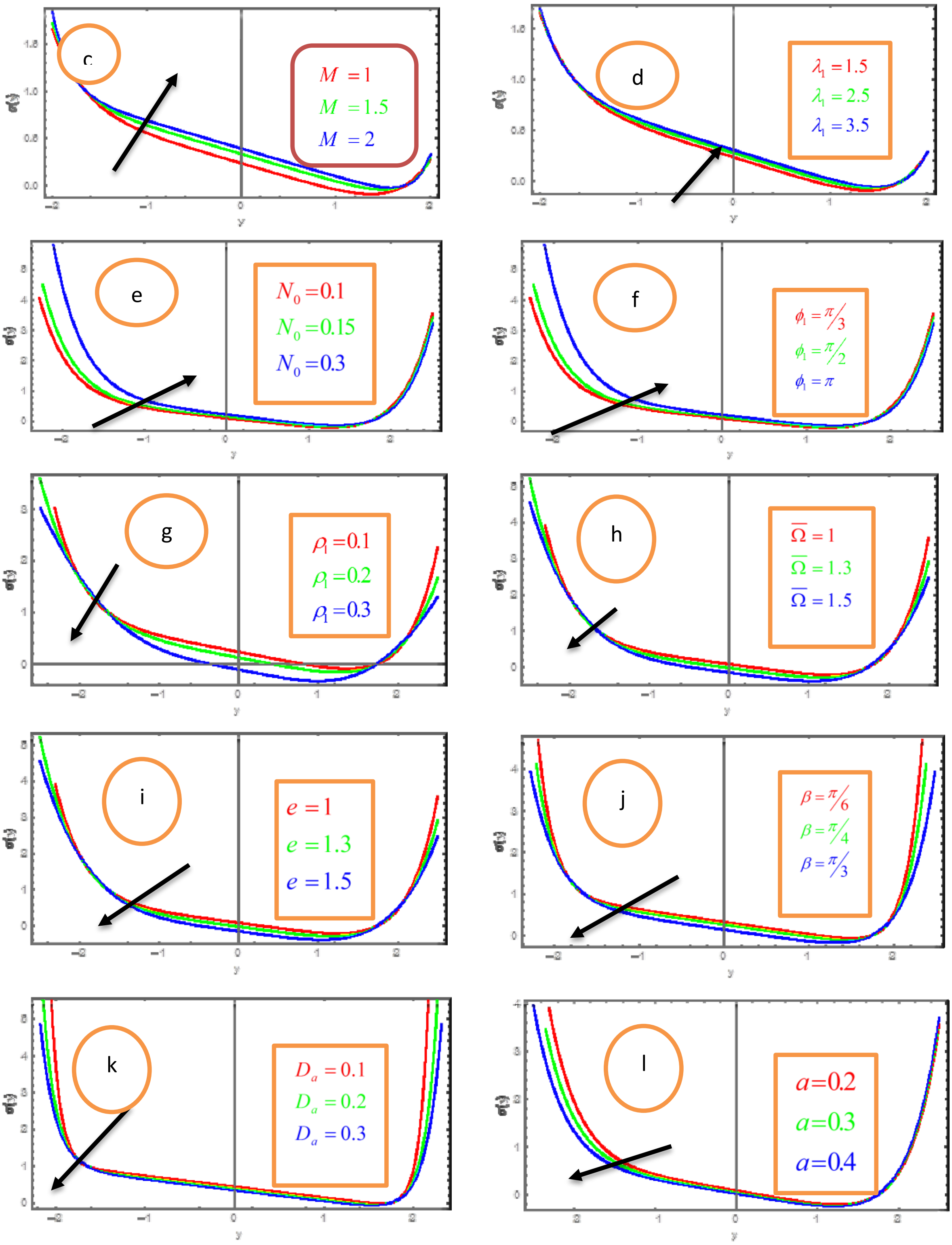
Fig(2-a,b,…s): Effect of parameters on temperature profile

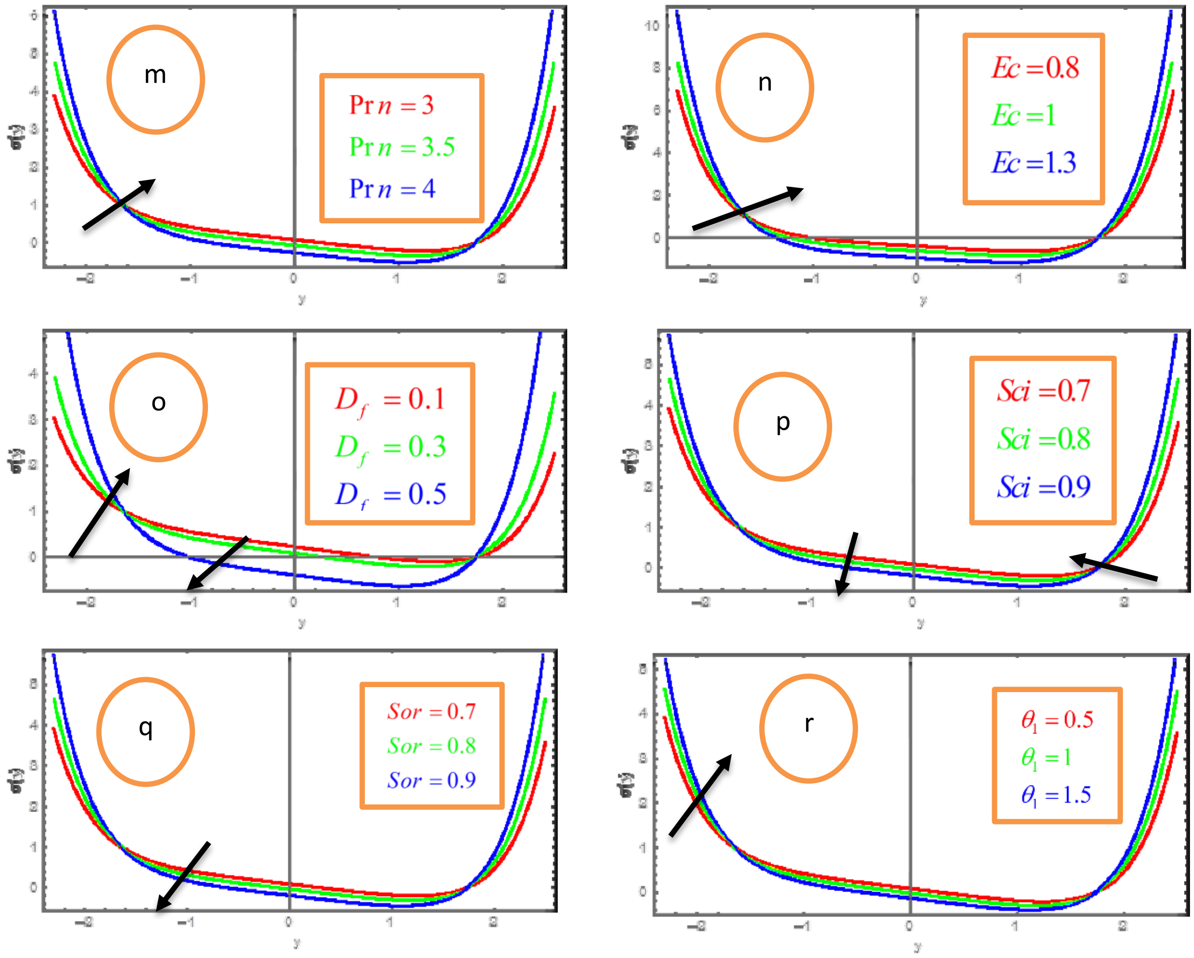
$t = 0.01, \phi_1 = \pi/6, a = 0.2, b = 0.3, M = 1,$
$\lambda_1 = 1.5, \rho_1 = 0.1, N_0 = 0.2, \overline{\Omega} = 1, e = 1, \beta_1 = 1.5,$
$\beta = \pi/6, \Pr n = 2.5, Ec = 0.5, D_f = 0.5, Sci = 0.5$
$D_a = 1, Sor = 0.5, \theta_1 = 0.5, m_1 = 1.5, x = 0.3$

5.3 Concentration's Profile

By equation (4-13), we can notice that the fluid's concentration is a function of y. figure (3)have drown to show the variation of concentration distribution for sundry value of $m_1, \phi_1, a, b, M, \lambda_1, \rho_1, N_0, \overline{\Omega}, e, \beta, \Pr n, Ec, D_f, Sci, D_a, Sor, \theta_1$ and β_1**. The concentration's profile is opposite of temperature profile and the variables have behaved inverse action on concentration than a fluid's heat. Figure (3-a) have depicted to state the effect of non-regularity parameter of channel** m_1 **on fluid's concentration, we have seen that the concentration decreases at the walls of the channel but it have started to increase by a slightly way at the upper wall when** $y \in (0.2, 1.5)$**. Opposite conduct on the impact of (b) which have shown in figure (3-b). figure (3-c,d,e,f) are displayed the efforts of** M, λ_1, N_0 and ϕ_1**, we have noted that the concentration is an increasing function of these parameters. figure (3-g,h,I,j,k,l) are sketched to clarify the actions of** $\rho_1, N_0, \overline{\Omega}, e, \beta, D_a$ and a **, we have observed that the concentration is an decreasing function of these parameters. the activity of** $\Pr n$ **on the fluid's concentration have formalized in figure (3-m), we have perceived that the concentration is less in the center of channel but it is more at the walls of the channel. Similar effectiveness for the influence of** Ec, D_f, Sci, Sor and θ_1 **and their efficacy have shown in figure (3-n, o, p, q), but we can see the inverse impress for the parameter** β_1 **and have seen it's influence in figure (3-s).**




c
$M = 1$
$M = 1.5$
$M = 2$
d
$\lambda_1 = 1.5$
$\lambda_1 = 2.5$
$\lambda_1 = 3.5$
e
$N_0 = 0.1$
$N_0 = 0.15$
$N_0 = 0.3$
f
$\phi_1 = \pi/3$
$\phi_1 = \pi/2$
$\phi_1 = \pi$
g
$\rho_1 = 0.1$
$\rho_1 = 0.2$
$\rho_1 = 0.3$
h
$\overline{\Omega} = 1$
$\overline{\Omega} = 1.3$
$\overline{\Omega} = 1.5$
i
$e = 1$
$e = 1.3$
$e = 1.5$
j
$\beta = \pi/6$
$\beta = \pi/4$
$\beta = \pi/3$
k
$D_a = 0.1$
$D_a = 0.2$
$D_a = 0.3$
l
$a = 0.2$
$a = 0.3$
$a = 0.4$



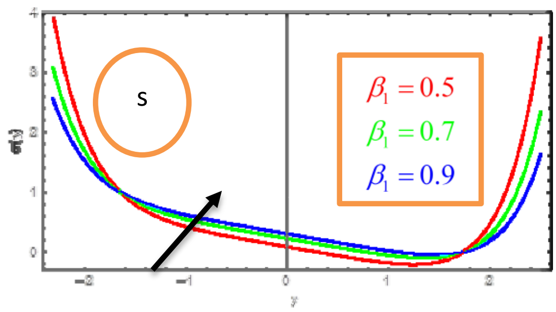


Fig (3-a,b,…,s): effect of parameter on concentration profile.

$t = 0.1, \phi_1 = \pi/6, a = 0.2, b = 0.3, M = 1,$

$\lambda_1 = 1.5, \rho_1 = 0.1, N_0 = 0.2, \overline{\Omega} = 1, e = 1,$

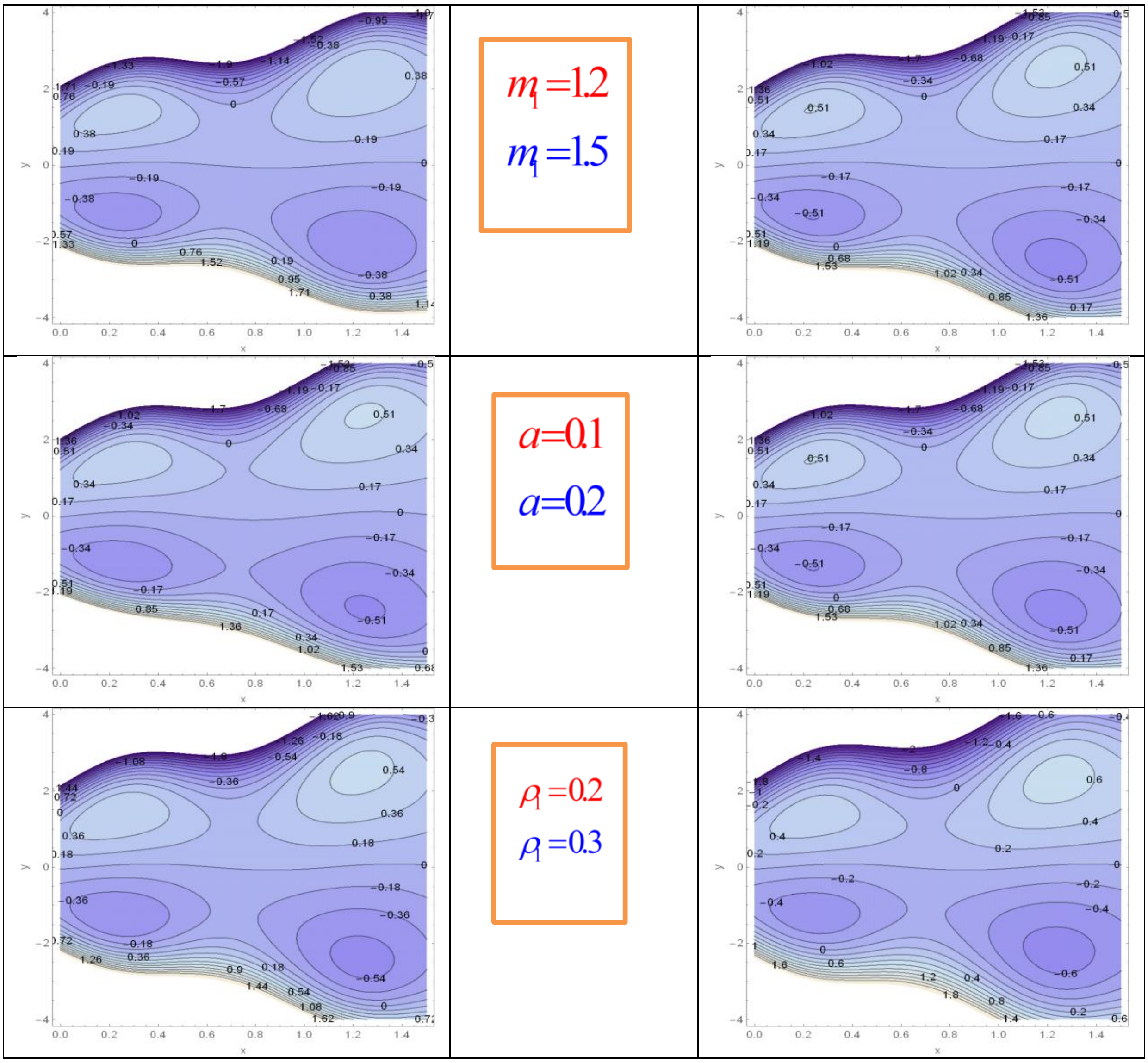
$\beta = \pi/6, \Pr n = 3, Ec = 0.8, D_f = 0.3, Sci = 0.7$

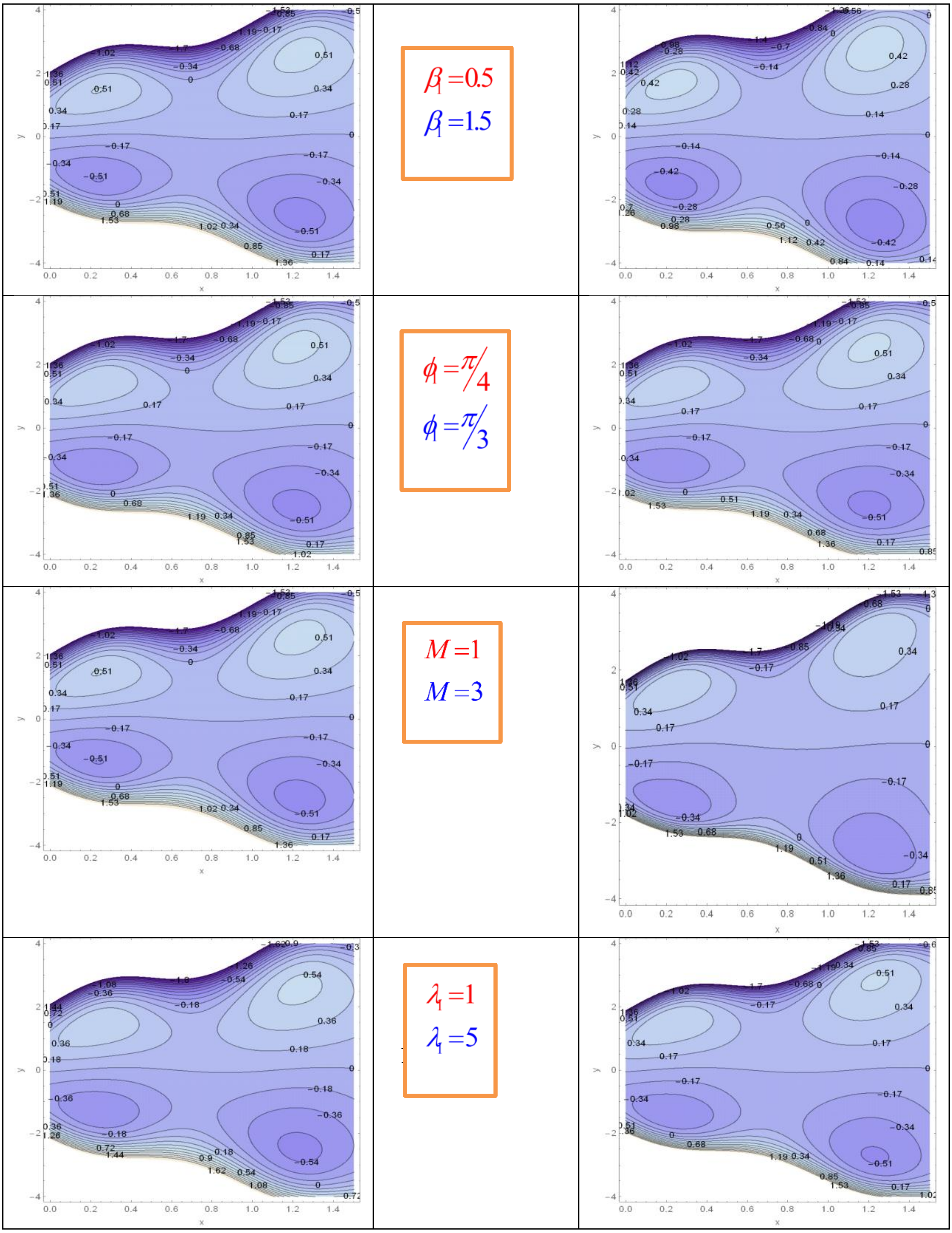
$D_a = 2, Sor = 0.7, \theta_1 = 0.5, m_1 = 1.5, x = 0.3$

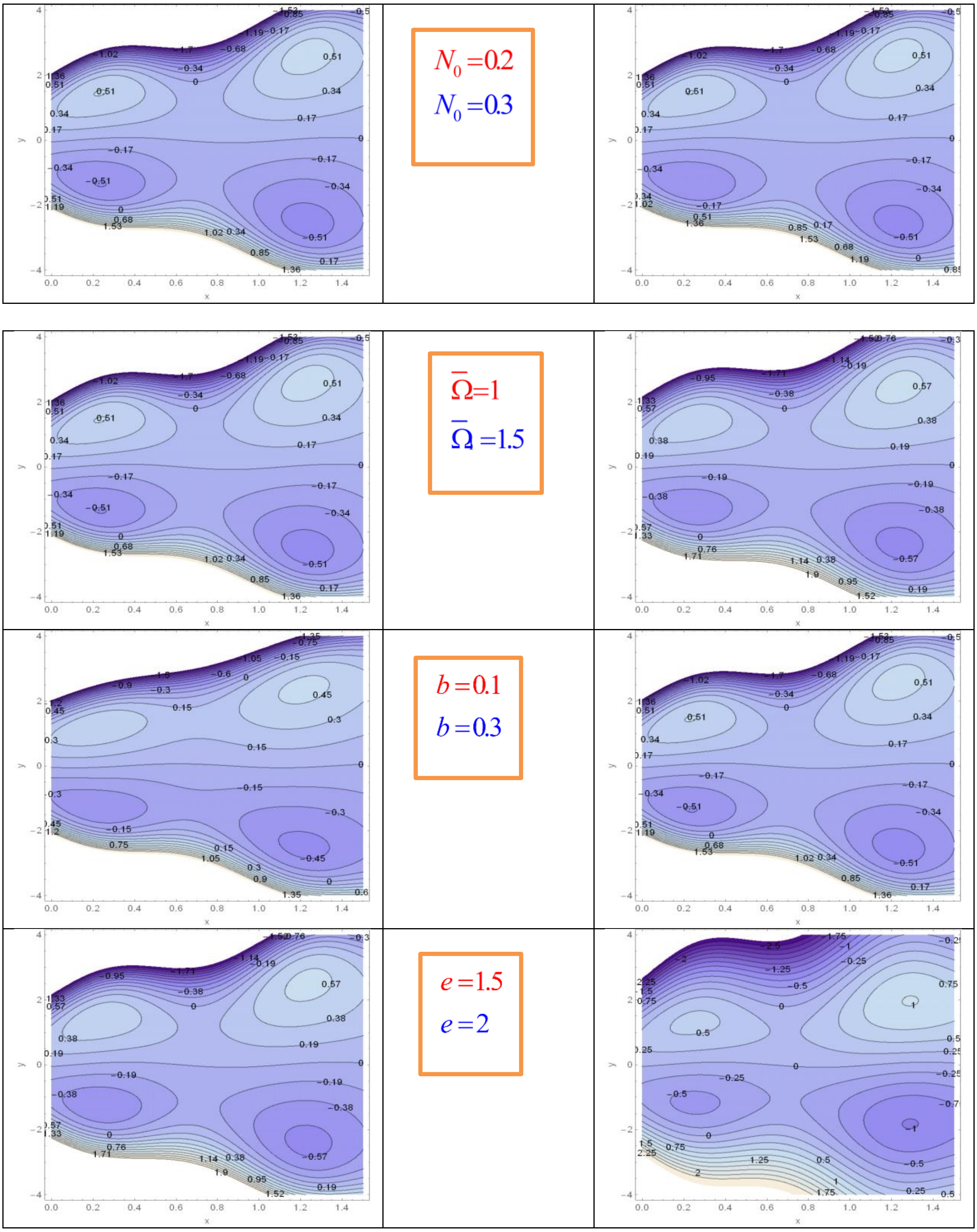
5.4 Phenomenon of fluid's waves stream

The phenomenon of fluid's trapping is an motivating them in wave's transporting of fluids. The formulation of an inwardly revolving bolus of fluid through enclosed stream lines is known by trapping and this trapping bolus is derived a head a long with the contracted waves. The impacts of various parameters like $m_1, \phi_1, a, b, M, \lambda_1, \rho_1, N_0, \overline{\Omega}, e, \beta, \Pr n, Ec, D_f, Sci, D_a, Sor, \theta_1$ and β_1 **on trapping have seen through the figures (4-17). Figures (4-a,b)-(7-a,b) show that the number and size of trapping bolus increase with an increase of** m_1, a, ρ_1 and β_1**. Inverse situation can noticed in the figures (8-a,b)-(12-a,b) for the actions of** $\phi_1, M, \lambda_1, N_0$ and $\overline{\Omega}$**. The effect of b is sketched in figure (13-a,b), at the beginning, we have noted that there is a connected wave but it have taken to separated different waves which**

is increasing in volume and number. The influence of e have illustrated in figure (14-a,b), it have observed there is an increasing in volume and number of bolus in the right side of channel when $0.8 < x < 1.5$ and there is a decreasing in the size and number of bolus in the left part of channel when $0 < x < 0.6$. Similar effect for the activity of β and D_a on the waves of fluid and their effect have represented in figure (15-a,b)-(16-a,b) respectively, and we have noticed that there is clear boost in number of bolus in the right wall of channel when $0.8 < x < 1.5$. Where as in figure (17-a,b), we have viewed the contrary demeanor for the work of θ_1 on the fluid's waves, we have recognized that the bolus of fluid have gone down in number for both sides of channel but they have enhanced in the size.




$\beta_1 = 0.5$
$\beta_1 = 1.5$
$\phi_1 = \pi/4$
$\phi_1 = \pi/3$
$M = 1$
$M = 3$
$\lambda_1 = 1$
$\lambda_1 = 5$



$N_0 = 0.2$
$N_0 = 0.3$
$\bar{\Omega} = 1$
$\bar{\Omega} = 1.5$
$b = 0.1$
$b = 0.3$
$e = 1.5$
$e = 2$

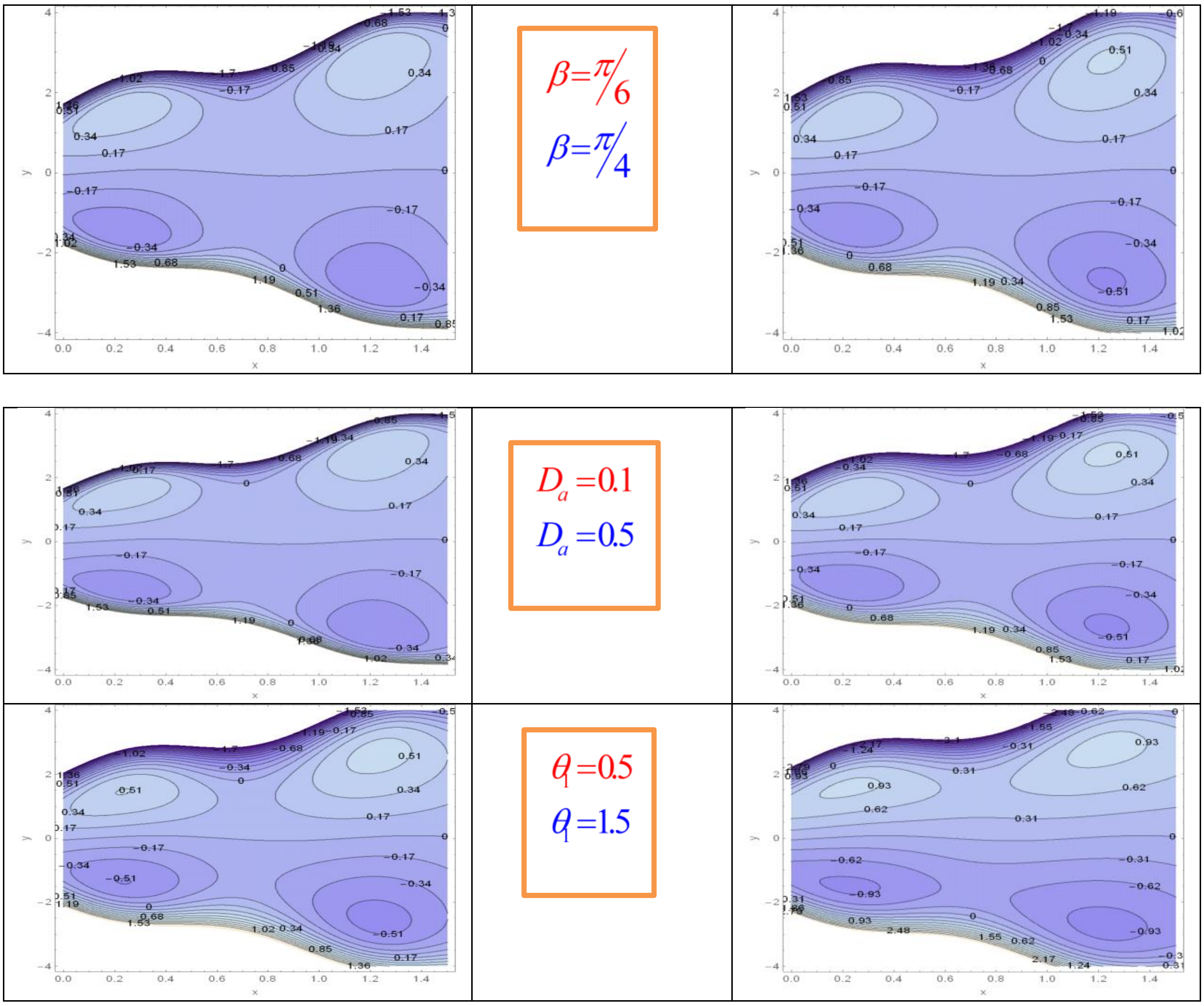


Fig (4-a,b,…,n): Effect of parameters on streamline

$m_1 = 1.5, t = 0.01, \phi_1 \to \pi/6, a = 0.2, b = 0.3, M = 3, \lambda_1 = 1.5,$

$\rho_1 = 0.1, N0 = 0.2, \bar{\Omega} = 1, e = 1, Da = 1, \theta_1 = 0.5, \beta_1 = 0.5$

6- Inferred notes for the problem

In the present study, we deal with the waveform transport of non-Newtonian fluid under the combined influence of inclined magnetic field and heat /mass transfer in the porous medium of non-symmetric inclined channel by using the effect of rotation parameter of the channel. Thus through our study we have conclude the following observations:

1. **On the velocity's distribution, there is an enhancement on it's profile with an increase values of non-uniform parameter (m_1) of the channel, amplitudes of channel (a& b), Hartmann number M, fluid's material parameter λ_1, fluid's viscosity N_0, volume flow rate of fluid θ_1 and slip parameter β_1. Opposite case is satisfied with an increase values of phase of fluid's wave (ϕ_1), fluid's density ρ_1. Rotation parameter of the channel $\bar{\Omega}$, half-width of channel (e) and slopping angle of magnetic field β.**
2. **On temperature's distribution ; there is an ascending on it's profile with an rising magnitude of left amplitude of wave (b), fluid's density ρ_1,darcy number D_a, rotation**

parameter $\overline{\Omega}$, half-width of channel (e) and slopping angle of magnetic field β, inverse status is achieved with an increase of Hartmann number M , fluid's material parameter λ_1 and fluid's viscosity N_0 .

3. **With an increase of right amplitude of wave a . There is clear increasing on fluid's heat on left wall of the channel and there is slight reducing in the middle part of the channel. We can see the opposite behavior for the influence of wave's phase.**
4. **With an increase of non-uniform parameter of channel and slip parameter β_1 . There is clear increasing on fluid's temperature on the walls of the channel and it decreases at the center of channel. The contrary case can be seen with an increase of prandtl number $\Pr n$, Eckert number Ec , Dufour number D_f , Schmidt number Sci ,soret numbe Sor and volume flow rate of the wave θ_1 .**
5. **There is a seriousness relationship between the distribution of velocity of fluid and it's temperature.**
6. **There is discrepant relationship between the distribution of fluid's temperature and it's concentration. So, we have noticed that the fluid's concentration is an ascending function of the parameters M, λ_1, N_0 and it is decreasing function of the parameters $\rho_1, \overline{\Omega}, e, \beta, D_a, a$.**
7. **With an increase of the following parameters $\Pr n, Ec, D_f, Sci, Sor, \theta_1$, the fluid's concentration have increased at the walls of the channel and have decreased at the center of the channel. We can observe the inverse case with an increase of (m_1, b, β_1) .**
8. **The number and size of the trapping bolus have increased with an increase of $m_1, (a,b), \rho_1$ *and* β_1 . Opposite p light with an increase of $\phi_1, M, \lambda_1, N_0$ *and* $\overline{\Omega}$.**
9. **With increase values of parameters, (e, β, D_a), there is clear increasing in size and number of bolus in the right side of channel and clear decreasing in it in the left side of channel.**
10. **The influence of volume flow rate θ_1 on the trapping bolus of fluid's waves have promoted basically the size of these bolus, but it have negative effect on their number on both sides of channels walls.**

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Intuitionistic fuzzy *S*-filter in *Q*-algebra

Habeeb Kareem Abdullah[1], Rakzan Shaker Naji[2]

Mathematics Department, University of Kufa

[1]habeebk.abdullah@uokufa.edu.iq

Mathematics Department, University of Kufa

[2]shakrakzan192@gmail.com

Abstract : This project aimed to concept types intuitionistic fuzzy $\mathcal{S}$-filter and intuitionistic fuzzy complete $\mathcal{S}$-filter of $\mathcal{Q}$-algebra. Showing the relationship between the different types of intuitionistic fuzzy filters and condition that must be put on $\mathcal{Q}$-algebra using an example putting forward to explain that .Explore the properties of the types of intuitionistic fuzzy filter, finally a chart has been drawn up showing the types and relationship between them .

Keywords: $\mathcal{Q}$-algebra,$\mathcal{S}$-filter, complete $\mathcal{S}$-filter, intuitionistic fuzzy $\mathcal{S}$-filter,, intuitionistic fuzzy complete $\mathcal{S}$-filter.

1. Introduction :

In 1965,Zadeh .L. [14] introduced in the real physical world the notion of fuzzy sub set of the set as a tool for verbal doubt. Atanassov K.T. [4,5] further described The generalization of Intuitionistic fuzzy, Takeuti .G and Titanti S.[13] have also intuitionistic fuzzy sets , but Titanti S. intuitionistic fuzzy mysterious logic in the narrow sense and they derive from the set theory of logic which they said to by (Intuitionistic fuzzy set theory). In 2001 Neggers .J. and Ahn SS,KimHS,[11] We introduced a new idea ,called $\mathcal{Q}$-algebra, $\mathcal{Q}$-algebra considered generalization of some types algebras (BCK/ BCH/BCI-algebras).In this work, we introduce the idea of (Intuitionistic fuzzy $\mathcal{S}$-filter,Intuitionistic fuzzy $\mathcal{C}$- $\mathcal{S}$-filter) of $\mathcal{Q}$-algebra, also some properties , relationship and condition between different Intuitionistic fuzzy filters of $\mathcal{Q}$-algebra.

2. Background:

In this part of our subject, we have provided some basic concepts of $\mathcal{Q}$-algebra, types of filters and we need in our work.

Definition 2.1:[11]

A set $\mathcal{X}$ is called $\mathcal{Q}$- algebra with a binary operation " $*$ " and constant "0", if $\forall x, y, z \in \mathcal{X}$,then

$\mathcal{Q}_1$- $x * x = 0$

$\mathcal{Q}_2$- $x * 0 = x$

$\mathcal{Q}_3$- $(x * y) * z = (x * z) * y$

A binary relation denoted $\preccurlyeq$ we will define on $\mathcal{X}$, then $x \preccurlyeq y \Leftrightarrow x * y = 0$,$\forall x, y \in \mathcal{X}$.

Definition 2.2:[2]

Let $(\mathcal{X},*,0)$ be a $\mathcal{Q}$ -algebra , if there is a special element $e \in \mathcal{X}$ $if x \preccurlyeq e$, $for\ all\ x \in \mathcal{X}$,then e is called an unit of $\mathcal{X}$. A $\mathcal{Q}$- algebra with unit is called the bounded

Remark2.3:[2]

1-we denoted $e * x$ by x^* $for\ each\ x \in \mathcal{X}$,such that $\mathcal{X}$ a bounded $\mathcal{Q} -$ algebra.

2- In a bounded $\mathcal{Q}$-algebra , then $x^* * y = y^* * x, if\ x, y \in \mathcal{X}$

Remark2.4:

From now on ,all $\mathcal{Q} -$ algebra **is a bounded with unite is unique . also the sets** $\mathcal{X}\ and\ \mathcal{Y}\ are\ \mathcal{Q} -$ algebra .

Definition2.5:[11]

If $\mathfrak{f}: (\mathcal{X},*,0) \longrightarrow (\mathcal{Y},*,0)$ is a mapping, then$\mathfrak{f}$ *is said to be*

(1) *Homomorphism* . $if\ \mathfrak{f}(x * y) = \mathfrak{f}(x) * \mathfrak{f}(y), for\ all\ x, y \in \mathcal{X}$.

(2)*Monorphism*. *if* $\mathfrak{f}$ *is an injective homomorphism* .

(3)*Epimorphismif* $\mathfrak{f}$ *is a surjective homomorphism* .

(4)*Isomorphisomif* $\mathfrak{f}$ *is a surjective and injective homomorphism.*

Proposition 2.6:[8]

Let $\mathfrak{f}: (\mathcal{X},*,0) \longrightarrow (\mathcal{Y},*,0)$ be a mapping epimorphism, then:

(1) $\mathfrak{f}(x^*) = (\mathfrak{f}(x))^*, for\ each\ x \in \mathcal{X}$
(2) if $e\ is$ a units of $\mathcal{X}$ and e' is a units of $\mathcal{Y}$, then $\mathfrak{f}(e) = e'$.

Definition2.7: [2]

If $x^{**} = x$ then $x\ it$ is called an involution , such that $x \in \mathcal{X}$. If for all $x \in \mathcal{X}$ is an involution then $\mathcal{X}$ is called an involutory .

Proposition2.8:[12]

In an involutory Q- algebra , for all $x, y \in \mathcal{X}$, then .
1- If $x \preccurlyeq y^*$ then $y \preccurlyeq x^*$
2- $x * y^* = y * x^*$

Definition2.9: [**4**]

The intuitionistic fuzzy sets (shortly, IFS) are defined on a non-empty set $\mathcal{X}$,as objects having the form $\mathcal{A} = \{(x, \vartheta_{\mathcal{A}}(x), \Omega_{\mathcal{A}}(x)) : x \in \mathcal{X}\}$, where the functions $\vartheta_{\mathcal{A}} : \mathcal{X} \to [0,1]$, $\Omega_{\mathcal{A}} : \mathcal{X} \to [0,1]$ mean the degree of membership and mean the degree of nonmember ship ,correspondingly , such that $0 \preccurlyeq \vartheta_{\mathcal{A}}(x) + \Omega_{\mathcal{A}}(x) \preccurlyeq 1$.For ease the form is used $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$.

Definition2.10: [7]

An IFS $\mathcal{A} = (\vartheta_{\mathcal{A}}(x), \Omega_{\mathcal{A}}(x))$ of a non-empty set $\mathcal{X}$. Then

(1)$\lozenge\ \mathcal{A} = \{(x, 1 - \Omega_{\mathcal{A}}(x), \Omega_{\mathcal{A}}(x)) : x \in \mathcal{X}\} = \{\left(x, \overline{\Omega_{\mathcal{A}}}(x), \Omega_{\mathcal{A}}(x)\right) : x \in \mathcal{X}\}$

(2) $\square\mathcal{A} = \left\{(x, \vartheta_{\mathcal{A}}(x), 1 - \vartheta_{\mathcal{A}}(x)) : x \in \mathcal{X}\right\} = \{\left(x, \vartheta_{\mathcal{A}}(x), \overline{\vartheta_{\mathcal{A}}}(x)\right) : x \in \mathcal{X}\}$

Definition2.11:[7]

Let $\mathcal{A} = (\vartheta_{\mathcal{A}}(x), \Omega_{\mathcal{A}}(x))$ and $\mathcal{K} = (\vartheta_{\mathcal{K}}(x), \Omega_{\mathcal{K}}(x))$ $the\ sets\ of\ \mathcal{X}$,then

(1)$\mathcal{A} \cup \mathcal{K} = \left\{(x, \vartheta_{\mathcal{A}}(x) \vee \vartheta_{\mathcal{K}}(x), \Omega_{\mathcal{A}}(x) \wedge \Omega_{\mathcal{K}}(x)) : x \in \mathcal{X}\right\}$
$= \{(x, max(\vartheta_{\mathcal{A}}(x), \vartheta_{\mathcal{K}}(x)), min(\Omega_{\mathcal{A}}(x), \Omega_{\mathcal{K}}(x)) : x \in \mathcal{X}\}$.

(2)$\mathcal{A} \cap \mathcal{K} = \left\{(x, \vartheta_{\mathcal{A}}(x) \wedge \vartheta_{\mathcal{K}}(x), \Omega_{\mathcal{A}}(x) \vee \Omega_{\mathcal{K}}(x)) : x \in \mathcal{X}\right\}$.
$= \{(x, min(\vartheta_{\mathcal{A}}(x), \vartheta_{\mathcal{K}}(x)), max(\Omega_{\mathcal{A}}(x), \Omega_{\mathcal{K}}(x))) : x \in \mathcal{X}\}$

Definition2.12:[7]

Let $\{\mathcal{A}_i, i \in \Delta\}$ by a family of IFS in set $\mathcal{X}$. then

1- $\cap \mathcal{A}_i = \{(x, \vee\ \vartheta_{\mathcal{A}\mathrm{i}}(x), \wedge\ \Omega_{\mathcal{A}\mathrm{i}}(x)) : x \in \mathcal{X}\}$.

2- $\cup \mathcal{A}_i = \{(x, \vee\ \vartheta_{\mathcal{A}\mathrm{i}}(x), \wedge\ \Omega_{\mathcal{A}\mathrm{i}}(x)) : x \in \mathcal{X}\}$.

Where $(\wedge \mathcal{A}_i)(x) = inf\ \{\delta_{\mathcal{A}_i}(x), \mathrm{i} \in \Delta\}$, and$(\vee\ \delta_{\mathcal{A}_i})(x) = sup\{\delta_{\mathcal{A}_i}(x), \mathrm{i} \in \Delta\}$

Definition2.13:[4]

If $\mathfrak{f} : \mathcal{X} \longrightarrow \mathcal{Y}$ is a mapping, IFS $\mathcal{K} = \left\{< y, \vartheta_{\mathcal{K}}(y), \Omega_{\mathcal{K}}(y) >: y \in \mathcal{Y}\right\}$ in $\mathcal{Y}$,the $\mathfrak{f}^{-1}(\mathcal{A})$ pre-image of $\mathcal{K}$ under $\mathfrak{f}$ is the IFS in $\mathcal{X}$ denoted by $\mathfrak{f}^{-1}(\mathcal{K})$is the IFS in $\mathcal{X}$ defined by :
$\mathfrak{f}^{-1}(\mathcal{K}) = \{(x, \mathfrak{f}^{-1}(\vartheta_{\mathcal{K}}(x), \mathfrak{f}^{-1}(\Omega_{\mathcal{K}}(x))), x \in \mathcal{X}\}$, such that :
$\mathfrak{f}^{-1}(\Omega_{\mathcal{K}}(x)) = \Omega_{\mathcal{K}}(\mathfrak{f}(x))$, $\mathfrak{f}^{-1}\left(\vartheta_{\mathcal{K}}(x)\right) = \vartheta_{\mathcal{K}}(\mathfrak{f}(x))$.

If IFS $\mathcal{A} = \{(x, \vartheta_{\mathcal{A}}(x), \Omega_{\mathcal{A}}(x)) : x \in \mathcal{X}\}$ in , the image of $\mathcal{X}$ under $\mathfrak{f}$ denoted by :
$\mathfrak{f}(\mathcal{A}) = \{(y, \mathfrak{f}_{sup}(\vartheta_{\mathcal{A}}(y)), \mathfrak{f}_{\inf}(\Omega_{\mathcal{A}}(y))) : y \in \mathcal{Y}\}$, where

$$\mathfrak{f}_{sup}(\vartheta_{\mathcal{A}}(y)) = \begin{cases} sup_{x \in \mathfrak{f}^{-1}(y)} \delta_{\mathcal{A}}(x) & , \text{ if } \mathfrak{f}^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}, \text{ and}$$

$$\mathfrak{f}_{\inf}(\Omega_{\mathcal{A}}(y)) = \begin{cases} inf_{\ x \in \mathfrak{f}^{-1}(y)} \gamma_{\mathcal{A}}(x) & , \text{ if } \mathfrak{f}^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}, \ for\ all\ y \in \mathcal{Y}$$

Definition2.14:[12]

If $\mathcal{N} \sqsubseteq \mathcal{X}, then\ \mathcal{N}$ is said to be a Q-filter of $\mathcal{X}$,if $for\ all\ x, y \in \mathcal{X}$,then
F_1 - $e \in \mathcal{N}$ F_2 - $(x^* * y^*)^* \in \mathcal{N}$, $y \in \mathcal{N}$ implies $x \in \mathcal{N}$.

Definition2.15:[1]

An IFS $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ is called intuitionistic fuzzy Q-filter of $\mathcal{X}$,(briefly IFS-Q -filter), if:

I_1- $\vartheta_{\mathcal{A}}(e) \geqslant \vartheta_{\mathcal{A}}(x)$, and $\Omega_{\mathcal{A}}(e) \leqslant \Omega_{\mathcal{A}}(x), \forall x \in \mathcal{X}$

I_2- $\vartheta_{\mathcal{A}}(x) \geqslant min\{\vartheta_{\mathcal{A}}((x^* * y^*)^*, \vartheta_{\mathcal{A}}(y)\}$,

I_3- $\Omega_{\mathcal{A}}(x) \leqslant max\{\Omega_{\mathcal{A}}((x^* * y^*)^*, \Omega_{\mathcal{A}}(y)\}, \forall x, y \in \mathcal{X}$.

Definition2.16:[12]

If $\mathcal{F} \subseteq \mathcal{X}$, $\mathcal{F}$ it is called complete Q-filter of $\mathcal{X}$,(shortly ,$\mathcal{C} - Q - filter$) , if :

C_1- $e \in \mathcal{F}$ C_2 - $(x^* * y^*)^* \in \mathcal{F}$,$\forall\ y \in \mathcal{F}$ implies $x \in \mathcal{F}, for\ all\ x, y \in \mathcal{X}$,

Definition2.17:[1]

If $\mathcal{F}$ is a $\mathcal{C}$-Q-filter of $\mathcal{X}$. An $IFS\ \mathcal{A} = (\vartheta_{\mathcal{F}}, \Omega_{\mathcal{F}})$ is called intuitionist fuzzy $\mathcal{C}$-Q-filter at $\mathcal{F}$ (briefly, IFS-$\mathcal{C}$-Q-filter), if :

I_{C1}- $\vartheta_{\mathcal{F}}(e) \geqslant \vartheta_{\mathcal{F}}(x)$, and $\Omega_{\mathcal{F}}(e) \leqslant \Omega_{\mathcal{F}}(x)$, $\forall\ x \in \mathcal{X}$

I_{C2}- $\vartheta_{\mathcal{F}}(x) \geqslant min\{\vartheta_{\mathcal{F}}((x^* * y^*)^*), \vartheta_{\mathcal{F}}(y)\}, \forall y \in \mathcal{F}$

I_{C3} - $\Omega_{\mathcal{F}}(x) \leqslant max\{\Omega_{\mathcal{F}}((x^* * y^*)^*), \Omega_{\mathcal{F}}(y)\}, \forall y \in \mathcal{F}$.

Definition2.18: [12]

Let $\mathcal{N} \subseteq \mathcal{X}$, $\mathcal{N}$ is called a $\mathcal{S}$- filter of $\mathcal{X}, if$:

S_1- $e \in \mathcal{N}$. S_2- $(y^{**} * x^*)^* \in \mathcal{N}, y \in \mathcal{N}\ implies\ x^* \in \mathcal{N}$.

Proposition2.19:[12]

Every Q-filter is an $\mathcal{S}$-filter.

Definition2.20:[12]

If $\mathcal{F} \subseteq \mathcal{X}$, then $\mathcal{F}$ is called a complete $\mathcal{S}$- filter ,($\mathcal{C}$- $\mathcal{S}$-filter), if :

$\mathcal{C}_1$- $e \in \mathcal{F}$, $\mathcal{C}_2$ - $(y^{**} * x^*)^* \in \mathcal{F}, \forall\ y \in \mathcal{F}\ implies\ x^* \in F; \forall x, y \in \mathcal{X}$

Proposition2.21:[12]

I- Every Q-filter is $\mathcal{C}$- $\mathcal{S}$-filter**.**

II-Every $\mathcal{C}$- Q-filter is $\mathcal{C}$- $\mathcal{S}$-filter.

III- Every $\mathcal{S}$-filter is $\mathcal{C}$- $\mathcal{S}$-filter

Proposition2.22:[12]

Every $\mathcal{C}$-$\mathcal{S}$-filter an involuntary Q-algebra $\mathcal{C}$-Q-filter.

Definition2.23:[14]

A fuzzy set ϑ in set $\mathcal{X}$ is a function $\vartheta: \mathcal{X} \to [0,1]$. If $\vartheta\ and\ \Omega$ are two fuzzy subset of $\mathcal{X}$, then by $\vartheta \leqslant \Omega$, we mean $\vartheta(x) \leqslant \Omega(x), \forall x \in \mathcal{X}$.

The complement of ϑ [symbolize it, $\overline{\vartheta}$] is the fuzzy set in $\mathcal{X}$ by : $\overline{\vartheta}(x) = 1 - \vartheta(x), \forall x \in \mathcal{X}$.

Definition2.24:[10]

If $\mathcal{X} \neq \emptyset$ and a fuzzy set ϑ in $\mathcal{X}, for\ any\ m \in [0,1]$,the sets

I) $L(\Omega; m) = \{x: \Omega(x) \leqslant m\}$,it's said to be lower m-level cut of $\mathcal{X}$.

II) $U(\vartheta; m) = \{x: \vartheta(x) \geqslant m\}$,it's said to be upper m-level cut of$\mathcal{X}$..

Definition2.25: **[12]**

A fuzzy subset ϑ in $\mathcal{X}$ is called a fuzzy $\mathcal{S}$-filter (briefly F-$\mathcal{S}$-filter), if

1- $\vartheta(e) \geqslant \vartheta(x), \forall\ x \in \mathcal{X}$,

2- $\vartheta(x^*) \geqslant min\{\vartheta((j^{**} * x^*)^*), \vartheta(j)\}, \forall\ x, j \in \mathcal{X}$.

3. Intuitionistic fuzzy $\mathcal{S}$-filter

In this section ,we provide a description of Intuitionistic fuzzy $\mathcal{S}$- filter, and we are studying its relationship with Intuitionistic fuzzy Q -filter in Q-algebra .

Definition3.1:

In IFS $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ in $\mathcal{X}$ is said to be an Intuitionistic fuzzy $\mathcal{S}$-filter of $\mathcal{X}$,(briefly,
IFS- $\mathcal{S}$-filter),if :

$\mathcal{S}_1$- $\vartheta_{\mathcal{A}}(e) \geqslant \vartheta_{\mathcal{A}}(x)$, and $\Omega_{\mathcal{A}}(e) \leqslant \Omega_{\mathcal{A}}(x)$, for all $x \in \mathcal{X}$

$\mathcal{S}_2$- $\vartheta_{\mathcal{A}}(x^*) \geqslant min\{\vartheta_{\mathcal{A}}((y^{**} * x^*)^*) , \vartheta_{\mathcal{A}}(y)\}$

$\mathcal{S}_3$- $\Omega_{\mathcal{A}}(x^*) \leqslant \max\{\Omega_{\mathcal{A}}((y^{**} * x^*)^*), \Omega_{\mathcal{A}}(y)\}, \forall x, y \in \mathcal{X}$

Example3.2:

Let $\mathcal{X} = \{0, r, s, t, m\}$,then $(\mathcal{X},*,0)$ $is\ Q-algebra$,with unit t, as shown table :

Table 1.

$*$	0	r	s	t	m
0	0	0	0	0	0
r	r	0	r	0	0
s	s	s	0	0	0
t	t	t	0	0	t
m	m	m	r	0	0

If ,

$$\vartheta_{\mathcal{A}}(x) = \begin{cases} 0.6 & if\ x = t, s \\ 0.1 & if\ x = 0, r, m \end{cases} \qquad \Omega_{\mathcal{A}}(x) = \begin{cases} 0.3 & if\ x = t, s \\ 0.7 & if\ x = 0, r, m \end{cases}$$

Then $IFS\ \mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ is an IFS-$\mathcal{S}$-filter of $\mathcal{X}$, since:

$\vartheta_{\mathcal{A}}(e) \geqslant \vartheta_{\mathcal{A}}(x)$,and $\Omega_{\mathcal{A}}(e) \leqslant \Omega_{\mathcal{A}}(x)$, $\forall\ x \in \mathcal{X}$

$\vartheta_{\mathcal{A}}(s^*) = 0.1 \geqslant min\ \{\vartheta_{\mathcal{A}}((t^{**} * s^*)^*, \vartheta_{\mathcal{A}}(t)\} = 0.1$

$\vartheta_{\mathcal{A}}(s^*) = 0.1 \geqslant min\{\vartheta_{\mathcal{A}}\left((s^{**} * s^*)^*\right), \vartheta_{\mathcal{A}}(s)\} = 0.1$

$\vartheta_{\mathcal{A}}(t^*) = 0.1 \geqslant min\ \{\vartheta_{\mathcal{A}}((t^{**} * t^*)^*, \vartheta_{\mathcal{A}}(t)\} = 0.1$

$\vartheta_{\mathcal{A}}(t^*) = 0.1 \geqslant min\ \{\vartheta_{\mathcal{A}}((s^{**} * t)^*, \vartheta_{\mathcal{A}}(s)\} = 0.1$

$\Omega_{\mathcal{A}}(s^*) = 0.7 \leqslant max\{\Omega_{\mathcal{A}}\left((s^{**} * s^*)^*\right), \Omega_{\mathcal{A}}(s)\} = 0.7$

$\Omega_{\mathcal{A}}(s^* = 0.7 \leqslant max\{\Omega_{\mathcal{A}}\left((t^{**} * s^*)^*\right), \Omega_{\mathcal{A}}(t)\} = 0.7$

$\Omega_{\mathcal{A}}(t^*) = 0.7 \leqslant max\{\Omega_{\mathcal{A}}\left((t^{**} * t^*)^*\right), \Omega_{\mathcal{A}}(t)\} = 0.7$

$\Omega_{\mathcal{A}}(t^*) = 0.7 \leqslant max\{\Omega_{\mathcal{A}}\left((s^{**} * t^*)^*\right), \Omega_{\mathcal{A}}(s)\} = 0.7$

And, if

$$\vartheta_{\mathcal{K}}(v) = \begin{cases} 0.7 & \text{if } v = t \\ 0.3 & \text{if } v = r, s \\ 0.5 & \text{if } v = 0, m \end{cases} \qquad \Omega_{\mathcal{K}}(v) = \begin{cases} 0.2 & \text{if } v = t \\ 0.4 & \text{if } v = r, s \\ 0.3 & \text{if } v = 0, m \end{cases}$$

Then $IFS\ \mathcal{K} = (\vartheta_{\mathcal{K}}, \Omega_{\mathcal{K}})$ is not IFS- $\mathcal{S}$-filter of $\mathcal{X}$,since :

$\vartheta_{\mathcal{K}}(s^*)$= 0.5 $\ngeq$ min $\{\vartheta_{\mathcal{K}}((r^{**} * s^*)^*), \vartheta_{\mathcal{K}}(r)\}$=0.3.

Proposition 3.3:

Every IFS- Q-filter of Q-algebra $(\mathcal{X},*,0)$ is IFS- $\mathcal{S}$-filter.

Proof :

Let $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ be an IFS- Q-filter of $\mathcal{X}$,then by Definition (2.14), we have :

$\mathcal{S}_1$- $\vartheta_{\mathcal{A}}(e) \geqslant \vartheta_{\mathcal{A}}(x)$, and $\Omega_{\mathcal{A}}(e) \leqslant \Omega_{\mathcal{A}}(x)$, $\forall x \in \mathcal{X}$

$\mathcal{S}_2$ - $\vartheta_{\mathcal{A}}(x) \geqslant min\ \{\vartheta_{\mathcal{A}}((x^* * z^*)^*, \vartheta_{\mathcal{A}}(z)\}, \forall x, z \in \mathcal{X}$, then

$\vartheta_{\mathcal{A}}(x^*) \geqslant min\ \{\vartheta_{\mathcal{A}}((x^{**} * z^*)^*, \vartheta_{\mathcal{A}}(z)\}$, [by Remark(2.3)2] , we will get:

$\vartheta_{\mathcal{A}}(x^*) \geqslant min\ \{\vartheta_{\mathcal{A}}((z^{**} * x^*)^*), \vartheta_{\mathcal{A}}(z)\}, \forall x, z \in \mathcal{X}$

$\mathcal{S}_3$-$\Omega_{\mathcal{A}}(x) \leqslant max\{\Omega_{\mathcal{A}}((x^* * z^*)^*, \Omega_{\mathcal{A}}(z)\}, \forall\ x, z \in \mathcal{X}$, then

$\Omega_{\mathcal{A}}(x^*) \leqslant max\{\Omega_{\mathcal{A}}((x^{**} * z^*)^*, \Omega_{\mathcal{A}}(z)\}$, [by Remark(2.3),2)], we will get:

$\Omega_{\mathcal{A}}(x^*) \leqslant max\{\Omega_{\mathcal{A}}((z^{**} * x^*)^*), \Omega_{\mathcal{A}}(z)\}, \forall x, z \in \mathcal{X}$

Then $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ is IFS- $\mathcal{S}$-filter of $\mathcal{X}$.

Remark 3.4:

The converse Proposition (3.3) is not generally true as the next example.

Example3.5:

Let $\mathcal{X} = \{0, \mathfrak{r}, d, \mathfrak{t}, \mathfrak{m}\}$,note that $(\mathcal{X},*,0)$ $is\ Q-algebra$,and $\mathfrak{m}$ is unit of $\mathcal{X}$, by the table :

Table 2.

$*$	0	$\mathfrak{r}$	d	$\mathfrak{t}$	$\mathfrak{m}$
0	0	0	0	0	0
$\mathfrak{r}$	$\mathfrak{r}$	0	0	0	0
d	d	0	0	0	0

$\mathfrak{t}$	$\mathfrak{t}$	0	0	0	0
$\mathfrak{m}$	$\mathfrak{m}$	$\mathfrak{t}$	$\mathfrak{t}$	d	0

If ,

$\vartheta_{\mathcal{A}}(x) = \begin{cases} 0.5 & if\ x = 0, d, \mathfrak{m} \\ 0.2 & if\ x = \mathfrak{r}, \mathfrak{t} \end{cases}$ $\quad\quad \Omega_{\mathcal{A}}(x) = \begin{cases} 0.4 & if\ x = 0, d, \mathfrak{m} \\ 0.6 & if\ x = \mathfrak{r}, \mathfrak{t} \end{cases}$

Then $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ is IFS-$\mathcal{S}$-filter but not IFS-Q-filter, since :

$\vartheta_{\mathcal{A}}(\mathfrak{t}) = 0.2 \ngeqslant min\{\vartheta_{\mathcal{A}}((\mathfrak{t}^* * \mathfrak{m}^*)^*), \vartheta_{\mathcal{A}}(\mathfrak{m})\} = min\{\vartheta_{\mathcal{A}}(\mathfrak{t}), \vartheta_{\mathcal{A}}(\mathfrak{m})\} = 0.5$

Proposition 3.6:

Every IFS- $\mathcal{S}$-filter on an involutory Q-algebra $(\mathcal{X}, *, 0)$ is IFS-Q-filter.

Proof :

If IFS-$\mathcal{S}$-filter $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ of an involutory Q-algebra, then

1-$\vartheta_{\mathcal{A}}(e) \geqslant \vartheta_{\mathcal{A}}(x)$, and $\Omega_{\mathcal{A}}(e) \leqslant \Omega_{\mathcal{A}}(x)$, $\forall\ x \in \mathcal{X}$

2- $\vartheta_{\mathcal{A}}(x) = \vartheta_{\mathcal{A}}(x^{**}) \geqslant min\{\vartheta_{\mathcal{A}}((z^{**} * x^{**})^*), \vartheta_{\mathcal{A}}(z)\}$,[by Proposition(2.8,1)]

$= min\{\vartheta_{\mathcal{A}}((x^{***} * z^*)^*), \vartheta_{\mathcal{A}}(z)\}$

$= min\{\vartheta_{\mathcal{A}}((x^* * z^*)^*), \vartheta_{\mathcal{A}}(z)\}$

3-$\Omega_{\mathcal{A}}(x) = \Omega_{\mathcal{A}}(x^{**}) \leqslant max\{\Omega_{\mathcal{A}}((z^{**} * x^{**})^*), \Omega_{\mathcal{A}}(z)\}$

$= max\{\Omega_{\mathcal{A}}((x^{***} * z^*)^*), \Omega_{\mathcal{A}}(z)\}$

$= max\{\Omega_{\mathcal{A}}((x^* * z^*)^*), \Omega_{\mathcal{A}}(z)\}, for\ every\ x, z \in \mathcal{X}$

thus $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ is IFS-Q-filter.

Proposition3.7 :

Let $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ be IFS-$\mathcal{S}$-filter of $\mathcal{X}$. Then $\vartheta_{\mathcal{A}}(x^*) \geqslant \vartheta_{\mathcal{A}}(y)$ and $\Omega_{\mathcal{A}}(x^*) \leqslant \Omega_{\mathcal{A}}(y)$,if $y^{**} \leqslant x^*$,$\forall x, y \in \mathcal{X}$

Proof :-

$if\ y^{**} \leqslant x^*$,then $y^{**} * x^* = 0, such\ that\ x, y \in \mathcal{X}$. Since $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ is IFS- $\mathcal{S}$-filter of $\mathcal{X}$, then

$\vartheta_{\mathcal{A}}(x^*) \geqslant min\{\vartheta_{\mathcal{A}}((y^{**} * x^*)^*), \vartheta_{\mathcal{A}}(y)\} = min\{\vartheta_{\mathcal{A}}((0)^*), \vartheta_{\mathcal{A}}(y)\}$

$= min\{\vartheta_{\mathcal{A}}(e), \vartheta_{\mathcal{A}}(y)\}$, $[since\ \vartheta_{\mathcal{A}}(e) \geqslant \vartheta_{\mathcal{A}}(x)]$

$= \vartheta_{\mathcal{A}}(y)$

$\Omega_{\mathcal{A}}(x^*) \leqslant max\{\Omega_{\mathcal{A}}((y^{**} * x^*)^*), \Omega_{\mathcal{A}}(y)\} = max\{\Omega_{\mathcal{A}}((0)^*), \Omega_{\mathcal{A}}(y)\}$

$= max\{\Omega_{\mathcal{A}}(e), \Omega_{\mathcal{A}}(y)\}$, $since\ \Omega_{\mathcal{A}}(e) \leqslant \Omega_{\mathcal{A}}(x)$

$= \Omega_{\mathcal{A}}(y)$.

Corollary3.8 :

If $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ is a IFS-$\mathcal{S}$-filter of involutory Q-algebra $\mathcal{X}$,then $\vartheta_{\mathcal{A}}(x^*) \geqslant \vartheta_{\mathcal{A}}(y)$ and $\Omega_{\mathcal{A}}(x^*) \leqslant \Omega_{\mathcal{A}}(y)$ if $x \leqslant y^*$ for every $x, y \in \mathcal{X}$.

Proof :-

Let if $x \leqslant y^*$, then $y^{**} \leqslant x^*$, by Proposition ((2.8),2 and by Proposition (3.7),we have hence $\vartheta_{\mathcal{A}}(x^*) \geqslant \vartheta_{\mathcal{A}}(y)$ and $\Omega_{\mathcal{A}}(x^*) \leqslant \Omega_{\mathcal{A}}(y)$

Proposition3.9 :

If $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ is an IFS- $\mathcal{S}$-filter of $\mathcal{X}$, then :-

1- $\vartheta_{\mathcal{A}}(x^*) \geqslant \vartheta_{\mathcal{A}}(0)$, $\Omega_{\mathcal{A}}(x^*) \leqslant \Omega_{\mathcal{A}}(0)$, $\forall x \in \mathcal{X}$.

2- $\vartheta_{\mathcal{A}}(x^{**}) \geqslant \vartheta_{\mathcal{A}}(x)$, $\Omega_{\mathcal{A}}(x^{**}) \leqslant \Omega_{\mathcal{A}}(x)$,for all $x \in \mathcal{X}$.

Proof :-

Let $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ be an IFS- $\mathcal{S}$-filter then :-

$\vartheta_{\mathcal{A}}(x^*) \geqslant min\{\vartheta_{\mathcal{A}}((0^{**} * x^*)^*), \vartheta_{\mathcal{A}}(0)\}$

$= min\{\vartheta_{\mathcal{A}}((e^* * x^*)^*), \vartheta_{\mathcal{A}}(0)\}$

$= min\ \{\vartheta_{\mathcal{A}}\ ((0 * x^*)^*), \vartheta_{\mathcal{A}}\ (0)\}$
$= min\ \{\vartheta_{\mathcal{A}}\ ((0\ )^*), \vartheta_{\mathcal{A}}\ (0)\}$
$= min\{\vartheta_{\mathcal{A}}\ (e), \vartheta_{\mathcal{A}}\ (0)\}$ [since $\vartheta_{\mathcal{A}}\ (e) \geqslant \vartheta_{\mathcal{A}}\ (x)$]
$= \vartheta_{\mathcal{A}}\ (0)$
Similarly $\Omega_{\mathcal{A}}\ (x^*) \leqslant \Omega_{\mathcal{A}}\ (0), \forall x \in \mathcal{X}$
2- $\vartheta_{\mathcal{A}}\ (x^{**}) \geqslant min\{\ \vartheta_{\mathcal{A}}\ ((x^{**} * x^{**})^*), \vartheta_{\mathcal{A}}\ (x)\}$
$= \ min\ \{\ \vartheta_{\mathcal{A}}\ ((0)^*)\ , \vartheta_{\mathcal{A}}\ (x)\}$
$= min\ \{\vartheta_{\mathcal{A}}\ (e), \vartheta_{\mathcal{A}}\ (x)\}$ [since $\vartheta_{\mathcal{A}}\ (e) \geq \vartheta_{\mathcal{A}}\ (x)$]
$= \vartheta_{\mathcal{A}}\ (x)$
Similarly $\Omega_{\mathcal{A}}\ (x^{**}) \leqslant \Omega_{\mathcal{A}}\ (x)$.

Proposition3.10 :

Let $\mathcal{N} \subseteq \mathcal{X}$ and $a, d \in [0,1]$ such that $a < d$ and $0 \leq a + d \leq 1$,if $\mathcal{A} = (\vartheta_{\mathcal{A}}\ , \Omega_{\mathcal{A}}\ )$ is IFS ,defined the

$\vartheta_{\mathcal{A}}\ (x) = \begin{cases} d & \text{if } x \in \ \mathcal{N} \\ a & \text{if o.w} \end{cases}$ $\Omega_{\mathcal{A}}\ (x) = \begin{cases} a & \text{if } x \in \mathcal{N} \\ d & \text{if o.w} \end{cases}$

Then $\mathcal{N}$ is $\mathcal{S}$-filter of $\mathcal{X}$ if and only if $\mathcal{A}$ is IFS-$\mathcal{S}$-filter

Proof :-

Suppose that $\mathcal{N}$ is a $\mathcal{S}$-filter and $\mathcal{A} = (\vartheta_{\mathcal{A}}\ , \Omega_{\mathcal{A}}\ )$ is not IFS-$\mathcal{S}$-filter, $\exists\ x, j \in \mathcal{X}$, such that $\vartheta_{\mathcal{A}}\ (x^*) < min\{\ \vartheta_{\mathcal{A}}\ ((j^{**} * x^*)^*), \vartheta_{\mathcal{A}}$ (j)}$= d$, thus $(j^{**} * x^*)^*$, j∈ $\mathcal{N}$ [since $\mathcal{N}$ is a $\mathcal{S}$-filter] ,then $x^* \in \mathcal{N}$, the implies $\vartheta_{\mathcal{A}}\ (x^*) = d$,it need to contradict .
or ,
$\Omega_{\mathcal{A}}\ (x^*) > max\ \{\Omega_{\mathcal{A}}\ ((j^{**} * x^*)^*), \Omega_{\mathcal{A}}$ (j)}$= a$,thus $(j^{**} * x^*)^*$, j∈ $\mathcal{N}$ [since $\mathcal{N}$ is $\mathcal{S}$-filter] then $x^* \in \mathcal{N}$ the implies $\Omega_{\mathcal{A}}\ (x^*) = a$, it need to contradict .
Thus $\mathcal{A} = (\vartheta_{\mathcal{A}}\ , \Omega_{\mathcal{A}}\ )$ is IFS-$\mathcal{S}$-filter of .
Conversely, let $\mathcal{A} = (\vartheta_{\mathcal{A}}\ , \Omega_{\mathcal{A}}\ )$ be IFS- $\mathcal{S}$-filter of $\mathcal{X}$, and $(j^{**} * x^*)^* \in \mathcal{N}, j \in \mathcal{N}$,
$\vartheta_{\mathcal{A}}\ (x^*) \geqslant min\{\vartheta_{\mathcal{A}}\ ((j^{**} * x^*)^*), \vartheta_{\mathcal{A}}$ (j)}$= d$ [since $\mathcal{N}$ is IFS- $\mathcal{S}$-filter],then $x^* \in \mathcal{N}$
And, $\Omega_{\mathcal{A}}\ (x^*) \leqslant max\{\Omega_{\mathcal{A}}((j^{**} * x^*)^*, \Omega_{\mathcal{A}}$ (j)}$= a$
$\Omega_{\mathcal{A}}\ (x^*) \leqslant a$, then $x^* \in \mathcal{N}$,hence $\mathcal{N}$ is a $\mathcal{S}$-filter

Proposition 3.11:

An IFS $\mathcal{A} = (\vartheta_{\mathcal{A}}\ , \Omega_{\mathcal{A}}\ )$ of $\mathcal{X}$ is IFS-$\mathcal{S}$-filter if and only if $\vartheta_{\mathcal{A}}\ and\ \overline{\Omega_{\mathcal{A}}}$ are F-$\mathcal{S}$-filter .

Proof:

Let $\mathcal{A} = (\vartheta_{\mathcal{A}}\ , \Omega_{\mathcal{A}}\ )$ be IFS-$\mathcal{S}$-filter of $\mathcal{X}$,cleary, $\vartheta_{\mathcal{A}}$ is F-$\mathcal{S}$-filter
$\forall x$, j $\in \mathcal{X}, so\ \overline{\Omega_{\mathcal{A}}}\ (e) = 1 - \Omega_{\mathcal{A}}\ (e) \geqslant 1 - \Omega_{\mathcal{A}}\ (x) = \ \overline{\Omega_{\mathcal{A}}}\ (x)$
$\overline{\Omega_{\mathcal{A}}}\ (x^*) = 1 - \Omega_{\mathcal{A}}\ (x^*) \geqslant max\{\Omega_{\mathcal{A}}((j^{**} * x^*)^*, \Omega_{\mathcal{A}}$ (j)}
$= min\{1 - \Omega_{\mathcal{A}}((j^{**} * x^*)^*, 1 - \Omega_{\mathcal{A}}$ (j)}
$= min\{\ \overline{\Omega_{\mathcal{A}}}\ ((j^{**} * x^*)^*), \overline{\Omega_{\mathcal{A}}}$ (j)},then $\overline{\Omega_{\mathcal{A}}}$ is F- $\mathcal{S}$-filter.
Conversely, let $\vartheta_{\mathcal{A}}\ and\ \overline{\Omega_{\mathcal{A}}}$ be F-$\mathcal{S}$-filter of $\mathcal{X}$,$\forall x$, j $\in \mathcal{X}$,then
$\mathcal{S}_1$- $\vartheta_{\mathcal{A}}\ (e) \geqslant \vartheta_{\mathcal{A}}\ (x), and\ 1 - \Omega_{\mathcal{A}}\ (e) = \ \overline{\Omega_{\mathcal{A}}}(e) \geqslant \ \overline{\Omega_{\mathcal{A}}}\ (x) = 1 - \Omega_{\mathcal{A}}\ (x)$
$then\ \Omega_{\mathcal{A}}\ (e)\ \geqslant \Omega_{\mathcal{A}}\ (x)$
$\mathcal{S}_2$- $\vartheta_{\mathcal{A}}\ (x^*) \geqslant min\{\vartheta_{\mathcal{A}}\ ((j^{**} * x^*)^*), \vartheta_{\mathcal{A}}$ (j)}
$\mathcal{S}_3$-1 $-\ \Omega_{\mathcal{A}}\ (x^*) = \ \overline{\Omega_{\mathcal{A}}}\ (x^*) \geqslant min\{\overline{\Omega_{\mathcal{A}}}\ ((j^{**} * x^*)^*, \overline{\Omega_{\mathcal{A}}}$ (j)}
$= min\{1 - \Omega_{\mathcal{A}}\ (j^{**} * x^*)^*)$,$1 - \Omega_{\mathcal{A}}$ (j)}
$= 1 - max\{\Omega_{\mathcal{A}}((j^{**} * x^*)^*, \Omega_{\mathcal{A}}$ (j)}$, so$
$\Omega_{\mathcal{A}}\ (x) \leqslant max\{\Omega_{\mathcal{A}}((j^{**} * x^*)^*, \Omega_{\mathcal{A}}$ (j)}$, then\ \mathcal{A} = (\vartheta_{\mathcal{A}}\ , \Omega_{\mathcal{A}}\ )$ is IFS-$\mathcal{S}$-filter.

Corollary3.12 :-

If IFS $\mathcal{A} = (\vartheta_{\mathcal{A}}\ , \Omega_{\mathcal{A}}\ )$ in $\mathcal{X}$, than $\Box\mathcal{A} = (\vartheta_{\mathcal{A}}\ , \overline{\vartheta_{\mathcal{A}}}) and\ \lozenge\ \mathcal{A} = (\overline{\Omega_{\mathcal{A}}}\ , \Omega_{\mathcal{A}}\ )$ are IFS- $\mathcal{S}$-filter if and only if $\mathcal{A}$ is IFS-$\mathcal{S}$-filter of $\mathcal{X}$.

Proof :-

If $\square\mathcal{A}$ and $\lozenge\,\mathcal{A}$ are IFS- $\mathcal{S}$-filter of $\mathcal{X}$, than the fuzzy sets $\vartheta_{\mathcal{A}}\ and\ \overline{\Omega_{\mathcal{A}}}$ are F-$\mathcal{S}$-filter . Hence $\mathcal{A}=(\vartheta_{\mathcal{A}},\Omega_{\mathcal{A}})$ is an IFS- $\mathcal{S}$-filter.

Conversely, suppose $\mathcal{A}=(\vartheta_{\mathcal{A}},\Omega_{\mathcal{A}})$ is IFS-$\mathcal{S}$-filter of $\mathcal{X}$,than $\vartheta_{\mathcal{A}}=\overline{\overline{\vartheta_{\mathcal{A}}}}$ and $\overline{\Omega_{\mathcal{A}}}$ are F-$\mathcal{S}$-filter [by Proposition (3.11)] hence $\lozenge\,\mathcal{A}=(\overline{\Omega_{\mathcal{A}}},\Omega_{\mathcal{A}})\ and\ \square\mathcal{A}=(\vartheta_{\mathcal{A}},\overline{\vartheta_{\mathcal{A}}})$ are IFS- $\mathcal{S}$-filter.

Proposition 3.13:

Let $\mathcal{A}=(\vartheta_{\mathcal{A}},\Omega_{\mathcal{A}})be$ an IFS of $\mathcal{X}$ .then $\mathcal{A}$ is IFS- $\mathcal{S}$-filter of $\mathcal{X}$ if and only if the sets $\mathrm{U}(\vartheta_{\mathcal{A}};m)and\ \mathrm{L}(\Omega_{\mathcal{A}};n)$are $\mathcal{S}$-filter or it's empty of $\mathcal{X}$,$\forall m,n\in[0,1]$

Proof:

If $\mathcal{A}=(\vartheta_{\mathcal{A}},\Omega_{\mathcal{A}})$ is IFS- $\mathcal{S}$-filter of $\mathcal{X}$,and $m,n\in[0,1],\mathrm{U}(\vartheta_{\mathcal{A}};m)\neq\emptyset\neq\mathrm{L}(\Omega_{\mathcal{A}};n)$
by Definition (3.1), then $\vartheta_{\mathcal{A}}(e)\geqslant\vartheta_{\mathcal{A}}(x)\geqslant m, and\ \Omega_{\mathcal{A}}(e)\leqslant\Omega_{\mathcal{A}}(x)\leqslant n, for\ some\ x\in\mathcal{X},$
then $e\in\mathrm{U}(\vartheta_{\mathcal{A}};m)\cap\mathrm{L}(\Omega_{\mathcal{A}};n)$,let $x,j\in\mathcal{X}, and\ (j^{**}*x^{*})^{*},j\in\mathrm{U}(\vartheta_{\mathcal{A}};m), so$
$\vartheta_{\mathcal{A}}((j^{**}*x^{*})^{*})\geqslant m\ and\ \vartheta_{\mathcal{A}}(j)\geqslant m$,therefor $\vartheta_{\mathcal{A}}(x^{*})\geqslant\min\{\vartheta_{\mathcal{A}}((j^{**}*x^{*})^{*}),\vartheta_{\mathcal{A}}(j)\}\geqslant m$
Then $\mathrm{U}(\vartheta_{\mathcal{A}};m)\ is$ $\mathcal{S}$-filter.Similarly $\mathrm{L}(\Omega_{\mathcal{A}};n)$ is $\mathcal{S}$-filter.
Conversely,we imposed $\mathrm{U}(\vartheta_{\mathcal{A}};m)\ and\ \mathrm{L}(\Omega_{\mathcal{A}};n)$are $\mathcal{S}$-filter or it's empty of $\mathcal{X}$,if $n,m\in[0,1]$
If we take any $x\in\mathcal{X}, and\ \vartheta_{\mathcal{A}}(x)=m,\Omega_{\mathcal{A}}(x)=n$ we conclude that
$x\in\mathrm{U}(\vartheta_{\mathcal{A}};m)\cap\mathrm{L}(\Omega_{\mathcal{A}};n), so\mathrm{U}(\vartheta_{\mathcal{A}};m)\neq\emptyset\neq\mathrm{L}(\Omega_{\mathcal{A}};n), then\ \mathrm{U}(\vartheta_{\mathcal{A}};m)and\ \mathrm{L}(\Omega_{\mathcal{A}};n)$
are $\mathcal{S}$-filter concluded $e\in\mathrm{U}(\vartheta_{\mathcal{A}};m)\cap\mathrm{L}(\Omega_{\mathcal{A}};n)$,hence $\vartheta_{\mathcal{A}}(e)\geqslant m=\vartheta_{\mathcal{A}}(x)and$
$\Omega_{\mathcal{A}}(e)\leqslant\Omega_{\mathcal{A}}(x)=n,\forall x\in\mathcal{X}.$ Let $x,j\in\mathcal{X}$,if we take $m=\min\{\vartheta_{\mathcal{A}}((j^{**}*x^{*})^{*}),\vartheta_{\mathcal{A}}(j)\}$
$\vartheta_{\mathcal{A}}((j^{**}*x^{*})^{*})\geqslant m,\vartheta_{\mathcal{A}}(j)\geqslant m, so\ (j^{**}*x^{*})^{*},j\in\mathrm{U}(\vartheta_{\mathcal{A}};m)$[by Definition (2.23)]
then $x^{*}\in\mathrm{U}(\vartheta_{\mathcal{A}};m)$[since $\mathrm{U}(\vartheta_{\mathcal{A}};m)$is a $\mathcal{S}$-filter],
so $\vartheta_{\mathcal{A}}(x^{*})\geqslant m=min\{\vartheta_{\mathcal{A}}\left((j^{**}*x^{*})^{*}\right),\vartheta_{\mathcal{A}}(j)\}$.
Let $x,j\in\mathcal{X}\ if\ we\ take: n=max\{\Omega_{\mathcal{A}}(j^{**}*x^{*})^{*}),\Omega_{\mathcal{A}}(j)\}$,
then $\Omega_{\mathcal{A}}(j^{**}*x^{*})\leqslant n,\Omega_{\mathcal{A}}(j)\leqslant n,$
$so\ (j^{**}*x^{*})^{*},j\in\mathrm{L}(\Omega_{\mathcal{A}};n)$ [by Definition (2.23)], then $x^{*}\in\mathrm{L}(\Omega_{\mathcal{A}};n)$
[since $\mathrm{L}(\Omega_{\mathcal{A}};n)$is a $\mathcal{S}$-filter], so $\Omega_{\mathcal{A}}(x^{*})\leqslant n=max\{\Omega_{\mathcal{A}}(j^{**}*x^{*})^{*}),\Omega_{\mathcal{A}}(j)\}$
Then $\mathcal{A}=(\vartheta_{\mathcal{A}},\Omega_{\mathcal{A}})$ is an IFS- $\mathcal{S}$-filter.

Proposition3.14 :

If $\{\mathcal{A}_i, \mathrm{i}\in\Delta\}$ is an arbitrary family of IFS- $\mathcal{S}$-filter of $\mathcal{X}$, then $\cap\mathcal{A}_i$ is an IFS- $\mathcal{S}$-filter of $\mathcal{X}$.

Proof:

Let $\mathcal{A}_i$, $\mathrm{i}\in\Delta$ be IFS- $\mathcal{S}$-filter, such that $\mathcal{A}_i=\left(\vartheta_{\mathcal{A}_i},\Omega_{\mathcal{A}_i}\right)$then

$\mathcal{S}_1$- $\vartheta_{\mathcal{A}_i}(e)\geqslant\vartheta_{\mathcal{A}_i}(x)$, so $\wedge\vartheta_{\mathcal{A}_i}(e)\geqslant\wedge\vartheta_{\mathcal{A}_i}(x)$,and
$\Omega_{\mathcal{A}_i}(e)\leqslant\Omega_{\mathcal{A}_i}(x)$,so $\vee\Omega_{\mathcal{A}_i}(e)\leqslant\vee\Omega_{\mathcal{A}_i}(x)$,for all $x\in\mathcal{X}$and $\mathrm{i}\in\Delta$.

$\mathcal{S}_2$- $\vartheta_{\mathcal{A}_i}(x^{*})\geqslant min\{\vartheta_{\mathcal{A}_i}((j^{**}*x^{*})^{*}),\vartheta_{\mathcal{A}_i}(\mathrm{j})\}$
$\wedge\vartheta_{\mathcal{A}_i}(x^{*})\geqslant\wedge\{min\{\vartheta_{\mathcal{A}_i}(j^{**}*x^{*})^{*}),\vartheta_{\mathcal{A}_i}(\mathrm{j})\}$
$\wedge\vartheta_{\mathcal{A}_i}(x^{*})\geqslant min\{\wedge\vartheta_{\mathcal{A}_i}(j^{**}*x^{*})^{*}),\wedge\vartheta_{\mathcal{A}_i}(\mathrm{j})\}$

$\mathcal{S}_3$- $\Omega_{\mathcal{A}_i}(x^{*})\leqslant max\{\Omega_{\mathcal{A}_i}(j^{**}*x^{*})^{*}),\Omega_{\mathcal{A}_i}(\mathrm{j})\},\forall\ x,\mathrm{j}\in\mathcal{X}$
$\vee\Omega_{\mathcal{A}_i}(x^{*})\leqslant\vee\{max\{\Omega_{\mathcal{A}_i}((j^{**}*x^{*})^{*}),\Omega_{\mathcal{A}_i}(\mathrm{j})\}\}$
$\vee\Omega_{\mathcal{A}_i}(x^{*})\leqslant max\{\vee\Omega_{\mathcal{A}_i}(j^{**}*x^{*})^{*}),\vee\Omega_{\mathcal{A}_i}(\mathrm{j})\},\forall x,j\in\mathcal{X}$

Then $\cap\mathcal{A}_i$ is IFS-$\mathcal{S}$-filter of $\mathcal{X}$.

Remark3.15:

In general, the union of two IFS- $\mathcal{S}$-filter is not needed ,as shown in the following example

Example3.16 :

If $\mathcal{A}=(\vartheta_{\mathcal{A}},\Omega_{\mathcal{A}})$ is IFS- $\mathcal{S}$-filter in Example (3.5) and if ,

$\vartheta_{\mathcal{K}}(x)=\begin{cases}0.6 & if\ x=\mathfrak{m}\\ 0.1 & if\ x=0,d,\mathfrak{r},\mathfrak{t}\end{cases}\qquad \Omega_{\mathcal{K}}(x)=\begin{cases}0.3 & if\ x=\mathfrak{m}\\ 0.8 & if\ x=0,d,\mathfrak{r},\mathfrak{t}\end{cases}$

Then $IFS\ \mathcal{K}=(\vartheta_{\mathcal{K}},\Omega_{\mathcal{K}})\ is$ IFS- $\mathcal{S}$-filter of $\mathcal{X}$, But

$$\vartheta_{\mathcal{A}} \cup \vartheta_{\mathcal{K}}(x) = \begin{cases} 0.6 \text{ if } x = \mathfrak{m} \\ 0.5 \text{ if } x = 0, d \\ 0.2 \text{ if } x = \mathfrak{r}, \mathfrak{t} \end{cases} \quad \Omega_{\mathcal{A}} \cup \Omega_{\mathcal{K}}(x) = \begin{cases} 0.3 \text{ if } x = \mathfrak{m} \\ 0.4 \text{ if } x = 0, d \\ 0.6 \text{ if } x = \mathfrak{r}, \mathfrak{t} \end{cases}$$

Then $\mathcal{A} \cup \mathcal{K} = (\vartheta_{\mathcal{A}} \cup \vartheta_{\mathcal{K}}, \Omega_{\mathcal{A}} \cup \Omega_{\mathcal{K}})$ *is not* IFS- $\mathcal{S}$-filter,were

$\vartheta_{\mathcal{A}} \cup \vartheta_{\mathcal{K}}(\mathfrak{r}^*) = 0.2 \ngeqslant min\{\vartheta_{\mathcal{A}} \cup \vartheta_{\mathcal{K}}((d^{**} * \mathfrak{r}^*)^*), \vartheta_{\mathcal{A}} \cup \vartheta_{\mathcal{K}}(d)\} = 0.5$.

Proposition 3.17 :

If $\mathfrak{f}$ is epimorphosim mapping from $(\mathcal{X}, *, 0)$ *into* $(\mathcal{Y}, *, 0)$,and $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ is an IFS- $\mathcal{S}$-filter of $\mathcal{Y}$,then $\mathfrak{f}^{-1}(\mathcal{A})$ is an IFS- $\mathcal{S}$-filter of $\mathcal{X}$.

Proof:

If $x, y \in \mathcal{X}, \mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ *is* IFS-$\mathcal{S}$-filter of $\mathcal{Y}$,

$\mathcal{S}_1$- $\vartheta_{\mathfrak{f}^{-1}(\mathcal{A})}(e) = \vartheta_{\mathcal{A}}(\mathfrak{f}(e)) \geqslant \vartheta_{\mathcal{A}}(\mathfrak{f}(x)) = \vartheta_{\mathfrak{f}^{-1}(\mathcal{A})}(x), \forall x \in \mathcal{X}$

$\Omega_{\mathfrak{f}^{-1}(\mathcal{A})}(e) = \Omega_{\mathcal{A}}(\mathfrak{f}(e) \leqslant \Omega_{\mathcal{A}}(\mathfrak{f}(x)) = \Omega_{\mathfrak{f}^{-1}(\mathcal{A})}(x), \forall x \in \mathcal{X}$, [since $\mathcal{A}$ is IFS- $\mathcal{S}$-filter] .

$\mathcal{S}_2$- $\vartheta_{\mathfrak{f}^{-1}(\mathcal{A})}(x^*) = \vartheta_{\mathcal{A}}(\mathfrak{f}(x^*) = \vartheta_{\mathcal{A}}((\mathfrak{f}(x))^*) \geqslant min\{\vartheta_{\mathcal{A}}((\mathfrak{f}(y))^{**} * \mathfrak{f}(x))^*)^*, \vartheta_{\mathcal{A}}(\mathfrak{f}(y))\}$

$= min\{\vartheta_{\mathcal{A}}(\mathfrak{f}(y^{**} * x^*)^*, \vartheta_{\mathcal{A}}(\mathfrak{f}(y))\}$

$= min\{\vartheta_{\mathfrak{f}^{-1}(\mathcal{A})}(y^{**} * x^*)^*, \vartheta_{\mathfrak{f}^{-1}(\mathcal{A})}(y)\}$

$\mathcal{S}_3$- $\Omega_{\mathfrak{f}^{-1}(\mathcal{A})}(x^*) = \Omega_{\mathcal{A}}(\mathfrak{f}(x^*) = \Omega_{\mathcal{A}}(\mathfrak{f}(x))^* \leqslant max\{\Omega_{\mathcal{A}}((\mathfrak{f}(y))^{**} * (\mathfrak{f}(x))^*)^*, \Omega_{\mathcal{A}}(\mathfrak{f}(y))\}$

$= max\{\Omega_{\mathcal{A}}(\mathfrak{f}(y^{**} * x^*)^*), \Omega_{\mathcal{A}}(\mathfrak{f}(y))\}$

$= max\{\Omega_{\mathfrak{f}^{-1}(\mathcal{A})}((y^{**} * x^*)^*), \Omega_{\mathfrak{f}^{-1}(\mathcal{A})}(y)\}$

Hence $\mathfrak{f}^{-1}(\mathcal{A})$ is an IFS-$\mathcal{S}$-filter of $\mathcal{X}$.

Proposition3.18 :-

Let $\mathfrak{f}$ be epimorphosim mapping from$(\mathcal{X}, *, 0)$into$(\mathcal{Y}, *, 0)$ and $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ is an IFS in $\mathcal{Y}$, such that $\mathfrak{f}^{-1}(\mathcal{A}) = (\vartheta_{\mathfrak{f}^{-1}(\mathcal{A})}, \Omega_{\mathfrak{f}^{-1}(\mathcal{A})})$ is an IFS-$\mathcal{S}$-filter of $\mathcal{X}$, then $\mathcal{A}$ is an IFS-$\mathcal{S}$-filter of $\mathcal{Y}$.

Proof :-

$\mathcal{S}_1$- $\forall y \in \mathcal{Y}$, $\exists x \in \mathcal{X}$,such that $\mathfrak{f}(x) = y$,then

$\vartheta_{\mathcal{A}}(\grave{e}) = \vartheta_{\mathcal{A}}(\mathfrak{f}(e)) = \vartheta_{\mathfrak{f}^{-1}(\mathcal{A})}(e) \geqslant \vartheta_{\mathfrak{f}^{-1}(\mathcal{A})}(x) = \vartheta_{\mathcal{A}}(\mathfrak{f}(x)) = \vartheta_{\mathcal{A}}(y), \forall y \in \mathcal{Y}$, and

$\Omega_{\mathcal{A}}(\grave{e}) = \Omega_{\mathcal{A}}(\mathfrak{f}(e)) = \Omega_{\mathfrak{f}^{-1}(\mathcal{A})}(e) \leqslant \Omega_{\mathfrak{f}^{-1}(\mathcal{A})}(x) = \Omega_{\mathcal{A}}(\mathfrak{f}(x)) = \Omega_{\mathcal{A}}(y)$.

$\mathcal{S}_2$- Let $\mathfrak{t}, y \in \mathcal{Y}$.Then $\mathfrak{f}(x) = y$, and $\mathfrak{f}(\mathfrak{s}) = \mathfrak{t}$, for some $x, \mathfrak{s} \in \mathcal{X}$. It follow that

$\vartheta_{\mathcal{A}}(y^*) = \vartheta_{\mathcal{A}}(\mathfrak{f}(x))^* = \vartheta_{\mathcal{A}}(\mathfrak{f}(x)^*) = \vartheta_{\mathfrak{f}^{-1}(\mathcal{A})}(x)^* \geqslant min\{\vartheta_{\mathfrak{f}^{-1}(\mathcal{A})}((\mathfrak{s}^{**} * x^*)^*), \vartheta_{\mathfrak{f}^{-1}(\mathcal{A})}(\mathfrak{s})\}$

$= min\{\vartheta_{\mathcal{A}}(\mathfrak{f}((\mathfrak{s}^{**} * x^*)^*), \vartheta_{\mathcal{A}}(\mathfrak{f}(\mathfrak{s}))\}$

$= min\{\vartheta_{\mathcal{A}}(\mathfrak{f}(\mathfrak{s}^{**}) * \mathfrak{f}(x^*))^*, \vartheta_{\mathcal{A}}(\mathfrak{f}(\mathfrak{s}))\}$

$= min\{\vartheta_{\mathcal{A}}((\mathfrak{t}^{**} * y^*)^*), \vartheta_{\mathcal{A}}(\mathfrak{t})\}$.

$\mathcal{S}_3$-$\Omega_{\mathcal{A}}(y^*) = \Omega_{\mathcal{A}}(\mathfrak{f}(x))^* = \Omega_{(\mathcal{A})}(\mathfrak{f}(x^*) = \Omega_{\mathfrak{f}^{-1}(\mathcal{A})}(x)^*$

$\leqslant max\{\Omega_{\mathfrak{f}^{-1}(\mathcal{A})}((\mathfrak{s}^{**} * x^*)^*), \Omega_{\mathfrak{f}^{-1}(\mathcal{A})}(\mathfrak{s})\}$

$= max\{\Omega_{\mathcal{A}}(\mathfrak{f}(\mathfrak{s}^{**} * x^*)^*), \Omega_{\mathcal{A}}(\mathfrak{f}(\mathfrak{s}))\}$

$= max\{\Omega_{\mathcal{A}}(\mathfrak{f}(\mathfrak{s}^{**}) * \mathfrak{f}(x^*))^*, \Omega_{\mathcal{A}}(\mathfrak{f}(\mathfrak{s}))\}$

$= max\{\Omega_{\mathcal{A}}((\mathfrak{t}^{**} * y^*)^*), \Omega_{\mathcal{A}}(\mathfrak{t})\}$

Then $\mathcal{A}$ is an IFS-$\mathcal{S}$-filter of $\mathcal{Y}$.

Proposition3.19 :-

If $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ is IFS-$\mathcal{S}$-filter of $\mathcal{X}$,then $\mathcal{X}_{\vartheta} = \{x \in \mathcal{X}, \vartheta_{\mathcal{A}}(x) = \vartheta_{\mathcal{A}}(e)\}$ *and* $\mathcal{X}_{\Omega} = \{x \in \mathcal{X}, \Omega_{\mathcal{A}}(x) = \Omega_{\mathcal{A}}(e)\}$ are $\mathcal{S}$-filter of $\mathcal{X}$.

Proof :

Let $x, j \in \mathcal{X}$, *and let*$(j^{**} * x^*)^*, j \in \mathcal{X}_{\vartheta}$,then $\vartheta_{\mathcal{A}}((j^{**} * x^*)^*) = \vartheta_{\mathcal{A}}(e), \vartheta_{\mathcal{A}}(j) = \vartheta_{\mathcal{A}}(e)$

Since $\mathcal{A}$ is IFS-$\mathcal{S}$-filter ,so $\vartheta_{\mathcal{A}}(x^*) \geqslant min\{\vartheta_{\mathcal{A}}((j^{**} * x^*)^*), \vartheta_{\mathcal{A}}(j)\} = \vartheta_{\mathcal{A}}(e)$

but $\vartheta_{\mathcal{A}}(e) \geqslant \vartheta_{\mathcal{A}}(x^*)$, *then* $\vartheta_{\mathcal{A}}(x^*) = \vartheta_{\mathcal{A}}(e)$ thus $x^* \in \mathcal{X}_{\vartheta}$,hence $\mathcal{X}_{\vartheta}$ is $\mathcal{S}$-filter .

And, let $x, j \in \mathcal{X}$,and $(j^{**} * x^*)^*, j \in \mathcal{X}_\Omega$, then $\Omega_{\mathcal{A}}((j^{**} * x^*)^*) = \Omega_{\mathcal{A}}(e), \Omega_{\mathcal{A}}(j) = \Omega_{\mathcal{A}}(e)$
$\Omega_{\mathcal{A}}(x^*) \preccurlyeq max\{\Omega_{\mathcal{A}}((j^{**} * x^*)^*), \Omega_{\mathcal{A}}(j)\} = \Omega_{\mathcal{A}}(e)[since\ \mathcal{A}$ IFS-$\mathcal{S}$-filter],but
$\Omega_{\mathcal{A}}(e) \preccurlyeq \Omega_{\mathcal{A}}(x^*), so\ \Omega_{\mathcal{A}}(x^*) = \Omega_{\mathcal{A}}(e), so\ x^* \in \mathcal{X}_\Omega$,hence $\mathcal{X}_\Omega$ is $\mathcal{S}$-filter .

4. Intuitionistic fuzzy Complete-S-filter.

In this part ,we provide the definition of Intuitionistic fuzzy complete $\mathcal{S}$-filter, and study its relationship with the Intuitionistic fuzzy filters in Q-algebra .

Definition4.1 :

Let $\mathcal{F}$ be $\mathcal{C}$- $\mathcal{S}$-filter of $\mathcal{X}$. An IFS $\mathcal{A} = (\vartheta_{\mathcal{F}}, \Omega_{\mathcal{F}})$of $\mathcal{X}$ is called IFS complete $\mathcal{S}$-filter at $\mathcal{F}$ (briefly, IFS- $\mathcal{C}$- $\mathcal{S}$-filter).

$\mathcal{C}_1$- $\vartheta_{\mathcal{F}}(e) \succcurlyeq \vartheta_{\mathcal{F}}(\mathrm{s})$, and $\Omega_{\mathcal{F}}(e) \preccurlyeq \Omega_{\mathcal{F}}(\mathrm{s})$, $\forall \mathrm{s} \in \mathcal{X}$.

$\mathcal{C}_2$- $\vartheta_{\mathcal{F}}(\mathrm{s}^*) \succcurlyeq min\{\vartheta_{\mathcal{F}}((\mathrm{r}^{**} * \mathrm{s}^*)^*), \vartheta_{\mathcal{F}}(\mathrm{r})\}, \forall \mathrm{r} \in \mathcal{F}$.

$\mathcal{C}_3$- $\Omega_{\mathcal{F}}(\mathrm{s}^*) \preccurlyeq max\{\Omega_{\mathcal{F}}(\mathrm{r}^{**} * \mathrm{s}^*)^*), \Omega_{\mathcal{F}}(\mathrm{r})\}, \forall \mathrm{r} \in \mathcal{F}$.

Example4.2 :

Let $\mathcal{X} = \{0, \mathfrak{r}, g, \mathfrak{t}, m\}$, then $(\mathcal{X}, *, 0)$ is Q-algebra, m is a unit , as the shown table:

Table 3.

$*$	0	$\mathfrak{r}$	g	$\mathfrak{t}$	m
0	0	0	0	0	0
$\mathfrak{r}$	$\mathfrak{r}$	0	$\mathfrak{r}$	0	0
g	g	g	0	0	0
$\mathfrak{t}$	$\mathfrak{t}$	0	$\mathfrak{t}$	0	0
m	m	$\mathfrak{t}$	m	$\mathfrak{r}$	0

A sub set $\mathcal{F} = \{\mathfrak{r}, m\}$ is a $\mathcal{C}$-$\mathcal{S}$-filter $of\ \mathcal{X}$,if IFS $\mathcal{A} = (\vartheta_{\mathcal{F}}, \Omega_{\mathcal{F}})$ is IFS-$\mathcal{C}$-$\mathcal{S}$-filter, such that:

$\vartheta_{\mathcal{F}}(z) = \begin{cases} 0.9 & if\ z = \mathfrak{t}, m \\ 0.5 & if\ z = 0, \mathfrak{r}, g \end{cases}$ $\qquad \Omega_{\mathcal{F}}(z) = \begin{cases} 0.1 & if\ z = \mathfrak{t}, m \\ 0.4 & if\ z = 0, \mathfrak{r}, g \end{cases}$

but the set B =(ϑ_{F} ,λ_{F}) is not IFS- $\mathcal{C}$- $\mathcal{S}$-filter , such that

$\vartheta_{\mathrm{F}}(z) = \begin{cases} 0.8 & if\ z = \mathfrak{r}, m \\ 0.1 & if z = 0, g, \mathfrak{t} \end{cases}$ $\qquad \lambda_{\mathrm{F}}(z) = \begin{cases} 0.1 & if\ z = \mathfrak{r}, m \\ 0.6 & if\ z = 0, g, \mathfrak{t} \end{cases}$

since,

$\vartheta_{\mathrm{F}}(\mathfrak{r}^*) = \vartheta_{\mathrm{F}}(\mathfrak{t}) = 0.1 \ngeq min\{\vartheta_{\mathrm{F}}((\mathfrak{r}^{**} * \mathfrak{r}^*)^*), \vartheta_{\mathrm{F}}(\mathfrak{r})\}$
$= min\{\vartheta_{\mathrm{F}}(m), \vartheta_{\mathrm{F}}(\mathfrak{r})\} = 0.8.$

Proposition 4.3:

Every IFS-$\mathcal{S}$-filter of $\mathcal{X}$ is IFS-$\mathcal{C}$-$\mathcal{S}$-filter at any $\mathcal{C}$- $\mathcal{S}$-filter .

Proof :-

If $\mathcal{F}$ is $\mathcal{C}$- $\mathcal{S}$-filter of $\mathcal{X}$, and $\mathcal{A} = (\vartheta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ is IFS- $\mathcal{S}$-filter, then

$\mathcal{C}_1$- $\vartheta_{\mathcal{F}}(e) \succcurlyeq \vartheta_{\mathcal{F}}(z)$, and $\Omega_{\mathcal{F}}(e) \preccurlyeq \Omega_{\mathcal{F}}(z)$, $\forall z \in \mathcal{X}$.

$\mathcal{C}_2$- $\vartheta_{\mathcal{A}}(z^*) \succcurlyeq min\{\vartheta_{\mathcal{A}}((\mathrm{r}^{**} * z^*)^*), \vartheta_{\mathcal{A}}(\mathrm{r})\}$,for all $z, \mathrm{r} \in \mathcal{X}$,since $\mathcal{F} \subseteq \mathcal{X}$,then ,
$\vartheta_{\mathcal{F}}(z^*) \succcurlyeq min\{\vartheta_{\mathcal{F}}((\mathrm{r}^{**} * z^*)^*), \vartheta_{\mathcal{F}}(\mathrm{r})\}$, $\forall\ \mathrm{r} \in \mathcal{F}$.

$\mathcal{C}_3$- $\Omega_{\mathcal{A}}(z^*) \preccurlyeq max\{\Omega_{\mathcal{A}}((\mathrm{r}^{**} * z^*)^*), \Omega_{\mathcal{A}}(\mathrm{r})\}, \forall z, \mathrm{r} \in \mathcal{X}$,since $\mathcal{F} \subseteq \mathcal{X}$, then ,
$\Omega_{\mathcal{F}}(z^*) \preccurlyeq max\{\Omega_{\mathcal{F}}((\mathrm{r}^{**} * z^*)^*), \Omega_{\mathcal{F}}(\mathrm{r})\}, \forall \mathrm{r} \in \mathcal{F}$.

Thus $\mathcal{A} = (\vartheta_{\mathcal{F}}, \Omega_{\mathcal{F}})$is IFS-$\mathcal{C}$-$\mathcal{S}$-filter at $\mathcal{F}$in .

Remark 4.4:

In general, the inverse of Proposition (4.3) is not realized, can demonstrate this by the following example.

Example4.5 :

In Example (4.2),let $\mathcal{F} = \{\mathfrak{r}, m\}$ be $\mathcal{C}$- $\mathcal{S}$-filter of $\mathcal{X}$.if

$\vartheta_{\mathcal{F}}(z) = \begin{cases} 0.6 & if\ z = 0, g, m \\ 0.2 & if\ z = \mathfrak{r}, \mathfrak{t} \end{cases}$ $\qquad \Omega_{\mathcal{F}}(z) = \begin{cases} 0.3 & if\ z = 0, g, m \\ 0.5 & if\ z = \mathfrak{r}, \mathfrak{t} \end{cases}$

then $\mathcal{A} = (\vartheta_{\mathcal{F}}, \Omega_{\mathcal{F}})$ is IFS-$\mathcal{C}$-$\mathcal{S}$-filter at $\mathcal{F}$,but $\mathcal{A}$ is not IFS- $\mathcal{S}$-filter,because

$\vartheta_{\mathcal{F}}(\mathrm{t}^*) = \vartheta_{\mathcal{F}}(\mathrm{r}) = 0.2 \ngeqslant min\{\vartheta_{\mathcal{F}}((0^{**} * \mathrm{t}^*)^*), \vartheta_{\mathcal{F}}(0)\} = min\{\vartheta_{\mathcal{F}}(m), \vartheta_{\mathcal{F}}(0)\} = 0.6$

Corollary4.6 :

Every IFS-Q-filter of Q- algebra $\mathcal{X}$ is IFS-$\mathcal{C}$-$\mathcal{S}$-filter at any $\mathcal{C}$- $\mathcal{S}$-filter .

Proof :

By using Proposition (3.3) and using Proposition (4.3).

Proposition4.7 :

Every IFS-$\mathcal{C}$-Q-filter at $\mathcal{C}$-Q-filter $\mathcal{F}$ of $\mathcal{X}$ is IFS-$\mathcal{C}$-$\mathcal{S}$-filter at $\mathcal{C}$-Q-filter $\mathcal{F}$.

Proof :

Let $\mathcal{A} = (\vartheta_{\mathcal{F}}, \Omega_{\mathcal{F}})$ be IFS- $\mathcal{C}$- Q-filter at $\mathcal{F}$,such that $\mathcal{F}$ is $\mathcal{C}$-Q-filter then by Proposition (2.17),3, $\mathcal{F}$ is $\mathcal{C}$-$\mathcal{S}$-filter on $\mathcal{X}$, by Definition (2.14) we have

$\mathcal{C}_1$- $\vartheta_{\mathcal{F}}(e) \geqslant \vartheta_{\mathcal{F}}(z)$, and $\Omega_{\mathcal{F}}(e) \leqslant \Omega_{\mathcal{F}}(z)$, $\forall z \in \mathcal{X}$.

$\mathcal{C}_2$- $\vartheta_{\mathcal{F}}(z) \geqslant min\{\vartheta_{\mathcal{F}}((z^* * \mathrm{r}^*)^*), \vartheta_{\mathcal{F}}(\mathrm{r})\}$, $\forall$ r $\in \mathcal{F}$. Thus

$\vartheta_{\mathcal{F}}(z^*) \geqslant min\{\vartheta_{\mathcal{F}}((z^{**} * \mathrm{r}^*)^*), \vartheta_{\mathcal{F}}(\mathrm{r})\}$, [by using Remark (2.3),2]

$\vartheta_{\mathcal{F}}(z^*) \geqslant min\{\vartheta_{\mathcal{F}}((\mathrm{r}^{**} * z^*)^*), \vartheta_{\mathcal{F}}(\mathrm{r})\}, \forall \mathrm{r} \in \mathcal{F}$.

$\mathcal{C}_3$- $\Omega_{\mathcal{F}}(z) \leqslant max\{\Omega_{\mathcal{F}}((z^* * \mathrm{r}^*)^*), \Omega_{\mathcal{F}}(\mathrm{r})\}, \forall \mathrm{r} \in \mathcal{F}$.Thus

$\Omega_{\mathcal{F}}(z^*) \leqslant max\{\Omega_{\mathcal{F}}((z^{**} * \mathrm{r}^*)^*), \Omega_{\mathcal{F}}(\mathrm{r})\}, \forall \mathrm{r} \in \mathcal{F}$.[by using Remark (2.3),2]

$\Omega_{\mathcal{F}}(z^*) \leqslant max\{\Omega_{\mathcal{F}}((\mathrm{r}^{**} * z^*)^*), \Omega_{\mathcal{F}}(\mathrm{r})\}, \forall \mathrm{r} \in \mathcal{F}$.

Then $\mathcal{A} = (\vartheta_{\mathcal{F}}, \Omega_{\mathcal{F}})$ is IFS-$\mathcal{C}$-$\mathcal{S}$-filter at $\mathcal{F}$.

Remark 4.8:

In general, IFS-$\mathcal{C}$-$\mathcal{S}$-filter at $\mathcal{F}$ is not IFS-$\mathcal{C}$-Q-filter an in the following example .

Example 4.9:

in Example (4.2) let $\mathcal{F} = \{\mathrm{t}, m\}$ be $\mathcal{C}$- $\mathcal{S}$-filter and $\mathcal{C}$-Q-filter of $\mathcal{X}$.

If

$$\vartheta_{\mathcal{F}}(z) = \begin{cases} 0.4 & if\ z = g \\ 0.7 & if\ z = 0, \mathrm{r}, \mathrm{t}, m \end{cases} \qquad \Omega_{\mathcal{F}}(z) = \begin{cases} 0.5 & if\ z = g \\ 0.2 & if\ g = 0, \mathrm{r}, \mathrm{t}, m \end{cases}$$

Then IFS $\mathcal{A} = (\vartheta_{\mathcal{F}}, \Omega_{\mathcal{F}})$ is IFS-$\mathcal{C}$-$\mathcal{S}$-filter at $\mathcal{F}$, but $\mathcal{A}$ is not IFS-$\mathcal{C}$-Q-filter,because

$\vartheta_{\mathcal{F}}(g) = 0.4 \ngeqslant min\{\vartheta_{\mathcal{F}}((g^* * \mathrm{t}^*)^*), \vartheta_{\mathcal{F}}(\mathrm{t})\} = 0.7$

Proposition 4.10:

Every IFS-$\mathcal{C}$-$\mathcal{S}$-filter at $\mathcal{C}$-$\mathcal{S}$-filter $\mathcal{F}$ in an involutory Q-algebra$\mathcal{X}$ is IFS-$\mathcal{C}$-Q-filter at $\mathcal{F}$.

Proof :

If $\mathcal{A} = (\vartheta_{\mathcal{F}}, \Omega_{\mathcal{F}})$ is IFS- $\mathcal{C}$- $\mathcal{S}$-filter at $\mathcal{F}$,such that $\mathcal{F}$ is $\mathcal{C}$-$\mathcal{S}$-filter [by Proposition (2.21)],then $\mathcal{F}$ is $\mathcal{C}$-Q-filter.By using Definition (4.1)

1-$\vartheta_{\mathcal{F}}(e) \geqslant \vartheta_{\mathcal{F}}(z)$, and $\Omega_{\mathcal{F}}(e) \leqslant \Omega_{\mathcal{F}}(z)$, $\forall z \in \mathcal{X}$.

2-$\vartheta_{\mathcal{A}}(z^*) \geqslant \min\{\vartheta_{\mathcal{A}}((\mathrm{r}^{**} * z^*)^*), \vartheta_{\mathcal{A}}(\mathrm{r})\}$

$$\begin{aligned} \vartheta_{\mathcal{A}}(z) = \vartheta_{\mathcal{F}}(z^{**}) &\geqslant min\{\vartheta_{\mathcal{A}}((\mathrm{r}^{**} * z^{**})^*), \vartheta_{\mathcal{A}}(\mathrm{r})\} \\ &= min\{\vartheta_{\mathcal{A}}((z^{***} * \mathrm{r}^*)^*), \vartheta_{\mathcal{A}}(\mathrm{r})\} \\ &= min\{\vartheta_{\mathcal{A}}((z^* * \mathrm{r}^*)^*), \vartheta_{\mathcal{A}}(\mathrm{r})\}. \end{aligned}$$

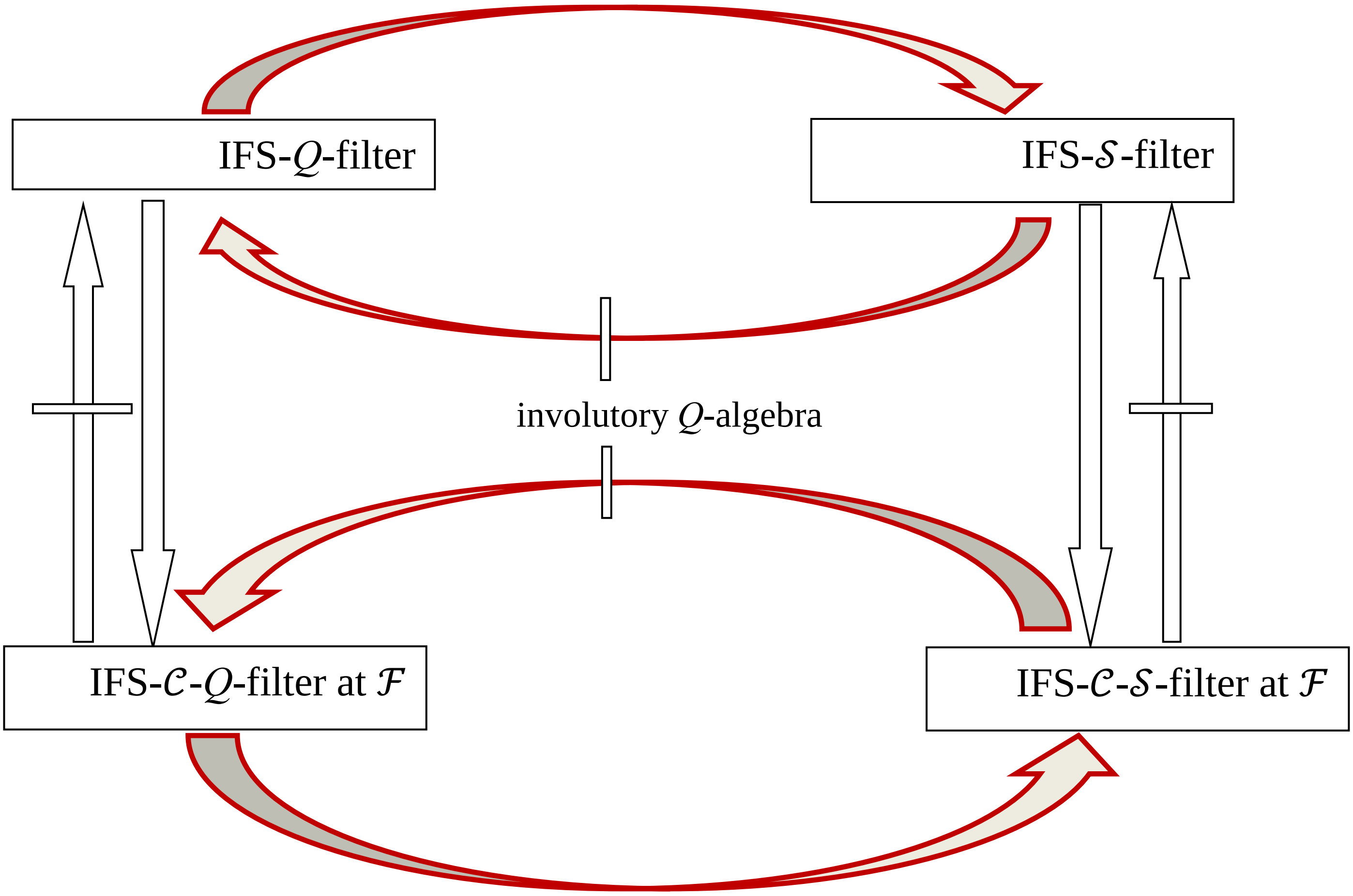
3- $\Omega_{\mathcal{F}}(z^*) \leqslant \max\{\Omega_{\mathcal{F}}((\mathrm{r}^{**} * z^*)^*), \Omega_{\mathcal{F}}(\mathrm{r})\}$

$$\begin{aligned} \Omega_{\mathcal{F}}(z) = \Omega_{\mathcal{F}}(z^{**}) &\leqslant max\{\Omega_{\mathcal{F}}((\mathrm{r}^{**} * z^{**})^*), \Omega_{\mathcal{F}}(\mathrm{r})\} \\ &= max\{\Omega_{\mathcal{F}}((\mathrm{r}^{**} * z^*)^*), \Omega_{\mathcal{F}}(\mathrm{r})\} \\ &= max\{\Omega_{\mathcal{F}}((z^{***} * \mathrm{r}^*)^*), \Omega_{\mathcal{F}}(\mathrm{r})\} \\ &= max\{\Omega_{\mathcal{F}}((z^* * \mathrm{r}^*)^*), \Omega_{\mathcal{F}}(\mathrm{r})\} \end{aligned}$$

Thus$\mathcal{A}$ is IFS-$\mathcal{C}$-Q-filter at $\mathcal{F}$of $\mathcal{X}$.

Remark 4.11:

The next diagram shows the relationship between different types Intuitionistic fuzzy filters (IFS-Q-filter, IFS- $\mathcal{C}$-Q-filter at $\mathcal{F}$, IFS- $\mathcal{S}$-filter and IFS- $\mathcal{C}$- $\mathcal{S}$-filter at $\mathcal{F}$).



5. Conclusion

This work is study some types of intuitionistic fuzzy filters which is called ($\mathcal{S}$-filter and $\mathcal{C}$- $\mathcal{S}$-filter) on Q-algebra, which is generalizing the concept of fuzzy filters, we added some important characteristics and equivalents definition in Q-algebra, and the relations related to them .

6. REFRENES

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Lie group Method for Solving System of Stochastic Differential Equations

Zainab Mohammed Alwan

Lect., Dr., Department of Mathematics, College of Science,
Mustansiriyah University, Iraq
lionwight_2009@uomustansiriyah.edu.iq

Abstract. In the current work, we realize Lie group method for system of stochastic differential equations(SDE). To comprehend this method which is used the vector field in the function and solved system by associated with Fokker-Planck equations(FPE). For more accurate, we inserted some applications of system solved by this method.

Keywords: Lie group , SDE ,FPE ,Vector field , Wiener process.

Introduction

Lie group (L.G) method of (ODEs) is will sense in [1,2,3,4] and exercised many important applications for sense (DEs). The idea Lie's classical tactic settled on ruling a symmetry group (symg) correlating with the (DEs). The inequality to the deterministic (DEs) only a few effort have been made to dilate (L.G) theory to (SDE) . It is shown in [6], how to calculation get (sym) of the (FPE) which is an equation for probability density from those of (SDE) which is the equation for space variable χ(t), depending on Wiener process(W.P). Lie symmetries of Wiener process (SDE) in [5,6,9,7,12,8].

(L.G),(SDEs),[10]

In the subsidiary section, discussed the SDEs in the Ito brew :

$$dx^i = \Theta^i(x,t)\,dt + \Upsilon^i_k(x,t)\,d\varpi^k \tag{2.1}$$

Where Θ^i and Υ^i_k are fine functions , Υ a nonzero matrix and ϖ^k are distinct identical gauge (W.P) , satisfactory:

$$\left\langle \left| \varpi^i(t) - \varpi^i(t) \right|^2 \right\rangle = \rho^{ij}\rho(t-s)$$

(2.2)

It is famous that (2.1) is the (Ito eq) is correlating a diffusion (F-P or Chapman-Kolmogorov) (eq) which write as:

$$G_t + g^{ij}G_{ij} + h^i G_i + DG = 0 \tag{2.3}$$

$$\gamma^{ij} = -\frac{1}{2}\left(\varpi\ \varpi^{T}\right)^{ij}$$
$$\hbar^{i} = \Theta^{i} - \partial_{j}\left(\varpi\ \varpi^{T}\right)^{ij}$$
$$\hbar = \left(\partial_{i}.\Theta^{i}\right) - \frac{1}{2}\partial^{2}_{ij}\left(\varpi\ \varpi^{T}\right)^{ij} \tag{2.4}$$

Is settled , to invention for (2.3) the 2- extension of the(sym) worker is :

$$\Pi^{[2]} = \tau(t)\frac{\partial}{\partial t} + \varsigma_{i}\frac{\partial}{\partial x_{i}} + \eta\frac{\partial}{\partial u} + \xi_{t}\frac{\partial}{\partial u_{t}} + \xi_{i}\frac{\partial}{\partial u_{i}} + \xi_{ik}\frac{\partial}{\partial u_{ik}} \tag{2.5}$$

The extended infinitesimals are:

$$\xi_{t} = D_{(t)}(\eta) - u_{t}\ D_{(t)}(\tau) - u_{j}\ D_{(t)}(\varsigma_{j}) \tag{2.6}$$

$$\xi_{i} = D_{(i)}(\eta) - u_{t}\ D_{(i)}(\tau) - u_{j}\ D_{(i)}(\varsigma_{j}) \tag{2.7}$$

$$\xi_{ik} = D_{(k)}(\varsigma_{i}) - u_{it}\ D_{(k)}(\tau) - u_{ij}\ D_{(k)}(\varsigma_{j}) \tag{2.8}$$

anywhere:

$$D_{(t)} = \frac{\partial}{\partial t} + u_{t}\frac{\partial}{\partial u} + u_{(tj)}\frac{\partial}{\partial u_{j}} + \ldots + u_{(t_{i1}\to M)}\frac{\partial}{\partial\ u_{(t_{i1}\to M)}} \tag{2.9}$$

$$D_{(i)} = \frac{\partial}{\partial x_{i}} + u_{i}\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_{j}} + \ldots + u_{(ii_{1}\to M)}\frac{\partial}{\partial\ u_{(i_{1}\to M)}} \tag{2.10}$$

Then the determining (eq) of an SDE associated with the FPE as :

$$\frac{\partial(\tau\gamma_{ik})}{\partial t} + \left(\varsigma_{r}\frac{\partial\gamma_{ik}}{\partial x_{r}} - \gamma_{ir}\frac{\partial\varsigma_{k}}{\partial x_{r}} - \gamma_{rk}\frac{\partial\varsigma_{i}}{\partial x_{r}}\right) = 0 \tag{2.11}$$

$$\frac{\partial(\varsigma_{i} - \tau\Theta_{i})}{\partial t} + \Theta_{r}\frac{\partial\varsigma_{i}}{\partial x_{r}} - \varsigma_{r}\frac{\partial\Theta_{i}}{\partial x_{r}} - \gamma_{rk}\frac{\partial^{2}\varsigma_{i}}{\partial x_{r}\partial x_{k}} = 0 \tag{2.12}$$

17. Applications

In the following, we discuses some examples to show this method.

Application(3.1): Consider $\Theta = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2\ell} \end{pmatrix}$, $\Upsilon = \begin{pmatrix} y \\ -\ell y \end{pmatrix}$, record as :

$dx = y dt$, $dy = -\ell y dt + \sqrt{\ell}\, dW(t)$, where ℓ is (+ constant)
(3.1)

Corresponding (FPE) :

$$\frac{\partial u}{\partial t} = \ell \frac{\partial^2 u}{\partial y^2} - y \frac{\partial u}{\partial x} + \ell y \frac{\partial u}{\partial y} + \ell u$$
(3.2)

Let

$$\Pi = \tau(t)\frac{\partial}{\partial t} + \varsigma^1(t,x,y)\frac{\partial}{\partial x} + \varsigma^2(t,x,y)\frac{\partial}{\partial y} + \eta(t,x,y)\frac{\partial}{\partial u} \tag{3.3}$$

The 2-prolongation:

$$\Pi^{[2]} = \Pi + \xi_t \frac{\partial}{\partial u_t} + \xi_x \frac{\partial}{\partial u_x} + \xi_y \frac{\partial}{\partial u_y} + \xi_{tt}\frac{\partial}{\partial u_{tt}} + \xi_{tx}\frac{\partial}{\partial u_{tx}} + \xi_{ty}\frac{\partial}{\partial u_{ty}}$$
$$+ \xi_{xx}\frac{\partial}{\partial u_{xx}} + \xi_{xy}\frac{\partial}{\partial u_{xy}} + \xi_{yy}\frac{\partial}{\partial u_{yy}}$$
(3.4)

The determining equation is:

$$\Pi^{[2]}\left(\frac{\partial u}{\partial t} - \ell \frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial x} - \ell y \frac{\partial u}{\partial y} - \ell u\right)\Bigg|_{(3.1)} = 0$$
(3.5)

We get:

$$\xi_t - \ell \xi_{yy} + \xi^2 u_x + y\xi_x - \ell \xi^2 u_y - \ell y \xi_y - \ell \eta = 0$$
(3.6)

By using expansions for ((2.6)-(2.8)) and replacing $\frac{\partial u}{\partial t}$ by $\ell \frac{\partial^2 u}{\partial y^2} - y \frac{\partial u}{\partial x} + \ell y \frac{\partial u}{\partial y} + \ell u$

We result :

$$\eta_t + \left(\ell u_{yy} - y u_x + \ell y u_{yy} + \ell u\right)\left(\eta_u - \tau_t\right) - u_x \xi_t^1 - u_y \xi_t^2 - \ell\left(\eta_{yy} + 2u_y \eta_{yu} + u_{yy}\eta_u + 2u_y \eta_{uu} - 2u_{yy}\xi_y^2 - u_y \xi_{yy}^2\right)$$
$$+ \xi^2 u_x + y\left(\eta_x + u_x \eta_u - u_x \xi_x^2\right) - \ell \xi^2 u_y - \ell y\left(\eta_y + u_y \eta_y - u_y \xi_y^2\right) - \ell \eta = 0 \tag{3.7}$$

Solved by separation of the coefficient we obtain the general solution :

$$\tau = c_1$$
$$\xi^1 = c_5 + c_3 e^{-\ell t}\ell^{-1} + c_4 t + c_6 e^{\ell t}\ell^{-1}$$
$$\xi^2 = c_4 - c_3 e^{-\ell t} + c_6 e^{\ell t}$$
$$\eta = \left(c_2 - \frac{1}{2}c_4(y + \ell t) - c_6 y e^{\ell t}\right) + \alpha(t, x, y)$$

(3.8)

Where C_i are constant , we obtain the following:

$$\mathbf{P}_1 = \frac{\partial}{\partial t}$$
$$\mathbf{P}_2 = u\frac{\partial}{\partial u}$$
$$\mathbf{P}_3 = e^{-\ell t}\left[\ell^{-1}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]$$
$$\mathbf{P}_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - \frac{1}{2}(y + \ell x)u\frac{\partial}{\partial u}$$
$$\mathbf{P}_5 = \frac{\partial}{\partial x}$$
$$\mathbf{P}_6 = e^{\ell t}\left[\ell^{-1}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - y u\frac{\partial}{\partial u}\right]$$
$$\mathbf{P}_\alpha = \alpha(t, x, y)\frac{\partial}{\partial u} \qquad (3.9)$$

Now, the symmetry generators of (3.1) , when using ((2.11)-(2.12)) we find:

$$\frac{\partial \xi^1}{\partial y} = 0$$
$$\frac{\partial \xi^2}{\partial y} = 0$$
$$\frac{\partial \tau}{\partial t} = 0$$
$$\frac{\partial \xi^1}{\partial t} + y\frac{\partial \xi^1}{\partial x} + \xi^2 = 0$$
$$\frac{\partial \xi^2}{\partial t} + y\frac{\partial \xi^2}{\partial x} + \ell\,\xi^2 = 0$$

(3.10)

By solving above system we get the general solution:

$$\tau(t) = c_1$$
$$\xi^1 = c_2\ell^{-1}e^{-\ell t} + c_3 \qquad (3.11)$$
$$\xi^2 = -c_2 e^{-\ell t}$$

The symmetry generators are Π_1, Π_3 and Π_5.

Application(3.2): Consider $\Theta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\Upsilon = \begin{pmatrix} \frac{1}{x} \\ 1 \end{pmatrix}$,

record as:

$$dx = \frac{1}{x} dt + dW_1(t) \ , \ dy = dt + dW_2(t)$$

(3.12)

The associated with FPE is:

$$u_t = \frac{1}{2}\left(u_{xx} + u_{yy}\right) + \frac{1}{x^2} u - \frac{1}{x} u_x - u_y$$

(3.13)

By using (3.3) and applied 2-prolongation (3.4) on (3.13) as:

$$\Pi^{[2]} \left(u_t - \frac{1}{2}\left(u_{xx} + u_{yy}\right) - \frac{1}{x^2} u + \frac{1}{x} u_x + u_y \right)\Bigg|_{(3.13)} = 0$$

(3.14)

$$\xi_t - \frac{1}{2}(\xi_{xx} + \xi_{yy}) - x^{-2}\eta + 2x^{-3}u + x^{-1}\xi_x - x^{-2}u_x + \xi_y = 0$$

(3.15)

$$\eta_t + u_t(\eta_u - \tau_t) - u_x \xi^1_t - u_y \xi^2_t - \frac{1}{x^2}\eta + \frac{2}{x^3} u - \frac{1}{x^2} u_x + (\eta_y + u_y \eta_u - u_y \xi^2_y)$$
$$+ \frac{1}{x}(\eta_x + u_x \eta_u - u_x \xi^1_x) - \frac{1}{2}\eta_{xx} - \eta_{xu} u_x - \frac{1}{2} u_{xx} \eta_u - \frac{1}{2} u_{x^2} \eta_{uu} + u_{xx} \xi^1_x$$
$$+ \frac{1}{2} u_x \xi^1_{xx} - \frac{1}{2}\eta_{yy} - u_y \eta_{yu} - \frac{1}{2} u_{yy} \eta_u - \frac{1}{2} u_{y^2} \eta_{uu} + u_{yy} \xi^2_y + \frac{1}{2} u_y \xi^2_{yy} = 0$$

(3.16)

Solved (3.16) by separation of the coefficient yields the general solution:

$$\tau = c_1 + 2t c_4 - t c_5 + t^2 c_6$$
$$\xi^1 = x c_4 + t x c_6$$
$$\xi^2 = c_3 + (y + t) c_4 + t y c_6$$
$$\eta = \left(c_2 - 2c_4 + c_5 (y - t) + c_6 t y - \frac{1}{2}(x^2 + y^2 t^2) \right) u + \alpha(t, x, y)$$

(3.17)

We obtain:

$$\Pi_1 = \frac{\partial}{\partial t}$$

$$\Pi_2 = u\frac{\partial}{\partial u}$$

$$\Pi_3 = \frac{\partial}{\partial y}$$

$$\Pi_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + (y+t)\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}$$

$$\Pi_5 = -t\frac{\partial}{\partial y} + (y-t)u\frac{\partial}{\partial u}$$

$$\Pi_6 = t\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + \left(ty - \frac{1}{2}(x^2 + y^2 + t^2)\right)u\frac{\partial}{\partial u}$$

$$\Pi_\alpha = \alpha(t,x,y)\frac{\partial}{\partial u}$$

(3.18)

We find the determining equation by using ((2.11)-(2.12)) as:

$$\frac{\partial \xi^1}{\partial y} = 0$$

$$\frac{\partial \xi^1}{\partial x} - \frac{1}{2}\frac{\partial \tau}{\partial t} = 0$$

$$\frac{\partial \xi^2}{\partial x} = 0$$

$$\frac{\partial \xi^2}{\partial y} - \frac{1}{2}\frac{\partial \tau}{\partial t} = 0$$

$$\frac{\partial \xi^1}{\partial t} + \frac{1}{x}\frac{\partial \xi^1}{\partial x} + \frac{1}{x^2}\xi^1 - \frac{1}{x}\frac{\partial \tau}{\partial t} + \frac{1}{2}\frac{\partial^2 \xi^1}{\partial x^2} = 0$$

$$\frac{\partial \xi^2}{\partial t} + \frac{\partial \xi^2}{\partial y} - \frac{\partial \tau}{\partial t} + \frac{1}{2}\frac{\partial^2 \xi^2}{\partial y^2} = 0$$

(3.19)

By solving system (3.19) we find the general solution as:

$$\tau = c_1 t + c_2$$

$$\xi^1 = \frac{1}{2}c_1 x$$

$$\xi^2 = \frac{1}{2}c_1 y + \frac{1}{2}c_1 t + c_3$$

(3.20)

The (sym) generators of (3.12) generate by Π_1, Π_3 and

$$\tilde{\Pi}_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + (y+t)\frac{\partial}{\partial y}$$

(3.21)

Which is a projection of Π_4 to (t,x,y)-space.

Conclusion

In this paper, introduced Lie group method for solving system of stochastic differential equations(SDE). Also studied techniques for this method which is used to solve system by associated with Fokker-Planck equations(FPE) show that by give some applications about this method.

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Some Results on (N, k)-Hyponormal Operators

Nidaa Muraeh Atheab[1] and Salim Dawood Mohsen[2]

[1]Wasit University/ College of Education /Department of Mathematics
[2]AL-Mustansirya University/ College of Education /Department of Mathematics

Nadaaalbadry1981@yahoo.com

dr_salim2015@yahoo.com

Abstract. In this paper, we introduce a new generalization for hyponormal operators which is (N, k)-hyponormal operators, also we study some properties of these operators. In addition, we given the solvability of the λ- commuting operator equation $ST = \lambda TS$,where $\lambda \in \mathbb{C}$, and S , T are bounded (N, k)- hyponormal operators.

1. Introduction

The first to study the concept of hyponormal operators was P.R.Halmos in (1950)[6], .In (1962), J.G.Stampfli [10] was studied some properties of hyponormal operators. In (1972) Shila Devi[9] defined a new generalization for hyponormal operators which call quasihyponormal operators. In (1974), B. L. Wadhwa[12] introduced the M-hyponormal operators. In (1979) Kevin Clancy [3] introduced three equivalent formulas for hyponormal operators. In (2009) N.L.Braha [2] given a new formula for hyponormal operators.

The purpose of this paper is to present a study on the (N, k)-hyponormal operators. In this study we explain that the inverse of invertible (N, k)-hyponormal operator is not necessarily be (N, k)-hyponormal. Also we explain that the sum and the product of two (N, k)-hyponormal operators need not be (N, k)-hyponormal.

During this paper, $\mathbb{H}$ represents the Hilbert space, and every operator defined on ć is bounded linear operator.

Finally, we give the following theorem:

Theorem

Let S , T: $\mathbb{H} \to \mathbb{H}$ be operators on $\mathbb{H}$ such that $ST = \lambda TS \neq 0$, $\lambda \in \mathbb{C}$. Let $N_1, N_2: \mathbb{H} \to \mathbb{H}$ be non-zero positive operators on $\mathbb{H}$, such that $N_1T = TN_1$ and $N_2S = SN_2$, then:

i. If S^* is (N1,k)-hyponormal operator and T is (N_2 ,k)-hyponormal operator , then $|\lambda| \leq (\|N_1\|.\|N_2\|)^{\frac{1}{2}}$.
ii. If S is (N1,k)-hyponormal operator and T* is (N_2 ,k)-hyponormal operator , then $|\lambda| \geq (\|N_1\|.\|N_2\|)^{-\frac{1}{2}}$.

5. Preliminaries

In this section, we given some essential definitions and propositions, we will need in this paper. Let us start by the definition of self – adjoint operator.

2.1. Definition [11]
Let T: $\mathbb{H} \rightarrow \mathbb{H}$ be an operator on $\mathbb{H}$, then T is called self-adjoint operator if $T^* = T$.

2.2. Definition [1 , P. 2]
Let T: $\mathbb{H} \rightarrow \mathbb{H}$ be a self-adjoin operator on $\mathbb{H}$, then T is called **positive** , written $T \geq 0$, if and only if $< Tx, x > \geq 0, \forall x \in \mathbb{H}$.

2.3.Definition [5]
Let T: $\mathbb{H} \rightarrow \mathbb{H}$ be an operator on $\mathbb{H}$, then T is called **normal** if $T^* T = T T^*$, that is:
$< T^* T x, x > = < T T^* x, x >, \forall x \in H$.

2.4.Definition [3, P. 1],[7]
Let T: $\mathbb{H} \rightarrow \mathbb{H}$ be an operator on $\mathbb{H}$, then T is called **hyponormal** if $T^* T \geq T T^*$, that is:
$< T^* T x, x > \geq < T T^* x, x >, \forall x \in \mathbb{H}$.

The following proposition gives equivalent formulas for hyponormal operators:

2.5.Proposition [3, P. 3],[2]
Let T: $\mathbb{H} \rightarrow \mathbb{H}$ be an operator on $\mathbb{H}$, then the following arguments are equivalent:

i. $T^*T \geq TT^*$
ii. $T^*T + 2\lambda TT^* + \lambda^2 T^*T \geq 0, \forall \lambda \in \mathbb{R}$.
iii. $\|T^*x\| \leq \|Tx\|, \forall x \in \mathbb{H}$.
iv. $T^* = ST$,for some bounded linear operator S : $\mathbb{H} \rightarrow \mathbb{H}$, such that $\|S\| \leq 1$.

Now, we recall a few properties for hyponormal operators.

2.6.Proposition[7, P. 225], [10]

Let T: $\mathbb{H} \rightarrow \mathbb{H}$ be an operator on $\mathbb{H}$,then:

i) λT is hyponormal operator , for every $\lambda \in \mathbb{C}$.
ii) $(T - \lambda I)$ is hyponormal operator , for every $\lambda \in \mathbb{C}$.
iii) If T has inverse ,then the inverse of T is hyponormal operator .
iv) If $E \subset \mathbb{H}$ invariant under T, then $T|_E$ is hyponormal.

2.7.Proposition [8], [4]
Let S, T: $\mathbb{H} \rightarrow \mathbb{H}$ be hyponormal operators, then:

i. (S+T) is hyponormal operator if $TS^* = S^*T$ and $ST^* = T^*S$.
ii. (S T) is hyponormal operator if $ST^* = T^*S$.

6. Main Result

In the following ,we introduce the new generalization for the hyponormal operators:

3.1.Definition
Let T: $\mathbb{H} \rightarrow \mathbb{H}$ be an operator on $\mathbb{H}$, then T is called **(N, k)- hyponormal** if there exists a positive operator N: $\mathbb{H} \rightarrow \mathbb{H}$ such that $NT^*T^k \geq T^kT^*$, that is $< N T^* T^k x, x > \geq < T^k T^* x, x >, \forall x \in \mathbb{H}$ and for any positive integer k. To explain this definition consider the next example.

3.2.Example

The operator T = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is (N, k) – hyponormal , where N = $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$.

3.3.Proposition

Let T: $\mathbb{H} \to \mathbb{H}$ be an (N, k) – hyponormal operator, then :

i. If N = I (identity operator on $\mathbb{H}$), and k = 1, then T is hyponormal operator,

ii. If A is closed subspace of $\mathbb{H}$ and invariant under T , then $T|_A$ is (N, k) – hyponormal operator.

Proof:

i. Obvious

ii. Suppose that T is (N, k) – hyponormal operator, and $T_1 = T|_A$, then: $Tx = T_1x$,for all $x \in A$

Let $x \in A$, then

$< N(T_1)^* (T_1)^k x , x > = < N T^* T^k x , x > \geq < T^k T^* x, x > = < (T_1)^k T^* x, x >$,for all $x \in A$.

Hence, T_1 is (N, k) – hyponormal operator.

3.4.Remarks and Examples

Let T: $\mathbb{H} \to \mathbb{H}$ be an (N, k) – hyponormal operator, then :

i. λT is (N, k) - hyponormal operator , for every $\lambda \in \mathbb{C}$.

Proof:

Assume that T is (N, k) - hyponormal operator, then $NT^*T^k \geq T^kT^*$

Now,

$$\begin{aligned}(\lambda T)^k (\lambda T)^* &= (\lambda^k T^k)(\bar{\lambda} T^*) \\ &= (\lambda^k \bar{\lambda})(T^k T^*) \\ &\leq (\lambda^k \bar{\lambda})(N T^* T^k) \\ &= N(\bar{\lambda} T^*)(\lambda^k T^k) \\ &= N(\bar{\lambda} T^*)(\lambda^k T^k) \\ &= N(\lambda T)^* (\lambda T)^k\end{aligned}$$

Thus, λT is an (N, k) - hyponormal operator.

ii. $(T- \lambda I)$ is not (N, k)- hyponormal operator for every $\lambda \in \mathbb{C}\backslash\{0\}$ .To illustrate this consider the following example:

The operator T = $\begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix}$ is (N, k)-hyponormal, where N = $\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$.

(T- 2I) = $\begin{bmatrix} 2 & 6 \\ 2 & 2 \end{bmatrix}$ is not (N,k)-hyponormal ,when k = 1 and $\lambda = 2$ .Since

$N(T- 2I)^*(T-2I) – (T- 2I)(T-2I)^* = \begin{bmatrix} 16 & 104 \\ 8 & 48 \end{bmatrix}$ and $\begin{vmatrix} 16 & 104 \\ 8 & 48 \end{vmatrix} = - 64$

iii. If T has inverse , then the inverse of T is not necessarily be (N, k) – hyponormal operator. To show this consider the next example:

The operator T = $\begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix}$ is an (N, k)-hyponormal, where N = $\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$.

But $T^{-1} = \begin{bmatrix} 1 & -1.5 \\ -0.5 & 1 \end{bmatrix}$ is not (N, k)-hyponormal .Since when k =2, we have

$N(T^{-1})^*(T^{-1})^2 - (T^{-1})^2(T^{-1})^* = \begin{bmatrix} -0.125 & 8.375 \\ 4.125 & -1.375 \end{bmatrix}$ which is not positive.

iv. T^* is not necessarily be (N, k)- hyponormal operator. To explain this consider the following example:

The operator T = $\begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix}$ is an (N, k)-hyponormal, where N = $\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$.

But T^*= $\begin{bmatrix} 4 & 2 \\ 6 & 4 \end{bmatrix}$ is not (N, k)-hyponormal operator. Since when k = 1,we have

$$NT^*T - TT^* = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 272 & 148 \\ 52 & 0 \end{bmatrix}.$$

And the determinant $\begin{vmatrix} 272 & 148 \\ 52 & 0 \end{vmatrix} = - 7696$

Now, in the following proposition we give the conditions that make Remark(3.4)(iii) are true .

3.5. Proposition

Permit that T: $\mathbb{H} \rightarrow \mathbb{H}$ is (N, k) – hyponormal operator, then:

i. If T^{-1} and N^{-1} are exists and $NT^*T^k = T^*T^kN$,then T^{-1} is an (N, k)- hyponormal operator.
ii. If $T^kT^* = T^*T^k$, and $N^*(T^*T^k)^* = (T^*T^k)^*N^*$, then T^* is an (N,k)- hyponormal operator.

Proof:

i)

Assume that T is (N, k)- hyponormal operator, then :

$N\,T^*T^k \geq T^kT^*$

$T^*T^kN \geq T^kT^*$

$(T^kT^*)^{-1} \geq (T^*T^kN)^{-1}$

$(T^*)^{-1}(T^k)^{-1} \geq N^{-1}(T^k)^{-1}(T^*)^{-1}$

$N(T^*)^{-1}(T^k)^{-1} \geq N\,N^{-1}(T^k)^{-1}(T^*)^{-1}$

$N(T^*)^{-1}(T^k)^{-1} \geq I(T^k)^{-1}(T^*)^{-1}$

$N(T^*)^{-1}(T^k)^{-1} \geq (T^k)^{-1}(T^*)^{-1}$

$N(T^{-1})^*(T^{-1})^k \geq (T^{-1})^k(T^{-1})^*$

Hence , T^{-1} is (N, k)- hyponormal operator.

ii)

$(T^*)^k(T^*)^* = (T^*T^k)^*$

$= (T^kT^*)^*$

$\leq (N\,T^*T^k)^*$

$= (T^*T^k)^*N^*$

$= N^*(T^*T^k)^*$

$= N(T^*T^k)^*$

$= N(T^kT^*)^*$

$= N(T^*)^*(T^k)^*$

$= N(T^*)^*(T^*)^k$

Therefore, T^* is an (N, k)-hyponormal operator. ■

3.6. Remark

Let S,T : $\mathbb{H} \rightarrow \mathbb{H}$ be an (N, k)-hyponormal operators on $\mathbb{H}$, then (S+T) is not necessarily be (N, k)-hyponormal. To illustrate this consider the next example:

The operators T = $\begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$, and S = $\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$ are (N, k)-hyponormal operators, where N= $\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$.

But (S+T) = $\begin{bmatrix} 8 & 6 \\ 2 & 3 \end{bmatrix}$ is not (N , k)-hyponormal. Since, when k = 2

$N(S+T)^*(S+T)^k - (S+T)^k(S+T)^* = \begin{bmatrix} 2492 & 508 \\ 6 & -2 \end{bmatrix}$.

In the following theorem we will provide the conditions that make Remark (3.6.) correct.

3.7. Theorem

Let S,T : $\mathbb{H} \rightarrow \mathbb{H}$ be (N, k)-hyponormal operators on $\mathbb{H}$ such that $ST = TS = TS^* = S^*T = ST^* = T^*S = 0$,(0 is zero operator on $\mathbb{H}$), then (S+T) is (N, k)-hyponormal operator.

Proof:

$$\begin{aligned}(S + T)^k (S + T)^* &= (S^k + T^k)(S^* + T^*) \\ &= S^k S^* + S^k T^* + T^k S^* + T^k T^* \\ &= S^k S^* + T^k T^* \\ &\leq N S^* S^k + N T^* T^k \text{ (since S, T are (N, k)-hyponormal operators)} \\ &= N (S^* S^k + T^* T^k) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)\end{aligned}$$

On the other hand

$$\begin{aligned}N (S + T)^* (S + T)^k &= N (S^* + T^*) (S^k + T^k) \\ &= N (S^* S^k + S^* T^k + T^* S^k + T^* T^k) \\ &= N (S^* S^k + T^* T^k) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)\end{aligned}$$

By (1) and (2) , we get

(S + T) is (N, k)-hyponormal operator.

3.8. Remark

Let S,T : $\mathbb{H} \rightarrow \mathbb{H}$ be (N, k)-hyponormal operators on $\mathbb{H}$, then (ST) is not necessarily be (N, k)-hyponormal. To explain this consider the below example:

The operators T = $\begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ and S = $\begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix}$ are (N, k)-hyponormal operators ,where N = I.

But (ST) = $\begin{bmatrix} 8 & 3 \\ -3 & -1 \end{bmatrix}$ is not (N, k)-hyponormal operator, when k = 1.Since

$N (ST)^* (ST) - (ST) (ST)^* = \begin{bmatrix} 0 & 54 \\ 54 & 0 \end{bmatrix}$, and the determent $\begin{vmatrix} 0 & 54 \\ 54 & 0 \end{vmatrix} = -2916 < 0$.

Now, the following theorem give the conditions which make Remark(3.8.) is true.

3.9. Theorem

Let S,T : $\mathbb{H} \rightarrow \mathbb{H}$ be (N, k)-hyponormal operators on $\mathbb{H}$ such that
$ST = TS$, $ST^* = T^*S$, $TS^* = S^*T$, $NT^*T^k = T^*T^kN$ and $N^2 = N$, then (ST) is (N, k)-hyponormal operator.

Proof:

Suppose that S, T are (N, k)- hyponormal operators, then by hypothesis we have

$(S T)^k (S T)^* = (T S)^k (S T)^*$

$= T^k S^k T^* S^*$
$= T^k T^* S^k S^*$
$\leq N T^* T^k N S^* S^k$ (since S, T are (N, k)-hyponormal operators)
$= N^2 T^* T^k S^* S^k$
$= N T^* T^k S^* S^k$
$= N T^* S^* T^k S^k$
$= N (S T)^* (T S)^k$
$= N (S T)^* (S T)^k$.

Therefore, $N (S T)^* (S T)^k \geq (S T)^k (S T)^*$, which mean that (ST) is (N , k)-hyponormal operator.

In the following theorem we solve the equation ST = λTS, where S and T are (N, k)-hyponormal operators.

3.10.Theorem

Let $S,T : \mathbb{H} \to \mathbb{H}$ be operators on $\mathbb{H}$ such that $ST = \lambda TS \neq 0$, $\lambda \in \mathbb{C}$ and let $N_1,N_2 : \mathbb{H} \to \mathbb{H}$ be non-zero positive operators on $\mathbb{H}$ such that $N_1T = TN_1$ and $N_2S = SN_2$, then:

i. If S^* is (N_1,k)-hyponormal operator and T is (N_2,k)-hyponormal operator , then $|\lambda| \leq (\|N_1\|.\|N_2\|)^{\frac{1}{2}}$.
ii. If S is (N_1,k)-hyponormal operator and T^* is (N_2,k)-hyponormal operator , then $|\lambda| \geq (\|N_1\|.\|N_2\|)^{-\frac{1}{2}}$.

Proof :

i)

Suppose that S^* is (N_1,k)-hyponormal operator and T is (N_2,k)-hyponormal operator,
Since ST = λTS , then

$$
\begin{aligned}
|\lambda|\|TS\| &= \|\lambda TS\| \\
&= \|ST\| \\
&= \|(ST)(ST)^*\|^{\frac{1}{2}} \\
&= \|ST\,T^*S^*\|^{\frac{1}{2}} \\
&\leq \|SN_2\,T^*\,T\,S^*\|^{\frac{1}{2}} \\
&= \|N_2S\,T^*\,T\,S^*\|^{\frac{1}{2}} \\
&\leq \|N_2\|^{\frac{1}{2}}.\|(S\,T^*)(ST^*)^*\|^{\frac{1}{2}} \\
&= \|N_2\|^{\frac{1}{2}}.\|ST^*\| \\
&= \|N_2\|^{\frac{1}{2}}.\|(ST^*)^*\,(S\,T^*)\|^{\frac{1}{2}} \\
&= \|N_2\|^{\frac{1}{2}}.\|TS^*ST^*\|^{\frac{1}{2}} \\
&\leq \|N_2\|^{\frac{1}{2}}.\|TN_1SS^*T^*\|^{\frac{1}{2}} \\
&= \|N_2\|^{\frac{1}{2}}.\|N_1TSS^*T^*\|^{\frac{1}{2}} \\
&\leq (\|N_1\|\,\|N_2\|)^{\frac{1}{2}}.\|(TS)(TS)^*\|^{\frac{1}{2}} \\
&= (\|N_1\|\,\|N_2\|)^{\frac{1}{2}}\|TS\|
\end{aligned}
$$

Hence $|\lambda|\,\|TS\| \leq (\|N_1\|\,\|N_2\|)^{\frac{1}{2}}\,\|TS\|$ and $|\lambda| \leq (\|N_1\|\,\|N_2\|)^{\frac{1}{2}}$.

ii)

Suppose that S is (N_1,k)-hyponormal operator and T^* is (N_2,k)-hyponormal operator.
Since ST = λTS, then

$$
\begin{aligned}
\|ST\| &= \|\lambda TS\| \\
&= |\lambda|\ \|TS\| \\
&= |\lambda|\ \|(TS)(TS)^*\|^{\frac{1}{2}} \\
&= |\lambda|\ \|TSS^*T^*\|^{\frac{1}{2}} \\
&\le |\lambda|\ \|TN_1S^*ST^*\|^{\frac{1}{2}} \\
&= |\lambda|\ \|N_1TS^*ST^*\|^{\frac{1}{2}} \\
&\le |\lambda|\ \|N_1\|^{\frac{1}{2}}\ \|TS^*ST^*\|^{\frac{1}{2}} \\
&= |\lambda|\ \|N_1\|^{\frac{1}{2}}\ \|(TS^*)(TS^*)^*\|^{\frac{1}{2}} \\
&= |\lambda|\ \|N_1\|^{\frac{1}{2}}\ \|TS^*\| \\
&= |\lambda|\ \|N_1\|^{\frac{1}{2}}\ \|(TS^*)^*\ (TS^*)\|^{\frac{1}{2}} \\
&= |\lambda|\ \|N_1\|^{\frac{1}{2}}\ \|ST^*TS^*\|^{\frac{1}{2}} \\
&= |\lambda|\ \|N_1\|^{\frac{1}{2}}\ \|SN_2TT^*S^*\|^{\frac{1}{2}} \\
&= |\lambda|\ \|N_1\|^{\frac{1}{2}}\ \|N_2STT^*S^*\|^{\frac{1}{2}} \\
&= |\lambda|\ \|N_1\|^{\frac{1}{2}}\ \|N_2STT^*S^*\|^{\frac{1}{2}} \\
&\le |\lambda|\ (\|N_1\|\ \|N_2\|)^{\frac{1}{2}}\ \|(ST)(ST)^*\|^{\frac{1}{2}} \\
&= |\lambda|\ (\|N_1\|\ \|N_2\|)^{\frac{1}{2}}\ \|ST\|
\end{aligned}
$$

Hence, $\|ST\| = |\lambda|\ (\|N_1\|\ \|N_2\|)^{\frac{1}{2}}\ \|ST\|$ and $|\lambda| \ge\ (\|N_1\|\ \|N_2\|)^{-\frac{1}{2}}$.

3.11. Corollary

Let S,T : $\mathbb{H} \to \mathbb{H}$ be operators on $\mathbb{H}$ such that ST = λTS ≠ 0, $\lambda \in \mathbb{C}$ .Let $N_1,N_2:\mathbb{H} \to \mathbb{H}$ be positive non-zero operators on $\mathbb{H}$ such that $N_1T = TN_1$ and $N_2S = SN_2$, then:

i. If S^* and T are (N ,k)-hyponormal operators , then $|\lambda| \le \|N\|$.
ii. If S and T^* are (N ,k)-hyponormal operator , then $|\lambda| \ge (\|N\|)^{-1}$.

Proof:
Obvious.

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Uniqueness Solution of Abstract Fractional Order Nonlinear Dynamical Control Problems

Sameer Qasim Hasan
Department of Mathematics, Al-Mustansiriyah University, Baghdad, Iraq
dr.sameer_kasim@yahoo.com
dr.sameerqasim@uomustansiriyah.edu.iq
https://orcid.org/0000-0002-2613-2584

Abstract. The aim of this paper is to investigate the Uniqueness solution of Abstract Cauchy Problem represented for fractional order nonlinear dynamical control system involving certain control input and their approach of investigated depended on commutative composite semigroup and some certain conditions in certain space.

1. Introduction

The semilinear and nonlinear equations appearing in variety of theories and applications ,in particular in the theory of fractional ordinary and fractional partial differential equations as well as integral equations with different types of derivatives have recently been addressed by several researchers for different problems and provided excellent tool for the description of memory and hereditary properties of various materials and processes.

In [12],[14],[15],[17],[19],[20], the authors had been studied some classes of nonlinear and semiliner equation without ordinary or fractional derivatives with projectively compact and which among others contains completely continuous , quasi compact and monotone operators with general fixed point theorems as well as the nonlinear and semilinear equation studied with closed linear operator in Hilbert space, self adjoint operator also some time with perturbed operator that has densely defined domain in Banach space , moreover studied with monotonicity and compactness of the linear operator on reflexive Banach space ,the strongly positive operator and maximal monotonicity linear operator with nonlinear functions presented with existence and uniqueness approach.
In [7],[1],[11],[21],[16],[5],[6],[13],[2], the authors had been studied the solvability of fractional order nonlinear and semilinear control differential equations by using fractional integral formulation with properties of calculus of fractional derivative and integration and the existence and uniqueness obtained by using classical fixed point theorems with initial values as well as boundary values and integral boundary condition also some of them involving nonlocal initial condition
Our intersect in this paper to study the fractional order nonlinear dynamical feedback control system involve sum of N- unbounded operators with feedback perturbation as a generators of N-semigroup with new definitions depended on no expansive prosperity , maximal accretive, maximal monotone, resolvent set , fractional derivative and fixed point theorem also presented some results for solvability without using fractional calculus and equivalent integral formulation . main interest on nonlinear functional analysis and some new properties defined on special space,
$\boldsymbol{L_2^{\alpha}[0,T]=\{x: x\in L_2[0,T],\ {}^{C}D^{\alpha}x\in L_2[0,T]\}}$, $\boldsymbol{T>0}$. Also appear the role of feedback control operator as a perturbation for the generators still a challenge for many researchers up to our knowledge.

Our aim establish necessary and sufficient conditions on sum of nonlinearity operator interacts suitably their system:

$\sum_{i=1}^{n} \boldsymbol{F_i}\,(\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{D_a^{\alpha} x}) = \sum_{i=1}^{n} \boldsymbol{A_i x} + \sum_{i=1}^{n} \boldsymbol{B_i u_i}$

(1) $\boldsymbol{u_i}$=$\boldsymbol{K_i x}$, for all $\boldsymbol{x} \in \bigcap_{i=1}^{n} \boldsymbol{D}(\boldsymbol{A_i})$

(2)

Where $\boldsymbol{A_i}: \boldsymbol{D}(\boldsymbol{A_i}) \subseteq \boldsymbol{L_2^{\alpha}} \to \boldsymbol{L_2^{\alpha}}, \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots, \boldsymbol{n}$ are linear unbounded operators generators of C_0-semigroups $\boldsymbol{T_i}(\boldsymbol{t}): \boldsymbol{L_2^{\alpha}} \to \boldsymbol{L_2^{\alpha}}, i = 1,2,\dots,n$, $\mathbf{0} < \alpha \leq 1$, $\boldsymbol{B_i}: \boldsymbol{D}(\boldsymbol{B_i}) \subseteq \boldsymbol{L_2^{\alpha}} \to \boldsymbol{L_2^{\alpha}}$. $\boldsymbol{F_i}: \boldsymbol{R_0^+} \times \boldsymbol{L_2} \times \boldsymbol{L_2} \to \boldsymbol{L_2^{\alpha}}$, $i = 1,2,\dots,n$, are nonlinear operators. The input control functions $\boldsymbol{u_i}(.) \in \boldsymbol{L_2^{\alpha}}[\mathbf{0}, \boldsymbol{T}]$ such that $\boldsymbol{K_i}: \boldsymbol{L_2^{\alpha}} \to \boldsymbol{L_2^{\alpha}}$ is a feedback linear operators, $i = 1,2,\dots,n$.

2. Preliminaries

Some necessary mathematical concepts for semigroup theory as well as some non-linear fractional calculus concepts have been presented.

Definition (2.1), [18]:

The family of bounded linear operators $T(t)$, $0 \leq t < \infty$ defined on the Banach space X is a semigroup if $T(0) = I$. I is identity operator on X, and $\boldsymbol{T}(\boldsymbol{t} + \boldsymbol{s}) = \boldsymbol{T}(\boldsymbol{t})\boldsymbol{T}(\boldsymbol{s})$ for every t, $s \geq 0$.

Definition (2.2) ,[18]:

Let $T(t)$ be a semigroup then $T(t)$ is called strongly continuous and which denoted by C_0 on a Banach space X if $\boldsymbol{lim}_{t\downarrow 0} \|\boldsymbol{T}(\boldsymbol{t})\boldsymbol{x} - \boldsymbol{I}\boldsymbol{x}\|_{\boldsymbol{X}} = \mathbf{0}$.

Definition (2.3) ,[18]:

The domain of the linear operator A is defined as follows:

$D(A) = \{x \in X \; : \lim_{t\downarrow 0} \frac{T(t)x - x}{t}$ ***exists}*** and (3)

$\boldsymbol{Ax} = \left.\frac{\boldsymbol{dT(t)}}{\boldsymbol{dt}}\right|_{\boldsymbol{t=0}} = \boldsymbol{lim}_{\boldsymbol{t\downarrow 0}} \frac{\boldsymbol{T(t)x - x}}{\boldsymbol{t}}$ for $x \in D(A)$. A is the generator of the semigroup $T(t)$.

Remarks (2.4),[18]:

There exists a constant $w \geq 0$, such that

$\|T(t)\|_{L(X)} \leq M\, e^{wt}$, *for* $M \geq 1$.

The family of linear operator $t \longrightarrow T(t)$ is differentiable which is

$\frac{\boldsymbol{dT(t)}}{\boldsymbol{dt}} = \boldsymbol{AT}(\boldsymbol{t}) = \boldsymbol{T}(\boldsymbol{t})\boldsymbol{A}$

Lemma (2.5), [18]:

A bounded linear operator A is the generator of a uniformly continuous semigroup.

A strongly continuous semigroup of bounded linear operators on a Banach space X will be called a semigroup of class $C_{\circ}$.

Theorem (2.6) ,[3]:

A linear (unbounded) operator A is the generator of a strong semigroup of contraction family $\{T(t)\}_{t\geq 0}$ if and only if:

(i) A is closed and densely defined, and

(ii) The resolvent set $\rho(A)$ of A contains R^+ and $\|(\boldsymbol{\lambda I} - \boldsymbol{A})^{-\mathbf{1}}\| \leq \frac{\mathbf{1}}{\boldsymbol{\lambda}}$ for every $\lambda > 0$.

Remark (2.7), [18]:

1. If B is a bounded linear operator on ***X***, then ***A+B*** with ***D(A+B)=D(A)*** is the generator of C_0-semigroup **S(t)** on X. satisfying $\|\boldsymbol{S}(\boldsymbol{t})\| \leq \boldsymbol{M}\boldsymbol{e}^{(\boldsymbol{w}+\boldsymbol{M}\|\boldsymbol{B}\|)\boldsymbol{t}}$ for $t \geq 0$.

2. For $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s) x \, ds = T(s)x, h \in (0, t)}$.

Definitions (2.8), [22]:

1. Let $\boldsymbol{X}$ be a real Banach space and let $\boldsymbol{A{:}X \to X^*}$ be an operator . Then $\boldsymbol{A}$ is called monotone if $\boldsymbol{\langle Ax - Ay, x - y \rangle \geq 0}$ foe all x,y∈ X.
2. Assume operator $\boldsymbol{A{:}\ D(A) \subseteq H \to H}$ defined on real Hilbert space H.

a. $\boldsymbol{A}$ is called maximal monotone if $\boldsymbol{A}$ is monotone and $\boldsymbol{\langle b - Ay, x - y \rangle \geq 0}$ for $\boldsymbol{y} \in \boldsymbol{D(A)}$. Implies $\boldsymbol{Ax=b}$ which is **A** has no proper monotone extension.

b. $\boldsymbol{A}$ is accretive if $(I+\boldsymbol{\mu A})$: $\boldsymbol{D(A) \to H}$ is injective also $\boldsymbol{(I + \mu A)^{-1}}$is nonexpansive for $\mu > 0$.

c. $\boldsymbol{A}$ is maximal accretive if A is accretive also $\boldsymbol{(I + \mu A)^{-1}}$exists on H for $\boldsymbol{\mu} > 0$.

Definition (2.9), [8]:

The For a function $g : [0,\infty) \longrightarrow R$, the Caputo derivative of fractional order α is defined as

$${}^cD^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s) ds, \ \alpha > 0, \quad n-1 < \alpha < n,$$

where Γ denotes the gamma function

Definition (2.10), [10]:

The Riemann-Liouville fractional integral of order α for a function g is defined as

$I^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} ds, \ \alpha > 0,$ provided the right hand side is pointwise defined on $(0,\infty)$.

Lemma(2.11),[22]:

Let an operator $\boldsymbol{A}: \boldsymbol{D(A) \subseteq H \to H}$ on the real Hilbert space H . the statements are equivalent:

A is monotone and $\boldsymbol{R(I\text{-}A)=H}$ ⟺ $\boldsymbol{A}$ is maximal accretive ⟺ **A** is maximal monotone.

Lemma(2.12),[22]:

Let a linear operator $\boldsymbol{A}: \boldsymbol{D(A) \subseteq H \to H}$ on real Hilbert space $\boldsymbol{H}$

1. $\boldsymbol{A}$ is the generator of a linear nonexpansive semigroup.
2. $-A$ maximal accretive and $\overline{\mathbf{D(A)}} = \mathbf{X}$.

Lemma(2.13),[4]:

Let A be the generator of C_0- semigroup of contraction (nonexpansive semigroup) on a Banach space $\boldsymbol{X}$. A bounded linear operator B is a perturbation of A such that $\mathbf{D(A)} \subset \mathbf{D(B)}$ and

i. Let F denoted the duality on Y Banach space to Y* defined as
$\mathbf{F}(\boldsymbol{y})=\{\boldsymbol{g} \in \boldsymbol{Y}^*, \langle \boldsymbol{y}, \boldsymbol{g} \rangle = \|\boldsymbol{g}\|^2 = \|\boldsymbol{y}\|^2\}$ So for every $\boldsymbol{x} \in \boldsymbol{D}(\boldsymbol{\lambda I} - (\boldsymbol{A}+\boldsymbol{B}))$ there is $\mathbf{g} \in \boldsymbol{F}\left((\boldsymbol{\lambda I} - (\boldsymbol{A}+\boldsymbol{B}))\boldsymbol{x}\right)$, for every $\boldsymbol{y} \in \boldsymbol{Y}$, thus $\langle(-\|\boldsymbol{B}\|\boldsymbol{I})\boldsymbol{x}, \boldsymbol{g}\rangle \geq -\boldsymbol{c}\|\boldsymbol{x}\|^2$ $-\boldsymbol{a}\|\boldsymbol{\lambda I} - (\boldsymbol{A}+\boldsymbol{B}))\boldsymbol{x}\|\|\boldsymbol{x}\| - \boldsymbol{b}\|\boldsymbol{\lambda I} - (\boldsymbol{A}+\boldsymbol{B}))\boldsymbol{x}\|^2$

ii. $\boldsymbol{c}\left\|(\boldsymbol{\lambda I} - (\boldsymbol{A}+\boldsymbol{B}))^{-1}\right\| + \boldsymbol{a}\left\|(\boldsymbol{\lambda I} - (\boldsymbol{A}+\boldsymbol{B}))^{-1}\right\| + \boldsymbol{b} < 1$

iii. $\boldsymbol{\lambda} > \|\boldsymbol{B}\|$.Then $\boldsymbol{A+B}$ is the generator of C_0-semigroup of contraction (nonexpansive semigroup) in **X**.

Lemma(2.14),[22]:

Let the mapping **A.B:X → X*** be maximal monotone on the real reflexive Banach space **X**,(where **X*** is the dual space of **X**) and let $\mathbf{D(A)} \cap \mathbf{IntD(B)} \neq \emptyset$. Then the sum **A+B:X→ X*** is also maximal monotone.

Lemma(2.15),[9]:

Let f be a contraction on complete metric space **X**. Then f has a unique fixed point $\overline{\boldsymbol{x}} \in \boldsymbol{X}$.

Our problem investigated on the following space that which denoted by $\boldsymbol{L_2^\alpha}$,

$$L_2^\alpha[0,T] = \{x : x \in L_2[0,T],\ {}^C D^\alpha x \in L_2[0,T]\},\quad 0 < \alpha \le 1.$$

2.Main Results:

Lemma(3.1):

Let $A_i + B_iK_i : D(A_i) \subseteq L_2^\alpha \to L_2^\alpha, i = 1,2,3,\dots,n$ are linear unbounded operators generators of C_0-semigroups $S_i(t): L_2^\alpha \to L_2^\alpha, i = 1,2,\dots,n$, respectively $D(\sum_{i=1}^{n-1}(A_i + B_iK_i) \cap Int\ D(A_n + B_nK_n) = D(\sum_{i=1}^{n-1}(A_i) \cap Int\ D(A_n)) \neq \phi$, *for* $n \ge 2$. Then

$$\left\| \left(\left(\frac{1}{\lambda - \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}}\right) I - \sum_{i=1}^{n}(A_i + B_iK_i) \right)^{-1} \right\| \le \lambda - \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\ ,\text{forall } \lambda > \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}$$

(4)

Where $\frac{\sum_{i=1}^{n}\|B_iK_i\|}{n} = \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}$.

Proof:

From lemma(2.12) , we have that $-(A_i + B_iK_i)$ are a maximal monotone for $i=1,\dots n$. Since $D(\sum_{i=1}^{n-1}(A_i + B_iK_i) \cap Int\ D(A_n + B_nK_n) = D(\sum_{i=1}^{n-1}(A_i) \cap Int\ D(A_n)) \neq \phi$, then by lemma(2.15) *we have that* $-\sum_{i=1}^{n-1}(A_i + B_iK_i): D(\sum_{i=1}^{n-1}(A_i + B_iK_i)) \subseteq H \to H$; *a* maximal monotone, then by lemma(2.13) and definition(2.9), we get

$$\left\| \left(I - \left(\lambda - \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right) \sum_{i=1}^{n-1}(A_i + B_iK_i)\right)^{-1} \right\| \le 1 \ \text{ for } \lambda > \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}$$

$$\left(\lambda - \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^{-1} \left\| \left(\left(\lambda - \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^{-1} I - \sum_{i=1}^{n-1}(A_i + B_iK_i)\right)^{-1} \right\| \le 1 \ \text{ for } \lambda > \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}$$

Hence, $\left\| \left(\left(\lambda - \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^{-1} I - \sum_{i=1}^{n-1}(A_i + B_iK_i)\right)^{-1} \right\| \le \lambda - \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}$ *for* $\lambda > \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}$

Lemma (3.2):

Let $A_i + B_iK_i : D(A_i) \subseteq L_2^\alpha \to L_2^\alpha, i = 1,2,3,\dots,n$ are linear unbounded operators generators of C_0-semigroups $S_i(t): L_2^\alpha \to L_2^\alpha, i = 1,2,\dots,n$, respectively and $B_i : D(B_i) \subseteq L_2^\alpha \to L_2^\alpha$ satisfies the following condition for every $x \in D(\lambda I - (A_i + B_iK_i))$ there is $g \in F(\lambda I - (A_i + B_iK_i)x)$ such that

$$\langle(-\|B_iK_i\|I)x, g\rangle \ge -c\|x\|^2 - a\|(\lambda I - (A_i + B_iK_i))x\| - b\|(\lambda I - (A_i + B_iK_i))x\|^2$$

, for $\lambda > \|B_iK_i\|$ such that $D(\sum_{i=1}^{n} A_i) \subset D(\sum_{i=1}^{n} B_iK_i)$

$D(\sum_{i=1}^{n-1}(A_i + B_iK_i) \cap Int\ D(A_n + B_nK_n) = D(\sum_{i=1}^{n-1}(A_i) \cap Int\ D(A_n)) \neq \phi$ *, for* $n \ge 2$. *Then*

$$\left\| \left(\left(\lambda - \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right) - \sum_{i=1}^{n}(A_i + B_iK_i)\right)^{-1} \right\| \le \left(\lambda - \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^{-1} ,\text{for all } \lambda > \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}$$

(5)

Where $\frac{\sum_{i=1}^{n}\|B_iK_i\|}{n} = \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}$.

Proof:

Since$(A_i + B_iK_i): D(A_i + B_iK_i) = D(A_i) \subseteq L_2^\alpha \to L_2^\alpha, i = 1,2,\dots,n$, are generators of perturbed C_0-semigroups, then from remark(2.8), we have that

$$\|((\lambda - (A_i + B_iK_i))^{-1}\| \le (\lambda - \|B_iK_i\|)^{-1},\ i = 1,2,\dots,n \ \text{ for } \lambda > \|B_iK_i\|$$

(6)

By using Lemma (2.14), we get

$$\|(\lambda - \|B_iK_i\|)I - (A_i + B_iK_i))^{-1}\| \le \frac{1}{\lambda - \|B_iK_i\|},\ i = 1,2,\dots,n \ \text{ for } \lambda > \|B_iK_i\|. \tag{7}$$

Thus, the operators $(A_i + B_i K_i)$ are generators of nonexpansive semigroup. Then from theorem (2.12) and lemma (2.13) we have the operators $-(A_i + B_i K_i)$ are maximal monotone for $i = 1,2,...,n$, hence

$$\left\| \left(\left(I - \frac{1}{\lambda - \|B_i K_i\|} (A_i + B_i K_i) \right) \right)^{-1} \right\| \leq 1 \text{ , } i = 1,2,...,n \text{ for } \lambda > \|B_i K_i\|$$
(8)

Since $D(\sum_{i=1}^{n-1}(A_i + B_i K_i) \cap Int\, D(A_n + B_n K_n) = D(\sum_{i=1}^{n-1}(A_i) \cap Int\, D(A_n) \neq \phi$

and for $\lambda > \|B_i K_i\|$, *we get* $n\lambda > \sum_{i=1}^{n}\|B_i K_i\| \Longrightarrow \frac{\sum_{i=1}^{n}\|B_i K_i\|}{n} = \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}$

Then by lemma (2.15) we have that

$$-\sum_{i=1}^{n}(A_i + B_i K_i) : D(\sum_{i=1}^{n}(A_i + B_i K_i)) \subseteq H \to H$$
(9)

is also a maximal monotone, then by lemma (3.1), we have that

$$\left\| \left(I - \left(\lambda - \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^{-1} \right) \sum_{i=1}^{n}(A_i + B_i K_i) \right)^{-1} \right\| \leq 1 \text{ , for } \lambda > \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|},$$

Thus,

$$(\lambda - \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}) \|((\lambda - \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}) I - \sum_{i=1}^{n}(A_i + B_i K_i)))^{-1}\| \leq 1 \text{ for} \lambda > \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}.$$

Hence,

$$\left\| ((\lambda - \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}) I - \sum_{i=1}^{n}(A_i + B_i K_i))^{-1} \right\| \leq \left(\lambda - \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^{-1} , \quad \text{for } \lambda > \quad \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}$$

Lemma (3.3):

Let $F_i: R_0^+ \times L_2 \times L_2 \to L_2^\alpha$, $\quad i = 1,2,...,n$, are nonlinear operators satisfy the following

1. $\langle F_i(t, x, D_a^\propto x) - F_i(t, y, D_a^\propto y), x - y \rangle \geq m_i(\|x - y\| + \|D_a^\propto x - D_a^\propto y\|)^2$
   $\geq m\|x - y\|_{L_2^\alpha}{}^2$, $m = min\{m_i, i = 1, ..., n\}$
2. $\langle D_a^\propto F_i(t, x, D_a^\propto x) - D_a^\propto F_i(t, y, D_a^\propto y), x - y \rangle \geq m_i^*(\|x - y\| + \|D_a^\propto x - D_a^\propto y\|)^2 \geq m^*\|x - y\|_{L_2^\alpha}^2$
   for all $x, y \in H$ ***and some*** $m_i^* > 0$; $m^* = min\{m_i, i = 1, ..., n\}$,
3. $\|F_i(t, x, D_a^\propto x) - F_i(t, x, D_a^\propto x)\|_{L_2^\alpha} \leq K_i(\|x - y\| + \|D_a^\propto x - D_a^\propto y\|) \leq K$,
   $(\|x - y\| + \|D_a^\propto(x - y)\|) \leq K\|x - y\|_{L_2^\alpha}$ ***for all*** $x, y \in H$, and some $K_i > 0$.

   $K = min\{K_i, i = 1, ..., n\}$
4. $\|D_a^\propto F_i(t, x, D_a^\propto x) - D_a^\propto F_i(t, x, D_a^\propto x)\|_{L_2^\alpha} \leq K_i^*(\|x - y\| + \|D_a^\propto x - D_a^\propto y\|) \leq K^*\|x - y\|_{L_2^\alpha}$,
   $K^* = min\{K_i^*, i = 1, ..., n\}$

Then there exists interval of λ such that $\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|} < \lambda < min\{ \frac{2m}{nK^2} + \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|} , \frac{2m^*}{nK^{*2}} + \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|} \}$ for some $m^*, k^* > 0$, $n \in N$ such that $S_\lambda: L_2^\alpha \longrightarrow L_2^\alpha$.

$$S_\lambda(x) = x - (\lambda - \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}) \sum_{i=1}^{n} F_i(t_1, x, D_a^\propto x))$$
$$D_a^\propto S_\lambda(x) = x - (\lambda - \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}) \sum_{i=1}^{n} D_a^\propto F_i(t_1, x, D_a^\propto x))$$

$S_\lambda(x)$ is a contraction operator in $L_2^\alpha -$ space.

Proof:

We have $\|S_\lambda(x)-S_\lambda(y)\|^2 = \langle x-\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\sum_{i=1}^{n}F_i(t,x,D_a^\propto x)-(y-\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\sum_{i=1}^{n}F_i(t,y,D_a^\propto y)), x-\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\sum_{i=1}^{n}F_i(t,x,D_a^\propto x)-(y-\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\sum_{i=1}^{n}F_i(t,y,D_a^\propto y))\rangle$

$= \langle x-\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\sum_{i=1}^{n}F_i(t,x,D_a^\propto x)-y+\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\sum_{i=1}^{n}F_i(t,y,D_a^\propto y)), x-(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|})\sum_{i=1}^{n}F_i(t,x,D_a^\propto x)-y+\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\sum_{i=1}^{n}F_i(t,y,D_a^\propto y))\rangle=$ $\langle x-y-\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\sum_{i=1}^{n}F_i(t,x,D_a^\propto x)+\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\sum_{i=1}^{n}F_i(t,y,D_a^\propto y)), x-y-(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|})\sum_{i=1}^{n}F_i(t,x,D_a^\propto x)+\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\sum_{i=1}^{n}F_i(t,y,D_a^\propto y))\rangle$ $\langle x-y,x-y\rangle-\langle x-y,\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\sum_{i=1}^{n}F_i(t,x,D_a^\propto x)-\overline{(\lambda-\sum_{\iota=1}^{n}\|B_\iota K_\iota\|)}\sum_{i=1}^{n}F_i(t,y,D_a^\propto y)\rangle-\langle\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\sum_{i=1}^{n}F_i(t,x,D_a^\propto x)-(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|})\sum_{i=1}^{n}F_i(t,y,D_a^\propto y), x-y\rangle+\langle\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\sum_{i=1}^{n}F_i(t,x,D_a^\propto x)-(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|})\sum_{i=1}^{n}F_i(t,y,D_a^\propto y), \overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|})\sum_{i=1}^{n}F_i(t,x,D_a^\propto x)-(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|})\sum_{i=1}^{n}F_i(t,y,D_a^\propto y)\rangle$

Thus,

$\|S_\lambda(x)-S_\lambda(y)\|^2=\|x-y\|^2-2\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\langle\sum_{i=1}^{n}F_i(t,x,D_a^\propto x)-\sum_{i=1}^{n}F_i(t,y,D_a^\propto y), x-y\rangle+(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|})\|\sum_{i=1}^{n}F_i(t,x,D_a^\propto x)-\sum_{i=1}^{n}F_i(t,y,D_a^\propto y)\|^2$

Also

$\|D_a^\propto S_\lambda(x)-D_a^\propto S_\lambda(y)\|^2=\|D_a^\propto(x-y)\|^2-2\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\langle\sum_{i=1}^{n}D_a^\propto F_i(t,x,D_a^\propto x)-\sum_{i=1}^{n}D_a^\propto F_i(t,y,D_a^\propto y), D_a^\propto(x-y)\rangle+(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|})\|\sum_{i=1}^{n}D_a^\propto F_i(t,x,D_a^\propto x)-\sum_{i=1}^{n}D_a^\propto F_i(t,y,D_a^\propto y)\|^2$

From conditions (1-4), we obtain

$$\|S_\lambda(x)-S_\lambda(y)\|_{L_2^\alpha}\le\left(1-2\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)nm)+\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^2(nK)^2\right)^{1/2}+\left(1-2\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)nm^*)+\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^2(nK^*)^2\right)^{1/2}(\|x-y\|+\|D_a^\propto(x-y)\|)=\|x-y\|_{L_2^\alpha}$$

(10)

We claim that

$$\left(1-2\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)nm)+\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^2(nK)^2\right)^{1/2}+\left(1-2\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)nm^*)+\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^2(nK^*)^2\right)^{1/2}<1$$

So,

$$0<2\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)nm)+\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^2(nK)^2<1$$

and

$$0<2\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)nm^*)+\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^2(nK^*)^2<1$$

$$\Rightarrow\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^2(nK)^2<2\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)nm)$$

$$\Rightarrow\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^2(nK^*)^2<2\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)nm^*$$

(11)

Then by (11), we have that $<min\{\frac{2m}{nK^2}+\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}, \frac{2m^*}{nK^{*2}}+\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\}$ and then the interval of λ is

$$\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}<\lambda<min\{\frac{2m}{nK^2}+\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}, \frac{2m^*}{nK^{*2}}+\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\}$$

. *To be*, $\|S_\lambda(x)-S_\lambda(y)\|_{L_2^\alpha}\le b(\|x-y\|+\|D_a^\propto x-D_a^\propto y\|)=b\,\|x-y\|_{L_2^\alpha}$.

Where

$b=\left(1-2\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)nm\right)+\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^2(nK)^2\Big)^{1/2}+$

$\left(1-2\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)nm^*\right)+\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^2(nK^*)^2\Big)^{1/2}<1$

Hence $S_\lambda(x)$ is contraction operator in $\mathbf{L_2^\alpha}$ space.

Consider the following semilinear of sum of N-perturbed unbounded operators equations discussed in the following equations.

Theorem (3.4):

Let $A_i+B_iK_i: D(A_i)\subseteq L_2^\alpha\to L_2^\alpha, i=1,2,\dots,n$, are linear unbounded operators generators of C_0-semigroups $S_i(t): L_2^\alpha\to L_2^\alpha\, i=1,2,\dots,n$, respectively and $\mathbf{B_iD(B_i)}\subseteq \mathbf{L_2^\alpha}\to\mathbf{L_2^\alpha}$satisfies the following condition for every $\mathbf{x}\in\mathbf{D(\lambda I-(A_i+B_iK_i))}$ there is $g\in F(\lambda I-(A_i+B_iK_i)x)$ such that

$$\langle(-\|B_iK_i\|I)x,g\rangle\geq -c\|x\|^2-a\|(\lambda I-(A_i+B_iK_i))x\|-b\|(\lambda I-(A_i+B_iK_i))x\|^2$$

,for $\lambda>\|B_i\|$, such that

, for $\boldsymbol{\lambda}>\|\mathbf{B_iK_i}\|$ such that $\mathbf{D}(\sum_{i=1}^{n}A_i)\subset\mathbf{D}(\sum_{i=1}^{n}B_iK_i)$

$\mathbf{D}(\sum_{i=1}^{n-1}(A_i+B_iK_i)\cap \mathbf{Int\ D}(A_n+B_nK_n)=\mathbf{D}(\sum_{i=1}^{n-1}(A_i)\cap\mathbf{Int\ D}(A_n)\neq\phi$ **, for n** ≥ 2,

and Let $F_i: R_0^+\times L_2\times L_2\to L_2^\alpha$, $i=1,2,\dots,n$, are nonlinear operators satisfy the following

1. $\langle F_i(t,x,D_a^\propto x)-F_i(t,y,D_a^\propto y),x-y\rangle\geq m_i(\|x-y\|+\|D_a^\propto x-D_a^\propto y\|)^2$
   $\geq m\|x-y\|_{L_2^\alpha}{}^2$, $m=\min\{m_i, i=1,\dots,n\}$
2. $\langle D_a^\propto F_i(t,x,D_a^\propto x)-D_a^\propto F_i(t,y,D_a^\propto y),x-y\rangle\geq m_i^*(\|x-y\|+\|D_a^\propto x-D_a^\propto y\|)^2$
   $\geq m^*\|x-y\|_{L_2^\alpha}{}^2$

   for all $x,y\in H$ **and some** $m_i^*>0$; $m^*=\min\{m_i, i=1,\dots,n\}$,
3. $\|F_i(t,x,D_a^\propto x)-F_i(t,x,D_a^\propto x)\|_{L_2^\alpha}\leq K_i(\|x-y\|+\|D_a^\propto x-D_a^\propto y\|)$
   $\leq K(\|x-y\|+\|D_a^\propto(x-y)\|)\leq K\|x-y\|_{L_2^\alpha}$

   for all $x,y\in L_2^\alpha$ $x,y\in H$ **, and some** $K_i>0$. $K=\min\{K_i, i=1,\dots,n\}$
4. $\|D_a^\propto F_i(t,x,D_a^\propto x)-D_a^\propto F_i(t,x,D_a^\propto x)\|_{L_2^\alpha}\leq K_i^*(\|x-y\|+\|D_a^\propto x-D_a^\propto y\|)\leq K^*\|x-y\|_{L_2^\alpha}$
   , $K^*=\min\{K_i^*, i=1,\dots,n\}$ **for all x,y** ϵH.

Then the following equation

$$\sum_{i=1}^{n}F_i(t,x,D_a^\propto x)=\sum_{i=1}^{n}A_ix+\sum_{i=1}^{n}B_iu_i$$

$$u_i=K_i\,x, \text{ for all } x\in\bigcap_{i=1}^{n}D(A_i) \qquad (12)$$

has an unique solution.

Proof:

The Equation (12) can be written as

$(\mathbf{I}-\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\sum_{i=1}^{n}(A_i+B_iK_i))x-(x-(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|})\sum_{i=1}^{n}F_i(t,x,D_a^\propto x))=\mathbf{0}$,for $\lambda>\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}$ and $\mathbf{x}\in\mathbf{H}$. Or

$-\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)\sum_{i=1}^{n}(A_i+B_iK_i))x=S_\lambda(x)$,for $\lambda>\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}$ and $\mathbf{x}\in\mathbf{H}$. (13)

Where $S_\lambda(x)=x-(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|})\sum_{i=1}^{n}F_i(t,x,D_a^\propto x)$.

$\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)(I(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|})^{-1}-\sum_{i=1}^{n}(A_i+B_iK_i))x=S_\lambda(x)$,

for $\lambda>\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}$ and $\mathbf{x}\in\mathbf{H}$.

$(I(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|})^{-1}-\sum_{i=1}^{n}(A_i+B_iK_i))x=\left(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|}\right)^{-1}S_\lambda(x)$ for

$(\lambda-\overline{\sum_{\iota=1}^{n}\|B_\iota K_\iota\|})^{-1}>0$

From lemma (2.15), we have

$$\mathbf{x} = (\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}((\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}\mathbf{I} - \textstyle\sum_{i=1}^{n}(A_i + B_iK_i))^{-1}\,S_\lambda(\mathbf{x})$$
(14)

To show that, $(\lambda - \overline{\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}((\lambda - \overline{\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}\mathbf{I} - \sum_{i=1}^{n}(A_i + B_iK_i))^{-1}\,S_\lambda(\mathbf{x})$ is a contraction operator

$$(\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}((\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}\mathbf{I} - \textstyle\sum_{i=1}^{n}(A_i + B_iK_i))^{-1}\,S_\lambda(\mathbf{x}) -$$
$$(\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}((\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}\mathbf{I} - \textstyle\sum_{i=1}^{n}(A_i + B_iK_i))^{-1}\,S_\lambda(\mathbf{y})$$
$$= (\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}((\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}\mathbf{I} - \textstyle\sum_{i=1}^{n}(A_i + B_iK_i))^{-1}\,(S_\lambda(\mathbf{x}) - S_\lambda(\mathbf{y}))$$
$$\leq (\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}\left\|((\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}\mathbf{I} - \textstyle\sum_{i=1}^{n}(A_i + B_iK_i))^{-1}\right\|\,\|(S_\lambda(\mathbf{x}) - S_\lambda(\mathbf{y}))\|$$

By lemmas(3.1) and equation(3.2), we get

$$(\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}\Big\|((\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}\mathbf{I} - \textstyle\sum_{i=1}^{n}(A_i + B_iK_i))^{-1}\,S_\lambda(\mathbf{x}) -$$
$$((\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}\mathbf{I} - \textstyle\sum_{i=1}^{n}(A_i + B_iK_i))^{-1}\,S_\lambda(\mathbf{y})\Big\| \leq$$
$$\big((\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}\big)(\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|})\mathbf{b}\|\mathbf{x} - \mathbf{y}\|$$

Hence, $(\lambda - \overline{\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}\Big\|((\lambda - \overline{\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}\mathbf{I} - \sum_{i=1}^{n}(A_i + B_iK_i))^{-1}\,S_\lambda(\mathbf{x}) - ((\lambda - \overline{\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}\mathbf{I} - \sum_{i=1}^{n}(A_i + B_iK_i))^{-1}\,S_\lambda(\mathbf{y})\Big\| \leq \mathbf{b}\|\mathbf{x} - \mathbf{y}\|$

(15)

From lemma (3.2), we have

$$\boldsymbol{b} = \Big(1 - 2\big(\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|}\big)(nm^*) + \big(\lambda - \overline{\textstyle\sum_{i=1}^{n}\|B_iK_i\|}\big)^2(nK^*)^2\Big)^{-1} < 1$$

for $\overline{\sum_{i=1}^{n}\|B_iK_i\|} < \lambda < \frac{2m^*}{nK^{*2}} + \overline{\sum_{i=1}^{n}\|B_iK_i\|}$ by theorem (1.7.8), we have that $((\lambda - \overline{\sum_{i=1}^{n}\|B_iK_i\|}\,)^{-1}\mathbf{I} - \sum_{i=1}^{n}(A_i + B_iK_i))^{-1}\,S_\lambda(\mathbf{x})$ has an unique fixed point, thus (14) and consequently (12) has an unique solution.

Definition (3.5):

Let X be a real separable Banach space a one-parameter family $\{S_n(t)S_{n-1}(t)..S_1(t)\} \subset L(X)$, $t \in [0,\infty)$ of a perturbed C_0-semigroups of bounded linear operators $\{S_i(t)\}_{i=1}^{n} \subset L(X)$ are commutative and generated by$(A_i + B_iK_i)$ for $i = 1,...,n$ respectively and $t \in [0,\infty)$ is called commutative composite perturbed semigroup if

1. $S_n(0)S_{n-1}(0)..S_1(0) = I$ **, (I is the identity operator on X).**
2. $S_n(t+s)S_{n-1}(t+s)..S_1(t+s) = \big(S_n(t)S_{n-1}(t)..S_1(t)\big)\big(S_n(s)S_{n-1}(s)..S_1(s)\big)$
   for every $t,s \geq 0$.

Definition (3.6):

The generator $\sum_{i=1}^{n}(A_i + B_iK_i)$)of a semigroup of commutative composite perturbed semigroups $\{S_n(t)S_{n-1}(t)..S_1(t)\}_{t\geq 0}$, on a real separable Banach space X, defined as the Limit

$$\textstyle\sum_{i=1}^{n}(A_i + B_iK_i)) = \lim_{t\downarrow 0}\frac{S_n(t)S_2(t)\ldots S_1(t)x - Ix}{t} \quad , x \in D(\sum_{i=1}^{n}(A_i + B_iK_i)) = D(A_1 + B_1K_1)\cap D(A_2 + B_2K_2)\ldots\cap D(A_n + B_nK_n) = D(A_1)\cap D(A_2)\ldots\cap D(A_n)$$

where $\mathbf{D}(\sum_{i=1}^{n}(A_i + B_iK_i)) \subset \mathbf{X}$ is a domain of $\sum_{i=1}^{n}(A_i + B_iK_i)$ has a countable subset which is dense in X and defined as follows

$$D(\textstyle\sum_{i=1}^{n}(A_i + B_iK_i)) = \{x \in X : \lim_{t\downarrow 0}\frac{S_n(t)S_2(t)\ldots S_1(t)x - Ix}{t}\ \textit{exist in}\ X\}$$

Lemma (3.7):

Let **H** be a real separable Hilbert space, and

$\sum_{i=1}^{n}(A_i + B_iK_i): \cap_{i=1}^{n} D(A_i) \subseteq H \to H$
be a generator of a semigroup of a commutative composite perturbed semigroups.Then
$\left\| (\lambda I - \sum_{i=1}^{n}(A_i + B_iK_i))^{-1} x \right\| \le \frac{1}{\lambda - \sum_{i=1}^{n}\|B_iK_i\|} \|x\|$ **and** $x \in \cap_{i=1}^{n} D(A_i)$ **for** $i = 1,...,n$.
(16)

Proof:
$F_p(\lambda)x = L(S_n(t)S_{n-1}(t) \dots, S_1(t)x) = \int_0^\infty e^{-\lambda t} S_n(t)S_{n-1}(t) \dots, S_1(t)x\,dt$, **,for** $\lambda > \sum_{i=1}^{n}\|B_iK_i\|$ and $x \in X$.
(17)

Since $t \longrightarrow S_i(t)x$ are continuous for $i = 1,2,...,n$ the integral exists and defines a bounded linear operator $F_p(\lambda)$ satisfying

$$\|F_p(\lambda)x\| \le \int_0^\infty e^{-\lambda t} \|S_n(t)S_{n-1}(t)...S_1(t)x\| dt$$

$$\|F_p(\lambda)x\| \le \int_0^\infty e^{-\lambda t} \|S_n(t)\|\|S_{n-1}(t)\|...\|S_1(t)\|\|x\| dt$$

But $\|S_i(t)\| \le e^{\|B_iK_i\|t}$**, for** $i = 1,2,...,n$ **, then**

$$\|F_p(\lambda)x\| \le \int_0^\infty e^{-\lambda t} e^{e^{\|B_nK_n\|t}} e^{e^{\|B_{n-1}K_{n-1}\|t}} \dots e^{e^{\|B_1K_1\|t}} \|x\| dt$$

$$\|F_p(\lambda)x\| \le \int_0^\infty e^{-(\lambda - \sum_{i=1}^{n}\|B_iK_i\|)t} \|x\| dt$$

$$\|F_p(\lambda)x\| \le \frac{1}{\lambda - \sum_{i=1}^{n}\|B_iK_i\|} \|x\|$$

(18)
Furthermore, for $h > 0$

$$\frac{S_n(h)S_{n-1}(h)..S_1(h) - I}{h} F(\lambda)x = \frac{S_n(h)S_{n-1}(h)..S_1(h) - I}{h} \int_0^\infty e^{-\lambda t} S_n(t)S_{n-1}(t)..S_1(t)x\,dt$$

$$= \frac{1}{h}\int_0^\infty e^{-\lambda t} \left((S_n(h)S_{n-1}(h)..S_1(h))(S_n(t)S_{n-1}(t)..S_1(t))x - S_n(t)S_{n-1}(t)..S_1(t)x \right) dt$$

Since $S_i(t)S_j(t)$ are commutative then

$$= \frac{1}{h}\int_0^\infty e^{-\lambda t} \left(S_n(t+h)S_{n-1}(t+h)..S_1(t+h)x - S_n(t)S_{n-1}(t)..S_1(t)x \right) dt$$

$$= \frac{1}{h}\int_0^\infty e^{-\lambda t} S_n(t+h)S_{n-1}(t+h)..S_1(t+h)x\,dt - \frac{1}{h}\int_0^\infty e^{-\lambda t} S_n(t)S_{n-1}(t)..S_1(t)x\,dt$$

Let $\delta = t + h \Rightarrow d\delta = dt$, if $0 \le t \le \infty$ then $h \le \delta \le \infty$, we get

$$= \frac{1}{h}\int_h^\infty e^{-\lambda(\delta - h)} S_n(\delta)S_{n-1}(\delta)..S_1(\delta)x\,d\delta - \frac{1}{h}\int_0^\infty e^{-\lambda t} S_n(t)S_{n-1}(t)..S_1(t)x\,dt$$

$$= \frac{e^{\lambda h}}{h}\int_0^\infty e^{-\lambda\delta} S_n(\delta)S_{n-1}(\delta)..S_1(\delta)x\,d\delta - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda\delta} S_n(\delta)S_{n-1}(\delta)..S_1(\delta)x\,d\delta$$

$$- \frac{1}{h}\int_0^\infty e^{-\lambda t} S_n(t)S_{n-1}(t)..S_1(t)x\,dt$$

We get

$$= \frac{e^{\lambda h} - 1}{h}\int_0^\infty e^{-\lambda t} S_n(t)S_{n-1}(t)..S_1(t)x\,dt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t} S_n(t)S_{n-1}(t)..S_1(t)x\,dt \cdot$$

(19)
As $h \downarrow 0$, from (17) and remarks (2.4) the right-hand side of (19) converges to $\lambda F_p(\lambda)x - x$.

This implies that for every $x \in H$ and $\lambda > 0$, $F_p(\lambda)x \in D(\sum_{i=1}^{n}(A_i + B_iK_i))$ and $\sum_{i=1}^{n}(\mathbf{A_i} + \mathbf{B_iK_i})\mathbf{F_p}(\boldsymbol{\lambda}) = \boldsymbol{\lambda}\,\mathbf{F_p}(\boldsymbol{\lambda}) - \boldsymbol{I}$

or

$(\boldsymbol{\lambda I} - \sum_{i=1}^{n}(A_i + B_iK_i))\, F_p(\lambda) = \mathbf{I}$ (20)

For $x \in D(\sum_{i=1}^{n}(A_i + B_iK_i))$ we have
$F_p(\lambda) \sum_{i=1}^{n}(A_i + B_iK_i)\, x = \int_0^{\infty} e^{-\lambda t} S_n(t)S_{n-1}(t) \dots, S_1(t) \sum_{i=1}^{n}(A_i + B_iK_i)\, xdt$,
(21)
From remarks (2.4), the Equation (21) become

$$F_p(\lambda) \sum_{i=1}^{n}(A_i + B_iK_i)\, x = \int_0^{\infty} e^{-\lambda t} \sum_{i=1}^{n}(A_i + B_iK_i)\, S_n(t)S_{n-1}(t) \dots, S_1(t) \sum_{i=1}^{n}(A_i + B_iK_i)xdt$$

$$= \sum_{i=1}^{n}(A_i + B_iK_i) \int_0^{\infty} e^{-\lambda t} S_n(t)S_{n-1}(t) \dots, S_1(t)xdt$$

$$= \sum_{i=1}^{n}(A_i + B_iK_i)F_p(\lambda)x$$
(22)
From (20) and (22) it follows that
$F_p(\lambda)\, (\lambda I - \sum_{i=1}^{n}(A_i + B_iK_i)x \to x$ **for** $x \in D(\sum_{i=1}^{n}(A_i + B_iK_i))$
Thus, $F_p(\lambda)$ is the inverse of $\lambda I - \sum_{i=1}^{n}(A_i + B_iK_i)$, it exists for all $\lambda > \sum_{i=1}^{n}\|B_iK_i\|$.

Theorem (3.8):

Let H be a real separable Hilbert space, $\sum_{i=1}^{n}(A_i + B_iK_i)$ be a generator of commutative composite perturbed semigroup $\{S_n(t)S_{n-1}(t)..S_1(t)\}_{t\geq 0}$ and $F_i : H \to H$, $i = 1,2,...,n$ are nonlinear operators and there exist $m_i, k_i > 0, i = 1,2,...,n$ such that

1. $\langle F_i(t_1, x, D_a^{\propto}x) - F_i(t_2, y, D_a^{\propto}y), x - y\rangle \geq m_i \|x - y\|$ **for all** $x, y \in H$ **and some** $m_i > 0$;
2. $\|F_i(t_1, x, D_a^{\propto}x) - F_i(t_2, x, D_a^{\propto}x)\| \leq K_i(\|t_1 - t_2\| + \|x - y\| + \|D_a^{\propto}x - D_a^{\propto}y\|)$
$\leq K_i(\|t_1 - t_2\| + \|x - y\| + \|D_a^{\propto}(x - y)\|) \leq K_i(\|t_1 - t_2\| + \|x - y\| + \|D_a^{\propto}(x - y)\|)$

for all $x, y \in H$. hence the equation

$\sum_{i=1}^{n} F_i\,(t, x, D_a^{\propto}x) = \sum_{i=1}^{n} A_ix + \sum_{i=1}^{n} B_iu_i, u_i = K_i\, x$, for all $x \in \bigcap_{i=1}^{n} D(A_i)$ (23)
has an unique solution.

Proof:

The Equation (23) can be equivalently written as
$(\lambda I - \sum_{i=1}^{n}(A_i + B_iK_i))x - (\lambda I - \sum_{i=1}^{n} F_i\,(t, x, D_a^{\propto}x)) = 0$,
Or $\lambda I - \sum_{i=1}^{n}(A_i + B_iK_i)x = T_\lambda(x)$
(24)
Where $T_\lambda(x) = \lambda x - \sum_{i=1}^{n} F_i\,(t, x, D_a^{\propto}x)$, we have

$$\|T_\lambda(x) - T_\lambda(y)\|^2 = \langle \lambda x - \sum_{i=1}^{n} F_i\,(t, x, D_a^{\propto}x) - \left(\lambda y - \sum_{i=1}^{n} F_i\,(t, y, D_a^{\propto}y)\right), \lambda x - \sum_{i=1}^{n} F_i\,(t, x, D_a^{\propto}x) - (\lambda y - \sum_{i=1}^{n} F_i\,(t, y, D_a^{\propto}y))\rangle$$

$$= \langle \lambda x - \sum_{i=1}^{n} F_i\,(t, x, D_a^{\propto}x) - \lambda y + \sum_{i=1}^{n} F_i\,(t, y, D_a^{\propto}y))\, , \lambda x - \sum_{i=1}^{n} F_i\,(t, x, D_a^{\propto}x) - \lambda y + \sum_{i=1}^{n} F_i\,(t, y, D_a^{\propto}y))\rangle = \langle \lambda x - \lambda y - \sum_{i=1}^{n} F_i\,(t, x, D_a^{\propto}x) + \sum_{i=1}^{n} F_i\,(t, y, D_a^{\propto}y), \lambda x - \lambda y - \sum_{i=1}^{n} F_i\,(t, x, D_a^{\propto}x)) + \sum_{i=1}^{n} F_i\,(t, y, D_a^{\propto}y))\rangle$$

$$= \langle \lambda x - \lambda y, \lambda x - \lambda y\rangle - \langle \lambda x - \lambda y - \sum_{i=1}^{n} F_i\,(t, x, D_a^{\propto}x) - \sum_{i=1}^{n} F_i\,(t, y, D_a^{\propto}y)\rangle - \langle \sum_{i=1}^{n} F_i\,(t, x, D_a^{\propto}x) - \sum_{i=1}^{n} F_i\,(t, y, D_a^{\propto}y), \lambda x - \lambda y\rangle + \langle \sum_{i=1}^{n} F_i\,(t, x, D_a^{\propto}x) - \sum_{i=1}^{n} F_i\,(t, y, D_a^{\propto}y), \sum_{i=1}^{n} F_i\,(t, x, D_a^{\propto}x) - \sum_{i=1}^{n} F_i\,(t, y, D_a^{\propto}y)\rangle$$

$\|T_\lambda(x) - T_\lambda(y)\|^2 = \lambda^2\|x-y\|^2 - 2\lambda\langle \sum_{i=1}^n F_i(t,x,D_a^\propto x), \sum_{i=1}^n F_i(t,y,D_a^\propto y), x-y\rangle + \|\sum_{i=1}^n F_i(t,x,D_a^\propto x) - \sum_{i=1}^n F_i(t,y,D_a^\propto y)\|^2$

Also, $\|D_a^\propto T_\lambda(x) - D_a^\propto T_\lambda(y)\|^2 = \lambda^2\|D_a^\propto x - D_a^\propto y\|^2 - 2\lambda\langle \sum_{i=1}^n D_a^\propto F_i(t,x,D_a^\propto x), \sum_{i=1}^n D_a^\propto F_i(t,y,D_a^\propto y), D_a^\propto x - D_a^\propto y\rangle + \|\sum_{i=1}^n D_a^\propto F_i(t,x,D_a^\propto x) - \sum_{i=1}^n D_a^\propto F_i(t,y,D_a^\propto y)\|^2$

From conditions(1)(2), we obtain

$\|T_\lambda(x) - T_\lambda(y)\|_{L_2^\alpha} \le \left(\lambda^2 - 2\lambda(nm^*) + (nK^*)^2\right)^{1/2} + \left((\lambda^2 - 2\lambda(nm^*) + (nK^*)^2)\right)^{1/2}\left(\|x-y\| + \|D_a^\propto x - D_a^\propto y\| = \|x-y\|_{L_2^\alpha}\right)$

(25)

From lemma (3.4.36) the operator $\sum_{i=1}^n(A_i + B_iK_i))$ is generator of a family of linear commutative composite perturbed semigroup.

Then the operator $\lambda I - \sum_{i=1}^n(A_i + B_iK_i)$ is invertible and

$$\left\|(\lambda I - \sum_{i=1}^n(A_i + B_iK_i))^{-1}\right\| \le (\lambda - \sum_{i=1}^n\|B_iK_i\|\ \ )^{-1}, \text{for } \lambda > \sum_{i=1}^n\|B_iK_i\| \tag{26}$$

Now, Equation (24) is equivalent with

$$x = (\lambda I - \sum_{i=1}^n(A_i + B_iK_i))^{-1}\, T_\lambda(x) \tag{27}$$

To show that $x = (\lambda I - \sum_{i=1}^n(A_i + B_iK_i))^{-1}\, T_\lambda(x)$ is a contraction operator

$$\left\|(\lambda I - \sum_{i=1}^n(A_i + B_iK_i))^{-1}\, T_\lambda(x) - (\lambda I - \sum_{i=1}^n(A_i + B_iK_i))^{-1}\, T_\lambda(y)\right\|$$

$$= \left\|(\lambda I - \sum_{i=1}^n(A_i + B_iK_i))^{-1}\,(T_\lambda(x) - (T_\lambda(y))\right\|$$

$$\le \left\|(\lambda I - \sum_{i=1}^n(A_i + B_iK_i))^{-1}\right\|\|T_\lambda(x) - T_\lambda(y)\|_{L_2^\alpha}$$

By (24) and (25), we get

$\le (\lambda - \overline{\sum_{\iota=1}^n\|B_\iota K_\iota\|}\ )^{-1}\left(\lambda^2 - 2\lambda(nm) + (nK)^2\right)^{1/2} + (\lambda^2 - 2\lambda(nm^*) + (nK^*)^2)^{1/2})\|x-y\|_{L_2^\alpha}$

for all $x, y \in H$.

Now we find when the following inequality is hold

$(\lambda - \overline{\sum_{\iota=1}^n\|B_\iota K_\iota\|}\ )^{-1}\left(\lambda^2 - 2\lambda(nm) + (nK)^2\right)^{1/2} + (\lambda^2 - 2\lambda(nm^*) + (nK^*)^2)^{1/2} < 1$

$\left(\lambda^2 - 2\lambda(nm) + (nK)^2\right)^{1/2} + (\lambda^2 - 2\lambda(nm^*) + (nK^*)^2)^{1/2} < (\lambda - \sum_{i=1}^n\|B_iK_i\|)$

$$\left(\lambda^2 - 2\lambda(nm) + (nK)^2\right)^{1/2} < (\lambda - \sum_{i=1}^n\|B_iK_i\|)$$

$(\lambda^2 - 2\lambda(nm^*) + (nK^*)^2)^{1/2} < (\lambda - \sum_{i=1}^n\|B_iK_i\|)$

Hence

$\lambda^2 - 2\lambda(nm) + (nK)^2) < (\lambda - \sum_{i=1}^n\|B_iK_i\|)^2$

$(\lambda^2 - 2\lambda(nm^*) + (nK^*)^2) < (\lambda - \sum_{i=1}^n\|B_iK_i\|)^2$

$(\lambda^2 - 2\lambda(nm) + (nK)^2) < \lambda^2 - 2\lambda\sum_{i=1}^n\|B_iK_i\| + (\sum_{i=1}^n\|B_iK_i\|)^2$

$(\lambda^2 - 2\lambda(nm^*) + (nK^*)^2) < \lambda^2 - 2\lambda\sum_{i=1}^n\|B_iK_i\| + (\sum_{i=1}^n\|B_iK_i\|)^2$

$-\lambda(2nm - 2\sum_{i=1}^n\|B_iK_i\|) < -(nK)^2 + (\sum_{i=1}^n\|B_iK_i\|)^2$

$\lambda > \left((nK)^2 - (\sum_{i=1}^n\|B_iK_i\|)^2\right)(2nm - 2\sum_{i=1}^n\|B_iK_i\|)^{-1}$

Also

$-\lambda(2nm - 2\sum_{i=1}^n\|B_iK_i\|) < -(nK)^2 + (\sum_{i=1}^n\|B_iK_i\|)^2$

$\lambda > \left((nK^*)^2 - (\sum_{i=1}^n\|B_iK_i\|)^2\right)(2nm^* - 2\sum_{i=1}^n\|B_iK_i\|)^{-1}$

Let us choose

$\lambda > \max\{\sum_{i=1}^n\|B_iK_i\|, \left((nK)^2 - (\sum_{i=1}^n\|B_iK_i\|)^2\right)(2nm - 2\sum_{i=1}^n\|B_iK_i\|)^{-1}, \left((nK^*)^2 - (\sum_{i=1}^n\|B_iK_i\|)^2\right)(2nm^* - 2\sum_{i=1}^n\|B_iK_i\|)^{-1}\},$

it result that

$$(\boldsymbol{\lambda}-\sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}}\|\boldsymbol{B}_{\boldsymbol{i}}\boldsymbol{K}_{\boldsymbol{i}}\|))^{-\boldsymbol{1}}\left(\left(\boldsymbol{\lambda}^{\boldsymbol{2}}-\boldsymbol{2\lambda}(\boldsymbol{nm})+(\boldsymbol{nK})^{\boldsymbol{2}}\right)^{\boldsymbol{1}/\boldsymbol{2}}+\left(\boldsymbol{\lambda}^{\boldsymbol{2}}-\boldsymbol{2\lambda}\left(\boldsymbol{nm}^{*}\right)+(\boldsymbol{nK}^{*})^{\boldsymbol{2}}\right)^{\boldsymbol{1}/\boldsymbol{2}}\right)<1$$

Therefore, $(\boldsymbol{\lambda I}-\sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}}(\boldsymbol{A}_{\boldsymbol{i}}+\boldsymbol{B}_{\boldsymbol{i}}\boldsymbol{K}_{\boldsymbol{i}}))^{-\boldsymbol{1}}\boldsymbol{T}_{\boldsymbol{\lambda}}(\boldsymbol{x})$ is a contraction in $\boldsymbol{L}_{\boldsymbol{2}}^{\boldsymbol{\alpha}}$. Then by theorem (2.16) the Equation (27) and consequently (23) has a unique solution.

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Computations in the Pre-Bloch group

D. Al Baydli and A. Rahm

1-University of Wasit, College of Education for Pure Science,Department of mathematics.

2- Mathematics Research Unit of Université du Luxembourg, Luxembourg.

E-mail: 1) daheralbaydli@uowasit.edu.iq

2) alexander.rahm@uni.lu

Abstract. For compute the five term relations in the pre-Bloch group for specify an infinite-order- element
in $K_3\ (Q(\sqrt{-m})),\ \ m\ \in\ N$ square- free. For the quadratic imaginary number fields F of discriminant
(-1;-2;-3;-7;-17;-19). We use the GAP Programming software to implement our method.

1. Introduction

Let R be an associative ring with unit. The higher algebraic K-group of R are defined to be the homotopy groups $Kn(R) := \pi_n(K(R))$ for a space $K(R)$ that is constructed as follows
Type equation here.

where the union is formed using the inclusions $GL_n(R) \longrightarrow GL_{n+1}(R);\ A \longrightarrow \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. for a group G, the commutator subgroup is $[G;G] := gfghg^{-1}h^{-1}|g;h \in G$, then as an Abelian group, the first K-group is

Any ring map $R \longrightarrow S$ induces a natural map $GL(R) \longrightarrow GL(S)$, and hence a map $K_1(R) \longrightarrow K_1(S)$.Therefore, K_1 is a functor from rings to Abelian groups.

Definition 1.1 If $i \neq j \in$ N and $r \in$ R, then the elementary matrix $e_{i,j}(r)$ is the matrix in GL(R) which has diagonal entries all 1, (i; j)-entery r, and 0 elsewhere.
Setting $E_n(R) :=< e_{i,j}(r)|1 \leq i;\ j \leq n\ ; r \in R >$,
$E(R) := \cup_{n\in N} E_n(R)$, the subgroup E(R) of elementary matrices in GL(R), equals the commutator : $E(R) = [GL(R); GL(R)]$: Note that for a field F, En(F) = SLn(F):

1.1. Quillen's space $BGL(R)^+ = K(R)$

The space $BGL(R)^+ = K(R)$ that we are going to construct is also called Quillen's " + "-construction ofthe space $K(R)$ defining algebraic K-theory.
For a group G, there exist a space BG with $\pi_n(BG) = G, \pi_n(BG) = 0$, for all $n \geq 2$.

So there is a theoretical construction of $BGL(R)$ -the classifying space for group homology.

Definition 1.2 The notation $BGL(R)^+$ will denote any CW-complex X which has a distinguished map $BGL(R) \longrightarrow BGL(R)^+$ such that the following are true:

1) $\pi_1 BGL(R)^+ \cong K1(R)$, and the natural map $GL(R) = p1BGL(R) -! p1BGL(R)+$ is surjective with kenerl E(R):
2) $H_*(BGL(R); M) \cong H_*BGL(R)^+; M)$ for every $K1(R)$ −module M. Such a space X is called a model for $BGL(R)^+$

Definition 1.3 An R- module P is called projective if there exist an R-module Q such that $P \oplus Q$ is free (it has a basis). The set PR of isomorphism classes of finitely generated projective R-module, together with direct sum and identity 0, forms an abelian monoid.

$K_0(R) := (PR)^{-1}P(R)$ is the Grothendieck group completion. K(R) is the disjoint union of copies of $BGL(R)^+ = K(R) := K_0(R) \times BGL(R)^+ = ;$

because $BGL(R)^+$ is a connected space. we recover $K1(R)$ with the definition $K_n(R) := \pi_n(K(R))$, for all $n \in N \geq 0$. Note that for all n ≥ 1, $K_n(R) = \pi_n(BGL(R)^+)$; because in $\coprod_{p \in K_0(R)} BGL(R)^+$, all connected components are identical, so it dose not matter where we place the basepoint.

Now we have a theoretical construction of the higher algebraic K-groups, but we do not know yet how any non-trivial element in them looks like.

A theorem of Borel implies that for an imaginary quadratic field F, $K_3(F) \cong Z \oplus Z = \omega_2(F)Z$ for a natural number $\omega_2(F) \in N_geq1$ which is constructed using Tate twists(we will not go into the details of that

construction, because for the present purposes, we are not interested in the torsion).

Question. Can we specify an infinite-order- element in $K_3(Q(\sqrt{-m}))$, $m \in N$ square- free? For this purpose, we use the Bloch group, and work of de Jen,Gangl, Rahm and Yasaki.

2. The Bloch group

for an Abelian group A, let $\widetilde{\Lambda}^2 A$ denote the quotient of the group $A \otimes A$ by the subgroup generated by all $a \otimes b + b \otimes a$:

$$\widetilde{\Lambda}^2 A := A \otimes A = < a \otimes b + b \otimes a | a; b \in A >$$

Definition 2.1 :[7] For any field F, the pre-Bloch group P(F) denote the abelian group presented with generator symbols [x] for $x \in F \backslash \{0\}$ with relations $[1] = [0] = [\infty] = 0$ and for $x \neq y$ in $F \backslash \{0; 1\}$, the "five-term relations":

$[x] - [y] + [y/x] - [\frac{1-1/x}{1-1/y}] + [\frac{1-x}{1-y}] = 0$, In [4] refer the five term relation is different because of the different definition of the cross- ratio for more details see [[3],[5]]. In addition, for [Proposition 2.14 [3]] illustrates the equivalence between the two relation of five term relations. If we have [r]+[r−1] = 0, r > 0 and [r1]−[r2]+[r1=r2]−[1 1− −1 1= =r r1 2]+[11−−rr12] = 0, where 1 < r1 < r2 and [r] 6= [¥;0;1], r > 1, can be translated to defining relation in terms of generators ［s］, satisfying ［s1］−［s2］+［s1=s2］−［1 1− −1 1= =ss1 2 ］+［11−−ss12 ］ = 0. Setting s1 = 11−−xy x and s2 = 1y−−xy xy in the relation above, we obtain

Definition 2.4 The 6-fold symmetry ［x］ = ［1−（1=x）］ = ［1=（1−x）］ = −［1=x］ = −［1−x］ = −［−x=（1−x）］

and similar with ［y］. Also if we have −［x］ it does not mean ［−x］.

Example 2.5 Show ［2］ − ［1=2］ = 0.

•∑2 i=1 mi［xi］=1［x1］ − 1［x2］=1［2］+ 1［1=2］

[2] [1/2]

1 1

•0 = ［xi］ − ［xj］+F3 −F4 +F5

［xi］ ［xj］ F3 F4 F5

1 -1 1 -1 1

•Choose m1 = 1；m2 = 1.

F3 = ［xy］ = ［1= 22］ = ［1 4］, F4 = ［11−−11==xy］ = ［1− 1−1= 22］ = ［−21］, F5 = ［11−−xy］ = ［11 −−12 =2］ = ［1−=12］ = ［−2］, the 6-fold symmetry ［x］ = ［2］ = ［1=2］ = ［−1］ = ［−1=2］ = ［1］ = −［2］, since ［2］+［2］ = 2［2］ = 0：

［2］ ［1=2］ ［1=4］ ［−1=2］ ［−2］

1 1 0 0 0

-1 1 -1 1 -1

•We can merge column with 6-fold symmetry.

•Choose i=3, j=4. F3 = ［xy］ = ［−−12 =2］ = ［4］, F4 = ［11−−11==xy］ = ［1−1−1=1−=−1=22］ = ［2］, F5 = ［11−−xy］ = ［1+1+1=22］ = ［1 2］

［2］=［1=2］ ［1=4］ ［−1=2］ =−［−2］

1 + (-1) 0 0+ 0

-1 + 1 -1 1+1

0 -1 2

•Choose i=3, j=4. F3 = ［xy］ = ［−11==42］ = ［−21］, F4 = ［11−−11==xy］ = ［11−−11==−11==42］ = ［−1］, F5 = ［11−−xy］ = ［11++11==24］ = ［2］,

［−1=2］ 2	［1=4］ -1	［−1=2］ 0	［−1］ 0	［2］ 0
-1	1	-1	1	-

				1

•We can merge column with 6-fold symmetry.

[−1=2]=[−1=2] [1=4] [−1] =[2]

2 + 0 -1 0+ 0

-1 + (-1) 1 1+(-1)

0 0 0

Algorithm 2.1 Algorithm for the check [x]−[y] = 0 in P(F)

Input: A difference [x]−[y] = ∑k i=1 mi[xi], where x1; : : : ;xk 2 F, m1; : : : ;mk 2 Z.

Output: Either a list of 5-term relations with which [x]−[y] can be seen to be zero in P(F). Or return
"fail" if the algorithm cannot find such 5-term relations.

Procedure:

1: Write the vector [x]−[y] in the space < [x1]; : : : ;[x2] >≈ Zk,

2: check if there are two coefficients mi;mj with the same absolute value.

3: Choose two coefficients with high absolute values jmij; jmjj, (assume jmij ≥ jmjj).
F3 = [xxij],
F4 = [11−−11==xxij], F5 = [11−−xxij].

4: Pick [x1] and [y1] with the biggest prime in their denominators: x1 = p=q, q = pm 1 1; : : : ; pm r r prime
factorisation of q, p1; : : : ; pr prime mi 2 N.

5: Add the 5-term relations 0 = [xi]−[xj]+ F3 −F4 + F5

6: We just keep the sum: If we instead take m3 times the row, then we get
[x1] [x2] : : : [xk] F3 F4 F5
m1 m2 + m3 : : : mk m3 −m3 m3
Here we have to keep track of the sign, so we can enter the coefficient with the correct sign.

7: Merge rows using the 6-fold symmetry.

8: If we arrive at a final row ∑ = 0, then run the program a second time and print the 5-term relations
that have been used.

9: If the number of non-zero columns exceeds a limit that has been defined in advance (10m) then return
"fail".

Example 2.6 Show 2[3]−[−3] = 0.

To prove the difference class [3] with coefficient 2 and [−3] with coefficient 1, need to find the five terms
relations from these classes.

•F(xi;xj) = [xi]−[xj]+ F3 −F4 + F5

•We add F(3;−3). F3 = [xy] = [−33] = [−1], F4 = [11−−11==xy] = [11−+11==33] = [12], F5 = [11−−xy] = [11−+33] = [−21],

[3] [−3] [−1] [1=2] [−1=2]

2 -1 0 0 0

-1 1 -1 1 -1

Using the 6-fold symmetry we find [3]=[−1=2] and [−1] =[1=2],so we can merge these columns.

[3]=[−1=2]	[−3]	[−1] = [1=2]
2	-1	0+ 0
-1+(-1)	1	-1 +1
0	0	0

Hence +2[3] +-1[-3]= 0+-1F(3,-3), as claimed

[2] [1=2] [1=4] [−1=2] [−2]
1 1 0 0 0
-1 1 -1 1 -1
•We can merge column with 6-fold symmetry.
•Choose i=3, j=4. F3 = [xy] = [−−12 =2] = [4], F4 = [11−−11==xy] = [1−1−1=1−=−1=22] = [2], F5 = [11−−xy] = [1+1+1=22] = [1 2]
[2]=[1=2] [1=4] [−1=2] =−[−2]
1 + (-1) 0 0+ 0
-1 + 1 -1 1+1
0 -1 2
•Choose i=3, j=4. F3 = [xy] = [−11==42] = [−21], F4 = [11−−11==xy] = [11−−11==−11==42] = [−1], F5 = [11−−xy] =
[11++11==24] = [2],

[−1=2] 2	[1=4] -1	[−1=2] 0	[−1] 0	[2] 0
-1	1	-1	1	-1

•We can merge column with 6-fold symmetry.
[−1=2]=[−1=2] [1=4] [−1] =[2]
2 + 0 -1 0+ 0
-1 + (-1) 1 1+(-1)
0 0 0

Algorithm 2.1 Algorithm for the check [x]−[y] = 0 in P(F)
Input: A difference [x]−[y] = ∑k i=1 mi[xi], where x1;:::;xk 2 F, m1;:::;mk 2 Z.
Output: Either a list of 5-term relations with which [x]−[y] can be seen to be zero in P(F). Or return
"fail" if the algorithm cannot find such 5-term relations.
Procedure:
1: Write the vector [x]−[y] in the space < [x1];:::;[x2] >≈ Zk,
2: check if there are two coefficients mi;mj with the same absolute value.
3: Choose two coefficients with high absolute values jmij;jmjj, (assumejmij ≥ jmjj).

F3 = ［xxij ］,
F4 = ［11−−11==xxij ］, F5 = ［11−−xxij ］.
4: Pick［x1］ and［y1］ with the biggest prime in their denominators: x1 = p=q, q = pm 1
1; : : : ; pm r r prime
factorisation of q, p1; : : : ; pr prime mi 2 N.
5: Add the 5-term relations 0 = ［xi］−［xj］+ F3 −F4 + F5
6: We just keep the sum: If we instead take *m*3 times the row, then we get
［*x*1］［*x*2］ : : : ［*xk*］ *F*3 *F*4 *F*5
*m*1 *m*2 + *m*3 : : : *mk* *m*3 −*m*3 *m*3
Here we have to keep track of the sign, so we can enter the coefficient with the correct sign.
7: Merge rows using the 6-fold symmetry.
8: If we arrive at a final row ∑ = 0, then run the program a second time and print the 5-term relations
that have been used.
9: If the number of non-zero columns exceeds a limit that has been defined in advance（10*m*） then return
"fail".

Example 2.6 *Show* 2［3］−［−3］ = 0.
To prove the difference class ［3］ *with coefficient* 2 *and* ［−3］ *with coefficient* 1, *need to find the five terms*
relations from these classes.
•*F*（*xi* ;*xj*） = ［*xi*］−［*xj*］+ *F*3 −*F*4 + *F*5
•*We add F*（3 ;−3）. *F*3 = ［*xy*］ = ［−33］ = ［−1］, *F*4 = ［11−−11==*xy*］ = ［11−+11==33］ = ［12］, *F*5 = ［11−−*xy*］ = ［11−+33］ = ［−21］,
［3］［−3］［−1］［1=2］［−1=2］
2 -1 0 0 0
-1 1 -1 1 -1
Using the 6-fold symmetry we find ［3］=［−1=2］ *and* ［−1］ =［1=2］,*so we can merge these columns.*

［3］=［−1=2］	［−3］	［−1］ = ［1=2］
2	-1	0+ 0
-1+(-1)	1	-1 +1
0	0	0

Hence +2[3] +-1[-3]= 0+-1F(3,-3), as claimed

Example 2.7 *we can implement a GAP function* **CheckEquivalence**（**x** ;**y** ;**Cx** ;**Cy**）, *which inputs x,y the coefficient of* ［*x*］ *and the coefficient of* ［*y*］, *and output five term relation*

GAP session
gap>L:=[2,1/2];;H:=[1,-1];; gap> CertifyEquivalence(H,L); We want to show that +1[2] +-1[1/2]is zero,

in case that this is possible for us.The terms 1[2] and -1[1/2]are being merged because 1/2 has been found in the class [[2], [1/2], [-1], [1/2], [-1], [2]] For j = 2 we get 0 Success: +1[2] +-1[1/2]=0 "success"

Example 2.8 *For example we 2.6 we can computation by use gap function* **CheckEquivalence**（**x**；**y**；**Cx**；**Cy**）, *which inpute x,y and coefficient of x,coefficient of y, and the output five term relation.*

GAP session
gap>L:=[3,-3];;H:=[2,-1];; gap> CheckEquivalence(L,H); We want to show that +2[3] +-1[-3] is zero, in case that this is possible for us. +2[3] +-1[-3] The terms 1[3] and -1[-1/2]are being merged because -1/2 has been found in the class [[3], [2/3], [-1/2], [1/3], [-2], [3/2]] For j = 3 we get 0 The terms -1[-1] and 1[1/2]are being merged because 1/2 has been found in the class [[-1], [2], [1/2], [-1], [2], [1/2]] For j = 3 we get 0 Success: +2[3] +-1[-3] = 0+-1F(3,-3) Success: +2[3] +-1[-3] = 0+-1F(3,-3) "success"

Algorithm 2.2 Algorithm for picking the biggest prime
Input: The list of coefficient and list of classes .
Output: The list of pick two terms as [Cx1,Cy1,x1,y1].
Procedure:
I = [] the list *I* is going to contain the maximum of absolute value.
for *j* ∈ 1∴*N*= Length(list of classes) **do**

p=Numerator rational (list of classes[*j*]). *q*=Denominator rational(list of classes[*j*]). absolute values of primes=[]; **for** *x* in union(prime factors(*p*),prime factors(*q*)) **do**. Add(absolute values of primes,|*x*|).

end for
pj=Maximum(absolute values of primes).
Add(I, pj) each element of list of classes produces an element of I, at the same index j.
end for
for $i \in I$ **do**
if $I[i]$= maximum(I) **then**
$j1 = i$;
end if
end for
$x1$= list of classes $[j1]$.
$Cx1$=List of coefficient $H[j1]$.
for $i \in I$ **do**
if not $j1 = i$ **then**
Insert the element $I[i]$ into reduced list.
end if
end for
for $i \in I$ **do**
if $I[i]$= maximum(reduced list) and not not $j1 = i$ **then**
Add (L-reduced, list of classes $[i]$).
Add (H-reduced, $H[i]$).
end if
end for
Apply Algorithem 0.3 to (H-reduced,L-reduced)and return the output.
EndProcedure:

we use the command gap **PickBiggestPrime(H,L)** which is function input the list of coefficient and list of classes and the output the list of [Cx1,Cy1,x1,y1], where cx1,Cy1 the coefficient of x1 and y1 respectively. the algorithm above describe how can pick the biggest prime.

By merging duplication we can computation for the discriminant -3 case with which we prove that the algebraic and geometric elements.

Example 2.9 *Let we have the algebraic element* $[[-3, -1/2 * x - 1/2]]$ *and geometric element [[2, w]], where delta=-3 mod 4, d= delta/4 and w= Sqrt(d). The GAP session below prove the discriminant -3.*

We want to show that

$-2[-z32] - 3[z32] = 0$

we can rewrite

$-2[-z32] - 2[z32] - 1[z32] = 0$

2.1 Cross- ratios of ideal hyperbolic tetrahedra 8

By merging duplication

$-2[-z32] -2[z32] = -1[z3]$

we have
−1[z3]−1[z32] = 0
Applying the 6-fold symmetries has yielded
The terms −1[z3] *and* −1[z32] *are being merged because* z32
has been found in the class [[z3];[−z3 −2∗z32];[−1=3∗z3 −2=3∗z32];[z32];[−2∗z3 −
z32];[−2=3∗z3 −
1=3∗z32]]
Then we have
+0[z3] = 0

GAP session
We want to show that +-2[-E(3)^2] +-3[E(3)^2] is zero, in case that this is possible for us.Then we have +-2[-E(3)^2] +-3[E(3)^2] Applying the 6-fold symmetries has yielded +-2[-E(3)^2] +-3[E(3)^2] Inserting duplication relations has yielded +-1[E(3)] +-1[E(3)^2] The terms -1[E(3)] and -1[E(3)^2]are being merged because E(3)^2 has been found in the class [[E(3)], [-E(3)-2*E(3)^2], [-1/3*E(3)-2/3*E(3)^2], [E(3)^2], [-2*E(3)-E(3)^2], [-2/3*E(3)-1/3*E(3)^2]] Then we have +0[E(3)] Applying the 6-fold symmetries has yielded +0[E(3)] Success: +-2[-E(3)^2] +-3[E(3)^2] = 0

In The GAP session below we can computation for the discriminant -7,with algebraic element and
geometric element.

GAP session
gap> Read("./desktop/Bloch.g.txt"); Over the imaginary quadratic field of discriminant -7, we compare thegeometric Bloch group element +8[-E(7)^3-E(7)^5-E(7)^6] +2[-1/4*E(7)-1/4*E(7)^2-1/2*E(7)^3- 1/4*E(7)^4-1/2*E(7)^5-1/2*E(7)^6] +-2[-1/2*E(7)-1/2*E(7)^2 -1/4*E(7)^3-1/2*E(7)^4-1/4*E(7)^5-1/4*E(7)^6] with j times the algebraic Bloch group element +-2[-3/11*E(7)-3/11*E(7)^2-1/11*E(7)^3-3/11*E(7)^4-1/11*E(7)^5 -1/11*E(7)^6] +2[-E(7)-E(7)^2-5/6*E(7)^3-E(7)^4-5/6*E(7)^5-5/6* E(7)^6] +-2[-1/5*E(7)-1/5*E(7)^2-1/15*E(7)^3-1/5*E(7)^4-1/15* E(7)^5-1/15*E(7)^6]+2[-3/4*E(7)-3/4*E(7)^2-5/8*E(7)^3-3/4*E(7)^4 -5/8*E(7)^5-5/8*E(7)^6] +-2[-7/8*E(7)-7/8*E(7)^2-3/4*E(7)^3-7/8 *E(7)^4-3/4*E(7)^5-3/4*E(7)^6] +2[1/90*E(7)^3+1/90*E(7)^5+1/90* E(7)^6]+2[-1/2] +-2[1/3*E(7)+1/3*E(7)^2+E(7)^3+1/3*E(7)^4+

E(7)^5+E(7)^6] +2[-1/2*E(7)-1/2*E(7)^2-1/2*E(7)^4] +-2[E(7)+
E(7)^2+3/2*E(7)^3 +E(7)^4+3/2*E(7)^5+3/2*E(7)^6] +2[2*E(7)+
2*E(7)^2+5/2*E(7)^3+2*E(7)^4+5/2*E(7)^5+5/2*E(7)^6]
+2[-1/4] +-2[2/11*E(7)+2/11*E(7)^2+5/22*E(7)^3+2/11*E(7)^4+
5/22*E(7)^5+5/22*E(7)^6]+2[-3/5*E(7)-3/5*E(7)^2-1/5*E(7)^3
-3/5*E(7)^4-1/5*E(7)^5-1/5*E(7)^6]
+-2[-1/2*E(7)-1/2*E(7)^2-1/4*E(7)^3-1/2*E(7)^4-1/4*E(7)^5
-1/4*E(7)^6] +-2[-5/4*E(7)-5/4*E(7)^2-E(7)^3-5/4*E(7)^4-E(7)^5
-E(7)^6] +-2[-11/5]+-2[-15/22*E(7)-15/22*E(7)^2-6/11*E(7)^3
-15/22*E(7)^4-6/11*E(7)^5-6/11*E(7)^6] +-2[-5/2*E(7)-5/2*E(7)^2
-5/2*E(7)^4] +-2[-E(7)-E(7)^2+3/2*E(7)^3-E(7)^4+3/2*E(7)^5+3/2*
E(7)^6]+-2[-7/2*E(7)-7/2*E(7)^2 -9/4*E(7)^3-7/2*E(7)^4-9/4*
E(7)^5-9/4*E(7)^6]2[-1/6*E(7)^3-1/6*E(7)^5-1/6*E(7)^6]+
2[1/8*E(7)+1/8*E(7)^2-1/4*E(7)^3+1/8*E(7)^4-1/4*E(7)^5
-1/4*E(7)^6] +-2[2*E(7)+2*E(7)^2-E(7)^3+2*E(7)^4-E(7)^5-E(7)^6].
j = -3 yields 22 remaining terms.
j = -2 yields 1 remaining terms.
geobelt = 2*algbelt +
+22[-1]
, where geobelt is the geometric Bloch group element and algbelt
the algebraic Bloch group element.
We observe the 6-fold symmetries [[-1], [2], [1/2], [-1],
[2],[1/2]],
which might allow us to indentify the remainder term as torsion.
We want to show that
+22[-1] is zero, in case that this is possible for us.Applying
the 6-fold
symmetries has yielded
+22[-1] Inserting duplication relations has yield
Success:
+22[-1]
=[0]=[1]

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Topological Dynamics and the Space of Continuous Mappings

[1] Mohammed Nokhas Murad Kaki , [2] Reyadh.D.Ali

[1] Department of Mathematics, College of Science,University of Sulaimani
[2] Department of Mathematics, College of Education for Pure Science, University of Karbala
Email: mohammed.murad@univsul.edu.iq , reyadhdelphi@gmail.com

Abstract. The aim of this paper is define and investigate some new forms of transitive maps, minimal systems and chaotic maps on the space of all continuous maps from a space M to a space N, denoted by $Co(M,N)$. Also, we introduce some new definitions namely point-wise convergence transitive, compact-open transitive, uniform convergence topological transitive, chaotic maps defined on spaces up to product of uniform convergence spaces. In addition, we study the relationship between these new definitions.

Keywords: Uniform convergence, topological transitive, compact-open topology, minimal systems

1. Introduction

Let (M,Γ_1) and (N,Γ_2) be topological spaces, consider that $Co(M,N)=\{h:h;M\to N\,continuous\quad map\}$. The properties of $Co(M,N)$, and many of those of transitivity on this set interrelated. We have to study dynamics properties in topologies defined on $Co(M,N)$.

A topology may be introduced on the set $Co(M,N)$ [1] as on any other, in different ways. We have studied two kinds of topologies and shown which of them weaker dynamics than other one is. So we can introduce in transitivity, mixing, chaos and exactness on this set in different ways (for more knowledge about chaos, weakly mixing and exactness cf ([2],[3], [4], and [5]). First we have to study two maps h_1 , h_2 are said to be near if $h_1(u)$ *and* $h_2(u)$ $\forall u\in M$ are near in N . Let N be a metric space then these notions are expressed in terms of metric on N . Hence, various topologies and thus we can introduce various types of, transitivity, minimal systems and chaos on the $Co(M,N)$ via the topology of point-wise convergence, the compact-open, Uniform Topology [1], etc.

2. Uniform convergence Topology

In this section we have introduced and studied some definitions and theorems as follows:

Definition 2.1 [6] $h:M\to N$ is a homeomorphism map between two topological spaces M and N , if h is continuous, bijective and h^{-1} is continuous.

Definition 2.2 [1] If (N,ρ) is a metric space and M is compact, then $Co(M,N)$ is equipped with a metric μ thus:

$$\mu(h_1,h_2)=\sup_{u\in M}\mu(h_1(u),h_2(u)),\ h_1,h_2\in Co(M,N).$$

Definition 2.3 [1] The topology Γ_1 on $Co(M,N)$ is called the uniform convergence topology, if Γ_1 is determined by the metric μ. Any open set in Γ_1 is uc-open set and $(Co(M,N),\Gamma_1)$ is uc – space. The compliment of uc –closed set is uc -open set

If $(Co(M,N),\Gamma_1)$ is a uniform convergence-topological space, and H: $Co(M,N) \to Co(M,N)$ be a function. Then $(Co(M,N),H)$ is called uniform convergence system, in short uc-system.

Definition 2.4 Let $(Co(M,N),\Gamma_1)$ be a uc-topological space. Given $h \in Co(M,N)$, $O_G(h) = \{h, G(h), G^2(h),...\}$ refers to its forwaed orbit and $\omega_G(h)$ refers to its ω -limit set, i.e. $h, G(h), G^2(h),...G^n(h),...$ the set of limit functions of the sequence.

A set $B \subseteq Co(M,N)$ is called a uc-*minimal* if $B \neq \phi$, invariant uc-closed also let no proper subset of B has three of these properties.

The following conditions are equivalent:

- $(Co(M,N),\Gamma_1)$ is uc-minimal,
- every orbit is uc-dense $Co(M,N)$,
- $\omega_G(h) = Co(M,N)$ for every $h \in Co(M,N)$.

Definition 2.5 A function $F: Co(M,N) \to Co(M,N)$ is called uc-irresolute if the inverse image of each uc-open set is a uc-open set in $Co(M,N)$.

Definition 2.6 A function $H: Co(M,N) \to Co(M,N)$ is ucr -homeomorphism if H is surjective, injective and thus invertible ,also H and H^{-1} are both uc -irresolute.

The systems $F: Co(\mathrm{M,M}) \to Co(\mathrm{M,M})$ *and* $G: Co(\mathrm{N,N}) \to Co(\mathrm{N,N})$ are topologically ucr-conjugate if $\exists$ $H: Co(\mathrm{M,M}) \to Co(\mathrm{N,N})$ is a ucr-homeomorphism such that $H \circ F = G \circ H$

If $(Co(M,N),\Gamma_1)$ is a uc-topological space. We define the uc-closure of B by $Cl_{uc}(\mathrm{B}) = \bigcap_i F_i$

$$\ni \quad F_i \quad uc-closed \quad set\, of \quad (Co(M,N),\Gamma_1) \quad and \quad \mathrm{B} \subseteq F_i \quad \forall \quad i.$$

Definition 2.7 If $(Co(M,N),\Gamma_1)$ is a uniform convergence-topological space.

The map $H: Co(M,N) \to Co(M,N)$ is said to have uc-dense *orbit* if $\exists$ $h \in Co(M,N) \ni$ $Cl_{uc}(O_H(h)) = Co(M,N)$, where $O_H(h)$ is the orbit passing through h and Cl_{uc} is th uniform convergence closure of this orbit.

Definition 2.8

Let $(Co(M,N),\Gamma_1)$ be a uniform convergence-topological space, $H: Co(M,N) \to Co(M,N)$ be a uc- continuous map, then $(Co(M,N),H)$ is the uc –system and H is a uniform-convergence-transitive (uc-transitive) map if $\forall O$ and V are uc-open sets in $(Co(M,N),\Gamma_1)$ $\exists$ n is a positive integer $\ni$ $H^n(O) \cap V$ is not empty.

Lemma 2.9 A map $F: Co(M,N) \to Co(M,N)$ is topologically uc-transitive if $\omega_F(g) = Co(M,N)$ for some $g \in Co(M,N)$

Proof: Suppose that $\omega_F(g) = Co(M,N)$ for some $g \in Co(M,N)$. Then for every pair of non-empty, uc-open $D, W \subset Co(M,N)$ there are integers $n > m > 0$ such that $F^m(g) \in D$ *and* $F^n(g) \in W$ Hence $F^{n-m}(D) \cap W \neq \phi$ and $F: Co(M,N) \to Co(M,N)$ is topologically uc-transitive.

Theorem 2.10 If H is a ucr-homeomorphism, then $(Co(M,N),H)$ is topologically uc-transitive then every non empty backward invariant uc-open subset of $Co(M,N)$ is uc-dense.

Proof: Suppose that the map H is uc-transitive, $U \subset Co(M,N)$ is uc-open and $f^{-1}(U) \subset U$. Assume that $U \neq \phi$ and U is not uc-dense in $Co(M,N)$ (i.e. $Cl_{uc}(U) \neq Co(M,N)$). Then there exists a non-empty uc-open $V = Co(M,N) \setminus Cl_{uc}(U)$, since $Cl_{uc}(U)$ is uc-closed, such that $U \cap V = \phi$. Further $H^{-n}(U) \cap V = \phi$ for all n ϵ N. This implies $U \cap H^{n}(V) = \phi$ for all n ϵ N, a contradiction to H being uc-transitive map. Therefore U is uc-dense in $Co(M,N)$.

Theorem 2.11 Suppose that $(Co(M,N),\Gamma_1)$ is a uc –compact space without isolated point and $G: Co(M,N) \to Co(M,N)$ is a map. If there exists uc-dense orbit , that is there exists $f_0 \in Co(M,N)$ such that the set $O_G(f_0)$ is uc- dense then the map G is uc- transitive .

Proof : Let $f_0 \in Co(M,N)$ be such that $O_G(f_0)$ is uc-dense in $Co(M,N)$. Given any pair U, V of uc-open subsets of $Co(M,N)$, by uc-density there exists n such that $G^{n}(f_0) \in U$, but $O_G(f_0)$ is uc- dense implies that $O_G(G^{n}(f_0))$ is uc-dense, so it intersects V, i.e. Tthere exists m such that $G^{m}(G^{n}(f_0)) \in V$. Therefore $G^{m+n}(f_0) \in G^{m}(U) \cap V$ That is $G^{m}(U) \cap V \neq \varphi$. So G is uc-transitive.

Definition2.12 If $(Co(M,N),\Gamma_1)$ is a uniform convergence-topological space, also the map $G: Co(M,N) \to Co(M,N)$ be a uc-irresolute then $B \subseteq Co(M,N)$ is called uc-transitive set if $\forall U$ and V are non-empty uc-open sets in $Co(M,N)$ with $B \cap U \neq \phi$ and $B \cap V \neq \phi$ $\exists$ $n \in \mathbf{N}$ $\ni G^{n}(U) \cap V \neq \phi$.

Theorem 2.13 *Let* $\mathrm{B} \neq \phi$ *be a* uc-*closed invariant subset of* $(Co(M,N),\Gamma_1)$. *So*

(a) B *is uc- transitive set of* $(Co(M,N),\Gamma_1)$. $\Leftrightarrow$ (b) (B,H) *is uc- transitive*.

Proof:

(a) $\Longrightarrow$(b): Let $V_1 \neq \phi$ *and* $U_1 \neq \phi$ be uc-open subsets of B . For a uc-open subset $U_1 \neq \phi$ of B, $\exists$ a uc- open set O of M $\ni U_1 = O \cap \mathrm{B}$.Since B is a uc-transitive set of $(Co(M,N),H)$, $\exists$ $n \in N$ $\ni H(V_1) \cap O \neq \phi$. Moreover, B is invariant, *i.e.*, $H(\mathrm{B}) \subset \mathrm{B}$. Therefore, $H(V_1) \cap \mathrm{B} \cap O \neq \phi$, i.e. $H(V_1) \cap U_1 \neq \phi$. Hence (B,H) is uc-transitive.

(b) $\Longrightarrow$(a): Suppose $V_1 \neq \phi$ is a uc-open set of B and $O \neq \phi$ is a uc-open set of $(Co(M,N),\Gamma_1)$ with $\mathrm{B} \cap O \neq \phi$, Since O is an uc-open set of $(Co(M,N),\Gamma_1)$ and $\mathrm{B} \cap O \neq \phi$, therefore $\mathrm{B} \cap O$ is a uc-open set of B . Since (B,H) is topologically uc-transitive, $\exists$ $n \in N$ $\ni H(V_1) \cap (\mathrm{B} \cap O) \neq \phi$, so $H(V_1) \cap O \neq \phi$. Hence B is a uc-transitive set of $(Co(M,N),H)$.

Definition2.14

(1) If $(Co(M,N),\Gamma_1)$ is a uniform convergence-topological space, $H: Co(M,N) \to Co(M,N)$ is a uc-irresolute , $B \subseteq Co(M,N)$ and given $U,V \subseteq Co(M,N)$ any nonempty uc-open with $B \cap U \neq \phi$ and $B \cap V \neq \phi$ then $\exists$ $N > 0$ $\ni H^{n}(U) \cap V \neq \phi$ $\forall$ $n > N$.

In this case B is called topologically uc-mixing set.

(2) $B \subseteq Co(M,N)$ is a weakly uc-mixing of $(Co(M,N),H)$ if $\forall V_1$ and V_2 are non-empty uc-open subsets of B and nonempty uc –open subsets U_1 *and* U_2 of $Co(M,N)$ with $B \cap U_1 \neq \phi$ *and* $B \cap U_2 \neq \phi$ $\exists$ $n \in N$ $\ni H^n(V_1) \cap U_1 \neq \phi$ *and* $H^n(V_1) \cap U_2 \neq \phi.$

(3) $(Co(M,N),H)$ is a topologically uc-mixing ,if given O and V any nonempty uc-open sets in $Co(M,N)$, $\exists N$ is an integer $\ni$ $\forall n > N$, one has $. H^n(O) \cap V \neq \phi.$

Theorem 2.15 topologically uc- mixing $\Rightarrow$ weakly uc- mixing $\Rightarrow$ uc- transitive

Chaos in product uc-topological spaces: If $(Co(M,N),F)$ is uc-system. $F: Co(M,N) \to Co(M,N)$ is called uc-chaotic map, if it is uc- transitive and, its periodic points are uc-dense in $Co(M,N)$**,** each uc-open non-empty subset of $Co(M,N)$ contains a periodic point. ($f \in Co(M,N)$ is called periodic if $\exists n \geq 1$ *with* $F^n(f) = f$). $Per(F). = \{f \in CO(M,N): f \text{ is periodic point}\}.$ Given two uc-topological spaces $Co(M_1,N_1)$ and $Co(M_2,N_2)$, their product is the set $Co(M_1,N_1) \times Co(M_2,N_2)$ $= \{(f,g): f \in Co(M_1,N_1) \text{ and } g \in Co(M_2,N_2)\}$, we can define a topology on $Co(M_1,N_1) \times Co(M_2,N_2)$ by saying that a basis consists of the subsets $D \times W$ as D ranges over open sets in $Co(M_1,N_1)$ and W ranges over open sets in $Co(M_2,N_2)$. The criterion for a family of subsets to be a basis for a topology is satisfied since $(D_1 \times W_1) \cap (D_2 \times W_2) = (D_1 \cap D_2) \times (W_1 \cap W_2).$ This is called the **product topology** on $Co(M_1,N_1) \times Co(M_2,N_2)$

Now, given two maps $G: Co(M_1,N_1) \to Co(M_1,N_1)$ and $H: Co(M_2,N_2) \to Co(M_2,N_2)$ on uc-topological spaces $Co(M_1,N_1)$ and $Co(M_2,N_2)$. respectively, consider their product $G \times H: Co(M_1,N_1) \times Co(M_2,N_2) \to Co(M_1,N_1) \times Co(M_2,N_2)$, $(G \times H)(f,g) = (G(f),H(g))$, with product topology on $Co(M_1,N_1) \times Co(M_2,N_2)$

Lemma 2.16 Let $(Co(M_1,N_1),H)$, $(Co(M_2,N_2),L)$ be uc-topological systems. *Then* the following are equivalent:

(a) The set of periodic points of $H \times \mathrm{L}$ is uc-dense in $Co(M_1,N_1) \times Co(M_2,N_2)$.

(b) For both of H and L , the sets of periodic points in $Co(M_1,N_1)$ *and* $Co(M_2,N_2)$ are uc-dense in $Co(M_1,N_1)$, respectively $Co(M_2,N_2)$.

Proof: (b) $\Longrightarrow$(a): Suppose that the set of periodic points of H is uc-dense in $Co(M_1,N_1)$ (i.e. $Cl_{uc}(Per(H)) = Co(M_1,N_1)$) and the set of periodic points of L is uc-dense in $Co(M_2,N_2)$ (i.e. $Cl_{uc}(Per(L)) = Co(M_2,N_2)$. We can prove this the set of periodic points of $H \times \mathrm{L}$ is uc-dense in $Co(M_1,N_1) \times Co(M_2,N_2)$. Let $\mathrm{E} \neq \phi \subset Co(M_1,N_1) \times Co(M_2,N_2)$ be any uc-open set. Then $\exists$ uc-open sets $O \neq \phi \subset Co(M_1,N_1)$ and $V \neq \phi \subset Co(M_2,N_2)$ with $O \times V \subset \mathrm{E}$. By assumption, $\exists$ a point $h \in O \ni$ $H^n(h) = h$, $n \geq 1$. Similarly, $\exists$ $l \in V$ such that $L^m(l) = l$, $m \geq 1$. For $\mathrm{q} = (\mathrm{r,s}) \in \mathrm{W}$ and $k = m \times n$ we get

$$(H \times L)^k(q) = (H \times L)^k(h,l) = ((H^k(h), L^k(l)) = (h,l) = q$$

Therefore E contains a periodic point and thus the set of periodic points of $H \times \mathrm{L}$ is uc-dense in $Co(M_1,N_1) \times Co(M_2,N_2)$.

(a) $\Longrightarrow$(b): let $O \neq \phi \subset Co(M_1,N_1)$ and $V \subset Co(M_2,N_2)$ be non-empty uc-open subsets. Then $O \times \mathrm{V} \neq \phi$ is a uc-open subset of $Co(M_1,N_1) \times Co(M_2,N_2)$. As the set of the periodic points of $H \times \mathrm{L}$ is uc-dense in $Co(M_1,N_1) \times Co(M_2,N_2)$, $\exists$ a point $\mathrm{q} = (\mathrm{h,l}) \in \mathrm{O} \times \mathrm{V}$ $\ni$ $(H \times L)^n(h,L) = ((H^n(l), L^n(l)) = (h,l)$ for some n. From the last equality we obtain $H^n(h) = h$ for $h \in O$ and $L^n(l) = l$ for $l \in V$.

Lemma 2.17 Let $(Co(M_1,N_1),H)$, $(Co(M_2,N_2),L)$ be topological systems and H , L be topologically uc- mixing maps, then $H \times L$ is topologically uc- mixing.

Proof: Given $W_1,W_2 \subset Co(M_1,N_1)\times Co(M_2,N_2)$, $\exists$ uc-open sets $O_1,O_2 \subset Co(M_1,N_1)$ and $V_1,V_2 \subset Co(M_2,N_2)\ni$ $O_1 \times V_1 \subset E_1$ and $O_2 \times V_2 \subset E_2$. By assumption there exist n_1 *and* $n_2 \ni$ $H^k(O_1) \cap O_2 \neq \phi$ *for* $n \geq n_1$ *and* $L^k(V_1) \cap V_2 \neq \phi$ *for* $n \geq n_2$.

$n_0 = \max\{n_1 , n_2\}$

For $n \geq n_0$

we're having

$[(HxL)^k(O_1 \ x\ V_1)] \cap (O_2 \ x\ V_2) = [H^k(O_1) \ x\ L^k(V_1)] \cap (O_2 \ x\ V_2)$

$= [H^k(O_1) \cap O_2] \ x\ [L^k(V_1) \cap V_2] \neq \phi$

Which means that $H \times L$ is topologically uc- mixing.

Definition 2.18 The Function $H : Co(M_1,N_1) \to Co(M_1,N_1)$ is called uc -chaotic if it is topologically uc-transitive and has uc-dense orbit.

Now we afford some sufficient conditions for a product map to be uc- chaotic and Let us clarify the condition to be uc-mixing as illustrated in the following theorem:

Theorem 2.19 Let $H : Co(M_1,N_1) \to Co(M_1,N_1)$ and $L : Co(M_2,N_2) \to Co(M_2,N_2)$ be uc- chaotic and topologically uc-mixing maps. Then $H \times L : Co(M_1,N_1)\times Co(M_2,N_2) \to Co(M_1,N_1)\times Co(M_2,N_2)$ is uc -chaotic.

Proof: By Lemma 2.16, $H \times L$ has uc-dense periodic points and by Lemma 2.17, $H \times L$ is topologically uc mixing . Therefore topologically uc-transitive. Therefore the two conditions of uc-chaos are satisfied.

3 Definition and Theorems of point wise- convergence Topology

In this section, we have introduced some new definitions of maps called pc –irresolute map. pcr homeomorphism, pcr-conjugate and pc-minimal maps and some new definions of sets called up-closure, pc-transitive and pc-mixing sets

Definition 3.1 Consider in $Co(M,N)$ the sets

$$\{m_i,V_i\}_{i=1}^k = \{h \in Co(M,N) : h(m_i) \in V_i, i = 1,...,k,\}$$

, *where* $m_1,...,m_k \in M, V_1,...,V_k$ *are open sets in* N

Γ_2 is topology generated by these sets in their capacity as a subset is called the topology of point-wise convergence on $Co(M,N)$.

Any open set in Γ_2 is called pc-open set and $(Co(M,N),\Gamma_2)$ is pc- topological space. The compliment of pc-open set is called pc-closed set.

Definition 3.2 A function $H : Co(M,N) \to Co(M,N)$ is called pc-irresolute if the inverse image of each pc-open set is a pc-open set in $Co(M,N)$.

Definition 3.3 A function $H : Co(M,N) \to Co(M,N)$ is pcr -homeomorphism if it is surjective, injective and thus invertible , also H , H^{-1} are both pc -irresolute.

The systems $H:Co(M,M)\to Co(M,M)$ and $L:Co(N,N)\to Co(N,N)$ are topologically pcr-conjugate if $\exists$ $G:Co(M,M)\to Co(N,N)$ is a pcr-homeomorphism $\ni G\circ H = L\circ G$.

If $(Co(M,N),\Gamma_2)$ is a pc-topological space. We define the up-closure of B by $Cl_{pc}(\mathrm{B})=\bigcap_i F_i$

$\ni$ F_i $pc-closed$ $set\ of$ $(Co(M,N),\Gamma_2)$ and $\mathrm{B}\subseteq F_i$ $\forall$ i.

Definition 3.4 If $(Co(M,N),\Gamma_2)$ is a uniform convergence-topological space.

The map $H:Co(M,N)\to Co(M,N)$ is pc-dense *orbit* if $\exists$ $h\in Co(M,N)\ni$ $Cl_{pc}(O_H(h))=Co(M,N)$.

Definition 3.5 Let $(Co(M,N),\Gamma_2)$ be a pc-topological space, and $G:Co(M,N)\to Co(M,N)$ be a pc-irresolute map, then G is a point-wise- converge –transitive (shortly pc- transitive) map if $\forall V$ and O are pc-open non-empty sets in $(Co(M,N),\Gamma_2)$ $\exists n$ is a positive integer $\ni G^n(V)\cap O\neq\phi$.

Definition 3.6 If $(Co(M,N),\Gamma_2)$ is a point wise convergence-topological space, and $H:Co(M,N)\to Co(M,N)$ be a pc-irresolute then the set $B\subseteq Co(M,N)$ is called pc-type transitive set if $\forall V$ and O are pc-open non-empty sets in $(Co(M,N),\Gamma_2)$ with $B\cap V\neq\phi$ and $B\cap O\neq\phi$ $\exists n$ is a positive integer $\ni H^n(V)\cap O\neq\phi$.

Definition 3.7

(1) If $(Co(M,N),\Gamma_2)$ is a point-wise convergence-topological space, and $G:Co(M,N)\to Co(M,N)$ be a pc-irresolute then the set $B\subseteq Co(M,N)$ is topologically pc-mixing set if, given $U,V\subseteq Co(M,N)$ any nonempty pc-open with $B\cap U\neq\phi$ and $B\cap V\neq\phi$ then $\exists N>0$ $\ni G^n(U)\cap V\neq\phi$ $\forall$ $n>N$.

(2) $B\subseteq Co(M,N)$ is a weakly pc- mixing set of $(Co(M,N),G)$ if $\forall V_1$ and V_2 are non-empty pc-open subsets of B and nonempty pc –open subsets U_1 *and* U_2 of $Co(M,N)$ with $B\cap U_1\neq\phi$ *and* $B\cap U_2\neq\phi$ $\exists$ $n\in N$ $\ni G^n(V_1)\cap U_1\neq\phi$ *and* $G^n(V_1)\cap U_2\neq\phi$

(3) $(Co(M,N),G)$ is topologically pc-mixing, if given $U\neq\phi$ and $V\neq\phi$ any pc-open sets in $Co(M,N)$, $\exists N$ is an integer $\ni$ $\forall n>N$, one has $G^n(U)\cap V\neq\phi$.

In addition, we have studied the compact-open topology. The compact-open topology is a topology defined on $Co(M,N)$. This topology is applied in homotopy theory and functional analysis.

Given a compact subset C of M and an open subset U of N, let $V(C,U)=\{h\in Co(M,N):h(C)\subset U\}$.

The following definition supplies a compact-open topology on $Co(M,N)$.

Definition 3.8 [1] Let $V(C,U)=\{h\in Co(M,N):h(C)\subset U\}$. The topology Γ_3 generated by $V(C,U)$ as a subbase of a topology which is called the compact-open topology on $Co(M,N)$.(Does not always this collection form a base for a topology on $Co(M,N)$).

Note that if we define another new definition:

$V_\alpha(C,U)=\{h\in Co(M,N):h(C)\subset U\}$ where C is compact and U is $\alpha-$open in N. The topology Γ_3^α generated by $V_\alpha(C,U)$ as a subbase of a topology which is called compact-$\alpha-$open topology on $Co(M,N)$.

Any open set in Γ_3 is called co-open set and $(Co(M,N),\Gamma_3)$ is called co- topological space. The compliment of co-open set is called co-closed set.

Definition 3.9

If $(Co(M,N),\Gamma_3)$ is a co-topological space, and $G: Co(M,N) \to Co(M,N)$ be a co-irresolute, so G is a compact-open–transitive (shortly co- transitive) if $\forall O$ and V are co-open sets in $(Co(M,N),\Gamma_3)$ $\exists$ n is a positive integer $\ni$ $G^n(O) \cap V \neq \phi$.

Definition 3.10

(1) Let $(Co(M,N),\Gamma_3)$ be a co-topological space, and $G: Co(M,N) \to Co(M,N)$ be a co-irresolute then $B \subseteq Co(M,N)$ is topologically co-mixing set, if given $U,V \subseteq Co(M,N)$ any nonempty uc-open with $B \cap U \neq \phi$ and $B \cap V \neq \phi$ then $\exists$ $N > 0$ $\ni G^n(U) \cap V \neq \phi$ $\forall n > N$.

Theorem3.11 For $(Co(M,N),G)$,

(a) G is pc-minimal map .
(b) If S is pc-closed subset of X with $G(S) \subset S$. Then S= ϕ or $S = Co(M,N)$.
(c) If D is pc-open and nonempty set in $Co(M,N)$, then $\bigcup_{n=0}^{\infty} G^{-n}(D) = Co(M,N)$.

Proof:

(a) $\Rightarrow$(b): let S ≠ ϕ and h ϵ S. Since S is invariant and pc-closed, i.e. $Cl_{pc}(S) = S$ so $Cl_{pc}(O_G(h)) \subset S$. But $Cl_{pc}(O_G(h)) = Co(M,N)$. Therefore, we have $S = Co(M,N)$.

(b)$\Rightarrow$(c) Let S= $Co(M,N) \setminus \bigcup_{n=0}^{\infty} G^{-n}(D)$. Since D is nonempty, $Co(M,N)$ ≠ S and Since D is pc-open and G is pc irresolute, S is pc-closed. Also $G(S) \subset S$, so S must be ϕ.

(c) $\Rightarrow$(a): Let h ϵ $Co(M,N)$ and D be a nonempty pc-open subset of $Co(M,N)$. Since h ϵ $Co(M,N)$ $= \bigcup_{n=0}^{\infty} G^{-n}(D)$. Therefore h ϵ $G^{-n}(D)$ for some n>0. So $G^n(h) \in D$.

Theorem 3.12 Let $Co(M,N)$ be a compact space without isolated point, if there is a co-dense orbit, that is there is $h_0 \in Co(M,N)$ such that $O_H(h_0)$ is co-dense then H is co-transitive .

Proof .Let h_0 be such that $O_H(h_0)$ is co-dense. Given D, W of co-open sets, by co-density $\exists$ a positive integer n $\ni$ $H^n(h_0) \in D$, but $O_H(h_0)$ is co- dense implies that $O_H(H^n(h_0))$ is co-dense, so, there is k such that $H^k(H^n(h_0)) \in W$. Therefore $H^{k+n}(h_0) \in H^k(D) \cap W$ That is $H^k(D) \cap W \neq \phi$ So H is topological co-transitive.

4. CONCLUSION

The main results are the following:

Every uniform-convergence-transitive implies compact - open –transitive.,Every uc-mixing implies co–mixing which implies pc –mixing,Every weakly uc-mixing implies weakly co–mixing., If a map is a ucr-homeomorphism on the set of all continuous functions then, it is topologically uc-transitive iff every non empty invariant uc-open subset of that space is uc-dense.

Let $B \neq \phi$ *be a* uc-*closed invariant subset of* $(Co(M,N),\Gamma_1)$. *Then*
(a) B *is uc- transitive set of* $(Co(M,N),\Gamma_1)$. $\Leftrightarrow$ (b) (B,H) *is uc- transitive*.
We have also shown that every uc-mixing implies uc-transitive.
For $(Co(M,N),G)$
(a) G is pc-minimal map . $\Leftrightarrow$ (b) If S is pc-closed subset of M with $G(S)\subset S$.
Then S= ϕ or $S = Co(M,N)$. $\Leftrightarrow$ (c) If D is pc-open and nonempty set in $Co(M,N)$, then
$\bigcup_{n=0}^{\infty} G^{-n}(D) = Co(M,N)$. Fortheremore, If $(Co(M,N),F)$ is a compact system without isolated point, and
there is a co-dense orbit, then F is co-transitive . And we have proved that the product of two topologically uc-mixing maps is a topologically uc-mixing map.

Suppose that $(Co(M,N),\Gamma_1)$ is second countable and has a Baire property. If $(Co(M,N),G)$ is uc-transitive then there exists uc-dense orbit. A map $F: Co(M,N) \to Co(M,N)$ is topologically uc-transitive if $\omega_F(g) = Co(M,N)$ for some $g \in Co(M,N)$.

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Study about fuzzyω-paracompact space in fuzzy topological space

***Prof.Dr.Munir Abdul Khalik AL-Khafaji* and Gazwan Haider Abdulhusein**

Department of Mathematics, AL-Mustinsiryah University, Baghdad, Iraq

mnraziz@uomustansiriyah.edu.iq and haider.gazwan@uomustansiriyah.edu.iq

Abstract: The purpose of this paper is to introduce a new class of fuzzy paracompact space is named fuzzy **ω**- paracompact space on fuzzy topological space also study the relationships with fuzzy **ω**- separation axioms and we give some characterization on fuzzy **ω**- paracompact space by using fuzzy countable set also we study the fuzzy **ω**- paracompact subspace and consider some relationship between fuzzy paracompact space and fuzzy **ω**- paracompact space by using certain types of fuzzy **ω**- continuous functions.

1. Introduction

The concept, which we will be considered in this paper, is the so called "fuzzy sets" which is totally different from the classical concept which is called "a crisp set". The recent concept is introduced by Zadeh in 1965 [15], in which he defines fuzzy sets as a class of objects with a continuum of grades of membership and such a set is characterized by a membership function that assigns to each object a grade of membership ranging between zero and one, In (1968) Chang [2] introduced the definition of fuzzy topological spaces and extended in a straight forward manner some concepts of crisp topological spaces to fuzzy topological spaces. Later Lowen [10] (1976) redefined what is now known as stratified fuzzy topology.While Wong [13] in 1974 discussed and generalized some properties of fuzzy topological spaces. The note on paracompact space has been introduced by Ernest Michael [4] in (1953). Qutaiba Ead Hassanin [9] in (2005) introduced characterizations of fuzzy paracompactness. In this paper we introduce the concepts of fuzzy **ω**-open set and fuzzy **ω**-paracompact space and fuzzy **ω**-paracompact subspace on fuzzy topological space, and studied the relationships with fuzzy **ω**-separation axioms also we presented some types of fuzzy **ω**-continuous function and we give some characterization. And we obtained several properties.

2. Preliminaries

2.1 Definition [15]

Let X be a non empty set, and let I be the unit interval i.e I=[0,1], a fuzzy set in X is a function from X into the unit interval I ,$\tilde{A} : X \to [0,1]$ be a function A fuzzy set $\tilde{A}$ in X can be represented by the set of pairs: $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)): x \in X\}$ the family of all fuzzy sets in X is denoted by I^X.

2.2 Definition [6]

A fuzzy point x_r is a fuzzy set such that :

$\mu_{x_r}(y) = r > 0$ if $x = y$, $\forall\ y \in X$ and

$\mu_{x_r}(y) = 0$ if $x \neq y$, $\forall\ y \in X$, The family of all fuzzy points of $\tilde{A}$ will be denoted by $FP(\tilde{A})$

2.3 Definition [13]

A fuzzy point x_r is said to belong to a fuzzy set $\tilde{A}$ in X (denoted by : $x_r \in \tilde{A}$) if and only if $\mu_{x_r} \leq \mu_{\tilde{A}}(x)$

2.4 Proposition[13]

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets in X with membership functions $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$ respectively, then for all

$x \in X$: -

1. $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$.
2. $\tilde{A} = \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$.
3. $\tilde{C} = \tilde{A} \cap \tilde{B}$ if and only if $\mu_{\tilde{C}}(x) = \min\{ \mu_{\tilde{A}}(x) , \mu_{\tilde{B}}(x) \}$.
4. $\tilde{D} = \tilde{A} \cup \tilde{B}$ if and only if $\mu_{\tilde{D}}(x) = \max\{ \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$.

2.5 Definition [7]

The support of a fuzzy set $\widetilde{A}$, Supp ($\widetilde{A}$), is the crisp set of all $x \in X$, such that $\mu_{\tilde{A}}(x) > 0$.

2.6 Definition [2]

A fuzzy topology is a family $\widetilde{T}$ of fuzzy subsets in X, satisfying the following conditions:

(a) $\emptyset$, $1_X \in \widetilde{T}$.

(b) If $\widetilde{A}, \widetilde{B} \in \widetilde{T}$, then $\widetilde{A} \cap \tilde{B} \in \widetilde{T}$.

(c) If $\tilde{A}_i \in \widetilde{T}$, $\forall$ $i \in J$, where J is any index set, then $\bigcup_{i \in J} \tilde{A}_i \in \widetilde{T}$.

$\widetilde{T}$ is called fuzzy topology for $\tilde{X}$, and the pair (X, $\widetilde{T}$) is a fuzzy topological space. Every member of $\widetilde{T}$ is called open fuzzy set ($\widetilde{T}$-open fuzzy set). A fuzzy set $\tilde{C}$ in 1_X is called closed fuzzy set ($\widetilde{T}$-closed fuzzy set) if and only if its complement $\tilde{C}^c$ is $\widetilde{T}$-open fuzzy set.

2.7 Definition [8]

If $\tilde{B} \in (X,\tilde{T})$,the complement of $\tilde{B}$ referred to 1_X denoted by $\tilde{B}^c$, is defined by $\tilde{B}^c = 1_X - \tilde{B}$

2.8 Definition [1]

An fuzzy open set $\tilde{A}$ in a fuzzy topological space (X,$\tilde{T}$) is said to be clopen if its complement $1_X - \tilde{A}$ is an fuzzy open.

2.9 Definition [2]

A fuzzy set $\tilde{B}$ in a fuzzy topological space ($\tilde{A}$,$\tilde{T}$) is said to be a fuzzy neighborhood of a fuzzy point x_r in $\tilde{A}$ if there is a fuzzy open set $\tilde{G}$ in $\tilde{A}$ such that $\mu_{x_r}(x) \leq \mu_{\tilde{G}}(x) \leq \mu_{\tilde{B}}(x)$, $\forall x \in X$.

2.10 Definition [11]

Let (X, $\tilde{T}$) be a fuzzy topological space and $\tilde{B} \in P(1_X)$,then the relative fuzzy topology for $\tilde{B}$ defined by $\tilde{T}_B = \{ \tilde{B} \cap \tilde{G} : \tilde{G} \in \tilde{T} \}$.The corresponding ($\tilde{B}$, $\tilde{T}_B$) is called fuzzy subspace of (X, $\tilde{T}$) .

2.11 Definition [3]

Let (X,T̃) be a fuzzy topological space a family Z̃ of fuzzy sets is open cover of a fuzzy set $\tilde{A}$ if and only if $\tilde{A} \subseteq \cup\{\tilde{G} : \tilde{G} \in \tilde{Z}\}$ and each member of Z̃ is a fuzzy open set.

2.12 Definition [12]

Let B = $\{\tilde{B}_\alpha : \alpha \in \Lambda\}$, **C** = $\{\tilde{C}_\beta : \beta \in \Lambda\}$ ($\beta < \alpha$) be any two collection of fuzzy sets in (X,T̃) , then C is a refinement of B if for each $\beta \in \Lambda$ there exist $\alpha \in \Lambda$ such that $\mu_{\tilde{C}_\beta}(x) \leq \mu_{\tilde{B}_\alpha}(x)$.

2.13 Definition [5]

A fuzzy topological space (X,T̃) is said to be fuzzy connected, if it has no proper fuzzy clopen set. Otherwise it is called fuzzy disconnected.

2.14 Definition [15]

Let f be a function from universal set X to universal set Y. Let $\tilde{B}$ be a fuzzy subset in 1_Y with membership function $\mu_{\tilde{B}}(y)$. Then, the inverse of $\tilde{B}$, written as $f^{-1}(\tilde{B})$, is a fuzzy subset of 1_X whose membership function is defined by $\mu_{f^{-1}(\tilde{B})}(x) = \mu_{(\tilde{B})}(f(x))$, for all x in X.If $\tilde{A}$ be a fuzzy subset in 1_X with membership function $\mu_{\tilde{A}}(x)$. The image of $\tilde{A}$, written as $f(\tilde{A})$, is a fuzzy subset in 1_Y whose membership function is defined by

$\mu_{f(\tilde{A})}(y) = \begin{cases} \sup\limits_{z\in f^{-1}(y)}\{\mu_{\tilde{A}}(z)\} & if\ f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$, for all y in Y, where $f^{-1}(y)$, = {x| f(x) = y }.

From the above it is clear that:

1. If f is injective then $\mu_{f(\tilde{A})}(y) = \begin{cases} \sup\limits_{z\in f^{-1}(y)}\{\mu_{\tilde{A}}(z)\} & if\ f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$

2. If f is surjective then $\forall$ x ∈ X then $\mu_{\tilde{B}}(f(x)) = \mu_{\tilde{B}}(y)$ $\forall$ y ∈ Y, x ∈ $f^{-1}(y)$

3. If f is bijective then $\mu_{f(\tilde{A})}(y) = \mu_{\tilde{A}}(x)$ $\forall$ x = $f^{-1}(y)$ $\mu_{f^{-1}(\tilde{B})}(x) = \mu_{(\tilde{B})}(y)$, $\forall$ y ∈ Y, y = f(x)

3. Fuzzy *ω*-Open Set In Fuzzy Topological Space

***3.1 Definition* [14]**

A fuzzy set Ã in a fuzzy topological space (X,$\tilde{T}$) is called a fuzzy uncountable if and only if supp(Ã) is an uncountable subset of X

3.2 Definition

A fuzzy point x_r of a fuzzy topological space $(X,\tilde{T})$ is called a fuzzy condensation point of $\tilde{A} \subseteq 1_X$ if $\tilde{B} \cap \tilde{A}$ is fuzzy uncountable for each fuzzy open set $\tilde{B}$ containing x_r. And the set of all fuzzy condensation point of $\tilde{A}$ is denoted by Cond($\tilde{A}$)

3.3 Definition

A fuzzy subset $\tilde{A}$ in a fuzzy topological space $(X,\tilde{T})$ is called a fuzzy ω-closed set if it contains all its fuzzy condensation point. The complement fuzzy ω-closed sets are called fuzzy ω-open sets.

3.4 Theorem

A fuzzy subset $\tilde{G}$ of a fuzzy topological space $(X,\tilde{T})$ is fuzzy ω-open set if and only if $x_r \in \tilde{G}$ there exist a fuzzy open set $\tilde{U}$ such that $x_r \in \tilde{U}$ and $\tilde{U} - \tilde{G}$ is countable.

Proof: $\tilde{G}$ is fuzzy ω-open set if and only if $1_X - \tilde{G}$ is fuzzy ω-closed set, And $1_X - \tilde{G}$ is fuzzy ω-closed set if and only if $\text{Cond}(1_X - \tilde{G}) \subseteq 1_X - \tilde{G}$, And $\text{Cond}(1_X - \tilde{G}) \subseteq 1_X - \tilde{G}$ if and only if each $x_r \in \tilde{G}$, $x_r \notin \text{Cond}(1_X - \tilde{G})$, Thus $x_r \notin \text{Cond}(1_X - \tilde{G})$ there exist a fuzzy open set $\tilde{U}$ such that $x_r \in \tilde{U}$ and $\tilde{U} \cap (1_X - \tilde{G}) = \tilde{U} - \tilde{G}$ is countable

3.5 Theorem

A fuzzy subset $\tilde{G}$ of a fuzzy topological space $(X,\tilde{T})$ is ω-open set if and only if for each $x_r \in \tilde{G}$ there exist an fuzzy open set $\tilde{U}$ containing x_r and countable fuzzy subset $\tilde{C}$ of 1_X such that $\tilde{U} - \tilde{C} \subseteq \tilde{G}$.

Proof: ($\Longrightarrow$) suppose $\tilde{G}$ is fuzzy ω-open set and let $x_r \in \tilde{G}$, Then there exist a fuzzy open set $\tilde{U}$ and $x_r \in \tilde{U}$ and $\tilde{U} - \tilde{G}$ is countable, Set $\tilde{C} = \tilde{U} - \tilde{G}$, then $\tilde{C}$ is countable and $x_r \in \tilde{U} - \tilde{C} = \tilde{U} - (\tilde{U} - \tilde{G}) \subseteq \tilde{G}$

($\Longleftarrow$) let $x_r \in \tilde{G}$ then by assumption there exist fuzzy open set $\tilde{U}$ containing x_r and countable fuzzy subset $\tilde{C}$ of 1_X such that $\tilde{U} - \tilde{C} \subseteq \tilde{G}$, since $\tilde{U} - \tilde{G} \subseteq \tilde{C}$ then $\tilde{U} - \tilde{G}$ is countable, hence $\tilde{G}$ is fuzzy ω-open set

3.6 Proposition

Every fuzzy open set is fuzzy ω-open set

Proof: Let $\tilde{G}$ be fuzzy open set and $x_r \in \tilde{G}$, Set $\tilde{U} = \tilde{G}$, $\tilde{C} = \emptyset$, then $\tilde{U}$ is fuzzy open set and $\tilde{C}$ countable set, Such that $x_r \in \tilde{U} - \tilde{C} \subseteq \tilde{G}$, thus $\tilde{G}$ is fuzzy ω-open set

Remark
The converse of **(3.6 proposition)** is not true in general as the following examples show:-
3.8 Example: let X = { a , b , c } and $\tilde{A}$, $\tilde{B}$ are fuzzy subset in 1_X where
1_X= { (a , 1) , (b , 1) , (c , 1) }, $\tilde{A}$ = { (a , 0.6) , (b , 0.6) , (c , 0.7) }

$\tilde{B}$ = { (a , 0.5) , (b , 0.5) , (c , 0.4) }, Let $\tilde{T} = \{\emptyset , 1_X , \tilde{A}\}$ be a fuzzy topology on X, Then the fuzzy set $\tilde{B}$ is a fuzzy ω – open set but not fuzzy open set

3.9 Definition

Let $\tilde{B}$ be a fuzzy set in a fuzzy topological space $(X,\tilde{T})$ then, The **ω**-interior of $\tilde{B}$ is denoted by

ω-Int($\tilde{B}$) and defined by $\omega\text{-Int}(\tilde{B}) = \cup \{ \tilde{G} : \tilde{G} \text{ is a fuzzy } \omega\text{-open set in } 1_X , \tilde{G} \subseteq \tilde{B} \}$

3.10 Definition

Let $\tilde{B}$ be a fuzzy set in a fuzzy topological space $(X,\tilde{T})$ then, The **ω**-closure of $\tilde{B}$ is denoted by

ω-cl($\tilde{B}$) and defined by $\omega\text{-cl}(\tilde{B}) = \bigcap \{ \tilde{G} : \tilde{G}$ is a fuzzy ω-closed set in 1_X , $\tilde{B} \subseteq \tilde{G} \}$

3.11Theorem

Let $\tilde{A}$ be fuzzy subset of a fuzzy topological space $(X,\tilde{T})$ then $(\tilde{T}_{\tilde{A}})^{\omega} = \tilde{T}_{\tilde{A}}^{\omega}$

Proof: To prove $(\tilde{T}_{\tilde{A}})^{\omega} \subseteq \tilde{T}_{\tilde{A}}^{\omega}$, let $\tilde{B} \in (\tilde{T}_{\tilde{A}})^{\omega}$ and $x_r \in \tilde{B}$ **,** by **(3.5 Theorem)**, There exist fuzzy open set $\tilde{V}$ of $\tilde{T}_{\tilde{A}}$ and $\tilde{C}$ countable subset of $\tilde{T}_{\tilde{A}}$ such that $x_r \in \tilde{V} - \tilde{C} \subseteq \tilde{B}$, choose $\tilde{U} \in \tilde{T}$ such that $\tilde{V} = \tilde{U} \cap \tilde{A}$, Then $\tilde{U} - \tilde{C} \in \tilde{T}^{\omega}$, $x_r \in \tilde{U} - \tilde{C}$ and $\tilde{U} - \tilde{C} \cap \tilde{A} = \tilde{V} - \tilde{C} \subseteq \tilde{B}$, Therefore $\tilde{B} \in \tilde{T}_{\tilde{A}}^{\omega}$, To prove $\tilde{T}_{\tilde{A}}^{\omega} \subseteq (\tilde{T}_{\tilde{A}})^{\omega}$, let $\tilde{G} \in \tilde{T}_{\tilde{A}}^{\omega}$ then there exist $\tilde{H} \in \tilde{T}^{\omega}$ such that $\tilde{G} = \tilde{H} \cap \tilde{A}$ if $x_r \in \tilde{G}$ then $x_r \in \tilde{H}$ and there exist fuzzy open set $\tilde{U}$ of $\tilde{T}$ and $\tilde{D}$ countable subset of $\tilde{T}$ such that $x_r \in \tilde{U} - \tilde{D} \subseteq \tilde{H}$, We put $\tilde{V} = \tilde{U} \cap \tilde{A}$, then $\tilde{V} \in \tilde{T}_{\tilde{A}}$ and $x_r \in \tilde{V} - \tilde{D} \subseteq \tilde{G}$, It follows that $\tilde{G} \in (\tilde{T}_{\tilde{A}})^{\omega}$

3.12 Definition

The fuzzy family $\{\tilde{B}_{\alpha} : \alpha \in \Lambda\}$ of subset of a fuzzy topological space $(X,\tilde{T})$ is called

1- Fuzzy ω-locally finite if for each $x_r \in 1_X$ there exist an fuzzy ω-open set $\tilde{G}$ containing x_r such that the set $\{\tilde{G} \cap \tilde{B}_{\alpha} \neq \emptyset : \alpha \in \Lambda\}$ is finite
2- Fuzzy ω-discrete if for each $x_r \in 1_X$ there exist an fuzzy ω-open set $\tilde{G}$ containing x_r such that the set $\{\tilde{G} \cap \tilde{B}_{\alpha} \neq \emptyset : \alpha \in \Lambda\}$ has at most one member

3.13 proposition

Every fuzzy locally finite (resp.fuzzy discrete) family of any fuzzy topological space $(X,\tilde{T})$ is fuzzy ω-locally finite (resp.fuzzy ω-discrete)

Proof:Follows from the fact (every fuzzy open set is fuzzy ω-open set)

3.14 Definition

A fuzzy topological space $(X,\tilde{T})$ is called a fuzzy anti-locally-countable if each nonempty fuzzy open subset of 1_X is uncountable.

3.15 Definition

A fuzzy topological space $(X,\tilde{T})$ is said to be

1- ω-$\tilde{T}_0$ if for each pair of distinct fuzzy point x_r and y_t of 1_X there exist fuzzy ω-open set $\tilde{G}$ such that either $x_r \in \tilde{G}$ and $y_t \notin \tilde{G}$ or $y_t \in \tilde{G}$ and $x_r \notin \tilde{G}$.

2- ω-$\tilde{T}_1$ if for each pair of distinct fuzzy point x_r and y_t of 1_X there exist fuzzy ω-open sets $\tilde{G}$ and $\tilde{H}$ such that $x_r \in \tilde{G}$ and $y_t \notin \tilde{G}$ and $y_t \in \tilde{H}$ and $x_r \notin \tilde{H}$.

3- ω-$\tilde{T}_2$ if for each pair of distinct fuzzy point x_r and y_t of 1_X there exist disjoint fuzzy ω-open sets $\tilde{G}$ and $\tilde{H}$ containing x_r and y_t respectively .

3.18 Definition

A fuzzy topological space $(X,\tilde{T})$ is called a fuzzy ω-regular space if for each fuzzy ω-closed subset $\tilde{B}$ of 1_X and a fuzzy point x_r in 1_X such that $x_r \notin \tilde{B}$, there exist disjoint fuzzy ω-open sets $\widetilde{U}$ and $\tilde{V}$ containing x_r and $\tilde{B}$ respectively

3.19 Definition

A fuzzy topological space $(X,\tilde{T})$ is called a fuzzy ω-Normal space if for each pair of disjoint fuzzy ω-closed sets $\tilde{A}$ and $\tilde{B}$ in 1_X there exist disjoint fuzzy ω-open sets $\widetilde{U}$ and $\tilde{V}$ containing $\tilde{A}$ and $\tilde{B}$ respectively

3.20 Theorem

A fuzzy topological space $(X,\tilde{T})$ is fuzzy ω-Normal if for each pair of fuzzy ω-open sets $\tilde{G}$ and $\widetilde{H}$ in 1_X such that $1_X = \tilde{G} \cup \widetilde{H}$ there are fuzzy ω-closed sets $\widetilde{U}$ and $\tilde{V}$ contained in $\tilde{G}$ and $\widetilde{H}$ respectively such that $1_X = \widetilde{U} \cup \tilde{V}$

Proof: Obvious

3.21 Theorem

Every fuzzy ω-closed subspace of fuzzy ω-Normal space is fuzzy ω-Normal space.

Proof: Obvious

3.22 Proposition

Every fuzzy ω-regular space is fuzzy ω-$\tilde{T}_2$ space

Proof : Let x_r and y_t be pair of fuzzy distinct points in a fuzzy ω-regular space 1_X,Then x_r is a fuzzy point of 1_X which is not in the fuzzy ω-closed subset $\{y_t\}$ of 1_X so by fuzzy ω-regularity of 1_X there exist fuzzy disjoint ω-open sets $\widetilde{U}$ and $\tilde{V}$ containing x_r and y_t respectively, Hence 1_X is fuzzy ω-$\tilde{T}_2$ space.

3.23 Proposition

If $(X,\tilde{T})$ is fuzzy anti-locally countable topological space and $\tilde{A}$ fuzzy ω-open subset of 1_X then ω-cl$(\tilde{A})$ = cl$(\tilde{A})$.

Proof: Clearly ω-cl$(\tilde{A}) \subseteq$ cl$(\tilde{A})$. On the other hand, let $x_r \in$ cl$(\tilde{A})$ and $\tilde{G}$ be an fuzzy ω-open subset containing x_r then by **(3.5 Theorem)**There exist an fuzzy open set $\widetilde{H}$ containing x_r and countable set $\tilde{C}$ such that $\widetilde{H} - \tilde{C} \subseteq \tilde{G}$, thus $(\widetilde{H} - \tilde{C}) \cap \tilde{A} \subseteq \tilde{G} \cap \tilde{A}$ and so $\widetilde{H} \cap \tilde{A} - \tilde{C} \subseteq \tilde{G} \cap \tilde{A}$. As $x_r \in \widetilde{H}$ and $x_r \in$ cl$(\tilde{A})$, $\widetilde{H} \cap \tilde{A} \neq \emptyset$. And then as $\widetilde{H}$ and $\tilde{A}$ are fuzzy ω-open sets , $\widetilde{H} \cap \tilde{A}$ is fuzzy ω-open set and as 1_X is fuzzy anti-locally countable , $\widetilde{H} \cap \tilde{A}$ is fuzzy uncountable and so is $(\widetilde{H} \cap \tilde{A}) - \tilde{C}$. Thus $\tilde{G} \cap \tilde{A}$ is uncountable therefore $\tilde{G} \cap \tilde{A} \neq \emptyset$ which means that $x_r \in$ ω-cl$(\tilde{A})$

3.24 Corollary

If $(X,\tilde{T})$ is fuzzy anti-locally countable topological space and $\tilde{A}$ fuzzy ω-open subset of 1_X then ω-Int$(\tilde{A})$ = Int$(\tilde{A})$.

Proof: Obvious

3.25 Theorem

If a fuzzy topological space (X,$\tilde{T}$) is fuzzy anti-locally-countable space then every fuzzy ω-Normal space is fuzzy Normal space.

Proof: Let $\tilde{F}$ and $\tilde{H}$ be two disjoint fuzzy closed subset of fuzzy anti-locally-countable ω-Normal space 1_X , then there are fuzzy ω- open sets $\tilde{U}$ and $\tilde{V}$ such that $\tilde{F} \subseteq \tilde{U}$ and $\tilde{H} \subseteq \tilde{V}$ and $\tilde{U} \cap \tilde{V} = \emptyset$ this implies that ω-cl($\tilde{U}$) $\cap \tilde{V} = \emptyset$ and $\tilde{U} \cap$ ω-cl($\tilde{V}$) = $\emptyset$ since 1_X is fuzzy anti-locally-countable so by **(3.23 *Proposition)*** we get cl($\tilde{U}$) $\cap \tilde{V} = \emptyset$ and $\tilde{U} \cap$ cl($\tilde{V}$) = $\emptyset$ since Int(cl($\tilde{U}$)) ⊆ cl($\tilde{U}$) and Int(cl($\tilde{V}$)) ⊆ cl($\tilde{V}$) then Int(cl($\tilde{U}$)) $\cap \tilde{V} = \emptyset$ and $\tilde{U} \cap$ Int(cl($\tilde{V}$)) = $\emptyset$, And this implies that Int(cl($\tilde{U}$)) ∩ cl($\tilde{V}$) = $\emptyset$ and Int(cl($\tilde{V}$)) ∩ cl($\tilde{U}$) = $\emptyset$ thus Int(cl($\tilde{U}$)) ∩ Int(cl($\tilde{V}$)) = $\emptyset$, hence Int(cl($\tilde{U}$)) and Int(cl($\tilde{V}$)) are disjoint fuzzy open sets in 1_X containing $\tilde{F}$ and $\tilde{H}$ respectively hence (X,$\tilde{T}$) is fuzzy Normal space

3.26 Definition

Two fuzzy families $\{\tilde{A}_\lambda\}_{\lambda\in\Lambda}$ and $\{\tilde{B}_\lambda\}_{\lambda\in\Lambda}$ of subset of a fuzzy space 1_X are said to be similar if for every finite subset Δ of Λ the fuzzy sets $\bigcap_{\lambda\in\Delta}\tilde{A}_\lambda$ and $\bigcap_{\lambda\in\Delta}\tilde{B}_\lambda$ are either empty or nonempty.

3.27 Definition

Let (X,$\tilde{T}$) be a fuzzy topological space a family W of fuzzy sets is ω-open cover of a fuzzy set $\tilde{A}$ if and only if $\tilde{A} \subseteq \cup\{\tilde{G} : \tilde{G} \in W\}$ and each member of W is a fuzzy ω-open set. A sub cover of W is a sub family which is also cover.

3.28 Definition

A function f: (X,$\tilde{T}$) ⟶ (Y,$\tilde{\sigma}$) is said to be fuzzy ω-continuous at a fuzzy point $x_r \in 1_X$ if for each fuzzy open subset $\tilde{V}$ in 1_Y containing $f(x_r)$ there exists an fuzzy ω-open subset $\tilde{U}$ of 1_X that containing x_r such that $f(\tilde{U}) \subseteq \tilde{V}$ and f is called fuzzy ω-continuous if it is fuzzy ω-continuous at each fuzzy point

3.29 Definition

A function f: (X,$\tilde{T}$) ⟶ (Y,$\tilde{\sigma}$) is said to be

1- fuzzy pre- ω-open, if image of each fuzzy ω-open set is fuzzy ω-open
2- fuzzy ω-irresolute if $f^{-1}(\tilde{F})$ is fuzzy ω-closed in 1_X for each fuzzy ω-closed subset $\tilde{F}$ of 1_Y

4. Fuzzy *ω*-Paracompact space

4.1 Definition

A fuzzy topological space (X,$\tilde{T}$) is said to be :

Fuzzy paracompact space if for each fuzzy open covering of 1_X has a fuzzy locally finite open refinement. [9]

Fuzzy **ω** -paracompact space if for each fuzzy ω-open covering of 1_X has a fuzzy ω-locally finite ω-open refinement

4.2 Propositions

If a fuzzy topological space $(X,\tilde{T})$ is a fuzzy locally countable space then $(X,\tilde{T}^{\omega})$ is fuzzy paracompact space.

Proof : Follows from the fact every fuzzy discrete space is fuzzy locally finite and A fuzzy topological space $(X,\tilde{T})$ is fuzzy locally countable if and only if $\tilde{T}^{\omega} = \tilde{T}_{dis}$

4.3 Propositions

If a fuzzy covering $\{\tilde{U}_{\lambda}\}_{\lambda\in\Lambda}$ of a fuzzy topological space $(X,\tilde{T})$ has a fuzzy locally-finite (fuzzy ω-locally finite) ω-open refinement then there exist a fuzzy locally-finite (fuzzy ω-locally finite) ω-open covering $\{\tilde{G}_{\lambda}\}_{\lambda\in\Lambda}$ of 1_X such that $\tilde{G}_{\lambda} \subseteq \tilde{U}_{\lambda}$ for each $\lambda \in \Lambda$.

Proof: Let $\{\tilde{V}_{\gamma}\}_{\gamma\in\Gamma}$ be the fuzzy locally-finite (fuzzy ω-locally finite) ω-open refinement $\{\tilde{U}_{\lambda}\}_{\lambda\in\Lambda}$ therefore there exist a function $\beta:\Gamma\to\Lambda$ such that $\tilde{V}_{\gamma} \subseteq \tilde{U}_{\beta(\gamma)=\lambda}$ for each $\gamma\in\Gamma$. Let $\tilde{G}_{\lambda} = \bigcup_{\gamma\in\Gamma,\beta(\gamma)=\lambda} \tilde{V}_{\gamma}$ then the family $\{\tilde{G}_{\lambda}\}_{\lambda\in\Lambda}$ is fuzzy ω-open covering of 1_X with the property that $\tilde{G}_{\lambda} \subseteq \tilde{U}_{\lambda}$ for each $\lambda \in \Lambda$. Also $\{\tilde{G}_{\lambda}\}_{\lambda\in\Lambda}$ is fuzzy locally-finite (fuzzy ω-locally finite).

If $x_r \in 1_X$ there is an fuzzy open (ω-open) set $\tilde{W}$ containing x_r such that the set
$\Gamma_0=\{\gamma \in \Gamma : \tilde{W} \cap \tilde{V}_{\lambda} \neq \emptyset\}$ is finite. But since $\tilde{W} \cap \tilde{G}_{\lambda} \neq \emptyset$
If and only if $\lambda=\beta(\gamma)$ for some $\gamma \in \Gamma_0$ so the set $\{\lambda \in \Lambda : \tilde{W} \cap \tilde{G}_{\lambda} \neq \emptyset\}$ is finite

4.4 Corollary

A fuzzy topological space $(X,\tilde{T})$ is fuzzy ω-paracompact space if and only if for every fuzzy ω-open covering $\{\tilde{U}_{\lambda}\}_{\lambda\in\Lambda}$ of 1_X there exist an fuzzy ω-locally finite ω-open covering $\{\tilde{V}_{\lambda}\}_{\lambda\in\Lambda}$ of 1_X such that $\tilde{V}_{\lambda} \subseteq \tilde{U}_{\lambda}$ for each $\lambda \in \Lambda$.

4.5 Propositions

Let $(X,\tilde{T})$ be a fuzzy ω-paracompact space and let $\tilde{H}$ be a fuzzy subset of 1_X and $\tilde{F}$ be an fuzzy ω-closed of 1_X which disjoint from $\tilde{H}$, if for every $x_r \in \tilde{F}$ there exist disjoint fuzzy ω-open set $\tilde{U}_{x_r}$ and $\tilde{V}_H$ containing x_r and $\tilde{H}$ respectively then there are disjoint ω-open set $\tilde{U}$ and $\tilde{V}$ containing $\tilde{F}$ and $\tilde{H}$ respectively.

Proof : Consider the fuzzy ω-open covering $\{\tilde{U}_{x_r}\}_{x_r\in\tilde{F}} \cup \{1_X - \tilde{F}\}$ of an fuzzy ω-paracompact space $(X,\tilde{T})$ then by ***(4.4 Corollary)*** there exist an fuzzy ω-locally finite ω-open covering $\{\tilde{G}_{x_r}\}_{x_r\in\tilde{F}} \cup \tilde{G}$ of 1_X such that $\tilde{G} \subseteq 1_X - \tilde{F}$ and $\tilde{G}_{x_r} \subseteq \tilde{U}_{x_r}$ for each $x_r \in \tilde{F}$. if $\tilde{U}_{x_r} \cap \tilde{V}_H = \emptyset$ then $\tilde{G}_{x_r} \cap \tilde{V}_H = \emptyset$ so ω-cl$(\tilde{G}_{x_r}) \cap \tilde{V}_H = \emptyset$ for each $x_r \in \tilde{F}$ then the fuzzy sets $\tilde{U} = \bigcup_{x_r\in\tilde{F}} \tilde{G}_{x_r}$ and $\tilde{V} = 1_X - \bigcup_{x_r\in\tilde{F}} \omega - \mathrm{cl}(\tilde{G}_{x_r})$ are the required ω-open sets of 1_X

4.6 Propositions

Each fuzzy ω-paracompact fuzzy ω-regular (resp. fuzzy ω-$\tilde{T}_2$) space is fuzzy ω-Normal space.

Proof: Let $(X,\tilde{T})$ be an fuzzy ω-paracompact ω-$\tilde{T}_2$ space and let x_r be any fuzzy point in 1_X which is not in an arbitrary fuzzy ω-closed set $\tilde{F}$ of 1_X therefore for each $y_t \in \tilde{F}$ there are disjoint fuzzy ω-open sets $\tilde{U}_{y_t}$ and $\tilde{V}_{x_r}$ containing y_t and $\{x_r\}$ respectively so by ***(4.5 Propositions)*** there exist disjoint fuzzy ω-open sets $\tilde{U}$ and $\tilde{V}$ containing $\tilde{F}$ and x_r respectively this shows that $(X,\tilde{T})$ is fuzzy ω-regular space, thus we have $(X,\tilde{T})$ fuzzy ω-paracompact fuzzy ω-regular. Let $\tilde{F}$ and $\widetilde{H}$ be any fuzzy two disjoint fuzzy ω-closed subset of 1_X , since $\widetilde{H}$ is fuzzy ω-closed so by fuzzy ω-regularity of 1_X for each $y_t \in \tilde{F}$ there exist disjoint fuzzy ω-open sets $\widetilde{U}_{y_t}$ and $\tilde{V}_H$ containing y_t and $\widetilde{H}$ respectively therefore By ***(4.5 Propositions)*** there exist disjoint fuzzy ω-open sets $\widetilde{U}$ and $\tilde{V}$ containing $\tilde{F}$ and $\widetilde{H}$ this showed that $(X,\tilde{T})$ is fuzzy ω-Normal space

4.7 Corollary

Every fuzzy ω-paracompact $\tilde{T}_2$ space is an fuzzy ω-Normal space.

Proof: Follows by the fact(Every fuzzy $\tilde{T}_2$ space is an fuzzy ω-$\tilde{T}_2$ space) and ***(4.5 Propositions)***

4.8 Proposition

If $(X,\tilde{T})$ is an fuzzy anti-locally countable fuzzy ω-paracompact $\tilde{T}_2$-(resp. ω-$\tilde{T}_2$, ω-regular , ω-Normal) space then it is fuzzy paracompact.

Proof: From ***4.6 Propositions*** **and** ***4.7 Corollary*** we have only to assume that 1_X is an fuzzy ω-paracompact ω-Normal space. Therefore by ***3.24 Corollary and 3.25 Theorem*** $(X,\tilde{T})$ is fuzzy paracompact

4.9 Theorem

A fuzzy topological space $(X,\tilde{T})$ is fuzzy ω-paracompact ω-Normal space if and only if every fuzzy ω-open covering of 1_X has a fuzzy ω-locally finite ω-closed refinement.

Proof: (⟹) Let $\{\widetilde{U}_\lambda\}_{\lambda\in\Lambda}$ be a fuzzy ω-open covering of a fuzzy ω-paracompact ω-Normal space $(X,\tilde{T})$ so by ***(4.4 Corollary)*** there exist an fuzzy ω-locally finite ω-open covering $\{\tilde{V}_\lambda\}_{\lambda\in\Lambda}$ of 1_X such that $\tilde{V}_\lambda \subseteq \widetilde{U}_\lambda$ for each $\lambda \in \Lambda$, since $(X,\tilde{T})$ is fuzzy ω-Normal space then there exist an fuzzy ω-locally finite ω-closed refinement of $\{\tilde{V}_\lambda\}_{\lambda\in\Lambda}$ which also fuzzy covers of 1_X

(⟸) Let $(X,\tilde{T})$ be a fuzzy topological space with the property that every fuzzy ω-open covering of it has fuzzy ω-locally finite ω-closed refinement, thus $(X,\tilde{T})$ is fuzzy ω-Normal space, it remains only to show $(X,\tilde{T})$ is fuzzy ω-paracompact. For this let $\{\widetilde{W}_\lambda\}_{\lambda\in\Lambda}$ be a fuzzy ω-open covering of 1_X and $\{\tilde{F}_\lambda\}_{\lambda\in\Gamma}$ be fuzzy ω-locally finite ω-closed refinement of $\{\widetilde{W}_\lambda\}_{\lambda\in\Lambda}$ therefore for each $x_r \in 1_X$ there exist fuzzy ω-open set $\widetilde{U}_{x_r}$ containing x_r such that the fuzzy set $\{ \gamma \in \Gamma : \widetilde{U}_{x_r} \cap \tilde{F}_\gamma \neq \emptyset \}$ is finite. Consider $\{\tilde{E}_\upsilon\}_{\upsilon\in\vartheta}$ is fuzzy ω-locally finite ω-closed refinement of the fuzzy ω-open covering $\{\widetilde{U}_{x_r}\}_{x_r\in 1_X}$ of 1_X then for each $\upsilon \in \vartheta$ the fuzzy set $\{ \gamma \in \Gamma : \tilde{E}_\upsilon \cap \tilde{F}_\gamma \neq \emptyset \}$ is finite so there exist fuzzy ω-locally finite family $\{\tilde{G}_\gamma : \gamma \in \Gamma\}$of fuzzy ω-open set of 1_X such that $\tilde{F}_\gamma \subseteq \tilde{G}_\gamma$for each $\gamma \in \Gamma$ which also fuzzy cover of 1_X , since $\{\tilde{F}_\lambda\}_{\lambda\in\Gamma}$ is fuzzy refinement of $\{\widetilde{W}_\lambda\}_{\lambda\in\Lambda}$ so for each $\gamma \in \Gamma$ there is $\lambda(\gamma) \in \Lambda$ such that $\tilde{F}_\lambda \subseteq \widetilde{W}_{\lambda(\gamma)}$ therefore $\{\tilde{G}_\gamma \cap \widetilde{W}_{\lambda(\gamma)} : \gamma \in \Gamma\}$ is fuzzy ω-locally finite ω-open refinement of $\{\widetilde{W}_\lambda\}_{\lambda\in\Lambda}$ Hence $(X,\tilde{T})$ fuzzy ω-paracompact space

4.10 Proposition

Let $\{\widetilde{H}_\lambda\}_{\lambda\in\Lambda}$ be an fuzzy ω-locally finite family of fuzzy ω-closed sets of fuzzy ω-paracompact ω-Normal space $(X,\tilde{T})$ then there exists an fuzzy ω-locally finite family $\{\widetilde{U}_\lambda\}_{\lambda\in\Lambda}$ of fuzzy ω-open subset of 1_X such that $\widetilde{H}_\lambda \subseteq \widetilde{U}_\lambda$ for each $\lambda \in \Lambda$ and the fuzzy families $\{\widetilde{H}_\lambda\}_{\lambda\in\Lambda}$ and $\{\omega\text{-}cl(\widetilde{U}_\lambda)\}_{\lambda\in\Lambda}$ are similar

Proof: Let $\{\widetilde{H}_\lambda\}_{\lambda\in\Lambda}$ be fuzzy ω-locally finite family of fuzzy ω-closed sets of fuzzy ω-paracompact ω-Normal space $(X,\tilde{T})$. for each $x_r \in 1_X$ there exist fuzzy ω-open set $\tilde{G}_{x_r}$ containing x_r such that $\tilde{G}_{x_r}$

intersects only finite number of $\widetilde{H}_\lambda$ and clearly the fuzzy family $\{\tilde{G}_{x_r}\}_{x_r\in 1_X}$ forms fuzzy ω-open covering of 1_X, therefore by ***(4.9 Theorem)*** $\{\tilde{G}_{x_r}\}_{x_r\in 1_X}$ has a fuzzy ω-locally finite ω-closed refinement $\{\tilde{F}_\gamma\}_{\gamma\in\Gamma}$ and $\tilde{F}_\gamma$ intersects only finite number of $\{\widetilde{H}_\lambda\}_{\lambda\in\Lambda}$ for each $\gamma \in \Gamma$,
so there exist fuzzy ω-locally finite family $\{\tilde{V}_\lambda : \lambda \in \Lambda\}$of fuzzy ω-open set of 1_X such that $\widetilde{H}_\lambda \subseteq \tilde{V}_\lambda$ for each $\lambda \in \Lambda$, hence there exist an fuzzy ω-locally finite family $\{\widetilde{U}_\lambda : \lambda \in \Lambda\}$ of fuzzy ω-open sets such that $\widetilde{H}_\lambda \subseteq \widetilde{U}_\lambda \subseteq$ ω-cl($\widetilde{U}_\lambda$) $\subseteq \tilde{V}_\lambda$ for each $\lambda \in \Lambda$ and the fuzzy families $\{\widetilde{H}_\lambda\}_{\lambda\in\Lambda}$ and $\{\omega\text{-}cl(\widetilde{U}_\lambda)\}_{\lambda\in\Lambda}$ are similar

5. Fuzzy ω-Paracompact subset

5.1 Proposition

Every fuzzy ω-paracompact subset of a fuzzy topological space (X,T̃) is fuzzy ω-paracompact subspace.

Proof: Let $\widetilde{H}$ be a fuzzy ω-paracompact subset of a fuzzy topological space (X,T̃) and let $\{\widetilde{U}_\lambda\}_{\lambda\in\Lambda}$ be fuzzy covering of $\widetilde{H}$ by fuzzy ω-open subset of $\widetilde{H}$. By **(*3.11 Theorem*)** there exist an fuzzy ω-open subset $\tilde{V}_\lambda$ of 1_X such that $\widetilde{U}_\lambda = \tilde{V}_\lambda \cap \widetilde{H}$ for each $\lambda \in \Lambda$, then $\{\tilde{V}_\lambda\}_{\lambda\in\Lambda}$ is fuzzy covering of $\widetilde{H}$ by fuzzy ω-open subset of 1_X. so by hypothesis there exist fuzzy ω-locally-finite ω-open refinement $\{\tilde{G}_\gamma\}_{\gamma\in\Gamma}$ of the fuzzy family $\{\tilde{V}_\lambda\}_{\lambda\in\Lambda}$ which covers $\widetilde{H}$ also. Therefore $\{\tilde{G}_\gamma \cap \widetilde{H}\}_{\gamma\in\Gamma}$ is fuzzy ω-locally-finite ω-open refinement of $\{\widetilde{U}_\lambda\}_{\lambda\in\Lambda}$ in $\widetilde{H}$, Thus $\widetilde{H}$ is fuzzy ω-paracompact subspace of (X,T̃)

5.2 Proposition

An fuzzy ω-closet subset of fuzzy ω-paracompact space is fuzzy ω-paracompact subspace.

Proof: Let $\tilde{F}$ be fuzzy ω-closet subset of fuzzy ω-paracompact space 1_X and let $\{\widetilde{U}_\lambda\}_{\lambda\in\Lambda}$ be be fuzzy covering of $\tilde{F}$ by fuzzy ω-open set of 1_X, then $\{\widetilde{U}_\lambda\}_{\lambda\in\Lambda} \cup \{1_X - \tilde{F}\}$ is fuzzy covering of 1_X then by hypothesis and in virtue of **(*4.4 Corollary)*** there exist an fuzzy ω-locally-finite ω-open covering $\{\tilde{G}_\lambda\}_{\lambda\in\Lambda} \cup \tilde{G}$ of 1_X such that $\tilde{G} \subseteq 1_X - \tilde{F}$ and $\tilde{G}_\lambda \subseteq \widetilde{U}_\lambda$ for each $\lambda \in \Lambda$ therefore $\{\tilde{G}_\lambda\}_{\lambda\in\Lambda}$ is fuzzy ω-locally-finite ω-open refinement of $\{\widetilde{U}_\lambda\}_{\lambda\in\Lambda}$ which cover $\tilde{F}$ this show that $\tilde{F}$ fuzzy ω-paracompact subset to 1_X and by ***(5.1 Proposition)*** we obtain $\tilde{F}$ fuzzy ω-paracompact subspace

5.3 Proposition

If a fuzzy topological space (X,T̃) is fuzzy ω-$\tilde{T}_2$ space and has a fuzzy subset $\tilde{F}$ which is fuzzy ω-paracompact subset to 1_X then for each $x_r \in 1_X - \tilde{F}$ there exist two disjoint fuzzy ω-open sets of 1_X containing x_r and $\tilde{F}$

Proof: Let $\tilde{F}$ be fuzzy ω-paracompact subset of fuzzy ω-$\tilde{T}_2$ space (X,T̃) and let x_r be any fuzzy point of $1_X - \tilde{F}$ then for each $y_t \in \tilde{F}$ there exist fuzzy ω-open sets $\widetilde{U}_{y_t}$ and $\tilde{V}_{x_r}$ such that $y_t \in \widetilde{U}_{y_t}$ and $x_r \in \tilde{V}_{x_r}$ and $\widetilde{U}_{y_t} \cap \tilde{V}_{x_r} = \emptyset$ this implies that ω-cl($\widetilde{U}_{y_t}$) $\cap \tilde{V}_{x_r} = \emptyset$ hence $x_r \notin$ ω-cl($\widetilde{U}_{y_t}$) for each $y_t \in \tilde{F}$. Now $\{\widetilde{U}_{y_t}\}_{y_t\in\tilde{F}}$ is fuzzy cover of $\tilde{F}$ by fuzzy ω-open subset of 1_X thus by hypothesis and in virtue of **(*4.4 Corollary)*** there exist an fuzzy ω-locally finite covering $\{\tilde{G}_{y_t}\}_{y_t\in\tilde{F}}$ of $\tilde{F}$ such that for each $y_t \in \tilde{F}$, $\tilde{G}_{y_t}$ is fuzzy ω-open set in 1_X and $\tilde{G}_{y_t} \subseteq \widetilde{U}_{y_t}$ therefore $x_r \notin$ ω-cl($\tilde{G}_{y_t}$) for each $y_t \in \tilde{F}$. Hence $\widetilde{U} = \bigcup_{y_t\in\tilde{F}} \tilde{G}_{y_t}$ and $\tilde{V} = 1_X - \bigcup_{y_t\in\tilde{F}} \omega-\mathrm{cl}(\tilde{G}_{y_t})$ Therefore there exist two disjoint fuzzy ω-open sets of 1_X containing x_r and $\tilde{F}$

5.4 Corollary

If $\tilde{F}$ is fuzzy ω-paracompact subset of a fuzzy topological ω-$\tilde{T}_2$ space (X,T̃) then $\tilde{F}$ is fuzzy ω-Normal subspace of 1_X.

Proof : Obvious

5.5 Proposition

If a fuzzy topological space (X,T̃) is fuzzy ω-regular space and $\tilde{F}$ is fuzzy subset of 1_X which is fuzzy ω-paracompact subset of 1_X then for each fuzzy ω-open set $\tilde{U}$ containing $\tilde{F}$ there exist fuzzy ω-closed set $\tilde{H}$ containing $\tilde{F}$ and it is contained in $\tilde{U}$ furthermore $\tilde{F}$ is is fuzzy ω-Normal subspace of 1_X.

Proof: Since a fuzzy topological space (X,T̃) is fuzzy ω-regular space so by ***(3.22 Proposition)*** and ***(5.3 Proposition)*** $\tilde{F}$ fuzzy ω-closed subset of 1_X. And by ***(5.4 Corollary)*** it is fuzzy ω-Normal subspace of 1_X, therefore for each $x_r \in \tilde{F}$ there exist fuzzy ω-open set $\tilde{U}_{x_r}$ such that $x_r \in \tilde{U}_{x_r} \subseteq$ ω-cl($\tilde{U}_{x_r}$) $\subseteq \tilde{U}$ since $\tilde{F}$ is fuzzy ω-paracompact subset of 1_X so there exist an fuzzy ω-locally finite family $\{\tilde{G}_\gamma\}_{\gamma\in\Gamma}$ of $\tilde{F}$ by fuzzy ω-open sets of 1_X which refines $\{\tilde{U}_{x_r}\}_{x_r\in\tilde{F}}$ and covers $\tilde{F}$ therefore $\tilde{H}$= $\bigcup_{\gamma\in\Gamma} \omega-\mathrm{cl}(\tilde{G}_\gamma)$ is the required fuzzy ω-closed set

5.6 Theorem

Let (X,T̃) be a fuzzy ω-disconnected space then the statements are equivalent:

1- (X,T̃) is fuzzy ω-paracompact space
2- Every fuzzy proper ω-closed subset of 1_X is fuzzy ω-paracompact subset of 1_X
3- Every fuzzy proper ω-closed subset of 1_X is fuzzy ω-paracompact subspace
4- Every fuzzy proper ω-clopen subset of 1_X is fuzzy ω-paracompact
5- There exist a fuzzy proper ω-clopen subset $\tilde{F}$ of 1_X such that both $\tilde{F}$ and 1_X - $\tilde{F}$ are fuzzy ω-paracompact.

Proof :(1⟹2) Follows from ***5.2 Proposition***

(2⟹3) Follows from ***5.1 Proposition***
(3⟹4) Obvious
(4⟹5) Clear.
(5⟹1) let (X,T̃) be a fuzzy topological space contains a fuzzy proper ω-clopen subset $\tilde{F}$ in which both $\tilde{F}$ and 1_X - $\tilde{F}$ are fuzzy ω-paracompact and let $\{\tilde{G}_\gamma\}_{\gamma\in\Gamma}$ be any fuzzy ω-open cover of 1_X , then $\{\tilde{F} \cap \tilde{G}_\gamma\}_{\gamma\in\Gamma}$ and $\{1_X - \tilde{F} \cap \tilde{G}_\gamma\}_{\gamma\in\Gamma}$ Covering $\tilde{F}$ and 1_X - $\tilde{F}$ respectively therefore there exist fuzzy ω-locally finite refinement $\{\tilde{V}_\lambda\}_{\lambda\in\Lambda}$ and $\{\tilde{V}_\upsilon\}_{\upsilon\in\vartheta}$ of $\{\tilde{F} \cap \tilde{G}_\gamma\}_{\gamma\in\Gamma}$ and $\{1_X - \tilde{F} \cap \tilde{G}_\gamma\}_{\gamma\in\Gamma}$ Covering $\tilde{F}$ and 1_X - $\tilde{F}$ respectively such that $\tilde{V}_\lambda$ is fuzzy ω-openset in $\tilde{F}$ for each $\lambda \in \Lambda$ and $\tilde{V}_\upsilon$ is fuzzy ω-open set in 1_X - $\tilde{F}$ for each $\upsilon \in \vartheta$,
then both $\tilde{V}_\lambda$ and $\tilde{V}_\upsilon$ are fuzzy ω-open sets in 1_X for each $\lambda \in \Lambda$ and $\upsilon \in \vartheta$
Therefore $\{\tilde{V}_\beta\}_{\beta\in\Lambda\cup\vartheta}$ is fuzzy ω-locally finite ω-open refinement of $\{\tilde{G}_\gamma\}_{\gamma\in\Gamma}$ which covers 1_X , hence (X,T̃) is fuzzy ω-paracompact space

Remark

In the above theorem if (X,T̃) is fuzzy ω-connected space then the only fuzzy ω-clopen subset of 1_X are fuzzy empty set and 1_X itself so the condition that (X,T̃) is fuzzy ω-disconnected space is essential.

5.8 Proposition

Let $\tilde{G}$ be a fuzzy ω-clopen subset of a fuzzy topological space (X,T̃) then $\tilde{G}$ is fuzzy ω-paracompact subset if and only if $\tilde{G}$ is fuzzy ω-paracompact subspace.

Proof: In view of ***(5.1 Proposition)***, we need only to prove the only if part.

Let $\tilde{G}$ be a fuzzy ω-clopen ω-paracompact subspace of a fuzzy topological space (X,T̃) and let $\{\tilde{U}_\lambda\}_{\lambda\in\Lambda}$ be fuzzy covering of $\tilde{G}$ by fuzzy ω-open subset of 1_X , then $\{\tilde{G} \cap \tilde{U}_\lambda\}_{\lambda\in\Lambda}$ is a fuzzy covering of $\tilde{G}$ by fuzzy ω-open subset of $\tilde{G}$, since $\tilde{G}$ be a fuzzy ω-paracompact subspace of a fuzzy topological space (X,T̃) therefore by ***(4.4 Corollary)*** there exist an fuzzy ω-locally finite ω-open covering $\{\tilde{V}_\lambda\}_{\lambda\in\Lambda}$ of $\tilde{G}$ such that for each λ ∈ Λ , $\tilde{V}_\lambda \subseteq \tilde{G} \cap \tilde{U}_\lambda \subseteq \tilde{U}_\lambda$ and $\tilde{V}_\lambda$ is fuzzy ω-open set in $\tilde{G}$ so for each λ ∈ Λ $\tilde{V}_\lambda$ is fuzzy ω-open set in 1_X, since $\tilde{G}$ and 1_X - $\tilde{G}$ are fuzzy ω-open sets in 1_X this implies that $\{\tilde{V}_\lambda\}_{\lambda\in\Lambda}$ is fuzzy ω-locally finite in 1_X

5.9 Proposition

Let $\tilde{G}$ and $\tilde{H}$ be two fuzzy subset of a fuzzy topological space (X,T̃) if $\tilde{G}$ is fuzzy ω-closed and $\tilde{H}$ is fuzzy ω-paracompact subset to 1_X then $\tilde{G} \cap \tilde{H}$ is fuzzy ω-paracompact subset to 1_X furthermore it is fuzzy ω-paracompact subset to $\tilde{H}$

Proof: Let $\{\tilde{V}_\lambda\}_{\lambda\in\Lambda}$ be any fuzzy covering of $\tilde{G} \cap \tilde{H}$ by fuzzy ω-open subset of 1_X since 1_X - $\tilde{G}$ is fuzzy

ω-open set in 1_X and $\tilde{H}$ - $\tilde{G} \subseteq 1_X$ - $\tilde{G}$ then for each $x_r \in \tilde{H}$ - $\tilde{G}$ there exist fuzzy ω-open set $\tilde{W}$ in 1_X such that $x_r \in \tilde{W} \subseteq \tilde{H}$ - $\tilde{G}$ and $\{\tilde{V}_\lambda\}_{\lambda\in\Lambda} \cup \{\tilde{W}\}_{x_r\in\tilde{H} - \tilde{G}}$ is a fuzzy covering of $\tilde{H}$ by fuzzy ω-open subset of 1_X, since $\tilde{H}$ is fuzzy ω-paracompact subset to 1_X, Therefore this cover has fuzzy

ω-locally finite refinement $\{\tilde{Z}_\gamma\}_{\gamma\in\Gamma}$, Which covers $\tilde{H}$ and $\tilde{Z}_\gamma$ is fuzzy ω-open set in 1_X for each γ ∈ Γ that is the fuzzy ω-locally finite subfamily $\{\tilde{Z}_\gamma\}_{\gamma\in\Gamma_1}$ where Γ_1={ γ ∈ Γ; $\tilde{Z}_\gamma \subseteq \tilde{V}_\lambda$ for some $\lambda \in \Lambda$} is fuzzy ω-open refinement of $\{\tilde{V}_\lambda\}_{\lambda\in\Lambda}$ which covers $\tilde{G} \cap \tilde{H}$ too, thus $\tilde{G} \cap \tilde{H}$ is fuzzy

ω-paracompact subset to 1_X, since $\tilde{H}$ is fuzzy ω-paracompact subset to 1_X so by ***5.1 Proposition*** it is fuzzy ω-paracompact subspace of 1_X since $\tilde{G}$ fuzzy ω-closed in 1_X hence $\tilde{G} \cap \tilde{H}$ is fuzzy

ω-closed subset of $\tilde{H}$ and then by ***5.2 Proposition*** $\tilde{G} \cap \tilde{H}$ is fuzzy ω-paracompact subset to $\tilde{H}$

5.10 Proposition

Let f: (X,$\tilde{T}$) ⟶ (Y,$\tilde{\sigma}$) be a fuzzy ω-continuous surjection which maps Fuzzy ω-open sets onto Fuzzy open sets, if $\tilde{G}$ is fuzzy ω-paracompact subset to 1_X then $f(\tilde{G})$ is fuzzy paracompact subset to 1_Y.

Proof: Let $\{\tilde{V}_\lambda\}_{\lambda\in\Lambda}$ be any fuzzy covering of $f(\tilde{G})$ by fuzzy open sets of 1_Y since f is fuzzy ω-continuous function then $\{f^{-1}(\tilde{V}_\lambda)\}_{\lambda\in\Lambda}$is fuzzy covering of $\tilde{G}$ by fuzzy ω-open subset of 1_X. But $\tilde{G}$ is fuzzy ω-paracompact subset to 1_X therefore there exist a fuzzy ω-locally finite ω-open family $\{\tilde{Z}_\gamma\}_{\gamma\in\Gamma}$ of subset of 1_X which refines $\{f^{-1}(\tilde{V}_\lambda)\}_{\lambda\in\Lambda}$ and cover $\tilde{G}$ since f surjection and maps fuzzy ω-open sets onto fuzzy open sets then $\{f(\tilde{Z}_{\gamma})\}_{\gamma\in\Gamma}$ is fuzzy locally finite open family $\{\tilde{Z}_{\gamma}\}_{\gamma\in\Gamma}$ of subset of 1_Y which refines $\{\tilde{V}_{\lambda}\}_{\lambda\in\Lambda}$ and cover $f(\tilde{G})$ this shows that $(\tilde{G})$ is fuzzy paracompact subset to 1_Y

5.11 Corollary

Let f: (X,$\tilde{T}$) ⟶ (Y,$\tilde{\sigma}$) be a fuzzy ω-continuous surjection which maps fuzzy open sets onto fuzzy open sets, if (X,$\tilde{T}$) is fuzzy ω-paracompact space then (Y,$\tilde{\sigma}$) is fuzzy paracompact space.
Proof : Obvious

5.12 Proposition

Let f: (X,$\tilde{T}$) ⟶ (Y,$\tilde{\sigma}$) be a fuzzy ω-irresolute pre- ω-open surjection function if $\tilde{G}$ is fuzzy ω-paracompact subset to 1_X then $f(\tilde{G})$ is fuzzy ω-paracompact subset to 1_Y.
Proof: Similar to the proof of **5.10 Proposition.**

5.13 Corollary

Let f: (X,$\tilde{T}$) ⟶ (Y,$\tilde{\sigma}$) be a fuzzy ω-irresolute open surjection function if $\tilde{G}$ is fuzzy ω-paracompact subset to 1_X then $f(\tilde{G})$ is fuzzy ω-paracompact subset to 1_Y.
Proof : Obvious

5.14 Corollary

Let f: (X,$\tilde{T}$) ⟶ (Y,$\tilde{\sigma}$) be a fuzzy ω-irresolute (pre- ω-open) open surjection function if (X,$\tilde{T}$) is fuzzy ω-paracompact space then (Y,$\tilde{\sigma}$) is fuzzy ω-paracompact space.
Proof: Obvious

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Normed Space Of Measurable Functions With Some Of Their Properties

Asawer J.Al-Afloogee
Noori F. Al-Mayahi

Department of Mathematics, College of Science Al-Qadisiya University.
Department of Mathematics, College of Science Al-Qadisiya University

Nafm60 @yahoo.com

Abstract Let $L^0(\Omega, F, \mu)$ be the space of measurable functions defined on measure space (Ω, F, μ) ,where we consider any two functions in which are equal almost everywhere (a .e). Then $L^0(\Omega, F, \mu)$ is complete metric space with respect to metric functions defined by $d(f\ ,g) = \int_\Omega \frac{|f-g|}{1+|f-g|} d\mu$ for all $f, g\ \in L^0(\Omega, F, \mu)$.This paper includes two main parts , the first part we prove this space $L^0(\Omega, F, \mu)$ in general is not a normed space, and second we prove norm on $L^0(\Omega, F, \mu)$ achieved if and only if she was Ω is the finite union of disjoint atom .

1. Introduction

In the measure theory ,we deal with different types of convergence of sequences of measurable functions ,especially convergence in measure and convergence almost everywhere(a. e) ,and study the relationships between them ,for example ,is that each sequence convergence measure is convergence (a .e)?and is the converse is true? ,and under what condition is that achieved ?There are many sources that have studied this topic from them Marczewiski was showed in [4] 1955 convergence in measure implies convergence everywhere (a. e) and thomasian proved in [8] (1957) convergence in metric equivalent to convergence a .s (in probability) if and only if Ω is the union of finite of disjoint atoms. Eugene was introduced in [2] 1975 several different definitions for the stochastic on convergence of sequence of random variables . And Jordan was proved in[3] 2015 $L^0(\Omega, \mathrm{F}, \mu)$ is a complete metric space . Noori and Asawer were proved in [6] 2020 $L^0(\Omega, F, \mu)$ is a complete metric space using another metric function .In this paper ,we are discussed the relationship between convergence in measure and convergence almost everywhere (a.e.) ,and what condition that must be set for equivalence to be achieved between them . After that we set with proof the necessary and sufficient condition for the existence of the norm on $L^0(\Omega, F, \mu)$ **.**

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2.Topology of convergence in measure

Let $L^0(\Omega, F, \mu)$ be the space of measurable functions defined on measure space (Ω, F, μ) are equle almost everywhere (a .e). Then $L^0(\Omega\ ,\mathrm{F},\mu)$ is a linear space under the following addition and scalar multiplication

1.$(f + g)(x) = f(x) + g(x)$ for all $f, g \in L(\Omega)$

2.$(\lambda f)(x) = \lambda(x)$ for all $f \in L(\Omega)$ and for $\lambda \in R$

Theorem(2.1)

Let $L^0(\Omega, F, \mu)$ be the space of measurable functions which is defined on measure space (a .e)

Define $\|.\|: L^0(\Omega,F,\mu) \to \mathbb{R}$ by $\|f\|_0 = \int_\Omega \frac{|f|}{1+|f|} d\mu$ for all f$\in L^0(\Omega,F,\mu)$,then

1.$\|f\|_0 \geq 0$ for all f$\in L^0(\Omega,F,\mu)$

2.$\|f\|_0$ =0 iff f=0 a .e .

3.$\|f+g\|_0 \leq \|f\|_0 + \|g\|_0$ for all f ,g $\in L^0(\Omega,F,\mu)$

Proof:

1.Since $|f| \geq 0$ for all $f \in L^0(\Omega,F,\mu)$,then $\frac{|f|}{1+|f|} \geq 0$ for all $f \in L^0(\Omega,F,\mu) \Rightarrow \int_\Omega \frac{|f|}{1+|f|} d\mu \geq 0 \Rightarrow \|f\|_0 \geq 0$

2. let $f \in L^0(\Omega,F,\mu)$

If f=0 a .e . i .e $\mu\{x \in \Omega: f(x) \neq 0\} = 0 \Rightarrow \frac{|f|}{1+|f|} = 0\ a.e \Rightarrow \int_\Omega \frac{|f|}{1+|f|} d\mu = 0 \Rightarrow \|f\| = 0$

If $\|f\|_0 = 0$ then $\int_\Omega \frac{|f|}{1+|f|} d\mu = 0$, since $\frac{|f|}{1+|f|} \geq 0 \Rightarrow \frac{|f|}{1+|f|} = 0\ a.e. \Rightarrow f = g\ \ a.e$

3.Let f , g $\in L^0(\Omega,F,\mu)$

Since $\frac{|f|}{1+|f|} + \frac{|g|}{1+|g|} \geq \frac{|f|}{1+|f|+|g|} + \frac{|g|}{1+|f|+|g|} = \frac{|f|+|g|}{1+|f|+|g|}$

$$\Rightarrow \frac{|f|}{1+|f|} + \frac{|g|}{1+|g|} \geq \frac{1}{\frac{1}{|f|+|g|}+1} \geq \frac{1}{\frac{1}{|f-g|}+1} = \frac{|f-g|}{1+|f-g|} \Rightarrow \frac{|f-g|}{1+|f-g|} \leq \frac{|f|}{1+|f|} + \frac{|g|}{1+|g|}$$

$$\Rightarrow \int_\Omega \frac{|f-g|}{1+|f-g|} d\mu \leq \int_\Omega \frac{|f|}{1+|f|} d\mu + \int_\Omega \frac{|g|}{1+|g|} d\mu \Rightarrow \|f+g\|_0 \leq \|f\|_0 + \|g\|_0$$

Remark : $\|.\|$ is not norm on $L^0(\Omega,F,\mu)$,since if $f \in L^0(\Omega,F,\mu)$,then $\|\lambda f\|_0 = \int_\Omega \frac{|\lambda f|}{1+|\lambda f|} d\mu = \int_\Omega \frac{|\lambda|\|f\|}{1+|\lambda|\|f\|} d\mu \neq \lambda\|f\|_0$

In order to discuss the compatibility of convergence in measure and a norm we have to introduce a definition from the theory of summability

Theorem(2.2) :[6]

The metric space $L^0(\Omega,F,\mu)$ is complete

Definition(2.3) : [2]

The sequence of real numbers $\{x_n\}$ is called Cesaro summable of order 1 to x and write $x_n \xrightarrow{(c,1)} x$ if $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i = x$.

The following result is very important

Lemma (2.4)

Let $\{x_n\}$ be a convergent sequence of real numbers if $x_n \to x$, then $x_n \xrightarrow{(c,1)} x$ .The converse not true

Proof:

Let $\varepsilon > 0$, since $x_n \to x$,then is $k \in \mathbb{Z}^+$ such that $|x_n - x| < \frac{\varepsilon}{2}$ for all $n \geq k$.

Let $y_n = \frac{1}{n}\sum_{i=1}^{n} x_i$,then $y_n - x = \frac{1}{n}\sum_{i=1}^{n} x_i - x = \frac{1}{n}\sum_{i=1}^{k}(x_i - x) + \frac{1}{n}\sum_{i=k+1}^{n}(x_i - x)$

Let $m = \max\{x, \max x_i\}$ and select n so large that $\frac{1}{n} < \frac{\varepsilon}{4km}$, then

$$|y_n - x| < \frac{\varepsilon}{4km} k(2m) + \frac{\varepsilon}{2}\left(\frac{n-k}{n}\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore $y_n \to x$.

Example(2.5):[2]

Let $x_n = \frac{1+(-1)^{n-1}}{2}$ for all $n \in \mathbb{N}$.

Clearly $x_{2n} = 0$, $x_{2n-1} = 1$,so that the sequence is divergent ,but $x_n \xrightarrow{(c,1)} \frac{1}{2}$

Remark :

In similar manner we can introduce (c ,1)_summability for sequence of measurable functions (c,1) to f ,

and write $f_n \xrightarrow{(c,1)-s} f$ if $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f_i = f$ .Here f may be proper or a degenerate measurable function

Theorem (2.6):

If $L^0(\Omega, F, \mu)$ is a normed space which is compatible with s-convergence ,and $\{f_n\}$ is a sequence in $L^0(\Omega, F, \mu)$ such that $f_n \xrightarrow{s} 0$,then $f_n \xrightarrow{(c,1)-s} 0$.

Proof:

Let $g_n = \frac{1}{n}\sum_{i=1}^{n} f_i \Rightarrow \|g_n\| = \left\|\frac{1}{n}\sum_{i=1}^{n} f_i\right\| \leq \frac{1}{n}\sum_{i=1}^{n} \|f_i\|$

Since $f_n \xrightarrow{s} 0 \Rightarrow \lim_{i\to\infty}\|f_i\| = 0$ by theorem (2.4) ,we have $\|f_n\| \xrightarrow{(c,1)} 0$,so that $\lim_{n\to n} \frac{1}{n}\sum_{i=1}^{n}\|f_i\| = 0 \Rightarrow \lim_{n\to n}\|g_n\| = 0$

Using the equivalent of norm convergence and s-convergence ,we conclude that $g_n \xrightarrow{s} 0$,therefore $f_n \xrightarrow{(c,1)-s} 0$

Remark :

We construct an example (example 2.5) of a sequence $\{f_n\}$ which converge in measure to zero but for which f_n not converge to zero in (c ,1)-$\mu . i . e.$

If $f_n = \frac{1+(-1)^{n-1}}{2}$ for all $n \in \mathbb{N}$ .Then $f_n \xrightarrow{\mu} 0$,but f_n not converge to zero in (c ,1)-μ

We can then use theorem (2.6) to prove the following statement

Theorem (2.7): [2]

Convergence in measure is in general incompatible with the existence of a norm .

The reason is that the existence of a norm which is compatible with converge in measure ispossible if the basic measure space (Ω, F, μ)has a certain property

3.The necessary and sufficient condition for the existence of the norm on $L^0(\Omega, F, \mu)$

Definition (3.1):[4]

1.A set $A \in F$ is called an atom ,if there no proper subset B of A such that $B \in F$

2.An atom of a measure space (Ω, F, μ) is set $A \in F$ with $\mu(A) > 0$ such that $B \subseteq A$ and $B \in F$ imply that either $\mu(B) = 0$ *or* $\mu(B) = \mu(A)$ .i.e

A set $A \in F$ is called atom of μ if 0< $\mu(A) < \infty$ and for every $B \subseteq A$ with $B \in F$ either $\mu(B)$

$$0 \ or \ \mu(B) = \mu(A)$$

A set $A \in F$ is called atom of μ if $\mu(A) > 0$ and for any $B \in F$ and $B \subseteq A$ with $\mu(A) < \mu(B)$,then

$$\mu(B) = 0$$

3. A measure without any atoms is called nonatomic (or atomless or diffuse).In other words

A measure μ is called nonatomic or diffuse ,if there are no atoms .

A measure μ is nonatomic if for any $A \in F$ with $\mu(A) > 0$ there exists $B \in F$ and $B \subseteq A$ such that

$$\mu(A) > \mu(B) > 0.$$

A measure μ is nonatomic if there are no atoms for μ . This means that every measurable set of positive measure can be split in to two disjoint measurable sets ,each having positive measure .

4. μ is called purely atomic or simply atomic if every measurable set of positive measure contains an atom .In other words A measure space (Ω,F, μ) ,or the measure μ is called purely atomic if there is a family g of atoms of μ such that for each $A \in F$, $\mu(A)$ is the sum of the numbers of $\mu(B)$ for all $B \in g$ such that $\mu(A \cap B) = \mu(A)$.

5. Let (Ω, F, μ) be a measure space such that all singleton $\{x\} \in F$ .A point $x \in \Omega$ Is called an atom for the measure $\mu(\{x\}) > 0$.

Example(3.2):[4]

1.Let $\Omega=\{1,2,3,\ldots,10\}$ and let $F = P(X)$ be the power set of Ω .Define the measure μ of a set to be cardinality ,that is , the number of elements in the set . Then ,each of the singletons $\{x\}$ for $x \in \Omega$ is an atom.

2.The singleton $\{x\}$ with positive finite measure are atoms of μ .

3.If $A \in F$ is an atom for μ and $\mu(A \cap B) > 0$, then $A \cap B$ is also an atom for μ .

4. A set of positive finite measure is an atom if its only measurable subsets are itself and $\emptyset$.

Here is a less trivial atom.

5.Let Ω be an uncountable set and let $\mathcal{F}$ be family of set which either countable ,with $\mu(A) = 0$ or have countable complement ,with $\mu(A) = 1$ .then μ is a measure and Ω is an atom .

6.Lebesgue measure is nonatomic .

7.If μ isσ $-$finite measure ,the set of atom of μ is countable .

8.The zero measure is the only measure which is both purely atomic and nonatomic

Theorem(3.3):

Let (Ω, F, μ) be a measure space

1.A measurable function is a. e. constant on an atom .

2. There is decomposition of Ω in to disjoint sets ,$\Omega = \bigcup_{n=0}^{\infty} A_n$ where A_0 is either empty or an atomless set of positive measure ,and each of the sets $A_1, A_2, \ldots$ is either a empty set or an atom

3. If μ is atomless ,then every $A \in F$,and every number c with $0 < c < \mu(A)$,there is a set $B \in F$ such that $B \subseteq A$ and $\mu(B) = 0$

4. If μ is atomless and $\mu(\Omega) = 1$,then for every sequence P_n with $0 \le P_n \le 1$,there exists a sequence $\{A_n\}$ of stochastically independent sets with $\mu(A_n) = P_n$.

Proof:

1.let (Ω, F, μ) be a measure space and $f : \Omega \to \mathbb{R}$ be a measurable function

If $A \in F$ is called atom of μ ,then f is constant on A

If $y \in \mathbb{R}$ and $\mu(\{x \in A: f(x) < y\}) = 0$,then $\mu(\{x \in A: f(x) < z\}) = 0$ for all $z \le y$.

Let $k = \sup\ \{y \in \mathbb{R}: \mu(\{x \in A: f(x) < y\})\} = 0$. Then $\mu(\{x \in A: f(x) < k\}) = \mu(\bigcup_{r \in Q, r<k} \{x \in A: f(x) \ge r\}) = 0$

If $y > k$,then $\mu(\{x \in A : f(x) < y\}) > 0$, hence $\mu(\{x \in A : f(x) \ge y\}) = 0$ since A is an atom of .Thus $\mu(\{x \in A : f(x) > k\}) = \mu(\bigcup_{r \in Q, r>k} \{x \in A: f(x) \ge r\}) = 0$

It follows that $f = k$ a .e. on A.

4.Let $A^0 = A^C = \Omega\, |A$, $A^2 = \Omega$ for every $A \subseteq \Omega$

BY (3) , there is a set A_1 with $\mu(A_1) = P_1$. If A_i are already defined for $i \le n$,and if they are stochastically independents sets with $\mu(A_i) = P_i$, then there is ,in view of (3) ,a set A_{n+1} such that $(\bigcap_{i=1}^{n+1} A_i^{K_i}) = P_{n+1} \mu(\bigcap_{i=1}^{n} A_{i=1}^{k_i})$ for every system $k_1, k_2, \ldots, k_n$ of number 0 and I .it is easy that $\{A_n\}$ is the required sequence

Definition(3,4)[7]

Let $\{A_n\}$ be a sequence of subsets of a set Ω .The set of all points which belong to infinitely many sets of the sequence $\{A_n\}$ is called the upper limit (or limit superior)of $\{A_n\}$ and is denoted by A^* and defined by $A^* = \lim_{n\to\infty} \sup A_n = \{x \in A_n: for\ infinitely\ many\ n\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \lim_{n\to\infty} \bigcup_{k=n}^{\infty} A_k$

Thus $x \in A_*$ iff for all n ,then $x \in A_k$ for some k$\geq n$

The lower limit (or limit inferior) of $\{A_n\}$, denoted by A_* is the set of all points which belong to almost all sets of the sequence $\{A_n\}$, and defined by

$A_* = \lim_{n\to\infty} \inf A_n = \{x \in A_* : for\ all\ but\ finiteiy\ many\ n\} = \bigcap_{n=1}^{\infty} \bigcup_{n=k}^{\infty} A_k = \lim_{n\to\infty} \bigcup_{n=k}^{\infty} A_k$

Thus $x \in A_*$ iff for some n ,then x $\in A_n$ for all K$\geq n$

Definition (3.5):[1]

A sequence $\{A_n\}$ of subsets of a set Ω is said to

1.converge if $\lim_{n\to\infty} sup_n A_n = \lim_{n\to\infty} inf_n A_n = A$, and A is said to be the limit of $\{ A_n\}$,we write

$A = \lim_{n\to\infty} A_n$ or$A_n \to A$. in other terms ,$A_n \to A$ iff $I_{A_n} \to I_A$

2.Converges in measure to set A ,write $A_n \overset{\mu}{\to} A$ if $I_{A_n} \overset{\mu}{\to} A$. in other terms ,if $\mu(A\Delta A_n)$

0

3.Converges a .e to set A ,write $A_n \overset{a.e}{\longrightarrow} A$ if $I_{A_n} \overset{a.e}{\longrightarrow} I_A$ .in other terms ,if $\mu(A\Delta(\lim_{n\to\infty} supA_n)) = \mu(A\Delta(\lim_{n\to\infty} infA_n)) = 0$

Theorem(3.6)

1 .If $f_n \overset{\mu}{\to} f$,then $f_n \overset{a.e}{\to}$ f on every atom set A of μ

2.If μ is atomless ,then there is a sequence $\{A_n\}$ of measurable sets convergence to the void set in measure and such that $\lim_{n\to\infty} infA_n = \emptyset$,$\lim_{n\to\infty} supA_n = \Omega$.

3.If the sequence convergence in measure on measurable sets implies their convergence a .e. ,then μ is purely atomic .

Proof:

1.Let $\{f_n\}$ be a sequence of measurable sequence defined on (Ω, F, μ) such that $f_n \overset{\mu}{\to} f$

Let A be an atom of μ ,then there is an atom $A^* \subseteq A$ such that $\mu(A|A^*) = 0$ and that $f_1, f_2, f_3, ...$ are constant on A^* ,then f(x)=c , $f_n(x) = c_n$ for $x \in A^*$

That $f_1, f_2, f_3, ...$ are constant on A^* ,then $f(x) = c$, $f_n(x) = c_n$ for $x \in A^*$

Let $\varepsilon > 0$,since $f_n \overset{\mu}{\to} f$,then there is $k \in \mathbb{Z}^+$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > k$ outside a

set Z_n with $\mu(Z_n) < \mu(A^*)$

Consequently $|c_n - c| < \varepsilon$ for all $n > k$ which implies $f_n \overset{a.e}{\longrightarrow} f$ on A.

2.Without any loss of generally we may suppose that $\mu(\Omega) = 1$.

By(4) theorem (3.3), there exists a sequence $\{B_n\}$ of stochastically independent sets with $\mu(B_n) = \frac{1}{2}$ for all n .the sequence

$A_1 = B_1$, $A_2 = B_1^C$, $A_3 = B_1 \cap B_2$, $A_4 = B_1^C \cap B_2$, $A_5 = B_1 \cap B_1^C$, $A_6 = B_1^C \cap B_2^C$, $A_7 = B_1 \cap B_2 \cap B_3$, $A_8 = B_1^C \cap B_2 \cap B, ...$

Obviously satisfies the conditions of (2) .

Theorem(3.7)

If μ is finite ,then $\Omega = A \cup (\bigcup_{n=1}^{\infty} A_n)$,where all of the sets in the decomposition are disjoint and each A_n is the empty set or an atom ,and for every measurable subset B of A , μ takes every value between 0 and $\mu(B)$ for measurable subset of B .

Proof :

There is only a countable numbers of μ −equivalence classes of such A_i of these classes and let $B \subseteq A = \Omega | \bigcup_{i=1}^{\infty} A_i$. Select representation inductively sets $C_n \in g_n$ such that $\mu(C_n) > \sup \mu(C) - \frac{1}{n}$ for all C$\in g_n$,where g_n is the class of all $C \subseteq B | \bigcup_{i=1}^{n-1} C_i$ for with $\mu(C) \leq C - \mu(\bigcup_{i=1}^{n-1} C_i)$. Then $\mu(C) = c$,for C=$\bigcup_{n=1}^{\infty} C_n$.

Definition(3.8)[1]

1.Converge in norm is said to be equivalent to convergence a .e .if for every sequence $\{f_n\}$ in $L^0(\Omega, F, \mu)$,$\|f_n\| \to 0$,iff $f_n \overset{a.e}{\to} 0$.

2.Converge in norm is said to be equivalent to convergence in measure if ,for every sequence $\{f_n\}$ in $l^0(\Omega, F, \mu)$,$\|f_n\| \to 0$ iff $f_n \overset{\mu}{\to} 0$.

Theorem(3.9)

If (Ω, F, μ) is finite measure .Then there exists a norm on $L^0(\Omega, F, \mu)$ which is compatible with convergence in measure iff **Ω** is the finite union of disjoint atoms .

Proof:

Suppose there exists a norm $\|.\|$ on $L^0(\Omega, F, \mu)$ which is compatible with convergence in measure Assume that **Ω** is not finite union of disjoint atoms .

Then there exists a sequence $\{A_n\}$ in **Ω** with 0< $\mu(A_n) \to 0$

Let f_n be the in indicator function of the set A_n ,i.e. $f_n = I_{A_n}$

If $\|f_{n_0}\| = 0$,then $f_{n_0} \overset{\mu}{\to} 0$,contradicting $\mu(A_{n_0}) > 0$,then $\|f_n\| \neq 0$ for all n

Since$\left\|\frac{f_n}{\|f_n\|}\right\| = \frac{\|f_n\|}{\|f_n\|} = 1$ for all n ,so that the sequence of measurable function $g_n = \frac{f_n}{\|f_n\|}$ cannot converge to 0 in measure .However ,it must ,because $\mu(A_n) \to 0$ contradiction

Conversely suppose that Ω is the finite union of disjoint atoms .

Define $\|.\|$:$L^0(\Omega, F, \mu) \to \mathbb{R}$ by $\|f\|$=$\int_{\Omega} |f| d\mu$ for all $f \in L^0(\Omega, F, \mu)$

In clear $\|.\|$ is a norm on $L^0(\Omega, F, \mu)$

Theorem(3.10)

If (Ω, F, μ) is finite measure .Then convergence in measure implies almost everywhere convergence for all sequence in $L^0(\Omega, F, \mu)$ iff Ω is the union of countable number of disjoint atoms .

Proof :

Suppose that convergence in measure implies almost everywhere convergence for all sequence in $L^0(\Omega, F, \mu)$.

Assume that **Ω** is not finite union of disjoint atoms

Thus in the decomposition of theorem (2.4) $\mu(A) > 0$ and for each n , $A = \bigcup_{k=1}^{n} A_{nk}$,

Where $\mu(A_{nk})$=$\frac{1}{n}\mu(A)$ for k=1,2,…,n ,and $A_{n1}, A_{n2}, \dots, A_{nk}$ are disjoint .

Let f_{nk} be the indicator function of the set A_{nk} . The sequence of measurable function $\{f_{nk}\}$ converge to 0 in measure but not a .e . .This contradiction .

Conversely : suppose that **Ω** is the finite union of disjoint atoms

Let $\{f_n\}$ in $L^0(\Omega, F, \mu)$ such that $f_n \overset{\mu}{\to} 0$. To prove $f_n \overset{a.e.}{\longrightarrow} 0$

By theorem (3.9) , there exists a norm on $L^0(\Omega, F, \mu)$ which is compatible with convergence in measure If $\|f_n\| \nrightarrow 0$ then there exists a subsequence $\{f_{nk}\}$,and an $\varepsilon > 0$ such that $\|f_{nk}\| > \varepsilon$. But $f_{nk} \overset{\mu}{\to} 0$ so that it has a subsequence $f_{nk} \overset{a.e.}{\longrightarrow} 0$ Thus $\|f_{nk}\| \overset{a.e.}{\longrightarrow} 0$ contradicting $\|f_{nk}\| > \varepsilon$ therefore ,$\|f_n\|$ must converge to 0 ,hence $f_n \overset{a.e.}{\longrightarrow} 0$.

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δ-Fuzzy measure on fuzzy δ-Algebra

Alaa M Jasim [1] and Noori F Al-Mayahi [2]

"Department of Mathematics, College of Science, University of Al-Qadisiyah,Iraq[1]"
,Diwaniyah Education Directorate,Iraq[2]
Email: sybzqoa@gmail.com[1], : nfam60@yahoo.com[2]

Abstract. The objective of this paper are , first , a new study of fuzzy δ-algebra and we discuss the properties of this family , second, introduce concepts related to the fuzzy δ-algbra such as fuzzy measure on fuzzy δ-algebra , and we obtained some important results deal with these concepts .

1. Introduction

Sugeno in (1975) [5] discusses many details about fuzzy measure define on σ-field and prove some important results in fuzzy measure theory , Ralescu and Adams in (1980) [2] generalized the concepts of fuzzy measure . the concept of fuzzy σ-field was studied by (1980) [3],(1987) [7] , where $\mathcal{F}$ is a family of fuzzy sets defined on a nonempty set Ω , satisfied the conditions : $\Omega, \emptyset \in \mathcal{F}$ and $\mathcal{F}$ closed under complement and countable union , this paper is organized as follows : in section 2 we give the essential definitions and results pertinent to fuzzy δ-algbra . In section 3 we introduce the notion of fuzzy measure defined on fuzzy δ-algebra and investigate some of their properties.

2. Main Results

The main results of this paper is to introduce and study the concept of fuzzy δ-algebra , fuzzy measure defined on fuzzy δ-algebra and we give basic properties and examples of these concepts.

2.1.*fuzzy δ- algebra*

In this section, we will discuss concept of fuzzy δ-algbra and we give basic properties and examples of these concepts .

Definition 2.1.1. A family $\mathcal{F}$ of a fuzzy set on a set Ω is called fuzzy δ-algebra on a set Ω if

a.$\emptyset \in \mathcal{F}$

b.if A is a nonempty fuzzy set in $\mathcal{F}$ and $A \subset B$,and B is a fuzzy set on Ω, then $B \in \mathcal{F}$

c. if $A_1, A_2, \ldots\ldots \in \mathcal{F}$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

A $\delta -$ fuzzy measurable space is a pair $(\Omega, \mathcal{F})$ where Ω is a non- empty set and $\mathcal{F}$ is a fuzzy $\delta -$ algebra on Ω

A fuzzy set A on Ω is called $\delta -$fuzzy measurable ($\delta -$fuzzy measurable with respect to the fuzzy δ-algbra if $A \in \mathcal{F}$) i.e any member of $\mathcal{F}$ is called a $\delta -$fuzzy measurable set .

*Example 2.1.2***.** The family $\mathcal{F}$ of all fuzzy sets on the set Ω is a fuzzy δ-algebra

Solution .Suppose that $\mathcal{F} = \{A : A \text{ is fuzzy set on } \Omega\}$

a. since $\emptyset$ and Ω is fuzzy set onΩ ,then $\emptyset, \Omega \in \mathcal{F}$

b. let $A \in \mathcal{F}$, such that $\emptyset \neq A \subset B$. and B fuzzy set on Ω, hence $B \in \mathcal{F}$.

c. let $A_1, A_2, \ldots\ldots \in \mathcal{F}$,hence $A_1, A_2, \ldots$ are fuzzy sets on Ω, Consequently, we have $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$ and hence $\mathcal{F}$ *is a fuzzy* δ-algbra .

*Remark 2.1.3***.** The family $\mathcal{F}=\{\emptyset , \Omega\}$ is a fuzzy δ-algebra**.**

Theorem 2.1.4. Let $\{\mathcal{F}_i\}_{i\in I}$ be a collection of fuzzy δ-algbra on Ω . then $\bigcap_{i \in I} \mathcal{F}_i$ is a fuzzy $\delta-$ algebra on Ω

Proof . (1) since $\mathcal{F}_i$ is fuzzy δ-algebra $\forall i \in I$, then $\emptyset ,\Omega \in \mathcal{F}_i \forall i \in I$, hence $\emptyset,\Omega \in \cap_{i=1}^{\infty} \mathcal{F}_i$

(2)let $A \in \cap_{i\in I} \mathcal{F}_i$ such that $\emptyset \neq A \subset B$, and B is fuzzy set on Ω. hence $A \in \mathcal{F}_i$ $\forall i \in I$, but $A \subset B$, and $\mathcal{F}_i$ $fuzzy\ \delta - algbra\ \forall\ i \in I, so$ we get $B \in \mathcal{F}_i$ $\forall i \in I$, hence $B \in \cap_{i\in I} \mathcal{F}_i$

(3)let $A_1 , A_2 , \ldots\ldots \in \cap_{i\in I} \mathcal{F}_i$, then $A_1 , A_2 , \ldots\ldots . \in \mathcal{F}_i$ $\forall i \in I$,since $\mathcal{F}_i\ fuzzy\ \delta - algebra\ \forall i \in I$, hence $\cap_{j=1}^{\infty} A_j \in \mathcal{F}_i$ $\forall\ i \in I$,

it follows that $\cap_{j=1}^{\infty} A_j \in \cap_{i\in I} \mathcal{F}_i$

thus $\cap_{i\in I} \mathcal{F}_i$ is a fuzzy δ-algebra .

Remark 2.1.5. The union of fuzzy δ–algebra not necessary to be fuzzy δ-algebra as in the next example.

Example 2.1.6. Let Ω=[0,1] and A, B, C are fuzzy sets on a set Ω such that

$$A(x)=\begin{cases} 0 & 0 \leq X \leq \frac{1}{2} \\ 1 & \frac{1}{2} < X \leq 1 \end{cases}$$

$$B(x)=\begin{cases} X & 0 \leq X \leq \frac{1}{2} \\ 1 & \frac{1}{2} < X \leq 1 \end{cases}$$

$$C(x)=\begin{cases} 1-X & 0 \leq X \leq \frac{1}{2} \\ 1 & \frac{1}{2} < X \leq 1 \end{cases}$$

Let $\mathcal{F}_1 = \{ \emptyset, A , B , \Omega \}$, $\mathcal{F}_2 = \{ \emptyset , A , C , \Omega \}$ are two fuzzy $\delta -$ algbra , but $\mathcal{F}_1 \cup \mathcal{F}_2$ is not fuzzy $\delta - algebra$

Solution: First, we must prove that $\mathcal{F}_1$ and $\mathcal{F}_2$ are fuzzy δ-algebra .

To prove $\mathcal{F}_1$ is fuzzy δ-algebra.

1. $\emptyset \in \mathcal{F}_1$, $\Omega \in \mathcal{F}_1$

2. (i) $A \in \mathcal{F}_1$ $\ni \emptyset \neq A \subset B$, $B \in \mathcal{F}_1$.

   (ii) $B \in \mathcal{F}_1$ $\ni$ $\emptyset \neq B \subset \Omega$,and $\Omega \in \mathcal{F}_1$

3. (i) if $0 \leq x \leq \frac{1}{2}$

$(A\cap B)\ (x) = \min\{A(x), B(x)\} = 0$

(a)if x=0

$(A\cap B)\ (0) = \min \{A(0), B(0)\} = 0 = \emptyset(x) \in \mathcal{F}_1$

(b)if $x=\frac{1}{2}$

$(A\cap B)(\frac{1}{2}) = \min\{A\left(\frac{1}{2}\right), B(\frac{1}{2})\}$

$= 0 = \emptyset(x) \in \mathcal{F}_1$

3. (ii) if $\frac{1}{2} < x \leq 1$

$(A \cap B)(x) = \min\{A(x), B(x)\} = 1 = \Omega(x) \in \mathcal{F}_1$
Then $\mathcal{F}_1$ is a fuzzy δ-algebra.
In the same way. we can prove that $\mathcal{F}_2$ is fuzzy $\delta -$ algebra .
Now to prove that $\mathcal{F}_1 \cup \mathcal{F}_2$ is , not fuzzy δ- algebra.
$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\emptyset, A, B, C, \Omega\}$
(i) if $0 \leq x \leq \frac{1}{2}$
$(B \cap C)(x) = \min\{B(x), C(x)\}$
$= \min\{x, 1-x\} = x$
(a) if $x = \frac{1}{2}$
$(B \cap C)\left(\frac{1}{2}\right) = \min\{B(\frac{1}{2}), C(\frac{1}{2})\} = \frac{1}{2} \notin \mathcal{F}_1 \cup \mathcal{F}_2$.
Hence $\mathcal{F}_1 \cup \mathcal{F}_2$ is not fuzzy δ-algbra .

Definition 2.1.7. Let Ω be a nonempty set and let G be a family of fuzzy sets on Ω .then the intersection of all fuzzy δ –algebra of Ω which contain G he claims the fuzzy δ-algebra generated by G and symbolizeit δ(G) that is

$\delta(G) = \cap\{\mathcal{F}_i : \mathcal{F}_i \text{is a fuzzy } \delta - \text{algebra of } \Omega \text{ and } G \subseteq \mathcal{F}_i, \forall i \in I\}$.

Lemma 2.1.8. Let G be a family of fuzzy sets on Ω ,then δ(G) is the smallest fuzzy δ –algebra of Ω which contain G.

Proof**:** Since $\delta(G) = \cap\{\mathcal{F}_i : \mathcal{F}_i \text{ is a fuzzy } \delta - \text{algebra of } \Omega \text{ and } G \subseteq \mathcal{F}_i, \forall i \in I\}$.
it follows that δ(G) is fuzzy $\delta -$ algebra of Ω by theorem (2.1.4)
T.p $G \subseteq \delta(G)$
Since $\mathcal{F}_i$ is a fuzzy $\delta -$ algebra of Ω and $G \subseteq \mathcal{F}_i \ \forall i \in I$.
Hence $G \subseteq \cap_{i \in I} \mathcal{F}_i$, there for $G \subseteq \delta(G)$.
Now let $\mathcal{F}$ is a fuzzy $\delta -$ algbra of Ω such that $G \subseteq \mathcal{F}$.
Then $\delta(G) = \cap\{\mathcal{F}_i : \mathcal{F}_i \text{is a fuzzy } \delta - \text{algebra of } \Omega$ and
$G \subseteq \mathcal{F}_i, \forall i \in I\}$.
Hence , $\delta(G) \subseteq \mathcal{F}$, there for δ(G) is the smallest fuzzy $\delta -$ algebra.
Of Ω which contain G.
In the example (2.1.6) , Ω=[0,1] assume G={A} then $\delta(G) = \{\emptyset, A, \Omega\}$ is the smallest fuzzy $\delta -$ algbra of a set Ω which contain G .

Proposition 2.1.9. Let G be a family of fuzzy sets on Ω ,then G is a fuzzy $\delta -$ algbra of a set Ω if and only if $G = \delta(G)$.

Proof: assume G is a fuzzy $\delta -$ algebra of a set Ω .
Since δ(G) is a fuzzy $\delta -$ algbra of a set Ω which contain G it follows that $G \subseteq \delta(G)$, But G is a fuzzy $\delta -$ algebra of a set Ω and δ(G)is the smallest fuzzy $\delta -$ algbra of a set Ω it follows that $\delta(G) \subseteq G$, and thus G=δ(G).
Conversely: let G be a family of fuzzy sets of Ω and
Let G=δ(G). Since δ(G) is a fuzzy $\delta -$ algebra of a set Ω
It follows that G is a fuzzy $\delta -$ algebra of a set Ω.

Definition 2.1.10. Let $\mathcal{F}$be a fuzzy $\delta-$ algebra of Ω and let A be a nonempty fuzzy set on Ω ,then the restriction of $\mathcal{F}$on A is symbolizeit $\mathcal{F}_A$ and define as :

$\mathcal{F}_A = \{ D: D = A \cap N , N \in \mathcal{F}\}$.

Theorem 2.1.11. Let $\mathcal{F}$be a fuzzy $\delta-$ algebra of a set Ω and $A \in \mathcal{F}$. Then

$\mathcal{F}_A= \{ N \subseteq A : N \in \mathcal{F}\}$

Proof : Let $D \in \mathcal{F}_A$, then $D = A \cap N$, $N \in \mathcal{F}$.thus $D \in \mathcal{F}$. Hence , $D \in \{ N \subseteq A : N \in \mathcal{F}\}$ and $\mathcal{F}_A \subseteq \{N \subseteq A: N \in \mathcal{F}\}$.

Let $C \in \{ N \subseteq A : N \in \mathcal{F}\}$, it follows that $C \subseteq A$ and $C \in \mathcal{F}$

Thus C=C $\cap$ A ,but $C \in \mathcal{F}$,then $C \in \mathcal{F}_A$which implies that $\{N \subseteq A : N \in \mathcal{F}\} \subseteq \mathcal{F}_A$

There fore, $\mathcal{F}_A = \{N \subseteq A: N \in \mathcal{F}\}$.

Corollary 2.1.12. Let $\mathcal{F}$ be a fuzzy $\delta-$ algbra of a set Ω and A be a non empty fuzzy set of Ω $\ni$ $A \in \mathcal{F}$. then $\mathcal{F}_A \subseteq \mathcal{F}$.

Proof: by theorem (2.1.11)

$\mathcal{F}_A=\{N \subseteq A: N \in \mathcal{F}\}$. let $C \in \mathcal{F}_A$. Then $C \subseteq A$ and $C \in \mathcal{F}$, hence $\mathcal{F}_A \subseteq \mathcal{F}$.

Proposition 2.1.13. Let $\mathcal{F}$be a fuzzy $\delta-$ algbra of a set Ω and let A be a non-empty fuzzy set of Ω $\ni A \in \mathcal{F}$ then $\mathcal{F}_A$ is a fuzzy $\delta-$ algbra on A.

Proof**:** 1**.**since $\mathcal{F}$is a fuzzy $\delta-$ algbra of Ω ,then $\emptyset,\Omega \in \mathcal{F}$,

since $A \subseteq \Omega$, then $A = A \cap \Omega$,hence $A \in \mathcal{F}_A$.

Since $\emptyset = \emptyset \cap A$ .then $\emptyset \in \mathcal{F}_A$.

2. let $B \in \mathcal{F}_A$ such that $\emptyset \neq B \subset D \subseteq A$. then by corollary (2.1.12) we get $B \in \mathcal{F}$,but $B \subset D \subseteq A$ and A fuzzy set on Ω and $\mathcal{F}$is a fuzzy $\delta-$ algbra of a set Ω ,it follows that $D \in \mathcal{F}$, and $D \subseteq A$ Then by theorem (2.1.11) we have $D \in \mathcal{F}_A$

3. let $B_1, B_2, \dots \in \mathcal{F}_A$, then there exist $N_1, N_2, \dots \in \mathcal{F}$ such that $B_i = N_i \cap A$, where $i = 1,2,\dots$. now $\cap_{i=1}^{\infty} B_i = (\cap_{i=1}^{\infty} N_i) \cap A$, but $\mathcal{F}$ is a fuzzy δ-algebra , then $\cap_{i=1}^{\infty} N_i \in \mathcal{F}$, hence $\cap_{i=1}^{\infty} B_i \in \mathcal{F}_A$ there for $\mathcal{F}_A$ is a fuzzy δ −algbra of a set A .

In the example (2.1.6) , let $B = \{B(x)\}$

then $\mathcal{F}_B = \{\emptyset, A(x), B(x)\} \Longrightarrow \mathcal{F}_B$ is a fuzzy $\delta-$ algbra of a set B and $\mathcal{F}_B \subseteq \mathcal{F}$.

*Definition 2.1.14***.** Let Ω be a nonempty set and G be a family of fuzzy set of Ω and $\emptyset \neq$ A and A fuzzy set on Ω ,then the restriction of G on A is symbolizeit G_A and define as :

$G_A = \{ D: D = A \cap N , N \in G \}$.

Proposition 2.1.15. Let Ω be a nonempty set and G be a family of fuzzy set of Ω and $\emptyset \neq A$, *and* A is a fuzzy set of Ω, if $\mathcal{F}$ is a fuzzy δ −algbra of Ω which contain G and $A \in \mathcal{F}$ then $\delta(G)_A$ is a fuzzy $\delta-$ algbra of A

Proof: the proof by using (2.1.8) and (2.1.13) .

Theorem 2.1.16. Let Ω be anon empty set and G is family of the fuzzy set of Ω and $\emptyset \neq A$ such that A is a fuzzy set of Ω and G_A is the restriction of G on A then $\delta(G_A)$ is the smallest fuzzy $\delta-$ algbra of a set A which contain G_A where

$\delta(G_A) = \cap \{ \mathcal{F}_{iA} : \mathcal{F}_{iA}$ is a fuzzy $\delta-$ algbra of A and $G_A \subseteq \mathcal{F}_{iA} \ \forall i \in I\}$.

Proof :From lemma (2.1.8) we get $\delta(G_A)$ is a fuzzy $\delta-$ algbra of a set A

T. P $G_A \subseteq \delta(G_A)$

Since for each $\mathcal{F}_{iA}$ is a fuzzy $\delta-$ algbra of a set A and

$G_A \subseteq \mathcal{F}_{iA} \ \forall i \in I$, then $G_A \subseteq \cap_{i \in I} \mathcal{F}_{iA}$, thus $G_A \subseteq \delta(G_A)$

Let $\mathcal{F}_A$ is a fuzzy δ-algbra of a set A $\ni$ $G_A \subseteq \mathcal{F}_A$. thus $\delta(G_A) \subseteq \mathcal{F}_A$
There for $\delta(G_A)$ is the smallest fuzzy $\delta -$ algbra of a set A contain G_A .

Lemma 2.1.17. Let Ω be a nonempty set and G be a family of the fuzzy set of Ω and Ø≠ K ,and K fuzzy set on Ω define the family $\mathcal{F}^*$as : $\mathcal{F}^* = \{ A \in I^{\Omega} : A \cap N \in \delta(G_N)\}$.
Then $\mathcal{F}^*$is a fuzzy $\delta -$ algbra of a set Ω.
Proof: 1. $\delta(G_N)$is a fuzzy $\delta -$ algbra of a set N , hence
,$N \in \delta(G_N)$. Since $N \subset \Omega$,it follows that $N = N \cap \Omega$, hence $\Omega \in \mathcal{F}^*$,
also $\emptyset = \emptyset \cap N$. then $\emptyset \in \mathcal{F}^*$
2.let $A \in \mathcal{F}^*$ such that $\emptyset \neq A \subset B$ and B fuzzy set on Ω .
Then $(A\cap N) \in \delta(G_N)$. since $A \subset B$, then $(A \cap N) \subset (B \cap N)$
But $\delta(G_N)$ is a fuzzy $\delta -$ algbra of a set N, then $(B\cap N) \in \delta(G_N)$,
hence $B \in \mathcal{F}^*$
3. if $A_1, A_2, \ldots \in \mathcal{F}^*$. then $A_1, A_2 \ldots \in I^{\Omega}$, and $(A_j \cap N) \in \delta(G_N)$ For all j=1,2,… hence $\cap_{j=1}^{\infty} A_j \in I^{\Omega}$ *and*, $(\cap_{j=1}^{\infty} A_j \cap N) \in \delta(G_N)$, hence $\cap_{j=1}^{\infty} A_j \in \mathcal{F}^*$.
therefor $\mathcal{F}^*$ is fuzzy $\delta -$ algbra of a set Ω .
Proposition 2.1.18. Let Ω be a non-empty set and G be a family of a fuzzy set of Ω and Ø≠ $N \in I^{\Omega}$ and $\delta(G)_N$ is a fuzzy $\delta -$ algbra of a set N then $\delta(G_N) = \delta(G)_N$.
Proof :let $B \in G_N$. then $B = A \cap N$. $A \in G$.
But G⊆ $\delta(G)$. hence $A \in \delta(G)$, $B \in \delta(G)_N$, hence $G_N \subseteq \delta(G)_N$
,But $\delta(G_N)$is the smallest fuzzy $\delta -$ algbra of a set N, which contain G_N and $\delta(G)_N$ is a fuzzy $\delta -$ algbra of a set N Which contain G_N , then $\delta(G_N) \subseteq \delta(G)_N$
Assume that $\mathcal{F}$={A⊆ N: A⋂N $\in \delta(G_N)$} . from lemma (2.1.17) we get
$\mathcal{F}$ is a fuzzy $\delta -$ algbra of a set Ω. let $A \in G$, then
$(A\cap N) \in G_N$, but $G_N \subseteq \delta(G_N)$, it follows that
$(A\cap N) \in \delta(G_N)$, thus A∈ $\mathcal{F}$ and G⊆ $\mathcal{F}$, Let B∈ $\delta(G)_N$,
Then B=A∩ N , A∈ $\delta(G)$,But $\delta(G) \subseteq \mathcal{F}$,then
A∈ $\mathcal{F}$,thus B∈ $\delta(G_N)$,and $\delta(G)_N \subseteq \delta(G_N)$, hence $\delta(G_N) = \delta(G)_N$.

2.2. δ-Fuzzy Measure

In this section, we will introduce the notion related with respect to fuzzy δ-algbra such as fuzzy measure on fuzzy δ-algbra .
Definition 2.2.1.[5]. Let $(\Omega, \mathcal{F})$ be a "δ-fuzzy measurable space" a set function
$\mu: \mathcal{F} \to [0, \infty]$ is said to be a "δ-fuzzy measure" on $(\Omega,\mathcal{F})$ if it
Satisfied the following properties:
1.$\mu(\emptyset) = 0$.
2.if $A \in \mathcal{F}$ and $A \subset B$ and B fuzzy set on Ω , then
$\mu(A) \leq \mu(B)$
■ A $\delta -$ fuzzy measure space is a triple $(\Omega, \mathcal{F}, \mu)$, where
$(\Omega, \mathcal{F})$ is $\delta -$fuzzy measurable space and μ is a $\delta -$fuzzy measure On $(\Omega, \mathcal{F})$.
■ A $\delta -$ fuzzy measure μ on $(\Omega, \mathcal{F})$ he claims regular if
$\mu(\Omega) = 1$.

Remark 2.2.2. Every measure on a measurable space $(\Omega, \mathcal{F})$ is a δ −fuzzy measure
But the converse need not true as follows:
Let Ω=[0,1], and A, B fuzzy sets on Ω define as follows

$$A(x)=\begin{cases} 0 & 0 \le x \le \frac{1}{2} \\ 1 & \frac{1}{2} < x \le 1 \end{cases} \quad , \quad B(x)=\begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ 0 & \frac{1}{2} < x \le 1 \end{cases}$$

Then $\mathcal{F}$={Ø, A, B, Ω } is fuzzy δ-algebra ,(Ω,$\mathcal{F}$) is a δ-measurable space .
Define $\mu: \mathcal{F} \to [0,\infty]$ by $\mu(\emptyset) = \mu(A) = \mu(B) = 0$
$\mu(\Omega) = 1$. μ is δ −fuzzy measure but not measure on (Ω,$\mathcal{F}$)
because of A,B disjoint sets in $\mathcal{F}$ and
$\mu(A \cup B) = \mu(\max\ \{A(x), B(x)\}\ ) = \mu(\Omega) = 1$
$\mu(A) + \mu(B) = 0 + 0 = 0$, *hence* $\mu(A \cup B) \neq \mu(A) + \mu(B)$.

Definition 2.2.3. [1]. Let $(\Omega, \mathcal{F})$ be a measurable space a set function $\mu: \mathcal{F} \to [0,\infty]$
Is said to be :
1.finite , if $\mu(A) < \infty \quad \forall A \in \mathcal{F}$.
2.Semi-finite ,if $\forall A \in \mathcal{F}$ with $\mu(A) = \infty$ there exists $B \in \mathcal{F}$
With $B \subseteq A$ and $0 < \mu(B) < \infty$
3.Bounded ,if $\sup\{|\mu(A)|: A \in \mathcal{F}\} < \infty$.
4.$\sigma -$ finite ,if $\forall A \in \mathcal{F}$,there is sequence $\{A_n\}$ of sets in $\mathcal{F}$
$\ni A \subset \cup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty\ \forall n$.
5.Additive ,if $\mu(A \cup B) = \mu(A) + \mu(B)$
Whenever A,B $\in \mathcal{F}$,and $A \cap B = \emptyset$
6. Finitly additive if $\mu\ (\cup_{k=1}^{n} A_k) = \Sigma_{k=1}^{n} \mu(A_k)$
Whenever $A_1, A_2, \ldots\ldots A_n$ are disjoint sets in $\mathcal{F}$
7. σ − additive (sometimes called completely additive or a countable additive) if $\mu\ (\cup_{k=1}^{\infty} A_k) = \Sigma_{k=1}^{\infty} \mu(A_k)$
Whenever $\{A_k\}$ is a sequence of disjoint sets in $\mathcal{F}$
8.Null additive if $\mu(A \cup B) = \mu(A)$,whenever A,B∈ $\mathcal{F}$ such that $A \cap B = \emptyset$ and $\mu(B) = 0$
9.Measure if μ is $\sigma -$ additive and $(A) \geq 0$,$\forall A \in \mathcal{F}$
10.probability if μ is a measure and $\mu(\Omega) = 1$
11.continuous from below at A∈ $\mathcal{F}$ if

$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$

Whenever $\{A_n\}$ is a sequence of sets in $\mathcal{F}$ and $A_n \uparrow A$
12.continuous from above at A∈ $\mathcal{F}$ if

$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$

Whenever $\{A_n\}$ is a sequence of sets in $\mathcal{F}$ and $A_n \downarrow A$
13.continuous at A∈ $\mathcal{F}$ if it is continuous both from below and from above at A.

Theorem 2.2.4. [6]. Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space if μ is a finite fuzzy measure , then we have

$$\lim_{n\to\infty} \mu(A_n) = \mu(\lim_{n\to\infty} A_n)$$

For any sequence $\{A_n\}$ of sets is$\mathcal{F}$ whose lim it exists.

Theorem 2.2.5. Let $(\Omega, \mathcal{F})$ be δ −fuzzy measurable space and μ, ω be a δ −fuzzy measure on Ω, then $\mu + \omega$ which defined by $(\mu + \omega)(A) = \mu(A) + \omega(A)$ is a δ −fuzzy measure on Ω

proof :(1) '

since μ , ω be δ −fuzzy measures , then $\mu(\emptyset) = 0$ and $\omega(\emptyset) = 0$

Hence $(\mu + \omega)(\emptyset) = \mu(\emptyset) + \omega(\emptyset) = 0$

(2) if $B \in \mathcal{F}$ and $B \subset D \in I^{\Omega}$, then $D \in \mathcal{F}$

Since μ , ω are δ −fuzzy measure then

$\mu(B) \le \mu(D)$……….(1)

$\omega(B) \le \omega(D)$………(2) hence

$(\mu + \omega)(B) = \mu(B) + \omega(B) \le \mu(D) + \omega(D)$

$= (\mu + \omega)(D)$

So $\mu + \omega$ is a δ −fuzzy measure .

Theorem 2.2.6. Let $(\Omega, \mathcal{F})$ be a δ −measurable space ,μ be a "δ −fuzzy measure" on Ω and $\lambda \in (0, \infty)$ define a set function

$(\lambda\, \mu)(A) = \lambda\, \mu(A)$,then $\lambda\, \mu$ is a δ −fuzzy measure on Ω.

Proof: 1.since is a $\delta -$ fuzzy measure , we have $\mu(\phi) = 0$

And $\in (0, \infty)$, then $(\lambda\, \mu)(\phi) = \lambda\, \mu(\phi) = 0$

2.if $A \in \mathcal{F}$, $A \subset B \in I^{\Omega}$, hence $B \in \mathcal{F}$

Since μ is δ −fuzzy measure, then $\mu(A) \le \mu(B)$

$(\lambda\, \mu)(A) = \lambda\, \mu(A) \le \lambda\mu(B) = (\lambda\mu)(B)$, So $\lambda\, \mu$ is a δ −fuzzy measure.

Corollary 2.2.7. Let $\mu_1, \mu_2, \ldots, \mu_n$ are δ −fuzzy measure on $\mathcal{F}$and

$\lambda_i \in (0, \infty)\ \forall\ i = 1,2, \ldots, n$

If $\Sigma_{i=1}^{n} \lambda_i\, \mu_i : \mathcal{F} \to [0, \infty]$ is defined by

$(\Sigma_{i=1}^{n} \lambda_i \mu_i)(A) = \Sigma_{i=1}^{n} \lambda_i \mu_i(A)\ \forall\ A \in \mathcal{F}$,then

$\Sigma_{i=1}^{n} \lambda_i \mu_i$ is a "δ −fuzzy measure" on $\mathcal{F}$.

Remark 2.2.8. Let μ be a "δ −fuzzy measure" on $\mathcal{F}$and let A,B fuzzy set then

1.$\mu(A \cup B) \ge \mu(A)$and $\mu(A \cup B) \ge \mu(B)$

Whenever A$\in \mathcal{F}$ and B $\in \mathcal{F}$.

2.$\mu(A \cap B) \le \mu(A)$and $\mu(A \cap B) \le \mu(B)$.

Whenever $A, B \in \mathcal{F}$

Proposition 2.2.9. Let $\mu: \mathcal{F} \to [0, \infty]$ be set function if μ is $\delta -$ fuzzy

Measure then μ is non-negative .

Proof: Let $A \in \mathcal{F} \to \emptyset \subset A \in I^{\Omega}$

Since μ is "δ −fuzzy measure" then $\mu(\emptyset) \le \mu(A)$

$\mu(A) \ge 0$, then μ is non-negative.

Definition 2.2.10. [4]. Let $(\Omega, \mathcal{F})$ be a δ-fuzzy measurable space .a set function μ is called :

1. Upper semi-continuous "δ-fuzzy measure" if and only if

   $lim_{n\to\infty}\mu(A_n) = \mu(\cup_{n=1}^{\infty} A_n)$

   Whenever $\{A_n\}$ is increasing sequence.

2. Lower semi-continuous δ-fuzzy measure if and only if

$lim_{n\to\infty}\mu(A_n) = \mu(\cap_{n=1}^{\infty} A_n)$

whenever $\{A_n\}$ is decreasing sequence .

3. Semi-continuous δ-fuzzy measure if it is both upper and lower semi-continuous δ-fuzzy measure.

Theorem 2.2.11. Let $(\Omega,\mathcal{F})$ be a δ-fuzzy measurable space and let $\mu:\mathcal{F} \to [0,\infty]$ be a function , if μ is additive ,non-decreasing and upper semi-continuous ,then μ is δ-fuzzy measure .

Proof :1. Since A=A$\cup$ $\emptyset$, also μ is additive we have .

$$\mu(A) = \mu(A \cup \emptyset) = \mu(A) + \mu(\emptyset)$$

$$\therefore \ \mu(\emptyset) = 0$$

2. let A$\in \mathcal{F}$,such that A$\subset B$ then B$\in \mathcal{F}$ .we have B=A$\cup$ (B/A) and A$\cap$ $(B/A) = \emptyset$, since μ *is* additive we have ,

$\mu(B) = \mu(A) + \mu(B/A) \geq \mu(A)$

Consequently $\mu(A) \leq \mu(B)$

So μ is δ-fuzzy measure.

Theorem 2.2.12. Let $(\Omega,\mathcal{F})$ be a δ-fuzzy measurable space , let $\{A_n\}$ be sequence of disjoint fuzzy sets in $\mathcal{F}$ and it is decreasing ,if $\mu(A_n) < \infty$ and μ is lower semi-continuous δ-fuzzy measure at $\emptyset$,

$$then \lim_{n\to\infty}\mu(A_n) = 0$$

Proof**:** Since $\{A_n\}$ is lower continuous δ-fuzzy measure at $\emptyset$, we have $\lim_{n\to\infty} \mu(A_n) = \mu(\emptyset)$, But $\mu(\emptyset) = 0$ consequently we have $\lim_{n\to\infty} \mu(A_n) = 0$ **.**

Definition 2.2.13.[4]. Let $(\Omega,\mathcal{F})$ be a "δ-fuzzy measurable space" .a set function $\mu: \mathcal{F} \to [0,\infty]$ is said to be .

1. Exhaustive if $\lim_{n\to\infty} \mu(A_n) = 0$,for any sequence $\{A_n\}$ of disjoint sets in $\mathcal{F}$.
2. Order-continuous if $\lim_{n\to\infty} \mu(A_n) = 0$,whenever $A_n \in \mathcal{F}$, n=1,2.. and $A_n \downarrow \emptyset$.

Theorem 2.2.14. Let $(\Omega,\mathcal{F})$ be a δ-fuzzy measurable space . if μ is a finite upper semi-continuous δ $-$fuzzy measure , then it is exhaustive .

Proof: Let $\{A_n\}$ be a disjoint sequence of sets in $\mathcal{F}$ if we write $M_n = \cup_{k=n}^{\infty} A_k$,then $\{M_n\}$ is a decreasing sequence of sets in $\mathcal{F}$ and, $\lim_{n\to\infty} M_n = \cap_{n=1}^{\infty} M_n$ =$\lim_{n\to\infty} \sup A_n = \emptyset$, since μ is a finite upper semi-continuous "δ-fuzzy measure" , then by using the finiteness and the continuity from above of μ , we have $\lim_{n\to\infty} \mu(M_n) = \mu(\lim_{n\to\infty} M_n) = \mu(\emptyset) = 0$, Noting that $0\leq \mu(A_n) \leq \mu(M_n)$

We obtain $\lim_{n\to\infty} \mu(A_n) = 0$. So μ is exhaustive .

Theorem 2.2.15. [6]. Let $(\Omega,\mathcal{F})$ be a measurable space .if $\mu: \mathcal{F} \to [0,\infty]$ is a non decreasing set function , then the following statement are equivalent :

1. μ is null additive .
2. $\mu(A \cup B) = \mu(A)$ whenever A,B $\in \mathcal{F}$ and $\mu(B) = 0$
3. $\mu(A/B) = \mu(A)$ whenever A,B$\in \mathcal{F}$ such that B$\subseteq A$ and $\mu(B) = 0$
4. $\mu(A/B) = \mu(A)$ whenever A,B $\in \mathcal{F}$ and $\mu(B) = 0$
5. $\mu(A\Delta B) = \mu(A)$ whenever A,B $\in \mathcal{F}$ and $\mu(B) = 0$.

Theorem2.2.16. Let $(\Omega,\mathcal{F})$ be a δ-measurable space ,A$\in \mathcal{F}$ if μ is null additive ,then $\lim_{n\to\infty} \mu(A \cup A_n) = \mu(A)$ for any decreasing sequence $\{A_n\}$ of sets in $\mathcal{F}$ for which $\lim_{n\to\infty} \mu(A_n) = 0$ and there exists at least one positive integer n_0 such that $\mu(A \cap A_{n_0}) < \infty \ as \ \mu(A) < \infty$.

Proof: it is sufficient to prove this theorem for $\mu(A) < \infty$.

If we write B=$\cap_{n=1}^{\infty} A_n$,we have $\mu(B) = \lim_{n\to\infty} \mu(A_n) = 0$. Since A∪ $A_n \uparrow A \cup B$, it follows ,from the continuity and null additivity of μ ,that $\lim_{n\to\infty} \mu(A \cup A_n) = \mu(A \cup B) = \mu(A)$.

Theorem2.2.17. Let (Ω,$\mathcal{F}$) be a δ-fuzzy measurable space ,A∈ $\mathcal{F}$ if μ is null additive ,then $\lim_{n\to\infty} \mu(A/A_n) = \mu(A)$ for any decreasing sequence $\{A_n\}$ of sets in $\mathcal{F}$ for which $\lim_{n\to\infty} \mu(A_n) = 0$

Proof :Since A / $A_n \uparrow \boldsymbol{A}$ / ($\cap_{n=1}^{\infty}$ A_n) and $\mu(\cap_{n=1}^{\infty} A_n) = 0$ by the theorem (2.2.15) continuity of μ , it follows that $\lim_{n\to\infty} \mu(A/A_n) = \mu(A/(\cap_{n=1}^{\infty} A_n)) = \mu(A)$.

Definition 2.2.18 [7]. Let (Ω,$\mathcal{F}$) be a δ-fuzzy measurable space .a set function $\mu: \mathcal{F} \to [-\infty, \infty]$ is said to be

1. Autocontinuous from above ,if $\lim_{n\to\infty} \mu(A \cup A_n) = \mu(A)$ Whenever A∈ $\mathcal{F}, A_n \in \mathcal{F}$,A∩ $A_n = \emptyset$, n=1,2 … and $\lim_{n\to\infty} \mu(A_n) = 0$.
2. Autocontinuous from below ,if $\lim_{n\to\infty} \mu(A/A_n) = \mu(A)$ whenever A∈ $\mathcal{F}$, $A_n \in \mathcal{F}, A_n \subseteq A$, n=1,2 … and $\lim_{n\to\infty} \mu(A_n) = 0$.
3. Autocontinuous , if it is both autocontinuous from above and autocontinuous from below .

Theorem 2.2.19.let (Ω, $\mathcal{F}$) be δ-fuzzy measurable space ,and $\mu: \mathcal{F} \to [-\infty, \infty]$ be a set function .if there exists $\varepsilon > 0$ such that $|\mu(A)| \geq \varepsilon$ for any A∈ $\mathcal{F}$, A $\neq \emptyset$ then μ is autocontinuous .

proof: under the condition of this theorem ,if $\{A_n\}$ is a sequence of sets in $\mathcal{F}$ such that $\lim_{n\to\infty} \mu(A_n) = 0$,then there must be some n_0 such that $A_n = \emptyset$ whenever n≥ n_0 ,and therefore $\lim_{n\to\infty} \mu(A \cup A_n) = \lim_{n\to\infty} \mu(A/A_n) = \lim_{n\to\infty} \mu(A) = \mu(A)$

Theorem 2.2.20. let (Ω,$\mathcal{F}$) be δ-fuzzy measurable space ,if $\mu: \mathcal{F} \to [-\infty, \infty]$ is autocontinuous from above ,then it is null additive .

proof:For any A,B ∈ $\mathcal{F}$,A∩ B=∅ and $\mu(B) = 0$,take $A_n = B$, n=1,2…, we have $\lim_{n\to\infty} \mu(A_n) = \mu(B) = 0$, since μ is autocontinuous from above , then $\mu(A \cup B) = \lim_{n\to\infty} \mu(A \cup A_n) = \mu(A)$,and μ is null additive as well .

Theorem 2.2.21. Let (Ω,$\mathcal{F}$) be δ-fuzzy measurable space , and let $\mu: \mathcal{F} \to [-\infty, \infty]$ be non decreasing set function , then μ *is* autocontinuous if and only if $\lim_{n\to\infty} \mu(A \Delta A_n) = \mu(A)$ whenever $\{A_n\}$ is a sequence of sets in $\mathcal{F}$ such that $\lim_{n\to\infty} \mu(A_n) = 0$

Proof: Suppose that μ is autocontinuous

For any A∈ $\mathcal{F}$ and $\{A_n\}$ is a sequence of sets in $\mathcal{F}$ such that $\lim_{n\to\infty} \mu(A_n) = 0$,noting A/ $A_n \subseteq A \vartriangle A_n \subseteq A \cup A_n$, by monotonicity of μ , we have

$\mu(A/A_n) \subseteq \mu(A \Delta A_n) \subseteq \mu(A \cup A_n)$, since μ is both autocontinuous from above and autocontinuous from below m we have $\lim_{n\to\infty} \mu(A \cup A_n) = \mu(A)$ *and*

$\lim_{n\to\infty} \mu(A/A_n) = \mu(A)$ .thuse we have

$\lim_{n\to\infty} \mu(A \Delta A_n) = \mu(A)$

Conversely .for any A∈ $\mathcal{F}$ and $\{A_n\}$ is a sequence of sets in $\mathcal{F}$ such that $\lim_{n\to\infty} \mu(A_n) = 0$,we have $A_n / A \in \mathcal{F}$ and $\mu(A_n/A) \leq \mu(A_n)$. so we have $\lim_{n\to\infty} \mu(A_n/A) = 0$

and there fore , by the condition given in this theorem , we have

$\lim_{n\to\infty} \mu(A \cup A_n) = \lim_{n\to\infty} \mu\big(A\Delta(A_n/A)\big) = \mu(A)$

that is ,μ aoutocontinuous from above . similarly , from

$\lim_{n\to\infty} \mu(A_n \cap A) = 0$ *it follows that* $\lim_{n\to\infty} \mu(A/A_n) = \lim_{n\to\infty} \mu\big(A\Delta(A_n \cap A)\big) = \mu(A)$,

that is μ aoutocontinuous from below .

Remark 2.2.22. The following theorem indicates the relation between the outocontinuity and the continuity of nonnegative set function .

Theorem 2.2.23. Let $(\Omega,\mathcal{F})$ be δ-fuzzy measurable space .if $\mu: \mathcal{F} \to [0,\infty]$ is continuous from above at $\emptyset$ and autocontinuous from above , then μ is continuous from above

Proof: If $\{A_n\}$ is a decreasing sequence of sets in $\mathcal{F}$ and $\cap_{n=1}^{\infty} A_n = A, then\ A_n/A \downarrow \emptyset$. from the finiteness and the continuity from above at $\emptyset$ of $\mu, we\ now$ $\lim_{n\to\infty} \mu(A_n/A) = 0$ and therefore by using the autocontinuity from above of μ we have

$\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu\big(A \cup (A_n/A)\big) = \mu(A)$ **,** that is μ is continuous from above s not fuzzy δ-field.

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Weak* topology on modular space and some properties

[a]Murtda Mohammed Jabber and [b]Prof. Dr Noori Farhan Al-Mayahi

[a]Department of Mathematics, College of Science Al-Qadisiya University.
[b]Department of Mathematics, College of Science Al-Qadisiya University
murtda01@gmail.com

Nafm60 @yahoo.com

Abstract: This study aims to redefine the weak topology $\sigma(X_M^*, X_M^{**})$ on a specific topological dual space (modular dual space ${X_M}^*$) this is weak topology generated by all linear bounded functional on ${X_M}^*$, but we interested in a subspace of this topology generated by X_M called weak* topology on ${X_M}^*$, it follows from that the modular space X_M over the field K can be embedded in X_M^{**} by using the canonical map, we denoted to this topology by $\sigma(X_M^*, X_M)$. After that, we checked the weak* topology is Hausdorff and investigated some properties, finally, we showed that under which condition the strong topology and the weak* topology coincided.

Keyword: weak topology on modular space, weak-star topology, weak* topology on modular space, modular space, weak topology,

1. Introduction

The initials definitions and basic concepts of modular space, weak topology and weak topology on modular spaces were indicated in preliminaries. Nakano's assumption of modular functions appeared in the 1950s [13], who introduced a family of functions from any vector space V over a field K (where $K = R \ or \ C$) into the interval $[0, \infty]$ $M: V \to [0, \infty]$ with conditions, the vector space V with modular M is called modular space [4]. It will be metric space when the distance between any two points v, u in V_M is defined by $d(v, u) = M(v - u)$ [2]. That is, d generates a topology for V_M . After that, The preliminaries introduced weak topology on any set in a general view, see coherent topology as well as [7,16]. Moreover, there is research that especially talks about weak topology on modular space which has recently been published 2020, see [12]. Finally, the weak topology on a specific topological dual space (modular dual space X_M^*) is redefined by the family X_M^{**} of all linear bounded functions from X_M^* into R. But, the interest will be focused on weak topology generated by a subspace X_M of X_M^{**}; this weak-star topology of X_M^*

2. Preliminaries:

This section is divided into three parts, let's begin with:

2.1. Modular space

In this part, basic definitions and descriptions of the concept of the modular space

2.1.1 Definition : [4]

Let X be a linear space over afield K. A map $M: X \to [0, \infty]$ called a modular if 1. $M(m) = 0$ if and only if $m = 0$.

2. $M(\lambda m) = M(m)$ with $|\lambda| = 1$, for $\lambda \in K, m \in X$

3. $M(\alpha m + \beta r) \leq M(m) + M(r)$ when $\alpha, \beta \geq 0,\ m \in X$ and $\alpha + \beta = 1$

Space is given by $X_M = \{m \in X : \ M(\lambda m) \to 0$ when $\lambda \to 0\}$ is called modular space, as follows.

If condition 3 above replaced by

$M(\alpha m + \beta r) \leq \alpha M(m) + \beta M(r)$, for $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ for all $m, r \in M$, then M called a convex modular.

If $r = 0$ then $M(\alpha x) = M\left(\frac{\alpha_1}{\beta}\ \beta m\right) \leq M(\beta x), \alpha, \beta \in K$, $0 < \alpha < \beta$. Thus M increasing map..

2.1.2. Remarks:

1. If X_M is a modular space, then X_M is a metric space by defined the distance function as follow $d(m, r) = \ M(m - r)$, for all$m, r \in X$. **See [2-4]**

2. every modular space is topological vector space and it is Hausdorff **[11]**

Now, For the definition of topological vector space

2.1.3. Definition: [1,2,10]

In the modular space X_M

1- The M-open ball $B_\varepsilon(m)$ with centre $m \in X_M$ and radius $\varepsilon > 0$ as

$B_\varepsilon(m) = \{r \in X_M : M(m - r) < \varepsilon\}$.

2- The M-closed ball $\underline{B_\varepsilon(m)}$ centred $m \in X_M$ with radius $\varepsilon > 0$ as

$\underline{B_r(m)} = \{r \in X_M : M(m - r) \leq \varepsilon\}$.

3- The family of all M-balls in X_M generates the topology makes X_M Hausdorff

4- Since every M-ball is convex, then every modular space is locally convex topological linear space.

5- Let X_M be a modular space and $E \subseteq X_M$ we say that E is M −open set if for every $m \in E$ there exist $\varepsilon > 0, \ni B_\varepsilon(m) \subset E$.

6- A subset E of X_M is said to be M −closed if its complement is M −open, that is, $E^c = X_M - E$ is M −open.

2.1.4. Definition:

Let X_M be a modular space over the field K, then the space of all continuous linear functional from X_M into the field K called the dual modular space and denoted by ${X_M}^*$

2.1.5. Remark:

The space ${X_M}^*$ is also modular space.

By defining $M^*: {X_M}^* \longrightarrow [0, \infty]$ as $M^*(f) = sup\{M(f(m): M(m) = 1, m \in X_M\}$

2.2. The weak topology

In this part, introduced notion of weak topology and some properties we needed it

2.2.1. Definition:[9]

Let A be a nonempty set and let $\{(A_\alpha, \tau_\alpha): \alpha \in \Delta\}$ be a nonempty family of topological spaces. For each $\alpha \in \Delta$, let f_α be a map of A into A_α. Then the topology τ on A generated by the family $G = \{f_\alpha^{-1}(G): G \in \tau_\alpha, \alpha \in \Delta\}$ is called the initial (weak) topology on A determined by the family $\{f_\alpha: \alpha \in \Delta\}$. G is defining subbase of τ and the family β of all finite intersections of members of G is called a basis of τ.

2.2.2. Remark:[7]

Let A a nonempty set with $\{(A_\alpha, \tau_\alpha): \alpha \in \Delta\}$ be a nonempty collection of topological spaces indexed by Δ. The weak (initial) topology generated by a collection of functions $F = \{f_\alpha: A \to A_\alpha, \alpha \in \Delta\}$ is the topology generated by the subbasis $G = \{f_\alpha^{-1}(G_\alpha): G_\alpha \in \tau_\alpha, \alpha \in \Delta\}$. Denoted to the topology generated by F on A by $\sigma(A, F)$.

2.2.3. Definition: [8]

A set G in A is said to be open in a topology $\sigma(A, F)$ if for all $z \in G$, there exists a finite subset I of Δ and open sets $\{G_\alpha\}_{\alpha \in I}$ such that $G_\alpha \subseteq A_\alpha$ for all $\alpha \in I$,$z \in \bigcap_{i=1}^{n} f_i^{-1}(G_i)$ that means that $\forall i \in I$, $f_i(z) \in G_i$.

2.2.4. Definition: [7]

In this part, X_M modular space over the field K where, $K = R$ or $K = C$, we don't assume that it is X_M complete.

suppose $f_\alpha: X_M \longrightarrow X_\alpha$ be a function and $X_\alpha = K$ and let $F = \{f_\alpha: \alpha \in \Delta\}$, and let $G = \{I \subseteq \Delta: I$ finite }.

Then the weak topology on X_M denoted by $\sigma(X_M, {X_M}^*)$ such that generated by F has the defining

$$\beta = \{\bigcap_{\alpha\in\Gamma} ⟦f_\alpha^{-1}(-\epsilon,\epsilon): I \in G , \epsilon > 0 \}⟧$$

So, a set E is a weak open in X_M if and only if given E, there exists $\alpha_1.\alpha_2, \dots, \alpha_n \in \Delta$ with $x \in \bigcap_{i=1}^{n} f_{\alpha_i}^{-1}(-\epsilon,\epsilon) \subseteq E$ that is $|f_{\alpha_i}(x)| < \epsilon$ for all $i = 1,2, \dots, n$

A subbasis of the weak open set containing $x_0 \in X_M$ is of the form

$$f_\alpha^{-1}(f_\alpha(x_0) - \epsilon, f_\alpha(x) - \epsilon)$$

for all $\alpha \in \Lambda$ and each $\varepsilon > 0$. Hence it can be as the form

$\beta(x_0; f_{.1}, f_2 \dots, f_n; \epsilon) = \{ y \in X: |f_\alpha(y) - f_\alpha(x_0)| < \epsilon\}$ for $f_i \in F, i = 1,2, \dots, n\ n \geq 1,\ \epsilon > 0.$

3. The main resulet

Let X_M be any modular space over a field K (where $K = R$ or $K = C$), then by definition of modular in X_M^* and by remark (2.1.5) X_M^* is a modular space. Therefore, a weak topology can be defined on X_M^* and generated by the family of all bounded linear function from X_M^* in to the field K ; that's nothing but the weak topology $\sigma(X_M^*, X_M^{**})$. But**,** we interested in a weak topology generated by X_M i.e. the topology $\sigma(X_M^* , X_M)$, where X_M is a subspace embedded in X_M^{**} such that every element of X_M is written as a bounded linear function from $\sigma(X_M^*, X_M^{**})$ into K by the canonical map $\Psi: X_M \longrightarrow X_M^{**}$ and given by $\Psi(x) = p_x$ where $p_x(f) = f(x)$ for every $f \in X_M^*$ with $M(p_x) = sup\{|f(x)|: f \in S_{X^*}\} = M(x)$ for each $x \in X_M$. Since Ψ is an isometry, then can be concluded that X_M is isometrically-isomorphic $\Psi(X_M)$.

If $\Psi(X_M) = X_M^{**}$, then X_M called reflexive.

In the following, we introduced a definition of the open and close sets in $\sigma(X_M^* , X_M)$.

3.1. Definition: A set E in the modular space X_M^* is said to be weak-star open set (W^*-open) if and only if for each function $f \in E$ there is $\varepsilon > 0$ and $x_1, x_2, \dots, x_n \in X_M$ such that $\{g \in X_M^*: |(g - f)(x_i)| < \varepsilon\} \subseteq E$ where $i = 1,2, \dots, n$ and $n \geq 1$. A set E is called w^*-closed if the complement is w^*-open set.

3.2. Definition: Let X_M be a modular space, the weak topology $\sigma(X_M^* , X_M)$ consist of all weak-star open sets in X_M^* is called weak-star topology (w^*-topology) on X_M^* and denoted by $\sigma(X_M^* , X_M)$.

Note that: Since $X_M \subseteq X_M^{**}$, then the w^*-topology $\sigma(X_M^* , X_M)$ is weaker than the topology $\sigma(X_M^*, X_M^{**})$

3.3. Remark: If X_M is reflexive, then then the weak topology on X_M^* and the weak-star topology of X_M^*, are the same; $\sigma(X_M^*, X_M^{**}) = \sigma(X_M^*, X_M)$.

Now we introduce the local base of the weak-topology of X_M^* in next theorem

3.4.Theorem: Let $f_0 \in X_M^*$. A local base of f_0 for the weak-star topology of X_M^* is given by the collection of open balls of the form $\beta(\varepsilon, x_1, x_2, \dots, x_n) = \{f \in X_M^*: for\ all\ i = 1,2,\dots,n, |f(x_i) - f_0(x_i) < \varepsilon\}, \varepsilon > 0, |\ x_1, x_2, \dots, x_n \in X_M$

Proof: Since the weak-star topology of X_M^* generated by X_M has the basis $\beta = \{\bigcap_{\alpha\in\Gamma} ⟦x_\alpha^{-1}(-\varepsilon, \varepsilon): \Gamma \in G, \varepsilon > 0\}⟧$ where $G = \{\Gamma \subseteq \Delta: \Gamma$ finite $\}$. and Δ any index for X_M. Thus a set E is w^*-open in X_M^* iff given E, there exists $\alpha_1.\alpha_2, \dots, \alpha_n \in \Delta$ with $f \in \bigcap_{i=1}^n x_{\alpha_i}^{-1}(-\varepsilon_i, \varepsilon_i) \subseteq E$ implies that $|x_{\alpha_i}(f)| < \varepsilon_i$ for $i = 1,2,\dots,n$. A sub basis open set containing a point $f_0 \in X_M$ is of the form $x_\alpha^{-1}(x_\alpha(f_0) - \varepsilon, x_\alpha(f_0) + \varepsilon)$ for all $\alpha \in \Lambda$ and each $\varepsilon > 0$. Hence it can be of the form $\beta(\varepsilon, x_1, x_2, \dots, x_n) = \{f \in X_M^*: for\ all\ i = 1,2,\dots,n, |f(x_i) - f_0(x_i) < \varepsilon\}, \varepsilon > 0, |, x_1, x_2, \dots, x_n \in X_M$

The following theorem is very important to study the properties of the topology $\sigma(X_M^*, X_M)$ because it shows whether the limit point is unique or not.

3.5. Theorem: Let X_M be a modular space over the field K, then the w^*-topology of X_M^* is Hausdorff.

Proof: Let $f, g \in X_M^*$ with $f \neq g$, then $f(x) \neq g(x)$ for some $x \in X_M$.

Let $y \in \{x \in X_M: f(x) \neq g(x)\}$, then either $f(y) < g(y)$ or $f(y) > g(y)$ and in both cases can be founded $\gamma \in R$ such that either $f \in y^{-1}((-\infty, \gamma))$ and $g \in y^{-1}((\gamma, \infty))$ or converse.

Thus there are two disjoint sets in $\sigma(X_M^*, X_M)$ separate f and g, hence the weak-star topology is Hausdorff space.

3.6. Definition: a sequence $\{f_n\}$ in the dual modular space X_M^* is w^*-convergent to a function f and denoted by $f_n\ w^* \to f$ if it converges to f in the topology $\sigma(X_M^*, X_M)$.

The next theorem is to redefine the convergence property in w^*-topology of X_M^*.

3.7. Theorem: Let X_M be a modular space, a sequence $\{f_n\}$ in the dual space of the modular space X_M is said to be w^*-convergent to a function $f \in X_M^*$ if and only if for every $\varepsilon > 0$ and for each element x in X_M, there exists $k \in Z^+$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > k$; this $f_n\ w^* \to f$ if and only if $f_n(x) = f(x)$.

Proof: suppose that $\{f_n\}$ is a sequence in X_M^*.

Firstly, take $f_n\ w^* \to f$. Let $\varepsilon > 0$ and $E \in \sigma(X_M^*, X_M)$ s.t. $E = \{h \in X_M^*: |h(x) - f(x)| < \varepsilon\}$ for each x in X_M. Since $f_n\ w^* \to f$, then by definition of w^*-convergent can be shown there is $k \in Z^+$

such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > k$ and each element in X_M. Thus for each w^*-open set E containing $f(x)$, there is $k \in Z^+$ with $f_n(x) \in E$ for all $n > k$; $f_n\ (x) = f(x)$.

Conversely, when $f_n\ (x) = f(x)$ for all x in X_M. Let $E \in \sigma(X_M^*\ , X_M)$ such that E containing $f(x)$. There exists $\varepsilon > 0$ and a finite number of elements $x_1, x_2, \dots, x_r$ of X_M with $\{h \in X_M^*: |h(x_i) - f(x_i)| < \varepsilon, i = 1,2, \dots, r\} \subseteq E$.

Since $f_n(x_i) \longrightarrow f(x_i)$ for $i = 1,2, \dots, r$, then there exists $k_i \in Z^+$where $i = 1,2, \dots, r$ with $|f_n(x_i) - f(x_i)| < \varepsilon$ for all $n > k_i$.

By choosing $k = \{k_1, k_2, \dots, k_r\}$. Then for each $i = 1,2, \dots, r$, we have $|f_n(x_i) - f(x_i)| < \varepsilon$ for all $n > k$. Thus $f_n \in E$ for all $n > k$: that is $f_n\ w^* \rightarrow\ f$.

Here some properties of w^*-topology of X_M^*

3.8. Proposition: let X_M be a modular linear space and $\{f_n\}$ be a sequence in X_M^* then the following properties are holding;

1. if $f_n \longrightarrow f$ in X_M^*, then $f_n\ \longrightarrow f$ in w^*-topology.

2. if $f_n\ w^* \rightarrow f$ and $\{x_n\}$ a sequence in X_M with $x_n \longrightarrow x \in X_M$, then $f_n(\ x_n) \longrightarrow f(x)$ as $n \longrightarrow \infty$.

Proof:

1. suppose that $\{f_n\}$ be a sequence in the dual modular space X_M^* with $f_n \longrightarrow f$, that's mean for all $x \in X_M$ the limit point by $\{f_n\}$ is exists, unique and equal to $f(x)$. Thus by **(3.6)** $f_n\ w^* \rightarrow f$

2. suppose that $\{f_n\} \subseteq X_M^*$ and $\{\ x_n\} \subseteq X_M$ such that $f_n\ w^* \rightarrow f$ and $x_n \longrightarrow x$. Let $\varepsilon > 0$ then there exists k_1 and $k_2 \in Z^+$ such that $|x_n - x| < \varepsilon, n >\ k_1$

and for all $x \in X_M$ $|f_n(x) - f(x)| < \varepsilon$. Choose $k = max\{\ k_1, k_2\}$, then we have $|f_n(\ x_n) - f(x)| < \varepsilon$. Thus $f_n(\ x_n) \longrightarrow f(x)$.

And the next theorem showed that under which condition the strong modular topology and the w^*-topology are coincided, as following

3.9. Theorem: let X_M be a modular space, if X_M finite-dimensional, then the weak-star topology of X_M^* and the modular topology on X_M^* are coinciding.

Proof: Since the weak –star topology of X_M^* is weaker than the modular topology on X_M^*, The **proof:** will be limited to proving the opposite side: every open set in the modular space X_M^* is w^*-open set.

Let E be open in the modular space X_M^* and $g \in\ E$. Then can be founded $\varepsilon > 0$ such that $g + B_{X_M^*}(\varepsilon) \subseteq\ E$ where $B_{X_M^*}(\varepsilon)$ is an open ball at the origin with radius $\varepsilon > 0$ in X_M^*. Since X_M finite-dimensional, then it has a basis β consists of an only finite number of elements. Now define $M^*(f) = max\{|f(e)|, e \in \beta\}$ for all $f \in X_M^*$, then $M^*: X_M^* \longrightarrow [0, \infty)$ is a modular space. Since all modulars on a finite-dimensional are equivalent, there is $\delta > 0$ with $M^*(f) < \delta$, we have $M(f) < \varepsilon$. Then the w^*-open $\{f \in X_M^*: max|f(e) - g(e)| < \varepsilon, e \in \beta\}$ is contained in $\{f \in X_M^*: M(f-g) < \varepsilon\}$. Hence E is w^*-open.

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On Nano soft-$\mathcal{J}$-semi-g-closed sets

R.B. Esmaeel[1] and R.J.Mohammad[2]

[1,2]*Department of Mathematics, College of Education for Pure Sciences, Ibn Al – Haitham / University of Baghdad, / Baghdad / Iraq*

[1]ranamumosa@yahoo.com [2]raf44456@gmail.com

[1]ORCID ID: https://orcid.org/0000-0002-4743-6034

Abstract.

In this paper, the notions of *nano* soft closed sets were introduced by using *nano*soft ideal and *nano* soft semi-open sets, which are *nano* soft-$\mathcal{J}$-semi-g-closed sets "n-s$\mathcal{J}$sgclosed" where many of the properties of these sets were clarified. Using many figures and proposition have been studied the relationships among these kinds of *nano* soft sets with some examples were explained.

Keywords.
Nano soft space , *nano* soft open set , *nano* soft closed set, *nano* soft semi open set , n-s$\mathcal{J}sg$-$c(\chi)_{\mathcal{H}}$ and n-s$\mathcal{J}sg$-$o(\chi)_{\mathcal{H}}$.

1. *Introduction*

In 2011, Shaber [1] established introduced soft topological spaces .They various studies have been introduced to study many topological properties by using soft set like derived sets, compactness, separation axioms and other properties. [2], [3], [4]. Also, use the soft ideal which is a family of soft sets that meet hereditary and finite additively property of χ to study the notion of soft logical function [5], which was the starting point for studying the properties of soft ideal topological spaces $(\chi, \mathcal{T}, \mathcal{H}, \mathcal{I})$ and defined new types of near open soft sets and study their properties as [6], [7], [8]. The notion of nano topology was introduced by Lellis Thivagar [9]. Based on that, Benchalli et al [10] introduced the notion of nano soft topological spaces using soft set equivalence relation on the universal set. Also, the notion of nano soft continuity and weaker forms of nano soft open sets namely nano soft semi open, nano soft pre-open, nano soft α-open and nano soft β-open sets in nano soft topological spaces are introduced and studied in [10] and [11].

2. Preliminaries

Definition 2.1. [12] Let $\chi \neq \emptyset$ and $\mathcal{H}$ be a set of parameters. Such that is$\wp(\chi)$ *the* collection of χ and $\mathcal{P} \neq \emptyset$ such that $\mathcal{P} \subseteq \mathcal{H}$. $(\Gamma, \mathcal{H})$ (briefly $\Gamma_{\mathcal{H}}$) is a soft set over χ whenever, Γ is a function such that $\Gamma : \mathcal{H} \to \wp(\chi)$. So, $\Gamma_{\mathcal{H}} = \{ \Gamma(\hbar): \hbar \in \mathcal{P} \subseteq \mathcal{H}, \Gamma : \mathcal{H} \to \wp(\chi) \}$. The collection of each soft sets (briefly $\c{S}\c{S}(\chi)_{\mathcal{H}}$).

Definition 2.2. [7] Let $(\mathcal{K}, ҿ), (\mathcal{G}, \omega) \in \c{S}\c{S}(\chi)_{\mathcal{H}}$, $(\Gamma, \mathcal{H}) = (\mathcal{K}, ҿ) \ \tilde{\cup}\ (\mathcal{G}, \omega)$ where , $\mathcal{H} = ҿ \cup \omega$ and $\forall\ \hbar \in \mathcal{H}$,

$$\Gamma(\hbar) = \begin{cases} \mathcal{K}(\hbar), \hbar \in ꞔ - ω, \\ \mathcal{G}(\hbar), \hbar \in ω - ꞔ, \\ \mathcal{K}(\hbar) \cup \mathcal{G}(\hbar), \hbar \in ꞔ \cap ω. \end{cases}$$

Definition 2.3. [7] Let $(\mathcal{K}, ꞔ), (\mathcal{G}, ω) \in ȘȘ(\chi)$, $(\Gamma, \mathcal{H}) = (\mathcal{K}, ꞔ) \tilde{\cap} (\mathcal{G}, ω)$. Where, $\mathcal{H} = ꞔ \cap ω$ and for each $\hbar \in \mathcal{H}$, $\Gamma(\hbar) = \mathcal{K}(\hbar) \cap \mathcal{G}(\hbar)$.

Definition 2.4. [1] Let $\mathcal{T}$ be a collection of soft sets over χ with same $\mathcal{H}$, then $\mathcal{T} \in ȘȘ(\chi)_{\mathcal{H}}$ is a soft topology on χ if;

i. $\tilde{X}, \tilde{\emptyset} \in \mathcal{T}$ where, $\tilde{\emptyset}(\hbar) = \emptyset$ and $\tilde{\chi}(\hbar) = \chi$, for each $\hbar \in \mathcal{H}$

ii. $\tilde{\bigcup}_{\alpha \in \Lambda} (O_\alpha, \mathcal{H}) \in \mathcal{T}$ whenever, $(O_\alpha, \mathcal{H}) \in \mathcal{T}$ $\forall \alpha \in \Lambda$,

iii. $((\Gamma, \mathcal{H}) \tilde{\cap} (\mathcal{G}, \mathcal{H})) \in \mathcal{T}$ for each $(\Gamma, \mathcal{H}), (\mathcal{G}, \mathcal{H}) \in \mathcal{T}$.

$(\chi, \mathcal{T}, \mathcal{H})$ is a soft topological space if $(O, \mathcal{H}) \in \mathcal{T}$ then $(O, \mathcal{H})$ is an open soft set.

Definition 2.5. [5] Let $\mathcal{I} \neq \emptyset$, then $\mathcal{I} \tilde{\subseteq} ȘȘ(\chi)_{\mathcal{H}}$ is a soft ideal whenever,

i. If $(\Gamma, \mathcal{H}) \tilde{\in} \mathcal{I}$ and $(\mathcal{G}, \mathcal{H}) \tilde{\in} \mathcal{I}$ implies, $(\Gamma, \mathcal{H}) \tilde{\cup} (\mathcal{G}, \mathcal{H}) \tilde{\in} \mathcal{I}$.

ii. If $(\Gamma, \mathcal{H}) \tilde{\in} \mathcal{I}$ and $(\mathcal{G}, \mathcal{H}) \tilde{\subseteq} (\Gamma, \mathcal{H})$ implies, $(\mathcal{G}, \mathcal{H}) \tilde{\in} \mathcal{I}$.

Definition 2.6. [5] Any $(\chi, \mathcal{T}, \mathcal{H})$ with a soft ideal $\mathcal{I}$ is namely a soft ideal topological space (briefly $(\chi, \mathcal{T}, \mathcal{H}, \mathcal{I})$).

Definition 2.7. [12] Let $(\Gamma, \mathcal{H}), (\mathcal{G}, \mathcal{H}) \in ȘȘ(\chi)_{\mathcal{H}}$. Then $(\Gamma, \mathcal{H})$ is a soft subset of, $(\mathcal{G}, \mathcal{H})$, (briefly$(\Gamma, \mathcal{H}) \tilde{\subseteq}, (\mathcal{G}, \mathcal{H})$), if $\Gamma(\hbar) \tilde{\subseteq} \mathcal{G}(\hbar)$, for all $\hbar \in \mathcal{H}$. Now $(\Gamma, \mathcal{H})$ is a soft subset of , $(\mathcal{G}, \mathcal{H})$ and $(\mathcal{G}, \mathcal{H})$is a soft super set of $(\Gamma, \mathcal{H})$, $(\Gamma, \mathcal{H}) \tilde{\subseteq}, (\mathcal{G}, \mathcal{H})$.

Definition 2.8. [13] The complement of $(\Gamma, \mathcal{H})$ (briefly $(\Gamma, \mathcal{H})'$) $(\Gamma, \mathcal{H})' = (\Gamma', \mathcal{H})$, $\Gamma': \mathcal{H} \to \wp(\chi)$ is a function such that $\Gamma'(\hbar) = \chi - \Gamma(\hbar)$, for all $\hbar \in \mathcal{H}$ and Γ' is namely the soft complement of Γ.

Definition 2.9. [1] Let $(\Gamma, \mathcal{H})$ be a soft over χ and $x \in \chi$. Then $x \tilde{\in} (\Gamma, \mathcal{H})$, whenever, $x \in \Gamma(\hbar)$ for each $\hbar \in \mathcal{H}$.

Definition 2.10. [12] $(\Gamma, \mathcal{H})$ is a NULL soft set (briefly $\tilde{\emptyset}$ or $\emptyset_{\mathcal{H}}$) whenever, $\forall \hbar \in \mathcal{H}$, $\Gamma(\hbar) = \emptyset$.

Definition 2.11. [12] $(\Gamma, \mathcal{H})$ is an absolute soft set (briefly $\tilde{\chi}$ or $\chi_{\mathcal{H}}$) whenever, $\forall \hbar \in \mathcal{H}$, $\Gamma(\hbar) = \chi$.

Definition 2.12. [14]. Let $(\mathcal{P},\mathcal{A}),(\mathcal{W},\mathcal{B}) \in ȘȘ(\chi)_{\mathcal{H}}$, then the Cartesian product of $(\mathcal{P}, \mathcal{A})$ and $(\mathcal{W}, \mathcal{B})$ is defined as $(\mathcal{P}, \mathcal{A}) \times (\mathcal{W}, \mathcal{B}) = (Ħ, \mathcal{A}\times\mathcal{B})$, where $Ħ : \mathcal{A}\times\mathcal{B} \to \wp(\chi \times \chi)$ and for each $(a, b) \in \mathcal{A} \times \mathcal{B}$ ، $Ħ(a, b) = \{(h_i, h_j) : h_i \in \mathcal{P}(a)$ and $h_j \in \mathcal{W}(b)\} = \mathcal{P}(a) \times \mathcal{P}(b)$.

Theorem 2.13. [14]. Let $(\mathcal{P}, \mathcal{A})$ and $(\mathcal{W}, \mathcal{B})$ be two soft sets over a universe χ. Then a soft set relation from $(\mathcal{P}, \mathcal{A})$ to $(\mathcal{W}, \mathcal{B})$ is a soft subset of $(\mathcal{P}, \mathcal{A}) \times (\mathcal{W}, \mathcal{B})$. In other words, a soft set relation from $(\mathcal{P}, \mathcal{A})$ to $(\mathcal{W}, \mathcal{B})$ is of the form $(Ħ_1, S)$, where $S \subseteq \mathcal{A} \times \mathcal{B}$ and $Ħ_1(a, b) = Ħ(a, b)$, for all $(a, b) \in S$, where $(Ħ, \mathcal{A}\times\mathcal{B}) = (\mathcal{P}, \mathcal{A})\times(\mathcal{W}, \mathcal{B})$ as in the above definition. In an equivalent way, we can define the soft set relation $\mathfrak{R}$ on $(\mathcal{P}, \mathcal{A})$ in the parameterized form as follows: if $(\mathcal{P}, \mathcal{A}) = \{(a),(b),...\}$, then $\mathcal{P}(a)\mathfrak{R}\mathcal{P}(b) \Leftrightarrow \mathcal{P}(a) \times \mathcal{P}(b) \in \mathfrak{R}$.

Definition 2.14. [14]**.** Let $\mathfrak{R}$ be a relation on $(F,\mathcal{H})$.

- **i.** $\mathfrak{R}$ is reflexive, if $Ħ_1(a, a) \in \mathfrak{R}, \forall a \in \mathcal{H}$
- **ii.** $\mathfrak{R}$ is symmetric, if $Ħ_1(a, b) \in \mathfrak{R} \Rightarrow Ħ_1(b, a) \in \mathfrak{R}, \forall (a, b) \in \mathcal{H} \times \mathcal{H}$
- **iii.** $\mathfrak{R}$ is transitive, if $Ħ_1(a, b) \in \mathfrak{R}, Ħ_1(b, c) \in \mathfrak{R} \Rightarrow Ħ_1(a, c) \in \mathfrak{R}, \forall a, b, c \in \mathcal{H}$.

Definition 2.15. [14]. A soft set relation $\mathfrak{R}$ on a soft set $(\mathcal{P}, \mathcal{A})$ is called an equivalence relation, if it
is reflexive, symmetric and transitive.

Example 2.16. [14]. Consider a soft set $(\mathcal{P},\mathcal{H})$ over χ, where $\chi = \{c_1, c_2, c_3, c_4\}$ $\mathcal{H} = \{\hbar_1, \hbar_2\}$ and $F(\hbar_1) = \{c_1, c_3\}, F(\hbar_2) = \{c_2, c_4\}$**.** Consider a relation $\mathfrak{R}$ defined on $(\mathcal{P},\mathcal{H})$ as follows: $\mathfrak{R} = \{F(\hbar_1)\times F(\hbar_2), F(\hbar_2)\times F(\hbar_1), F(\hbar_1)\times F(\hbar_1), F(\hbar_2)\times F(\hbar_2)\}$. Then $\mathfrak{R}$ is a soft set equivalence relation.

Definition 2.17. [14]. Let $(\mathcal{P},\mathcal{H})$ be a soft set. Then equivalence class of $\mathcal{P}(a)$ denoted by $[\mathcal{P}(a)]$ is defined as follows:

$$[\mathcal{P}(a)] = \{\mathcal{P}(b) : \mathcal{P}(b)\mathfrak{R}\mathcal{P}(a)\}.$$

Definition 2.18. [10]. Let χ be a non-empty finite set which called the universe and $\mathcal{H}$ be a set of parameters. Let $\mathfrak{R}$ be a soft equivalence relation on χ. Then $(\chi, \mathfrak{R}, \mathcal{H})$ is called the soft approximation space. Let $\mathcal{A} \subseteq \chi$.

- **i.** The soft lower approximation of $\mathcal{A}$ w. r. t. $\mathfrak{R}$ and the set of parameters $\mathcal{H}$ is the set of all objects, which can be for certain classifieds $\mathcal{A}$ w. r. t. $\mathfrak{R}$ and it is denoted by $(L_{\mathfrak{R}}(\mathcal{A}),\mathcal{H})$, equivalently

$$(L_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}) = \cup\{\mathfrak{R}(x) : \mathfrak{R}(x) \subseteq \mathcal{A}\}$$

where $\mathfrak{R}(x)$ denotes the equivalence class determined by $x \in \chi$.

ii. The soft upper approximation of $\mathcal{A}$ w. r. t. $\mathfrak{R}$ and the set of parameters $\mathcal{H}$ is the set of all objects, which can be possibly classified as $\mathcal{A}$ w. r. t. $\mathfrak{R}$ and it is denoted by $(U_{\mathfrak{R}}(\mathcal{A}),\mathcal{H})$, equivalently

$$(U_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}) = \cup\{\mathfrak{R}(x) : \mathfrak{R}(x) \cap \mathcal{A} \neq \emptyset\}.$$

iii. The soft boundary region of $\mathcal{A}$ w. r. t. $\mathfrak{R}$ and the set of parameters $\mathcal{H}$ is the set of all objects , which can be classified neither inside $\mathcal{A}$ nor as outside $\mathcal{A}$ with respect to $\mathfrak{R}$ and is denoted by $(B_{\mathfrak{R}}(\mathcal{A}),\mathcal{H})$, equivalently

$$(B_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}) = (U_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}) - (L_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}).$$

Definition 2.19. [10]. Let $\chi \neq \emptyset$ and $\mathcal{H}$ be a set of parameters. Let $\mathfrak{R}$ be a soft equivalence relation on χ. Let $\mathcal{A} \subseteq \chi$ and let $\mathcal{T}_{\mathfrak{R}}(\mathcal{A}) = \{\tilde{\chi}, \tilde{\emptyset}, (L_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}), (U_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}), (B_{\mathfrak{R}}(\mathcal{A}),\mathcal{H})\}$. Then $\mathcal{T}_{\mathfrak{R}}(\mathcal{A})$ is a soft topology on $(\chi ,\mathcal{H})$. In this case, $\mathcal{T}_{\mathfrak{R}}(\mathcal{A})$ is called the *nano* soft topology with respect to A . Elements of the *nano* soft topology are known as the *nano* soft open sets and $(\mathcal{T}_{\mathfrak{R}}(\mathcal{A}),\chi ,\mathcal{H})$ is called a *nano* soft topological space. The complements of *nano* soft open sets are called as *nano* soft closed sets in $(\mathcal{T}_{\mathfrak{R}}(\mathcal{A}),\chi ,\mathcal{H})$.

Definition 2.20. [11]. Let $(\mathcal{T}_{\mathfrak{R}}(\mathcal{A}),\chi ,\mathcal{H})$ be a nano soft topological space and $(\mathcal{P},\mathcal{H})$ be any soft set over χ. Then $(\mathcal{P},\mathcal{H})$ is said to be *nano* soft semi-open if $(\mathcal{P},\mathcal{H}) \subseteq n\text{-}cl(n\text{-}int(\mathcal{P},\mathcal{H}))$. Here $n\text{-}int(\mathcal{P},\mathcal{H})$ is the *nano* soft interior of $(\mathcal{P},\mathcal{H})$, which is the union of all *nano* soft open sets contained in $(\mathcal{P},\mathcal{H})$ and $n\text{-}cl(\mathcal{P},\mathcal{H})$ is the *nano* soft closure of $(\mathcal{P},\mathcal{H})$, which is the intersection of all *nano* soft closed sets containing $(\mathcal{P},\mathcal{H})$. Also, here $n\text{-}ŞŞO(\chi ,\mathcal{H})$ denotes the family of all *nano* soft semi-open sets over χ with respect to an equivalence relation $\mathfrak{R}$ and parameter set $\mathcal{H}$.

Example 2.21. Let $\chi = \{1,2,3,4\}$, $\mathcal{H} = \{\hbar_1, \hbar_2, \hbar_3\}$ and let $(\mathcal{P},\mathcal{H}) = \{(\hbar_1,\{1\}),(m_2,\{3\}),(m_3,\{2,4\})\}$ be a soft set over χ. Let $\mathfrak{R}$ be a soft equivalence relation on $(\mathcal{P},\mathcal{H})$ defined as follows:
$\mathfrak{R} = \{ F(\hbar_1)\times F(\hbar_2) , F(\hbar_2)\times F(\hbar_1) , F(\hbar_1)\times F(\hbar_1) , F(\hbar_2)\times F(\hbar_2) , F(\hbar_3) \times F(\hbar_3) \}$. Then the soft equivalence classes are as follows:
$[F(\hbar_1)] = \{F(b) : F(b)\mathfrak{R}F(a)\} = \{F(\hbar_1),F(\hbar_2)\} = [F(\hbar_2)]$, and $[F(\hbar_3)] = \{F(\hbar_3)\}$. Now, let $\chi / \mathfrak{R} = \{F(\hbar_1) , F(\hbar_2) , F(\hbar_3)\} = \{\{1\},\{3\},\{2,4\}\}$. Let $\mathcal{A} = \{1,2\} \subseteq \chi$. Then $(L_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}) = \{(\hbar_1,\{1\}),(\hbar_2,\{1\}),(\hbar_3,\{1\})\}$, $(U_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}) = \{(\hbar_1,\{1,2,4\}),(\hbar_2,\{1,2,4\}),(\hbar_3,\{1,2,4\})\}$, $(B_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}) = \{(\hbar_1,\{2,4\}),(\hbar_2,\{2,4\}),(\hbar_3,\{2,4\})\}$.
Thus $(\mathcal{T}_{\mathfrak{R}}(\mathcal{A}), \chi , \mathcal{H}) =\{\tilde{\chi} , \tilde{\emptyset} , (L_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}) , (U_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}) , (B_{\mathfrak{R}}(\mathcal{A}),\mathcal{H})\}$. is a soft *nano* topology on χ. So soft *nano* open sets are $\{\tilde{\chi} , \tilde{\emptyset} , (L_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}), \mathcal{H}) , (U_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}) , (B_{\mathfrak{R}}(\mathcal{A}),\mathcal{H})\}$. and soft *nano* semi open sets are $\tilde{\chi}$, $\tilde{\emptyset}$,$(\mathcal{A}_1,\mathcal{H})$,$(\mathcal{A}_2,\mathcal{H})$,$(\mathcal{A}_3,\mathcal{H})$,$(\mathcal{A}_4,\mathcal{H})$ and $(\mathcal{A}_5,\mathcal{H})$ where

$(\mathcal{A}_1,\mathcal{H})$ = {($\hbar_1$,{1}) , ($\hbar_2$,{1}) , ($\hbar_3$,{1})}, $(\mathcal{A}_2,\mathcal{H})$ = {($\hbar_1$,{1,3}) , ($\hbar_2$,{1,3}) , ($\hbar_3$,{1,3})}, $(\mathcal{A}_3,\mathcal{H})$ = {($\hbar_1$,{2,4}) , ($\hbar_2$,{2,4}) , ($\hbar_3$,{2,4})}, $(\mathcal{A}_4,\mathcal{H})$ = {($\hbar_1$,{1,2,4}) , ($\hbar_2$,{1,2,4}) , ($\hbar_3$,{1,2,4})}, $(\mathcal{A}_5,\mathcal{H})$ = {($\hbar_1$,{2,3,4}) , ($\hbar_2$,{2,3,4}) , ($\hbar_3$,{2,3,4})}.

Example 2.22. Let χ = {1,2,3}, $\mathcal{H}$ = {$\hbar_1$,$\hbar_2$} and let $(\mathcal{P},\mathcal{H})$ = {($\hbar_1$,{1}),($\hbar_2$,{2})} be a soft set over χ. Let $\mathfrak{R}$ be a soft equivalence relation on $(\mathcal{P},\mathcal{H})$ defined as follows: $\mathfrak{R}$ = {F($\hbar_1$)×F($\hbar_2$) , F($\hbar_2$)×F($\hbar_1$) , F($\hbar_1$)×F($\hbar_1$) , F($\hbar_2$)×F($\hbar_2$)}. Now let χ /$\mathfrak{R}$ = {F($\hbar_1$),F($\hbar_2$)} = {{1},{2}}. Then as the following table:

$\mathcal{A}$	**($L_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}$)**	**($U_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}$)**	**($B_{\mathfrak{R}}(\mathcal{A}),\mathcal{H}$)**	$\mathcal{T}_{\mathfrak{R}}(\mathcal{A})$
∅	$\tilde{\emptyset}$	$\tilde{\emptyset}$	$\tilde{\emptyset}$	{ $\tilde{\emptyset}$, $\tilde{\chi}$ }
χ	$(\mathcal{A}_3,\mathcal{H})$	$(\mathcal{A}_3,\mathcal{H})$	$\tilde{\emptyset}$	{ $\tilde{\emptyset}$, $\tilde{\chi}$,$(\mathcal{A}_3,\mathcal{H})$}
{1}	$(\mathcal{A}_1,\mathcal{H})$	$(\mathcal{A}_1,\mathcal{H})$	$\tilde{\emptyset}$	{$\tilde{\emptyset}$, $\tilde{\chi}$, $(\mathcal{A}_1,\mathcal{H})$}
{2}	$(\mathcal{A}_2,\mathcal{H})$	$(\mathcal{A}_2,\mathcal{H})$	$\tilde{\emptyset}$	{$\tilde{\emptyset}$, $\tilde{\chi}$, $(\mathcal{A}_2,\mathcal{H})$}
{3}	$\tilde{\emptyset}$	$\tilde{\emptyset}$	$\tilde{\emptyset}$	{ $\tilde{\emptyset}$, $\tilde{\chi}$ }
{1,2}	$(\mathcal{A}_3,\mathcal{H})$	$(\mathcal{A}_3,\mathcal{H})$	$\tilde{\emptyset}$	{$\tilde{\emptyset}$, $\tilde{\chi}$, $(\mathcal{A}_3,\mathcal{H})$}
{2,3}	$(\mathcal{A}_2,\mathcal{H})$	$(\mathcal{A}_2,\mathcal{H})$	$\tilde{\emptyset}$	{$\tilde{\emptyset}$, $\tilde{\chi}$, $(A_2,\mathcal{H})$}
{1,3}	$(\mathcal{A}_1,\mathcal{H})$	$(\mathcal{A}_1,\mathcal{H})$	$\tilde{\emptyset}$	$\tilde{\emptyset}$, $\tilde{\chi}$, $(\mathcal{A}_1,\mathcal{H})$}

Table 1

Such that $(\mathcal{A}_1,\mathcal{H})$ = {($\hbar_1$, {1}),($\hbar_2$, {1})}, $(\mathcal{A}_2,\mathcal{H})$ = {($\hbar_1$, {2}),($\hbar_2$, {2})} and $(\mathcal{A}_3,\mathcal{H})$ = {($\hbar_1$, {1,2}),($\hbar_2$, {1,2})}.

3. On $nano\ soft\text{-}\mathcal{I}\text{-}semi\text{-}g\text{-}closed$ set.

Definition 3.1. In $(\chi, \mathcal{T}_{\mathfrak{R}}(\mathcal{A}), \mathcal{I})$, the subset $(\mathcal{B}, \mathcal{H}) \in ȘȘ(\chi)_{\mathcal{H}}$ is a $nano$ soft-$\mathcal{I}$-semi-g-closed set (briefly, n-s$\mathcal{I}sg$-$closed$**)**, if $cl((\mathcal{B}, \mathcal{H})) - (\mathfrak{D}, \mathcal{H}) \in \mathcal{I}$ whenever, $(\mathcal{B}, \mathcal{H}) - (\mathfrak{D}, \mathcal{H}) \in \mathcal{I}$ and $(\mathfrak{D}, \mathcal{H})$ is $nano$ soft semi-open set. The complement of $(\mathcal{B}, \mathcal{H})$ is $nano$ soft-$\mathcal{I}$-semi-g-open set (briefly, n-s$\mathcal{I}sg$-open**)**. The summaries n-s$\mathcal{I}sg$-$c(\chi)_{\mathcal{H}}$ and n-s$\mathcal{I}sg$-$o(\chi)_{\mathcal{H}}$ are the family of all n-s$\mathcal{I}sg$-closed and n-s$\mathcal{I}sg$-open sets respectively.

Example 3.2. From table 1 let $\mathcal{I} = \{\emptyset\}$ is the ideal, the family of all n-s$\mathcal{I}sg$-closed (respectively, n-s$\mathcal{I}sg$-$open$) sets can be determined, according to the given $\mathcal{T}_{\mathfrak{R}}(\mathcal{A})$ and n-ȘȘO(χ) in the previous table as the following table;

A	$\mathcal{T}_{\mathfrak{R}}$**(A)**	n-**ȘȘO(χ)**	n-s$\mathcal{I}sg$ **-c(χ)$_{\mathcal{H}}$**	n-s$\mathcal{I}sg$ **-o(χ)$_{\mathcal{H}}$**
∅	$\{\tilde{\emptyset}, \tilde{\chi}\}$	$\{\tilde{\emptyset}, \tilde{\chi}\}$	$ȘȘ(\chi)_{\mathcal{H}}$	$ȘȘ(\chi)_{\mathcal{H}}$
X	$\{\tilde{\emptyset}, \tilde{\chi}\}$	$\{\tilde{\emptyset}, \tilde{\chi}\}$	$ȘȘ(\chi)_{\mathcal{H}}$	$ȘȘ(\chi)_{\mathcal{H}}$
{1}	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{A}_1, \mathcal{H})\}$	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{P}, \mathcal{H});$ $1 \tilde{\in} \mathcal{P}(\hbar)\ \forall \hbar\}$	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{A}'_1, \mathcal{H})\}$	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{A}_1, \mathcal{H})\}$
{2}	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{A}_2, \mathcal{H})\}$	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{W}, \mathcal{H});$ $2 \tilde{\in} \mathcal{W}(\hbar)\ \forall \hbar\}$	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{A}'_2, \mathcal{H})\}$	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{A}_2, \mathcal{H})\}$
{3}	$\{\tilde{\emptyset}, \tilde{\chi}\}$	$\{\tilde{\emptyset}, \tilde{\chi}\}$	$ȘȘ(\chi)_{\mathcal{H}}$	$ȘȘ(\chi)_{\mathcal{H}}$
{1,2}	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{A}_3, \mathcal{H})\}$	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{M}, \mathcal{H});$ $\{1,2\} \tilde{\in} \mathcal{M}(\hbar)\ \forall \hbar\}$	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{A}'_3, \mathcal{H})\}$	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{A}_3, \mathcal{H})\}$
{2,3}	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{A}_2, \mathcal{H})\}$	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{N}, \mathcal{H});$ $\{2\} \tilde{\in} \mathcal{N}(\hbar)\ \forall \hbar\}$	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{A}'_2, \mathcal{H})\}$	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{A}_2, \mathcal{H})\}$
{1,3}	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{A}_1, \mathcal{H})\}$	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{Z}, \mathcal{H});$ $\{1\} \tilde{\in} \mathcal{Z}(\hbar)\ \forall \hbar\}$	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{A}'_1, \mathcal{H})\}$	$\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{A}_1, \mathcal{H})\}$

Table 2

Remark 3.3.

i. Every n-soft closed set in $(\chi, \mathcal{T}_{\mathfrak{R}}(\mathcal{A}), \mathcal{H})$ is n-s$\mathcal{I}sg$-closed in $(\chi, \mathcal{T}_{\mathfrak{R}}(\mathcal{A}), \mathcal{H}, \mathcal{I})$.

ii. Every n-soft open set in $(\chi, \mathcal{T}_{\mathfrak{R}}(\mathcal{A}), \mathcal{H})$ is n-s$\mathcal{I}sg$-open in $(\chi, \mathcal{T}_{\mathfrak{R}}(\mathcal{A}), \mathcal{H}, \mathcal{I})$.

Proof:

i. Let $(\mathcal{P}, \mathcal{H})$ be any soft closed set in $(\chi, \mathcal{T}_{\mathfrak{R}}(\mathcal{A}), \mathcal{H}, \mathcal{I})$ and $(\mathcal{O}, \mathcal{H})$ be a *nano* soft semi-open set such that $(\mathcal{P}, \mathcal{H}) - (\mathcal{O}, \mathcal{H}) \in \mathcal{I}$, but $cl(\mathcal{P}, \mathcal{H}) = (\mathcal{P}, \mathcal{H})$, since $(\mathcal{P}, \mathcal{H})$ is a soft closed set so, $cl(\mathcal{P}, \mathcal{H}) - (\mathcal{O}, \mathcal{H}) = (\mathcal{P}, \mathcal{H}) - (\mathcal{O}, \mathcal{H}) \in \mathcal{I}$ this implies $(\mathcal{P}, \mathcal{H})$ is a nano soft-$\mathcal{I}$-semi-g-closed set.

ii. Let $(\mathcal{O}, \mathcal{H})$ be any soft open set in $(\chi, \mathcal{T}_{\mathfrak{R}}(\mathcal{A}), \mathcal{H}, \mathcal{I})$ then $\tilde{\chi} - (\mathcal{O}, \mathcal{H})$ is a soft closed set this implies by (i) $(\tilde{\chi} - (\mathcal{O}, \mathcal{H}))$ is a n-s$\mathcal{I}$sg-closed set thus $(\mathcal{O}, \mathcal{H})$ is a n-s$\mathcal{I}$sg-open soft set .

In this remark, the opposite is not true. By Example 3.2 , if the set $\mathcal{A} = \chi$ then $\mathcal{T}_{\mathfrak{R}}(\mathcal{A}) = \{\tilde{\emptyset}, \tilde{\chi}\}$ and n-s$\mathcal{I}sg$-$c(\chi)_{\mathcal{H}} = ŞŞ(\chi)_{\mathcal{H}}$ and n-s$\mathcal{I}sg$-$C(\chi)_{\mathcal{H}} = ŞŞ(\chi)_{\mathcal{H}}$.

4. *On nano soft kernel of set.*

Definition 4.1. In $(\chi, \mathcal{T}_{\mathfrak{R}}(\mathcal{A}), \mathcal{H})$, if $(\mathcal{B}, \mathcal{H}) \subseteq \tilde{\chi}$, then n-s-$\mathcal{K}er((\mathcal{B}, \mathcal{H})) = \tilde{\bigcap}\{(\mathcal{O}, \mathcal{H})\ ;\ (\mathcal{B}, \mathcal{H}) \tilde{\subseteq} (\mathcal{O}, \mathcal{H}), (\mathcal{O}, \mathcal{H}) \in \mathcal{T}_{\mathfrak{R}}(\mathcal{A})\}$ which is shortcut for *nano* soft-*kernal* of $(\mathcal{B}, \mathcal{H})$.

Example 4.2. Let $\chi = \{1,2\}$, $\mathcal{H} = \{\hbar_1, \hbar_2\}$ and let $(\mathcal{P}, \mathcal{H}) = \{(\hbar_1, \{1\}), (m_2, \{2\})\}$ be a soft set over χ. Let $\mathfrak{R}$ be a soft equivalence relation on $(\mathcal{P}, \mathcal{H})$ defined as follows:

$\mathfrak{R} = \{F(\hbar_1)\times F(\hbar_2), F(\hbar_2)\times F(\hbar_1), F(\hbar_1)\times F(\hbar_1), F(\hbar_2)\times F(\hbar_2)\}$. Then the soft equivalence classes are as follows:

Now, let $\chi / \mathfrak{R} = \{F(\hbar_1), F(\hbar_2)\} = \{\{1\},\{2\}\}$. Let $\mathcal{A} = \{1\} \subseteq \chi$. Then $(L_{\mathfrak{R}}(\mathcal{A}), \mathcal{H}) = \{(\hbar_1, \{1\}), (\hbar_2, \{1\})\}$, $(U_{\mathfrak{R}}(\mathcal{A}), \mathcal{H}) = \{(\hbar_1, \{1\}), (\hbar_2, \{1\})\}$, $(B_{\mathfrak{R}}(\mathcal{A}), \mathcal{H}) = \{(\hbar_1, \emptyset), (\hbar_2, \emptyset)\}$. Thus $\mathcal{T}_{\mathfrak{R}}(\mathcal{A}) = \{\tilde{\chi}, \tilde{\emptyset}, (\mathcal{B}, \mathcal{H}) = \{(\hbar_1, \{1\}), (\hbar_2, \{1\})\}$, then according to the given $(\mathcal{B}, \mathcal{H}) \in ŞŞ(\chi)_{\mathcal{H}}$, n-s-$\mathcal{K}er(\mathcal{B})$ can be determined in the following table:

$(\mathcal{B}, \mathcal{H}) \in ŞŞ(\chi)_{\mathcal{H}}$	n-s-$\mathcal{K}er((\mathcal{B}, \mathcal{H}))$
$\tilde{\emptyset}$	$\tilde{\emptyset}$
$\tilde{\chi}$	$\tilde{\chi}$
$(\mathcal{B}_1, \mathcal{H}) = \{(\hbar_1, \{1\}), (\hbar_2, \{1\})\}$	$(\mathcal{B}_1, \mathcal{H})$
$(\mathcal{B}_2, \mathcal{H}) = \{(\hbar_1, \{2\}), (\hbar_2, \{2\})\}$	$\tilde{\chi}$

$(\mathcal{B}_3, \mathcal{H}) = \{(\hbar_1, \{\emptyset\}), (\hbar_2, \chi)\}$	$\tilde{\chi}$
$(\mathcal{B}_4, \mathcal{H}) = \{(\hbar_1, \{\emptyset\}), (\hbar_2, \{1\})\}$	$(\mathcal{B}_1, \mathcal{H})$
$(\mathcal{B}_5, \mathcal{H}) = \{(\hbar_1, \{\emptyset\}), (\hbar_2, \{2\})\}$	$\tilde{\chi}$
$(\mathcal{B}_6, \mathcal{H}) = \{(\hbar_1, \chi), (\hbar_2, \{\emptyset\})\}$	$\tilde{\chi}$
$(\mathcal{B}_7, \mathcal{H}) = \{(\hbar_1, \chi), (\hbar_2, \{1\})\}$	$\tilde{\chi}$
$(\mathcal{B}_8, \mathcal{H}) = \{(\hbar_1, \chi), (\hbar_2, \{2\})\}$	$\tilde{\chi}$
$(\mathcal{B}_9, \mathcal{H}) = \{(\hbar_1, \{1\}), (\hbar_2, \{\emptyset\})\}$	$(\mathcal{B}_1, \mathcal{H})$
$(\mathcal{B}_{10}, \mathcal{H}) = \{(\hbar_1, \{1\}), (\hbar_2, \{2\})\}$	$\tilde{\chi}$
$(\mathcal{B}_{11}, \mathcal{H}) = \{(\hbar_1, \{1\}), (\hbar_2, \chi)\}$	$\tilde{\chi}$
$(\mathcal{B}_{12}, \mathcal{H}) = \{(\hbar_1, \{2\}), (\hbar_2, \{\emptyset\})\}$	$\tilde{\chi}$
$(\mathcal{B}_{13}, \mathcal{H}) = \{(\hbar_1, \{2\}), (\hbar_2, \{1\})\}$	$\tilde{\chi}$
$(\mathcal{B}_{14}, \mathcal{H}) = \{(\hbar_1, \{2\}), (\hbar_2, \chi)\}$	$\tilde{\chi}$

Table 3

Definition 4.3. In $(\chi, \mathcal{T}_{\Re}(\mathcal{A}), \mathcal{H})$, if $(\mathcal{B}, \mathcal{H}) = n\text{-}s\text{-}\mathcal{K}er((\mathcal{B}, \mathcal{H}))$, where given $(\mathcal{B}, \mathcal{H}) \in \c{S}\c{S}(\chi)_{\mathcal{H}}$, then $\mathcal{B}$ is namely *nano* soft-$\wp$ set and in briefly n-s-$\wp$ set.

From table 2 the sets $\tilde{\emptyset}$, $\tilde{\chi}$ and $(\mathcal{B}_1, \mathcal{H})$ are n-s-$\wp$ sets since every one of those sets is equal to it's *nano* soft-*kernal*.

Remark 4.4. For $(\chi, \mathcal{T}_{\Re}(\mathcal{A}), \mathcal{H})$, $(\mathcal{B}, \mathcal{H}) \in \c{S}\c{S}(\chi)_{\mathcal{H}}$, if $(\mathcal{B}, \mathcal{H})$ is a n-s-*open* set, then $(\mathcal{B}, \mathcal{H})$ is a n-s-$\wp$ set.

Definition 4.5. In $(\chi, \mathcal{T}_{\Re}(\mathcal{A}), \mathcal{H})$, if $(\mho, \mathcal{H}) = (\mathcal{M}, \mathcal{H}) \tilde{\cap} (\mathcal{B}, \mathcal{H})$ where $(\mho, \mathcal{H}) \in \tilde{\chi}$, $(\mathcal{M}, \mathcal{H})$ is n-s-closed set and $(\mathcal{B}, \mathcal{H})$ is n-$\wp$ set, then $(\mho, \mathcal{H})$ is namely *nano* soft-$\mathfrak{Q}$-closed set and in briefly n-s-$\mathfrak{Q}$-closed set.

From table 3 where $\mathcal{A} = \{1\}$ then $\mathcal{T}_{\Re}(\mathcal{A}) = \{\tilde{\chi}, \tilde{\emptyset}, (\mathcal{B}_1, \mathcal{H})\}$ then the family of all n-$\mathfrak{Q}$-closed sets is $\{\tilde{\chi}, \tilde{\emptyset}, (\mathcal{B}_1, \mathcal{H})\}$.

Proposition4.6.

i. Every n-s-$\wp$ set is n-s-$\mathfrak{Q}$-closed set.

ii. Every n-s-open set is n-s-Ꞷ-closed set.

iii. Every n-s-closed set is n-s-Ꞷ-closed set.

Proof:

i. Let $(\mathcal{B},\mathcal{H})$ is n-s-$\wp$ set. Since $\tilde{\chi} \in n$-s-closed set such that $(\mathcal{B},\mathcal{H}) = (\mathcal{B},\mathcal{H}) \tilde{\cap} \tilde{\chi}$, then $(\mathcal{B},\mathcal{H})$ is n-s-Ꞷ-closed set.

ii. Let $(\mathcal{B},\mathcal{H})$ is *nano* soft-open set, by remark 4.4 then $(\mathcal{B},\mathcal{H})$ is n-s-$\wp$ set then $(\mathcal{B},\mathcal{H})$ = n-s-$\mathcal{K}er((\mathcal{B},\mathcal{H}))$, and n-s-Ꞷ-closed set by (i).

iii. Let $(\mathcal{B},\mathcal{H}) \in n$-$s$-closed set. Since $\tilde{\chi}$ is n-s-$\wp$ set and $(\mathcal{B},\mathcal{H}) = (\mathcal{B},\mathcal{H}) \tilde{\cap} \tilde{\chi}$, then $(\mathcal{B},\mathcal{H})$ is n-s-Ꞷ-*closed* set.

The opposite of Proposition 4.6, is not true by the following example.

Example 4.7. From table 3 if $(\mathcal{B},\mathcal{H}) = (\mathcal{B}_2,\mathcal{H})$ where $\mathcal{A} = \{1\}$, then $\mathcal{T}_{\Re}(\mathcal{A}) = \{\tilde{\chi}, \tilde{\emptyset}, (\mathcal{B}_1,\mathcal{H})\}$ then n-s-$\mathcal{K}er((\mathcal{B}_2,\mathcal{H})) = \tilde{\chi}$, then $(\mathcal{B}_2,\mathcal{H})$ is not n-s-$\wp$ set and not n-s-open set, but $(\mathcal{B}_2,\mathcal{H})$ is n-s-Ꞷ-*closed* set since $(\mathcal{B}_2,\mathcal{H}) = (\mathcal{B}_2,\mathcal{H}) \tilde{\cap} \tilde{\chi}$. If we suggest $(\mathcal{B},\mathcal{H}) = (\mathcal{B}_1,\mathcal{H})$ with the same set then n-s-$\mathcal{K}er((\mathcal{B}_1,\mathcal{H})) = (\mathcal{B}_1,\mathcal{H})$ then $(\mathcal{B}_1,\mathcal{H})$ is n-s-$\wp$ set and $(\mathcal{B}_1,\mathcal{H})$ is n-s-Ꞷ-closed set since $(\mathcal{B}_1,\mathcal{H}) = (\mathcal{B}_1,\mathcal{H}) \tilde{\cap} \tilde{\chi}$, but $(\mathcal{B}_1,\mathcal{H})$ is not n-s-closed set.

Remark 4.8. In $(\chi, \mathcal{T}_{\Re}(\mathcal{A}), \mathcal{H})$, if $(\mho,\mathcal{H}) \subseteq \tilde{\chi}$; $(\mho,\mathcal{H})$ is n-s-Ꞷ-closed set, then $(\mho,\mathcal{H}) = n$-s-$\mathcal{K}er((\mho,\mathcal{H})) \tilde{\cap} (\mathcal{M},\mathcal{H})$, where $(\mathcal{M},\mathcal{H})$ is n-s-closed set.

Proof: Since $(\mho,\mathcal{H})$ is a n-s-Ꞷ-closed set, then $(\mho,\mathcal{H}) = (\mathcal{M},\mathcal{H}) \tilde{\cap} (\mathcal{B},\mathcal{H})$ such that $(\mathcal{M},\mathcal{H})$ is a n-s-closed set and $(\mathcal{B},\mathcal{H})$ is a n-s-$\wp$ set. Implies, $(\mho,\mathcal{H}) \tilde{\subseteq} (\mathcal{B},\mathcal{H}) = n$-$s$-$\mathcal{K}er\big((\mathcal{B},\mathcal{H})\big)$ and $(\mho,\mathcal{H}) \tilde{\subseteq} n$-$s$-$\mathcal{K}er\big((\mho,\mathcal{H})\big)$ which is the smallest n-s-open set containing $(\mho,\mathcal{H})$. So,
s-$\mathcal{K}er\big((\mho,\mathcal{H})\big) \subseteq n$-s-$\mathcal{K}er\big((\mathcal{B},\mathcal{H})\big) = (\mathcal{B},\mathcal{H})$and $(\mho,\mathcal{H}) = (\mathcal{M},\mathcal{H}) \tilde{\cap} (\mathcal{B},\mathcal{H})$. Therefore,
$(\mho,\mathcal{H}) = n$-s-$\mathcal{K}er((\mho,\mathcal{H}) \tilde{\cap} (\mathcal{M},\mathcal{H})$.

5. *On nano soft-$\mathcal{I}$-semi-g-kernal of set.*

Definition 5.1. In $(\chi, \mathcal{T}_{\Re}(\mathcal{A}), \mathcal{H}, \mathcal{I})$, if $(\mathcal{B},\mathcal{H}) \in \mathcal{SS}(\chi)_{\mathcal{H}}$, then n-$s\mathcal{I}sg$-$\mathcal{K}er((\mathcal{B},\mathcal{H})) = \bigcap\{(\mho,\mathcal{H}) ; (\mathcal{B},\mathcal{H}) \tilde{\subseteq} (\mho,\mathcal{H}), (\mho,\mathcal{H}) \in n$-s$\mathcal{I}sg$-$o(\chi)\}$ which is shortcut for *nano* soft-$\mathcal{I}$-semi-g-kernal of $(\mathcal{B},\mathcal{H})$. It is clear that if $(\mathcal{B},\mathcal{H}) \in \mathcal{SS}(\chi)$ is n-s$\mathcal{I}sg$-open set, then $(\mathcal{B},\mathcal{H})$ = n-$s\mathcal{I}sg$-$\mathcal{K}er((\mathcal{B},\mathcal{H}))$.

Example 5.2.

From the Example 4.2, if the set $\mathcal{A} = \{1\}$ then $\mathcal{T}_{\Re}((\mathcal{B},\mathcal{H})) = \{\tilde{\chi}, \tilde{\emptyset}, (\mathcal{B}_1,\mathcal{H}) = \{(\hbar_1, \{1\}),$

$(\hbar_2,\{1\})\}$ and $\mathcal{I} = \{\widetilde{\emptyset},(\mathcal{B}_4,\mathcal{H}),(\mathcal{B}_{12},\mathcal{H}),(\mathcal{B}_{13},\mathcal{H})\}$. Then ṢṢO(χ) = $\{\{\widetilde{\emptyset}, \tilde{\chi}, (\mathcal{P},\mathcal{H}); 1 \tilde{\in} \mathcal{P}(\hbar)\ \forall \hbar\}$. Then

***n-s𝕴sg*-c(χ)$_\mathcal{H}$** = $\{\tilde{\chi}, \widetilde{\emptyset}, (\mathcal{B}_2,\mathcal{H}),(\mathcal{B}_3,\mathcal{H}),(\mathcal{B}_5,\mathcal{H}),(\mathcal{B}_8,\mathcal{H}),(\mathcal{B}_{10},\mathcal{H}),(\mathcal{B}_{11},\mathcal{H}),(\mathcal{B}_{14},\mathcal{H})\}$ and

***n-s𝕴sg*-o(χ)$_\mathcal{H}$** = $\{\tilde{\chi}, \widetilde{\emptyset}, (\mathcal{B}_1,\mathcal{H}),(\mathcal{B}_6,\mathcal{H}),(\mathcal{B}_7,\mathcal{H}),(\mathcal{B}_4,\mathcal{H}),(\mathcal{B}_{13},\mathcal{H}),(\mathcal{B}_{12},\mathcal{H}),(\mathcal{B}_9,\mathcal{H})\}$.

Example 5.3.

From the Example 5.2, if the set $\mathcal{A} = \{1\}$ then $\mathcal{T}_{\mathfrak{R}}(\mathcal{A}) = \{\tilde{\chi}, \widetilde{\emptyset}, (\mathcal{B}_1,\mathcal{H}) = \{(\hbar_1,\{1\}), (\hbar_2,\{1\})\}\}$ then $n\text{-}s\mathcal{I}sg\text{-}o(\chi) = \{\widetilde{\emptyset}, \tilde{\chi}, (\mathcal{B}_1,\mathcal{H}),(\mathcal{B}_6,\mathcal{H}), (\mathcal{B}_7,\mathcal{H}),(\mathcal{B}_4,\mathcal{H}),(\mathcal{B}_{13},\mathcal{H}), (\mathcal{B}_{12},\mathcal{H}), (\mathcal{B}_9,\mathcal{H})\}$ according to the given $(\mathcal{B},\mathcal{H}) \in$ ṢṢ(χ), we can determine $n\text{-}s\mathcal{I}sg\text{-}\mathcal{K}er((\mathcal{B},\mathcal{H}))$ in the following table:

$(\mathcal{B},\mathcal{H}) \in$ ṢṢ$(\chi)_\mathcal{H}$	$n\text{-}s\text{-}\mathcal{K}er((\mathcal{B},\mathcal{H}))$	$n\text{-}s\mathcal{I}sg\text{-}\mathcal{K}er((\mathcal{B},\mathcal{H}))$
$\widetilde{\emptyset}$	$\widetilde{\emptyset}$	$\widetilde{\emptyset}$
$\tilde{\chi}$	$\tilde{\chi}$	$\tilde{\chi}$
$(\mathcal{B}_1,\mathcal{H}) = \{(\hbar_1,\{1\}),(\hbar_2,\{1\})\}$	$(\mathcal{B}_1,\mathcal{H})$	$(\mathcal{B}_1,\mathcal{H})$
$(\mathcal{B}_2,\mathcal{H}) = \{(\hbar_1,\{2\}),(\hbar_2,\{2\})\}$	$\tilde{\chi}$	$\tilde{\chi}$
$(\mathcal{B}_3,\mathcal{H}) = \{(\hbar_1,\{\emptyset\}),(\hbar_2,\chi)\}$	$\tilde{\chi}$	$(\mathcal{B}_{14},\mathcal{H})$
$(\mathcal{B}_4,\mathcal{H}) = \{(\hbar_1,\{\emptyset\}),(\hbar_2,\{1\})\}$	$(\mathcal{B}_1,\mathcal{H})$	$(\mathcal{B}_4,\mathcal{H})$
$(\mathcal{B}_5,\mathcal{H}) = \{(\hbar_1,\{\emptyset\}),(\hbar_2,\{2\})\}$	$\tilde{\chi}$	$(\mathcal{B}_2,\mathcal{H})$
$(\mathcal{B}_6,\mathcal{H}) = \{(\hbar_1,\chi),(\hbar_2,\{\emptyset\})\}$	$\tilde{\chi}$	$(\mathcal{B}_6,\mathcal{H})$
$(\mathcal{B}_7,\mathcal{H}) = \{(\hbar_1,\chi),(\hbar_2,\{1\})\}$	$\tilde{\chi}$	$(\mathcal{B}_7,\mathcal{H})$
$(\mathcal{B}_8,\mathcal{H}) = \{(\hbar_1,\chi),(\hbar_2,\{2\})\}$	$\tilde{\chi}$	$\tilde{\chi}$
$(\mathcal{B}_9,\mathcal{H}) = \{(\hbar_1,\{1\}),(\hbar_2,\{\emptyset\})\}$	$(\mathcal{B}_1,\mathcal{H})$	$(\mathcal{B}_9,\mathcal{H})$
$(\mathcal{B}_{10},\mathcal{H}) = \{(\hbar_1,\{1\}),(\hbar_2,\{2\})\}$	$\tilde{\chi}$	$\tilde{\chi}$
$(\mathcal{B}_{11},\mathcal{H}) = \{(\hbar_1,\{1\}),(\hbar_2,\chi)\}$	$\tilde{\chi}$	$\tilde{\chi}$
$(\mathcal{B}_{12},\mathcal{H}) = \{(\hbar_1,\{2\}),(\hbar_2,\{\emptyset\})\}$	$\tilde{\chi}$	$(\mathcal{B}_{12},\mathcal{H})$
$(\mathcal{B}_{13},\mathcal{H}) = \{(\hbar_1,\{2\}),(\hbar_2,\{1\})\}$	$\tilde{\chi}$	$(\mathcal{B}_{13},\mathcal{H})$
$(\mathcal{B}_{14},\mathcal{H}) = \{(\hbar_1,\{2\}),(\hbar_2,\chi)\}$	$\tilde{\chi}$	$(\mathcal{B}_{14},\mathcal{H})$

Table 4

Proposition 5.4. In $(\chi, \mathcal{T}_{\Re}(\mathcal{A}), \mathcal{H}, \mathcal{I})$, if $(\mathcal{B}, \mathcal{H}) \in$ ŞŞ(χ), then n-$s\mathcal{I}sg$-$\mathcal{K}er((\mathcal{B}, \mathcal{H})) \subseteq n$-$s$-$\mathcal{K}er((\mathcal{B}, \mathcal{H}))$.

Proof: Let $x \tilde{\notin} n$-s-$\mathcal{K}er((\mathcal{B}, \mathcal{H}))$.; $x \in \chi$ then $x \tilde{\notin} \tilde{\bigcap}\{(\mho, \mathcal{H})$; $(\mathcal{B}, \mathcal{H}) \tilde{\subseteq} (\mho, \mathcal{H})$, $(\mho, \mathcal{H}) \in \mathcal{T}_{\Re}(\mathcal{A})\}$. Implies, $\exists$ $(\mho, \mathcal{H}) \in \mathcal{T}_{\Re}(\mathcal{A})$, $(\mathcal{B}, \mathcal{H}) \tilde{\subseteq} (\mho, \mathcal{H})$; $x \notin (\mho, \mathcal{H})$. Then there exist $(\mho, \mathcal{H}) \in n$-$s\mathcal{I}sg$-o$(\chi)_{\mathcal{H}}$, $(\mathcal{B}, \mathcal{H}) \tilde{\subseteq} (\mho, \mathcal{H})$; $x \tilde{\notin} (\mho, \mathcal{H})$, so $x \tilde{\notin} \tilde{\bigcap}\{(\mho, \mathcal{H})$; $(\mathcal{B}, \mathcal{H}) \tilde{\subseteq} (\mho, \mathcal{H})$, $(\mho, \mathcal{H}) \in n$-$s\mathcal{I}sg$-o$(\chi)_{\mathcal{H}}\}$. Hence $x \tilde{\notin} n$-$s\mathcal{I}sg$-$\mathcal{K}er((\mathcal{B}, \mathcal{H}))$.

The phrase $(n$-s-$\mathcal{K}er((\mathcal{B}, \mathcal{H})) \tilde{\subseteq} n$-$s\mathcal{I}sg$-$\mathcal{K}er((\mathcal{B}, \mathcal{H})))$ is not true by table 4 if we suggest the set $(\mathcal{B}, \mathcal{H}) = (\mathcal{B}_5, \mathcal{H})$ then n-s-$\mathcal{K}er((\mathcal{B}_5, \mathcal{H})) = \tilde{\chi}$, but n-$s\mathcal{I}sg$-$\mathcal{K}er((\mathcal{B}_5, \mathcal{H})) = (\mathcal{B}_2, \mathcal{H})$ then n-s-$\mathcal{K}er((\mathcal{B}, \mathcal{H})) \tilde{\nsubseteq} n$-$s\mathcal{I}sg$-$\mathcal{K}er((\mathcal{B}, \mathcal{H}))$.

Remark 5.5. For $(\chi, \mathcal{T}_{\Re}(\mathcal{A}), \mathcal{H}, \mathcal{I})$, if χ is a finite space then $(\mathcal{B}, \mathcal{H}) \in$ ŞŞ$(\chi)_{\mathcal{H}}$ is a n-$s\mathcal{I}sg$-$open$ set, if and only if $(\mathcal{B}, \mathcal{H}) = n$-$s\mathcal{I}sg$-$\mathcal{K}er((\mathcal{B}, \mathcal{H}))$.

Definition 5.6. In $(\chi, \mathcal{T}_{\Re}(\mathcal{A}), \mathcal{H}, \mathcal{I})$, if $(\mathcal{B}, \mathcal{H}) = n$-$s\mathcal{I}sg$-$\mathcal{K}er((\mathcal{B}, \mathcal{H}))$, where $(\mathcal{B}, \mathcal{H}) \in$ ŞŞ$(\chi)_{\mathcal{H}}$, then $(\mathcal{B}, \mathcal{H})$ is namely *nano* soft-$\mathcal{I}$-semi-g-$\wp$ set and in (briefly n-$s\mathcal{I}sg$-$\wp$ set).

From Example 5.3, the sets $\{\tilde{\emptyset}, \tilde{\chi}, (\mathcal{B}_i, \mathcal{H})$ and $i = \{,1.4,6,7,9,12,13,14\}$ are n-$s\mathcal{I}sg$-$\wp$ sets.

Remark 5.7. For $(\chi, \mathcal{T}_{\Re}(\mathcal{A}), \mathcal{H}, \mathcal{I})$, $(\mathcal{B}, \mathcal{H}) \in$ ŞŞ$(\chi)_{\mathcal{H}}$, if $(\mathcal{B}, \mathcal{H})$ is a n-$s\mathcal{I}sg$-open set, then $(\mathcal{B}, \mathcal{H})$ is n-$s\mathcal{I}sg$-$\wp$ set.

Definition 5.8. In $(\chi, \mathcal{T}_{\Re}(\mathcal{A}), \mathcal{H}, \mathcal{I})$, if $(\mho, \mathcal{H}) = (\mathcal{M}, \mathcal{H}) \tilde{\cap} (\mathcal{B}, \mathcal{H})$ where $(\mho, \mathcal{H}) \in \tilde{\chi}$, $(\mathcal{M}, \mathcal{H})$ is n-$s\mathcal{I}sg$-closed set and $(\mathcal{B}, \mathcal{H})$ is n-$s\mathcal{I}sg$-$\wp$ set, then $(\mho, \mathcal{H})$ is namely *nano* soft-$\mathcal{I}$-semi-g-Ꝏ-closed set and briefly n-$s\mathcal{I}sg$-Ꝏ-closed set.

Example 5.9.

From Example 5.3, where $\mathcal{A} = \{1\}$ then $\mathcal{T}_{\Re}(\mathcal{A}) = \{\tilde{\chi}, \tilde{\emptyset}, (\mathcal{B}_3, \mathcal{H}) = \{(\hbar_1, \{1\}), (\hbar_2, \{1\})\}\}$ then n-$s\mathcal{I}sg$-c$(\chi) = \{\tilde{\chi}, \tilde{\emptyset}, (\mathcal{B}_2, \mathcal{H}), (\mathcal{B}_3, \mathcal{H}), (\mathcal{B}_5, \mathcal{H}), (\mathcal{B}_8, \mathcal{H}), (\mathcal{B}_{10}, \mathcal{H}), (\mathcal{B}_{11}, \mathcal{H}), (\mathcal{B}_{14}, \mathcal{H})\}$, then every subset $(\mho, \mathcal{H})$ of $\tilde{\chi}$ is n-$s\mathcal{I}sg$-Ꝏ-closed since $(\mho, \mathcal{H}) = (\mathcal{M}, \mathcal{H}) \tilde{\cap} (\mathcal{B}, \mathcal{H})$, such that $(\mathcal{M}, \mathcal{H})$ is n-$s\mathcal{I}sg$-closed set and $(\mathcal{B}, \mathcal{H})$ is n-$s\mathcal{I}sg$-$\wp$ set.

Theorem 5.10.

i. Every n-s$\mathcal{I}sg$-$\wp$ set is n-s$\mathcal{I}sg$-$\mathbb{Q}$-closed set.

ii. Every n-s$\mathcal{I}sg$-open set is n-s$\mathcal{I}sg$-$\mathbb{Q}$-closed set.

iii. Every n-s$\mathcal{I}sg$-closed set is n-s$\mathcal{I}sg$-$\mathbb{Q}$-closed set.

Proof:

i. Let $(\mathcal{B},\mathcal{H})$ is n-s$\mathcal{I}sg$-$\wp$ set. Since $\tilde{\chi} \in n$-s$\mathcal{I}sg$-$c(\chi)$ such that $(\mathcal{B},\mathcal{H}) = (\mathcal{B},\mathcal{H}) \tilde{\cap} \tilde{\chi}$, then $(\mathcal{B},\mathcal{H})$ is n-s$\mathcal{I}sg$-$\mathbb{Q}$-closed set.

ii. Let $(\mathcal{B},\mathcal{H})$ is nano soft-$\mathcal{I}$-semi-g-open set, by remark 5.5 then $(\mathcal{B},\mathcal{H}) = n$-s$\mathcal{I}sg$-$\mathcal{K}er$ $((\mathcal{B},\mathcal{H}))$, then $(\mathcal{B},\mathcal{H})$ is n-s$\mathcal{I}sg$-$\wp$ set and n-s$\mathcal{I}sg$-$\mathbb{Q}$-closed set by (i).

iii. Let $(\mathcal{B},\mathcal{H}) \in n$-s$\mathcal{I}sg$-$c(\chi)$. Since $\tilde{\chi}$ is n-s$\mathcal{I}sg$-$\wp$ set and $(\mathcal{B},\mathcal{H}) = (\mathcal{B},\mathcal{H}) \tilde{\cap} \tilde{\chi}$, then $(\mathcal{B},\mathcal{H})$ is n-s$\mathcal{I}sg$-$\mathbb{Q}$-$closed$ set.

The opposite of Theorem 5.10, is not true.

Example 5.11. From Example 5.2 if $(\mathcal{B},\mathcal{H}) = (\mathcal{B}_1,\mathcal{H})$ where $\mathcal{A} = \{1\}$ and $\mathcal{T}_{\Re}(\mathcal{A}) = \{\tilde{\chi}, \tilde{\emptyset}, (\mathcal{B}_3,\mathcal{H}) = \{(\hbar_1,\{1\}), (\hbar_2,\{1\})\}\}$. Then n-s$\mathcal{I}sg$-c$(\chi) = \{\tilde{\chi}, \tilde{\emptyset}, (\mathcal{B}_2,\mathcal{H}), (\mathcal{B}_3,\mathcal{H}), (\mathcal{B}_5,\mathcal{H}), (\mathcal{B}_8,\mathcal{H}), (\mathcal{B}_{10},\mathcal{H}), (\mathcal{B}_{11},\mathcal{H}), (\mathcal{B}_{14},\mathcal{H})\}$, n-s$\mathcal{I}sg$-$\mathcal{K}er((\mathcal{B}_1,\mathcal{H})) = (\mathcal{B}_1,\mathcal{H})$. Thus $(\mathcal{B}_1,\mathcal{H})$ is neither n-s$\mathcal{I}sg$-$\wp$ set nor n-s$\mathcal{I}sg$-open set, but $(\mathcal{B}_1,\mathcal{H})$ is a n-s$\mathcal{I}sg$-$\mathbb{Q}$-closed set since $(\mathcal{B}_1,\mathcal{H}) = (\mathcal{B}_1,\mathcal{H}) \tilde{\cap} \tilde{\chi}$. In other hand; if $(\mathcal{B},\mathcal{H}) = (\mathcal{B}_1,\mathcal{H})$ with the same set A then n-s$\mathcal{I}sg$-$\mathcal{K}er((\mathcal{B}_1,\mathcal{H})) = (\mathcal{B}_1,\mathcal{H})$. Implies, $(\mathcal{B}_1,\mathcal{H})$ is n-s$\mathcal{I}sg$-$\wp$ set, so $(\mathcal{B}_1,\mathcal{H})$ is n-s$\mathcal{I}sg$-$\mathbb{Q}$-closed set but $(\mathcal{B}_1,\mathcal{H})$ is not n-s$\mathcal{I}sg$-closed set.

Proposition 5.12. In $(\chi, \mathcal{T}_{\Re}(\mathcal{A}), \mathcal{H}, \mathcal{I})$, if χ is a finite set and $(\mho,\mathcal{H}) \subseteq \tilde{\chi}$; $(\mho,\mathcal{H})$ is a n-s$\mathcal{I}sg$-$\mathbb{Q}$-closed set, then $(\mho,\mathcal{H}) = n$-s$\mathcal{I}sg$-$\mathcal{K}er((\mho,\mathcal{H})) \tilde{\cap} (\mathcal{M},\mathcal{H})$ where $(\mathcal{M},\mathcal{H})$ is n-s$\mathcal{I}sg$-closed set.

Proof: Since $(\mho,\mathcal{H})$ is a n-s$\mathcal{I}sg$-$\mathbb{Q}$-closed set, then $(\mho,\mathcal{H}) = (\mathcal{M},\mathcal{H}) \tilde{\cap} (\mathcal{B},\mathcal{H})$ such that $(\mathcal{M},\mathcal{H})$ is a n-s$\mathcal{I}sg$-closed set and $(\mathcal{B},\mathcal{H})$ is a n-s$\mathcal{I}sg$-$\wp$ set. Implies, $(\mho,\mathcal{H}) \tilde{\subseteq} n$-s$\mathcal{I}sg$-$\mathcal{K}er\big((\mathcal{B},\mathcal{H})\big) = (\mathcal{B},\mathcal{H})$ and $(\mho,\mathcal{H}) \tilde{\subseteq} n$-s$\mathcal{I}sg$-$\mathcal{K}er\big((\mho,\mathcal{H})\big)$ which is the smallest n-s$\mathcal{I}sg$-open set containing $(\mho,\mathcal{H})$. So, $ns\mathcal{I}sg$-$\mathcal{K}er\big((\mho,\mathcal{H})\big) \subseteq n$-s$\mathcal{I}sg$-$\mathcal{K}er\big((\mathcal{B},\mathcal{H})\big) = (\mathcal{B},\mathcal{H})$ and $(\mho,\mathcal{H}) = (\mathcal{M},\mathcal{H}) \tilde{\cap} (\mathcal{B},\mathcal{H})$. Therefore, $(\mho,\mathcal{H}) = (\mathcal{M},\mathcal{H}) \tilde{\cap} n$-s$\mathcal{I}sg$-$\mathcal{K}er((\mho,\mathcal{H}))$.

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Soft Simply Connected Spaces And Soft Simply Paracompact Spaces

S. Noori[1] and Y. Y. Yousif[2]

[1] Department of Mathematics, College of Education for Pure Sciences

(Ibn- Al-Haitham), University of Baghdad

[2] Department of Mathematics, College of Education for Pure Sciences

(Ibn- Al-Haitham), University of Baghdad

sarah.asas90@gmail.com

yoyayuosif@yahoo.com

Abstract: We introduce in this paper some new concepts in soft topological spaces such as soft simply separated, soft simply disjoint, soft simply division, soft simply limit point and we define soft simply connected spaces, and we presented soft simply Paracompact spaces and studying some of its properties in soft topological spaces. In addition to introduce a new types of functions known as soft simply pu-continuous which are defined between two soft topological spaces.

Keywords: soft simply-connected, soft simply pu-continuous, soft simply limit point, soft simply Paracompact spaces.

MSC2010: 54A05, 54A∅10, 54D05, 54D10.

1. Introduction:

In 1999 the concept of soft set theory was used for the first time as a mathematical tool by$Molodtsov$ [1] to deal with confusion. He determinant the primal upshots of this new theory and successfully applied the soft set theory in many ways such as theory of measurement smoothness of functions, game theory, etc. In last year research work on soft set theory is taking place rapidly. In 2003 $Maji$ et al, presented many basic notions of soft set theory like universe soft set and empty soft set [2]. In 2011 $Shabir$ and Naz discussed the theory of soft topological spaceand many fundamental concepts of soft topological spaces including soft open, soft closed sets, soft nbd oft subspace, and soft separation axioms [3]. In 2012 $Aygünoğlu$ and $Aygün$ mentioned soft continuity of soft function, and theystudied soft product topology, etc in soft topological spaces [4]. In 2011 Min discussed some findings on soft topological spaces [5].In 1975 the concept of simply-open sets was introduced by $Neubrunnova$ [6] if ($H = K \cup N$ such that K is open set and N is $nowhere\ dense$(N is $nowhere\ dense$ if ($cl(int\ N) = \emptyset$ [7])), it symbolizes by $S^M O(X)$.In 2013 El. $sayed$ and $Noamman$ presented transformed definition of simply open set [8] if ($O \subset (X,\tau)$ is simply open set if $int\big(cl(O)\big) \subseteq cl(int(O))$. In 2017 $El.Sayed$ and $El.Bably$ introduce a new class of simply open sets as a generalization and modification for soft open sets called soft simply open set [9]. In 2014$J.Subhashinin$ et al [10] have studied soft connectedness in soft topological spaces and $Bin\ Chen$ [11] continued studying some properties of soft semi-open sets. We built on some of the results in [15], [16], [17], [18], [19]. [20] and [21].

The purpose of this paper is to introduce new concepts in soft topological spaces like soft simply disjoint, soft simply separated, soft simply division,$SS^M - connected$, soft simply pu-continuous, soft simply limit point, and defined soft simply Paracompact spaces.

1.preliminaries:

The following concepts and definition with some results are need it later on

Definition 1.1: [1] Let Udefined as a universe set and E as a parameter set with power set of U is denotes by $P(U)$and $A \, c \, E$. Then (F, A) is said to be a soft set, such that $F: A \to P(U)$**;** $F(a) \in P(U), \forall a \in A$.

Definition 1.2:[2]We say $(F, A)$is a null set and it symbolizes by$\widetilde{\Phi}$, if$F(a) = \emptyset$, $\forall a \in A$.

Definition 1.3:[2] We say (F, A) is a absolute soft set and it symbolizes by$\tilde{A}, if \; F(a) = U, \forall a \in A$.

Definition 1.4:[2]Let $(F, A)$and (G, B) are two soft set then $(F, A) \,\widetilde{\cup}\, (\mathrm{G}, \mathrm{B})$=$(H, C)$; ($i.e$ the union of these sets are also soft set), where $C = A \,\widetilde{\cup}\, B$ and for each $e \in C$

$$H(e)=\begin{cases} F(e) & if \; e \in A - B \\ G(e) & if \; e \in B - A \\ F(e) \cup G(e) & if \; e \in A \cap B \end{cases}$$

Definition 1.5:[2] Let $(F, A)$and (G, B) be two soft set then $(F, A) \,\widetilde{\cap}\, (\mathrm{G}, \mathrm{B})$= (H, C); ($i.e$ the intersection of these sets are also soft set), where $C = A \,\widetilde{\cap}\, B$ and for each $e \in C$such that$H(e) = F(e) \cap G(e)$.

Definition 1.6:[2] Let (F, A) and (G, B) be two soft sets over U, then$(F, A) \,\widetilde{\subset}\, (\mathrm{G}, \mathrm{B})$**,** if$A \subset B$ and $F(e) \subset G(e) \; \forall \; e \in A$,

Definition 1.7:[12] The soft topology $\tilde{\tau}$ defined as follows:

1. $\widetilde{U}$ and $\widetilde{\emptyset} \in \tilde{\tau}$
2. Thesoft union of any number of soft sets in $\tilde{\tau} \in \tilde{\tau}$.
3. The soft intersection of any two soft sets in $\tilde{\tau} \in \tilde{\tau}$.

Then the triplet $(U, \tilde{\tau}, E)$is said to be a soft topological space, and the elements of $\tilde{\tau}$are called soft open and their complements are soft closedand we denoted of each closed soft sets by $\tilde{\mathcal{F}}$.

Definition 1.8:[12] Assume that (F, E) be a soft set of $(U, \tilde{\tau}, E)$ is called soft neighborhood (briefly soft nbd) subset(H, E)if $\exists$ $(K, E) \,\widetilde{\in}\, \tilde{\tau}$; $(H, E) \,\widetilde{\subseteq}\, (K, E) \,\widetilde{\subseteq}\, (F, E)$.

Definition 1.9:[12] $(F, E)^o$ or $sint((F, E))$ is the soft interior of the set (F, E), is adefined as follows:

$$sint\big((F, E)\big) = \widetilde{\cup} \; \{(G, E); \; (F, E) \,\widetilde{\supseteq}\, (G, E), (G, E) \,\widetilde{\in}\, \tilde{\tau}\}.$$

Definition 1.10:[12] $\overline{(F, E)}$is a soft closure of a(F, E), is a soft set defined as follows:

$scl\big((F,E)\big) = \widetilde{\cap}\ \{(G,E);\ (F,E)\ \widetilde{\subseteq}\ (G,E), (G,E)^{C}\ \widetilde{\in}\ \tilde{\tau}\}$.

Definition 1.11:[12]We say$(U,\tilde{\tau},E)$ is a soft indiscrete space if $\tilde{\tau} = \{\widetilde{U},\widetilde{\emptyset}\}$, and it symbolizes by $\tilde{\tau}_{ind}$.

Definition 1.12:[12] We say$(U,\tilde{\tau},E)$ is a soft discrete space if $\tilde{\tau}$ is the family of all soft sets that can be defined over $U$and it symbolizes by $\tilde{\tau}_{dis}$.

Definition 1.13:[4] A family$\delta$of soft set is called a cover of a soft set (F,E) if $(F,E)\ \widetilde{\subset}\widetilde{\cup}\ \{(F_i,E);\ (F_i,E)\ \widetilde{\in}\ \ \delta\ ; i\in I\}$. $\delta$is said to be soft open cover if every members of δ is a soft open set.

Definition 1.14:[4]We say $(U,\tilde{\tau},E)$ is a soft compact if every soft open cover has a finite sub cover $(U,\tilde{\tau},E)$.

Definition 1.15:[8]A soft subset (F,A) of soft topological space $(U,\tilde{\tau},E)$is called Soft simply-open (for short SS^{M}_open) set if $sint\left(scl\big((F,A)\big)\right)\ \widetilde{\subseteq}\ \ scl(sint\big((F,A)\big))$. It is symbolizes by $SS^{M}O(U)$. The complement of a soft simply open set is a soft simply closed set (for short, SS^{M}_closed), and it symbolizes by $SS^{M}C(U)$.

Definition 1.16:[13] We say $(U,\tilde{\tau},E)$is a $soft\ lindelöf$, if every cover of Uhas a countable sub cover.

Definition 1.17:[4]Let$(U,\tilde{\tau},E)$be a softtopological space. A sub collection ω of $\tau$is said to be a base for τ if every member of τ can be expressed as a union of members of ω.

Proposition 1.18:[4] Each soft compact is soft $lindelöf$ and each soft $lindelöf$ is soft paracompact.

Definition 1.19:[12] We say that$(U,\tilde{\tau},E)$ is a $soft\ T_2 - space$ if for any two distinct points $a\ , b\ \widetilde{\in}\ U$, there exist$(F,E)$ and $(G,E)\ \widetilde{\in}\ \tilde{\tau}$, such that $a\ \widetilde{\in}\ (F,E)$, $b\ \widetilde{\in}\ (G,E)$ and $(F,E)\ \widetilde{\cap}\ (G,E) = \widetilde{\emptyset}$.

Definition 1.20:[12] We say that$(U,\tilde{\tau},E)$is a $soft\ regular\ space$ if for all $(H,E)\ \widetilde{\in}\ \tilde{\tau}^{C}$ ($i.e(H,E)$ is soft closed in U) and any points $a\ \widetilde{\in}\ U$ such that $a\ \widetilde{\notin}\ (H,E)$,then there exist$(F,E)$ and $(G,E)\ \widetilde{\in}\ \tilde{\tau}$, such that [$a\ \widetilde{\in}\ (F,E)$ and$(H,E)\ \widetilde{\subset}\ (G,E)$ and $(F,E)\ \widetilde{\cap}\ (G,E) = \widetilde{\emptyset}$].

Definition 1.21:[12] We say that$(U,\tilde{\tau},E)$ is a$soft\ normal\ space$ if for each(H,E) and $(K,E)\ \widetilde{\in}\ \tilde{\tau}^{C}$ ($i.e(H,E)\ and\ (K,E)$ are soft closed in U) such that $(H,E)\ \widetilde{\cap}\ (K,E) = \widetilde{\emptyset}$, then there exist$(F,E)$ and $(G,E)\ \widetilde{\in}\ \tilde{\tau}$, such that [$(H,E)\ \widetilde{\subset}\ (F,E)$, $(K,E)\ \widetilde{\subset}\ (G,E)$ and $(F,E)\ \widetilde{\cap}\ (G,E) = \widetilde{\emptyset}$].

2. Soft Simply Connected Spaces:

In the section, we introduce a new concepts which is called soft simply connected spaces.

Definition 2.1: Let $(U,\tilde{\tau},E)$ be a soft topological space, and $(F,A)^{M}$,$(G,B)^{M}$ be twosoft simply setsover U. The soft simply sets are said soft simply disjoint (for short SS^{M}_dis) if $(F,A)^{M}\ \widetilde{\cap}^{M}\ (G,B)^{M} = \widetilde{\emptyset}$.

Definition 2.2: Let $(U,\tilde{\tau},E)$ be a soft topological space, and $(F,A)^M$,$(G,B)^M$ be twosoftsimply setsover U. The soft simply sets are said soft simply separated (for short SS^M-sep) if $(F,A)^M \,\tilde{\cap}^M\, SS^M(cl(G\,B)^M)=\tilde{\emptyset}$ and $SS^M(cl(F,A)^M)\,\tilde{\cap}^M\,(G,B)^M=\tilde{\emptyset}$.

Remark 2.3: Two disjoint soft simply open sets may not be a soft simply separated, for example:

Example 2.4 : Consider$U$={1,2,3} and E={e_1,e_2}, let $\tilde{\tau}$={$\tilde{\emptyset}$, $\tilde{U}$, ,$(F_1,E)^M$,$(F_2,E)^M$,$(F_3,E)^M$,$(F_4,E)^M$, $(F_5,E)^M$, $(F_6,E)^M$} are soft simply sets defined as follows:
$(F_1,E)^M$= {$(e_1,\{2\}),(e_2,\{1\})$}
$(F_2,E)^M$={$(e_1,\{3\}),(e_2,\{2\})$}
$(F_3,E)^M$={ $(e_1,\{2,3\}),(e_2,\{1,2\})$}
$(F_4,E)^M$={$(e_1,\{1,2\}),(e_2,\tilde{U}$}
$(F_5,E)^M$={ $(e_1,\{1,2\}),(e_2,\{1,3\})$}
$(F_6,E)^M$={ $(e_1,\tilde{\emptyset}),(e_2,\{2\})$}
Then the triplet$(U,\tilde{\tau},E)$ is a soft topological space, it is easy to see that $(F_1,E)^M\,\tilde{\cap}^M\,(F_2,E)^M=\emptyset$. Hence $SS^M(cl(F_1,E)^M)$ =$(F_6,E)^M$ and $SS^M(cl(F_1,E)^M)\,\tilde{\cap}^M\,(F_2,E)^M\neq\emptyset$.

Definition 2.5: Let $(U,\tilde{\tau},E)$ be a soft topological space. If there exist two non-empty soft simply separated sets $(F,A)^M$ and $(G,B)^M$ such that $(F,A)^M\,\tilde{\cup}^M\,(G,B)^M$=$(U,E)^M$, then $(F,A)^M$ and $(G,B)^M$ are said to be soft simply division(for short SS^M-div)for soft simply topological space $(U,\tilde{\tau},E)$.

Definition 2.6 : Let $(U,\tilde{\tau},E)$ be a soft topological space, then $(U,\tilde{\tau},E)$ is said to be soft simply disconnected spaces if $(U,\tilde{\tau},E)$ has a soft simply division. Otherwise $(U,\tilde{\tau},E)$ is said to be soft simply connected spaces.

Example 2.7 : It is easy to see that each soft simply indiscrete space is soft simply connected and that each soft simply discrete non-trivial space is not soft simply connected.

Theorem 2.8: Let $(U,\tilde{\tau},E)$ be a soft topological space. Then the following conditions are equivalent:

a) $(U,\tilde{\tau},E)$ has a soft simply division.
b) There exist two disjoint soft simply closed sets $(F,A)^M$ and $(G,B)^M$ such that$(F,A)^M\,\tilde{\cup}^M\,(G,B)^M$=$(U,E)^M$.
c) There exist two disjoint soft simply open sets $(F,A)^M$ and $(G,B)^M$ such that$(F,A)^M\,\tilde{\cup}^M\,(G,B)^M$=$(U,E)^M$.
d) $(U,\tilde{\tau},E)$ has a proper soft simply open and soft simply closed set in U.

Proof: (**a**) $\Longrightarrow$ (**b**)Let$(U,\tilde{\tau},E)$ have a soft simply division$(F,E)^M$ and $(G,E)^M$. Then

$$(F,E)^M\,\tilde{\cap}^M\,(G,E)^M=\emptyset$$

and

$$SS^M(cl(F,E)^M)=SS^M(cl(F,E)^M)\,\tilde{\cap}^M\,((F,E)^M\,\tilde{\cup}^M\,(G,E)^M)$$

$$=(SS^M(cl(F,E)^M)\,\tilde{\cap}^M\,(F,E)^M)\,\tilde{\cup}^M\,(SS^M(cl(F,E)^M)\,\tilde{\cap}^M\,(G,E)^M)$$

$= (F,E)^M$.

There for $(F,E)^M$ is a soft simply closed set in U . Similar , we can see that $(G,E)^M$ is also a soft simply closed set in U .

(**b**) $\Longrightarrow$ (**c**)Let$(U,\tilde{\tau},E)$ have a soft simply division$(F,E)^M$ and $(G,E)^M$ such that $(F,E)^M$ *and* $(G,E)^M$are soft simply closed. Then the soft simply complement of $(F,E)^M$ and $(G,E)^M$ are soft simply open in U. Then $(F,E)^{\complement^M} \tilde{\cap}^M (G,E)^{\complement^M} = \emptyset$ and $(F,E)^{\complement^M} \tilde{\cup}^M (G,E)^{\complement^M} = U$.

(**c**) $\Longrightarrow$ (**d**)Let$(U,\tilde{\tau},E)$ have a soft simply division$(F,E)^M$ and $(G,E)^M$ such that $(F,E)^M$ *and* $(G,E)^M$are soft simply open in U. Then $(F,E)^M$ and $(G,E)^M$ are also soft simply closed in U.

(**d**) $\Longrightarrow$ (**a**)Let$(U,\tilde{\tau},E)$ has a proper soft simply open and soft simply closed set$(F,E)^M$. Then $(F,E)^{\complement^M}$ and $(F,E)^M$are non-empty soft simply closed set, $(F,E)^{\complement^M} \tilde{\cap}^M (F,E)^M = \emptyset$ and $(F,E)^{\complement^M} \tilde{\cup}^M (F,E)^M = U$. Then $(F,E)^M$ and $(F,E)^{\complement^M}$ is a soft simply division of U.

Theorem 2.9 : Let $(U,\tilde{\tau},E)$ be a soft topological space. Then the following conditions are equivalent:

a) $(U,\tilde{\tau},E)$ has a soft simply connected.
b) There exist two disjoint soft simply closed sets $(F,E)^M$ and $(G,E)^M$ such that$(F,E)^M \tilde{\cup}^M (G,E)^M$=$(U,E)^M$.
c) There exist two disjoint soft simply open sets $(F,E)^M$ and $(G,E)^M$ such that$(F,E)^M \tilde{\cup}^M (G,E)^M = (U,E)^M$.
d) $(U,\tilde{\tau},E)$ at most has two soft simply open and soft simply closed sets in U,that is $\emptyset$ and $(U,E)^M$.

Remark 2.10: By (Theorem 2.9) , the soft topological space in Example 2.20 is a $SS^M - disconnected$ spaces since the soft simply set $(G,E)^M$ is soft simply open set and soft simply closed set in U.

Lemma 2.11: Let $(U,\tilde{\tau},E)$ be a soft topological spaceover U, and V be a non-empty subset of $(U,E)^M$. If $(F_1,E)^M$ and $(F_2,E)^M$ are soft simply sets in $(V,E)^M$, then $(F_1,E)^M$ and $(F_2,E)^M$ are a soft simply separation of$(U,E)^M$.

Proof: We have $[SS^M(cl(F_1,E)^M) \tilde{\cap}^M (V,E)^M] \tilde{\cap}^M (F_2,E)^M = SS^M(cl(F_1,E)^M \tilde{\cap}^M (F_2,E)^M$. Similar we have $[SS^M(cl(F_2,E)^M) \tilde{\cap}^M (V,E)^M] \tilde{\cap}^M (F_1,E)^M = SS^M(cl(F_2,E)^M \tilde{\cap}^M (F_1,E)^M$. Therefor the lemma is hold.

Lemma 2.12: Let $(U,\tilde{\tau},E)$ be a soft topological space over $(U,E)^M$, and V be a non-empty subset of Usuch that $(V,\tilde{\sigma},E)$ is soft simply connected. If $(F_1,E)^M$ and $(F_2,E)^M$ are soft simplyseparation of$(U,E)^M$ such that $(V,E)^M \tilde{\subseteq}^M (F_1,E)^M \tilde{\cup}^M (F_2,E)^M$,then $(V,E)^M \tilde{\subseteq}^M (F_1,E)^M$ or $(V,E)^M \tilde{\subseteq}^M (F_2,E)^M$.

Proof: Since$(V,E)^M \tilde{\subseteq}^M (F_1,E)^M \tilde{\cup}^M (F_2,E)^M$,we$\tilde{\cap}^M$have $((V,E)^M = (V,E)^M \tilde{\cap}^M (F_1,E)^M) \tilde{\cup}^M ((V,E)^M \tilde{\cap}^M (F_2,E)^M)$.By (Lemma 2.11)

$(V,E)^M \tilde{\cap}^M (F_1,E)^M$ and $(V,E)^M \tilde{\cap}^M (F_2,E)^M$ are a soft simply separation of$(V,E)^M$. Since $(V,\tilde{\sigma},E)$ is soft simply connected, we have $(V,E)^M \tilde{\cap}^M (F_1,E)^M = \emptyset$ or $(V,E)^M \tilde{\cap}^M (F_2,E)^M = \emptyset$. There for,$(V,E)^M \tilde{\subseteq}^M (F_1,E)^M$ or $(V,E)^M \tilde{\subseteq}^M (F_2,E)^M$.

Definition 2.13 : Let$(U,\tilde{\tau},E)$ be a soft topological space**,** $(F,E)^M$ be soft simply subset of $U$and $e_x^M \tilde{\in}^M U$. If every $SS^M - nbdofe_x^M$**soft simply intersects** $(F,E)^M$ in some point other thane_x^Mitself, then e_x^M is called soft simply limit point of $(F,E)^M$,(for short $SS^M - Limp$). We denoted of the set of all soft simply limit point of $(F,E)^M$**by** $(F,E)^{dM}$.

Lemma 2.14: Let $\{(U_\alpha, \tilde{\tau}_{U_\alpha}, E)\,; \alpha\epsilon I\}$be a family non-empty soft simply connected subspaces of soft topological space $(U,\tilde{\tau},E)$. If$\tilde{\cap}^M_{\alpha\epsilon I} (U_\alpha,E)^M \neq \emptyset$, then $(\tilde{\cup}^M_{\alpha\epsilon I} U_\alpha, \tilde{\tau}_{\tilde{\cup}^M_{\alpha\epsilon I} U_\alpha}, E)$ is a soft simply connectedsubspace of $(U,\tilde{\tau},E)$.

Proof: Let $S = \tilde{\cup}^M_{\alpha\epsilon I} U_\alpha$. Choose a soft simply point $e_x^M \in (S,E)^M$. Let $(W,E)^M$ and $(Z,E)^M$ be a soft simply division of $\left(\tilde{\cup}^M_{\alpha\epsilon I} U_\alpha, \tilde{\tau}_{\tilde{\cup}^M_{\alpha\epsilon I} U_\alpha}, E\right)$, then $e_x^M \in (W,E)^M$ or $e_x^M \in (Z,E)^M$. Without loss of generality, we may assume that $e_x^M \in (W,E)^M$, for each $\alpha\epsilon I$, since $\left(U_\alpha, \tilde{\tau}_{U_\alpha}, E\right)$ is a soft simply connected it follows from (Lemma 2.12) that $(U_\alpha,E)^M \tilde{\subseteq}^M (W,E)^M$ or $(U_\alpha,E)^M \tilde{\subseteq}^M (Z,E)^M$. Therefore, we have $(V,E)^M \tilde{\subseteq}^M (W,E)^M$ since $e_x^M \in (W,E)^M$, and then $(Z,E)^M = \emptyset$, which is a contradiction. Therefor $\left(\tilde{\cup}^M_{\alpha\epsilon I} U_\alpha, \tilde{\tau}_{\tilde{\cup}^M_{\alpha\epsilon I} U_\alpha}, E\right)$ is a soft simply connected subspace of $(U,\tilde{\tau},E)$.

Theorem 2.15: Let $\{(U_\alpha, \tilde{\tau}_{U_\alpha}, E)\,; \alpha\epsilon I\}$ be a family non-empty soft simply connected subspaces of soft simply topological space $(U,\tilde{\tau},E)$. If $U_\alpha \tilde{\cap}^M U_\beta \neq \emptyset$ for arbitrary $\alpha, \beta \tilde{\in}^M I$, then $(\tilde{\cup}^M_{\alpha\epsilon I} U_\alpha, \tilde{\tau}_{\tilde{\cup}^M_{\alpha\epsilon I} U_\alpha}, E)$ is a soft simply connected subspace of $(U,\tilde{\tau},E)$.

Proof: Fix an $\alpha_0 \in I$. For arbitrary $\beta \in I$, put $S_\beta = U_{\alpha_0} \tilde{\cup}^M U_\beta$,(by Lemma 2.14) each $(S_\beta, \tilde{\tau}_{S_\beta}, E)$ is soft simply connected. Then $\left\{(S_\beta, \tilde{\tau}_{S_\beta}, E)\,; \beta\epsilon I\right\}$ is a family non-empty soft simply connected subspaces of softtopological space $(U,\tilde{\tau},E)$, and $\tilde{\cap}^M_{\beta\epsilon I} S_\beta = \left(U_{\alpha_0}, E\right)^M \neq \emptyset$. Obvious, we have $\tilde{\cup}^M_{\alpha\epsilon I} U_\alpha = \tilde{\cup}^M_{\beta\epsilon I} S_\beta$.It follows from (Lemma 2.14) that $(\tilde{\cup}^M_{\alpha\epsilon I} U_\alpha, \tilde{\tau}_{\tilde{\cup}^M_{\alpha\epsilon I} U_\alpha}, E)$is a soft simply connected subspaceof$(U,\tilde{\tau},E)$.

Theorem 2.16 : Let $(U,\tilde{\tau},E)$ be a soft topological spaceover $X$and $(V,\tilde{\sigma},E)$ is soft simply connected subspace of $(U,\tilde{\tau},E)$. If $(V,E)^M \tilde{\subseteq}^M (A,E)^M \tilde{\subseteq}^M SS^M(cl(Y,E)^M)$, then $(A,\tilde{\tau}_A,E)$is asoft simply connectedsubspace of$(U,\tilde{\tau},E)$. In particular $SS^M(cl(Y,E)^M)$ isa soft simply connected subspace of$(U,\tilde{\tau},E)$.

Proof : Let $(W,E)^M$ and $(Z,E)^M$ be a soft simply division of $(A,\tilde{\tau}_A,E)$.By (Lemma 2.12) we have $(A,E)^M \tilde{\subseteq}^M (W,E)^M$ or $(A,E)^M \tilde{\subseteq}^M (Z,E)^M$.Without loss of generality, we may assume that$(A,E)^M \tilde{\subseteq}^M (Z,E)^M$. By (Lemma 2.11) we have$SS^M(cl(W,E)^M) \tilde{\cap}^M (Z,E)^M = \emptyset$, and hence $(A,E)^M \tilde{\subseteq}^M (Z,E)^M = \emptyset$, which is a contradiction.

Definition 2.17 : Let$(U,\tilde{\tau},E)$ and $(V,\tilde{\sigma},\grave{E})$ be two soft topological spaces, let $u: U \to V$ and $p: E \to \grave{E}$be a mapping, let $f_{pu}: (U,E)^M \to (V,\grave{E})^M$ be a function and $e_F^M \in (\tilde{U},E)^M$

a) f_{pu} is soft simply pu −continuous (for short $SS^M pu - cont$) at $e_F^M \in (\widetilde{U}, E)^M$, if for all $(A,\grave{E})^M \in \widetilde{N}^M_{\widetilde{\sigma}^M}\big(f_{pu}(e_F^M)\big)$, there exists a $(B,E)^M \in \widetilde{N}^M_{\widetilde{\tau}^M}(e_F^M)$ such that $f_{pu}(B,E)^M \widetilde{\subseteq}^M (A,\grave{E})^M$.

b) f_{pu} is$SS^M pu - cont$on $(\widetilde{U},E)^M$**,** if f_{pu} is $SS^M pu - cont$at each soft simply point in $(\widetilde{U},E)^M$.

Theorem 2.18 : The image of soft simply connected spaces under a soft simply continuous map are soft simply connected.

Proof: : Let$(U,\tilde{\tau},E)$ and $\big(V,\tilde{\sigma},\grave{E}\big)$ be two soft topological spaces, where $(U,\tilde{\tau},E)$ is soft simply connected and f be a $SS^M pu - cont$function from $(U,\tilde{\tau},E)$ to $\big(V,\tilde{\sigma},\grave{E}\big)$, the restricted function is soft simply continuous, and without loss of generality, we may assume that$u(U) = u(V)$ and $p(E) = \grave{E}$. Suppose that $\big(V,\tilde{\sigma},\grave{E}\big)$ is soft simply disconnected. By (Theorem2.9), there exists a proper soft simply open and soft simply closed set$(A,E)^M$in V. Since $f$soft simply continuous function then $f^{-1}(A,E)^M$ is a proper soft simply open and soft simply closed set in U by (Theorem 6.3 in [15]), which is a contradiction.

Proposition 2.19: [11] Let$(U,\tilde{\tau},E)$ be a soft topological space, then the collection $\tau_\alpha = \{F(\alpha){:}\ (F,E)^M \in \tilde{\tau}\}$ for each $\alpha \in E$, define a topology on U.

Remark 2.20: There exists soft simply connected soft topological space$(U,\tilde{\tau},E)$ such that $(U,\tilde{\tau}_\alpha,E)$ is a soft simply disconnected softtopological space for some $\alpha \in E$.

Example 2.21: Consider$U$={1,2,3} and E={e_1,e_2}, let $\tilde{\tau}$={$\widetilde{\emptyset}$, $\widetilde{U}$, ,$(F_1,E)^M$, $(F_2,E)^M$,$(F_3,E)^M$$(F_4,E)^M$, $(F_5,E)^M$, $(F_6,E)^M$, $(F_7,E)^M$} are soft simply sets defined as follows:

$(F_1,E)^M$= {$(e_1,\{1,2\}),(e_2,\widetilde{U})$}
$(F_2,E)^M$={$(e_1,\{1,3\}),(e_2,\widetilde{U})$}
$(F_3,E)^M$={ $(e_1,\{1\}),(e_2,\widetilde{U})$}
$(F_4,E)^M$={$(e_1,\{2,3\}),(e_2,\widetilde{U}$}
$(F_5,E)^M$={ $(e_1,\{1,2\}),(e_2,\{1,3\})$}
$(F_6,E)^M$={ $(e_1,\{3\}),(e_2,\widetilde{U})$}
$(F_7,E)^M$={ $(e_1,\widetilde{\emptyset}\ ),(e_2,\widetilde{U})$}

Then$\tilde{\tau}$is defines a soft topological on $\widetilde{U}$ and hence $(U,\tilde{\tau},E)$is a soft topological spaces over $\widetilde{U}$. Then $(U,\tilde{\tau},E)$ is a soft simply connected spaces, however $(U,\tilde{\tau}_1,E)$ is soft simplydiscrete spaces, then $(U,\tilde{\tau}_1,E)$ is soft simply disconnected.

Definition 2.22 : Let$(U,\tilde{\tau},E)$be a soft topological spaces. A sub collection $\widetilde{\omega}^M$ of $\tilde{\tau}$is said to be soft simply base for $\tilde{\tau}$ if every member of $\tilde{\tau}$can be expressed as a soft simply union of members of $\widetilde{\omega}^M$.

Definition 2.23: Let$\{(U^\alpha,\tilde{\tau}_\alpha,,E_\alpha)\}_{\alpha\in I}$be a family of soft topological spaces. Let us take as a basis for soft topology on the product spaces $(\prod_{\alpha\in I} U^\alpha, \prod_{\alpha\in I}\tilde{\tau}_\alpha, \prod_{\alpha\in I} E_\alpha)$ the collection of all soft simply sets$\{(\prod_{\alpha\in I} F_\alpha^M, \prod_{\alpha\in I} E_\alpha^M)$; there is a finite set $k \subset I$ such that $(F_\alpha,E_\alpha)^M = (U^\alpha,E_\alpha)^M$ for each $\alpha \in I\backslash k\}$.

Theorem 2.24: A finite product of soft simply connected spaces is soft simply connected.

Proof : We prove the theorem first for the product of two soft simply connected spaces $(U,\tilde{\tau},E)$ and $(V,\tilde{\sigma},\grave{E})$ choose a fix point $x\times y\in U\times V$. Obvious, $(U\times y,\tilde{\tau}\times\tilde{\sigma}|_{U\times y},E\times\grave{E})$ is a soft simply connected. For each $u\in U,(u\times V,\tilde{\tau}\times\tilde{\sigma}|_{u\times V},E\times\grave{E})$ is also soft simply connected, and put $H_u=(U\times y)\,\tilde{\cup}^M\,(u\times V)$, then each $(H_u,\tilde{\tau}\times\tilde{\sigma}|_{H_u},E\times\grave{E})$ is a soft simply connected (Lemma 2.14). Since $x\times y\in H_u;\forall u\in U$, it follows from (Theorem 2.15) that $(\tilde{\cup}^M_{u\in U}H_u,\tilde{\tau}\times\tilde{\sigma}|_{\tilde{\cup}^M_{u\in U}H_u},E\times\grave{E})$ is a soft simply connected. The proof for any finite product of soft simply connected spaces follows by induction, using the fact that $(\prod_{i=1}^{n}U_i,\ \prod_{i=1}^{n}\tilde{\tau}_i\ ,\prod_{i=1}^{n}E_i)$ is soft simply homeomorphic with $\left(\prod_{i=1}^{n-1}U_i\right)\times U_n,(\prod_{i=1}^{n-1}U_i\,(\tilde{\tau}_i)\times\tilde{\tau}_n,\left(\prod_{i=1}^{n-1}A_i\right)\times A_n)$.

Definition 2.25 : Let$(U,\tilde{\tau},E)$be a softtopological spaces, define an equivalence relation on U by setting $e_x^M\sim e_y^M$ if there exists a soft simply connected subspace of $(U,\tilde{\tau},E)$ containing both soft simply points $e_x^M\ and\ e_y^M$. The equivalence classes are called the soft simply components of U(for short $SS^M-\ component$) or (the soft simply connected components) of$U$. Reflexivity and symmetry of the relation are obvious. Transitivity follows by noting if A_E is a soft simply connected subspaces containing soft simply points $e_x^M\ and\ e_y^M$, and if B_E is a soft simply connected subspaces containing soft simply points $e_y^M\ and\ e_z^M$, then $A_E\,\tilde{\cup}^M\,B_E$ is a subspace containing soft simply points $e_x^M\ and\ e_z^M$, that is soft simply connected because $A_E\ and\ B_E$have the soft simply pointe_y^M in common.

Theorem 2.26: The soft simply components of soft topological space $(U,\tilde{\tau},E)$are soft simply connected disjoint soft simply subspace of U whose union is U such that each non-empty soft simply connected subspace of$U$intersects only one of them.

Proof: Being equivalence classes, the soft simply components of $U$are disjoint and their union is U. Let A_E be an arbitrary soft simply connected subspace. Then $A_E$intersects only one of them. For if A_E intersects the soft simply components G_E and D_E of U, say in soft simply points$e_x^M\ and\ e_y^M$, respectively, then by definition, this cannot happen unless $G_E=D_E$. Next we shall show the soft simply component $G_E$is soft simply connected. Choose a soft simply point $e_z^M$of G_E. For each soft simply point $e_x^M$of G_E, we know that$e_z^M\sim e_x^M$, hence there exists a soft simply connected subspace$L_E^{e_x^M}$ containinge_z^M and e_x^M. Obvious, each $L_E^{e_x^M}\,\tilde{\subseteq}^M\,G_E$.Therefore, $G_E=\tilde{\cup}^M_{e_x\epsilon G_E}L_E^{e_x^M}$. Since the soft simply subspace$L_E^{e_x^M}$ are soft simply connected and have the soft simply pointe_z^M in common, $G_E$is soft simply connected by Theorem 2.15.

3.SOFT SIMPLY PARACOMPACT SPACES:

In this section, we introduce a new concepts which is called soft simply paracompact spaces.

Definition 3.1: Let $(U,\tilde{\tau},E)$be a soft topological space and η be a collection of soft simply sets of $(U,E)^M$, then :

1. η is said to be soft simply locally finite in $(U,E)^M$ (for short SS^M-locally finite), if each soft simply point of $(U,E)^M$ has aSS^M-nbd that intersects only finitely many elements of η.

2. A collection σ of soft simply sets of $(U,E)^M$, is said to be a soft simply refinement(for short $SS^M - ref$)of η if for each element $B \in \sigma$, there exists an element$A \in \eta$ containing B, if the elements of $\sigma$are soft simply open sets, we call σ a soft simply open refinement of η, if they are soft simply closed, we call σ a soft simply closed refinement.

Proposition 3.2: Let η be a soft simply locally finite collection of soft subsetsof $(U,E)^M$. Then:

1) Any subcollection of η is soft simply locally finite .

2) The collection $\sigma = \{(SS^M(cl(F,E)^M) : (F,E)^M \in \eta\}$is soft simply locally finite .

3)$SS^M(cl(\tilde{\cup}^M{}_{(F,E)^M \in \eta}\ (F,E)^M)) = \tilde{\cup}^M{}_{(F,E)^M \in \eta}\ SS^M(cl(F,E)^M)$.

Proof: (1) Is trivial by definition of soft simply locally finite.

(2) Note that any soft simply open set $(A,E)^M$ that intersects the soft simply set $SS^M(cl(F,E)^M)$ necessarily intersects$(F,E)^M$. Thus if $(A,E)^M$is a $SS^M - nbd$of$SS^M - point\ e_x^M$that intersects only finitely many elements $(F,E)^M$of η, then $(F,E)^M$ can intersect at must the same number of soft simply sets of the collection σ.

(3) Let$\tilde{\cup}^M_{(F,E)^M \in \eta}\ (F,E)^M = (Y,E)^M$. Obvious $\tilde{\cup}^M_{(F,E)^M \in \eta}\ SS^M(cl(F,E)^M) = SS^M(cl(Y,E)^M)$. We prove the reverse inclusion under the assumption of soft simply locally finiteness. Let $e_x^M \in SS^M(cl(Y,E)^M)$, let $(A,E)^M$is a $SS^M - nbd$of $SS^M - point e_x^M$that intersects only finitely many elements $(F,E)^M$of η, say $(F_1,E)^M, \ldots, (F_k,E)^M$. Then e_x^Mbelongs to one of the soft simply sets $SS^M(cl(F_1,E)^M, \ldots, SS^M(cl(F_k,E)^M$. For otherwise, the soft simply set $(A,E)^M \tilde{\cap}^M (\tilde{\cup}^M \{SS^M(cl(F_1,E)^M, \ldots, SS^M(cl(F_k,E)^M\})^C$would be a $SS^M - nbd$of e_x^M that intersects no element of η, and therefore it does not intersect $(Y,E)^M$, which is a contradiction with $e_x^M \in SS^M(cl(Y,E)^M$.

Definition 3.3: Let $(U,\tilde{\tau},E)$be a soft topologicalspace is said to be soft simply paracompact (for short $SS^M - paracompact$) if each soft simply open covering η of $(U,E)^M$ has a soft simply locally finite soft simply open refinementσ that covers $(U,E)^M$.

Remark 3.4 : Any $SS^M - compact$ is $SS^M - lindel\ddot{o}f$, and any $SS^M - lindel\ddot{o}f$ is $SS^M - paracompact$.

Proposition 3.5 : Let $(U,\tilde{\tau},E)$be a$SS^M - paracompact$space**.** If$E = \{e\}$, then $(U,\tilde{\tau},E)$ is $SS^M - paracompact$ if and only if the collection $\eta = \{F(e) : (F,E)^M \in \tilde{\tau}\}$ is a $SS^M - paracompact$ topology on U.

It is well known that a $lindel\ddot{o}f$ spacemay not compact and a paracompact space may not $lindel\ddot{o}f$. Therefore, it follows from Proposition 3.5 that a $SS^M - lindel\ddot{o}f$ space may not $SS^M - compact$and a $SS^M - paracompact$ space may not $SS^M - lindel\ddot{o}f$.

Theorem 3.6 : Each$SS^M - paracompact$and $SS^M - T_2\ space$ is $SS^M - normal\ space$.

Proof: Let $(U,\tilde{\tau},E)$ be a $SS^M - paracompact$ and $SS^M - T_2\ space$. First one proves soft simply regularity. Let e_x^M be a $SS^M - Limp$ of $(U,E)^M$ and let $(A,E)^M$ be a $SS^M - closed$set of$(U,E)^M$

disjoint from e_x^M. The $SS^M - \mathrm{T}_2$condition enable us to take,$\forall SS^M - Limpe_y^M$ in $(A,E)^M$ an $SS^M -$ open set $\left(B^{e_y^M},E\right)^M$ about e_y^M whose$SS^M - closure$ is disjoint frome_x^M. Let $\eta = \{\left(B^{e_y^M},E\right)^M : e_y^M \in (A,E)^M\} \,\tilde{\cup}^M\, \{{(A,E)^M}^{\complement}\}$. Then η is a $SS^M - open$covering of $(U,E)^M$. Since $(U,\tilde{\tau},E)$ is a $SS^M - paracompact$ there exists a $SS^M - locally\ finite SS^M - \ open$ refinement σ that covers $(U,E)^M$. Form the subcollection μ of σ consisting of each element of σ that intersects $(A,E)^M$. Then μ covers $(A,E)^M$. Moreover, if $C \in \mu$, then the $SS^M - closure$ of C is disjoint from e_x^M Since C interects $(A,E)^M$ it lies in some $SS^M - open$ set $\left(B^{e_y^M},E\right)^M$, whose $SS^M - closure$ is disjoint frome_x^M. Let $(V,E)^M = \tilde{\cup}^M_{C\in\mu} C, (V,E)^M$ is a $SS^M - open$ in $(U,E)^M$ containing $(A,E)^M$. Since μ is $SS^M - locally\, finite$ $, SS^M(cl(V,E)^M) = \tilde{\cup}^M_{C\in\mu} SS^M(cl(C))$by (Proposition 3.2). Then $SS^M(cl(V,E)^M)$ is disjoint frome_x^M. Thus soft simply regularity is proved .

To prove soft simply normality, one only repeats the same argument, replacing $e_x^M$by a $SS^M - closed$ set throughout and replacing the $SS^M - \mathrm{T}_2$condition bysoft simply regularity.

Theorem 3.7 : Each$SS^M - closed$ subspace of a $SS^M - paracompact$ is$SS^M - paracompact$.

Proof: Let $(U,\tilde{\tau},E)$be a $SS^M - paracompact$ space, and $Y \,\tilde{\subseteq}^M\, U$ such that $(Y,E)^M$ is $SS^M - closed$ in $(U,E)^M$, let η be a soft simply covering of $(Y,E)^M$ by $SS^M - open$ in $(Y,E)^M$. For every $(A,E)^M \in \eta$, take $SS^M - open$ set $\left(\grave{A},E\right)^M$ of $(U,E)^M$ such that $\left(\grave{A},E\right)^M \,\tilde{\cap}^M\, (Y,E)^M = (A,E)^M$. Cover $(U,E)^M$ by the $SS^M - open\left(\grave{A},E\right)^M$,along with the $SS^M - open$ set ${(Y,E)^M}^{\complement}$. Suppose that σ is a $SS^M - locally\, finite SS^M - open$ refinement of this $SS^M - covering$ that covers $(U,E)^M$. Then the collection $\mu = \{(B,E)^M \,\tilde{\cap}^M\, (Y,E)^M : (B,E)^M \in \sigma\}$ is the required locally finite soft simply open refinement of η.

Remark 3.8 : By Proposition 3.5 , it is easy to see the following two facts:

1) A $SS^M - paracompact$ sub space of a$SS^M - \mathrm{T}_2\ space(U,\tilde{\tau},E)$need do not be $SS^M - closed$ in $(U,E)^M$.
2) A $SS^M - subspace$ of a $SS^M - paracompact$ need not by$SS^M - paracompact$.

Lemma 3.9: Let $(U,\tilde{\tau},E)$be a softtopological space.If each $SS^M - open$ covering of $(U,\tilde{\tau},E)$ has a SS^M −locally finite$SS^M - closed$ refinement, then every $SS^M - open$ covering of $(U,\tilde{\tau},E)$ has SS^M −locally finite$SS^M - open$refinement.

Proof: Let η be a $SS^M - open$ covering of $(U,\tilde{\tau},E)$, and let $\sigma = \{(F_s,E)^M : s \in S\}$,be a SS^M −locally finite$SS^M - closed$ refinement of η. For each $SS^M - pointe_x^M \in (U,E)^M$, choose a $SS^M - open\, nbh\left(V_{e_x^M},E\right)^M$of e_x^Msuch that $\left(V_{e_x^M},E\right)^M$ intersect finitely many elements of σ.Let$\mu = \left\{\left(V_{e_x^M},E\right)^M : e_x^M \in (E,E)^M\right\}$, and let $\mathcal{D}$ be a SS^M −locally finite$SS^M - closed$ refinement ofμ. For each$s \in S$,put $(W_s,E)^M = (\tilde{\cup}^M \{(D,E)^M : (D,E)^M \in \mathcal{D}, (D,E)^M \,\tilde{\cap}^M\, (F_s,E)^M = \emptyset\})^{\complement}$. Obvious, each $(W_s,E)^M$ is $SS^M - open$ and contains$(F_s,E)^M$. Moreover, for each $s \in S$ and each $(D,E)^M \in \mathcal{D}$, we have $(W_s,E)^M \,\tilde{\cap}^M\, (D,E)^M \neq \emptyset$ if and only if $(F_s,E)^M \,\tilde{\cap}^M\, (D,E)^M \neq \emptyset$. For each $s \in S$, choose a $(A_s,E)^M \in$ ηsuch that$(F_s,E)^M \,\tilde{\subseteq}^M\, (A_s,E)^M$, and let $(G_s,E)^M = (A_s,E)^M \,\tilde{\cap}^M\, (W_s,E)^M$. Then

$\{(G_s,E)^M : s \in S\}$ is a $SS^M - open$ covering and refines η. It is easy to see that each element of $\mathcal{D}$ intersects only finitely many $(G_s,E)^M$. Therefore $\{(G_s,E)^M : s \in S\}$ is a SS^M −locally finite.

Lemma 3.10 : Eachσ −locally finite soft simply open covering has a soft simply locally finite refinement.

Proof : Let $\mathcal{U} = \tilde{\cup}^M_{n\in N} \mathcal{U}_n$be a σ −locally finite soft simply open covering for some soft topological space, where each $\mathcal{U}_n$ is SS^M −locally finite. Put $\mathcal{V}_1 = \mathcal{U}_1$, $\mathcal{V}_n = \{(F,E)^M \tilde{\cap}^M (\tilde{\cup}^M_{k<n} \mathcal{U}^*_k)^{\complement} : (F,E)^M \in \mathcal{U}_n\}$, where $\mathcal{U}^*_k = \tilde{\cup}^M \{(F,E)^M : (F,E)^M \in \mathcal{U}_k\}$. Then it is easy to see that $\mathcal{V} = \tilde{\cup}^M_{n\in N} \mathcal{V}_n$ is a SS^M −locally finite soft simply open covering and refines $\mathcal{U}$.

Lemma 3.11 : Let$(U,\tilde{\tau},E)$ be a $SS^M - regular$, if each soft simply open covering of $(U,\tilde{\tau},E)$ has a SS^M −locally finite refinement, then it has a SS^M −locally finiteSS^M −closed refinement.

Proof: Let $\mathcal{U} = \{(F_\alpha,E)^M ; \alpha \in A\}$ be an arbitrary soft simply open covering. Then, for each $SS^M - Limtpe^M_x \in U$, there exists some $(F_\alpha,E)^M \in U$ such that $e^M_x \in (F_\alpha,E)^M$. By soft simply regularity, there is an $SS^M - nbh(\mathcal{V}_{e^M_x},E)$ such that $e^M_x \in (\mathcal{V}_{e^M_x},E) \tilde{\subseteq}^M SS^M(cl(\mathcal{V}_{e^M_x},E)^M \tilde{\subseteq}^M (F_\alpha,E)^M$. Put $\mathcal{V} = \{(\mathcal{V}_{e^M_x},E);\ e^M_x \in U\}$. Then $\mathcal{V}$ is a soft simply open covering and refines $\mathcal{U}$. By the assumption, there is a SS^M −locally finite soft simply covering $\mathcal{W} = \{(\mathcal{W}_\beta,E)^M;\ \beta \in B\}$, such that $\mathcal{W}$ refines $\mathcal{V}$. Then $\{SS^M(cl(\mathcal{W}_\beta,E)^M);\ \beta \in B\}$ is a SS^M −locally finite soft simply closed covering and refines U.

By Lemma 3.9, 3.10, and 3.11, we have the following theorem:

Theorem 3.12: Let$(U,\tilde{\tau},E)$ be a $SS^M - regular$**.** Then the following conditions on U are equivalent:

1) $(U,\tilde{\tau},E)$ is a $SS^M - paracompact$.
2) Every soft simply open covering has a σ −locally finite soft simply open refinement.
3) Every soft simply open covering has a locally finite soft simply refinement.
4) Every soft simply open covering has a locally finite soft simply closed refinement.

Conclusion:

The aim of this research is using the class of soft simply open set to define soft simply connected spaces. we study basic definitions and theorems about it. Further, we introduce the notion Soft Simply Paracompact Spaces, and we present soft simply pu-continuous defined between two soft topological spaces and study their properties in detail. Finally, we hope is togeneralize these notions by using other open sets.

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Fuzzy Precompact Space

Saad Mahdi Jaber

University of Wasit/ Education College for Pure Science/ Department of Mathematics

s.jaber@uowasit.edu.iq

Abstract. In this paper we study and introduce the concept of preompactness of a fuzzy topological space and also, we attain a number of important characterizations of a fuzzy precompact space. The notion of precompactness that can be extended to arbitrary fuzzy sets. So, this paper explains the relationship between fuzzy precompact space and fuzzy precompact subspace. Finally, we give necessary and sufficient conditions for a fuzzy pre regular space to be fuzzy precompact.
Key words and phrases: Fuzzy precompact space, Fuzzy pre q-nbd, Fuzzy pre cluster point.

1. **Introduction**

The fuzzy concept has invaded almost all branches of mathematics, since the introduction the fundamental concept of fuzzy sets by Zadeh [9] in 1965. Chang [4] in 1968, introduced the definition of fuzzy topological spaces and extended in a straight forward manner some concepts of crisp topological spaces to fuzzy topological spaces. The fuzzy topology was originating with Chang's article [9] in 1968, also may be considered as a new branch of mathematics, then many additional structures were studied by using fuzzy sets and the related problems in pure and applied mathematics. While Wong [16] in 1974 discussed and generalized some properties of fuzzy topological spaces. Ming, p.p. and Ming, L.Y. [11] in 1980 used fuzzy topology to define the neighborhood structure of fuzzy point. Shahna A. S. Bin [13] in 1991 defined the concept of pre open in fuzzy topological space.

In what follows, a fuzzy topological space (X, T) as defined by Chang [4], we shall denote for its by a fts (X, T) or simply by a fts X. The concepts closure [4], interior [4] and complement [15] of a set A in a fuzzy topological space (X, T) are denoted by $cl(A)$, $int(A)$ and $1 - A$ respectively. A fuzzy set A in X is said to be fuzzy pre open if $A \leq int(cl(A))$. The fuzzy pre closed $1 - A$ is a complement of a fuzzy pre open set A. The notation $pcl(A)$ stands for the fuzzy pre closure, which is the union of all fuzzy points x_α, when any fuzzy pre open set U containing x_α with $A \wedge U \neq 0$, every fuzzy open in a fts X is fuzzy pre open.

2. **Preiminaries**

First, we recall the following definitions, theorems, propositions, corollaries, remarks and lemmas that are needed in the next section.

2.1. Definition [10, P.211-220]

Let $X \neq \emptyset$ and let I be the unite interval, that means $I = [0,1]$. A fuzzy set A in X is a function from X into the unit interval I. (that means $A: X \rightarrow [0,1]$ be a function).

A fuzzy set A in X can be explain by the set of pairs: $A = \{(x, A(x)): x \epsilon X\}$. The notation I^X stand for the family of all fuzzy sets in X.

2.2. Definition [10, P.211-220]

Let f be a fuzzy mapping from a set X into Y. Let $A \in I^X$ and $B \in I^Y$.

a- The image of A under f, $f(A)$ is a fuzzy set in Y defined by for each $y \in Y$,

$$[f(A)](y)=\begin{cases} \sup A(x) \ \ if \ \ f^{-1}(y) \neq \emptyset \\ \qquad\qquad x \in f^{-1}(y) \\ 0 \qquad\qquad otherwise, \end{cases}$$

Where $f^{-1}(y) = \{x \in X | \, f(x) = y\}$.

b- The inverse image of B under f, $f^{-1}(B)$ is a fuzzy set in X defined by for each $x \in X$, $[f^{-1}(B)](x) = B(f(x))$.

2.3. Definition [10, P.211-220],[3]

A fuzzy point x_α in X is fuzzy set defined as follows:

$$x_{\alpha}(y)=\begin{cases} \alpha & if \ \ y = x \\ \\ 0 & if \ \ y \neq x \end{cases}$$

Where$0 < \alpha \leq 1$; α is called its value and x is support of x_α.

The set of all fuzzy points in X will be denoted by$FP(X)$.

2.4. Definition [10, P.211-220], [1]

A fuzzy point x_α in X is said to belong to a fuzzy set A (denoted by: $x_\alpha \in A$) if and only if $\alpha \leq A(x)$.

2.5. Definition [10, P.211-220],[1]

A fuzzy set A in X is called quasi–coincident with a fuzzy set B in X denoted by AqB if and only if $A(x) + B(x)$>1, for some $x \in X$. If A is not quasi–coincident with, then $A(x) + B(x)$≤1, for every $x \in X$ and denoted by $A\tilde{q}B$.

2.6. Lemma [3, P.137-150]

Let A and B are fuzzy sets in X. Then:

(a) If $A \wedge B = 0,\ then\ A\tilde{q}B$

(b) $A\tilde{q}B\ \ if\ and\ only\ if\ A \leq B^c$

2.7. Proposition [3, P. 137-150]

If A a fuzzy set in X, then $x_\alpha \in A$ if and only if $x_\alpha \tilde{q} A^c$.

2.8. Definition [4, P.182-190]

A fuzzy topology on a set X is a collection T of fuzzy sets in X satisfieding:

(1) $0 \in T\ and\ 1 \in T$,

(2) $If\ A$ and B belong to T then $A \wedge B \in T$,

(3) $If\ A_i$ belong to T for each $i \in I$, then so does $\bigvee_{i \in I} A_i$

If T is a fuzzy topology on X, then the pair (X, T) is called a fuzzy topological space. Members of T are called fuzzy open sets. Fuzzy sets of the forms $1 - A = A^c$, where A is fuzzy open set are called fuzzy closed sets.

*2.9. Definition [13, P.303-*308*], [17]*

Let (X, T) be a fuzzy topological space. Then:

i) The fuzzy interior of A, denoted by $int(A)$ is the union of all fuzzy open sets in X wich are contained in A. ($int(A) = \bigvee\{B: B \leq A, B \in T\}$)

ii) The fuzzy pre closure of A, denoted by $cl(A)$ is the intersection of all fuzzy closed sets in X contains A. ($cl(A) = \bigwedge\{B: A \leq B, B^c \in T\}$)

2.10. Definition [10, P.211-220]

A fuzzy set A in $fts(X)$ is called quasi-neighborhood of fuzzy point x_α in X if and only if there exists $B \in T$such that $x_\alpha qB \leq A$.

2.11. Definition [10, P.211-220]

Let (X, T) be a fuzzy topological space and x_α be a fuzzy point in X. Then the family $N_{x_\alpha}^{Q}$ consisting of all quasi-neighborhood (q-nbd) of a fuzzy point x_α is said to be the system of quasi-neighborhood of x_α.

2.12. Theorem [13, P.303-308], [17]

Let (X, T) be a fuzzy topological space and A,B are two fuzzy sets in X. Then:

i) $0 = cl(0)$,

ii) $cl(A \vee B) = cl(A) \vee cl(B)$ and $cl(A \wedge B) \leq cl(A) \wedge cl(B)$,

iii) $int(A \wedge B) = int(A) \wedge int(B), int(A \vee B) \geq int(A) \vee int(B)$,

iv) $cl\big(cl(A)\big) = cl(A), int\big(int(A)\big) = int(A)$,

v) $int(A) \leq A \leq cl(A)$,

vi) $If\ A \leq B\ then\ int(A) \leq int(B)\ and\ cl(A) \leq cl(B)$.

2.13. Remark

Let A, B are two fuzzy sets in $fts(X)$, then:

a- $int(A) = 1 - cl(1 - A)$,

b- $pint(A) = 1 - pcl(1 - A)$.

Proof: a- It is straightforward. b- It is straightforward.

2.14. Definition [4,P.182-190]

Let (X, T) be a fuzzy topological space and let A be any fuzzy set in X, A is called fuzzy pre open set if $A \leq int(cl(A))$. The complement of a fuzzy pre open set is called fuzzy pre closed set.
The family of all fuzzy pre open sets in X will be denoted by $FPO(X)$.

2.15. Definition [2, P.131-139]

A fuzzy set A in $fts(X)$ is said to be pre quasi-neighborhood (pre q-nbd) of $x_\alpha \in FP(X)$ if and only if there exists $B \in FPO(X)$ such that $x_\alpha q B \leq A$.

2.16. Definition [6, P.303-312]

A fuzzy set A in $fts(X)$ is said to be fuzzy pre quasi-neighborhood (pre q-nbd) of $x_\alpha \in FP(X)$, if there is a fuzzy pre open set B in X, such that $x_\alpha q B \leq A$. The family of all pre quasi-neighborhood of fuzzy point $x_\alpha$is said to be the system of pre quasi-neighborhood of x_α and denoted by $N_{x_\alpha}^{pQ}$.

2.17. Proposition

Let A be a fuzzy set in $fts(X)$. Then:

iii) The fuzzy pre interior of A, denoted by $pint(A)$ is the union of all pre open subsets of X wich are contained in A.

iv) The fuzzy pre closure of A, denoted by $pcl(A)$ is the intersection of all fuzzy pre closed subset of X contains A.

2.18. Proposition [12, P.1601-1608]

Let (X, T) be a fuzzy topological space and $A, B \leq X$.Then:

i- $int(A) \leq pint(A) \leq A$,

ii- $A \leq pcl(A) \leq cl(A)$,

iii- A is a fuzzy pre closed iff $pcl(A) = A$,

iv- $pcl\big(pcl(A)\big) = pcl(A)$,

v- If $A \leq B$, then $pcl(A) \leq pcl(B)$,

vi- $\vee_{j\in J}\, pcl(U_j) \leq pcl(\vee_{j\in J}\big(U_j\big))$,

vii- $x_\alpha \in pcl(A)$ iff $U \wedge A \neq 0$, $\forall U \in FPO(X), x_\alpha \in U$.

2.19. Remark

If A, B are fuzzy pre open sets, then $A \wedge B$ is fuzzy pre open.
Proof: It is clear.

2.20. Remark [8,P.111-12]
Let A be a fuzzy set in $fts(X)$. Then A is a fuzzy pre open if and only if A is a fuzzy pre quasi-neighborhood of its fuzzy points.

2.21. Proposition
Let A be a fuzzy set in $fts(X)$. Then a fuzzy point $x_\alpha \in pcl(A)$ if and only if every fuzzy pre open $B \in FPO(X)$, if $x_\alpha qB$ then AqB.
Proof: $\Longrightarrow$)Suppose that B be a fuzzy pre open set in X such that $x_\alpha qB$ and $A\tilde{q}B$. Then $A \leq (1-B)$, but $x_\alpha \notin (1-B)$ (since $x_\alpha qB$, then $\alpha \geq (1-B)(x)$) and $1-B$ is a fuzzy pre closed set in X. Thus $x_\alpha \in pcl(A)$.
($\Longleftarrow$ Suppose that $x_\alpha \notin pcl(A)$, then there exists a fuzzy pre closed set B in X such that $A \leq B$ and $x_\alpha \notin B$, therefor by (*2.7.Proposition*), we have $x_\alpha q 1-B$. Since $A \leq B$, then by (*2.6.ii.Lemma*), $A\tilde{q}1-B$. Hence $x_\alpha \in pcl(A)$ if $x_\alpha qB$ and AqB.

2.22. Definition [8, P.111-121]
In a $fts(X)$, a mapping $S: D \to FP(X)$ is said to be a fuzzy net and denoted by $\{S(n): n \in D\}$, D is directed set. If $S(n) = x^n_{\alpha_n}$ where $x \in X, n \in D$ and $\alpha_n \in (0,1]$, then we shall denote it by $\{x^n_{\alpha_n}: n \in D\}$ or simply $\{x^n_{\alpha_n}\}$.

2.23. Definition [8, P.111-121]
A fuzzy net $\Im = \{y^m_{\alpha_m}: m \in E\}$ in X is called a fuzzy subnet of fuzzy net $S = \{x^n_{\alpha_n}: n \in D\}$ if and only if there is a mapping $f: E \to D$ such that:

(a) $\Im = S\text{o}f$, that is $y^i_{\alpha_i} = x^{f(i)}_{\alpha_{f(i)}}$, $\forall i \in E$.

(b) $\forall n \in D$ there is some $m \in E$, such that $f(m) \geq n$.

A fuzzy sub net of a fuzzy net $\{x^n_{\alpha_n}: n \in D\}$ denoted by $\{x^{f(m)}_{\alpha_{f(m)}}, m \in E\}$.

2.24. Definition [8, P.111-121]
Let $S = \{x^n_{\alpha_n}: n \in D\}$ be a fuzzy net in a fuzzy topological space (X, T) and $A \in I^X$, then:

i- S is said to be eventually with A if and only if $\exists\, m \in D$ such that $x^n_{\alpha_n} qA$, $\forall\, n \geq m$.

ii- S is said to be frequently with A if and only if $\forall\, n \in D, \exists\, m \in D, m \geq n$ and $x^m_{\alpha_m} qA$.

2.25. Definition [8, P.111-121]
Let $S = \{x^n_{\alpha_n}: n \in D\}$ be a fuzzy net in a fuzzy topological space (X, T) and $x_\alpha \in FP(X)$, then:

(i) S is said to be convergent to x_α and denoted by $S \to x_\alpha$, if S is eventually with $A, \forall\, A \in N^Q_{x_\alpha}$, x_α is called a limit point of S.

(ii) S is said to be has a cluster point x_α and denoted by $S \propto x_\alpha$, if S is frequently with $A, \forall\, A \in N^Q_{x_\alpha}$.

2.26. Definition
Let $S = \{x^n_{\alpha_n}: n \in D\}$ be a fuzzy net in a fuzzy topological space (X, T) and $x_\alpha \in FP(X)$. Then:

(i) S is said to be p-convergent to x_α (denoted by: $S \xrightarrow{p} x_\alpha$), if S is eventually with $A, \forall\, A \in N^{pQ}_{x_\alpha}$, x_α is called a pre limit point of S.

(ii) S is said to be called has a fuzzy p-cluster point x_α (denoted by: $S \overset{p}{\propto} x_\alpha$), if S is frequently with $A, \forall A \in N^{pQ}_{x_\alpha}$.

2.27. Definition [8, P.111-121]
A fuzzy filterbase on X is a non-empty subset $\mathcal{F}$ of I^X such that:
(1) $0 \notin \mathcal{F}$,
(2) If $A_1, A_2 \in \mathcal{F}$, then $\exists\, A_3$ such that $A_3 \le A_1 \wedge A_2$.

2.28. Definition [8, P.111-121]
A fuzzy point x_α in a fuzzy topological space (X,T) is said to be a fuzzy pre cluster point of a fuzzy filterbase $\mathcal{F}$ on X if $x_\alpha \in pcl(F)$, for all $F \in \mathcal{F}$.

2.29. Definition
A fuzzy topological space (X,T) is called fuzzy pre Hausdorff or pre T_2-space if and only if for every pair of distinct fuzzy points x_r, y_s in X, there exist $A \in N^{pQ}_{x_r}$, $B \in N^{pQ}_{y_s}$ such that $A \wedge B = 0$.

2.30. Definition
Let B be a fuzzy set in a fuzzy topological space (X,T), then $T_B = \{A \wedge B : A \in \tau\}$ is called a fuzzy relative topology and (B, T_B) is said to be a fuzzy topological subspace of X.

2.31. Theorem
In a fuzzy topological space (X,T), if V is a fuzzy open set, then $V \wedge cl(A) \le cl(V \wedge A)$ for any fuzzy set A in X.
Proof: Let $x_\alpha \in FP(X)$ and V is a fuzzy open in X. If $x_\alpha \in V \wedge cl(A)$, then $x_\alpha \in V\ and\ U \wedge A \ne 0$, $\forall U \in T, x_\alpha \in U$. Since $U \wedge V$ is fuzzy open set, therefore $U \wedge (V \wedge A) \ne 0$ and $x_\alpha \in cl(V \wedge A)$.
Hence $V \wedge cl(A) \le cl(V \wedge A)$.

2.32. Definition
In a fuzzy topological space (X,T), if $A \le B < X$, then a fuzzy set A is called fuzzy pre open in B if there exist a fuzzy pre open H in X such that $A = H \wedge B$.

2.33. Proposition
In a fuzzy topological space (X,T), if $A \le B < X$, then a fuzzy set A is a fuzzy pre open in B, if A is a fuzzy pre open in X.
Proof: We have $A = A \wedge B$, but A is fuzzy pre open in X. Hence, by (*2.32.Definition*) A is a fuzzy pre open in B.

2.34. Proposition
Let $A \le B < X$, where (X,T) is a fuzzy topological space and B is a fuzzy pre open in X. Then A is a fuzzy pre open in B if and only if $A = S \wedge B$, where S is a fuzzy open in X.
Proof: ⇒) To prove A is a fuzzy pre open in B, we must prove $S \wedge B$ is a fuzzy pre open in X (i.e. $S \wedge B \le int(cl(S \wedge B))$).
Since $S \wedge B \le S \wedge int\big(cl(B)\big) = int\big(int(S)\big) \wedge int\big(cl(B)\big) = int(\,int(S) \wedge cl(B))$
by (*2.31.Theorem*) $\le int(cl(int(S) \wedge B)) \le int(cl(S \wedge B))$. Thus $S \wedge B$ is pre open in X. Hence A is a fuzzy pre open in B by (*2.33.Proposition*).
(⇐ We have $A = S \wedge B$, since S is fuzzy open in X, then S is a fuzzy pre open. Hence by (*2.30.Definition*) A is a fuzzy pre open in B.

2.35. Definition [2,P.131-139]
A family V of fuzzy sets has the finite intersection property if and only if the intersection of the members of the each finite subfamily of V is a non-empty.

2.36. Definition [8,P.111-121]
A family B of a fuzzy sets in a fuzzy topological space(X,T) is said to be a fuzzy pre open cover of a

fuzzy set A if and only if $A \leq \bigvee\{G: G \in B\}$ and each member of B is pre open fuzzy set. A sub cover of B is a sub family which is also cover.

2.37. Definition [5, P.39-49]
In a fuzzy topological space (X, T), a fuzzy set D is said to be fuzzy dense if there exists no fuzzy closed set B in X, such that $D<B<1$. That is, $cl(D) = 1$.

2.38. Definition [5, P.39-49]
In a fuzzy topological space (X, T), a fuzzy set D is said to be fuzzy pre dense if there exists no fuzzy pre closed set B in X, such that $D<B<1$. That is, $cl(D) = 1$.

2.39. Theorem
Let (X, T) be a fuzzy topological space. A fuzzy set D in X is a fuzzy dense if and only if it is a fuzzy pre dense, with $int(D) \neq 0$.
Proof:$\Longrightarrow$) Suppose that D is a fuzzy dense in X. Let $x_\alpha \in FP(X)$ and $x_\alpha \in cl(D)$. but $x_\alpha \notin pcl(D)$. So $x_\alpha \in (1 - pcl(D))$ by (*2.13.Remark*) implies that $x_\alpha \in pint(1 - D) \leq (1 - D) \leq cl(1 - D)$. So $x_\alpha \in (1 - int(D))$ by (*2.13.Remark*). Thus, $x_\alpha \notin int(D)$ so there is no fuzzy open U (containing x_α) such that $U \leq D$, and so $U \wedge D = 0$, a contradiction that D is a fuzzy dense set. Therefor $x_\alpha \in pcl(D)$ and $1 \leq pcl(D)$. Hence, $pcl(D) = 1$.
($\Longleftarrow$It is straightforward.

2.40. Definition [6, P.303-312]
A fuzzy topological space (X, T) is said to be fuzzy pre regular if for each fuzzy point x_t and each fuzzy pre q-nbd U of x_t, there exists a fuzzy pre open set V in X such that $x_t qV \leq pcl(V) \leq U$.

2.41. Theorem
In a $fts(X)$, $x_\alpha \in FP(X)$ and $A \in I^X$. Then $x_\alpha \in pcl(A)$ if and only if there exists a fuzzy net in A pre converge to x_α .
Proof: $\Longrightarrow$) Suppose that $x_\alpha \in pcl(A)$, then for every $B \in N_{x_\alpha}^{pQ}$there is

$$x_B(y)=\begin{cases} A(x_\alpha) & if \;\; y = x_B \\ 0 & if \;\; y \neq x_B, \end{cases} \qquad \text{such that } A(x_\alpha) + B(x_\alpha) > 1.$$

Notice that $(N_{x_\alpha}^{pQ}, \geq)$ is directed set, therefor $S: N_{x_\alpha}^{pQ} \rightarrow FP(X)$ is a fuzzy net in A and defined as $S(B) = x_B^A$. To show that $S \rightarrow x_\alpha$. Let $W \in N_{x_\alpha}^{pQ}$, then there is $F \in T$ such that $x_\alpha qF$ and $F \leq W$. Since $F(x_F^A) + x_F^A > 1$ and $F \leq W$ then $W(x_F^A) + x_F^A > 1$. Thus $x_F^A qW$. Let $E \geq F$, therefor $E \leq F$. Since $E(x_E^A) + x_E^A > 1$ and $F \leq W$, then $W(x_F^A) + x_F^A > 1$. Thus $x_E^A qW$, $\forall E \geq F$. Therefor $S \xrightarrow{p} x_\alpha$.
($\Longleftarrow$ Suppose that $\{x_{\alpha_n}^n : n \in D\}$ is a fuzzy net in A where $(D, \geq)$ is directed set, such that $x_{\alpha_n}^n \xrightarrow{p} x_\alpha$.Then for every $\in N_{x_\alpha}^{pQ}$, there exists $m \in D$ such that $x_{\alpha_n}^n qW$ for all $n \geq m$. Since $x_{\alpha_n}^n \in A$, then by (*2.7.Proposition*) $x_{\alpha_n}^n \tilde{q} A^c$, thus AqW and $x_\alpha \in pcl(A)$.

2.42. Lemma
In a fts X, a fuzzy point $x_\alpha$is a fuzzy pre cluster point for the fuzzy net $\{S(n): n \in D\}$ with a directed set $(D, \geq)$ if and only if it has a fuzzy subnet which fuzzy pre converges to x_α.

Proof: $\Longrightarrow$) Suppose that a fuzzy net $\{x_{\alpha_n}^n : n \in D\}$ has the pre cluster point x_α. Let $N_{x_\alpha}^{pQ}$ be the collection of all fuzzy pre q-nbd of x_α. Thus, for any $W \in N_{x_\alpha}^{pQ}$ there exists $\{x_{\alpha_n}^n\}$ such that $\{x_{\alpha_n}^n\}qW$. All ordered pairs (n, W) with the above character forms the set $\mathcal{O}$, that means $n \in D, W \in N_{x_\alpha}^{pQ}$ and $\{x_{\alpha_n}^n\}qW$. Now, we will define a relation "$\mathcal{E}$" on $\mathcal{O}$ given by $(m, U)\mathcal{E}(n, V)$ iff $m \geq n$ in D and $U \leq V$, then $(\mathcal{O}, \mathcal{E})$ is a directed set and it is clear to see that $\mathfrak{J}: \mathcal{O} \to FP(X)$ given by $\mathfrak{J}(m, U) = \{x_{\alpha_m}^m\}$ is a fuzzy subnet of the assumed fuzzy net. W is a pre q-nbd of x_α thus, there exists $n \in D$ and therefor $\{x_{\alpha_n}^n\}qW$ when $(n, W) \in \mathcal{O}$. Now, $(m, W) \in \mathcal{O}$ and $(m, U)\mathcal{E}(n, W) \Rightarrow \mathfrak{J}(m, U) = \{x_{\alpha_m}^m\}qU$ and $U \leq W \Rightarrow \mathfrak{J}(m, U)qW$. Hence $\mathfrak{J}$ is pre converges to x_α.
($\Longleftarrow$ Suppose that a fuzzy net $\{x_{\alpha_n}^n : n \in D\}$ has not a pre cluster point. Therefor, for every fuzzy point x_α there exists a pre q-nbd of x_α such that $x_{\alpha_m}^m \tilde{q} U$ for all $m \geq n, n \in D$. Hence, clear no fuzzy net pre converge to x_α.

2.43. Proposition
In a fuzzy pre Hausdorff space X, any pre convergent fuzzy net has a unique limit point .
Proof:$\Longrightarrow$) Suppose that $x_{\alpha_n}^n$ is a fuzzy net on X with directed set D, such that $x_{\alpha_n}^n \xrightarrow{p} x_\alpha$, $x_{\alpha_n}^n \xrightarrow{p} y_\beta$ and $x \neq y$. Since $x_{\alpha_n}^n \xrightarrow{p} x_\alpha$, we have $\forall A \in N_{x_\alpha}^{pQ}, \exists m_1 \in D$, such that $\{x_{\alpha_n}^n\}qA$, $\forall n \geq m_1$. Also, $x_{\alpha_n}^n \xrightarrow{p} y_\beta$, we have $\forall B \in N_{y_\beta}^{pQ}, \exists m_2 \in D$, such that, $\{x_{\alpha_n}^n\}qB$, $\forall n \geq m_2$. Now, then there exists $m \in D$, such that, $m \geq m_1$and $m \geq m_2$ then $\{x_{\alpha_n}^n\}q(A \wedge B), \forall n \geq m$. Therefore $A \wedge B \neq 0$. Hence X is not fuzzy pre Hausdorff.
($\Longleftarrow$ Let X be a not fuzzy pre Hausdorff space, then there is $x_\alpha, y_\beta \in FP(X)$, such that $x \neq y$ and $A \wedge B \neq 0, \forall A \in N_{x_\alpha}^{pQ}, \forall B \in N_{y_\beta}^{pQ}$. Put $N_{x_\alpha, y_\beta}^{pQ} = \{A \wedge B : A \in N_{x_\alpha}^{pQ}, N_{y_\beta}^{pQ}\}$. Therefore $\forall D \in N_{x_\alpha, y_\beta}^{pQ}$, there exists $x_D qD$, then $\{x_D\}_{D \in N_{x_\alpha}^{y_\beta}}$is a fuzzy net in X. To prove that $x_D \xrightarrow{p} x_\alpha$ and $x_D \xrightarrow{p} y_\beta$. Let $W \in N_{x_\alpha}^{pQ}$, then $W \in N_{x_\alpha, y_\beta}^{pQ}$ (since $W = W \wedge X \neq 0$). Thus $x_D qW, \forall D \geq W$, thus $x_D \xrightarrow{p} x_\alpha$ and $x_D \xrightarrow{p} y_\beta$. Hence, $\{x_D\}_{D \in N_{x_\alpha}^{y_\beta}}$has two fuzzy limit point.

2.44. Definition
A fuzzy space X is called fuzzy precompact if every fuzzy pre open of cover X has finite sub cover.

2.45. Theorem [8, P.111-121]
A fuzzy topological space (X, T) is a fuzzy compact if and only if every fuzzy filter base on X has a fuzzy cluster point.

3. Fuzzy precompact space

3.1. Theorem
A fuzzy topological space (X, T) is a fuzzy precompact, if and only if any collection $\{B_j : j \in J\}$of fuzzy pre closed sets in X having the finite intersection property.
Proof: $\Longrightarrow$) Suppose that X is fuzzy precompact space and $\{B_j : j \in J\}$is collection of fuzzy pre closed sets of X with the finite intersection property. To show $\{B_j : j \in J\}$ has a non-empty intersection (i.e to show $\bigwedge_{j \in J} B_j \neq 0$).
Assume that $\bigwedge_{j \in J} B_j = 0$, then $\bigvee_{j \in J} B_j^c = 1$ and each B_j^c is fuzzy pre open set, thus there exist $j_1, j_2, \dots, j_n$such that $\bigvee_{i=1}^n B_{ji}^c = 1$ by (*2.44.Definition*), therefor $\bigwedge_{i=1}^n B_{ji} = 0$ which is contradiction and therefor $\bigwedge_{j \in J} B_j \neq 0$.

($\Longleftarrow$ Conversely, let $\{A_j : j \in J\}$be a fuzzy pre open cover of X and every collection of fuzzy pre closed sets in X with the finite intersection property has a non-empty. To show that X is a fuzzy precompact space. Since $\bigvee_{j\in J} A_j = 1$, then $\bigwedge_{j\in J} A_j^c = 0$ and each A_j^c is fuzzy pre closed set which implies that $\{A_j^c : j \in J\}$collection of fuzzy pre closed sets with empty intersection and so by hypothesis this collection does not have the finite intersection property. Thus, there exist a finite member of fuzzy sets A_{ji}^c, $i = 1,2,\dots,n$, such that $\bigwedge_{i=1}^n A_{ji}^c = 0$, which implies $\bigvee_{i=1}^n A_{ji} = 1$ and $\{A_{ji} : i = 1,2,\dots,n\}$is finite sub cover of the space X belong to a fuzzy pre open cover $\{A_j : j \in J\}$. Hence, X is a fuzzy precompact space.

3.2. Theorem

A fuzzy topological space (X,T) is a fuzzy precompact, if and only if for every fuzzy filterbase on X has a fuzzy pre cluster point.

Proof: $\Rightarrow$)Suppose that X is a fuzzy precompact and $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ is a fuzzy filterbase on X having no fuzzy pre cluster point. Let $x \in X$, then for each $n \in N(N\ is\ natural\ number)$, there exists a pre q-nbd U_x^n of $x_{\frac{1}{n}} \in FP(X)$ and $F_x^n \in \mathcal{F}$ such that $U_x^n \tilde{q} F_x^n$. Now, $U_x^n(x) > 1 - 1/n$, since we have $U_x(x) = 1$, where $U_x = \bigvee\{U_x^n : n \in N\}$. Therefore $\mathcal{O} = \{U_x^n : n \in N, x \in X\}$ is a fuzzy pre open cover of X. When X is fuzzy precompact, then there exists $U_{x_1}^{n_1}, U_{x_2}^{n_2}, \dots, U_{x_k}^{n_k}$ of $\mathcal{O}$ such that $\bigvee_{i=1}^k U_{x_i}^{n_i} = 1$. If $F \in \mathcal{F}$ such that $F \le F_{x_1}^{n_1} \wedge F_{x_2}^{n_2} \wedge \dots \wedge F_{x_k}^{n_k}$, then $F\tilde{q}1$ Consequently, $F = 0$ and this contradicts the definition of a fuzzy filterbase.

($\Longleftarrow$ Suppose that every fuzzy filterbase have a fuzzy pre cluster point. To prove that X is fuzzy precompact. Acollection of fuzzy pre closed sets $\beta = \{F_\alpha : \alpha \in \Lambda\}$ having finite intersection property. Now, the set of finite intersections of members of β forms a fuzzy filterbase $\mathcal{F}$ on X. By assumed condition $\mathcal{F}$ has a fuzzy pre cluster point, which is x_α. Thus, $x_\alpha \in \bigwedge_{\alpha\in\Lambda} pcl\,(F_\alpha) = \bigwedge_{\alpha\in\Lambda} F_\alpha$and $\bigwedge\{F : F \in \mathcal{F}\} \ne 0$. Hence by (*3.1.Theorem*), X is a fuzzy precompact.

3.3. Theorem

A fuzzy topological space (X,T) is a fuzzy pre compact if and only if for every fuzzy net in X has a fuzzy pre cluster point.

Proof : $\Longrightarrow$) Suppose that X is a fuzzy precompact and $\{S(n) : n \in D\}$ is a fuzzy net in X which has no pre cluster point. Thus, for any fuzzy point x_α, there is a fuzzy pre q-nbds U_{x_α} of x_α and an $n_{U_{x_\alpha}} \in D$ such that, for each $m \in D$, $S_m \tilde{q} U_{x_\alpha}$ with $m \ge n_{U_{x_\alpha}}$. Since $x_\alpha q U_{x_\alpha}$ then $S_m \ne 0$, $\forall\ m \ge n_{U_{x_\alpha}}$. Let $\mathcal{U}$ be a symbol for the collection of all U_{x_α} and x_α is symbol for all fuzzy points $FP(X)$. Now, to shwo that $V = \{1 - U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\}$ is a family of fuzzy pre closed sets in X having finite intersection property. At first notice that there exists $k \ge U_{x_{\alpha_1}}, U_{x_{\alpha_2}}, \dots, U_{x_{\alpha_m}}$ such that $S_p \tilde{q} U_{x_{\alpha_i}}$for $i = 1,2,\dots,m$ and for all $p \ge k$ $(p \in D)$, that means $S_p \in 1 - \bigvee_{i=1}^m U_{x_{\alpha_i}} = \bigwedge_{i=1}^m (1 - U_{x_{\alpha_i}})$ for all $p \ge k$. Hence $\bigwedge\{1 - U_{x_{\alpha_i}} : i = 1,2,\dots,m\} \ne 0$. Since X is a fuzzy precompact, then by (*3.1.Theorem*), there is $y_\beta \in FP(X)$ such that, $y_\beta \in \bigwedge\{1 - U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\} = 1 - \bigvee\{U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\}$. Therefor, $y_\beta \in$1- U_{x_α}, for all $U_{x_\alpha} \in \mathcal{U}$ and $y_\beta \in 1 - U_{y_\beta}$, that means $y_\beta \tilde{q} U_{y_\beta}$. Since, for each fuzzy point x_α, there is $U_{x_\alpha} \in \mathcal{U}$ Such that $x_\alpha q U_{x_\alpha}$, then we get a contradiction.

($\Longleftarrow$ By (*3.2.Theorem*) we prove the converse, since every fuzzy filterbase on X has a fuzzy pre cluster point. Let $\mathcal{F}$ be a fuzzy filterbase in X, then for each $0 \ne F \in \mathcal{F}$, we can select $x_F \in FP(X)$ such that $x_F \in F$. Let $S = \{x_F : F \in \mathcal{F}\}$ with the relation "≥" be defined as follows $F_\alpha \ge F_\beta$ if and only if $F_\alpha \le F_\beta$ in X, for $F_\alpha, F_\beta \in \mathcal{F}$. Thus $(\mathcal{F}, \ge)$ is directed set. Ttus, S is a fuzzy net when $(\mathcal{F}, \ge)$ is directed set for its. From assumption, S has a cluster point x_t. Therefor, for every fuzzy pre q–nbd N of x_t and for each $F \in \mathcal{F}$, there is $G \in \mathcal{F}$ with $G \ge F$ such that $x_G q N$. As $x_G \le G \le F$. It follows that FqN for each $F \in \mathcal{F}$, then by (*2.21.Proposition*), $x_t \in pcl(F)$. Hence x_t is a fuzzy pre cluster point of $\mathcal{F}$.

3.4. Corollary
A fuzzy topological space (X, T) is a fuzzy precompact if and only if for every fuzzy net in X has a pre convergent fuzzy subnet.
Proof : By (*2.42. Lemma*) and (*3.3. Theorem*).

3.5. Proposition
Let (X, T) be a fuzzy topological space. If G and H are two Fuzzy precompact in X, then $H \vee G$ is also fuzzy precompact.
Proof: Suppose that $\{A_j : j \in J\}$ is a fuzzy pre open cover of $H \vee G$, then $H \vee G \leq \bigvee_{j \in J} A_j$. Since $G \leq H \vee G$ and $H \leq H \vee G$, thus $\{A_j : j \in J\}$ is a fuzzy pre open cover of G and fuzzy pre open cover of H. But G and H are two fuzzy precompact sets, thus there exists a finite sub cover $\{A_{j_1}, A_{j_2}, \dots, A_{j_n}\}$ of $\{A_j : j \in J\}$ which covering$G$ and a finite sub cover $\{A_{j_1}, A_{j_2}, \dots, A_{j_n}\}$ of $\{A_j : j \in J\}$ which covering H such that $G \leq \bigvee_{i=1}^{m} A_{ji}$ and $H \leq \bigvee_{k=1}^{n} A_{jk}$, therefor, $H \vee G \leq \bigvee_{t=1}^{m+n} A_{jt}$. Hence $H \vee G$ is fuzzy precompact.

3.6. Proposition
Every fuzzy precompact space is a fuzzy compact.
Proof: Suppose that $\mathcal{A} = \{A_j : j \in J\}$ is a fuzzy open cover of fuzzy space X and $X = \bigvee_{j \in J} A_j$. But, every fuzzy open set in X is a fuzzy pre open and X is a fuzzy precompact space, then there exists $j_1, j_2, \dots, j_n \epsilon J$ such that $X = \bigvee_{i=1}^{n} A_{j_i}$, thus X is fuzzy compact space.

3.7. Corollary
Let (X, T) be a fuzzy topological space. If G is a fuzzy precompact in X, then G is fuzzy compact.
Proof: It is straightforward.

3.8. Proposition
Let (X, T) be a fuzzy topological space. If B is a fuzzy set in X and $A \leq B$, then A is a fuzzy precompact in X if and only if A is a fuzzy precompact in B.
Proof: ⟹) Suppose that $\mathcal{A} = \{A_j : j \in J\}$ is a fuzzy cover of A by pre open sets in B. By (*2.32.Definition*), $A_j = S_j \wedge B$ for each $j \in J$, where S_j is a fuzzy pre open in X. Thus $\mathcal{S} = \{S_j : j \in J\}$ is a fuzzy cover of A by pre open sets in X, but A is a fuzzy pre compact in X, so there exists $j_1, j_2, \dots, j_n \epsilon J$ such that $A \leq \bigvee_{i=1}^{n} (S_{j_i} \wedge B) = \bigvee_{i=1}^{n} (A_{j_i})$. Hence, A is a fuzzy precompact in B.
(⟸ It is straightforward.

3.9. Proposition
Let (X, T) be a fuzzy topological space. If B is a fuzzy pre open set in X and $A \leq B$, then A is a fuzzy compact in X if and only if A is a fuzzy precompact in B.
Proof: ⟹) Suppose that $\mathcal{A} = \{A_j : j \in J\}$ is a fuzzy pre open cover of A in B. By (*2.34.Proposition*), $A_j = S_j \wedge B$ for each $j \in J$, where S_j is a fuzzy open in X. Thus $\mathcal{S} = \{S_j : j \in J\}$ is a fuzzy cover of A by fuzzy open sets in X, but A is a fuzzy compact in X, so there exists $j_1, j_2, \dots, j_n \in J$ such that $A \leq \bigvee_{i=1}^{n} (S_{j_i} \wedge B) = \bigvee_{i=1}^{n} (A_{j_i})$. Hence, A is a fuzzy precompact in B.
(⟸ Suppose that $\mathcal{S} = \{S_j : j \in J\}$ is a fuzzy open cover of A in X.Then $\mathcal{A} = \{S_j \wedge B : j \in J\}$ is a fuzzy cover of A. But, S_j is a fuzzy open in X for all $j \in J$ and B is a fuzzy pre open in X, then by (*2.34.Proposition*) $S_j \wedge B$ is a fuzzy pre open in B for all $j \in J$. By assumption A is a fuzzy precompact in B, then there exists $j_1, j_2, \dots, j_n \epsilon J$ such that $A \leq \bigvee_{i=1}^{n} (S_{j_i} \wedge B) \leq \bigvee_{i=1}^{n} (S_{j_i})$. Hence, A is a fuzzy compact in X.

3.10. Proposition
Let (X, T) be a fuzzy topological space. If B is a fuzzy pre open set in X and $A \leq B$, then A is a fuzzy compact in X if and only if A is a fuzzy compact in B.
Proof: By (*3.8. Proposition*), (*3.9. Proposition*) and (*3.7. Corollary*).

3.11. Proposition

Let (X,T) be a fuzzy topological space. If B is a fuzzy set in X and $A \leq B$, then A is a fuzzy compact in X if A is a fuzzy compact in B.

Proof: Suppose that $\mathcal{S} = \{S_j : j \in J\}$ is a fuzzy open cover of A in X. Since $A \leq B$and $A \leq S_j$, then $\mathcal{A} = \{S_j \wedge B : j \in J\}$ is a fuzzy cover of A. But, S_j is a fuzzy open in X for all $j \in J$, then by (*2.30. Definition*) $S_j \wedge B$ is a fuzzy open in B for all $j \in J$, by assumption A is a fuzzy compact in B, so there exists $j_1, j_2, \ldots, j_n \epsilon J$ such that $A \leq \bigvee_{i=1}^{n}(S_{j_i} \wedge B) \leq \bigvee_{i=1}^{n}(S_{j_i})$. Hence, A is a fuzzy compact in X.

3.12. Proposition

A fuzzy pre closed subset of a fuzzy precompact space (X,T) is a fuzzy precompact.

Proof: Suppose that G is a fuzzy pre closed subset of a fuzzy precompact space X and $\{A_j : j \in J\}$ is a fuzzy open cover of G in X, which implies that $G \leq \bigvee_{j \in J} A_j$. Thus, G has a fuzzy pre open cover $\{A_j : j \in J\}$. Since $G^c$is pre open, then the family $\{A_j : j \in J\} \vee G^c$ is a fuzzy pre open cover of X, which is a fuzzy recompact space. Thus there exists $j_1, j_2, \ldots, j_n$ such thst $\bigvee_{i=1}^{n} A_{ji} \vee \{G^c\} = 1$. Since $\{A_{j_1}, A_{j_2}, \ldots, A_{j_n}, G^c\}$ is finite subcover of X and $G \leq 1 = \bigvee_{i=1}^{n} A_{ji} \vee \{G^c\}$, but $G \nleq G^c$, therefor $G \leq \bigvee_{i=1}^{n} A_{ji}$. Hence, G is a fuzzy precompact.

3.13. Corollary

A fuzzy closed subset of a fuzzy precompact space (X,T) is fuzzy pre compact.

Proof: It is clear.

3.14. Corollary

A fuzzy closed subset of a fuzzy pre compact space (X,T) is fuzzy compact.

Proof: It is clear.

3.15. Theorem

Every fuzzy precompact subset of a fuzzy pre Hausdroff topological space is fuzzy pre closed.

Proof: Suppose that $x_\alpha \in pcl(A)$, then by (*2.41.Theorem*) there exists a fuzzy net $x_{\alpha_n}^n$ such that $x_{\alpha_n}^n \xrightarrow{p} x_\alpha$. Since A is fuzzy precompact and X is fuzzy pre Hausdroff space, then by (*3.4.Corollary*) and (*2.43.Proposition*), we have $x_\alpha \in A$ which implies that $pcl(A) \leq A$. Hence A is fuzzy pre closed set.

3.16. Theorem

In any fuzzy space, the intersection of a fuzzy precompact set with a fuzzy pre closed set is fuzzy precompact.

Proof: Suppose that A, B are two fuzzy sets such that A is a fuzzy precompact and B is a fuzzy pre closed. We must prove that $A \wedge B$ is a fuzzy precompact. Let $x_{\alpha_n}^n$ is fuzzy net in A, since A is fuzzy precompact, then by (*3.4.Corollary*), $x_{\alpha_n}^n \xrightarrow{p} x_\alpha$ for some $x_\alpha \in FP(X)$ and by (*2.41.Proposition*), $x_\alpha \in pcl(A)$. Since B is fuzzy pre closed, then $x_\alpha \in B$. Hence $x_\alpha \in A \wedge B$ and. Thus $A \wedge B$ is fuzzy precompact.

3.17. Definition

In a $fts\ X$, a fuzzy set G is said to be precompactly fuzzy pre closed if $G \wedge K$ is fuzzy precompact, for every fuzzy precompact set K in X.

3.18. Proposition

Every fuzzy pre closed subset of a fuzzy topological space X is precompactly fuzzy pre closed.

Proof: Suppose that G is a fuzzy pre closed subset of a fuzzy space X and let K be a fuzzy precompact set. Then by (*3.16.Theorem*), $G \wedge K$ is a fuzzy precompact. Thus G is a precompactly fuzzy pre closed set .

3.19. Theorem
In a fuzzy pre Hausdorff space X, a fuzzy set G is precompactly fuzzy pre closed if and only if G is fuzzy pre closed.
Proof: ⇒) Suppose that G is a precompactly fuzzy pre closed and $x_\alpha \in pcl(G)$. Then, by (*2.41.Proposition*), there is a fuzzy net $x_{\alpha_n}^n$ in G, such that $x_{\alpha_n}^n \xrightarrow{p} x_\alpha$, then by (*3.4.Corollary*), $B = \{ x_{\alpha_n}^n, x_\alpha\}$ is a fuzzy precompact set. But G is precompactly fuzzy pre closed, then $G \wedge B$ is a fuzzy precompact set, also X is a fuzzy pre Hausdroff space by assumption, then by (*3.16.Theorem*),$G \wedge B$ is fuzzy pre closed. Since $x_{\alpha_n}^n \xrightarrow{p} x_\alpha$ and $x_{\alpha_n}^n \in G \wedge B$, then by (*2.41.Theorem*) $x_\alpha \in G \wedge B$, so $x_\alpha \in G$. Therefore, $pcl(G) \leq G$. Hence G is a fuzzy pre closed set.
(⇐ By (*3.18. Proposition*).

3.20. Theorem
A fuzzy pre regular space X is a fuzzy precompact if and only if there exist a fuzzy dense D of X such that any fuzzy filterbase in D have a fuzzy pre cluster point in X, with $int(D) \neq 0$.
Proof: ⟹) By (*3.2.Theorem*).
(⟸ we prove if there exist a fuzzy dense D in X such that any fuzzy filterbase in D have a fuzzy pre cluster point in X, then X is a fuzzy precompact. Let D be a fuzzy dense set and X is not fuzzy precompact, then there exist a cover $\{U_j: j \in J\}$of fuzzy pre open set in X with no finite fuzzy subcover. Since X is a fuzzy pre regular, then there exists fuzzy pre open cover $\{V_i: i \in I\}$ of X such that for each j there exist i such that $pcl(V_i) \leq U_j$. By (*2.39.Theorem*) $X = Pcl(D)$. Now, $\{V_i: i \in I\}$ is a fuzzy pre open cover of $pcl(D)$ with no finite subcover. Therefore, the collection $\mathcal{B} = \{D \wedge (1 - \vee V_{i_k}), k = 1,2, \dots, n\}$ is a fuzzy filterbase in D. But, $\mathcal{B}$ has a fuzzy pre cluster point x_α. Then $x_\alpha \in pcl(D)$ implies $x_\alpha \in V_i$ for some i and so V_i is a fuzzy pre open set containing x_α. Then $(D \wedge (1 - V_i)) \wedge V_i = 0$ contradicts the fact that x_α is a fuzzy pre cluster point of $\mathcal{B}$. Hence $pcl(D) = X$ is a fuzzy precompact.

3.21. Corollary
A fuzzy pre regular space X is a fuzzy compact if and only if there exist a fuzzy dense D of X such that any fuzzy filterbase in D have a fuzzy pre cluster point in X, with $int(D) \neq 0$.
Proof: ⟹) By (*2.45. Theorem*) and (*2.39. Theorem*).
(⟸ By (*3.20.Theorem*) and (*2.7. Corollary*).

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Generalized Rough Digraphs and Related Topologies

Samah Sarmad[1] and Yousif Yaqoub Yousif[2]

[1] Department of Mathematics, College of Education for Pure Sciences

(Ibn- Al-Haitham), University of Baghdad

[2] Department of Mathematics, College of Education for Pure Sciences

(Ibn- Al-Haitham), University of Baghdad

samahsarmad0@gmail.com yoyayuosif@yahoo.com

Abstract. The primary objective of this paper, is to introduce eight types of topologies on a finite digraphs and state the implication between these topologies. Also we used supra open digraphs to introduce a new types for approximation rough digraphs.

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Key words. *J*-degree spaces, *J*-supra lower digraphs, *J*-supra upper digraphs, *J*-exact digraphs, *J*-rough digraphs.

1. Introduction.

Rough set theory was introduced by Zdzislow Pawlak in 1982 [1]. He presented the conception of rough set inherently as a mathematical method to manipulate inexactness, uncertainty and vagueness in datagram analyses. This theory is an stretch of set theory for the studying of clearheaded systems diacritical by inadequate and incompletely information. The theory has found implementation in many domains, such as medicine, pharmacy, engineering and others. Furthermore, the prospered implementing of rough set theory in a diversity of problems has abundantly shown its benefit. A specific using of the theory is that property depreciating in databases. Giving a dataset with discretionary property weightings, it is tolerable to existing a subset of the original property that are the bulk informative. Rough set theory treated with the approximating of an arbitrarily subset of universe by depending on two observable or defined subsets, these subsets are named lower approximation and upper approximations, by utilization the terminology of these subsets in rough set theory knowing furtive in info regimes may be unraveled and manifested in the format of resolution norms [2]. We built on some of the results in [3], [4], [5], [6], [7], [8] and [9].

2. Preliminaries

In this part, we present some of essential notions in rough theory and peculiarities of lower approximation and upper approximation which are useful for our study.

Definition 2.1. [10] Let X be non-empty set and τ be a collection of subsets of $P(X)$, the collection τ is said to be a topology on X if τ satisfies:
(a) $X \in \tau$, $\emptyset \in \tau$.
(b) τ is closed within finite intersection.
(c) τ is closed within arbitrarily union.

If τ is a topology on X, then the pair (X, τ) is called a topological space. in this space, the subsets of X which belong to τ are dubbed open sets, while the closed sets is represented by the supplement of the subsets of X which belong to τ (that is the complement of open sets).

The approximation of lower and upper of a set is the basic conception of rough set theory, the approximation of space is the formalized categorization of acquaintance regarding the interesting domain. The partitioning represents a topological space, that topological space named approximation space and symbolized by $K = (X, R)$, so that X is a set named space or universe while $R \subseteq X \times X$ is represented by an indescribable equivalence relation [2]. In the relation R, the equivalence classes are savvied blocks, grained or primary sets too. The equivalence class which includes $x \in X$ denoted by R_x.

Definition 2.2. [11] Let $K = (X, R)$ be an approximation space and S is a subset of X, then the lower and the upper approximation of S denoted consecutively by $L(S)$, $U(S)$ and defined by

$$L(S) = \{x \in X; R_x \subseteq S\}, U(S) = \{x \in X; R_x \cap S \neq \emptyset\}.$$

According to the lower and upper approximations of a subset S of X. X can be dichotomizes in to three discrete areas, positive area (briefly $POS_R(S)$), negative area (briefly $NEG_R(S)$) and boundary area (briefly $B_R(S)$), where they are defined by

$$POS_R(S) = L(S), NEG_R(S) = X - U(S), B_R(S) = U(S) - L(S)$$

If $K = (X, R)$ be an approximation space, where S and F be two subsets of the universe X, the following properties of the Pawlak's rough sets [1, 12].

$(L1)$ $L(S) = [U(S^c)]^c$	$(U1)$ $U(S) = [L(S^c)]^c$
$(L2)$ $L(X) = X$	$(U2)$ $U(X) = X$
$(L3)$ $L(S \cap F) = L(S) \cap L(F)$	$(U3)$ $U(S \cup F) = U(S) \cup U(F)$
$(L4)$ $L(S \cup F) \supseteq L(S) \cup L(F)$	$(U4)$ $U(S \cap F) \subseteq U(S) \cap U(F)$
$(L5)$ If $S \subseteq F$ then, $L(S) \subseteq L(F)$	$(U5)$ If $S \subseteq F$ then, $U(S) \subseteq U(F)$
$(L6)$ $L(\emptyset) = \emptyset$	$(U6)$ $U(\emptyset) = \emptyset$
$(L7)$ $L(S) \subseteq S$	$(U7)$ $S \subseteq U(S)$
(L8) $L(L(S)) = L(S)$	$(U8)$ $U(U(S)) = U(S)$
$(L9)$ $L(U(S)) = L(S)$	$(U9)$ $U(L(S)) = U(S)$

Definition 2.3. [1] Let $K = (X, R)$ be an approximation space and $S \subseteq X$ then the accuracy measure of E is symbolized by the symbol $\Lambda_R(S)$ and is predefined by

$$\Lambda_R(S) = 1 - \frac{|L(S)|}{|U(S)|}, \text{ wherein } |U(S)| \neq 0.$$

Also, the accuracy measure dubbed accuracy of approximation.

Definition 2.4. [13] A directed graph (briefly d.g.) express a pair $D = (V(D), E(D))$ such that $V(D)$ named vertex set which is non-empty set and $E(D)$ named edge set represented by ordered pairs of elements of $V(D)$.

Definition 2.5. [14] A subdigraph $Q = (V(Q), E(Q))$ of a directed graph $D = (V(D), E(D))$ written $Q \subseteq D$ if $V(Q) \subseteq V(D)$ and $E(Q) \subseteq E(D)$.

3. Generalized Rough Digraphs and Related Topologies

In this section, we present some of definitions and propositions anent a new types of topologies and the implication among them. Also we give many results, examples were provided.

Definition 3.1. Let $D = (V(D), E(D))$ is a finite digraph. The *J*-degree of r, where $r \in V(D)$, for all $J \in$ {*O*, *I*, ∩, ∪, <*O*>, <*I*>, <∩>, <∪>} defined by

(a) $O\text{-}D(r) = \{u \in V(D);\ (r, u) \in E(D)\}$,
(b) $I\text{-}D(r) = \{u \in V(D);\ (u, r) \in E(D)\}$,
(c) $\cap\text{-}D(r) = O\text{-}D(r) \cap I\text{-}D(r)$,
(d) $\cup\text{-}D(r) = O\text{-}D(r) \cup I\text{-}D(r)$,
(e) $<O>\text{-}D(r) = \cap_{r \in O-D(r)}\ O\text{-}D(r)$,
(f) $<I>\text{-}D(r) = \cap_{r \in I-D(r)}\ I\text{-}D(r)$,
(g) $<\cap>\text{-}D(r) = <O>\text{-}D(r) \cap <I>\text{-}D(r)$,
(h) $<\cup>\text{-}D(r) = <O>\text{-}D(r) \cup <I>\text{-}D(r)$.

Definition 3.2. Let $D = (V(D), E(D))$ is a finite digraph and θ_J: $V(D) \longrightarrow P(V(D))$ be a mapping which assigns for all $r \in V(D)$ its *J*-degree in $P(V(D))$. The pair (D, θ_J) is namable *J*-degree space (concisely *J-DS*).

Theorem 3.3. If (D, θ_J) is *J-DS*, then the a family

$$\tau_J = \{V(Q) \subseteq V(D);\ \text{for each } r \in V(Q),\ J\text{-}D(r) \subseteq V(Q)\},$$

for all $J \in$ {*O*, *I*, ∩, ∪, <*O*>, <*I*>, <∩>, <∪>} is a topology on *D*.
Proof. For all $J \in$ {*O*, *I*, ∩, ∪, <*O*>, <*I*>, <∩>, <∪>}. Clearly, $V(D), \emptyset \in \tau_J$.

Let $M, Q \in \tau_J$ and $r \in V(M) \cap V(Q)$, then $r \in V(M)$ and $r \in V(Q)$, which implies that $J\text{-}D(r) \subseteq V(M)$ and $J\text{-}D(r) \subseteq V(Q)$, therefore $J\text{-}D(r) \subseteq V(M) \cap V(Q)$ and then $M \cap Q \in \tau_J$.

Let $Q_i \in \tau_J$ for each $i \in I$, and $r \in \cup_i V(Q_i)$, which mean that there exists $i_o \in I$ where $r \in V(Q_{i_o}) \subseteq \cup_i V(Q_i)$, therefore $J\text{-}D(r) \subseteq V(Q_{i_o}) \subseteq \cup_i V(Q_i)$ this implies $J\text{-}D(r) \subseteq \cup_i V(Q_i)$ and so $\cup_i V(Q_i) \in \tau_J$.

Example 3.4. If $D = (V(D), E(D))$ is a finite digraph such that $V(D) = \{r_1, r_2, r_3, r_4\}$, $E(D) = \{(r_1, r_1), (r_1, r_4), (r_2, r_1), (r_2, r_3), (r_3, r_3), (r_3, r_4), (r_4, r_1)\}$.

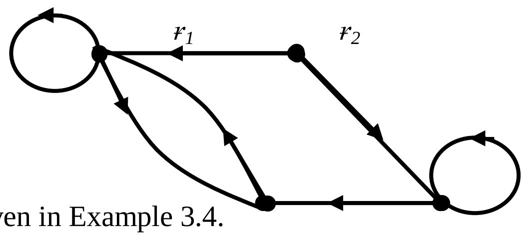

Figure 1: digraph given in Example 3.4.

Then, $O\text{-}D(r_1) = \{r_1, r_4\}$, $O\text{-}D(r_2) = \{r_1, r_3\}$, $O\text{-}D(r_3) = \{r_3, r_4\}$, $O\text{-}D(r_4) = \{r_1\}$.

$I\text{-}D(r_1) = \{r_1, r_2, r_4\}$, $I\text{-}D(r_2) = \emptyset$, $I\text{-}D(r_3) = \{r_2, r_3\}$, $I\text{-}D(r_4) = \{r_1, r_3\}$.

$\cap\text{-}D(r_1) = \{r_1, r_4\}$, $\cap\text{-}D\text{-}(r_2) = \emptyset$, $\cap\text{-}D(r_3) = \{r_3\}$, $\cap\text{-}D(r_4) = \{r_1\}$.

$\cup\text{-}D(r_1) = \{r_1, r_2, r_4\}$, $\cup\text{-}D(r_2) = \{r_1, r_3\}$, $\cup\text{-}D(r_3) = \{r_2, r_3, r_4\}$, $\cup\text{-}D(r_4) = \{r_1, r_3\}$.

$<O>\text{-}D(r_1) = \{r_1\}$, $<O>\text{-}D(r_2) = \emptyset$, $<O>\text{-}D(r_3) = \{r_3\}$, $<O>\text{-}D(r_4) = \{r_4\}$.

$<I>\text{-}D(r_1) = \{r_1\}$, $<I>\text{-}D(r_2) = \{r_2\}$, $<I>\text{-}D(r_3) = \{r_3\}$, $<I>\text{-}D(r_4) = \{r_1, r_2, r_4\}$.

$<\cap>\text{-}D(r_1) = \{r_1\}$, $<\cap>\text{-}D(r_2) = \emptyset$, $<\cap>\text{-}D(r_3) = \{r_3\}$, $<\cap>\text{-}D(r_4) = \{r_4\}$.

$<\cup>\text{-}D(r_1) = \{r_1\}$, $<\cup>\text{-}D(r_2) = \{r_2\}$, $<\cup>\text{-}D(r_3) = \{r_3\}$, $<\cup>\text{-}D(r_4) = \{r_1, r_2, r_4\}$.

$\tau_O = \{V(D), \emptyset, \{r_1, r_4\}, \{r_1, r_3, r_4\}\}$, $\tau_I = \{V(D), \emptyset, \{r_2\}, \{r_2, r_3\}\}$, $\tau_\cap = \{V(D), \emptyset, \{r_2\}, \{r_3\}, \{r_1, r_4\}, \{r_2, r_3\}, \{r_1, r_2, r_4\}, \{r_1, r_3, r_4\}\}$, $\tau_\cup = \{V(D), \emptyset\}$, $\tau_{<O>} = P(V(D))$, $\tau_{<I>} = \{V(D), \emptyset, \{r_1\}, \{r_2\}, \{r_3\}, \{r_1, r_2\}, \{r_1, r_3\}, \{r_2, r_3\}, \{r_1, r_2, r_3\}, \{r_1, r_2, r_4\}\}$, $\tau_{<\cap>} = P(V(D))$, $\tau_{<\cup>} = \{V(D), \emptyset, \{r_1\}, \{r_2\}, \{r_3\}, \{r_1, r_2\}, \{r_1, r_3\}, \{r_2, r_3\}, \{r_1, r_2, r_3\}, \{r_1, r_2, r_4\}\}$.

Remark 3.5. From the above results, the implication among different topologies τ_J are explained in the following diagram (where $\longrightarrow$ implies $\subseteq$)

Diagram 1

By using the above topologies, we present eight methods for approximation rough diagraphs using interior and closure of the topologies τ_J for all $J \in \{O, I, \cap, \cup, <O>, <I>, <\cap>, <\cup>\}$.

Definition 3.6. Let (D, θ_J) be *J-DS*. The subgraph $Q \subseteq D$ is called *J*-open graph if $V(Q) \in \tau_J$. While the complement of *J*-open graph is called *J*-closed graph. The family of every *J*-closed graphs of a *J-DS* is predefined by:

$$\Gamma_J = \{V(K) \subseteq V(D); [V(K)]^c \in \tau_J\}.$$

Definition 3.7. Let (D, θ_J) be *J-DS* and $Q \subseteq D$. The *J*-lower approximation of Q and *J*-upper approximation of Q are predefined consecutively by

$$L_J(Q) = \cup \{V(M) \in \tau_J : V(M) \subseteq V(Q)\} = J\text{-interior of } Q.$$

$$U_J(Q) = \cap \{V(M) \in \Gamma_J : V(Q) \subseteq V(M)\} = J\text{-closure of } Q.$$

Definition 3.8. Let (D, θ_J) be *J-DS* and $Q \subseteq D$. The, *J*-positive, *J*-negative and *J*-boundary areas of Q are defined as

$$POS_J(V(Q)) = L_J(V(Q)),\ NEG_J(V(Q)) = V(D) - U_J(V(Q)),$$

$$B_J(V(Q)) = U_J(V(Q)) - L_J(V(Q))$$

Definition 3.9. Let (D, θ_J) be *J-DS*. The subgraph Q is dubbed *J*-exact (definable) graph if

$$L_J(V(Q)) = U_J(V(Q)) = V(Q).$$

Otherwise is called *J*-rough graph.

Definition 3.10. Let (D, θ_J) is *J-DS*. The *J*-accuracy of the approximation of $Q \subseteq D$ is predefined by

$$\Lambda_J(V(Q)) = \frac{|LJ(V(Q))|}{|UJ(V(Q))|}, \text{ where } |UJ(V(Q)| \neq 0.$$

Remark 3.11. Clear that $0 \le \Lambda_J(V(Q)) \le 1$ and Q is J-exact graph if $B_J(V(Q)) = \emptyset$ and $\Lambda_J(V(Q)) = 0$. Otherwise Q is J-rough.

Remark 3.12. From above results, we have a concluding that using of $\tau\cap$ in construction the approximations of graphs is minutest than τ_O, τ_I and $\tau_\cup$. Also the using of $\tau_{<\cap>}$ in construction the approximations of graphs is minutest than $\tau_{<O>}$, $\tau_{<I>}$ and $\tau_{<\cup>}$. Moreover, the topologies $\tau_\cap$ and $\tau_{<\cap>}$ are not necessarily comparable.

Now, some properties of the operators *J*-lower approximation and *J*-upper approximation, will be presented in the next proposition.

Proposition 3.13. If (D, θ_J) is *J-DS* and $M, Q \subseteq D$. Then

$(L1)$ $L_J(V(Q) = [U_J(V(Q^c))]^c$	$(U1)$ $U_J(V(Q)) = [L_J(V(Q^c))]^c$
$(L2)$ $L_J(V(D)) = V(D)$, $L_J(\emptyset) = \emptyset$	$(U2)$ $U_J(V(D)) = V(D)$, $U_J(\emptyset) = \emptyset$
$(L3)$ If $V(M) \subseteq V(Q)$ then, $L_J(V(M)) \subseteq L_J(V(Q))$	$(U3)$ If $V(M) \subseteq V(Q)$ then, $U_J(V(M)) \subseteq U_J(V(Q))$
$(L4)$ $L_J(V(M) \cap V(Q)) = L_J(V(M) \cap L_J(V(Q))$	$(U4)$ $U_J(V(M) \cap V(Q)) \subseteq U_J(V(M)) \cap U_J(V(Q))$
$(L5)$ $L_J(V(M) \cup V(Q) \supseteq L_J(V(M)) \cup L_J(V(Q))$	$(U5)$ $U_J(V(M) \cup V(Q)) = U_J(V(M)) \cup U_J(V(Q))$
$(L6)$ $L_J(V(Q)) \subseteq V(Q)$	$(U6)$ $V(Q) \subseteq U_J(V(Q))$
$(L7)$ $L_J(L_J(V(Q))) = L_J(V(Q))$	$(U7)$ $U_J(U_J(V(Q))) = U_J(V(Q))$

Proof. The proof is evident, by employing peculiarities of closure and interior.

The next example explains the comparison between our approach and approach in Yousif and Sara approach [15, 16].

Example 3.14. Let (D, θ_J) be *J-DS* where $D = (V(D), E(D))$, $V(D) = \{r_1, r_2, r_3, r_4\}$ and $E(D) = \{(r_1, r_3), (r_2, r_2), (r_3, r_1), (r_4, r_1)\}$

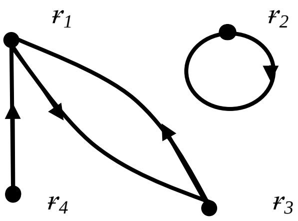

Figure 2: Digraph given in Example 3.14

$\cap$-$D(r_1)$ = $\{r_3\}$, $\cap$-$D(r_2)$ = $\{r_2\}$, $\cap$-$D(r_3)$ = $\{r_1\}$, $\cap$-$D(r_4)$ = $\emptyset$.

$\cup$-$D(r_1)$ = $\{r_3, r_4\}$, $\cup$-$D(r_2)$ = $\{r_2\}$, $\cup$-$D(r_3)$ = $\{r_1\}$, $\cup$-$D(r_4)$ = $\{r_1\}$.

$<O>$-$D(r_1)$ = $\{r_1\}$, $<O>$-$D(r_2)$ = $\{r_2\}$, $<O>$-$D(r_3)$ = $\{r_3\}$, $<O>$-$D(r_4)$ = $\emptyset$.

$<I>$-$D(r_1)$ = $\{r_1\}$, $<I>$-$D(r_2)$ = $\{r_2\}$, $<I>$-$D(r_3)$ = $\{r_3, r_4\}$, $<I>$-$D(r_4)$ = $\{r_3, r_4\}$

$<\cap>$-$D(r_1)$ = $\{r_1\}$, $<\cap>$-$D(r_2)$ = $\{r_2\}$, $<\cap>$-$D(r_3)$ = $\{r_3\}$, $<\cap>$-$D(r_4)$ = $\emptyset$.

$<\cup>$-$D(r_1)$ = $\{r_1\}$, $<\cup>$-$D(r_2)$ = $\{r_2\}$, $<\cup>$-$D(r_3)$ = $\{r_3, r_4\}$, $<\cup>$-$D(r_4)$ = $\{r_3, r_4\}$.

$\tau_{<O>} = P(V(D)), \Gamma_{<O>} = P(V(D))$.

$\tau_{<I>} = \{V(D), \emptyset, \{r_1\}, \{r_2\}, \{r_1, r_2\}, \{r_3, r_4\}, \{r_1, r_3, r_4\}, \{r_2, r_3, r_4\}\}$, $\Gamma_{<I>} = \{V(D), \emptyset, \{r_1\}, \{r_2\}, \{r_1, r_2\}, \{r_3, r_4\}, \{r_1, r_3, r_4\}, \{r_2, r_3, r_4\}\}$.

From Yousif and Sara approach [15, 16], we have

$\mathcal{F}_{\xi m} = \{V(D), \emptyset, \{r_2\}, \{r_3\}, \{r_4\}, \{r_1, r_3\}, \{r_2, r_4\}, \{r_1, r_2, r_3\}\}$, $\Omega_{\xi m} = \{V(D), \emptyset, \{r_4\}, \{r_1, r_3\}, \{r_2, r_4\}, \{r_1, r_2, r_3\}, \{r_1, r_2, r_4\}, \{r_1, r_3, r_4\}\}$.

Table 1: $L_M(V(Q))$, $U_M(V(Q))$, $L_{<O>}(V(Q))$, $U_{<O>}(V(Q))$, $L_{<I>}(V(Q))$ and $U_{<I>}(V(Q))$

for all $Q \subseteq$, Exact graph and Rough graph.

$P(V(D))$	$L_M(V(Q))$	$U_M(V(Q))$	$L_{<O>}(V(Q))$	$U_{<O>}(V(Q))$	$L_{<I>}(V(Q))$	$U_{<I>}(V(Q))$
$\{r_1\}$	$\emptyset$	$\{r_1, r_3\}$	$\{r_1\}$	$\{r_1\}$	$\{r_1\}$	$\{r_1\}$
$\{r_2\}$	$\emptyset$	$\{r_2\}$	$\{r_2\}$	$\{r_2\}$	$\{r_2\}$	$\{r_2\}$
$\{r_3\}$	$\emptyset$	$\{r_3\}$	$\{r_3\}$	$\{r_3\}$	$\emptyset$	$\{r_3, r_4\}$
$\{r_4\}$	$\{r_4\}$	$\{r_4\}$	$\{r_4\}$	$\{r_4\}$	$\emptyset$	$\{r_3, r_4\}$
$\{r_1, r_2\}$	$\emptyset$	$\{r_1, r_2, r_3\}$	$\{r_1, r_2\}$	$\{r_1, r_2\}$	$\{r_1, r_2\}$	$\{r_1, r_2\}$
$\{r_1, r_3\}$	$\{r_1, r_3\}$	$\{r_1, r_3\}$	$\{r_1, r_3\}$	$\{r_1, r_3\}$	$\{r_1\}$	$\{r_1, r_3, r_4\}$
$\{r_1, r_4\}$	$\{r_4\}$	$V(D)$	$\{r_1, r_4\}$	$\{r_1, r_4\}$	$\{r_1\}$	$\{r_1, r_3, r_4\}$
$\{r_2, r_3\}$	$\emptyset$	$\{r_1, r_2, r_3\}$	$\{r_2, r_3\}$	$\{r_2, r_3\}$	$\{r_2\}$	$\{r_2, r_3, r_4\}$

$\{r_2, r_4\}$	$\{r_2, r_4\}$	$\{r_2, r_4\}$	$\{r_2, r_4\}$	$\{r_2, r_4\}$	$\{r_2\}$	$\{r_2, r_3, r_4\}$
$\{r_3, r_4\}$	$\{r_4\}$	$V(D)$	$\{r_3, r_4\}$	$\{r_3, r_4\}$	$\{r_3, r_4\}$	$\{r_3, r_4\}$
$\{r_1, r_2, r_3\}$	$\{r_1, r_2, r_3\}$	$\{r_1, r_2, r_3\}$	$\{r_1, r_2, r_3\}$	$\{r_1, r_2, r_3\}$	$\{r_1, r_2\}$	$V(D)$
$\{r_1, r_2, r_4\}$	$\{r_1, r_2, r_4\}$	$V(D)$	$\{r_1, r_2, r_4\}$	$\{r_1, r_2, r_4\}$	$\{r_1, r_2\}$	$V(D)$
$\{r_1, r_3, r_4\}$	$\{r_1, r_3, r_4\}$	$V(D)$	$\{r_1, r_3, r_4\}$	$\{r_1, r_3, r_4\}$	$\{r_1, r_3, r_4\}$	$\{r_1, r_3, r_4\}$
$\{r_2, r_3, r_4\}$	$\{r_2, r_4\}$	$V(D)$	$\{r_2, r_3, r_4\}$	$\{r_2, r_3, r_4\}$	$\{r_2, r_3, r_4\}$	$\{r_2, r_3, r_4\}$
$V(D)$	$V(D)$	$V(D)$	$V(D)$	$V(D)$	$V(D)$	$V(D)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Remark 3.14. The above proposition and example can be considered as one of the difference between our approaches and Yousif and Sara approach [15]. So, we can say that our approach is the actual circularization of Yousif and Sara approach because the numbers of exact graph in our approach more than Yousif and Sara approach.

Definition 3.15. Let (D, θ_J) be *J-DS*. Then for each $J \in \{O, I, \cap, \cup, <O>, <I>, <\cap>, <\cup>\}$, the subgraph $Q \subseteq D$ is named:

(a) J-regular open (shortly R_J-open) if $V(Q) = Int_J(Cl_J(V(Q)))$
(b) J-pre-open(shortly P_J-open) if $V(Q) \subseteq Int_J(Cl_J(V(Q)))$
(c) J-semi-open(shortly S_J-open) if $V(Q) \subseteq Cl_J(Int_J(V(Q)))$
(d) α_J-open if $V(Q) \subseteq Int_J(Cl_J(Int_J(V(Q))))$
(e) b_J-open if $V(Q) \subseteq Int_J(Cl_J(V(Q))) \cup Cl_J(Int_J(V(Q)))$
(f) β_J-open if $V(Q) \subseteq Cl_J(Int_J(Cl_J(V(Q))))$

Remark 3.16.

(a) The above graphs are dubbed J-supra open graphs and the collection of J-supra open graphs of D symbolized by the symbol $K_JO(D)$ for every $K = R, P, S, b, \alpha, \beta$.
(b) The J-supra closed graphs is the complement of the J-supra open graphs where the families of J-supra closed graphs of D symbolized by the symbol $K_JC(D)$ for every $K = R, P, S, b, \alpha, \beta$.
(c) The family $\alpha_JO(D)$ idealizes a topology on D, furthermore, the J-supra interior and the J-supra closure idealizes the J-interior and the J-closure respectively.

Remark 3.17. The implication between the topologies τ_J(consecutively Γ_J) and the precedent collection of J-supra open graphs (consecutively J-supra closed graphs) are explained the next diagram (where $\longrightarrow$ implies $\subseteq$)

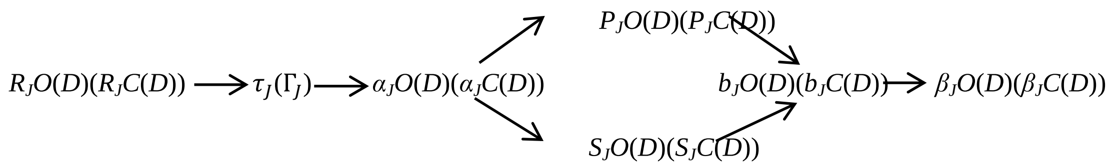

Diagram 2

By usage the J-supra open graph, we can present new causeways for approximation rough graphs using the J-supra interior and the J-supra closure for all topology of τ_J as the next definitions

Definition 3.18. Let (D, θ_J) is J-DS and $Q \subseteq D$. Then for all $J \in \{O, I, \cap, \cup, <O>, <I>, <\cap>, <\cup>\}$ and $K \in \{R, P, b, S, \alpha, \beta\}$, the J-supra lower approximation of Q and J-supra upper approximation of Q are predefined consecutively by

$$L_J^K(V(Q)) = \cup \{V(M) \in K_J O(D);\ V(M) \subseteq V(Q)\} = J\text{-supra interior of } Q,$$

$$U_J^K(V(Q)) = \cap \{V(M) \in K_J C(D);\ V(Q) \subseteq V(M)\} = J\text{-supra closure of } Q.$$

Definition 3.19. Let (D, θ_J) be J-DS and $Q \subseteq D$. Then for all $J \in \{O, I, \cap, \cup, <O>, <I>, <\cap>, <\cup>\}$ and $K \in \{R, P, b, S, \alpha, \beta\}$, the J-supra positive, J-supra negative and J-supra boundary areas of Q are predefined consecutively by

$$POS_J^K(V(Q)) = L_J^K(V(Q)),\ NEG_J^K(V(Q)) = V(D) - U_J^K(V(Q)),$$

$$B_J^K(V(Q)) = U_J^K(V(Q)) - L_J^K(V(Q))$$

Definition 3.20. Let (D, θ_J) be J-DS and $Q \subseteq D$. Then for all $J \in \{O, I, \cap, \cup, <O>, <I>, <\cap>, <\cup>\}$ and $K \in \{R, P, b, S, \alpha, \beta\}$, the J-supra accuracy of the J-supra approximations of $Q \subseteq D$ is predefined by

$$\Lambda_J^K(V(Q)) = \frac{|L_J^K(V(Q))|}{|U_J^K(V(Q))|}, \text{ where } |U_J^K(V(Q))| \neq 0.$$

It is clear that $0 \leq \Lambda_J^K(V(Q)) \leq 1$.

The essential properties of the J-supra approximations are mentioned in the next proposition.

Proposition 3.21. Let (D, θ_J) be J-DS and $Q, M \subseteq D$. Then, for every $J \in \{O, I, \cap, \cup, <O>, <I>, <\cap>, <\cup>\}$ and $K = R, P, b, S, \alpha, \beta$.

($L1$) $L_J^K(V(Q)) = [U_J^K(V(Q^c))]^c$,

($L2$) $L_J^K(V(D)) = V(D)$, $L_J^K(\emptyset) = \emptyset$,

($L3$) If $V(Q) \subseteq V(M)$ then,

$L_J^K(V(Q)) \subseteq L_J^K(V(M))$,

($L4$) $L_J^K(V(Q)) \cap V(M)) =$

$L_J^K(V(Q)) \cap L_J^K(V(M))$,

($L5$) $L_J^K(V(Q) \cup V(M)) \supseteq$

($U1$) $U_J^K(V(Q)) = [L_J^K(V(Q^c))]^c$,

($U2$) $U_J^K(V(D)) = V(D)$, $U_J^K(V(\emptyset) = \emptyset$,

($U3$) If $V(Q) \subseteq V(M)$ then,

$U_J^K(V(Q)) \subseteq U_J^K(V(M))$,

($U4$) $U_J^K(V(Q) \cap V(M)) \subseteq$

$U_J^K((V(Q) \cap U_J^K(V(M))$,

($U5$) $U_J^K(V(Q) \cup V(M)) =$

$L_J^K(V(Q)) \cup L_J^K(V(M))$ $U_J^K(V(Q)) \cup U_J^K(V(M))$

($L6$) $L_J^K(V(Q)) \subseteq V(Q)$, ($U6$) $V(Q) \subseteq U_J^K(V(Q))$,

($L7$) $L_J^K(L_J^K(V(Q))) = L_J^K(V(Q))$. ($U7$) $U_J^K(U_J^K(V(Q))) = U_J^K(V(Q))$.

Remark 3.21. The collections of all regular open graphs of D, $R_JO(D)$, are smaller than the topologies τ_J, (that is $R_JO(D)$ idealized a special case of the topologies τ_J) hence we will not using it in our approaches.

The J-supra approximations are extremely interesting in rough context because the it can assists in the detecting of unobserved information in datagram collected from real life implementations. Furthermore, the utilization of the J-supra formats can assists for more developments in the notional and implementations of rough graphs, because the boundary area will decreased or abolished by increasing the lower approximation and decreasing the upper approximation, as the following results explained.

Proposition 3.22. Let (D, θ_J) be J-DS and $Q \subseteq D$. Then, for every $J \in \{O, I, \cap, \cup, <O>, <I>, <\cap>, <\cup>\}$ and $K \in \{R, P, b, S, \alpha, \beta\}$ such that $K \neq R$,

$$L_J(V(Q)) \subseteq L_J^K(V(Q)) \subseteq V(Q) \subseteq U_J^K(V(Q)) \subseteq U_J(V(Q))$$

Proof. For each $J \in \{O, I, \cap, \cup, <O>, <I>, <\cap>, <\cup>\}$ and $K \in \{R, P, b, S, \alpha, \beta\}$ such that $K \neq R$, $L_J(V(Q)) = \cup\, \{V(M) \in \tau_J; V(M) \subseteq V(Q)\}$

$\subseteq \cup\, \{V(M) \in K_JO(D); V(M) \subseteq V(Q)\}$ since $\tau_J \subseteq K_JO(D)$

$= L_J^K(V(Q))$ (1)

By Proposition (2) $L_J^K(V(Q)) \subseteq V(Q) \subseteq U_J^K(V(Q))$ (2)

$U_J^K(V(Q)) = \cap\, \{V(F) \in K_JC(D); V(Q) \subseteq V(F)\}$

$\subseteq \cap\, \{V(F) \in \Gamma_J; V(Q) \subseteq V(F)\}$ since $K_JC(D) \subseteq \Gamma_J$

$= U_J(V(Q))$ (3)

From (1), (2) and (3) we get $L_J(V(Q)) \subseteq L_J^K(V(Q)) \subseteq V(Q) \subseteq U_J^K(V(Q)) \subseteq U_J(V(Q))$

Corollary 3.23. Let (D, θ_J) be J-DS and $Q \subseteq D$. Then, for each $J \in \{O, I, \cap, \cup, <O>, <I>, <\cap>, <\cup>\}$ and $K \in \{P, b, S, \alpha, \beta\}$ such that $K \neq R$

$$\text{(a) } B_J(V(Q)) \subseteq B_J^K(V(Q)), \text{ (b) } \Lambda_J(V(Q) \leq \Lambda_J^K(V(Q))$$

We will presenting the next example to explain the prominence of using J-supra conception in rough context and to expressing the precedent results.

Example 3.24. Let (D, θ_J) be J-DS where $D = (V(D), E(D))$, $V(D) = \{r_1, r_2, r_3, r_4\}$ and $E(D) = \{(r_1, r_1), (r_1, r_2), (r_2, r_1), (r_2, r_2), (r_3, r_1), (r_3, r_2), (r_3, r_3), (r_3, r_4), (r_4, r_4)\}$.

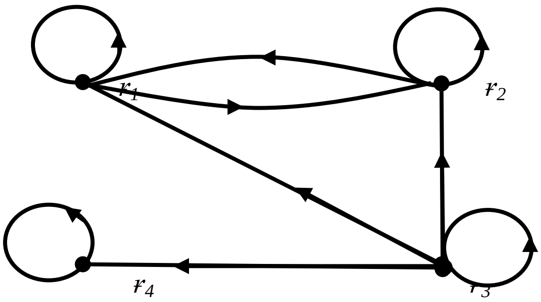

Figure 3: digraph given in Example 3.24.

$O\text{-}D(r_1) = \{r_1, r_2\}$, $O\text{-}D(r_2) = \{r_1, r_2\}$, $O\text{-}D(r_3) = V(D)$, $O\text{-}D(r_4) = \{r_4\}$.

$\tau_O = \{V(D), \emptyset, \{r_4\}, \{r_1, r_2\}, \{r_1, r_2, r_4\}\}$, and $\Gamma_O = \{V(D), \emptyset, \{r_3\}, \{r_3, r_4\}, \{r_1, r_2, r_3\}\}$.

We shall calculate the J-supra approximations for $J = O$ and $K = P, b, \beta$.

$P_OO(D) = \{V(D), \emptyset, \{r_1\}, \{r_2\}, \{r_4\}, \{r_1, r_2\}, \{r_1, r_4\}, \{r_2, r_4\}, \{r_1, r_2, r_4\}, \{r_1, r_3, r_4\}, \{r_2, r_3, r_4\}\}$.

$P_OC(D) = \{V(D), \emptyset, \{r_1\}, \{r_2\}, \{r_3\}, \{r_1, r_3\}, \{r_2, r_3\}, \{r_3, r_4\}, \{r_1, r_2, r_3\}, \{r_1, r_3, r_4\}, \{r_2, r_3, r_4\}\}$.

$boO(D) = \{V(D), \emptyset, \{r_1\}, \{r_2\}, \{r_4\}, \{r_1, r_2\}, \{r_1, r_4\}, \{r_2, r_4\}, \{r_3, r_4\}, \{r_1, r_2, r_3\}, \{r_1, r_2, r_4\}, \{r_1, r_3, r_4\}, \{r_2, r_3, r_4\}\}$.

$boC(D) = \{V(D), \emptyset, \{r_1\}, \{r_2\}, \{r_3\}, \{r_4\}, \{r_1, r_2\}, \{r_1, r_3\}, \{r_2, r_3\}, \{r_3, r_4\}, \{r_1, r_2, r_3\}, \{r_1, r_3, r_4\}, \{r_2, r_3, r_4\}\}$.

$\beta_OO(D) = \{V(D), \emptyset, \{r_1\}, \{r_2\}, \{r_4\}, \{r_1, r_2\}, \{r_1, r_3\}, \{r_1, r_4\}, \{r_2, r_3\}, \{r_2, r_4\}, \{r_3, r_4\}, \{r_1, r_2, r_3\}, \{r_1, r_2, r_4\}, \{r_1, r_3, r_4\}, \{r_2, r_3, r_4\}\}$.

$\beta_OC(D) = \{V(D), \emptyset, \{r_1\}, \{r_2\}, \{r_3\}, \{r_4\}, \{r_1, r_2\}, \{r_1, r_3\}, \{r_1, r_4\}, \{r_2, r_3\}, \{r_2, r_4\}, \{r_3, r_4\}, \{r_1, r_2, r_3\}, \{r_1, r_3, r_4\}, \{r_2, r_3, r_4\}\}$.

Table 2. [shaded] Exact graph and [unshaded] Rough graph.

$P(V(D))$	τ_O		P_O		b_O		β_O	
	$L_O(V(Q))$	$U_O(V(Q))$	$L^P_O(V(Q))$	$U^P_O(V(Q))$	$L^b_O(V(Q))$	$U^b_O(V(Q))$	$L^\beta_O(V(Q))$	$U^\beta_O(V(Q))$
$\{r_1\}$	$\emptyset$	$\{r_1, r_2, r_3\}$	$\{r_1\}$	$\{r_1\}$	$\{r_1\}$	$\{r_1\}$	$\{r_1\}$	$\{r_1\}$
$\{r_2\}$	$\emptyset$	$\{r_1, r_2, r_3\}$	$\{r_2\}$	$\{r_2\}$	$\{r_2\}$	$\{r_2\}$	$\{r_2\}$	$\{r_2\}$
$\{r_3\}$	$\emptyset$	$\{r_3\}$	$\emptyset$	$\{r_3\}$	$\emptyset$	$\{r_3\}$	$\emptyset$	$\{r_3\}$
$\{r_4\}$	$\{r_4\}$	$\{r_3, r_4\}$	$\{r_4\}$	$\{r_3, r_4\}$	$\{r_4\}$	$\{r_4\}$	$\{r_4\}$	$\{r_4\}$

$\{r_1, r_2\}$	$\{r_1, r_2\}$	$\{r_1, r_2, r_3\}$	$\{r_1, r_2\}$	$\{r_1, r_2, r_3\}$	$\{r_1, r_2\}$	$\{r_1, r_2\}$	$\{r_1, r_2\}$	$\{r_1, r_2\}$
$\{r_1, r_3\}$	$\emptyset$	$\{r_1, r_2, r_3\}$	$\{r_1\}$	$\{r_1, r_3\}$	$\{r_1\}$	$\{r_1, r_3\}$	$\{r_1, r_3\}$	$\{r_1, r_3\}$
$\{r_1, r_4\}$	$\{r_4\}$	$V(D)$	$\{r_1, r_4\}$	$\{r_1, r_3, r_4\}$	$\{r_1, r_4\}$	$\{r_1, r_3, r_4\}$	$\{r_1, r_4\}$	$\{r_1, r_4\}$
$\{r_2, r_3\}$	$\emptyset$	$\{r_1, r_2, r_3\}$	$\{r_2\}$	$\{r_2, r_3\}$	$\{r_2\}$	$\{r_2, r_3\}$	$\{r_2, r_3\}$	$\{r_2, r_3\}$
$\{r_2, r_4\}$	$\{r_4\}$	$V(D)$	$\{r_2, r_4\}$	$\{r_2, r_3, r_4\}$	$\{r_2, r_4\}$	$\{r_2, r_3, r_4\}$	$\{r_2, r_4\}$	$\{r_2, r_4\}$
$\{r_3, r_4\}$	$\{r_4\}$	$\{r_3, r_4\}$	$\{r_4\}$	$\{r_3, r_4\}$	$\{r_3, r_4\}$	$r_3, r_4\}$	$\{r_3, r_4\}$	$\{r_3, r_4\}$
$\{r_1, r_2, r_3\}$	$\{r_1, r_2\}$	$\{r_1, r_2, r_3\}$	$\{r_1, r_2\}$	$\{r_1, r_2, r_3\}$	$\{r_1, r_2, r_3\}$	$\{r_1, r_2, r_3\}$	$\{r_1, r_2, r_3\}$	$\{r_1, r_2, r_3\}$
$\{r_1, r_2, r_4\}$	$\{r_1, r_2, r_4\}$	$V(D)$	$\{r_1, r_2, r_4\}$	$V(D)$	$\{r_1, r_2, r_4\}$	$V(D)$	$\{r_1, r_2, r_4\}$	$V(D)$
$\{r_1, r_3, r_4\}$	$\{r_4\}$	$V(D)$	$\{r_1, r_3, r_4\}$	$\{r_1, r_3, r_4\}$	$\{r_1, r_3, r_4\}$	$\{r_1, r_3, r_4\}$	$\{r_1, r_3, r_4\}$	$\{r_1, r_3, r_4\}$
$\{r_2, r_3, r_4\}$	$\{r_4\}$	$V(D)$	$\{r_2, r_3, r_4\}$	$\{r_2, r_3, r_4\}$	$\{r_2, r_3, r_4\}$	$\{r_2, r_3, r_4\}$	$\{r_2, r_3, r_4\}$	$\{r_2, r_3, r_4\}$
$V(D)$	$V(D)$	$V(D)$	$V(D)$	$V(D)$	$V(D)$	$V(D)$	$V(D)$	$V(D)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

From the above table we can notice that:

(a) Implementing the *J*-supra approximations is extremely interesting for obliterating the abstruseness of rough graphs, and this would help to extract and detecting of furtive information in statements aggregated from real-life applications.

(b) The best *J*-supra approach is β_J, (since β_J is minutest than the other kinds of *J*-supra open graphs.

(c) There are many rough graphs in τ_O, but it is *J*-supra exact such as the shadowed graphs.

Conclusion.

By employing the *J*-supra open graph, a newfound ways for approximation rough graphs for each topology of τ_J are presented. Applying *J*-supra approximations helps to extract of unobserved information in datagram collected from real-life implementations. Example (3.24) show that there are many rough graphs in τ_O it is *J*-supra exact. β_J is the best *J*-supra approach since it is more accurate than the other types.

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MHD Peristaltic Flow of a Couple - Stress with varying Temperature for Jeffrey Fluid through Porous Medium

[1]Samah F. Jaber AL-khulaifawi and [2]Dheia Gaze Salih Al –Khafajy

[1,2]Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaneyah, Iraq.

[1]E-mail: samah85.samah85@gmail.com
[2]E-mail: dr.dheia.g.salih@gmail.com , dheia.salih@qu.edu.iq

Abstract
This paper is intended for investigating the effects of magnetohydrodynamic on the couple stress unsteady flow of incompressible Jeffrey fluid with varying temperature through a cylindrical porous channel. The analytical expression of the axial velocity, stream function and gradient pressure, was created taking into account the effect of thermal diffusion on the flow of the fluid. The analytical formulas of the velocity, temperature have been illustrated graphically for significant various parameters such as magnetic parameter, couple stress parameter, permeability parameter.

Keywords: MHD, Jeffrey Fluid, peristaltic flow, couple stress, porous medium.

List of symbols and meanings:

Symbol	The meaning
A	is the average radius of the undisturbed tube.
B	is the amplitude of the peristaltic wave.
$\mathcal{L}$	is the wavelength.
s	is the wave propagation speed.
$\bar{t}$	is the time.
$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$	is the Laplace operator.
$\bar{V}$	is the velocity field.
ρ	is the density.
μ	is the dynamic viscosity.
k^*	is the permeability.
$\bar{B} = (0, B_0, 0)$	is the inclined magnetic field.
μ_p	is the magnetic permeability.
$\bar{\sigma}$	is the Cauchy stress tensor.
$\bar{\zeta}$	is the constant associated with the couple stress.
T	is the temperature of the fluid.
T_c	is the thermal conductivity.
T_s	is the specific heat capacity at constant pressure.
∇V	is the fluid velocity gradient.
Q_r	is the radiation heat flux.
q	is the heat generation.
$\bar{p}$	is the pressure.

1. Introduction

Peristaltic flows received a broad study by researchers because of interest in physiology and industry. The movement of blood in the bodies of living organisms is one of the applications of peristaltic movement that occupied the ideas of many researchers of its importance in blood transfusion. The arterial segment was contracted and extended periodically by spreading the progressive wave. And as a result of this, the researchers presented their scientific results related to peristaltic flow engineering, and among the first of these researchers in this specialization are: Latham [1]. In [2] he presented a detailed analysis of the peristaltic flow fluid in circular cylindrical tubes, in [3] he along on experimental results with a long wave approximation is adopted to analyze the problem of peristaltic pumping in a circular cylindrical tube. Moreover, peristalsis subjected to magnetic field effects is important in the treatment of hyperthermia, arterial flow, cancer treatment, etc.

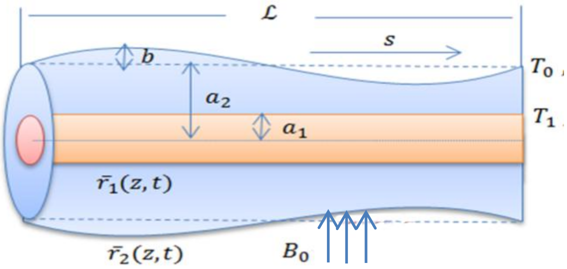
We can consider detailed explanation of peristaltic fluids as well as experimental results with a long wave approximation dependent on a round cylindrical tube. It is very important to cast a magnetic field on peristalsis in the treatment of hyperthermia, arterial flow, cancer treatment, etc. Where the magnet is important in healing diseases of the uterus, ulcers, infections and intestine. On the other hand, the role of permeability is very important for the movement of the fluid, as is the case in extracting oil from wells and absorbing food in the intestine ... etc. Many researchers presented a study on the combined effect of the magnetic field and the presence of permeability the fluid flow channel, see [4-8]. At the present time, interest began to study the effect of temperature on the movement of liquids through a channel, as most researchers agreed that increasing the temperature increases the velocity of the fluid, see [9-14] for more details.

The present analysis is interested in discussing the effects of MHD on a couple's stress on Jeffrey fluid through a cylindrical porous medium duct. To date, studies have not found the presence of a magnetic field and the effect of varying temperatures from a couple's stress on the flow of a Jeffrey liquid through a porous channel in the cylindrical coordinates. This paper was divided into seven sections. The first section contains the flow channel form with the formulation of the governing equations and the formula for the equation for liquids fluid. As for the second section, it includes reviewing the boundary conditions with including non-dimensional transformations to facilitate the governing equations that assume there is a very small number of Reynolds or a very large wavelength to solve. As for sections 3 and 4, it is to solve problems and find a formula for temperature, velocity function, high pressure, and frictional force using Bissell functions and the regular ultra-high pressure measurement function. Whereas, the fifth section includes a discussion of the effect of the parameters on temperature, speed velocity, and pressure through detailed illustrations. The sixth section examines the phenomenon of trapping and the factors affecting it, whether increasing or decreasing, and in the last section it briefly presents the most important factors affecting the shape.

2. Mathematical Formulation

Consider a peristaltic flow of an incompressible Jeffrey fluid in a coaxial uniform circular tube. The Jeffrey fluid is a non-Newtonian non-compressible liquid model and it is a real fluid in which shear stress does not match the shear stress rate (or velocity gradient). The cylindrical coordinates are considered, where R is along the radius of the tube and Z coincides with the axes of the tube as shown in figure 1.see [12].

Figure 1 Geometry of the problem



The geometry of wall surface is described as:

$$H(\bar{Z},\bar{t}) = a + b\sin\left[\frac{2\pi}{\mathcal{L}}(\bar{Z} - s\bar{t})\right] \quad (1)$$

The basic equations governing of the problem (continuity, momentum and temperature equations) are given by:

$$\nabla\bar{V} = 0 \quad (2)$$

$$\rho(\bar{V}.\nabla)\bar{V} = \nabla\bar{\sigma} + \mathcal{M}_e.\bar{J}\times\bar{B} - \frac{\mu}{k^*}\bar{V} + \rho g\beta_1(T - T_0) + \bar{\zeta}\,\nabla^4\bar{V}, \quad (3)$$

see. [4], [12]

$$T_s\,\rho(\bar{V}.\nabla)T = T_c.\nabla^2 T - \nabla.Q_r - q(T - T_0) \quad (4)$$

The constitutive equations for an incompressible Jeffrey fluid are given by:

$$\bar{\sigma} = -\bar{p}\bar{I} + \bar{S},. \quad (5)$$

$$\bar{S} = \frac{\mu}{1+\lambda_1}(\bar{\varkappa} + \lambda_2\,\bar{\ddot{\varkappa}}). \quad (6)$$

where $\bar{S}$ is the extra stress tensor, $\bar{p}$ is the pressure, $\bar{I}$ is the identity tensor, λ_1 is the ratio of relaxation to retardation times, $\bar{\varkappa}$ is the shear rate, $\bar{\ddot{\varkappa}}$ is material derivative, and λ_2 is the retardation time.

3. Method of solution

Let $\bar{U}$ and $\bar{W}$ be the respective velocity components in the radial and axial directions in the fixed frame, respectively. For the unsteady two - dimensional flow the velocity field, temperature function may be written as:

$$\bar{V} = \left(\bar{U}(\bar{r},\bar{z}), 0, \bar{W}(\bar{r},\bar{z})\right). \quad (7)$$

$$T = T(r,z), \quad (8)$$

By using the constitutive relations (5), (6) the equations of the problem (2)-(4) take the form:

$$\frac{\partial\bar{U}}{\partial\bar{R}} + \frac{\bar{U}}{\bar{R}} + \frac{\partial\bar{W}}{\partial\bar{Z}} = 0 \quad (9)$$

$$\rho\left(\frac{\partial\bar{U}}{\partial\bar{t}} + \bar{U}\frac{\partial\bar{U}}{\partial\bar{R}} + \bar{W}\frac{\partial\bar{U}}{\partial\bar{Z}}\right) = -\frac{\partial\bar{p}}{\partial\bar{R}} + \frac{1}{\bar{R}}\frac{\partial}{\partial\bar{R}}(\bar{R}\bar{S}_{\bar{R}\bar{R}}) + \frac{\partial}{\partial\bar{Z}}(\bar{S}_{\bar{Z}\bar{R}}) - \frac{\bar{S}_{\bar{\theta}\bar{\theta}}}{\bar{R}} - \frac{\mu}{k^*}\bar{U} - \sigma B_0^2\bar{U} - \bar{\zeta}\,\nabla^4\bar{U} \quad (10)$$

$$\rho\left(\frac{\partial\bar{W}}{\partial\bar{t}} + \bar{U}\frac{\partial\bar{W}}{\partial\bar{R}} + \bar{W}\frac{\partial\bar{W}}{\partial\bar{Z}}\right) = -\frac{\partial\bar{p}}{\partial\bar{Z}} + \frac{1}{\bar{R}}\frac{\partial}{\partial\bar{R}}(\bar{R}\bar{S}_{\bar{R}\bar{Z}}) + \frac{\partial}{\partial\bar{Z}}(\bar{S}_{\bar{Z}\bar{Z}}) - \sigma B_0^2\bar{W} - \frac{\mu}{k^*}\bar{W} + \rho g\beta_1(T - T_0) - \bar{\zeta}\,\nabla^4\bar{W} \quad (11)$$

$$\frac{\partial T}{\partial \bar{t}}+\bar{U}\frac{\partial T}{\partial \bar{R}}+\bar{W}\frac{\partial T}{\partial \bar{Z}}=\frac{T_c}{T_s\rho}\left(\frac{\partial^2 T}{\partial \bar{R}^2}+\frac{1}{\bar{R}}\frac{\partial T}{\partial \bar{R}}+\frac{\partial^2 T}{\partial \bar{Z}^2}\right)+\frac{16\sigma_0 T_2^E}{3k_0 T_s\,\rho}\left(\frac{1}{\bar{R}}\frac{\partial T}{\partial \bar{R}}+\frac{\partial^2 T}{\partial \bar{R}^2}\right)-\frac{q}{T_s\,\rho}(T-T_0)$$
(12)

The flow in the fixed coordinates ($\bar{R}$,$\bar{Z}$) between the two tubes is unsteady, it becomes steady at moving coordinates (r, z) when the wave is the same speed in the Z -direction. The Transformations between the two frames is given by:

$$\bar{r}=\ \bar{R}\ ,\bar{z}\ =\ \bar{Z}\ -\ s\bar{t},$$
(13)

$$\bar{u}=\bar{U}\ ,\bar{w}\ =\bar{W}-s,$$
(14)

Where ($\bar{u}$, $\bar{w}$) and ($\bar{U}$, $\bar{W}$) are the velocity components in the moving and fixed frames, respectively.
The appropriate boundary conditions are:

$$\left.\begin{aligned}&\bar{w}=-1,\ \bar{u}=0,\ T=T_1\ \text{ at }\ \bar{r}=\ \bar{r_1}\ =a_1\\&\bar{w}=-1,\ \bar{u}=0,\ T=T_0\ \text{ at }\ \bar{r}=\ \bar{r_2}(\bar{z},\bar{t})=a_2+b\,Sin(2\pi\bar{z})\end{aligned}\right\}$$
(15)

In order to simplify the governing equations of the problem, we may introduce the following dimensionless transformations as follows:

$$\left.\begin{aligned}&u=\frac{\bar{u}\,\mathcal{L}}{a_2 s},w=\frac{\bar{w}}{s},r=\frac{\bar{r}}{a_2},\ z=\frac{\bar{z}}{\mathcal{L}}\ ,S=\frac{a_2\bar{S}}{\mu\,s},\ \delta=\frac{a_2}{\mathcal{L}},Da=\frac{k}{a_2^2},\\&\mathcal{H}=\frac{T-T_0}{T_1-T_0},Rn=\frac{K_0\,T_s\mu}{4T_2^E\sigma_0},p=\frac{a_2^2\bar{p}}{\mu s\mathcal{L}},M^2=\frac{\sigma a_2^2 B_0^2}{\mu},Re=\frac{\rho s a_2}{\mu},\\&r_1=\frac{\bar{r_1}}{a_2}=\varepsilon<1,\ \emptyset=\frac{b}{a_2},\ r_2=\frac{\bar{r_2}}{a_2}=1+\emptyset\sin(2\pi\bar{z}),\\&\alpha=\bar{\alpha}a_2=\sqrt{\frac{\mu}{\zeta}}a_2\ ,Gr=\frac{\rho g\beta_1 a_2^2(T_1-T_0)}{\mu s},Pr\ =\ \frac{\mu T_s}{T_c},\Omega=\frac{q a_2^2}{\mu T_s}\end{aligned}\right\}$$
(16)

where $\emptyset$ the "amplitude ratio", $\bar{\alpha}$ the "couple stress" fluid parameter indicating the ratio of the tube radius (constant) to material characteristic length ($\sqrt{\mu/\zeta}$, has the dimension of length), Re the "Reynolds number is the ratio of inertia force to the viscous force", Pr the "Prandtl number is ratio of kinematic viscosity to the thermal diffusivity", Da the "Darcy number is the ratio of the permeability of the medium to the diameter of the particle", Rn the "thermal radiation parameter", Gr the "thermal Grashof number is a measure of buoyancy or free-convection effects in a flow", M^2 the "magnetic parameter is equal to the product of the square of the magnetic permeability, the square of the magnetic field strength, the electrical conductivity, and a characteristic length, divided by the product of the mass density and the fluid velocity", δ the "dimensionless wave number" and Ω "heat source/sink parameter".

After using these transformations equations (13)-(14), substituting dimensionless equations (16) into problem equations (9)-(12) and boundary conditions (15), we get:

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0$$
(17)

$$Re\delta^3\left(u\ \frac{\partial u}{\partial r}+(w+1)\frac{\partial u}{\partial z}\right)=-\ \frac{\partial p}{\partial r}\ +\delta\ \left[\frac{1}{r}\frac{\partial}{\partial r}(rS_{rr}\ )+\delta\ \frac{\partial}{\partial z}(S_{rz}\ )-\frac{S_{\theta\theta}}{r}-\frac{\delta}{\alpha^2}\ \nabla^4 u-\frac{\delta}{Da}u-\delta M^2 u\right]$$
(18)

$$Re\delta\left(\ u\ \frac{\partial w}{\partial r}+(w+1)\frac{\partial w}{\partial z}\right)=-\ \frac{\partial p}{\partial z}\ +\frac{1}{r}\frac{\partial}{\partial r}(rS_{zr}\ )+\delta\ \frac{\partial}{\partial z}(S_{zz}\ )-\frac{1}{\alpha^2}\nabla^4\mathrm{w}-\left(M^2+\frac{1}{Da}\right)w-\left(M^2+\frac{1}{Da}\right)+Gr\mathcal{H}$$
(19)

$$Re\delta\left(u\ \frac{\partial \mathcal{H}}{\partial r} + (w+1)\frac{\partial \mathcal{H}}{\partial z}\right) = \frac{1}{Pr}\left(\frac{\partial^2 \mathcal{H}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{H}}{\partial r} + \delta^2\frac{\partial^2 \mathcal{H}}{\partial z^2}\right) + \frac{4}{3Rn}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \mathcal{H}}{\partial r}\right) - \Omega\mathcal{H}$$
(20)

where

$$S_{rr} = \frac{2\delta}{1+\lambda_1}\ [1 + \frac{s\lambda_2\delta}{a_2}(u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z})](\frac{\partial u}{\partial r})$$
(21)

$$S_{rz} = \frac{1}{1+\lambda_1}\ [1 + \frac{s\lambda_2\delta}{a_2}(u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z})](\frac{\partial w}{\partial r} + \delta^2\ \frac{\partial u}{\partial z})$$
(22)

$$S_{\vartheta\vartheta} = \frac{2\delta}{1+\lambda_1}\left[\frac{u}{r} + \frac{s\lambda_2\delta}{a_2}(\frac{u}{r}\ \frac{\partial u}{\partial r} - \frac{u^2}{r^2}\ + \frac{w}{r}\frac{\partial u}{\partial z})\right]$$
(23)

$$S_{zz} = \frac{2\delta}{1+\lambda_1}\ [1 + \frac{s\delta}{a_2}(u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z})](\frac{\partial w}{\partial z})$$
(24)

The related boundary conditions regarding to the dimensionless variables in the wave frame are given by:

$$\left.\begin{array}{l} w = -1, u = 0, \mathcal{H} = 1\ \text{ at }\ r = r_1 = \varepsilon \\ w = -1, u = 0, \mathcal{H} = 0\ \text{ at }\ r = r_2 = 1 + \emptyset.Sin(2\pi z) \end{array}\right\}$$
(25)

It seems that the general solution of the equations (17) - (20) in the general case is impossible; therefore, we must limit the analysis to the assumption that the wavenumber is small ($\delta \ll 1$). Means, we studied long-wavelength approximation. Along with this assumption, equations (17) - (20) become:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$$
(26)

$$\frac{\partial p}{\partial r} = 0$$
(27)

$$\frac{1}{\alpha^2}\nabla^4 \mathrm{w} - \frac{1}{r}\frac{\partial}{\partial r}(rS_{zr}) + \left(M^2 + \frac{1}{Da}\right)w = -\ \frac{\partial p}{\partial z} - \left(M^2 + \frac{1}{Da}\right) + Gr\mathcal{H}$$
(28)

$$\left(\frac{1}{Pr} + \frac{4}{3Rn}\right)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \mathcal{H}}{\partial r}\right) - \Omega\mathcal{H} = 0$$
(29)

where$S_{rr} = S_{\vartheta\vartheta} = S_{zz} = 0, and\ S_{rz} = \frac{1}{1+\lambda_1}\left(\frac{\partial w}{\partial r}\right).$ (30)

Equation (27) shows that p dependents on z only, Replacing S_{rz} from equation (30) in equation (28), we have:

$$\frac{1}{\alpha^2}\nabla^4 \mathrm{w} - \frac{1}{1+\lambda_1}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \left(M^2 + \frac{1}{Da}\right)w = -\ \frac{\partial p}{\partial z} - \left(M^2 + \frac{1}{Da}\right) + Gr\mathcal{H}$$
(31)

Assuming the components of the couple stress tensor at the wall to be zero, when Couple-stress, denoted by $\bar{\alpha}$ is defined as the ratio of the tube radius (constant) to material characteristic length ($\sqrt{\frac{\eta}{\mu}}$ has the dimension of length), mathematically:

$$\bar{\alpha} = \alpha a_2 = \sqrt{\frac{\mu}{\eta}}\, a_2$$

(32)

Where, μ is the dynamic viscosity, η is constant associated with couple stress, we can write The Couple-stress $\bar{\eta}.\nabla^4\bar{\mathrm{V}}$, see. [7], we have the following dimensionless boundary conditions:

$$\left.\begin{aligned} w = -1,\ \frac{\partial^2 w}{\partial r^2} - \frac{\bar{\bar{\zeta}}}{r}\frac{\partial w}{\partial r} = 0 \ \text{ at } \ r = \varepsilon \\ w = -1,\ \frac{\partial^2 w}{\partial r^2} - \frac{\bar{\bar{\zeta}}}{r}\frac{\partial w}{\partial r} = 0 \ \text{ at } \ r = 1 + \emptyset.Sin(2\pi z) \end{aligned}\right\}$$

(33)

Where $\bar{\bar{\zeta}} = \frac{\bar{\zeta}}{\zeta}$ is a couple stress fluid parameter ($\bar{\zeta}$ and ζ are constants associated with the couple stress, when $\zeta \longrightarrow 1$ (i.e. $\bar{\zeta} \rightarrow \zeta$) no couple stress effects, see [4], [5], and [6]).

4. Solutions of the Temperature Equations

The temperature equation (29), can be written as;

$$\left(\frac{\partial^2 \mathcal{H}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{H}}{\partial r}\right) - \frac{\Omega}{(\frac{1}{Pr} + \frac{4}{3Rn})}\mathcal{H} = 0,$$

(34)

Set $\mathrm{A} = -\frac{\Omega}{(\frac{1}{Pr} + \frac{4}{3Rn})}$, the equation (34) takes the form:

$$r^2\frac{\partial^2 \mathcal{H}}{\partial r^2} + r\frac{\partial \mathcal{H}}{\partial r} + Ar^2\mathcal{H} = 0$$

(35)

The general solution of this equation "modified Bessel equation of zero-order", with the boundary conditions equation (25) is:

$$\mathcal{H} = B_1\, \mathrm{J}_0[r\sqrt{A}] + B_2\mathrm{Y}_0[r\sqrt{A}]$$

(36)

where $B_1 = \frac{\mathrm{Y}_0[h\sqrt{A}]}{J_0[\epsilon\sqrt{A}]\mathrm{Y}_0[h\sqrt{A}] - J_0[h\sqrt{A}]\mathrm{Y}_0[\epsilon\sqrt{A}]}$ and $B_2 = \frac{J_0[h\sqrt{A}]}{J_0[h\sqrt{A}]\mathrm{Y}_0[\epsilon\sqrt{A}] - J_0[\epsilon\sqrt{A}]\mathrm{Y}_0[h\sqrt{A}]}$

The general solution of motion equation (31) is:

$$w = B_1 I_0\left(\mathscr{b} r\sqrt{|s_1|}\right) + B_2 K_0\left(\mathscr{b} r\sqrt{|s_1|}\right) + B_3 I_0\left(\mathscr{b} r\sqrt{|s_2|}\right) + B_4 K_0\left(\mathscr{b} r\sqrt{|s_2|}\right) - \left(\frac{\frac{dp}{dz} - Gr\mathcal{H} + \left(M^2 + \frac{1}{Da}\right)}{M^2 + \frac{1}{Da}}\right)$$

(37)

Where $s_1 = -\frac{C_1}{\mathscr{b}^2} - \sqrt{\left(\frac{C_1}{\mathscr{b}^2}\right)^2 - 1}$, $s_2 = -\frac{C_1}{\mathscr{b}^2} + \sqrt{\left(\frac{C_1}{\mathscr{b}^2}\right)^2 - 1}, c_1 = \frac{\alpha^2}{2(1+\lambda_1)}$ $and\ \mathscr{b}^4 = \alpha^2\left(M^2 + \frac{1}{D_a}\right)$

Also I_0, K_0 are the modified Bessel functions of the first and second kind of zero order. By using the "MATHEMATICA 11" program and the boundary conditions equations (25) and (33) we have a constants $B_1,\ B_2,\ B_3$ and B_4.

5. Stream function

The corresponding stream functions $u = -\frac{1}{r}\frac{\partial \psi}{\partial z}$ and $w = \frac{1}{r}\frac{\partial \psi}{\partial r}$ is

$$\psi = \frac{A_1 r^2}{2} - \frac{B_2 r K_1(r u_1)}{u_1} - \frac{B_4 r K_1(r u_2)}{u_2} + \frac{1}{2} B_1 r^2 \;\; {}_0\tilde{F}_1 \;\; \left[2; \frac{r^2 u_1^2}{4}\right] + \frac{1}{2} B_3 r^2 \;\; {}_0\tilde{F}_1 \;\; \left[2; \frac{r^2 u_2^2}{4}\right]$$
(38)

$A_1 = -\left(\frac{\frac{dp}{dz} - Gr\mathcal{H} + \left(M^2 + \frac{1}{D_a}\right)}{M^2 + \frac{1}{D_a}}\right)$, $u_1 = \mathscr{b}\sqrt{|s_1|}$, $u_2 = \mathscr{b}\sqrt{|s_2|}$, K_1 ${}_0\tilde{F}_1$ are the modified Bessel function of the second kind and Hypergeometric regularized function, respectively.

The instantaneous volume flow rate $\mathrm{Q(z)}(= 2\int_{r_1}^{r_2} \mathrm{rwdr})$ is given by;

$$\frac{dp}{dz} = \left(Gr\mathcal{H} - \left(M^2 + \frac{1}{D_a}\right)\right) + \left(\frac{M^2 + \frac{1}{D_a}}{r_2^2 - r_1^2}\right)\Big\{\mathrm{Q(z)} + \frac{2B_2}{u_1}[r_1 K_1(r_1 u_1) - r_2 K_1(r_2 u_1)] + \frac{2B_4}{u_2}[r_1 K_1(r_1 u_2) - r_2 K_1(r_2 u_2)] - B_1\left\{r_2^2 \;\; {}_0\tilde{F}_1 \;\; \left[2; \frac{r_2^2 u_1^2}{4}\right] - r_1^2 \;\; {}_0\tilde{F}_1 \;\; \left[2; \frac{r_1^2 u_1^2}{4}\right]\right\} - B_3\left\{r_2^2 \;\; {}_0\tilde{F}_1 \;\; \left[2; \frac{r_2^2 u_2^2}{4}\right] - r_1^2 \;\; {}_0\tilde{F}_1 \;\; \left[2; \frac{r_1^2 u_2^2}{4}\right]\right\}\Big\}$$

(39)

Following the analysis given by Shapiro et al.[14], the mean volume flow, *q2* over a period is obtained as

$$q2 = Q + \frac{1}{2}\left(1 - \varepsilon^2 + \frac{\phi^2}{2}\right)$$
(40)

This on using Eq. (38) yields

$$\frac{dp}{dz} = \left(Gr\mathcal{H} - \left(M^2 + \frac{1}{D_a}\right)\right) + \left(\frac{M^2 + \frac{1}{D_a}}{r_2^2 - r_1^2}\right)\Big\{q2 - \frac{1}{2}\left(1 - \epsilon^2 + \frac{\emptyset^2}{2}\right) + \Big(\frac{2B_2}{u_1}[r_1 K_1(r_1 u_1) - r_2 K_1(r_2 u_1)] + \frac{2B_4}{u_2}[r_1 K_1(r_1 u_2) - r_2 K_1(r_2 u_2)] - B_1\left\{r_2^2 \;\; {}_0\tilde{F}_1 \;\; \left[2; \frac{r_2^2 u_1^2}{4}\right] - r_1^2 \;\; {}_0\tilde{F}_1 \;\; \left[2; \frac{r_1^2 u_1^2}{4}\right]\right\} - B_3\left\{r_2^2 \;\; {}_0\tilde{F}_1 \;\; \left[2; \frac{r_2^2 u_2^2}{4}\right] - r_1^2 \;\; {}_0\tilde{F}_1 \;\; \left[2; \frac{r_1^2 u_2^2}{4}\right]\right\}\Big)\Big\}$$
(41)

where Y_1 And ${}_0\tilde{F}_1$ are the modified Bessel function of the second kind and hypergeometric function, respectively.

The pressure rise Δp and the friction force (at the wall) on the inner and outer tubes are F^i and F^o, respectively, in a tube of length L, in their non-dimensional forms, are given by:

$$\Delta p = \int_0^1 \left(\frac{dp}{dz}\right) dz,$$
(42)

$$F^i = \int_0^1 r_2^2 \left(-\frac{dp}{dz}\right) dz,$$
(43)

$$F^o = \int_0^1 r_1^2 \left(-\frac{dp}{dz}\right) dz,$$
(44)

Substituting from equation (41) in equations (42) - (44) with $r_1 = \varepsilon$, $r_2 = 1 + \emptyset . \sin(2\pi z)$, and then evaluating the integrations by using the language of series for several values of the parameters included, by the "MATHEMATICA 11" program, and the obtained results are discussed in the next section.

6. Numerical Results and Discussion

In this section the numerical and computational results are discussed for the problem of an incompressible non- Newtonian Jeffrey fluid through porous medium with heat and mass transfer through the illustrations figures (2-39).

Based on equation (36), figures (2-3) shows that effects of the parameters ε , Ω , Rn and $\emptyset$ on the temperature function $\mathcal{H}$, in figure 2, we notice that $\mathcal{H}$ increases with increasing ε and Ω, while figure 3, illustrates the temperature function increases with increasing Rn and $\mathcal{H}$ decrease with increasing $\emptyset$.

Based on equation (37), figures (4-9), illustrate the effect of the parameters ε, Ω, α, η, $\emptyset$, $\lambda 1$, Gr, Da, M, $q2$, Pr and Rn on the velocity distribution w vs. r. We noticed that the velocity distribution starts to decrease and when it reaches point r=0.05 it starts to increase and for this, the general shape of the velocity distribution is a concave upward curve. Figure 4, illustrates the influence of the parameters ε and Ω on the velocity distribution function w vs. It is found that the velocity w increases with the increasing ε when $r < 0.07$, while w decreases with increasing of ε when $r > 0.07$, and w decreases with increasing Ω. In the fifth plot, shows the behavior of w under the variation of α and η, one can describe here that w increases with increasing of α and η at $r > 0.2$, while w decreases with increasing of α and η, at $r < 0.2$, Figure 6, we notice the rotation of the effects of the parameters $\lambda 1$ and $\emptyset$ on the velocity function, where the effect of parameter $\lambda 1$ is direct in the region $r < 0.2$, while in the region $r > 0.2$ the effect of parameter $\lambda 1$ is inversed, and vice versa for the parameter $\emptyset$, we notice the decrease in the velocity when increasing $\emptyset$ in the region $r < 0.2$ and the increase in the velocity with increasing $\emptyset$ in the region $r > 0.2$. Figure 7 contains the velocity profile behavior under the parameters Gr and $q2$, we see that the velocity profile goes down with the increases Gr and $q2$ when $r < 0.2$, and w increases with increasing of Gr and $q2$ when $r > 0.2$. We notice the effect of the magnetic field and permeability on the velocity function in shape 8, we get the velocity decreases with an increase in M and Da at $r > 0.2$, while the velocity w increase with an increase in M and Da at $r < 0.2$. In the ninth plot, It is found that the velocity w increases with the increasing Pr and Rn in the region $r > 0.2$, while w decreases with increasing of Pr and Rn in the region $r < 0.2$.

Based on equation (41), figures (10-15), illustrate the effect of the parameters ε, Ω, α, η, $\emptyset$, $\lambda 1$, Gr, $q2$, Da, M, Pr and Rn on the distribution of dp/dz vs. z. We noticed that dp/dz starts to increase and when it reaches point z=0.25 it starts to decrease and for this, the general diagram of the distribution of dp/dz is a concave downward curve. Figures 11, 13 and 14, illustrates the influence of the parameters α, η, Gr, $q2$, Da and M on dp/dz. It is found that dp/dz increases with the increasing α, η, Gr, $q2$, Da and M, respectively. Figures 10 and 15, illustrates the influence of the parameters Ω, ε, Pr and Rn on dp/dz. It is found that dp/dz decreases with the increasing Ω, ε, Pr and Rn, respectively. Figure 12, illustrates the influence of the parameters $\emptyset$ and $\lambda 1$, on dp/dz. It is found that dp/dz increases with the increasing $\emptyset$ while dp/dz decreases with the increasing $\lambda 1$.

Based on equation (42), figures (16-19) illustrates the effects of the parameters $\emptyset$, Ω, Da, ε, $\lambda 1$, $q2$, η, Gr, Rn and M on the pressure rise Δp. Figures (16-17) illustrates the effects of the parameters Ω, Da, ε and $\lambda 1$ on the Δp vs. $\emptyset$. We found that Δp increases with increasing Da, and Δp decreases with increasing Ω in figure 16. In figure 17 we notice that Δp decreases with increasing ε in the region (0,0.03) while Δp increases with increasing ε when $\emptyset > 0.03$, and Δp increases with increasing $\lambda 1$ when $\emptyset > 0.022$, while Δp decreases with increasing $\lambda 1$ when $\emptyset < 0.022$. Figures (18-19) illustrates the effects of the parameters η, Gr, M and Rn on the pressure rise Δp vs. $q2$, it is found that Δp increases with the increasing for each η, Gr, M and Rn.

Based on equation (43), figures (20-23) illustrates the effects of the parameters $\emptyset$, Ω, Da, ε, $\lambda 1$, $q2$, η, Gr, Rn and M on F^i. Figures (20-21) illustrates the effects of the parameters Ω, Da, ε and $\lambda 1$ on F^i vs. $\emptyset$. We found that F^i decreases with increasing Da, and F^i increases with increasing Ω in figure 20. In figure 21 we notice that F^i increases with increasing ε in the region (0,0.022) while F^i decreases with increasing ε when $\emptyset > 0.022$, the F^i decreases with increasing $\lambda 1$ when $0.021 < \emptyset < 0.1$ at $\varepsilon = 0.15$, and F^i increases with increasing $\lambda 1$ when $0 < \emptyset < 0.021$ at $\varepsilon = 0.175$, while F^i increases with increasing $\lambda 1$ otherwise. Figures (22-23) illustrates the effects of the parameters η, Gr, M and Rn on F^i vs. $q2$, it is found that F^i decreases with the increasing for each η, Gr, M and Rn.

Based on equation (44), figures (24-27) illustrates the effects of the parameters $\emptyset$, Ω, Da, ε, $\lambda 1$, $q2$, η, Gr, Rn and M on F^o. Figures (24-25) illustrates the effects of the parameters Ω, Da, ε and $\lambda 1$ on F^o vs. $\emptyset$. We found that F^o decreases with increasing Da, and F^o increases with increasing Ω in figure

24. In figure 25 we notice that F^o increases with increasing ε and $\lambda 1$ in the region (0,0.023) while F^o decreases with increasing ε and $\lambda 1$ when $\emptyset > 0.023$. Figures (26-27) illustrates the effects of the parameters η, Gr, M and Rn on F^o vs. $q2$, it is found that F^o decreases with the increasing for each η, Gr, M and Rn.

7. Trapping phenomena

The formation of an internally circulating bolus of fluid by closed streamlines is called trapping and this trapped bolus is pushed ahead along with the peristaltic wave. The effects of ε, Ω, $\emptyset$, $\lambda 1$, Rn, Pr Gr, M, Da, $q2$, α and η on trapping can be seen through 28-39. Figure 28 shows that the size of the trapped bolus decreases with the increase ε gradually in the middle of the channel while when we approach at the upper wall we notice the increase of the wave with the increase of ε.The wave near the upper wall of the channel decreases with an increase of Ω in figure 29. In the Thirty plot shows that the size of the trapped bolus located in the center of the channel increases with the increase $\emptyset$ while when it is close to the upper wall we notice the decrease of the wave with the increase $\emptyset$. By figure 31 the size of the trapped bolus grow increase of $\lambda 1$ when it is close to the upper wall of the channel gradually. The effect of parameter Rn on the trapped bolus in figure 32 is similar to the effect of parameter Ω on the trapped bolus in figure 29. By figure 33, we notice two trapped boluses, one in the center of the channel and the other at the upper wall both are decreases until it disappears with the increase Pr. In figure 34 the size of the trapped bolus decreases with the increase Gr gradually at the upper wall. In figure 35 we notice the emergence and growth of the size of the trapped boluses, in addition to an increase in the wave at the upper wall of the channel when the value of M increases. In figure 36, the size of the trapped bolus decreases with the increase Da gradually at the upper wall of the channel while its beginning to grow in the center with increase of Da. Figure 37 shows the effect of the parameter $q2$ on the trapped bolus, as with the increase of $q2$ the wave near the upper wall increases with the emergence of a new trapped bolus that caused the bolus to grow in the center of the channel. In figure 38 the size of the trapped bolus decreases with increase α in the wave near the upper wall. Finally in figure 39 we notice the effect of parameter η on trapped bolus similar to that of trapped bolus Da in figure 36.

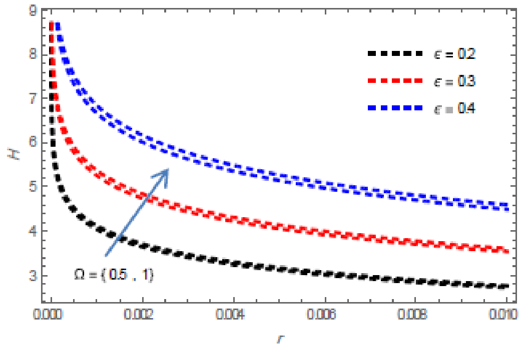


Figure 2: The variation of temperature $\mathcal{H}$ vs. r at $Pr = 1$, $\emptyset = 0.2$, $Rn = 0.5$, $z = 0.1$

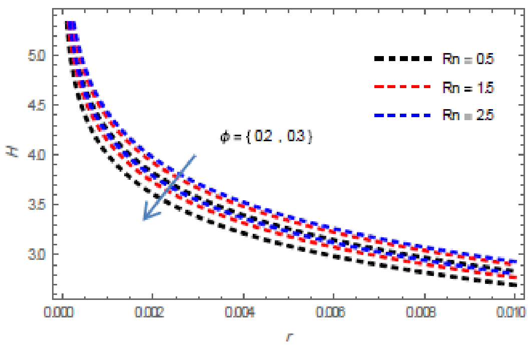


Figure 3: The variation of temperature $\mathcal{H}$ vs. r at $\epsilon = 0.2$, $\Omega = 1$, $Pr = 1$, $z = 0.1$

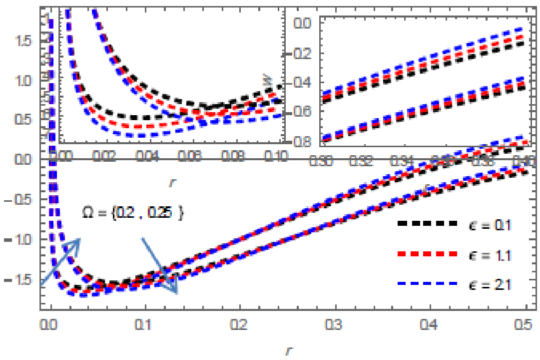


Figure 4: velocity distribution for various values of ε and Ω with $\eta = 0.5$, $\alpha = 3.75$, $\lambda 1 = 0.1$, $\emptyset = 0.2$, $Gr = 2$, $q2 = 0.5$, $M = 1.1$, $Da = 0.9$, $Sc = 0.5$, $Sr = 0.6$, $Pr = 2$, $Rn = 0.5$, $z = 0.1$.

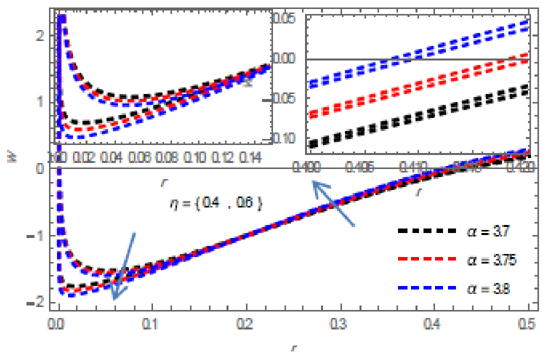


Figure 5: velocity distribution for various values of η and α with $\varepsilon = 0.2$, $\Omega = 0.9$, $\lambda 1 = 0.1$, $\emptyset = 0.2$, $Gr = 2$, $q2 = 0.5$, $M = 1.1$, $Da = 0.9$, $Sc = 0.5$, $Sr = 0.6$, $Pr = 2$, $Rn = 0.5$, $z = 0.1$.

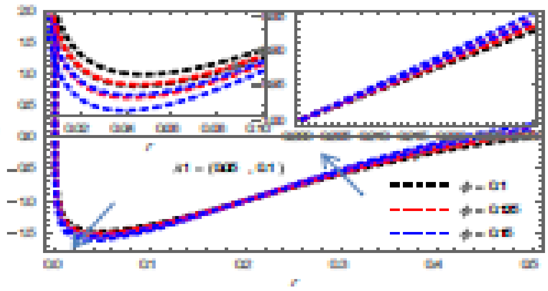


Figure 6: velocity distribution for various values of $\lambda 1$ and $\emptyset$ with $\eta = 0.5$, $\alpha = 3.75$, $\varepsilon = 0.2$, $\Omega = 0.9$, $Gr = 2$, $q2 = 0.5$, $M = 1.1$, $Da = 0.9$, $Sc = 0.5$, $Sr = 0.6$, $Pr = 2$, $Rn = 0.5$, $z = 0.1$.

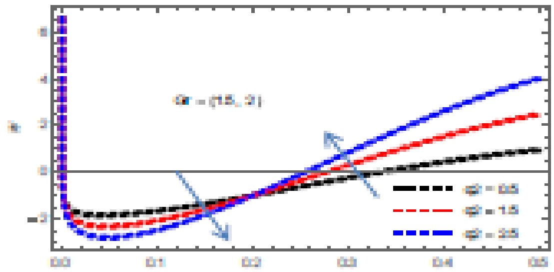


Figure 7: velocity distribution for various values of Gr and $q2$ with $\eta = 0.5$, $\alpha = 3.75$, $\lambda 1 = 0.1$, $\emptyset = 0.2$, $\varepsilon = 2$, $\Omega = 1$, $M = 1.1$, $Da = 0.9$, $Sc = 0.5$, $Sr = 0.6$, $Pr = 2$, $Rn = 0.5$, $z = 0.1$.

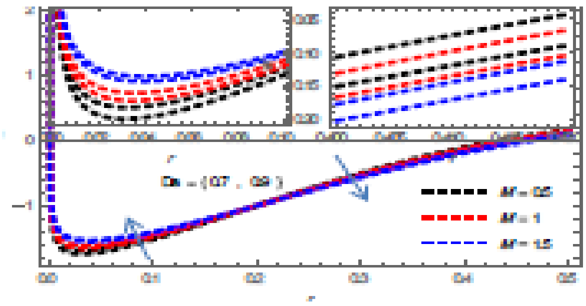


Figure 8: velocity distribution for various values of M and Da with $\eta = 0.5$, $\alpha = 3.75$, $\lambda 1 = 0.1$, $\emptyset = 0.2$, $Gr = 2$, $\varepsilon = 0.2$, $\Omega = 0.9$, $q2 = 0.5$, $Sc = 0.5$, $Sr = 0.6$, $Pr = 2$, $Rn = 0.5$, $z = 0.1$.

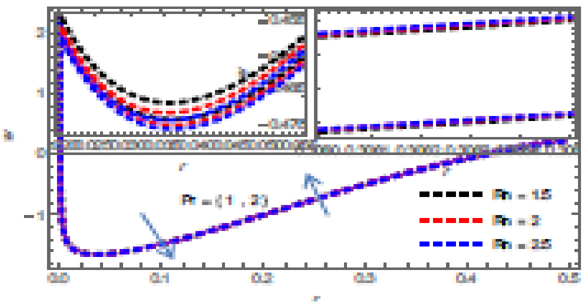


Figure 9: Velocity distribution for various values of Pr and Rn with $\eta = 0.5$, $\alpha = 3.75$, $\lambda 1 = 0.1$, $\emptyset = 0.2$, $Gr = 2$, $\varepsilon = 0.2$, $\Omega = 0.9$, $q2 = 0.5$, $M = 1.1$, $Da = 0.9$, $Sc = 0.5$, $Sr = 0.6$, $z = 0.1$.

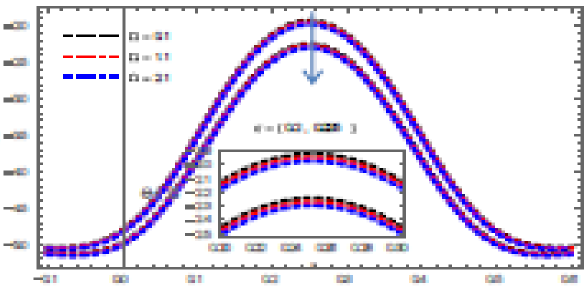


Figure 10: Distribution of $\frac{dp}{dz}$ vs. z for various values of Ω and ε with $\eta = 0.5$, $\alpha = 3.75$, $\lambda 1 = 0.1$, $\emptyset = 0.2$, $Gr = 2$, $Rn = 2$, $Pr = 1$, $q2 = 0.5$, $M = 1.1$, $Da = 0.9$, $Sc = 0.5$, $Sr = 0.6$, $z = 0.1$.

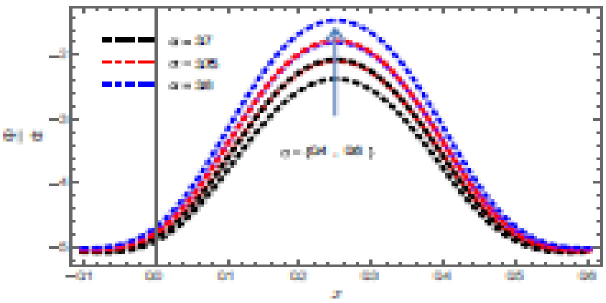


Figure 11: Distribution of $\frac{dp}{dz}$ vs. z for various values of α and η with $Sc = 0.5$, $Sr = 0.6$, $\lambda 1 = 0.1$, $\emptyset = 0.2$, $Gr = 2$, $\varepsilon = 0.2$, $\Omega = 0.9$, $q2 = 0.5$, $M = 1.1$, $Da = 0.9$, $Pr = 1$, $Rn = 2$, $z = 0.1$.

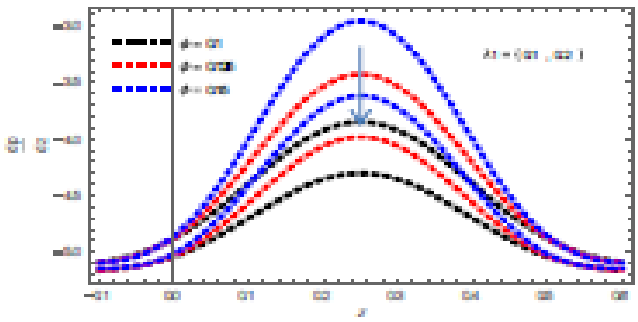


Figure 12: Distribution of $\frac{dp}{dz}$ vs. z for various values of $\emptyset$ and $\lambda 1$ with $\eta = 0.5$, $\alpha = 3.75$, $Sc = 0.5$, $Sr = 0.6$, $Gr = 2$, $\varepsilon = 0.2$, $\Omega = 0.9$, $q2 = 0.5$, $M = 1.1$, $Da = 0.9$, $Pr = 1$, $Rn = 2$, $z = 0.1$.

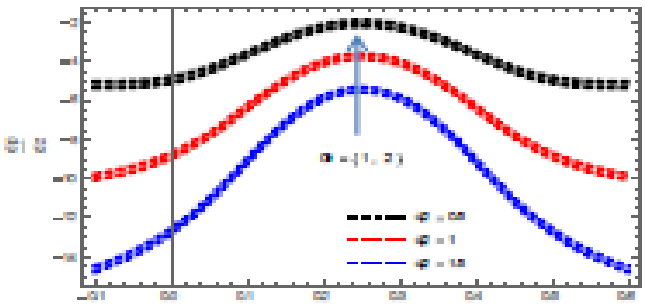


Figure 13: Distribution of $\frac{dp}{dz}$ vs. z for various values of $q2$ and Gr with $\eta = 0.5$, $\alpha = 3.75$, $Sc = 0.5$, $Sr = 0.6$, $\lambda 1 = 0.1$, $\emptyset = 0.2$, $\varepsilon = 0.2$, $\Omega = 0.9$, , $M = 1.1$, $Da = 0.9$, $Pr = 1$, $Rn = 2$, $z = 0.1$.

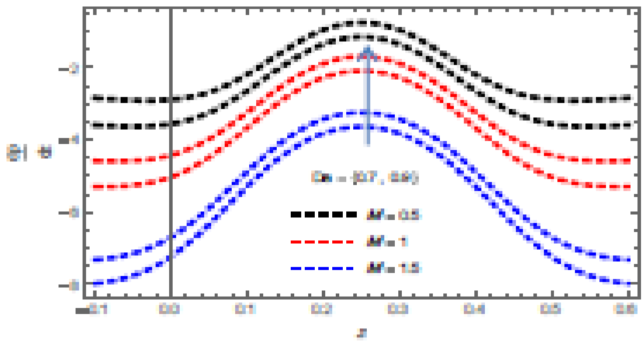


Figure 14: Distribution of $\frac{dp}{dz}$ vs. z for various values of M and Da with $\eta = 0.5$, $\alpha = 3.75$, $Sc = 0.5$, $Sr = 0.6$, $\lambda 1 = 0.1$, $\emptyset = 0.2$, $\varepsilon = 0.2$, $\Omega = 0.9$, $q2 = 0.5$, $Gr = 2$, $Pr = 1$. $Rn = 2$. $z = 0.1$.

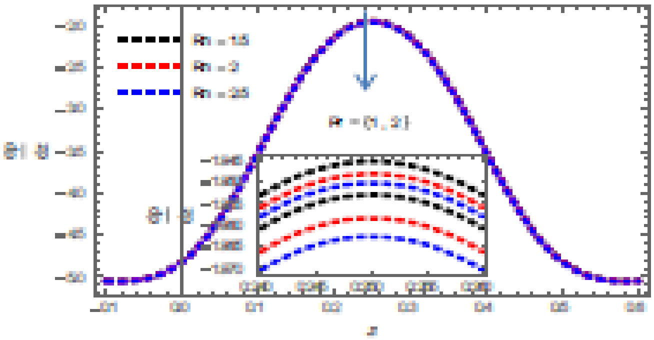


Figure 15: Distribution of $\frac{dp}{dz}$ vs. z for various values of Pr and Rn with $\eta = 0.5$, $\alpha = 3.75$, $Sc = 0.5$, $Sr = 0.6$, $\lambda 1 = 0.1$, $\emptyset = 0.2$, $\varepsilon = 0.2$, $\Omega = 0.9$, $q2 = 0.5$, $Gr = 2$, $M = 1.1$, $Da = 0.9$, $z = 0.1$.

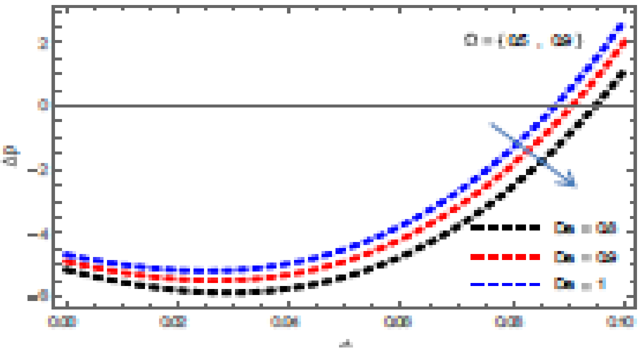


Figure 16: Distribution of Δp vs. $\emptyset$ for various values of Ω And Da with $\varepsilon = 0.2$, $\lambda 1 = 0.1$, $Rn = 2$, $Pr = 2$, $Sc = 0.5$, $Sr = 0.1$, $q2 = 0.5$, $Gr = 2$, $\alpha = 3.75$, $\eta = 0.5$, $M = 1.1$, $z = 0.1$.

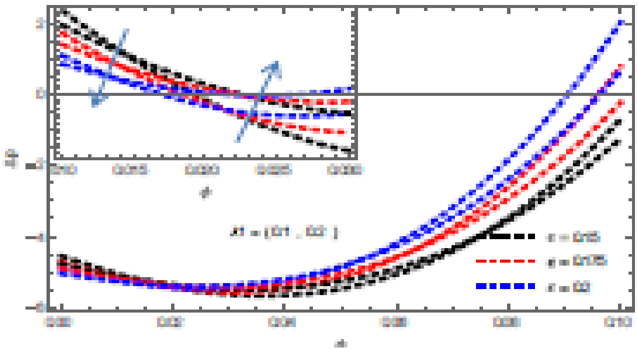


Figure 17: Distribution of Δp *vs.* $\emptyset$ for various values of $\lambda 1$ and ε with $\Omega = 0.9$, $Rn = 2$, $Pr = 2$, $Sc = 0.5$, $Sr = 0.1$, $Gr = 2$, $\alpha = 3.75$, $\eta = 0.5$, $M = 1.1$, $Da = 0.9$, $q2 = 0.5$, $z = 0.1$.

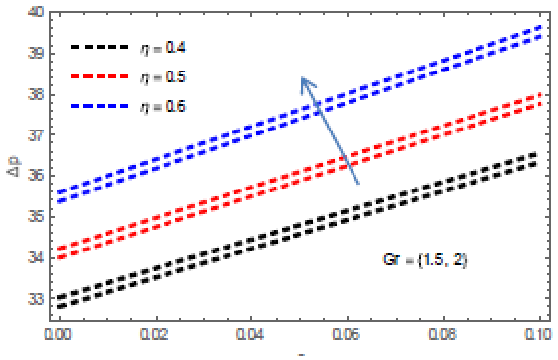


Figure 18: Distribution of Δp $vs.$ q2 for various values of Gr and η with $\varepsilon = 0.2$, $\Omega = 0.9$, $\phi = 0.2$, $\lambda 1 = 0.1$, $Rn = 2$, $Pr = 2$, $Sc = 0.5$, $Sr = 0.1$, $\alpha = 3.75$, $M = 1.1$, $Da = 0.9$, $z = 0.1$.

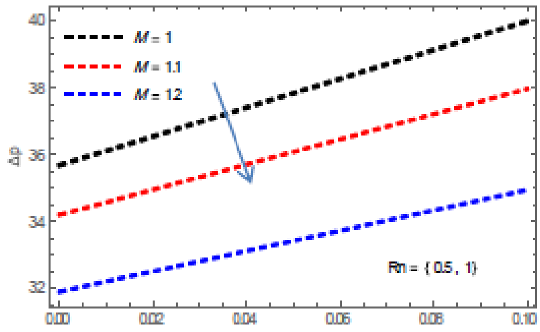


Figure 19: Distribution of Δp $vs.$ q2 for various values of M and Rn with $\varepsilon = 0.2$, $\Omega = 0.9$, $\phi = 0.2$, $\lambda 1 = 0.1$, $Pr = 2$, $Gr = 2$, $\eta = 0.5$, $Sc = 0.5$, $Sr = 0.1$, $\alpha = 3.75$, $Da = 0.9$, $z = 0.1$.

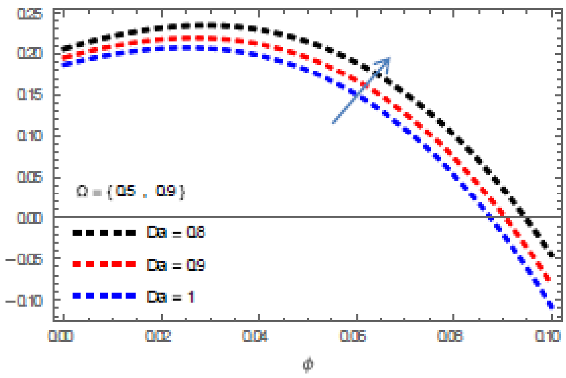


Figure 20: Distribution of F^l $vs.$ $\emptyset$ for various values of Ω And Da with $\varepsilon = 0.2$, $\lambda 1 = 0.1$, $Rn = 2$, $Pr = 2$, $Sc = 0.5$, $Sr = 0.1$, $q2 = 0.5$, $Gr = 2$, $\alpha = 3.75$, $\eta = 0.5$, $M = 1.1$, $z = 0.1$.

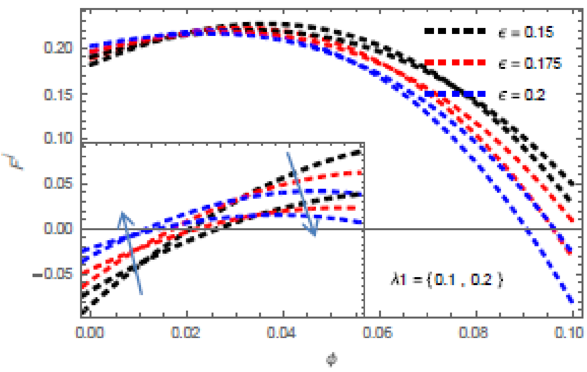


Figure 21: Distribution of F^l $vs.$ $\emptyset$ for various values of $\lambda 1$ and ε with $\Omega = 0.9$, $Rn = 2$, $Pr = 2$, $Sc = 0.5$, $Sr = 0.1$, $Gr = 2$, $\alpha = 3.75$, $\eta = 0.5$, $= 1.1$, $Da = 0.9$, $q2 = 0.5$, $z = 0.1$.

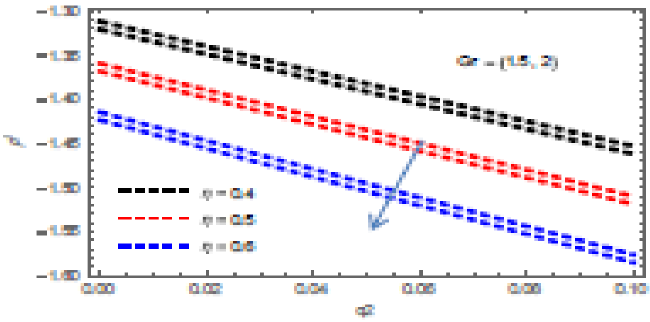


Figure 22: Distribution of F^1 *vs.* q2 for various values of Gr and η with $\varepsilon = 0.2$, $\Omega = 0.9$, $\phi = 0.2$, $\lambda 1 = 0.1$, $Rn = 2$, $Pr = 2$, $Sc = 0.5$, $Sr = 0.1$, $\alpha = 3.75$, $M = 1.1$, $Da = 0.9$, $q2 = 0.5$, $z = 0.1$.

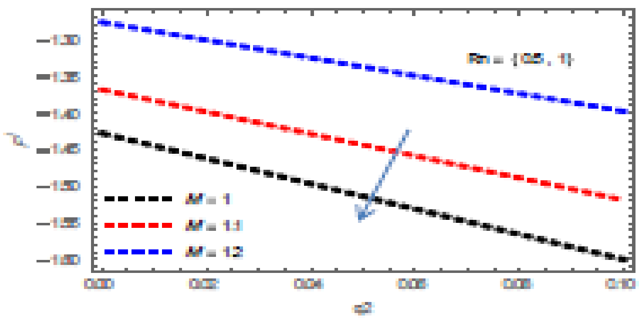


Figure 23: Distribution of F^1 *vs.* q2 for various values of M and Rn with $\varepsilon = 0.2$, $\Omega = 0.9$, $\phi = 0.2$, $\lambda 1 = 0.1$, $Pr = 2$, $Gr = 2$, $\eta = 0.5$, $Sc = 0.5$, $Sr = 0.1$, $\alpha = 3.75$, $Da = 0.9$, $q2 = 0.5$, $z = 0.1$.

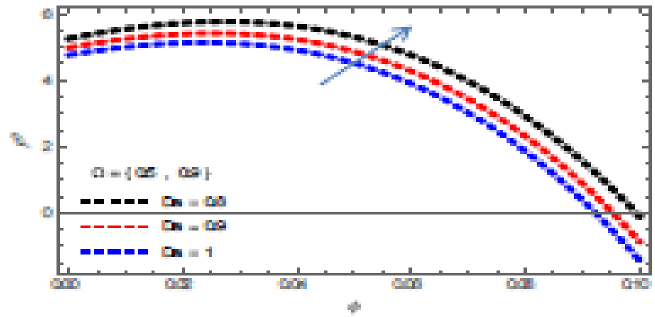


Figure 24: Distribution of F^o *vs.* Ø for various values of Ω And Da with $\varepsilon = 0.2$, $\lambda 1 = 0.1$, $Rn = 2$, $Pr = 2$, $Sc = 0.5$, $Sr = 0.1$, $q2 = 0.5$, $Gr = 2$, $\alpha = 3.75$, $\eta = 0.5$, $M = 1.1$, $z = 0.1$.

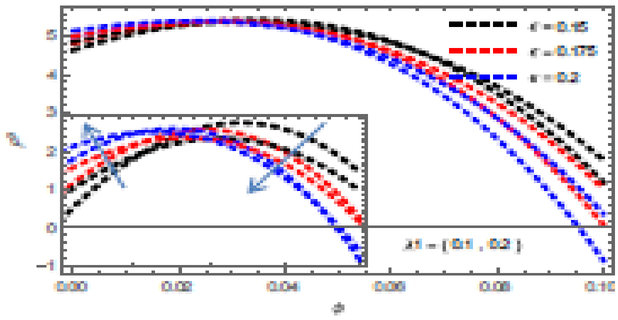


Figure 25: Distribution of F^o *vs.* Ø for various values of $\lambda 1$ and ε with $\Omega = 0.9$, $Rn = 2$, $Pr = 2$, $Sc = 0.5$, $Sr = 0.1$, $Gr = 2$, $\alpha = 3.75$, $\eta = 0.5$, $= 1.1$, $Da = 0.9$, $q2 = 0.5$, $z = 0.1$.

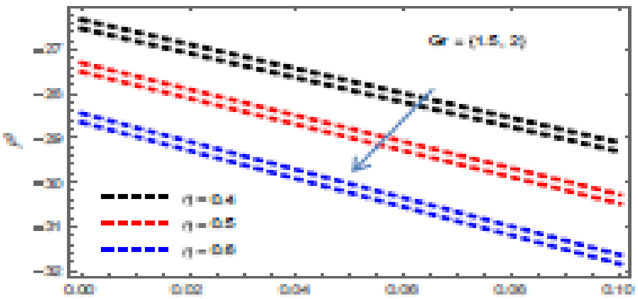


Figure 26: Distribution of F^o *vs.* q2 for various values of Gr and η with $\varepsilon = 0.2$, $\Omega = 0.9$, $\phi = 0.2$, $\lambda 1 = 0.1$, $Rn = 2$, $Pr = 2$, $Sc = 0.5$, $Sr = 0.1$, $\alpha = 3.75$, $M = 1.1$, $Da = 0.9$, $z = 0.1$.

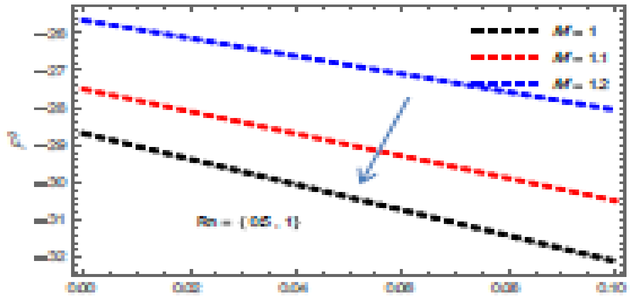


Figure 27: Distribution of F^o *vs.* q2 for various values of M and Rn with $\varepsilon = 0.2$, $\Omega = 0.9$, $\phi = 0.2$, $\lambda 1 = 0.1$, $Pr = 2$, $Gr = 2$, $\eta = 0.5$, $Sc = 0.5$, $Sr = 0.1$, $\alpha = 3.75$, $Da = 0.9$, $z = 0.1$.

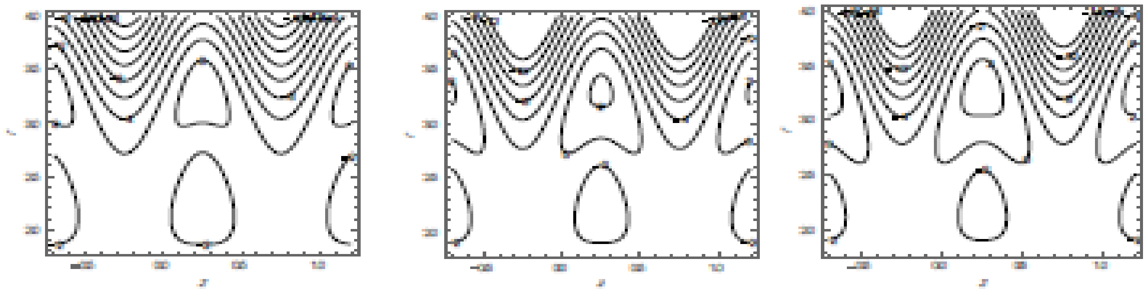


Figure 28: Streamlines in the wave frame for various values of $\varepsilon = \{0.2, 0.225, 0.25\}$ at $\Omega = 0.9$, $Da = 0.9$, $\phi = 0.2$, $\lambda 1 = 0.1$, $Rn = 2$, $Pr = 2$, $q2 = 0.5$, $Gr = 2$, $Sc = 0.5$, $Sr = 0.1$, $M = 1.1$, $\alpha = 3.75$, $\eta = 0.5$.

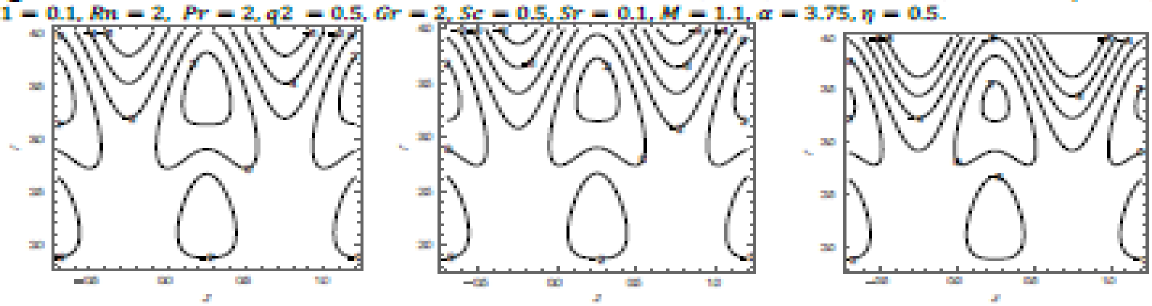


Figure 29: Streamlines in the wave frame for various values of $\Omega = \{0.3, 0.4, 0.5\}$ at $\varepsilon = 0.2$, $Da = 0.9$, $\phi = 0.2$, $\lambda 1 = 0.1$, $Rn = 2$, $Pr = 2$, $q2 = 0.5$, $Gr = 2$, $Sc = 0.5$, $Sr = 0.1$, $M = 1.1$, $\alpha = 3.75$, $\eta = 0.5$.

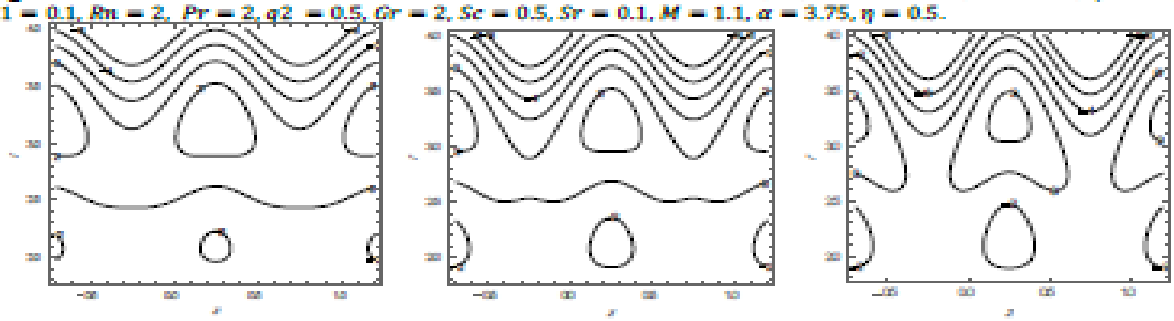


Figure 30: Streamlines in the wave frame for various values of $\phi = \{0.1, 0.125, 0.15\}$ at $\Omega = 0.9$, $\varepsilon = 0.2$, $Da = 0.9$, $\lambda 1 = 0.1$, $Rn = 2$, $Pr = 2$, $q2 = 0.5$, $Gr = 2$, $Sc = 0.5$, $Sr = 0.1$, $M = 1.1$, $\alpha = 3.75$, $\eta = 0.5$.

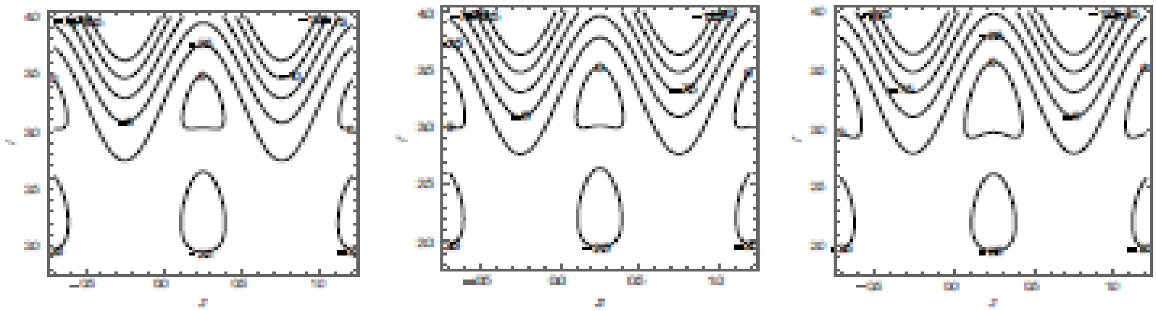


Figure 31: Streamlines in the wave frame for various values of $\lambda 1 = [0.07, 0.075, 0.08]$ at $\phi = 0.2$, $\Omega = 0.9$, $\varepsilon = 0.2$, $Da = 0.9$, $Rn = 2$, $Pr = 2$, $q2 = 0.5$, $Gr = 2$, $Sc = 0.5$, $Sr = 0.1$, $M = 1.1$, $\alpha = 3.75$, $\eta = 0.5$.

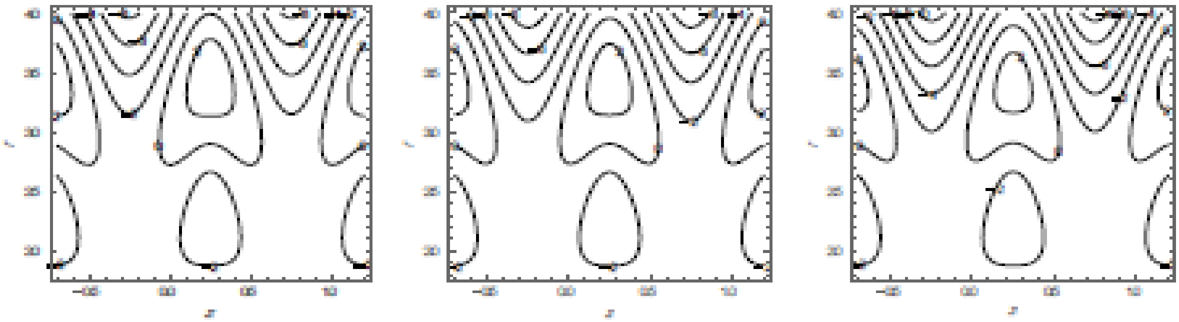


Figure 32: Streamlines in the wave frame for various values of $Rn = [0.5, 0.6, 0.7]$ at $\lambda 1 = 0.1$, $\phi = 0.2$, $\Omega = 0.9$, $\varepsilon = 0.2$, $Da = 0.9$, $Pr = 2$, $q2 = 0.5$, $Gr = 2$, $Sc = 0.5$, $Sr = 0.1$, $M = 1.1$, $\alpha = 3.75$, $\eta = 0.5$.

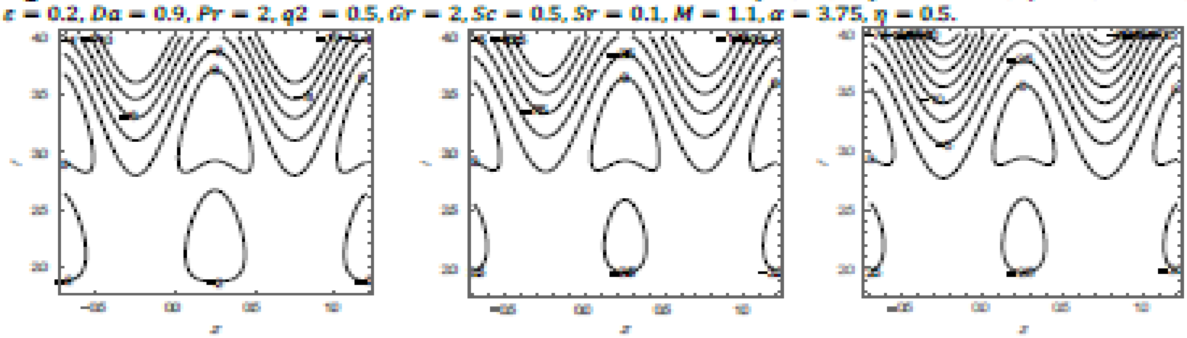


Figure 33: Streamlines in the wave frame for various values of $Pr = [1.5, 2, 2.5, 3]$ at $\lambda 1 = 0.1$, $\phi = 0.2$, $\Omega = 0.9$, $\varepsilon = 0.2$, $Da = 0.9$, $q2 = 0.5$, $Gr = 2$, $Sc = 0.5$, $Sr = 0.1$, $Rn = 2$, $M = 1.1$, $\alpha = 3.75$, $\eta = 0.5$.

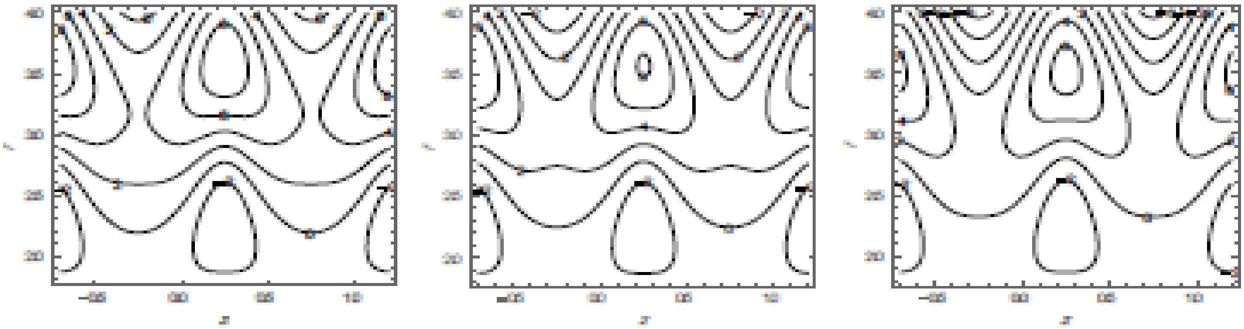


Figure 34: Streamlines in the wave frame for various values of $Gr = [0.5, 0.75, 1]$ at $\lambda 1 = 0.1$, $Rn = 2$, $\phi = 0.2$, $\Omega = 0.9$, $\varepsilon = 0.2$, $Da = 0.9$, $Pr = 2$, $q2 = 0.5$, $Sc = 0.5$, $Sr = 0.1$, $M = 1.1$, $\alpha = 3.75$, $\eta = 0.5$.

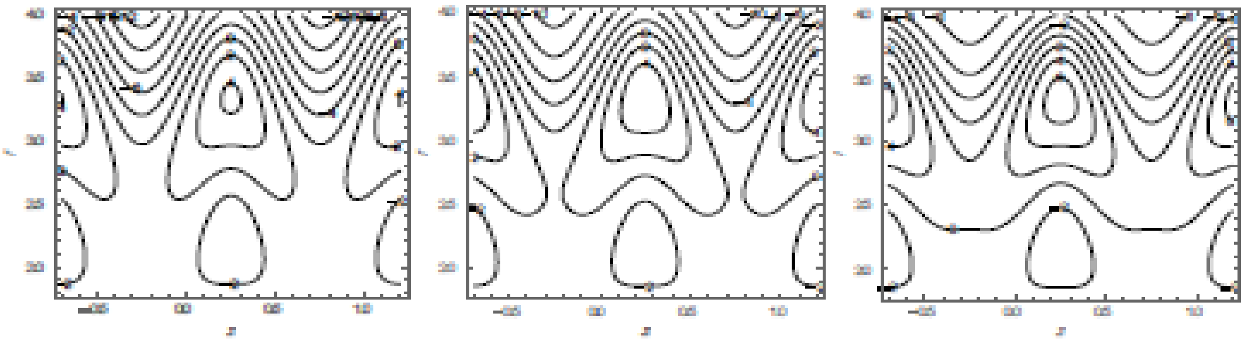


Figure 35: Streamlines in the wave frame for various values of $M = [1.3, 1.4, 1.5]$ at $Gr = 2$, $Pr = 2$ $Rn = 2$, $\lambda 1 = 0.1$, $\varepsilon = 0.2$, $\phi = 0.2$, $Da = 0.9$, $\Omega = 0.9$, $q2 = 0.5$, $Sc = 0.5$, $Sr = 0.1$, $\alpha = 3.75$, $\eta = 0.5$.

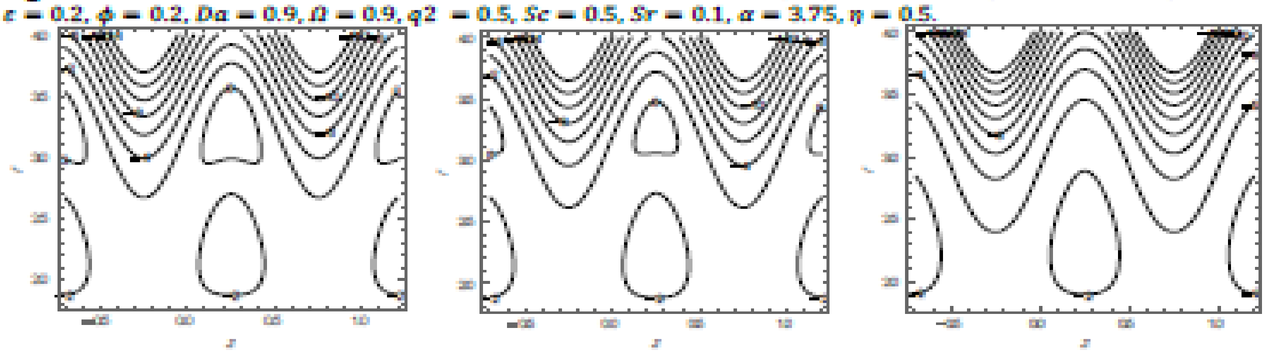


Figure 36: Streamlines in the wave frame for various values of $Da = [1, 1.1, 1.4]$ at $M = 1.1$, $Gr = 2$, $Pr = 2$ $Rn = 2$, $\lambda 1 = 0.1$, $\varepsilon = 0.2$, $\phi = 0.2$, $\Omega = 0.9$, $q2 = 0.5$, $Sc = 0.5$, $Sr = 0.1$, $\alpha = 3.75$, $\eta = 0.5$.

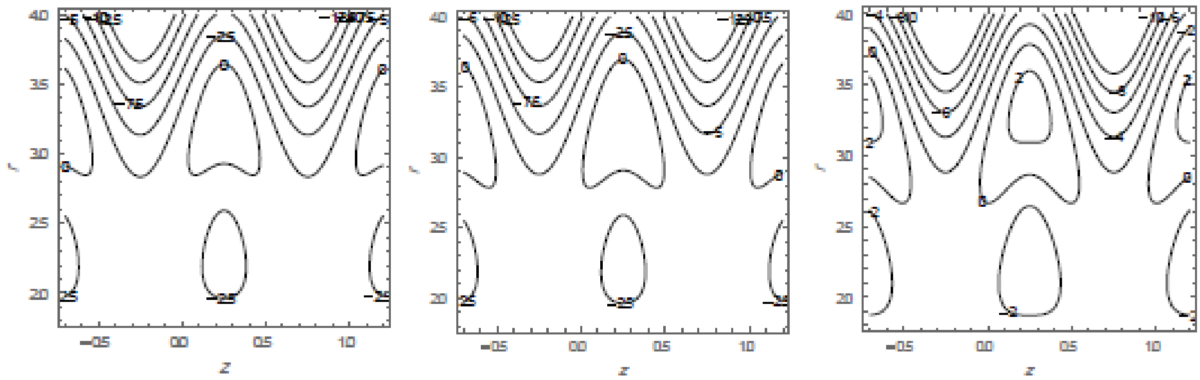


Figure 37: Streamlines in the wave frame for various values of $q2 = \{0.5, 0.52, 0.56\}$ at $\boldsymbol{Da} = \mathbf{0.9}$, $\boldsymbol{M} = \mathbf{1.1}$, $\boldsymbol{Gr} = \mathbf{2}$, $\boldsymbol{Pr} = \mathbf{2}$ $\boldsymbol{Rn} = \mathbf{2}$, $\boldsymbol{\lambda 1} = \mathbf{0.1}$, $\boldsymbol{\varepsilon} = \mathbf{0.2}$, $\boldsymbol{\phi} = \mathbf{0.2}$, $\boldsymbol{\Omega} = \mathbf{0.9}$, $\boldsymbol{Sc} = \mathbf{0.5}$, $\boldsymbol{Sr} = \mathbf{0.1}$, $\boldsymbol{\alpha} = \mathbf{3.75}$, $\boldsymbol{\eta} = \mathbf{0.5}$.

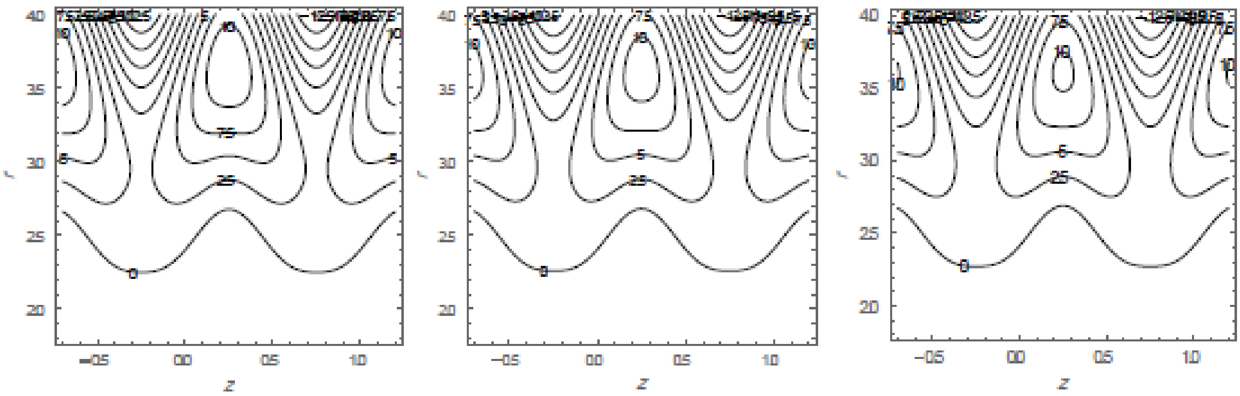


Figure 38: Streamlines in the wave frame for various values of $\alpha = \{3.1, 3.15, 3.2\}$ at $\boldsymbol{q2} = \mathbf{0.5}$, $\boldsymbol{Da} = \mathbf{0.9}$, $\boldsymbol{M} = \mathbf{1.1}$, $\boldsymbol{Gr} = \mathbf{2}$, $\boldsymbol{Pr} = \mathbf{2}$, $\boldsymbol{Rn} = \mathbf{2}$, $\boldsymbol{\lambda 1} = \mathbf{0.1}$, $\boldsymbol{\varepsilon} = \mathbf{0.2}$, $\boldsymbol{\phi} = \mathbf{0.2}$, $\boldsymbol{\Omega} = \mathbf{0.9}$, $\boldsymbol{Sc} = \mathbf{0.5}$, $\boldsymbol{Sr} = \mathbf{0.1}$, $\boldsymbol{\eta} = \mathbf{0.5}$, $\boldsymbol{z} = \mathbf{0.1}$.

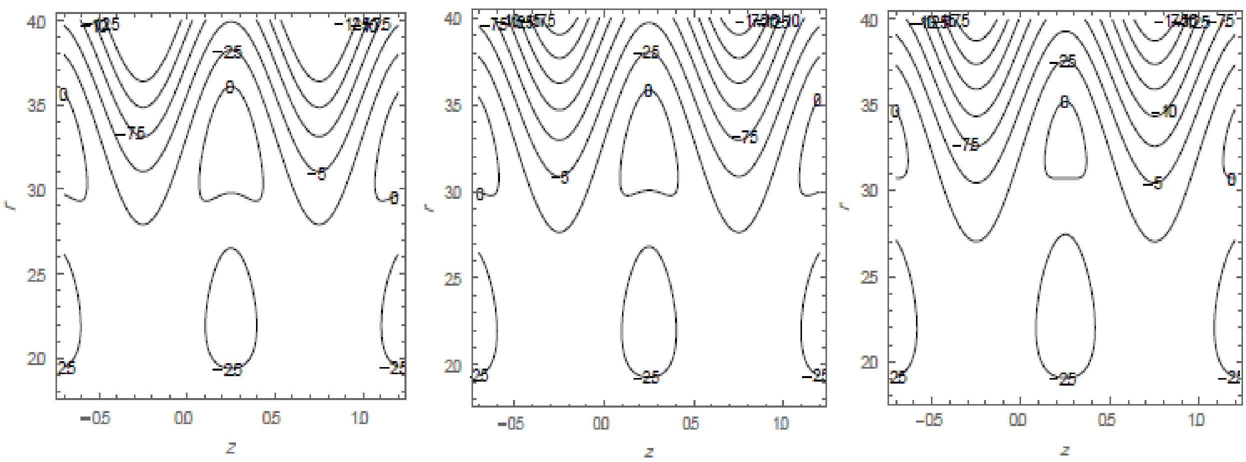


Figure 39: Streamlines in the wave frame for various values of $\eta = \{0.6, 0.65, 0.7\}$ at $\alpha = 3.75$, $q2 = 0.5$, $Da = 0.9$, $M = 1.1$, $Gr = 2$, $Pr = 2$ $Rn = 2$, $\lambda 1 = 0.1$, $\varepsilon = 0.2$, $\phi = 0.2$, $\Omega = 0.9$, $Sc = 0.5$, $Sr = 0.1$.

8. Concluding Remarks:

We briefly discuss the effect of different temperature on peristalsis MHD flow from a couple-stress Jeffrey fluid through the porous channel. Where we discussed the various parameters affecting the movement of the liquid and the pressure generated by the fluid movement, we list below the main points that we reached:

1. The velocity of the fluid increases with the increasing $\in$ and Ω when $r < 0.07$ and decreases otherwise.
2. The velocity of fluid decreases with the increasing , η, $\lambda 1$, ϕ, Gr, $q2$, Rn and Pr when $r < 0.2$ and increases otherwise.
3. The velocity of the fluid increases with the increasing M and Da when $r < 0.2$ and decreases otherwise.
4. The pressure variation dp/dz increases with the increasing α, η, Gr, $q2$, M and Da, while dp/dz decreases with the increasing ε, Ω, $\lambda 1$, ϕ, Pr and Rn.
5. The pressure rise Δp increases with the increasing η, Gr, M and Rn, Δp decreases with the increasing Ω and Da, while Δp decreases with the increasing $\in$ and $\lambda 1$ when $\emptyset < 0.03$, while Δp increases with the increasing $\in$ and $\lambda 1$ when $\emptyset > 0.03$.
6. The friction force at the wall F^i and F^o decreases with the increasing η, Gr, M and Rn, Δp increases with the increasing Ω and Da, while Δp increases with the increasing $\in$ and $\lambda 1$ when $\emptyset < 0.03$, and Δp decreases with the increasing $\in$ and $\lambda 1$ when $\emptyset > 0.03$.
7. The size of the trapped bolus decreases with the increasing ε, Ω and Pr gradually in the middle of the channel while when we approach at the upper wall we notice the increase of the wave with the increasing ε, $\lambda 1$, M and $q2$, respectively.
8. The size of the trapped bolus increases with the increasing $\emptyset$, Da, $q2$ and η in the middle of the channel while when we approach at the upper wall we notice the decrease of the wave with the increasing $\emptyset$, Ω, Pr, Gr, Da, Rn, α and η, respectively.

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Soft Closure Spaces

[1]Shahad Thaar Ekram, [1]Rasha Naser Majeed

University of Baghdad, Faculty of Education for Pure Sciences Abn Al-Haitham, Department of Mathematics,Baghdad, Iraq

rasha.n.m@ihcoedu.uobaghdad.edu.iq

Abstract. In this paper, the concept of soft closure spaces is defined and studied its basic properties. We show that the concept soft closure spaces are a generalization to the concept of $\check{C}$ech soft closure spaces introduced by Krishnaveni and Sekar. In addition, the concepts of subspaces and product spaces are extended to soft closure spaces and discussed some of their properties.

1. Introduction

There are many mathematical tools obtainable for dealing with an imperfect knowledge or for modelling complex systems such as probability theory, fuzzy set theory, rough set theory and also in computer science, engineering, physics, social sciences, economics, and medical sciences, etc. All these tools require the pre-specification of some parameters to start with. To conquer these obstacles, in 1999 Molodtsov [12] proposed a new mathematical tool, namely soft set theory to model uncertainty, which associates a set with a set of parameters. After Molodtsove's activity work, in 2003 Maji et al. [10] presented and studied several basic notions of soft set theory and some operation between two soft sets. The Applications of the theory of soft sets have been in many areas of mathematics. In 2011, Shabir and Naz [14] defined and studied the soft topological space. In 2014, El-Sheikh and Abd El-Latif [5] initiated the notion of supra soft topological spaces, which is wider and more general than the class of soft topological spaces.

The concept of closure space $(\mathcal{M},\mathcal{U})$ were introduced by $\check{C}$ech [3] in 1968, where $\mathcal{U}: P(\mathcal{M}) \to P(\mathcal{M})$ is a mapping defined on the power set $P(\mathcal{M})$ of a set $\mathcal{M}$ satisfying: $(C1)\mathcal{U}(\emptyset) = \emptyset,\ (C2)\mathcal{A} \subseteq \mathcal{U}(\mathcal{A})$ and $(C3)\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{U}(\mathcal{A}) \subseteq \mathcal{U}(\mathcal{B})$, the mapping $\mathcal{U}$ called closure operator on $\mathcal{M}$. A closure operator $\mathcal{U}$ is called $\check{C}$ech closure operator, if $\mathcal{U}$ satisfies: $(C4)\mathcal{U}(\mathcal{A} \cup \mathcal{B}) = \mathcal{U}(\mathcal{A}) \cup \mathcal{U}(\mathcal{B})$ and then $(\mathcal{M},\mathcal{U})$ is called $\check{C}$ech closure space. $\check{C}$ech closure spaces studied by several authors and in several directions. In 1985, Mashhour and Ghanim [11] introduced the concept of $\check{C}$ech closure spaces in fuzzy setting. Independently, in 2014, Gowri and Jegadeesan [7] and Krishnaveni and Sekar [8] defined and studied $\check{C}$ech closure spaces in soft setting. Recently, Majeed [9] using fuzzy soft sets to define the concept of $\check{C}$ech fuzzy soft closure spaces.

In this work, motivated by the theory of soft sets we introduced the notion of soft closure spaces. In Section 3, the concept of soft closure spaces is defined. Also, the notion of closed (respectively, open) soft sets in soft closure spaces is defined and give the basic properties of them with several examples to explain these concepts. In addition, we show our notion of soft closure space in more general than the notion of $\check{C}$ech soft closure spaces that defined by Krishnaveni and Sekar [8] (see Proposition 3.4). Moreover, we find for every soft closure space there exists a supra soft topology associative with it (see Remark 3.18). In Section 4, the soft closure subspace of a soft closure space is defined and studied with details. We discuss the relationships between the closed (respectively, open) soft sets in the soft-cs and its soft-c.subsp (see Proposition 4.7 and Theorems 4.10 and 4.12) Finally, Section 5 is devoted to introduce the notion of the product of soft closure spaces and studied its basic properties.

d) Preliminaries

In this section we recall some basic definitions and results of soft set theory defined and discussed by various authors. Throughout this paper, $\mathcal{M}$ refers to the initial universe, $P(\mathcal{M})$ denote the power set of $\mathcal{M}$ and R is the set of all parameters for $\mathcal{M}$.

Definition 2.1 [12] A soft set $\mathcal{F}_R = (\mathcal{F}, R)$ over the universe set $\mathcal{M}$ is defined by a mapping $\mathcal{F}: R \to P(\mathcal{M})$. Then $\mathcal{F}_R$ **can be represented by the set** $\mathcal{F}_R = \{(r, \mathcal{F}(r)): r \in R \; and \; \mathcal{F}(r) \in P(\mathcal{M})\}$. **We denote the family** of all soft sets over $\mathcal{M}$ is denoted by $\mathcal{SS}(\mathcal{M}, R)$.

Definition 2.2 [10**]** A null soft set, which denoted by $\widetilde{\Phi}_R$, is a soft set $\mathcal{F}_R$ over $\mathcal{M}$ such that for all $r \in R, \mathcal{F}(R) = \emptyset$ (empty set).

Definition 2.3 [10**]** An absolute soft set, which denoted by $\widetilde{\mathcal{M}}$, is a soft set $\mathcal{F}_R$ over $\mathcal{M}$ such that for all $r \in R, \mathcal{F}(r) = \mathcal{M}$.

Definition 2.4 [6] Let $\mathcal{F}_R$ and G_R be two soft sets over $\mathcal{M}$. Then, $\mathcal{F}_R$ is called a soft subset of G_R, denoted $\mathcal{F}_R \sqsubseteq G_R$, if $\mathcal{F}(r) \subseteq G(r)$ for all $r \in R$. $\mathcal{F}_R$ equals G_R, denoted by $\mathcal{F}_R = G_R$ if $\mathcal{F}_R \sqsubseteq G_R$ and $G_R \sqsubseteq \mathcal{F}_R$.

Definition 2.5 [10**]** The union of two soft sets $\mathcal{F}_R$ and G_R over $\mathcal{M}$ is the soft set $\mathcal{H}_R$ defined as b $\mathcal{H}(r) = \mathcal{F}(r) \cup G(r)$ for all $r \in R$. This is denoted by $\mathcal{F}_R \sqcup G_R$. And the soft intersection of $\mathcal{F}_R$ and G_R is the soft set $\mathcal{H}_R$ given by $\mathcal{H}(r) = \mathcal{F}(r) \cap G(r)$ for all $r \in R$ and denoted by, $\mathcal{F}_R \sqcap G_R$.

Definition 2.6 [14] Let $\mathcal{F}_R$ and G_R be two soft sets over $\mathcal{M}$, the difference $\mathcal{H}_R$ of $\mathcal{F}_R$ and G_R is denoted by $\mathcal{F}_R - G_R$, and defined as $\mathcal{H}(r) = \mathcal{F}(r) - G(r)$ for all $r \in R$.

Definition 2.7 [14] The relative complement of a soft set $\mathcal{F}_R$ is denoted by $\mathcal{F}_R^c$, where $\mathcal{F}^c: R \longrightarrow P(\mathcal{M})$ defined as $\mathcal{F}^c(r) = \mathcal{M} - \mathcal{F}(r)$, for all $r \in R$. Clearly, $\mathcal{F}_R^c = \widetilde{\mathcal{M}} - \mathcal{F}_R$.

Definition 2.8 [4, 15] The soft set $\mathcal{F}_R \in \mathcal{SS}(\mathcal{M}, R)$ is called soft point in $\mathcal{M}$, denoted by x_r, if for the element $r \in R, \mathcal{F}(r) = \{x\}$ and $\mathcal{F}(r') = \emptyset$ for every $r' \in R - \{r\}$.

Definition 2.9 [4, 15] The soft point x_r is said to be in the soft set G_R, denoted by $x_r \tilde{\in} G_R$, if for the element $r \in R$, we have $\{x\} \subseteq G(r)$.

Definition 2.10 [2] Let $\mathcal{F}_R \in \mathcal{SS}(\mathcal{M}, R)$ and $G_S \in \mathcal{SS}(\mathcal{Y}, S)$. The Cartesian product $\mathcal{F}_R \times G_S$ is defined by $(\mathcal{F} \times G)_{R\times S}$ where $(\mathcal{F} \times G)_{R\times S}(r, s) = \mathcal{F}(r) \times G(s)$, for all $(r, s) \in R \times S$.

According to this definition the soft set $\mathcal{F}_R \times G_S$ is a soft set over $\mathcal{M} \times \mathcal{Y}$ and its parameter universe is $R \times S$.

The pairs of projections $p_{\mathcal{M}}: \mathcal{M} \times \mathcal{Y} \longrightarrow \mathcal{M}$, $q_R: R \times S \longrightarrow R$ and $p_{\mathcal{Y}}: \mathcal{M} \times \mathcal{Y} \longrightarrow \mathcal{Y}$, $q_S: R \times S \longrightarrow S$ determine morphisms respectively $(p_{\mathcal{M}}, q_R)$ from $\mathcal{M} \times \mathcal{Y}$ to $\mathcal{M}$ and $(p_{\mathcal{Y}}, q_S)$ from $\mathcal{M} \times \mathcal{Y}$ to $\mathcal{Y}$, where

$(p_{\mathcal{M}}, q_R)(\mathcal{F}_R \times G_S) = p_{\mathcal{M}}(\mathcal{F} \times G)_{q_R(R\times S)}$ and $\left(p_{\mathcal{Y}}, q_S\right)(\mathcal{F}_R \times G_S) = p_{\mathcal{Y}}(\mathcal{F} \times G)_{q_S(R\times S)}$. [1].

Definition 2.11 [5] A supra soft topological space is the triple $(\mathcal{M}, \mathcal{T}^*, R)$, where $\mathcal{M}$ is universe set, R is the fixed set of parameters and $\mathcal{T}^*$ is the collection of soft sets over $\mathcal{M}$, which are satisfies:

1- $\tilde{\Phi}_R, \widetilde{\mathcal{M}} \in \mathcal{T}^*$,
2- The union of any number of soft sets in $\mathcal{T}^*$ belongs to $\mathcal{T}^*$.
The members of $\mathcal{T}^*$ are called supra open soft sets. A soft set $\mathcal{F}_R$ is called supra closed soft in $\mathcal{M}$ if, $\widetilde{\mathcal{M}} - \mathcal{F}_R \in \mathcal{T}^*$.

Definition 2.12 [14] Let $\mathcal{Y}$ be a non-empty subset of $\mathcal{M}$ and $\mathcal{F}_R$ be a soft set over $\mathcal{M}$. Then the subsoft set of $\mathcal{F}_R$ over $\mathcal{Y}$ denoted by $\mathcal{F}_R^{\mathcal{Y}}$ is defined as follows $\mathcal{F}^{\mathcal{Y}}(r) = \mathcal{Y} \cap \mathcal{F}(r)$ for all $r \in R$.

In other words that is $\mathcal{F}_R^{\mathcal{Y}} = \tilde{\mathcal{Y}} \sqcap \mathcal{F}_R$ where $\tilde{\mathcal{Y}}$ denotes to the soft set $\mathcal{Y}_R$ over $\mathcal{M}$ for which $\mathcal{Y}(r) = \mathcal{Y}$, for all $r \in R$.

Definition 2.13 [13] Let $(\mathcal{M}, \mathcal{T}^*, R)$ be a supra soft topological space and $\mathcal{Y}$ be a non-empty subset of $\mathcal{M}$. Then, $\mathcal{T}^*{}_{\mathcal{Y}} = \{\mathcal{F}_R^{\mathcal{Y}} : \mathcal{F}_R \in \mathcal{T}^*\}$ is called the supra soft relative topology on $\mathcal{Y}$ and $(\mathcal{Y}, \mathcal{T}^*{}_{\mathcal{Y}}, R)$ is called a supra soft subspace of $(\mathcal{M}, \mathcal{T}^*, R)$.

e) The Basic Structures of Soft Closure Spaces

This section is devoted to introduce the notion of soft closure spaces and discussed the basic properties of these spaces.

Definition 3.1 An operator $\tilde{u}: \mathcal{SS}(\mathcal{M}, R) \longrightarrow \mathcal{SS}(\mathcal{M}, R)$ is called a soft closure operator (soft-co, for short) on $\mathcal{M}$, if for all $\mathcal{F}_R, G_R \in \mathcal{SS}(\mathcal{M}, R)$ the following axioms are satisfied:

($\mathcal{C}1$) $\widetilde{\Phi}_R = \tilde{u}(\widetilde{\Phi}_R)$,

($\mathcal{C}2$) $\mathcal{F}_R \sqsubseteq \tilde{u}(\mathcal{F}_R)$,

($\mathcal{C}3$) $\mathcal{F}_R \sqsubseteq G_R \Longrightarrow \tilde{u}(\mathcal{F}_R) \sqsubseteq \tilde{u}(G_R)$.

The triple $(\mathcal{M}, \tilde{u}, R)$ is called a soft closure space (soft-cs, for short).

Next, we give two examples to explain the notion in Definition 3.1.

Example 3.2 Let $\mathcal{M} = \{a, b, c\}$ and $R = \{r_1, r_2\}$. Define a soft-co $\tilde{u}: \mathcal{SS}(\mathcal{M}, R) \longrightarrow \mathcal{SS}(\mathcal{M}, R)$ as follows:

$$\tilde{u}(\mathcal{F}_R) = \begin{cases} \widetilde{\Phi}_R & if\ \mathcal{F}_R = \widetilde{\Phi}_R, \\ \{(r_1, \{c\}), (r_{2,}\{b\})\} & if\ \mathcal{F}_R \sqsubseteq \{(r_1, \{c\}), (r_{2,}\{b\})\}, \\ \{(r_1, \{b\}), (r_{2,}\{c\})\} & if\ \mathcal{F}_R \sqsubseteq \{(r_1, \{b\}), (r_{2,}\{c\})\}, \\ \widetilde{\mathcal{M}} & other\ wise. \end{cases}$$

Clearly, the soft-co $\tilde{u}$ satisfies the three axioms of Definition 3.1. Hence $(\mathcal{X}, \tilde{u}, R)$ is a soft-cs.

Example 3.3 Let $\mathcal{M} = \{a, b, c\}$ and $R = \{r_{1,} r_2\}$. Define a soft-co $\tilde{u}: \mathcal{SS}(\mathcal{M}, R) \longrightarrow \mathcal{SS}(\mathcal{M}, R)$ as follows:

$$\tilde{u}(\mathcal{F}_R) = \begin{cases} \widetilde{\Phi}_R & if\ \mathcal{F}_R = \widetilde{\Phi}_R, \\ \{(r_{2,}\{c\})\} & if\ \mathcal{F}_R = \{(r_{1,}\{c\})\}, \\ \widetilde{\mathcal{M}} & other\ wise. \end{cases}$$

Then, it clear that the axiom ($\mathcal{C}2$) of Definition 3.1 is not hold because there exists $\mathcal{F}_R \in \mathcal{SS}(\mathcal{M}, R)$, where $\mathcal{F}_R = \{(r_{1,}\{c\}\}$ such that $\{(r_{1,}\{c\}\} \not\sqsubseteq \{(r_{2,}\{c\})\} = \tilde{u}(\mathcal{F}_R)$ and hence $(\mathcal{M}, \tilde{u}, R)$ is not soft-cs.

Now we give the relationship between our definition of soft-cs and the definition of $\check{C}$ech soft closure space introduced in [8].

Proposition 3.4 Every $\check{C}$ech soft closure space is a soft-cs.

Proof: Let $(\mathcal{M}, \tilde{u}, R)$ be a $\check{C}$ech soft-cs. To show $(\mathcal{M}, \tilde{u}, R)$ is soft-cs, it is sufficient to prove the soft-co $\tilde{u}$ satisfies the axioms $(\mathcal{C}3)$ in Definition 3.1. Now, let $\mathcal{F}_R, G_R \in \mathcal{SS}(\mathcal{M}, R)$ such that $\mathcal{F}_R \sqsubseteq G_R$. It is clear that $\tilde{u}(\mathcal{F}_R) \sqsubseteq \tilde{u}(\mathcal{F}_R) \sqcup \tilde{u}(G_R)$. By the axiom $(C3)$ of definition $\check{C}$ech soft closure operator we get, $\tilde{u}(\mathcal{F}_R) \sqsubseteq \tilde{u}(\mathcal{F}_R \sqcup G_R) = \tilde{u}(G_R)$. This implies $\tilde{u}(\mathcal{F}_R) \sqsubseteq \tilde{u}(G_R)$ and hence $\tilde{u}$ is a soft-co and $(\mathcal{M}, \tilde{u}, R)$ is soft-cs.

Remark 3.5 The convers of Proposition 3.4 is not true as the following example shows

Example 3.6 Let $\mathcal{M} = \{a, b\}\ and\ R = \{r_1, r_2\}$. Define a soft-co $\tilde{u}: \mathcal{SS}(\mathcal{M}, R) \longrightarrow \mathcal{SS}(\mathcal{M}, R)$ as follows:

$$\tilde{u}(\mathcal{F}_R) = \begin{cases} \widetilde{\Phi}_R & if \quad \mathcal{F}_R = \widetilde{\Phi}_R, \\ \{(r_1, \{a, b\})\} & if \quad \mathcal{F}_R = \{(r_1, \{a\})\}, \\ \{(r_1, \{b\})\} & if \quad \mathcal{F}_R = \{(r_1, \{b\})\}, \\ \{(r_2, \{b\})\} & if \quad \mathcal{F}_R = \{(r_2, \{b\})\}, \\ \widetilde{\mathcal{M}} & other\ wise. \end{cases}$$

Then, $(\mathcal{M}, \tilde{u}, R)$ is a soft-cs, but it is not čech soft closure space since there exist $\mathcal{F}_R, G_R \in \mathcal{SS}(\mathcal{M}, R)$, where $\mathcal{F}_R = \{(r_1, \{a\})\}$ and $G_R = \{(r_1, \{b\})\}$ such that $\tilde{u}(\mathcal{F}_R \sqcup G_R) \neq \tilde{u}(\mathcal{F}_R) \sqcup \tilde{u}(G_R)$.

Definition 3.7 Let $(\mathcal{M}, \tilde{u}, R)$ be a soft-cs. A soft subset $\mathcal{F}_R$ over $\mathcal{M}$ is said to be a closed soft set, if $\mathcal{F}_R = \tilde{u}(\mathcal{F}_R)$. A soft subset G_R over $\mathcal{M}$ is called an open soft set if it is soft complement $\widetilde{\mathcal{M}} - \mathcal{F}_R$ is closed soft set.

Example 3.8 In Example 3.6, it is clear that $\mathcal{F}_R = \{(r_1, \{b\})\}$. is a closed soft set and its complement $\widetilde{\mathcal{M}} - \mathcal{F}_R = \{(r_1, \{a\}), (r_2, \{a, b\})\}$ is an open soft set. While, the soft set $\mathcal{F}_R = \{(r_1, \{a\})\}$ is not a closed soft set neither open soft set.

Proposition 3.9 Let $(\mathcal{M}, \tilde{u}, R)$ be a soft-cs and $\mathcal{F}_R \in \mathcal{SS}(\mathcal{M}, R)$. If $\tilde{u}(\mathcal{F}_R) \sqsubseteq \mathcal{F}_R$, then $\mathcal{F}_R$ is a closed soft set in $(\mathcal{M}, \tilde{u}, R)$.

Proof: The proof obtained directly from hypothesis and Definition 3.1.

Theorem 3.10 Let $(\mathcal{M}, \tilde{u}, R)$ be a soft-cs and let $G_R \tilde{\in} \mathcal{SS}(\mathcal{M}, R)$. Then, $\tilde{u}(G_R) - G_R$ contains no non-empty open soft subset.

Proof: Let G_R be a soft subset in $(\mathcal{M}, \tilde{u}, R)$ and H_R be a nonempty open soft subset of $\tilde{u}(G_R) - G_R$. Then, there exists a soft point $x_r \tilde{\in} H_R \sqsubseteq \tilde{u}(G_R) - G_R$ this implies $x_r \tilde{\notin} \widetilde{\mathcal{M}} - H_R$. Which is a closed soft set. Therefore, $x_r \tilde{\notin} \widetilde{\mathcal{M}} - H_R = \tilde{u}(\widetilde{\mathcal{M}} - H_R)$. That means, $\tilde{u}(G_R)$ not contained in $\tilde{u}(\widetilde{\mathcal{M}} - H_R)$. Since $H_R \sqsubseteq \tilde{u}(G_R) - G_R$, then $G_R \sqsubseteq \tilde{u}(G_R) - H_R \sqsubseteq \widetilde{\mathcal{M}} - H_R$. From $(\mathcal{C}3)$, we get $\tilde{u}(G_R) \sqsubseteq \tilde{u}(\widetilde{\mathcal{M}} - H_R)$ and this is a contradiction. Therefore, $\tilde{u}(G_R) - G_R$ contains no non-empty open soft set.

Proposition 3.11 Let $(\mathcal{M}, \tilde{u}, R)$ be a soft-cs and $\{(\mathcal{F}_R)_\alpha : \alpha \in \mathcal{J}\}$ be a family of soft subsets over $\mathcal{M}$. Then:

1- $\sqcup_{\alpha\in\mathcal{J}} \tilde{u}((\mathcal{F}_R)_\alpha) \sqsubseteq \tilde{u}(\sqcup_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha)$.
2- $\tilde{u}(\sqcap_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha) \sqsubseteq \sqcap_{\alpha\in\mathcal{J}} \tilde{u}((\mathcal{F}_R)_\alpha)$.

Proof:

1- For all $\alpha \in \mathcal{J}$ we have, $(\mathcal{F}_R)_\alpha \sqsubseteq \sqcup_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha$. From $(\mathcal{C}2)$ of Definition 3.1, we get for all $\alpha \in \mathcal{J}$, $\tilde{u}((\mathcal{F}_R)_\alpha) \sqsubseteq \tilde{u}(\sqcup_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha)$. This implies, $\sqcup_{\alpha\in\mathcal{J}} \tilde{u}((\mathcal{F}_R)_\alpha) \sqsubseteq \tilde{u}(\sqcup_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha)$.
2- For all $\alpha \in \mathcal{J}$, since $\sqcap_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha \sqsubseteq (\mathcal{F}_R)_\alpha$. Then, by $(\mathcal{C}2)$ of Definition 3.1, we have $\tilde{u}(\sqcap_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha) \sqsubseteq \tilde{u}((\mathcal{F}_R)_\alpha)$ for all $\alpha \in \mathcal{J}$. Hence, $\tilde{u}(\sqcap_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha) \sqsubseteq \sqcap_{\alpha\in\mathcal{J}} \tilde{u}((\mathcal{F}_R)_\alpha)$.

Remark 3.12 The inclusion of Proposition 3.11 cannot be replaced by equalities in general as the following example shows.

Example 3.13 Let Let $\mathcal{M} = \{a, b, c\}$ and $R = \{r_1, r_2\}$. Define a soft-co $\tilde{u}: \mathcal{SS}(\mathcal{M}, R) \longrightarrow \mathcal{SS}(\mathcal{M}, R)$ as follows:

$$\tilde{u}(\mathcal{F}_R) = \begin{cases} \widetilde{\Phi}_R & if \quad \mathcal{F}_R = \widetilde{\Phi}_R, \\ \{(r_1, \{\mathrm{a}\})\} & if \quad \mathcal{F}_R = \{(r_1, \{a\})\}, \\ \{(r_1, \{b\})\} & if \quad \mathcal{F}_R = \{(r_1, \{b\})\}, \\ \{(r_2, \{c\})\} & if \quad \mathcal{F}_R = \{(r_2, \{c\})\}, \\ \widetilde{\mathcal{M}} & other\ wise. \end{cases}$$

Then, $(\mathcal{M}, \tilde{u}, R)$ is a soft-cs. Let $\mathcal{F}_R = \{(r_1, \{a\})\}$ and $G_R = \{(r_1, \{b\})\}$, then it is clear that $\tilde{u}(\mathcal{F}_R \sqcup G_R) = \widetilde{\mathcal{M}} \neq \{(r_1, \{a, b\})\} = \tilde{u}(\mathcal{F}_R) \sqcup \tilde{u}(G_R)$.

Also, if we take $\mathcal{F}_R = \{(r_1, \{a\})\}$ and $K_R = \{(r_1, \{b, c\})\}$, then $\tilde{u}(\mathcal{F}_R \sqcap K_R) = \widetilde{\Phi}_R \neq \{(r_1, \{a\})\} = \tilde{u}(\mathcal{F}_R) \sqcap \tilde{u}(K_R)$.

Proposition 3.14 The intersection of any collection of closed soft sets in a soft-cs is a closed soft set.

Proof: Let $\{(\mathcal{F}_R)_\alpha : \alpha \in \mathcal{J}\}$ be a family of closed sets in a soft-cs $(\mathcal{M}, \tilde{u}, R)$. We must prove $\tilde{u}(\sqcap_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha) = \sqcap_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha$. Since $\sqcap_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha \sqsubseteq (\mathcal{F}_R)_\alpha$ for all $\alpha \in \mathcal{J}$, then by $(\mathcal{C}3)$ of Definition 3.1, we get $\tilde{u}(\sqcap_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha) \sqsubseteq \tilde{u}((\mathcal{F}_R)_\alpha) = (\mathcal{F}_R)_\alpha$ (by $(\mathcal{F}_R)_\alpha$ is a closed soft set for all $\in \mathcal{J}$

). This implies $\tilde{u}(\sqcap_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha) \sqsubseteq \sqcap_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha$. On the other hand from $(\mathcal{C}2)$, it follows that $\sqcap_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha \sqsubseteq \tilde{u}(\sqcap_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha)$. Therefore, $\tilde{u}(\sqcap_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha) = \sqcap_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha$. Hence, the result.

Corollary 3.15 The union of any collection of open soft sets in a soft-cs is an open soft set.

Proof: Let $\{(\mathcal{F}_R)_\alpha : \alpha \in \mathcal{J}\}$ be a family of open sets in a soft-cs $(\mathcal{M}, \tilde{u}, R)$. Clearly the complement of $\sqcup_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha$ is $\widetilde{\mathcal{M}} - \sqcup_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha = \sqcap_{\alpha\in\mathcal{J}} (\widetilde{\mathcal{M}} - (\mathcal{F}_R)_\alpha)$. Since $(\mathcal{F}_R)_\alpha$ is an open soft set for all $\alpha \in \mathcal{J}$, then $\widetilde{\mathcal{M}} - (\mathcal{F}_R)_\alpha$ is a closed soft set. By Proposition 3.14, we have $\sqcap_{\alpha\in\mathcal{J}} \widetilde{\mathcal{M}} - (\mathcal{F}_R)_\alpha$ is a closed soft set. Therefore, $\sqcup_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha$ is an open soft set.

Corollary 3.16 Let $\{(\mathcal{F}_R)_\alpha : \alpha \in \mathcal{J}\}$ be a collection of closed soft sets in a soft-cs $(\mathcal{M}, \tilde{u}, R)$. Then, $\tilde{u}(\sqcap_{\alpha\in\mathcal{J}} (\mathcal{F}_R)_\alpha) = \sqcap_{\alpha\in\mathcal{J}} \tilde{u}((\mathcal{F}_R)_\alpha)$.

Proof: The proof follows from Proposition 3.14 and definition of closed soft set.

Remark 3.17 The intersection (respectively, union) of any family of open (respectively, closed) soft sets in a soft-cs $(\mathcal{M}, \tilde{u}, R)$ need not to be an open (respectively, closed) soft set.

To explain that, in Example 3.6, there exist $\mathcal{F}_R = \{(r_1, \{b\})\}$ and $G_R = \{(r_2, \{b\})\}$ are closed soft sets but their union is not a closed soft set. In addition, there exist $H_R = \{(r_1, \{a\}), (r_2, \{a, b\})\}$ and $K_R = \{(r_1, \{a, b\}), (r_2, \{a\})\}$ are open soft sets but $H_R \sqcap K_R = \{(r_1, \{a\}), (r_2, \{a\})\}$ is not an open soft set in $(\mathcal{M}, \tilde{u}, R)$.

Remark 3.18 From Corollary 3.15 and Remark 3.17, it follows for each soft- cs there exists an underlying supra soft topological space that can be defined in a natural way:

Let $(\mathcal{M}, \tilde{u}, R)$ be a soft-cs, we denote the associative supra soft topology on $\mathcal{M}$ by $\mathcal{T}_{\tilde{u}}$. That is
$\mathcal{T}_{\tilde{u}} = \{\widetilde{\mathcal{M}} - \mathcal{F}_R : \tilde{u}(\mathcal{F}_R) = \mathcal{F}_R\}$.

The members of $\mathcal{T}_{\tilde{u}}$ are called supra open soft sets and the complements are called supra closed soft sets.

i.e., $\mathcal{F}_\mathrm{R}$ is an open (respectively, closed) soft set in $(\mathcal{M}, \tilde{u}, R) \Leftrightarrow \mathcal{F}_R$ is a supra open (respectively, closed) soft set in $(\mathcal{M}, \mathcal{T}_{\tilde{u}}, R)$.

Example 3.19 In Example 3.2, the associative supra soft topology on $\mathcal{M}$ is $\mathcal{T}_{\tilde{u}} = \{\widetilde{\Phi}_R, \{(r_1, \{a, b\}), (r_2, \{a, c\})\}, \{(r_1, \{a, c\}), (r_2, \{a, b\})\}, \widetilde{\mathcal{M}}\}$ which is a supra soft topology on $\mathcal{M}$. In addition, $\mathcal{T}_{\tilde{u}}$ is not necessarily to be a soft topology on $\mathcal{M}$ since there exist $\mathcal{F}_R$, $G_R \in \mathcal{T}_{\tilde{u}}$, where $\mathcal{F}_R = \{(r_1, \{a, b\}), (r_2, \{a, c\})\}$ and $G_R = \{(r_1, \{a, c\}), (r_2, \{a, b\})\}$. However, $\mathcal{F}_R \sqcap G_R = \{(r_1, \{a\}), (r_2, \{a\})\} \notin \mathcal{T}_{\tilde{u}}$.

Definition 3.20 Let $\tilde{u}_1$ and $\tilde{u}_2$ be two soft-co's on $\mathcal{M}$. Then $\tilde{u}_1$ is said to be finer than $\tilde{u}_2$, or equivently, $\tilde{u}_2$ is coarser than $\tilde{u}_1$, if $\tilde{u}_1(\mathcal{F}_R) \sqsubseteq \tilde{u}_2(\mathcal{F}_R)$ for all $\mathcal{F}_R \in \mathcal{SS}(\mathcal{M}, R)$.

Now, we give an example to explain the above definition.

Example 3.21 Let $\mathcal{M} = \{a, b, c\}$, and $R = \{r_1, r_2\}$. Define $\tilde{u}_1, \tilde{u}_2 : \mathcal{SS}(\mathcal{M}, R) \longrightarrow \mathcal{SS}(\mathcal{M}, R)$ as follows:

$$\tilde{u}_1(\mathcal{F}_R) = \begin{cases} \widetilde{\Phi}_R & if\ \ \mathcal{F}_R = \widetilde{\Phi}_R, \\ \{(r_{1,}\{a\})\} & if\ \ \mathcal{F}_R = \{(r_{1,}\{a\})\}, \\ \{(r_{2,}\{b\})\} & if\ \ \mathcal{F}_R = \{(r_{2,}\{b\})\}, \\ \{(r_{2,}\{c\})\} & if\ \ \mathcal{F}_R = \{(r_{2,}\{c\})\}, \\ \widetilde{\mathcal{M}} & other\ wise. \end{cases}$$

And,

$$\tilde{u}_2(\mathcal{F}_R) = \begin{cases} \widetilde{\Phi}_R & if\ \ \mathcal{F}_R = \widetilde{\Phi}_R, \\ \{(r_{1,}\{a, b\})\} & if\ \ \mathcal{F}_R = \{(r_{1,}\{a\})\}, \\ \{(r_{2,}\{b, c\})\} & if\ \ \mathcal{F}_R = \{(r_{2,}\{b\})\}, \\ \{(r_{2,}\{c\})\} & if\ \ \mathcal{F}_R = \{(r_{2,}\{c\})\}, \\ \widetilde{\mathcal{M}} & other\ wise. \end{cases}$$

Then, it is easy to verify that $\tilde{u}_1$ and $\tilde{u}_2$ are soft-co's on $\mathcal{M}$ and $\tilde{u}_1$ is finer than $\tilde{u}_2$ since for all $\mathcal{F}_R \in \mathcal{SS}(\mathcal{M}, R)$, $\tilde{u}_1(\mathcal{F}_R) \sqsubseteq \tilde{u}_2(\mathcal{F}_R)$.

Theorem 3.22 Let $\tilde{u}_1$ and $\tilde{u}_2$ be two soft-co's on $\mathcal{M}$. Define $\tilde{u}_1 \sqcup \tilde{u}_2$, $\tilde{u}_1 \sqcap \tilde{u}_2 : \mathcal{SS}(\mathcal{M}, R) \longrightarrow \mathcal{SS}(\mathcal{M}, R)$ as follows: for all $\mathcal{F}_R \in \mathcal{SS}(\mathcal{M}, R)$, $(\tilde{u}_1 \sqcup \tilde{u}_2)(\mathcal{F}_R) = \tilde{u}_1(\mathcal{F}_R) \sqcup \tilde{u}_2(\mathcal{F}_R)$ and $(\tilde{u}_1 \sqcap \tilde{u}_2)(\mathcal{F}_R) = \tilde{u}_1(\mathcal{F}_R) \sqcap \tilde{u}_2(\mathcal{F}_R)$. Then, $\tilde{u}_1 \sqcup \tilde{u}_2$ and $\tilde{u}_1 \sqcap \tilde{u}_2$ are soft-co's on $\mathcal{M}$.

Proof: We prove $\tilde{u}_1 \sqcup \tilde{u}_2$ is a soft-co on $\mathcal{M}$ and similarly one can prove $\tilde{u}_1 \sqcap \tilde{u}_2$ is soft-co on $\mathcal{M}$. Now, we must prove $\tilde{u}_1 \sqcup \tilde{u}_2$ satisfies the axioms $(\mathcal{C}1)$, $(\mathcal{C}2)$ and $(\mathcal{C}3)$ of Definition 3.1.

($\boldsymbol{\mathcal{C}}\mathbf{1}$) $(\tilde{u}_1 \sqcup \tilde{u}_2)(\widetilde{\Phi}_R) = \tilde{u}_1(\widetilde{\Phi}_R) \sqcup \tilde{u}_2(\widetilde{\Phi}_R) = \widetilde{\Phi}_R \sqcup \widetilde{\Phi}_R = \widetilde{\Phi}_R$.

($\boldsymbol{\mathcal{C}}\mathbf{2}$) For all $\mathcal{F}_R \in \mathcal{SS}(\mathcal{M}, R)$. Since $\tilde{u}_1$ and $\tilde{u}_2$ are soft-co's on $\mathcal{M}$, then $\mathcal{F}_R \sqsubseteq \tilde{u}_1(\mathcal{F}_R)$ and $\mathcal{F}_R \sqsubseteq \tilde{u}_2(\mathcal{F}_R)$. This implies $\mathcal{F}_R \sqsubseteq \tilde{u}_1(\mathcal{F}_R) \sqcup \tilde{u}_2(\mathcal{F}_R) = (\tilde{u}_1 \sqcup \tilde{u}_2)(\mathcal{F}_R)$.

($\boldsymbol{\mathcal{C}}\mathbf{3}$) Let $\mathcal{F}_R, G_R \in \mathcal{SS}(\mathcal{M}, R)$ such that $\mathcal{F}_R \sqsubseteq G_R$. Since $\tilde{u}_1$ and $\tilde{u}_2$ are soft-co's on $\mathcal{M}$, then $\tilde{u}_1(\mathcal{F}_R) \sqsubseteq \tilde{u}_1(G_R)$ and $\tilde{u}_2(\mathcal{F}_R) \sqsubseteq \tilde{u}_2(G_R)$. It follows, $\tilde{u}_1(\mathcal{F}_R) \sqcup u_2(\mathcal{F}_R) \sqsubseteq \tilde{u}_1(G_R) \sqcup \tilde{u}_2(G_R)$ which implies, $(\tilde{u}_1 \sqcup u_2)(\mathcal{F}_R) \sqsubseteq (\tilde{u}_1 \sqcup \tilde{u}_2)(G_R)$. Hence, $\tilde{u}_1 \sqcup \tilde{u}_2$ is a soft-co on $\mathcal{M}$.

f) Soft closure subspaces

In this section we introduce the notion of soft closure subspace of a soft-cs and investigate some properties of its.

Theorem 4.1 Let $(\mathcal{M}, \tilde{u}, R)$ be a soft-cs and let $\mathcal{Y} \subseteq \mathcal{M}$. Let $\tilde{u}_{\mathcal{Y}}: \mathcal{SS}(\mathcal{Y}, R) \longrightarrow \mathcal{SS}(\mathcal{Y}, R)$ defined by $\tilde{u}_{\mathcal{Y}}(\mathcal{F}_R) = \widetilde{\mathcal{Y}} \sqcap \tilde{u}(\mathcal{F}_R)$. Then, $\tilde{u}_{\mathcal{Y}}$ is a soft-co on $\mathcal{Y}$.

Proof: We must prove $\tilde{u}_{\mathcal{Y}}$ satisfying the axioms $(\mathcal{C}1) - (\mathcal{C}3)$ of Definition 3.1.

($\boldsymbol{\mathcal{C}}\mathbf{1}$) $\tilde{u}_{\mathcal{Y}}(\widetilde{\Phi}_R) = \widetilde{\mathcal{Y}} \sqcap \tilde{u}_R(\widetilde{\Phi}_R) = \widetilde{\mathcal{Y}} \sqcap \widetilde{\Phi}_R = \widetilde{\Phi}_R$.

($\boldsymbol{\mathcal{C}}\mathbf{2}$) For all $\mathcal{F}_R \in \mathcal{SS}(\mathcal{Y}, R)$, we have $\mathcal{F}_R \sqsubseteq \widetilde{\mathcal{Y}}$ and $\mathcal{F}_R \sqsubseteq \tilde{u}(\mathcal{F}_R)$. This implies $\mathcal{F}_R \sqsubseteq \widetilde{\mathcal{Y}} \sqcap \tilde{u}(\mathcal{F}_R) = \tilde{u}_{\mathcal{Y}}(\mathcal{F}_R)$. Thus, $\mathcal{F}_R \sqsubseteq \tilde{u}_{\mathcal{Y}}(\mathcal{F}_R)$.

($\boldsymbol{\mathcal{C}}\mathbf{3}$) Let $\mathcal{F}_R, G_R \in \mathcal{SS}(\mathcal{Y}, R)$ such that $\mathcal{F}_R \sqsubseteq G_R$. Since $\tilde{u}$ is a soft-co, then $\tilde{u}(\mathcal{F}_R) \sqsubseteq \tilde{u}(G_R)$. Therefore, $\widetilde{\mathcal{Y}} \sqcap \tilde{u}(\mathcal{F}_R) \sqsubseteq \widetilde{\mathcal{Y}} \sqcap \tilde{u}(G_R)$ which means $\tilde{u}_{\mathcal{Y}}(\mathcal{F}_R) \sqsubseteq \tilde{u}_{\mathcal{Y}}(G_R)$.

Definition 4.2 Let $(\mathcal{M}, \tilde{u}, R)$ be a soft-cs, and let $\mathcal{Y} \subseteq \mathcal{M}$. The soft closure operator $\tilde{u}_{\mathcal{Y}}$ (defined in the Theorem 4.1) is called the relative soft closure operator on $\mathcal{Y}$ induced by $\tilde{u}$. The triple $(\mathcal{Y}, \tilde{u}_{\mathcal{Y}}, R)$ is called a soft closure subspace (soft-c.subsp, for short) of $(\mathcal{M}, \tilde{u}, R)$.

Remark 4.3 The soft-c.subsp $(\mathcal{Y}, \tilde{u}_{\mathcal{Y}}, R)$ is a closed (respectively, open) soft subspace if $\tilde{u}(\widetilde{\mathcal{Y}}) = \widetilde{\mathcal{Y}}$ (respectively, $\tilde{u}(\widetilde{\mathcal{M}} - \widetilde{\mathcal{Y}}) = (\widetilde{\mathcal{M}} - \widetilde{\mathcal{Y}})$).

In the next we give an example to explain the notion of soft-c.subsp.

Example 4.4 Let $(\mathcal{M}, \tilde{u}, R)$ be a soft-cs as defined in Example 3.2, where $\mathcal{M} = \{a, b, c\}$, $R = \{r_1, r_2\}$ and $\tilde{u}: \mathcal{SS}(\mathcal{M}, R) \longrightarrow \mathcal{SS}(\mathcal{M}, R)$ defined by

$$\tilde{u}(\mathcal{F}_R) = \begin{cases} \widetilde{\Phi}_R & if\ \mathcal{F}_R = \widetilde{\Phi}_R, \\ \{(r_1, \{c\}), (r_{2,}\{b\})\} & if\ \mathcal{F}_R \sqsubseteq \{(r_1, \{c\}), (r_{2,}\{b\})\}, \\ \{(r_1, \{b\}), (r_{2,}\{c\})\} & if\ \mathcal{F}_R \sqsubseteq \{(r_1, \{b\}), (r_{2,}\{c\})\}, \\ \widetilde{\mathcal{M}} & other\ wise. \end{cases}$$

Let $\mathcal{Y} = \{a, b\} \subseteq \mathcal{M}$, then $\tilde{u}_{\mathcal{Y}}: \mathcal{SS}(\mathcal{Y}, R) \longrightarrow \mathcal{SS}(\mathcal{Y}, R)$ defined as follows: for all $G_R \in \mathcal{SS}(\mathcal{Y}, R)$

$$\tilde{u}_{\mathcal{Y}}(G_R) = \begin{cases} \widetilde{\Phi}_R & if\ G_R = \widetilde{\Phi}_R, \\ \{(r_1, \{b\})\} & if\ G_R = \{(r_{1,}\{b\})\}, \\ \{(r_2, \{b\})\} & if\ G_R = \{(r_{2,}\{b\})\}, \\ \widetilde{\mathcal{Y}} & other\ wise. \end{cases}$$

Then, $(\mathcal{Y}, \tilde{u}_{\mathcal{Y}}, R)$ is soft-c.subsp of $(\mathcal{M}, \tilde{u}, R)$.

Remark 4.5 Let $(\mathcal{M}, \tilde{u}, R)$ be a soft-cs and $(\mathcal{Y}, \tilde{u}_{\mathcal{Y}}, R)$ be a soft-c.subsp of $(\mathcal{M}, \tilde{u}, R)$. If $(\mathcal{M}, \mathcal{T}_{\tilde{u}}, R)$ and $(\mathcal{Y}, \mathcal{T}_{\tilde{u}_{\mathcal{Y}}}, R)$ be the supra soft topological spaces induced form $(\mathcal{M}, \tilde{u}, R)$ and $(\mathcal{Y}, \tilde{u}_{\mathcal{Y}}, R)$ respectively. Then $(\mathcal{Y}, \mathcal{T}_{\tilde{u}_{\mathcal{Y}}}, R)$ is a supra soft subspace of the supra soft topological space $(\mathcal{M}, \mathcal{T}_{\tilde{u}}, R)$.

We can use Example 4.4 to explain Remark 4.5. Therefore,

$\mathcal{T}_{\tilde{u}} = \{\widetilde{\Phi}_R, \widetilde{\mathcal{M}}, \{(r_1, \{a, b\}), (r_2, \{a, c\})\},, \{(r_1, \{a, c\}), (r_2, \{a, b\})\}$ and since $\mathcal{T}_{\tilde{u}_{\mathcal{Y}}} = \{\mathcal{F}_R{}^{\mathcal{Y}}: \mathcal{F}_R \in \mathcal{T}_{\tilde{u}}\}$, then it follows $\mathcal{T}_{\tilde{u}_{\mathcal{Y}}} = \{\widetilde{\Phi}_R, \widetilde{\mathcal{Y}}, \{(r_1, \{a\}), (r_2, \{a, b\})\}, \{(r_1, \{a, b\}), (r_2, \{a\})\}$.

Theorem 4.6 Let $(\mathcal{M}, \tilde{u}, R)$ be a soft-cs and $\mathcal{Y} \subseteq \mathcal{M}$. Then the relative supra soft topology $(\mathcal{T}_{\tilde{u}})_{\mathcal{Y}}$ on $\mathcal{Y}$ induced by $\mathcal{T}_{\tilde{u}}$ is coarser than the associative supra soft topology $\mathcal{T}_{\tilde{u}_{\mathcal{Y}}}$ on $\mathcal{Y}$.

Proof: We must prove $(\mathcal{T}_{\tilde{u}})_{\mathcal{Y}} \subseteq \mathcal{T}_{\tilde{u}_{\mathcal{Y}}}$. Let $\mathcal{F}_R$ be a $(\mathcal{T}_{\tilde{u}})_{\mathcal{Y}}$-closed soft set over $\mathcal{Y}$. Then, there exists a $\mathcal{T}_{\tilde{u}}$-supra closed soft set G_R such that $\mathcal{F}_R = \widetilde{\mathcal{Y}} \sqcap G_R$. Since $\mathcal{F}_R \sqsubseteq G_R$, then $\tilde{u}(\mathcal{F}_R) \sqsubseteq \tilde{u}(G_R) = G_R$. This implies $\tilde{u}_{\mathcal{Y}}(\mathcal{F}_R) = \widetilde{\mathcal{Y}} \sqcap \tilde{u}(\mathcal{F}_R) \sqsubseteq \widetilde{\mathcal{Y}} \sqcap G_R = \mathcal{F}_R$. Therefore, $\tilde{u}_{\mathcal{Y}}(\mathcal{F}_R) = \mathcal{F}_R$ and this implies $\mathcal{F}_R$ is a supra closed soft set in $(\mathcal{Y}, \mathcal{T}_{\tilde{u}_{\mathcal{Y}}}, R)$. Hence, $(\mathcal{T}_{\tilde{u}})_{\mathcal{Y}} \sqsubseteq \mathcal{T}_{\tilde{u}_{\mathcal{Y}}}$.

Proposition 4.7 Let $(\mathcal{Y}, \tilde{u}_y, R)$ be a soft-c.subsp of $(\mathcal{M}, \tilde{u}, R)$. If $\mathcal{F}_R \in \mathcal{SS}(\mathcal{M}, R)$, and $\mathcal{F}_R$ is a closed soft set in $\mathcal{M}$, then $\mathcal{F}_R$ is a closed soft set in $(\mathcal{Y}, \tilde{u}_{\mathcal{Y}}, R)$.

Proof: Let $\mathcal{F}_R \in \mathcal{SS}(\mathcal{M}, R)$ such that $\tilde{u}(\mathcal{F}_R) = \mathcal{F}_R$. Now, $\tilde{u}_y(\mathcal{F}_R) = \tilde{\mathcal{Y}} \sqcap \tilde{u}(\mathcal{F}_R) = \tilde{\mathcal{Y}} \sqcap \mathcal{F}_R = \mathcal{F}_R$. Hence, $\mathcal{F}_R$ is a closed soft set in $(\mathcal{Y}, \tilde{u}_y, R)$.

Remark 4.8 The convers of Proposition 4.7 is not true as the following example shows.

Example 4.9 In Example 4.4, consider $G_R = \{(r_1, \{b\})\}$ which is a closed soft set in $(\mathcal{Y},\tilde{u}_y, R)$ but it is not a closed soft set in $\mathcal{M}$ since $\tilde{u}(G_R) = \{(r_1, \{b\}), (r_2, \{c\})\} \neq G_R$.

The following Theorem give the condition to be the converse of Proposition 4.7 is hold in general.

Theorem 4.10 Let $(\mathcal{M}, \tilde{u}, R)$ be a soft-cs and $(\mathcal{Y},\tilde{u}_y, R)$ be a closed soft subspace of $(\mathcal{M}, \tilde{u}, R)$. If $\mathcal{F}_R$is a closed soft set of $(\mathcal{Y}, \tilde{u}_y, R)$, then $\mathcal{F}_R$ is a closed soft set of $(\mathcal{M}, \tilde{u}, R)$.

Proof: To prove $\mathcal{F}_R$ is a closed soft set of $(\mathcal{M}, \tilde{u}, R)$ we must show $\tilde{u}(\mathcal{F}_R)$=$\mathcal{F}_R$. Since $\mathcal{F}_R$ is a closed soft set of $(\mathcal{Y}, \tilde{u}_y, R)$, then $\tilde{u}_y(\mathcal{F}_R) = \mathcal{F}_R$, which means $\tilde{\mathcal{Y}} \sqcap \tilde{u}(\mathcal{F}_R) = \mathcal{F}_R$. From hypothesis we have $\tilde{u}(\tilde{\mathcal{Y}}) = \tilde{\mathcal{Y}}$. Thus, it follows $\tilde{u}(\tilde{\mathcal{Y}}) \sqcap \tilde{u}(\mathcal{F}_R) = \mathcal{F}_R$. From Proposition 3.11(2), we have $\tilde{u}(\tilde{\mathcal{Y}} \sqcap \mathcal{F}_R) \sqsubseteq \tilde{u}(\tilde{\mathcal{Y}}) \sqcap \tilde{u}(\mathcal{F}_R) = \mathcal{F}_R$. This yield, $\tilde{u}(\mathcal{F}_R) \sqsubseteq \mathcal{F}_R$. On the other hand, $\mathcal{F}_R \sqsubseteq \tilde{u}(\mathcal{F}_R)$. Therefore, we obtain $\tilde{u}(\mathcal{F}_R)$=$\mathcal{F}_R$ and hence $\mathcal{F}_R$ is a closed soft set of $(\mathcal{M}, \tilde{u}, R)$.

Remark 4.11 In Theorem 4.10, the soft set $\tilde{\mathcal{Y}}$ is a closed soft set in $\mathcal{M}$ is a necessary condition for this theorem. We can explain that in more details. In Example 4.4, $\tilde{\mathcal{Y}} = \{(r_1, \{a, b\}), (r_2, \{a, b\})\}$ is not a closed soft set in $(\mathcal{M}, \tilde{u}, R)$ (because $\tilde{u}(\tilde{\mathcal{Y}}) \neq \tilde{\mathcal{Y}}$). Let $G_R = \{(r_1, \{b\})\}$ be a closed soft set $(\mathcal{Y},\tilde{u}_y, R)$. Then, it is clear that G_R is not a closed soft set in $(\mathcal{M}, \tilde{u}, R)$ since $\tilde{u}(G_R) = \{(r_1, \{b\}), (r_2, \{c\})\} \neq G_R$.

Theorem 4.12 Let $(\mathcal{Y},\tilde{u}_y, R)$ be a soft- c.subsp of a soft-cs $(\mathcal{M}, \tilde{u}, R)$. If G_R is an open soft set in $(\mathcal{M}, \tilde{u}, R)$, then $\tilde{\mathcal{Y}} \sqcap G_R$ is an open soft set in $(\mathcal{Y},\tilde{u}_y, R)$.

Proof: To prove $\tilde{\mathcal{Y}} \sqcap G_R$ is an open in $(\mathcal{Y},\tilde{u}_y, R)$, we must show $\tilde{\mathcal{Y}} - (\tilde{\mathcal{Y}} \sqcap G_R)$ is a closed soft set in $(\mathcal{Y},\tilde{u}_y, R)$, i.e., we must show $\tilde{u}_y\big(\tilde{\mathcal{Y}} - (\tilde{\mathcal{Y}} \sqcap G_R)\big) = \tilde{\mathcal{Y}} - (\tilde{\mathcal{Y}} \sqcap G_R)$. Now,

$$\tilde{u}_y\big(\tilde{\mathcal{Y}} - (\tilde{\mathcal{Y}} \sqcap G_R)\big) = \tilde{\mathcal{Y}} \sqcap \tilde{u}(\tilde{\mathcal{Y}}_R - (\tilde{\mathcal{Y}} \sqcap G_R)) \sqsubseteq \tilde{\mathcal{Y}} \sqcap \tilde{u}(\tilde{\mathcal{M}} - G_R)$$

$=\widetilde{\mathcal{Y}} \sqcap (\widetilde{\mathcal{M}} - G_R)$

$=\widetilde{\mathcal{Y}} - (\widetilde{\mathcal{Y}} \sqcap G_R)$.

Remark 4.13 The convers of Theorem 4.12 is not true as the following example shows.

Example 4.14 In Example 4.4, consider the soft set $G_R = \{(r_1, \{b\}), (r_2, \{a, b\})\}$ is an open soft set in $(\mathcal{Y}, \tilde{u}_{\mathcal{Y}}, R)$ since $\tilde{u}_{\mathcal{Y}}(\widetilde{\mathcal{Y}} - G_R) = \widetilde{\mathcal{Y}} - G_R$. But G_R is not an open soft set in $(\mathcal{M}, \tilde{u}, R)$. because $\widetilde{\mathcal{M}} - G_R$ is not a closed soft set in $\mathcal{M}$.

g) The product of soft closure spaces

In this section we define the product of a collection of soft-cs's and gives the properties of open and closed soft sets in the product soft-cs.

Theorem 5.1 Let $\{(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha): \alpha \in \mathcal{I}\}$ be a family of soft-cs's. Define a soft operator $\otimes\tilde{u}: \mathcal{SS}(\prod_{\alpha\in\mathcal{I}} \mathcal{M}_\alpha, \prod_{\alpha\in\mathcal{I}} R_\alpha) \longrightarrow \mathcal{SS}(\prod_{\alpha\in\mathcal{I}} \mathcal{M}_\alpha, \prod_{\alpha\in\mathcal{I}} R_\alpha)$, where $\prod_{\alpha\in\mathcal{I}} \mathcal{M}_\alpha$ and $\prod_{\alpha\in\mathcal{I}} R_\alpha$ denotes to the Cartesian product of the sets $\mathcal{M}_\alpha$ and R_α, $\alpha \in \mathcal{I}$, respectively as follows:

$\otimes\tilde{u}(\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha}) = \prod_{\alpha\in\mathcal{I}} \tilde{u}_\alpha\left(\left(p_{\mathcal{M}_\alpha}, q_{R_\alpha}\right)(\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha})\right), \forall\, \mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha} \in \mathcal{SS}(\prod_{\alpha\in\mathcal{I}} \mathcal{M}_\alpha, \prod_{\alpha\in\mathcal{I}} R_\alpha)$.

Then, the operator $\otimes\tilde{u}$ is a soft closure operator on $\prod_{\alpha\in\mathcal{I}} \mathcal{M}_\alpha$.

Proof: We must prove $\otimes\tilde{u}$ satisfies the axioms $(\mathcal{C}1)$- $(\mathcal{C}3)$ of Definition 3.1.

$(\mathcal{C}1)$ $\otimes\tilde{u}(\widetilde{\Phi}_{\prod_{\alpha\in\mathcal{I}} R_\alpha}) = \prod_{\alpha\in\mathcal{I}} \tilde{u}_\alpha\left(\left(p_{\mathcal{M}_\alpha}, q_{R_\alpha}\right)(\widetilde{\Phi}_{\prod_{\alpha\in\mathcal{I}} R_\alpha})\right) = \prod_{\alpha\in\mathcal{I}} \tilde{u}_\alpha(\widetilde{\Phi}_{R_\alpha}) = \prod_{\alpha\in\mathcal{I}} \widetilde{\Phi}_{R_\alpha} = \widetilde{\Phi}_{\prod_{\alpha\in\mathcal{I}} R_\alpha}$.

$(\mathcal{C}2)$ Let $\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha} \in \mathcal{SS}(\prod_{\alpha\in\mathcal{I}} \mathcal{M}_\alpha, \prod_{\alpha\in\mathcal{I}} R_\alpha)$. For all $\alpha \in \mathcal{I}$, since $\tilde{u}_\alpha$ is a soft-co on $\mathcal{M}_\alpha$, then it follows $\left(p_{\mathcal{M}_\alpha}, q_{R_\alpha}\right)(\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha}) \sqsubseteq \tilde{u}_\alpha\left(\left(p_{\mathcal{M}_\alpha}, q_{R_\alpha}\right)(\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha})\right)$. This implies $\prod_{\alpha\in\mathcal{I}}\left(p_{\mathcal{M}_\alpha}, q_{R_\alpha}\right)(\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha}) \sqsubseteq \prod_{\alpha\in\mathcal{I}} \tilde{u}_\alpha\left(\left(p_{\mathcal{M}_\alpha}, q_{R_\alpha}\right)(\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha})\right)$. Since $\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha} \sqsubseteq \prod_{\alpha\in\mathcal{I}}\left(p_{\mathcal{M}_\alpha}, q_{R_\alpha}\right)(\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha})$, then we have $\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha} \sqsubseteq \prod_{\alpha\in\mathcal{I}} \tilde{u}_\alpha\left(\left(p_{\mathcal{M}_\alpha}, q_{R_\alpha}\right)(\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha})\right) = \otimes\tilde{u}(\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha})$. Therefore, $\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha} \sqsubseteq \otimes\tilde{u}(\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha})$.

$(\mathcal{C}3)$ Let $\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha} \sqsubseteq G_{\prod_{\alpha\in\mathcal{I}} R_\alpha}$. Then, for all $\alpha \in \mathcal{I}$, $\left(p_{\mathcal{M}_\alpha}, q_{R_\alpha}\right)(\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha}) \sqsubseteq \left(p_{\mathcal{M}_\alpha}, q_{R_\alpha}\right)(G_{\prod_{\alpha\in\mathcal{I}} R_\alpha})$. This implies, $\tilde{u}_\alpha\left(\left(p_{\mathcal{M}_\alpha}, q_{R_\alpha}\right)(\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha})\right) \sqsubseteq \tilde{u}_\alpha\left(\left(p_{\mathcal{M}_\alpha}, q_{R_\alpha}\right)(G_{\prod_{\alpha\in\mathcal{I}} R_\alpha})\right)$.

Thus, $\prod_{\alpha\in\mathcal{I}} \tilde{u}_\alpha\left(\left(p_{\mathcal{M}_\alpha}, q_{R_\alpha}\right)(\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha})\right) \sqsubseteq \prod_{\alpha\in\mathcal{I}} \tilde{u}_\alpha\left(\left(p_{\mathcal{M}_\alpha}, q_{R_\alpha}\right)(G_{\prod_{\alpha\in\mathcal{I}} R_\alpha})\right)$ and that means $\otimes\tilde{u}(\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha}) \sqsubseteq \otimes\tilde{u}(G_{\prod_{\alpha\in\mathcal{I}} R_\alpha})$. Hence, we get the result.

Definition 5.2 Let $\{(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha): \alpha \in \mathcal{I}\}$ be a family of soft-cs's. and let $\otimes\tilde{u}$ be the soft-co defined as in Theorem 5.1. Then the triple $(\prod_{\alpha\in\mathcal{I}} \mathcal{M}_\alpha, \otimes\tilde{u}, \prod_{\alpha\in\mathcal{I}} R_\alpha)$ is said to be the product soft-cs of the family $\{(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha): \alpha \in \mathcal{I}\}$.

Example 5.3 Let $\mathcal{M}_1 = \{a, b\}$, $\mathcal{M}_{2=}\{x, y, z\}$, $R_1 = \{r_1, r_2\}$ and $R_2 = \{k_{1,} k_2\}$. Define soft-co's $\tilde{u}_1$ and $\tilde{u}_2$ on $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively as follows:

$\tilde{u}_1: \mathcal{SS}(\mathcal{M}_1, R_1) \longrightarrow \mathcal{SS}(\mathcal{M}_1, R_1)$ defined as

$$\tilde{u}_1(\mathcal{F}_{R_1}) = \begin{cases} \tilde{\Phi}_{R_1} & if\ \mathcal{F}_{R_1} = \tilde{\Phi}_{R_1}, \\ \{(r_{1,}\{a, b\})\} & if\ \mathcal{F}_{R_1} = \{(r_{1,}\{a\})\}, \\ \{(r_{1,}\{b\})\} & if\ \mathcal{F}_{R_1} = \{(r_{1,}\{b\})\}, \\ \{(r_{2,}\{b\})\} & if\ \mathcal{F}_{R_1} = \{(r_{2,}\{b\})\}, \\ \tilde{\mathcal{M}}_1 & otherwise. \end{cases}$$

And, $\tilde{u}_2: \mathcal{SS}(\mathcal{M}_2, R_2) \longrightarrow \mathcal{SS}(\mathcal{M}_2, R_2)$ defined as

$$\tilde{u}_2(\mathcal{F}_{R_2}) = \begin{cases} \tilde{\Phi}_{R_2} & if \mathcal{F}_{R_2} = \tilde{\Phi}_{R_2}, \\ \{(k_{1,}\{x\})\} & if\ \mathcal{F}_{R_2} = \{(k_{1,}\{x\})\}, \\ \tilde{\mathcal{M}}_2 & otherwise. \end{cases}$$

Then, $(\mathcal{M}_1, \tilde{u}_1, R_1)$ and $(\mathcal{M}_2, \tilde{u}_2, R_2)$ are soft-cs's. Let $p_{\mathcal{M}_1}: \mathcal{M}_1 \times \mathcal{M}_2 \longrightarrow \mathcal{M}_1$, $q_{R_1}: R_1 \times R_2 \longrightarrow R_1$ and $p_{\mathcal{M}_2}: \mathcal{M}_1 \times \mathcal{M}_2 \longrightarrow \mathcal{M}_2$, $q_{R_2}: R_1 \times R_2 \longrightarrow R_2$ be the projection maps. Then, $(\mathcal{M}_1 \times \mathcal{M}_2, \otimes\tilde{u}, R_1 \times R_2)$ is the product soft-cs of $(\mathcal{M}_1, \tilde{u}_1, R_1)$ and $(\mathcal{M}_2, \tilde{u}_2, R_2)$, where $\otimes\tilde{u}: \mathcal{SS}(\mathcal{M}_1 \times \mathcal{M}_2, R_1 \times R_2) \longrightarrow \mathcal{SS}(\mathcal{M}_1 \times \mathcal{M}_2, R_1 \times R_2)$ defined as: for all $\mathcal{F}_{R_1\times R_2} \in \mathcal{SS}(\mathcal{M}_1 \times \mathcal{M}_2, R_1 \times R_2)$, $\otimes\tilde{u}(\mathcal{F}_{R_1\times R_2}) = \tilde{u}_1((p_{\mathcal{M}_1}, q_{R_1})(\mathcal{F}_{R_1\times R_2})) \times \tilde{u}_2((p_{\mathcal{M}_2}, q_{R_2})(\mathcal{F}_{R_1\times R_2}))$. For example, if we take $\mathcal{F}_{R_1\times R_2} = \{((r_{1,}, k_{1,}), \{(a, x)\})\}$. Then,

$$\otimes\tilde{u}(\mathcal{F}_{R_1\times R_2}) = \tilde{u}_1((p_{\mathcal{M}_1}, q_{R_1})(\mathcal{F}_{R_1\times R_2})) \times \tilde{u}_2((p_{\mathcal{M}_2}, q_{R_2})(\mathcal{F}_{R_1\times R_2}))$$

$$= \tilde{u}_1(\{(r_{1,}\{a\})\}) \times \tilde{u}_2(\{(k_{1,}\{x\})\})$$

$$= \{(r_{1,}\{a, b\})\} \times \{(k_{1,}\{x\})\}$$

$$= \{((r_{1,}, k_{1,}), \{(a, x), (b, x)\})\}$$

It is clear that, $\mathcal{F}_{R_1\times R_2} \sqsubseteq \otimes\tilde{u}(\mathcal{F}_{R_1\times R_2})$.

Theorem 5.4 Let $\{(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha): \alpha \in \mathcal{I}\}$ be a family of soft-cs's. Then, $\mathcal{F}_{R_\alpha}$ is a closed soft set in $(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha)$ for all $\alpha \in \mathcal{I}$ if and only if $\prod_{\alpha\in\mathcal{I}}\mathcal{F}_{R_\alpha}$ is a closed soft set in $(\prod_{\alpha\in\mathcal{I}}\mathcal{M}_\alpha, \otimes\tilde{u}, \prod_{\alpha\in\mathcal{I}}R_\alpha)$.

Proof: Let $\alpha \in \mathcal{I}$ and $\mathcal{F}_{R_\alpha}$ be a closed soft set of $(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha)$. Then, $\tilde{u}_\alpha(\mathcal{F}_{R_\alpha}) = \mathcal{F}_{R_\alpha}$ for all $\alpha \in \mathcal{I}$. From the definition of soft projection map, it follows, $(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\prod_{\alpha\in\mathcal{I}}\mathcal{F}_{R_\alpha}) = \mathcal{F}_{R_\alpha}$. Hence, $\prod_{\alpha\in\mathcal{I}}\mathcal{F}_{R_\alpha} = \prod_{\alpha\in\mathcal{I}}\tilde{u}_\alpha(\mathcal{F}_{R_\alpha}) = \prod_{\alpha\in\mathcal{I}}\tilde{u}_\alpha((p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\prod_{\alpha\in\mathcal{I}}\mathcal{F}_{R_\alpha})) = \otimes\tilde{u}(\prod_{\alpha\in\mathcal{I}}\mathcal{F}_{R_\alpha})$. That means, $\prod_{\alpha\in\mathcal{I}}\mathcal{F}_{R_\alpha} = \otimes\tilde{u}(\prod_{\alpha\in\mathcal{I}}\mathcal{F}_{R_\alpha})$. Hence, $\prod_{\alpha\in\mathcal{I}}\mathcal{F}_{R_\alpha}$ is a closed soft set in $(\prod_{\alpha\in\mathcal{I}}\mathcal{M}_\alpha, \otimes\tilde{u}, \prod_{\alpha\in\mathcal{I}}R_\alpha)$.

Conversely, Let $\alpha \in \mathcal{I}$ and $\mathcal{F}_{R_\alpha} \in \mathcal{SS}(\mathcal{M}_\alpha, R_\alpha)$, to prove $\tilde{u}_\alpha(\mathcal{F}_{R_\alpha}) = \mathcal{F}_{R_\alpha}$. From hypothesis we have $\prod_{\alpha\in\mathcal{I}}\mathcal{F}_{R_\alpha}$ is a closed soft set in $(\prod_{\alpha\in\mathcal{I}}\mathcal{M}_\alpha, \otimes\tilde{u}, \prod_{\alpha\in\mathcal{I}}R_\alpha)$. This means $\prod_{\alpha\in\mathcal{I}}\mathcal{F}_{R_\alpha} = \otimes\tilde{u}(\prod_{\alpha\in\mathcal{I}}\mathcal{F}_{R_\alpha}) = \prod_{\alpha\in\mathcal{I}}\tilde{u}_\alpha\left((p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\prod_{\alpha\in\mathcal{I}}\mathcal{F}_{R_\alpha})\right)$. By compute the soft projection, we get $(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\prod_{\alpha\in\mathcal{I}}\mathcal{F}_{R_\alpha}) = (p_{\mathcal{M}_\alpha}, q_{R_\alpha})\left(\prod_{\alpha\in\mathcal{I}}\tilde{u}_\alpha\,(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\prod_{\alpha\in\mathcal{I}}\mathcal{F}_{R_\alpha})\right)$. It follows,

$\mathcal{F}_{R_\alpha} = \tilde{u}_\alpha\left((p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\prod_{\alpha\in\mathcal{I}}\mathcal{F}_{R_\alpha})\right) = \tilde{u}_\alpha(\mathcal{F}_{R_\alpha})$. Therefore, $\mathcal{F}_{R_\alpha}$ is a closed soft set in $(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha)$ for all $\alpha \in \mathcal{I}$.

Lemma 5.5 Let $\{(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha): \alpha \in \mathcal{I}\}$ be a collection of soft-cs's and $\nu \in \mathcal{I}$. If $G_{\prod_{\alpha\in\mathcal{I}}R_\alpha} \sqsubseteq \widetilde{\prod_{\alpha\in\mathcal{I}}\mathcal{M}_\alpha}$ and $((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{I}} \tilde{\in} G_{\prod_{\alpha\in\mathcal{I}}R_\alpha}$, then $\{x_{\nu_{r_\nu}}\} \times \prod_{\substack{\alpha\neq\nu \\ \alpha\in\mathcal{I}}}\{(p_{\mathcal{X}_\alpha}, q_{R_\alpha})((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{I}}\} \sqsubseteq \widetilde{\prod_{\alpha\in\mathcal{I}}\mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{I}}R_\alpha}$, for all $x_{\nu_{r_\nu}} \tilde{\in} \widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{I}}R_\alpha})$.

Proof: Let $G_{\prod_{\alpha\in\mathcal{I}}R_\alpha} \sqsubseteq \widetilde{\prod_{\alpha\in\mathcal{I}}\mathcal{M}_\alpha}$ and $((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{I}} \tilde{\in} G_{\prod_{\alpha\in\mathcal{I}}R_\alpha}$. Let $\nu \in \mathcal{I}$ and $x_{\nu_{r_\nu}} \tilde{\in} \widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{I}}R_\alpha})$. Then, $x_{\nu_{r_\nu}} \tilde{\notin} (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{I}}R_\alpha})$. Since $((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{I}} \tilde{\in} G_{\prod_{\alpha\in\mathcal{I}}R_\alpha}$, then $(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{I}}) \tilde{\in} (p_{\mathcal{M}_\alpha}, q_{R_\alpha})(G_{\prod_{\alpha\in\mathcal{I}}R_\alpha})$ for all $\alpha \in \mathcal{I}$. That means, $\prod_{\alpha\in\mathcal{I}}\{(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{I}})\} \sqsubseteq \prod_{\alpha\in\mathcal{I}}\{(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(G_{\prod_{\alpha\in\mathcal{I}}R_\alpha})\}$. Thus, $\{x_{\nu_{r_\nu}}\} \times \prod_{\substack{\alpha\neq\nu \\ \alpha\in\mathcal{I}}}\{(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{I}})\} \not\sqsubseteq (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{I}}R_\alpha}) \times \prod_{\substack{\alpha\neq\nu \\ \alpha\in\mathcal{I}}}\{(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(G_{\prod_{\alpha\in\mathcal{I}}R_\alpha})\} = \prod_{\alpha\in\mathcal{I}}\{(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(G_{\prod_{\alpha\in\mathcal{I}}R_\alpha})\}$. Clearly from the properties of the projection maps, $G_{\prod_{\alpha\in\mathcal{I}}R_\alpha} \sqsubseteq \prod_{\alpha\in\mathcal{I}}\{(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(G_{\prod_{\alpha\in\mathcal{I}}R_\alpha})\}$. Consequently, $\{x_{\nu_{r_\nu}}\} \times \prod_{\substack{\alpha\neq\nu \\ \alpha\in\mathcal{I}}}\{(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{I}})\} \not\sqsubseteq G_{\prod_{\alpha\in\mathcal{I}}R_\alpha}$. But, $\{x_{\nu_{r_\nu}}\} \times \prod_{\substack{\alpha\neq\nu \\ \alpha\in\mathcal{I}}}\{(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{I}})\}$ is a soft point. Thus, $\{x_{\nu_{r_\nu}}\} \times \prod_{\substack{\alpha\neq\nu \\ \alpha\in\mathcal{I}}}\{(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{I}})\} \tilde{\in} \widetilde{\prod_{\alpha\in\mathcal{I}}\mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{I}}R_\alpha}$. ■

Lemma 5.6 Let $\{(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha): \alpha \in \mathcal{I}\}$ be a collection of soft-cs's and let $\nu \in \mathcal{I}$. If $G_{\prod_{\alpha\in\mathcal{I}}R_\alpha} \sqsubseteq \widetilde{\prod_{\alpha\in\mathcal{I}}\mathcal{M}_\alpha}$, then $\widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{I}}R_\alpha}) \sqsubseteq (p_{\mathcal{M}_\nu}, q_{R_\nu})(\widetilde{\prod_{\alpha\in\mathcal{I}}\mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{I}}R_\alpha})$.

Proof: Let $\nu \in \mathcal{J}$ and $G_{\prod_{\alpha\in\mathcal{J}} R_\alpha} \sqsubseteq \widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha}$. If $G_{\prod_{\alpha\in\mathcal{J}} R_\alpha} = \widetilde{\Phi}_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$, then $(p_{\mathcal{M}_\nu}, q_{R_\nu})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}) = (p_{\mathcal{M}_\nu}, q_{R_\nu})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - \widetilde{\Phi}_{\prod_{\alpha\in\mathcal{J}} R_\alpha}) = \widetilde{\mathcal{M}}_\nu$. Since $\widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}) \sqsubseteq \widetilde{\mathcal{M}}_\nu$, then $\widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}) \sqsubseteq (p_{\mathcal{M}_\nu}, q_{R_\nu})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$. If $G_{\prod_{\alpha\in\mathcal{J}} R_\alpha} \neq \widetilde{\Phi}_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$, then there exists a soft point $((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}} \widetilde{\in} G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$. Let $x_{\nu_{r_\nu}} \widetilde{\in} \widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$. Then by Lemma 5.5 we have $\{x_{\nu_{r_\nu}}\} \times \prod_{\substack{\alpha\neq\nu \\ \alpha\in\mathcal{J}}} \{((p_{\mathcal{M}_\alpha}, q_{R_\alpha})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}})\} \sqsubseteq \widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$. It follows that $(p_{\mathcal{M}_\nu}, q_{R_\nu})(\{x_{\nu_{r_\nu}}\} \times \prod_{\substack{\alpha\neq\nu \\ \alpha\in\mathcal{J}}} \{(p_{\mathcal{M}_\alpha}, q_{R_\alpha})((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}})\}) \sqsubseteq (p_{\mathcal{M}_\nu}, q_{R_\nu})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$.

This implies $x_{\nu_{r_\nu}} \widetilde{\in} (p_{\mathcal{M}_\nu}, q_{R_\nu})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$. Therefore, $\widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}) \sqsubseteq (p_{\mathcal{M}_\nu}, q_{R_\nu})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$.

Theoorem 5.7 Let $\{(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha): \alpha \in \mathcal{J}\}$ be a family of soft-cs's. If $G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$ is an open soft set in the product soft closure space $(\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha, \otimes\tilde{u}, \prod_{\alpha\in\mathcal{J}} R_\alpha)$, then$(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$ is an open soft set in $(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha)$ for all $\alpha \in \mathcal{J}$.

Proof: Let $G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$ be an open soft set of $(\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha, \otimes\tilde{u}, \prod_{\alpha\in\mathcal{J}} R_\alpha)$. Then, $\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$ is a closed soft set in $(\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha, \otimes\tilde{u}, \prod_{\alpha\in\mathcal{J}} R_\alpha)$. That is mean, $\otimes\tilde{u}(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}) = \widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$. From the definition of $\otimes\tilde{u}$ we obtain, $\prod_{\alpha\in\mathcal{J}} \tilde{u}_\alpha((p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}) = \widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$.

Suppose that there exists $\nu \in \mathcal{J}$ such that$(p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$ is not open soft set in $(\mathcal{M}_\nu, \tilde{u}_\nu, R_\nu)$. Since $\widetilde{\mathcal{M}}_\nu$ is an open soft set in $(\mathcal{M}_\nu, \tilde{u}_\nu, R_\nu)$ and $(p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}) \sqsubseteq \widetilde{\mathcal{M}}_\nu$ this implies $(p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}) \neq \widetilde{\mathcal{M}}_\nu$, which means $\widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}) \neq \widetilde{\Phi}_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$. Hence, there exists a soft point $a_{\nu_{r_\nu}} \widetilde{\in} \widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$. From $(\mathcal{C}2)$ of Definition 3.1, we have $\widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}) \sqsubseteq \tilde{u}_\nu\left(\widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})\right)$. Thus, $\tilde{u}_\nu\left(\widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})\right)$ is not contained in $\widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$. Hence, there exists soft point $x_{\nu_{r'_\nu}}$ such that $x_{\nu_{r'_\nu}} \widetilde{\in} \tilde{u}_\nu\left(\widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})\right)$ and $x_{\nu_{r'_\nu}} \widetilde{\notin} \widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$, i.e., $x_{\nu_{r'_\nu}} \widetilde{\in} (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$. Hence, there exists a soft point $((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}} \widetilde{\in} G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$ such that $(p_{\mathcal{M}_\nu}, q_{R_\nu})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}}) = x_{\nu_{r'_\nu}}$. For all $a_{\nu_{r_\nu}} \widetilde{\in} \widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$ we have $\{a_{\nu_{r_\nu}}\} \times \prod_{\substack{\alpha\neq\nu \\ \alpha\in\mathcal{J}}} \{(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}})\} \sqsubseteq \widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$. By compute the soft projection for the last inclusion we get $(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\{a_{\nu_{r_\nu}}\} \times \prod_{\substack{\alpha\neq\nu \\ \alpha\in\mathcal{J}}} \{(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}})\}) \sqsubseteq (p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$ this implies

$(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}}) \sqsubseteq (p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}) \sqsubseteq \tilde{u}_\alpha\left((p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})\right)$. From Lemma 5.6, we have $\widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}) \sqsubseteq (p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} -$

$G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$. From $(\mathcal{C}3)$ of Definition 3.1, we have $\tilde{u}_\nu\left(\widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})\right) \sqsubseteq \tilde{u}_\nu((p_{\mathcal{M}_\nu}, q_{R_\nu})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}))$. Since $x_{\nu_{r'_\nu}} \widetilde{\in} \tilde{u}_\nu\left(\widetilde{\mathcal{M}}_\nu - (p_{\mathcal{M}_\nu}, q_{R_\nu})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})\right)$, then

$x_{\nu_{r'_\nu}} \widetilde{\in} \tilde{u}_\nu(p_{\mathcal{M}_\nu}, q_{R_\nu})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$. Thus, it follows $\{x_{\nu_{r'_\nu}}\} \times \prod_{\substack{\alpha\neq\nu\\ \alpha\in\mathcal{J}}} \{(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}})\} \sqsubseteq \prod_{\alpha\in\mathcal{J}} \tilde{u}_\alpha((p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}))$.
But $x_{\nu_{r'_\nu}} = (p_{\mathcal{M}_\nu}, q_{R_\nu})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}})$ this yields

$$\begin{aligned}
\{x_{\nu_{r'_\nu}}\} &\times \prod_{\substack{\alpha\neq\nu\\ \alpha\in\mathcal{J}}} \{(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}})\} \\
&= \{(p_{\mathcal{M}_\nu}, q_{R_\nu})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}}) \times \prod_{\substack{\alpha\neq\nu\\ \alpha\in\mathcal{J}}} \{(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}})\}\}
\end{aligned}$$

$$= \prod_{\alpha\in\mathcal{J}} \{(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}})\}$$

$$= \{((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}}\}$$

Consequently, $\{((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}}\} \sqsubseteq \prod_{\alpha\in\mathcal{J}} \tilde{u}_\alpha(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$. Therefore, $((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}} \widetilde{\in} \prod_{\alpha\in\mathcal{J}} \tilde{u}_\alpha(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}) = \widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$. But $((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}} \widetilde{\in} G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$, then $((x_\alpha)_{(r_\alpha)})_{\alpha\in\mathcal{J}} \widetilde{\notin} \widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$ which implies $\prod_{\alpha\in\mathcal{J}} \tilde{u}_\alpha((p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})) \not\sqsubseteq \widetilde{\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha} - G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$. That means $G_{\prod_{\alpha\in\mathcal{J}} R_\alpha}$ is not an open soft set in $(\prod_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha, \otimes\tilde{u}, \prod_{\alpha\in\mathcal{J}} R_\alpha)$which is a contradiction. Therefore, $(p_{\mathcal{M}_\alpha}, q_{R_\alpha})(G_{\prod_{\alpha\in\mathcal{J}} R_\alpha})$ is an open soft set in $(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha)$ for all $\alpha \in \mathcal{J}$.

Remark 5.8 The converse of Theorem 5.7 is not hold in general as the following example shows:

Example 5.9 Let $\mathcal{M}_1 = \{a, b, c\}, \mathcal{M}_2 = \{x, y, z\}$ and $R_1 = \{r_1, r_2\},\ R_2 = \{r_3, r_4\}$. Define soft-co's $\tilde{u}_1$ and $\tilde{u}_2$ on $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively as follows:

$$\tilde{u}_1(\mathcal{F}_{R_1}) = \begin{cases} \widetilde{\Phi}_{R_1} & if\ \ \mathcal{F}_{R_1} = \widetilde{\Phi}_{R_1}, \\ \mathcal{F}_{R_1} & if\ \ \mathcal{F}_{R_1} = \{(r_1,\{c\}), (r_2,\{c\})\}, \\ \mathcal{F}_{R_1} & if\ \ \mathcal{F}_{R_1} = \{(r_2,\{a\}), (r_2,\{a\})\}, \\ \widetilde{\mathcal{M}}_1 & other\ wise. \end{cases}$$

And

$$\tilde{u}_2(\mathcal{F}_{R_2}) = \begin{cases} \widetilde{\Phi}_{R_2} & if\ \mathcal{F}_{R_1} = \widetilde{\Phi}_{R_1}, \\ \mathcal{F}_{R_2} & if\ \mathcal{F}_{R_1} = \{(r_{3,}\{x\}), (r_{4,}\{x\})\}, \\ \mathcal{F}_{R_2} & if\ \mathcal{F}_{R_1} = \{(r_{3,}\{z\}), (r_{4,}\{z\})\}, \\ \widetilde{\mathcal{M}_2} & other\ wise. \end{cases}$$

Then, $(\mathcal{M}_1, \tilde{u}_1, R_1)$ and $(\mathcal{M}_2, \tilde{u}_2, R_2)$ are soft-cs's. Let $(p_{\mathcal{M}_1}, q_{R_1})$ and $(p_{\mathcal{M}_2}, q_{R_2})$ be the soft projection maps. Consider $G_{R_1 \times R_2} \in \mathcal{SS}(\mathcal{M}_1 \times \mathcal{M}_2, R_1 \times R_2)$, where $G_{R_1 \times R_2} = \{((r_1, r_3), \{(a,x), (a,y), (b,x), (b,y)\}), ((r_1, r_4), \{(a,x), (a,y), (b,x), (b,y)\}),$

$((r_2, r_3), \{(a,x), (a,y), (b,x), (b,y)\}), ((r_2, r_4), \{(a,x), (a,y), (b,x), (b,y)\})\}.$

Then, $(p_{\mathcal{M}_1}, q_{R_1})(G_{R_1 \times R_2}) = \{(r_1, \{a,b\}), (r_2, \{a,b\})\}$, and
$(p_{\mathcal{M}_2}, q_{R_2})(G_{R_1 \times R_2}) = \{(r_3, \{x,y\}), (r_4, \{x,y\})$ are open soft sets in $(\mathcal{M}_1, \tilde{u}_1, R_1)$ and $(\mathcal{M}_2, \tilde{u}_2, R_2)$, respectively. But $G_{R_1 \times R_2}$ is not an open in $(\mathcal{M}_1 \times \mathcal{M}_2, \otimes\tilde{u}, R_1 \times R_2)$. Since $\widetilde{\mathcal{M}_1 \times \mathcal{M}_2} - G_{R_1 \times R_2}$ is not closed soft set in $(\mathcal{M}_1 \times \mathcal{M}_2, \otimes\tilde{u}, R_1 \times R_2)$.

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Application of Algebraic Geometry In Three Dimensional projective space PG (3,7)

Ali Ahmed A. Abdulla1, **Nada Yassen Kasm Yahya2**

1Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Mosul, Iraq.

2Department of Mathematics College of Education for Pure Science, University of Mosul, Mosul, Iraq

drnadaqasim1@gmail.com

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Abstract

The main goal of this work is to construct surfaces and complete arcs in the projective 3 – space PG (3, q) over Galois fields GF (p), p=7. Which represents applications of algebraic geometry in three-dimensional projective space PG (3, P), where p=7 which is a (k, ℓ)-span. We get the following results. First, we found the points, lines and planes in PG (3,7) and we construct (k, ℓ)-span which is a set of k lines no two of which intersect. We prove that the maximum complete (k, ℓ)-span in PG (3,7) is (50, ℓ)-span, which is the equal to all the points of the space that is called a spread. Second in general we prove geometrical rule the total number of Spread in projective space PG (3, p) where p is prime, $P \geq 2 \; is \; p^2 + 1$.

1. Introduction

Hirschfeld, J.W.P. (1998) studied the basic definition and theorems of projective geometrics over finite fields[20]. In2008, Al-Mokhtar study the complete arcs and surface in three-dimensional projective space over Galois field GF(P), p=2, 3[3]. Kareem viewed (k, ℓ)-span in PG(3,p) over Galois field GF (p), p=4 in 2013[2]. In three-dimensional projective space, the control problem is how to construct and finding the whole space spread which is (50,ℓ)-span in PG (3, 7) and prove it in general when $P \geq 2 \; is \; p^2 + 1$.
This paper include three sections, first section consider the preliminaries of projective 3-space which contains some definition and theorems for the concept, whereas the second section consists of the subspace in PG (3, p). Finally, the third section construction on maximum complete (k, ℓ)-span in PG (3, 7) is spread, and in general prove that Geometric rule theorem (2.3) $P \geq 2$. The total number of (k, ℓ)-span in PG (3, q) is p^2+1, $P \geq 2$.

2. Preliminaries

Definition 1.1: "Plane π", [1]

A plane π in PG (3, p) is the set of all points $P(X_1, X_2, X_3, X_4)$ satisfying a linear equation $U_1X_1 + U_2X_2 + U_3X_3 + U_4X_4 = 0$. This plane is denoted by $\pi\,[U_1, U_2, U_3, U_4]$.
Space which consists of points, lines and planes with the incidence relation between them.

Theorem 1.2: [1]
A projective 3-space PG (3, k) over a field K is a 3-dimensional projective PG (3, k) satisfying the following axioms:
1. Any two distinct points are contained in a unique line.
2. Any three distinct non-collinear points, also any line and point not on the line are contained in a unique plane.
3. Any two distinct coplanar lines intersect in a unique point.
4. Any line not on a given plane intersects the plane in a unique point.

5. Any two distinct planes intersect in a unique line.
A projective space PG (3, p) over Galois field GF (p), where $p = q^m$ For some prime number q and some integer m, is a 3-dimensional projective space.

Any point in PG (3, p) has the form of a quadruple (X_1, X_2, X_3, X_4), where X_1, X_2, X_3, X_4 are elements in GF(p) with the exception of the quadrable consisting Lines must be arranged four zero elements. Two quadrable (X_1, X_2, X_3, X_4) and (y_1, y_2, y_3, y_4) represent the same point if there exists λ in GF(p)/{0} such that $(X_1, X_2, X_3, X_4) = \lambda(y_1, y_2, y_3, y_4)$. Similarly, any plane in PG (3, p) has the form of a quadrable $[X_1, X_2, X_3, X_4]$, where X_1, X_2, X_3, X_4, are elements in GF(p) with the exception of the quadrable consisting of four zero elements. Two quadrable $[X_1, X_2, X_3, X_4]$ and$[y_1, y_2, y_3, y_4]$ represent the same plane if there exists λ. in GF(p)\{0} such that $[X_1, X_2, X_3, X_4] = \lambda [y_1, y_2, y_3, y_4]$.
Finally, a point $P\ (X_1, X_2, X_3, X_4)$ is incident with the plane $\pi[a_1, a_2, a_3, a_4]$ iff
$a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 = 0$.

Theorem 1.3: [1,19]

The points of PG (3, p) have a unique forms which are
$(1,0,0,0), (x, 1,0,0), (x, y, 1,0), (x, y, z, 1)$ for all x, y, z in GF(p).
There exists one point of the form $(1,0,0,0)$.
There exists p points of the form $(x, 1,0,0)$.
There exists p^2 points of the form $(x, y, 1,0)$.
There exists p^3 points of the form $(x, y, z, 1)$.

Theorem 1.4: [19]

The planes of PG(3.p) have a unique forms which are: $[1,0,0,0], [x, 1,0,0], [x, y, 1,0], [x, y, z, 1]$ for all x, y, z in GF(p).
There exists one plane of the form $[1,0,0,0]$.
There exists p planes of the form $[x, 1,0,0]$.
There exists p^2 planes of the form $[x, y, 1,0]$.
There exists p^3 planes of the form $[x, y, z, 1]$.

Theorem 1.5: [1]

In PG (3.p) satisfies the following:
A) Every line contains exactly $p + 1$ points and every point is on exactly $p + 1$ lines.
B) Every plane contains exactly $p^2 + p + 1$ points (lines) and every point is on exactly $p^2 + p + 1$ planes.
C) There exist $p^3 + p^2 + p + 1$ of points and there exists $p^3 + p^2 + p + 1$ of planes.
D) Any two planes intersect in exactly $p + 1$ points and any line is on exactly $P + 1$ planes. So, any two points are on exactly $p + 1$ planes.

Theorem 1.6: [1]

There exists $(p^2 + 1)(p^2 + p + 1)$ of lines in PG (3, p).

Definition 1.7: [1,19]

A $(k,$ꞁ$)$-span, ꞁ $\geq$ 1 is a set of k spaces $\pi_{ꞁ}$ no two of which intersect.

Definition 1.8: [1,19]

A maximum $(k,$ꞁ$)$-span is a set of k spaces πꞁwhich are every points of
PG (3, p) lies in exactly one line of the, and every two lines of $\pi_{ꞁ}$ are disjoint.

Definition 1.9: [1,19]

Every maximum $(k,$ꞁ$)$ −span is a spread.

2-The projective space and the$(k,$ꞁ$)$ −span in PG(3,7).

2.1 The projective space in PG(3,7).

PG(3,7) contains 400 points and 400 planes such that each point is on 57 planes and every plane contains 57 points, any line contains 8 points and it is the intersection of 8 planes, all the points,planes and lines of PG(3,7) are given in table 2 and 3.

2.2 The(k,ꞁ)-span in PG(3,p).

In table (1) below Any two non-intersecting lines can be taken in PG(3,7).

Table (1) Spread in PG(3,7).

ţ$_i$	**ʅ$_i$**								**(k$_i$,ʅ)-span**
ξ	1	2	3	4	5	6	7	8	**(1,ʅ)-span**
ν	9	58	107	156	205	254	303	352	**(2,ʅ)-span**
μ	10	65	115	165	215	265	315	365	**(3,ʅ)-span**
θ	11	72	123	174	225	269	320	371	**(4,ʅ)-span**
η	12	79	131	183	228	280	325	377	**(5,ʅ)-span**
ζ	13	86	139	185	238	284	337	383	**(6,ʅ)-span**
ε	14	93	147	194	241	295	342	389	**(7,ʅ)-span**
δ	15	100	155	203	251	299	347	395	**(8,ʅ)-span**
γ	16	61	117	173	229	285	341	397	**(9,ʅ)-span**
β	17	68	125	182	239	289	346	354	**(10,ʅ)-span**
α	18	75	133	184	242	300	309	360	**(11,ʅ)-span**
ϑ	19	82	141	193	252	255	314	366	**(12,ʅ)-span**
ϐ	20	89	142	202	206	266	319	379	**(13,ʅ)-span**
ώ	21	96	150	162	216	270	324	385	**(14,ʅ)-span**
ύ	22	103	109	164	219	281	336	391	**(15,ʅ)-span**
ό	23	64	127	190	253	267	330	393	**(16,ʅ)-span**
ϋ	24	71	128	192	207	271	335	399	**(17,ʅ)-span**
ϊ	25	78	136	201	217	275	340	356	**(18,ʅ)-span**
ω	26	85	144	161	220	286	345	362	**(19,ʅ)-span**
ψ	27	92	152	163	230	290	308	368	**(20,ʅ)-span**
σ	28	99	111	172	233	301	313	374	**(21,ʅ)-span**
ς	29	106	119	181	243	256	318	380	**(22,ʅ)-span**
ϡ	30	60	130	200	221	291	312	382	**(23,ʅ)-span**
Ѣ	31	67	138	160	231	302	317	388	**(24,ʅ)-span**
ϟ	32	74	146	169	234	257	329	394	**(25,ʅ)-span**
Ϝ	33	81	154	171	244	261	334	358	**(26,ʅ)-span**
Ϙ	34	88	113	180	247	272	339	364	**(27,ʅ)-span**
ϖ	35	95	114	189	208	276	351	370	**(28,ʅ)-span**
ϱ	36	102	122	191	218	287	307	376	**(29,ʅ)-span**
Ϭ	37	63	140	168	245	273	350	378	**(30,ʅ)-span**
Ϫ	38	70	148	170	248	277	306	384	**(31,ʅ)-span**
Ϧ	39	77	149	179	209	288	311	390	**(32,ʅ)-span**
Ϣ	40	84	108	188	212	292	323	396	**(33,ʅ)-span**
Ͼ	41	91	116	197	222	296	328	353	**(34,ʅ)-span**
Б	42	98	124	199	232	258	333	359	**(35,ʅ)-span**
Ц	43	105	132	159	235	262	338	372	**(36,ʅ)-span**
Ў	44	59	143	178	213	297	332	367	**(37,ʅ)-span**
Ѝ	45	66	151	187	223	259	344	373	**(38,ʅ)-span**
Ћ	46	73	110	196	226	263	349	386	**(39,ʅ)-span**
Њ	47	80	118	198	236	274	305	392	**(40,ʅ)-span**
Љ	48	87	126	158	246	278	310	398	**(41,ʅ)-span**
Ѣ	49	94	134	167	249	282	322	355	**(42,ʅ)-span**
Л	50	101	135	176	210	293	327	361	**(43,ʅ)-span**

Щ	51	62	153	195	237	279	321	363	**(44,ʅ)-span**
Ю	52	69	112	204	240	283	326	369	**(45,ʅ)-span**
Ы	53	76	120	157	250	294	331	375	**(46,ʅ)-span**
ж	54	83	121	166	211	298	343	381	**(47,ʅ)-span**
ц	55	90	129	175	214	260	348	387	**(48,ʅ)-span**
ɑɭ	56	97	137	177	224	264	304	400	**(49,ʅ)-span**
GƆ	57	104	145	186	227	268	316	357	**(50,ʅ)-span**

In table(1) above any elements of the set $\text{ƫ}_{i=\{}$ ξ, ν, μ,………, GƆ} except the first element can be representing by union of below set and non- intersecting of them.

Finally, the line GƆ={57,104,145,186,227,268,316,357} cannot intersect any line of the set (ƫ_i) and (GƆ) is (50,ʅ)-span, which is the maximum (k,ʅ)-span of PG(3,7) can be obtained. Thus GƆ is called a Spread of fifty lines of PG(3,7) which partitions PG(3,7); that every point of PG(3,7) lies in exactly one line of $\text{ƫ}_{i..}$ and every line are disjoint. From the above results the number of the planes in the projective space

PG(3,7) are 400 planes and each plane contains 57 lines, therefore the total number of the lines in PG(3,7) are 22800.We found that the number of the lines do not intersect with some of them are fiftylines ,these lines contains the whole points of the projective space PG(3,7), and called him a (50,ʅ)-span ,i.e.

(50,ʅ)-span={ʅ_1,ʅ_2,..., ʅ_{50}}=PG(3,7)={1,2,3,…..,400}

Moreover, we found that a (50,ʅ)-span is a maximum (k, ʅ)-span inPG(3,7).

Table (2) Points and Plane of PG(3,7)

I	p_i	π_i
1	(1,0,0,0)	2 9 16 23 30 37 44 51 58 65 72 79 86 93 100 107 114 121 128 135 142 149 156 163 170 177 184 191 198 205 212 219 226 233 240 247 254 261 268 275 282 289 296 303 310 317 324 331 338 345 352 359 366 373 380 387 394
2	(0,1,0,0)	1 9 10 11 12 13 14 15 58 59 60 61 62 63 64 107 108 109 110 111 112 113 156 157 158 159 160 161 162 205 206 207 208 209 210 211 254 255 256 257 258 259 260 303 304 305 306 307 308 309 352 353 354 355 356 357 358
. . . . .	. . . . .	
400	(6,6,6,1)	8 15 21 27 33 39 45 51 59 65 78 84 90 96 102 107 120 126 132 138 144 150 162 168 174 180 186 192 198 210 216 222 228 234 240 253 258 264 270 276 282 295 301 306 312 318 324 337 343 349 354 360 366 379 385 391 397

Table (3) Plane and lines of PG(3,7)

	2	2	2	2	2	2	2	2	9	9	9	9	9	9	9	16	16	16	16	16	16	16	23	23	23	23	23	23	23	30	30	30	30	30	30	30
	9	58	107	156	205	254	303	352	58	65	72	79	86	93	100	58	65	72	79	86	93	100	58	65	72	79	86	93	100	58	65	72	79	86	93	100
	16	65	114	163	212	261	310	359	107	114	121	128	135	142	149	114	121	128	135	142	149	107	121	128	135	142	149	107	114	128	135	142	149	107	114	121
1	23	72	121	170	219	268	317	366	156	163	170	177	184	191	198	170	177	184	191	198	156	163	184	191	198	156	163	170	177	198	156	163	170	177	184	191
	30	79	128	177	226	275	324	373	205	212	219	226	233	240	247	226	233	240	247	205	212	219	247	205	212	219	226	233	240	219	226	233	240	247	205	212
	37	86	135	184	233	282	331	380	254	261	268	275	282	289	296	282	289	296	254	261	268	275	261	268	275	282	289	296	254	289	296	254	261	268	275	282
	44	93	142	191	240	289	338	387	303	310	317	324	331	338	345	338	345	303	310	317	324	331	324	331	338	345	303	310	317	310	317	324	331	338	345	303
	51	100	149	198	247	296	345	394	352	359	366	373	380	387	394	394	352	359	366	373	380	387	387	394	352	359	366	373	380	380	387	394	352	359	366	373

37	37	37	37	37	37	37	44	44	44	44	44	44	44	51	51	51	51	51	51	51
58	65	72	79	86	93	100	58	65	72	79	86	93	100	58	65	72	79	86	93	100
135	142	149	107	114	121	128	142	149	107	114	121	128	135	149	107	114	121	128	135	142
163	170	177	184	191	198	156	177	184	191	198	156	163	170	191	198	156	163	170	177	184
240	247	205	212	219	226	233	212	219	226	233	240	247	205	233	240	247	205	212	219	226
268	275	282	289	296	254	261	296	254	261	268	275	282	289	275	282	289	296	254	261	268
345	303	310	317	324	331	338	331	338	345	303	310	317	324	317	324	331	338	345	303	310
373	380	387	394	352	359	366	366	373	380	387	394	352	359	359	366	373	380	387	394	352

	1	1	1	1	1	1	1	1	9	9	9	9	9	9	9	10	10	10	10	10	10	10	11	11	11	11	11	11	11	12	12	12	12	12	12	12
	9	58	107	156	205	254	303	352	58	59	60	61	62	63	64	58	59	60	61	62	63	64	58	59	60	61	62	63	64	58	59	60	61	62	63	64
	10	59	108	157	206	255	304	353	107	108	109	110	111	112	113	108	109	110	111	112	113	107	109	110	111	112	113	107	108	110	111	112	113	107	108	109
2	11	60	109	158	207	256	305	354	156	157	158	159	160	161	162	158	159	160	161	162	156	157	160	161	162	156	157	158	159	162	156	157	158	159	160	161
	12	61	110	159	208	257	306	355	205	206	207	208	209	210	211	208	209	210	211	205	206	207	211	205	206	207	208	209	210	207	208	209	210	211	205	206
	13	62	111	160	209	258	307	356	254	255	256	257	258	259	260	258	259	260	254	255	256	257	255	256	257	258	259	260	254	259	260	254	255	256	257	258
	14	63	112	161	210	259	308	357	303	304	305	306	307	308	309	308	309	303	304	305	306	307	306	307	308	309	303	304	305	304	305	306	307	308	309	303
	15	64	113	162	211	260	309	358	352	353	354	355	356	357	358	358	352	353	354	355	356	357	357	358	352	353	354	355	356	356	357	358	352	353	354	355

13	13	13	13	13	13	13	14	14	14	14	14	14	14	15	15	15	15	15	15	15
58	59	60	61	62	63	64	58	59	60	61	62	63	64	58	59	60	61	62	63	64
111	112	113	107	108	109	110	112	113	107	108	109	110	111	113	107	108	109	110	111	112
157	158	159	160	161	162	156	159	160	161	162	156	157	158	161	162	156	157	158	159	160
210	211	205	206	207	208	209	206	207	208	209	210	211	205	209	210	211	205	206	207	208
256	257	258	259	260	254	255	260	254	255	256	257	258	259	257	258	259	260	254	255	256
309	303	304	305	306	307	308	307	308	309	303	304	305	306	305	306	307	308	309	303	304
355	356	357	358	352	353	354	354	355	356	357	358	352	353	353	354	355	356	357	358	352

	8	8	8	8	8	8	8	8	15	15	15	15	15	15	15	21	21	21	21	21	21	21	27	27	27	27	27	27	27	33	33	33	33	33	33	33
	15	59	107	162	210	258	306	354	59	65	78	84	90	96	102	59	65	78	84	90	96	102	59	65	78	84	90	96	102	59	65	78	84	90	96	102
	21	65	120	168	216	264	312	360	107	120	126	132	138	144	150	120	126	132	138	144	150	107	126	132	138	144	150	107	120	132	138	144	150	107	120	126
400	27	78	126	174	222	270	318	366	162	168	174	180	186	192	198	174	180	186	192	198	162	168	186	192	198	162	168	174	180	198	162	168	174	180	186	192
	33	84	132	180	228	276	324	379	210	216	222	228	234	240	253	228	234	240	253	210	216	222	253	210	216	222	228	234	240	222	228	234	240	253	210	216
	39	90	138	186	234	282	337	385	258	264	270	276	282	295	301	282	295	301	258	264	270	276	264	270	276	282	295	301	258	295	301	258	264	270	276	282
	45	96	144	192	240	295	343	391	306	312	318	324	337	343	349	343	349	306	312	318	324	337	324	337	343	349	306	312	318	312	318	324	337	343	349	306
	51	102	150	198	253	301	349	397	354	360	366	379	385	391	397	397	354	360	366	379	385	391	391	397	354	360	366	379	385	385	391	397	354	360	366	379

39	39	39	39	39	39	39	45	45	45	45	45	45	45	51	51	51	51	51	51	51
59	65	78	84	90	96	102	59	65	78	84	90	96	102	59	65	78	84	90	96	102
138	144	150	107	120	126	132	144	150	107	120	126	132	138	150	107	120	126	132	138	144
168	174	180	186	192	198	162	180	186	192	198	162	168	174	192	198	162	168	174	180	186
240	253	210	216	222	228	234	216	222	228	234	240	253	210	234	240	253	210	216	222	228
270	276	282	295	301	258	264	301	258	264	270	276	282	295	276	282	295	301	258	264	270
349	306	312	318	324	337	343	337	343	349	306	312	318	324	318	324	337	343	349	306	312
379	385	391	397	354	360	366	366	379	385	391	397	354	360	360	366	379	385	391	397	354

Conclusion: -

We have inferred and demonstrated the following geometrical rule.

Theorem 2.3

The total number of Spread in projective space $PG(3, p)$ where p is prime,$P \geq 2$ is $p^2 + 1$.

Proof :

In $PG(3, p)$, there exist $p^3 + p^2 + p + 1$ planes, but each line is on $p + 1$planes, then there exist exactly

$$\frac{(p^3+p^2+p+1)}{(p+1)} = (p^2 + 1) \text{ spread in PG(3,p).}$$

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The Reverse construction of complete (k, n)- arcs in three-dimensional projective space PG(3,4)

[1]Aidan Essa Mustafa Sulaimaan [2]Nada Yassen Kasm Yahya

[1] Ph.D. Student, Department of Mathematics, College of Computer Sciences and Mathematics, University of Mosul, Mosul-Iraq

[2] Department of Mathematics, College of Education for Pure Science, University of Mosul, Mosul-Iraq

eidanalkhatony@gmail.com

drnadaqasim1@gmail.com

Abstract : In this work, the complete (k, n) arcs in PG(3,4) over Galois field GF(4) can be created by removing some points from the complete arcs of degree m, where m = n + 1, 3 n q2 + q is used. In addition, where k ≤ 85, we geometrically prove that the minimum complete (k, n)—arc in PG(3,4) is (5,3)-arc. A(k, n)—arcs is a set of k points no n+1 of which collinear. A(k, n)—arcs is complete unless it is embedded in an arc (k+1,n).

1-Introduction:

This paper divided into three sections, section one consists of the basic theorems and definitions of a projective 3-space PG(3,q). In section two the addition's and multiplication operations of GF(4). The Reverse of complete (k, n)-arcs, for 3 ≤ n ≤ 21 explained in section three.

1.1 Definition1[3]:

PG(3,q)", A projective 3-space PG(3,q) over Galois field GF(q),where $q = p^m$ for some prime number (p) and some integer m is a three-dimensional projective space which consists of points, planes and lines with incidence relation between them. PG(3,q) is satisfying the following axioms:

a. Within a single line are found every two distinct points.

b. In a single plane are found all three distinct non-collinear points, even any line and point not on it.

c. Each two distinct lines of coplanar converge in one single point.

c. Any line which is not on a given plane intersects the plane at a single point.

e. The intersection of any two distinct planes in a single line.

Any point in PG(3,q) has the shape of a quadrable (U1,U2,U3,U4), where U1,U2,U3,U4 are elements in GF(q) except the quadrable composed of four zero elements. Two quadrables (U1,U2,U3,U4) and (V1,V2,V3,V4) represent the same point if, in GF(q)\{0}, there is (t) such that (U1,U2,U3,U4) = t(V1,V2,V3,V4). Similarly, every plane in PG(3,q) has the form of a quadrable [U1,U2,U3,U4], where U1,U2,U3,U4 are elements in GF(q) except the quadrable composed of four zero elements. Two quadrables [U1,U2,U3,U4] and [V1,V2,V3,V4] represent the same plane if, in GF(q)\{0}, there is (t) such that [U1,U2,U3,U4] = t[V1,V2,V3,V4]. A point N(U1,U2,U3,U4) is incident with the plan.

1.2 Definition2[3]: "Plane π"

A plane π in PG(3,q) is the set of all points $N(U_1,U_2,U_3,U_4)$ satisfying a linear equation $U_1X_1 + U_2X_2 + U_3X_3 + U_4X_4 = 0$. This plane is denoted by π $[X_1,X_2,X_3,X_4]$,where X_1,X_2,X_3,X_4 are elements in GF(q) with the exception of the quadrable consisting of four zero elements.

1.3 Theorem1[4]:

PG(3,q) points have special shapes that are (1,0,0,0), (U,1,0,0), (U, V,1,0) and (U, V, W,1) for all U, V, W in GF(q), which are (1,0,0,0) is one point, (U,1,0,0) q points, (U, V,1,0) q2 points, and (U, V, W,1) q3 points, for all U, V, W in PG(q) points.

1.4 Theorem2[4]:
The PG(3,q) planes have special shapes [1,0,0,0],[U,1,0,0], [U, V,1,0], [U, V, W,1] for all u, v,w in GF(q). which are [1,0,0,0] is one plane,[U,1,0,0] are q planes,[U,V,1,0] are q^2 planes, and [U,V,W,1] are q^3 planes, for all U, V, W in PG(q).

1.5 Corollary1[4]:
There exists $q^3+ q^2+ q + 1$ of points in PG (3,q) and there exist $q^3+ q^2+ q + 1$ of planes.

1.6 Theorem3[4]:
Every plane in PG(3,q) contains exactly $q^2+ q + 1$ points (lines) and every point is on exactly $q^2+ q + 1$ planes.

1.7 Theorem4[4]:
Every line in PG(3,q) contains exactly q + 1 points and every point is on exactly q + 1 lines.

1.8 Corollary2[4]:

Every two PG(3,q) aircraft intersects in exactly q + 1 points and every two points are on exactly q + 1 planes. Any line is also on precisely q + 1 planes.

1.9 Definition3[1] :"(k,n)-arcs"

A (k, n)—arc A in PG(3,q) is a set of k points such that at most n points of which lie in any plane, $n \geq 3$. n is called degree of the (k, n)—arc.

1.10 Definition4[1]:
In PG(3,q), if k is any k-set, then an n-secant of k is a line(a plane) ℓ such that $|\ell \cap k|=n$.
A 0—secant is called an external line (plane) of k, a 1—secant is called a unisecant line (plane),
a 2—secant is called a bisecant line and 3—secant is called a trisecant line.

1.11 Definition5[1]:
A point N not on a (k, n)—arc has index i if there are exactly i (n-secant) of K through N, one can denoted the number of point N of index i by C_i.

1.12 Remark1[2]: A (k, n)—arc A is complete iff $C_0=0$. Thus the k-set is complete iff every point of PG(3,q) lies on some n-secant of the (k, n)—set.

1.13 Definition6[2]:
Let T_i be the total number of the i-secant of a (k, n)—arc A, then the type of A with respect to its planes denoted by $(T_n, T_{n-1}, T_{n-2}, \ldots, T_0)$. One can also say that A is of type m where $m = m_i$; that is m is the smallest integer i for which $T_i \neq 0$.

1.14 Definition7[4]:
Let (k_1,n)-arc A is of type $(T_n, T_{n-1}, \ldots, T_0)$ and (k_2,n)-arc B of type $(S_n, S_{n-1}, \ldots, S_0)$, then A and B have the same type iff $T_i = S_i$, for all i, in this case they are projectively equivalent.

1.15 Theorem5[4]:
Let T_i represents the number of i-secants (planes) for the arc A in PG(3,q), that is T_2 is the number of bisecants, T_1 is the number of unisecants, and T_0 is the number of external line $b = q + 2 - k$, then ;

1. T_1 = k b
2. T_2 =k(k– 1)/2
3. T_3 =k(k– 1)(k– 2)/3!
4. T_n =k(k –1)… (k– n +1)/n!
5. $T_0 = q^3+ q^2+ q + 1 – k b –k(k– 1)/2–k(k– 1)(k– 2)/3!–… –k(k– 1)(k– 2)… (k– n +1)/n!$

1.16 Theorem6[4]:
Let C_i be the number of points of index i in PG(3,q) which are not on a (k, n)-arc A, then the constants C_i of A satisfy the following equations:
(1) $\sum_{\alpha}^{\beta} ci = q^3+q^2+ q + 1 – k$
(2) $\sum_{\alpha}^{\beta} ici$=k(k– 1)… (k– n+1) ($q^2$+ q + 1 – n) /n! where α is the smallest i for which Ci ≠ 0, β be the largest i for which Ci≠ 0.
1.17 Theorem7[1]:
A (k, n)-arc A is maximum if and only if every line in PG(3,q) is a 0—secant or n secant.
2- The Addition's and Multiplication's Operations of GF(4)[5]:
In order to find the addition and multiplication tables in GF(4), we have order pairs (U1,U2) so that U1,U2 in GF(2), as follows: 0≡(0,0), 1≡(1,0), 2≡(0,1), 3≡(1,1). Placed these points in one orbit, at (1,0) the first point and by (1,0) A^i, i=0,1,2,3 and A=$\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$, (1,0)A = (0,1) and (1,0)A^2= (1,1),
so
(1,0)=$\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$ $\begin{matrix}(0,1)\\(1,1)\end{matrix}$.
Currently, on the left of the table below, m is the multiplication operation and on the right n is the addition operation, on the multiplication side we write the numbering of points as second, and the addition side takes the usual sequence.

m(*)		(+)n = f(m)
1	(1,0)	0
2	(0,1)	1
3	(1,1)	2
Mod 3		

In addition table, we have the following relation:$(U_1,U_2) + (V_1,V_2) = (W_1,W_2)$ where $W_i = (U_i + V_i)$ mod(2), for i = 1, 2. In multiplication table, we have the following relation
$((1,0)A^{f(m_1)})A^{f(m_2)} \Leftrightarrow m_1 * m_2 = m_3$
$=(1,0)A^{(f(m_1)+f(m_2))(\text{mod } 3)}$
$= (U_1,U_2)$
For example: 2∗3=1 ⇔ $((1,0)A^1)A^2=(1,0)A^3=(1,0)A^0=(1,0)$
where (1,0) equal to 1 in multiplication side.
Now we have addition and multiplication tables:
Table(1) Table(2)

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

*	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

3. The Reverse construction of complete (k, n)–arcs in PG(3,4):
The complete (k, n)-arcs in PG(3,4) can be constructed from the complete arcs of degree m, m = n + 1,
3 ≤ n ≤ 21, through the following:

3.1 The complete (k,21) – arc in PG(3,4) :

The projective space PG (3,4) contains 85 points and 85 planes in such a way that each point is on 21 planes and each plane contains 21 points, each line has 5 points and is the intersection of 5 planes. And there's the maximal complete (k,21)–arc A exists when k = 85. This arc contains all the points of PG(3,4) since it intersects every plane in exactly 21 points and hence there arc no points of index zero for A.

So A = {1, …, 85} is the complete (85,21)–arc.

3.2 The Construction of Complete (k,20) – arc in PG(3,4) :

A complete (k,20)–arc B is constructed from the complete (85,21) – arc A by eliminating some points from A which are: 18,26,38,46,54,58,70,82. to obtain a complete (77,20)–arc B, since

1. B intersects any plane in at most 20 points.
2. every point not in B is on at least one 20 – secant of B,

B = {1,…,17,19,…,25,27,…,37,39,…,45,47,…,53,55,56,57,59,…,69,71,…,81,83,84,85}.

3.3 The Construction of Complete (k,19) – arc in PG(3,4) :

A complete (k,19) – arc C in PG (3,4) can be constructed from the complete (77,20) – arc B by eliminating some points from B, which are: 10,30,62,66,78.

Then a complete (72,19)–arc C is obtained, C={1,…,9,11,…,17,19,…,25,27,28,29,31,…,37,39,…,45,47,…,53,55,56,57,59,…61,63,64,65,67,68,69,71,…,77,79,80,81,83,84,85} since each point not in C is on at least one 19 – secant, hence there are no points of index zero for C and C intersects any plane of PG(3,4) in at most 19 points.

3.4 The Construction of Complete (k,18) – arc in PG(3,4) :

A complete (k,18)–arc D in PG(3,4) can be constructed from the complete (72,19)–arc C by eliminating four points from C, which are the points 14,34,50,74. then a complete (68,18)–arc D is obtained,

D={1,…,9,11,12,13,15,…,17,19,…,25,27,28,29,31,32,33,35,36,37,39,…,45,47,48,49,51,52,53,55,56,57,59,…,61,63,64,65,67,68,69,71,72,73,75,76,77,79,80,81,83,84,85} since each point not in D is on at least one

18 – secant of D and hence there are no points of index zero and D intersects each plane in at most 18 points.

3.5 The Construction of Complete (k,17) – arc in PG(3,4) :

A complete (k,17)–arc E in PG(3,4) can be constructed from the complete (68,18) – arc C by eliminating five points from D, which are the points 21,32,42,55,65. then a complete (63,17)–arc E is obtained,E={1,…,9,11,12,13,15,…,17,19,20,22,23,24,25,27,28,29,31,
33,35,36,37,39,40,41,43,44,45,47,48,49,51,52,53,56,57,59,…,61,63,64,67,68,69,
71,72,73,75,76,77,79,80,81,83,84,85} since each point not in E is on at least one 17–secant of E and hence there are no points of index zero and E intersects each plane in at most 17 points.

3.6 The Construction of Complete (k,16) – arc in PG(3,4) :

A complete (k,16) – arc F in PG(3,4) can be constructed from the complete (63,17) – arc E, by eliminating six points from E, which are: 8,25,45,71,80,85.then

F={1,…,7,9,11,12,13,15,16,17,19,20,22,23,24,27,28,29,31,33,35,36,37,39,40,41,43,44,47,48,49,51,52,53,56,57,59,60,61,63,64,67,68,69,72,73,75,76,77,79,81,83,84}, F is a complete (57,16)–arc, since1. F intersects any plane in at most 16 points and

2. every point not in F is on at least one 16 – secant of F.

3.7 The Construction of Complete (k,15) – arc in PG(3,4) :

A complete (*k*,15)–arc G constructed from the complete (57,16)–arc F, by eliminating four points from F, which are : 19,27,52,60. then G ={1,…,7,9,11,12,13,15,16,17,20,22,23,24, 28, 29, 31, 33, 35, 36, 37, 39, 40, 41,43,44,47,48,49,51,53,56,57,59,61,63,64,67,68,69,72,73,75,76,77,79,81,83,84}, G is a complete (53,15) – arc, since

G intersects any plane in at most 15 points and every point not in G is on at least one 15 – secant of G.

3.8 The Construction of Complete (k,14) – arc in PG(3,4) :

A complete (k,14)–arc H can be constructed from the complete (53,15)–arc G, by eliminating four points, which are: 5,20,59,77. from G, then H ={1,2,3,4,6,7,9,11,12,13,15,16,17,22,23,24, 28, 29, 31, 33, 35, 36, 37, 39,40,41,43,44,47,48,49,51,53,56,57,61,63,64,67,68,69,72,73,75,76,79,81,83,84},H is a complete (49,14)–arc, since

1. H intersects any plane in at most 14 points and
2. every point not in H is on at least one 14–secant of H.

3.9 The Construction of Complete (k,13) – arc in PG(3,4) :

A complete (k,13)–arc I can be constructed from the complete (49,14)–arc H, by eliminating five points from H, which are: 36,44,67,72,73. then I={1,2,3,4,6,7,9,11,12,13,15,16,17,22,23,24,28,29,31,33,35,37,39,40,41,43,47,48,49,51,53,56,57,61,63,64,68,69,75,76,79,81,83,84},I is a complete (44,13)–arc, since I intersects any plane in at most 13 points and every point not in I is on at least one 13–secant of I.

3.10 The Construction of Complete (k,12) – arc in PG(3,4) :

A complete (*k*,12)–arc J can be constructed from the complete (44,13)– arc I, by eliminating five points from I, which are: 17,28,41,51,79. then J ={1,2,3,4,6,7,9,11,12,13,15,16,22,23,24, 29,31,33,35,37,39,40,43,47,48,49,53,56,57,61,63,64,68,69,75,76,81,83,84},J is a complete (39,12)–arc, since J intersects any plane in at most 12 points and
every point not in J is on at least one 12–secant of J.

3.11 The Construction of Complete (k,11) – arc in PG(3,4) :
A complete (*k*,11)–arc K in PG(3,4) can be constructed from the complete (39,12)–arc K, by eliminating three points from J, which are : 16,35,64. then K={1,2,3,4,6,7,9,11,12,13,15,22,23,24,29,31,33, 37,39,40,43,47,48,49,53,56,57,61,63,68,69,75,76,81,83,84},K is a complete (36,11) – arc, since
K intersects any plane in PG(3,4) in at most 11 points and
every point not in K is on at least one 11– secant of K.

3.12 The Construction of Complete (k,10) – arc in PG(3,4) :
A complete (k,10)–arc L can be constructed from the complete (36,11)–arc K, by eliminating five points from K, which are : 9,23,31,33,69. then L={1,2,3,4,6,7,11,12,13,15,22,24,29, 37,39,40,43,47,48,49,53,56,57,61,63,68,75,76,81,83,84} is a complete (31,10)–arc, since
1. L intersects any plane in PG(3,4) in at most 10 points and
2. every point not in L is on at least one 10–secant of L.

3.13 The Construction of Complete (k,9) – arc in PG(3,4) :
A complete (k,9)–arc M can be constructed from the complete (31,10)–arc L, by eliminating three points from L, which are : 4,11,48. then M ={1,2,3,6,7,12,13,15,22,24,29,37,39,40,43,47, 49,53,56,57,61,63,68,75,76,81,83,84} is a complete (28,9)–arc, since
1. M intersects any plane in PG(3,4) in at most 9 points and
2. every point not in M is on at least one 9–secant of M.

3.14 The Construction of Complete (k,8)–arcs in PG(3,4) :
A complete (k,8)–arc N in PG(3,4) can be constructed from the complete (28,9)–arc M, by eliminating four points from M, which are : 13, 29, 39,,56. then
N ={1,2,3,6,7,12,15,22,24,37,40,43,47,49,53,57,61,63,68,75,76,81,83,84} is a complete (24,8)–arc, since1. N intersects any plane in PG(3,4) in at most 8 points and
2. every point not in N is on at least one 8–secant of N.

3.15 The Construction of Complete (k,7) – arcs in PG(3,4) :
A complete (k,7)–arc O in PG(3,4) can be constructed from the complete (24,8)–arc N, by eliminating four points from N, which are : 37,47,76,83, then
O ={1,2,3,6,7,12,15,22,24,40,43,49,53,57,61,63,68,75,81,84}is a complete(20,7)–arc, since
1. O intersects any plane in at most 7 points and
2. every point not in O is on at least one 7–secant of O.

3.16 The Construction of Complete (k,6) – arcs in PG(3,4) :
A complete (k,6)–arc P in PG(3,4) can be constructed from the complete (20,7)–arc O, by eliminating five points from O, which are : 12,24,53,61,84, then
P ={1,2,3,6,7,15,22,40,43,49,57,63,68,75,81} is a complete(15,6)– arc, since
1. P intersects any plane in at most 6 points and
2. every point not in P is on at least one 6–secant of P.

3.17 The Construction of Complete (k,5) – arcs in PG(3,4) :
A complete (k,5)–arc Q in PG(3,4) can be constructed from the complete (15,6)–arc P, by eliminating three points from P, which are : 3,7,81, then
Q ={1,2,6,15,22,40,43,49,57,63,68,75} is a complete(12,5)– arc, since
1. Q intersects any plane in at most 5 points and
2. every point not in Q is on at least one 5–secant of Q.

3.18 The Construction of Complete (k,4) – arcs in PG(3,4) :
A complete (k,4)–arc R in PG(3,4) can be constructed from the complete (12,5)–arc Q, by eliminating three points from Q, which are : 49,57,75, then
R ={1,2,6,15,22,40,43,63,68} is a complete(9,4)– arc, since
1. R intersects any plane in at most 4 points and

2. every point not in R is on at least one 4–secant of R.

3.19 The Construction of Complete (k,3) – arcs in PG(3,4) :
A complete (k,3)–arc S in PG(3,4) can be constructed from the complete (9,4)–arc R, by eliminating four points from R, which are : 15,40,63,68 then
S ={1,2,6,22,43} is a complete(5,3)– arc, since
1. S intersects any plane in at most 3 points and
2. every point not in S is on at least one 3–secant of S.(table below)

Conclusions :
Form the above results, the complete (k,n)-arcs in PG(3,4),21≥ n ≥3,as follows:
(k,21)–arc, where k=85, is a complete.
(k,20)–arc, where k=77, is a complete.
(k,19)–arc, where k=72, is a complete.
(k,18)–arc, where k=68, is a complete.
(k,17)–arc, where k=63, is a complete.
(k,16)–arc, where k=57, is a complete.
(k,15)–arc, where k=53, is a complete.
(k,14)–arc, where k=49, is a complete.
(k,13)–arc, where k=44, is a complete.
(k,12)–arc, where k=39, is a complete.
(k,11)–arc, where k=36, is a complete.
(k,10)–arc, where k=31, is a complete.
(k,9)–arc, where k=28, is a complete.
(k,8)–arc, where k=24, is a complete.
(k,7)–arc, where k=20, is a complete.
(k,6)–arc, where k=15, is a complete.
(k,5)–arc, where k=12, is a complete.
(k,4)–arc, where k=9, is a complete.
(k,3)–arc, where k=5, is a complete.

Notation: -
A (l, t)- blocking set S in PG(2, q) is a set of L points such that every line of PG(2, q) intersects S in at least n points, and there is a line intersecting S in exactly n points. Note that a (k, r)-arc is the complement of a (q^2+q+1-k, q+1− r)-blocking set in a projective plane and conversely. A linear code C of length n and dimension k over GF(q) is a k-dimensional subspace of V (n ,q). Such a code is called [n, k, d ;p]- code if its minimum Hamming distance is d. There is exists a relationship between (k, r)-arc in PG(2,q) and [n,3,d]q codes ,given by the following theorem .

Theorem [6]

There exists a projective [k,3,d]q code if and only if there exists an (n, n−d)-arc in PG(2, q).

Table for the related between (k,n)-arcs and{l,t}- blocking sets and linear codes

q	Arc	Blocking set	Linear code
	(85,21)–arc	…………………..	$[85,4,64]_4$
	(77,20)–arc	(8,1)–Blocking set	$[77,4,57]_4$
	(72,19)–arc	(13,2)–Blocking set	$[72,4,53]_4$
	(68,18)–arc	(17,3)–Blocking set	$[68,4,50]_4$
	(63,17)–arc	(22,4)–Blocking set	$[63,4,46]_4$
	(57,16)–arc	(28,5)–Blocking set	$[57,4,41]_4$
	(53,15)–arc	(32,6)–Blocking set	$[53,4,38]_4$
	(49,14)–arc	(36,7)–Blocking set	$[49,4,35]_4$
	(44,13)–arc	(41,8)–Blocking set	$[44,4,31]_4$
4	(39,12)–arc	(46,9)–Blocking set	$[39,4,27]_4$
	(36,11)–arc	(49,10)–Blocking set	$[36,4,25]_4$
	(31,10)–arc	(54,11)–Blocking set	$[31,4,21]_4$
	(28,9)–arc	(57,12)–Blocking set	$[28,4,19]_4$
	(24,8)–arc	(51,13)–Blocking set	$[24,4,16]_4$
	(20,7)–arc	(65,14)–Blocking set	$[20,4,13]_4$
	(15,6)–arc	(70,15)–Blocking set	$[15,4,9]_4$
	(12,5)–arc	(73,16)–Blocking set	$[12,4,7]_4$
	(9,4)–arc	(76,17)–Blocking set	$[9,4,5]_4$
	(5,3)–arc	(80,18)–Blocking set	$[5,4,2]_4$

Notation: -

The points of PG(3,q) have unique forms which are (1,0,0,0),(U,1,0,0), (U,V,1,0) and (U,V,W,1) for all U, V, W in GF(q).which are (1,0,0,0) is one point,(U,1,0,0) are q points,(U,V,1,0) are q2 points, and (U,V,W,1) are q3 points, for all U,V,W in PG(q).

Notation: -

There exists q^3+q^2+q+1 of points in PG(3,q) and there exist q^3+q^2+q+1 of planes.

Notation: -

Every plane in PG(3,q) contains exactly q2+q+1 points (lines) and every point is on exactly q^2+q+1 planes.

Notation: -

Every line in PG(3,q) contains exactly q + 1 points and every point is on exactly q + 1 lines.

Notation: -

Any two planes in PG(3,q) intersect in exactly q+1 points, and any two points are on exactly q + 1 planes. Also any line is on exactly q+1 planes.

The Points and Plans of PG(3,4)

L1	(1,0,0,0)	2	6	10	14	18	22	26	30	34	38	42	46	50	54	58	62	66	70	74	78	82
L2	(0,1,0,0)	1	6	7	8	9	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
L3	(1,1,0,0)	3	6	11	16	21	22	26	30	34	39	43	47	51	56	60	64	68	73	77	81	85
L4	(2,1,0,0)	5	6	13	15	20	22	26	30	34	41	45	49	53	55	59	63	67	72	76	80	84

L5	(3,1,0,0)	4	6	12	17	19	22	26	30	34	40	44	48	52	57	61	65	69	71	75	79	83
L6	(0,0,1,0)	1	2	3	4	5	22	23	24	25	38	39	40	41	54	55	56	57	70	71	72	73
L7	(1,0,1,0)	2	7	11	15	19	22	27	32	37	38	43	48	53	54	59	64	69	70	75	80	85
L8	(2,0,1,0)	2	9	13	17	21	22	29	31	36	38	45	47	52	54	61	63	68	70	77	79	84
L9	(3,0,1,0)	2	8	12	16	20	22	28	33	35	38	44	49	51	54	60	65	67	70	76	81	83
L10	(0,1,1,0)	1	10	11	12	13	22	23	24	25	42	43	44	45	62	63	64	65	82	83	84	85
L11	(1,1,1,0)	3	7	10	17	20	22	27	32	37	39	42	49	52	56	61	62	67	73	76	79	82
L12	(2,1,1,0)	5	9	10	16	19	22	29	31	36	41	42	48	51	55	60	62	69	72	75	81	82
L13	(3,1,1,0)	4	8	10	15	21	22	28	33	35	40	42	47	53	57	59	62	68	71	77	80	82
L14	(0,2,1,0)	1	18	19	20	21	22	23	24	25	46	47	48	49	66	67	68	69	74	75	76	77
L15	(1,2,1,0)	4	7	13	16	18	22	27	32	37	40	45	46	51	57	60	63	66	71	74	81	84
L16	(2,2,1,0)	3	9	12	15	18	22	29	31	36	39	44	46	53	56	59	65	66	73	74	80	83
L17	(3,2,1,0)	5	8	11	17	18	22	28	33	35	41	43	46	52	55	61	64	66	72	74	79	85
L18	(0,3,1,0)	1	14	15	16	17	22	23	24	25	50	51	52	53	58	59	60	61	78	79	80	81
L19	(1,3,1,0)	5	7	12	14	21	22	27	32	37	41	44	47	50	55	58	65	68	72	77	78	83
L20	(2,3,1,0)	4	9	11	14	20	22	29	31	36	40	43	49	50	57	58	64	67	71	76	78	85
L21	(3,3,1,0)	3	8	13	14	19	22	28	33	35	39	45	48	50	56	58	63	69	73	75	78	84
L22	(0,0,0,1)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
L23	(1,0,0,1)	2	6	10	14	18	23	27	31	35	39	43	47	51	55	59	63	67	71	75	79	83
L24	(2,0,0,1)	2	6	10	14	18	25	29	33	37	41	45	49	53	57	61	65	69	73	77	81	85
L25	(3,0,0,1)	2	6	10	14	18	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84
L26	(0,1,0,1)	1	2	3	4	5	26	27	28	29	42	43	44	45	58	59	60	61	74	75	76	77
L27	(1,1,0,1)	2	7	11	15	19	23	26	33	36	39	42	49	52	55	58	65	68	71	74	81	84
L28	(2,1,0,1)	2	9	13	17	21	25	26	32	35	41	42	48	51	57	58	64	67	73	74	80	83
L29	(3,1,0,1)	2	8	12	16	20	24	26	31	37	40	42	47	53	56	58	63	69	72	74	79	85
L30	(0,2,0,1)	1	2	3	4	5	34	35	36	37	50	51	52	53	66	67	68	69	82	83	84	85
L31	(1,2,0,1)	2	8	12	16	20	23	29	32	34	39	45	48	50	55	61	64	66	71	77	80	82
L32	(2,2,0,1)	2	7	11	15	19	25	28	31	34	41	44	47	50	57	60	63	66	73	76	79	82
L33	(3,2,0,1)	2	9	13	17	21	24	27	33	34	40	43	49	50	56	59	65	66	72	75	81	82
L34	(0,3,0,1)	1	2	3	4	5	30	31	32	33	46	47	48	49	62	63	64	65	78	79	80	81
L35	(1,3,0,1)	2	9	13	17	21	23	28	30	37	39	44	46	53	55	60	62	69	71	76	78	85
L36	(2,3,0,1)	2	8	12	16	20	25	27	30	36	41	43	46	52	57	59	62	68	73	75	78	84
L37	(3,3,0,1)	2	7	11	15	19	24	29	30	35	40	45	46	51	56	61	62	67	72	77	78	83
L38	(0,0,1,1)	1	6	7	8	9	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53
L39	(1,0,1,1)	3	6	11	16	21	23	27	31	35	38	42	46	50	57	61	65	69	72	76	80	84
L40	(2,0,1,1)	5	6	13	15	20	25	29	33	37	38	42	46	50	56	60	64	68	71	75	79	83
L41	(3,0,1,1)	4	6	12	17	19	24	28	32	36	38	42	46	50	55	59	63	67	73	77	81	85
L42	(0,1,1,1)	1	10	11	12	13	26	27	28	29	38	39	40	41	66	67	68	69	78	79	80	81
L43	(1,1,1,1)	3	7	10	17	20	23	26	33	36	38	43	48	53	57	60	63	66	72	77	78	83
L44	(2,1,1,1)	5	9	10	16	19	25	26	32	35	38	45	47	52	56	59	65	66	71	76	78	85
L45	(3,1,1,1)	4	8	10	15	21	24	26	31	37	38	44	49	51	55	61	64	66	73	75	78	84
L46	(0,2,1,1)	1	14	15	16	17	34	35	36	37	38	39	40	41	62	63	64	65	74	75	76	77

L47	(1,2,1,1)	3	8	13	14	19	23	29	32	34	38	44	49	51	57	59	62	68	72	74	79	85
L48	(2,2,1,1)	5	7	12	14	21	25	28	31	34	38	43	48	53	56	61	62	67	71	74	81	84
L49	(3,2,1,1)	4	9	11	14	20	24	27	33	34	38	45	47	52	55	60	62	69	73	74	80	83
L50	(0,3,1,1)	1	18	19	20	21	30	31	32	33	38	39	40	41	58	59	60	61	82	83	84	85
L51	(1,3,1,1)	3	9	12	15	18	23	28	30	37	38	45	47	52	57	58	64	67	72	75	81	82
L52	(2,3,1,1)	5	8	11	17	18	25	27	30	36	38	44	49	51	56	58	63	69	71	77	80	82
L53	(3,3,1,1)	4	7	13	16	18	24	29	30	35	38	43	48	53	55	58	65	68	73	76	79	82
L54	(0,0,2,1)	1	6	7	8	9	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85
L55	(1,0,2,1)	4	6	12	17	19	23	27	31	35	41	45	49	53	56	60	64	68	70	74	78	82
L56	(2,0,2,1)	3	6	11	16	21	25	29	33	37	40	44	48	52	55	59	63	67	70	74	78	82
L57	(3,0,2,1)	5	6	13	15	20	24	28	32	36	39	43	47	51	57	61	65	69	70	74	78	82
L58	(0,1,2,1)	1	18	19	20	21	26	27	28	29	50	51	52	53	62	63	64	65	70	71	72	73
L59	(1,1,2,1)	4	7	13	16	18	23	26	33	36	41	44	47	50	56	61	62	67	70	75	80	85
L60	(2,1,2,1)	3	9	12	15	18	25	26	32	35	40	43	49	50	55	60	62	69	70	77	79	84
L61	(3,1,2,1)	5	8	11	17	18	24	26	31	37	39	45	48	50	57	59	62	68	70	76	81	83
L62	(0,2,2,1)	1	10	11	12	13	34	35	36	37	46	47	48	49	58	59	60	61	70	71	72	73
L63	(1,2,2,1)	4	8	10	15	21	23	29	32	34	41	43	46	52	56	58	63	69	70	76	81	83
L64	(2,2,2,1)	3	7	10	17	20	25	28	31	34	40	45	46	51	55	58	65	68	70	75	80	85
L65	(3,2,2,1)	5	9	10	16	19	24	27	33	34	39	44	46	53	57	58	64	67	70	77	79	84
L66	(0,3,2,1)	1	14	15	16	17	30	31	32	33	42	43	44	45	66	67	68	69	70	71	72	73
L67	(1,3,2,1)	4	9	11	14	20	23	28	30	37	41	42	48	51	56	59	65	66	70	77	79	84
L68	(2,3,2,1)	3	8	13	14	19	25	27	30	36	40	42	47	53	55	61	64	66	70	76	81	83
L69	(3,3,2,1)	5	7	12	14	21	24	29	30	35	39	42	49	52	57	60	63	66	70	75	80	85
L70	(0,0,3,1)	1	6	7	8	9	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69
L71	(1,0,3,1)	5	6	13	15	20	23	27	31	35	40	44	48	52	54	58	62	66	73	77	81	85
L72	(2,0,3,1)	4	6	12	17	19	25	29	33	37	39	43	47	51	54	58	62	66	72	76	80	84
L73	(3,0,3,1)	3	6	11	16	21	24	28	32	36	41	45	49	53	54	58	62	66	71	75	79	83
L74	(0,1,3,1)	1	14	15	16	17	26	27	28	29	46	47	48	49	54	55	56	57	82	83	84	85
L75	(1,1,3,1)	5	7	12	14	21	23	26	33	36	40	45	46	51	54	59	64	69	73	76	79	82
L76	(2,1,3,1)	4	9	11	14	20	25	26	32	35	39	44	46	53	54	61	63	68	72	75	81	82
L77	(3,1,3,1)	3	8	13	14	19	24	26	31	37	41	43	46	52	54	60	65	67	71	77	80	82
L78	(0,2,3,1)	1	18	19	20	21	34	35	36	37	42	43	44	45	54	55	56	57	78	79	80	81
L79	(1,2,3,1)	5	8	11	17	18	23	29	32	34	40	42	47	53	54	60	65	67	73	75	78	84
L80	(2,2,3,1)	4	7	13	16	18	25	28	31	34	39	42	49	52	54	59	64	69	72	77	78	83
L81	(3,2,3,1)	3	9	12	15	18	24	27	33	34	41	42	48	51	54	61	63	68	71	76	78	85
L82	(0,3,3,1)	1	10	11	12	13	30	31	32	33	50	51	52	53	54	55	56	57	74	75	76	77
L83	(1,3,3,1)	5	9	10	16	19	23	28	30	37	40	43	49	50	54	61	63	68	73	74	80	83
L84	(2,3,3,1)	4	8	10	15	21	25	27	30	36	39	45	48	50	54	60	65	67	72	74	79	85
L85	(3,3,3,1)	3	7	10	17	20	24	29	30	35	41	44	47	50	54	59	64	69	71	74	81	84

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The Reverse construction of complete (k, n)- arcs in PG(2,q) where q=2,4,8 related with linear codes

[1]**Aidan Essa Mustafa Sulaimaan** [2]**Nada Yassen Kasm Yahya**

[1] Ph.D. Student, Department of Mathematics, College of Computer Sciences and Mathematics, University of Mosul, Mosul-Iraq

[2] Department of Mathematics, College of Education for Pure Science, University of Mosul, Mosul-Iraq

eidanalkhatony@gmail.com

drnadaqasim1@gmail.com

Abstract The aim of this work is to study The reverse construction of complete (k, n)- arcs in PG(2,q) where q=2,4,8 is related to linear codes, and n = q, q-1, ..., 2. And n = q, q-1, ... By removing points from the complete arc (K, n) to get a full arc (K, m) where m < n.

.**Introduction:**

A projective plane PG(2,q) above Galois field GF(q), Where q is a prime number, it shall consist of of
$q^2 + q + 1$ points and q2 + q + 1 lines; each line has q+1 points and each point is on q+1 lines[2]; each point of the plane has the shape of a triple line; (U0,U1,U2), where U0,U1,U2 are elements in PG (q) except a triple composed of three zero elements. If t occurs in GF(q)\{0}, s, then two triples (U0,U1,U2) and (V0,V1,V2) are the same. t. (V0,V1,V2)= t(U0,
U1,U2) Points have in PG(2,q) different shapes which are (1,0,0), (U,1,0), (U, V,1) for all GF(q) U and V. Similarly each line in PG(2,q) has one point of shape (1,0,0), q points of shape (U,1,0) and q2 points of shape (U, V,1).

Definition 1[8]:

A (K, n)-arc is a sequence of K points in PG(2,q) and there are no collinear n+1 points to them. A (K,2)- the arc known as the K- arc is a sequence of K arcs, and no three collinear axes exist.

Definition 2[8]:

A (K, n) -arc is complete except for an (K+1,n)- arc.

Definition 3 [6]:

The maximum number of points a(K,2)-arc holds is m(2,q), and an(K,2)-arc is an oval with that number of points. In the case of only finishing ovals.

Theory of relativity 1 [6]:
M(2,q) =q+1 for q is odd or M(2,q) =q +2 for q is even

Theorem 2 [2]:

In PG(2,q), every oval is a conic, with q odd.

Definition 4 Includes [8]:

The I of a (K, n)-arc is a line that intersects the arc in exactly I points, a 0-secant is called an external line from anywhere, a 1-secant is called a unisecant line, 2-secant is called a bisecant line and 3-secant is called a trisecant line.

Corollary 1 [8]:

A(K, n) –arc is a maximum if and only if each PG(2,q) line is a 0–secant or a n – secant.

Definition 5[8]:

Let Q be an element not on the PG(2,p) K-arc. Let Si(Q) be the one I list over Q. The number of bisecants S2(Q) is referred to as the Q index for q, and the unisecant number S1(Q) is referred to as the Q for q.

Lemma 1 [8]:

For any point Q in PG (2,p)\κ, then $S_1(Q) + 2S_2(Q)=k$.

Proof: Because any unisecant of the ubiquitous. Passes one point of the arc and each bisect passes through two arc points, the number of arc points is k, then S1(Q) + 2S2(Q)=k.

Lemma 2 [7]:

Let Ci be index Q number of points i. Then

1) $\sum_{\alpha}^{\beta}$ Ci=q^2+q+1-k

2) $\sum_{\alpha}^{\beta} i$ Ci = K(k-1)(q-1)/2, Of which α is smallest I Ci ≠ 0, and β is the largest i for which Ci≠ 0.

Proof: 1) ∑ Ci Its all the points of the aircraft not in k.because The total number of points on the plane is q^2+q+1, then $\sum_{\alpha}^{\beta}$ Ci = q^2+q+1-k. 2) $\sum_{\alpha}^{\beta} i C_i = C_1 + 2C_2 + 3C_3 + \dots$

This equation express the cardinality of the following set { (Q, ℓ) /Q € ℓ \κ, ℓ is abisecant of κ } each bisecant contains q-1 points not in κ. There are k!/ 2!(k-2)! Bisecant of κ. Then there exist k(k-1) (q-1) /2 of points satisfying the equation

$\sum_{\alpha}^{\beta} i$ C i= k(k-1) (q-1) /2.

Remark [2]:

The (k, n) –arc current configuration عه C0=0, Thus, if each point of PG(2,q) is on any n-secant of any κ.

Definition 6[12]:

A(k, n)-arc K in PG(2. p) is maximal arc if k =(n-1)p+ n.

Definition 7[8]:

The maximum number of points which could be a (K, 2)-arc in PG(2, p) is m(2, p)- this arc called an oval.

Definition 8 [12]:

A polynomial F in $k[X_1,X_2,\dots,X_n]$ is called homogenous or a form of degree d if all its terms have the same degree d. A subset V of PG (n, k) is variety over K if there exists forms $F_1,F_2,\dots,F_R$ in $k[X_1,X_2,\dots,X_n]$ such that $V = \{P(A) \text{ in } PG(n, k), F_1(A)=F_2(A)=\dots=F_R(A)=0\}=V(F_1,F_2,\dots,F_R)$.

Definition 9[12]:

A variety V(F) of PG(2,q) is a subset of PG(2,q) such that $V(F)=\{P(A) \in PG(2,q) \mid F(A) =0\}$.

Definition10 [12]:

A (k, n)-arc is complete unless it is found in an arc (k+1,n). The maximum number of points you can have (k,2)-arc is m(2,q) and this arc is an oval.

Definition 11[11]:

A (k, n)-arc is a set of k points of a projective plane such that some r, but no n+1 of them, are collinear.

Definition 12[11]:

A (l, t)-blocking set S in PG(2,q) is a set of l points such that each PG(2,q) line intersects S in at least t points, and a line intersects S in exactly t points.

Remember that a (k, n)-arc is a complement to a (q2 + q + 1 − k, q + 1 − n)-block set in a projective plane, and vice versa.

Theorem 5 [11]:

There exists a projective [n, 3, d]q code if and only if there exists an (n, n−d)-arc in PG(2,q)

1- The construction of complete (k,n) – arcs, where n=2,3,…,q+1, in PG(2,2) over GF(2)

The PG(2,2) projective plane contains (7) points and (7) lines and each line contains (3) points, with each point in (3) lines. In PG(2,2), you can construct any line using the variety v. let Ni and Li, i=1,2, ... 7 The points and lines given for in PG(2,2) shall be respectively. Let me reflect point Ni I for line Li, the co-ordinates of which are the same point Ni co-ordinates and all points and lines of PG(2,2) are given in table (1).

A- The construction of (k,3)-arc: If i=3, then m(3,2)=7 and (7,3)-arc is the maximum arc, since each line in PG(2,2) is a 3-section arc (K3,2). This arc covers all of the PG(2,2) plane stages, so it's a complete arc. We are going to create the (K, m) -arcs, now m=2,3.

B. The construction of (k,2) – arc, from the (k, 3) –arc:

We delete one line (K,3)–arc, say, from L7=[3,5,6]. On the other hand, every two distinct lines are intersected in a single point in the projective plane, the removing line intersects each line of PG(2,2) in exactly one point, so we subtract one point from each line in the plane PG(2,2). The line removed is a K2 0- secant, and the remaining (6) is the 2-sectants k2=[1,2,4,7] arc.so. In PG(2,2) we find: 1- K2 is a maximum (4,2)-arc, since each line in PG(2,2) is either 0–secant or 2–secant of K2, as shown in table (2).

2-K2 is a complete (4,2) –arc since there is no zero index point for ÿ2, i.e. The oval value is C0 = 0, and k2.

2-The construction of complete (k,n) – arcs, where n=2,3,…,q+1, in PG(2,4) over GF(4)

The projective plane PG(2,4) includes (21) points and (21) lines, each line having (5) points, and each point being on (5) lines. -- line can be constructed in PG(2,4) using variety v. Let Ni and Li, i=1,2, ... , 21, be the PG(2,4) Points and lines, respectively. Let me stand for point Ni[i] is for line Li, the coordinates of which are identical to point Ni, and all points and lines of PG(2,4) are given in table (1).

A. The construction of (k,5)-arc:

If i=5, then the maximum arc is m(5,4)=21 and (21,5)–arc, since each line in PG(2,4) is a 5 –secant of the (K5,4) –arc. This arc includes all of the PG(2,4) plane points, so it's a complete arc . Now we're going to create the (K, m)-arcs, m= 2,3,4,5.

B. The construction of (k,,4) – arc, from the (k, 5) –arc:

One row (K,5)–arc is subtracted, say, L12=[5,9,10,16,19]. In the other hand, if two distinct lines are intersected in a single point in the projective plane, each line of PG(2,4) is intersected in exactly one point by the removing line, so we deduct one point from each line in plane PG(2,4). The missing line is a K5 0-secant, and the remaining lines (20) are four arc-secants. We find: 1- K4 is a maximum (16,4) –arc in PG(2,4), since each line in PG(2,4) is either 0–secant or a 4–secant of K4, as shown in table (2).

2-K4 is a complete arc (16,4), since there is no zero index point for k4, i.e. C0 = No.

C. The construction of (k,,3) – arc, from the (k, 4) –arc:

By removing (7) points which are: 3,8, 13,14,15,17,21 we create a (k,3) –arc from K4. Then we find: 1- (9,3) –arc is not a full arc because there are some lines in PG(2,4) that are neither 3–secant nor 0–secant 2- The K3 is a complete (9,3)-arc since zero points are not indexed for K3, i.e., C0=0.

D. The construction of (k,,2) – arc, from the (k, 3) –arc:

By removing (5) points which are: 4,7,12,18,20 we create a(K,2)arc from K3.

So then K2 = [1,2,6,11]. We find: 1- K2 arc is not a complete arc because there are some 0–secant, 1–secant, and 2–secant lines in PG(2,4). 2. K2 is a complete (4,2) –arc As there is no index point zero for k2,i, e., C0=0, and k2 is oval.

3-The construction of complete (k,n) – arcs, where n=2,3,…,q+1, in PG(2,8) over GF(8)

The PG(2,8) projective plane contains (73) points and (73) lines, and each line has (9) points, and each point is on (9) lines. You can create any line in PG(2,8) using variety v. Let Ni and Li, i=1,2, ... The PG(2,8) points and lines shall be, respectively, 73. Let me represent point Ni[i] stands for line Li with the same point Ni coordinates, and all points and lines of PG(2,8) are shown in table (1).

<u>A. The construction of (k,9)–arc:</u>

If i=9, then m(9,8)=73 and (73,9)-arc is the maximum arc, since each line in PG(2,8) is a 9-section (K8,7) arc. This arc includes all of the PG(2,8) plane points, so it's a complete arc. Let's construct the (K, m)- arcs, m= 2,3,4,5,6,7,8,9.

<u>B. Building of (k,,8) – arc from the (k, 9) –arc:</u>

One segment we deduct, say, from (K,9)–arc L19=[3,11,18,32,38,45,57,60,71]. On the other hand, if two distinct lines are intersected in a single point in the projective plane, each line of PG(2,8) is intersected in exactly one point by the removal line, so that we deduct one point from each line in the PG(2,8). The missing line is a segment of K9 0 and the remaining lines (72) are the eight sections of the arc. We find: 1- K8 is a maximum (64,8) –arc in PG(2,8), since each line in PG(2,8) is either 0–secant or 8–secant of K8, as shown in table (2).

2-K8 is a complete (64,8) –arc k8, i.e., C0 = 0.

<u>C. Building (k,,7) – arc, from the (k, 8) –arc:</u>

By removing (15) points which are: 8,16,20,26,27,28,29,30,31,33,36,43,65,65,69, we create a (k,7) –arc from K8. Therefore we find: 1- (49,7) -arc is not a full arc since there are some lines in PG(2,8) which are neither 7–secant nor 0– secants 2- K7 is a complete (49,7)-arc K6, i.e. C0=0.

<u>D. Building (k,,6) – arc, from the (k, 7) –arc:</u>

By removing (8) points, we create a (k,6) –arc from K7 which are: 6, 15,25,34,48,52,61,70. Therefore we find: 1- (41,6) –arc is not a full arc since in PG(2,8) there are some lines that are neither 5–secant nor 0–secant 2-K6 is a complete (41,6) –arc K6, i.e. C0=0.

<u>E. Building (k,5) – arc, from the (k, 6) –arc:</u>

We construct a (k,5) –arc out of K6 by removing (9) points that are:7,13,24,40,46,50,58,59,67. Then we find: 1-(32,5) –arc is not a full arc because in PG(2,8) there are some lines that are neither 5–secant nor 0–secant 2-K5 is a complete (32,5)-arc K5, i.e. C0=0.

<u>F. Building (k,4) – arc, from the (k, 5) –arc:</u>

We Build a (k,4) –arc from K5 by removing (9) points that are:9,14,22,39,47,54,62,66.73.Then we find: 1- (23,4)–arc is not a full arc because there are some lines in PG(2,8) that are neither 4–sectant nor 0–sectors 2– K4 is a complete (23,4)–arc K4, i.e., C0=0.

<u>G. Building (k,3) – arc, from the (k,4) –arc:</u>

We Build a (k,3) –arc from K4 by removing (9) points that are:5,12,21,35,37,41,44,53,56.Then we find: 1- (14,3) –arc is not a full arc because there are some lines in PG(2,8) that are neither 3–sectant nor 0–sectors 2-K3 is a complete (14,3)-arc K3, i.e., C0=0.

<u>H. Building (k,2) – arc, from the (k, 3) –arc:</u>

We construct a (k,2) –arc from K3 by removing (7) points that are:4,17,23,42,51,64,72.so k2=[1,2,10,19,49,63,68].Then we find: 1- (7,2) –arc is not a full arc because there are some lines in PG(2,8) that are 0–sectant, 1–sectors and 2–sectors.

2- K2 is a complete K3 (7,2)-arc, i.e., C0=0. And then k2 is oval.

1-Tables for PG(2,2)

Table(1)

i	Ni	Li		
1	(1,0,0)	2	4	6
2	(0,1,0)	1	4	5

3	(1.1,0)	3	4	7
4	(0,0,1)	1	2	3
5	(1,0,1)	2	5	7
6	(0,1,1)	1	6	7
7	(1,1,1)	3	5	6

Table(2)

i	Ni	Li		
1	(1,0,0)	2	4	6
2	(0,1,0)	1	4	5
3	(1.1,0)	3	4	7
4	(0,0,1)	1	2	3
5	(1,0,1)	2	5	7
6	(0,1,1)	1	6	7
7	(1,1,1)	[illegible]	[illegible]	6

Tables for PG(2,4

Table(1)

i	Ni	Li				
1	(1,0,0)	2	6	10	14	18
2	(0,1,0)	1	6	7	8	9
3	(1,1,0)	3	6	11	16	21
4	(2,1,0)	5	6	13	15	20
5	(3,1,0)	4	6	12	17	19
6	(0,0,1)	1	2	3	4	5
7	(1,0,1)	2	7	11	15	19
8	(2,0,1)	2	9	13	17	21
9	(3,0,1)	2	8	12	16	20
10	(0,1,1)	1	10	11	12	13
11	(1,1,1)	3	7	10	17	20
12	(2,1,1)	5	9	10	16	19
13	(3,1,1)	4	8	10	15	21
14	(0,2,1)	1	18	19	20	21
15	(1,2.1)	4	7	13	16	18
16	(2,2,1)	3	9	12	15	18
17	(3,2,1)	5	8	11	17	18
18	(0,3.1)	1	14	15	16	17
19	(1,3,1)	5	7	12	14	21
20	(2,3,1)	4	9	11	14	20
21	(3,3,1)	3	8	13	14	19

Table(2)

i	Ni	Li				
1	(1,0,0)	2	6	10	14	18
2	(0,1,0)	1	6	7	8	9
3	(1,1,0)	3	6	11	16	21
4	(2,1,0)	5	6	13	15	20
5	(3,1,0)	4	6	12	17	19
6	(0,0,1)	1	2	3	4	5
7	(1,0,1)	2	7	11	15	19
8	(2,0,1)	2	9	13	17	21
9	(3,0,1)	2	8	12	16	20
10	(0,1,1)	1	10	11	12	13
11	(1,1,1)	3	7	10	17	20
12	(2,1,1)	5	9	10	16	19
13	(3,1,1)	4	8	10	15	21
14	(0,2,1)	1	18	19	20	21
15	(1,2.1)	4	7	13	16	18
16	(2,2,1)	3	9	12	15	18
17	(3,2,1)	5	8	11	17	18
18	(0,3.1)	1	14	15	16	17
19	(1,3,1)	5	7	12	14	21
20	(2,3,1)	4	9	11	14	20
21	(3,3,1)	3	8	13	14	19

3-Tables for PG(2,8)

Table(1)

i	Ni	Li								
1	(1,0,0)	2	10	18	26	34	42	50	58	66
2	(0,1,0)	1	10	11	12	13	14	15	16	17
3	(1,1.0)	3	10	19	28	37	46	55	64	73
4	(2,1,0)	9	10	25	27	36	45	54	63	72
5	(3,1,0)	8	10	24	33	35	44	53	62	71
6	(4,1,0)	7	10	23	32	41	43	52	61	70
7	(5,1,0)	6	10	22	31	40	49	51	60	69
8	(6,1,0)	5	10	21	30	39	48	57	59	68
9	(7,1,0)	4	10	20	29	38	47	56	65	67
10	(0,0,1)	1	2	3	4	5	6	7	8	9
11	(1,0,1)	2	11	19	27	35	43	51	59	67
12	(2,0,1)	2	17	25	33	41	49	57	65	73
13	(3,0,1)	2	16	24	32	40	48	56	64	72
14	(4,0,1)	2	15	23	31	39	47	55	63	71
15	(5,0,1)	2	14	22	30	38	46	54	62	70

16	(6,0,1)	2	13	21	29	37	45	53	61	69
17	(7,0,1)	2	12	20	28	36	44	52	60	68
18	(0,1,1)	1	18	19	20	21	22	23	24	25
19	(1,1,1)	3	11	18	32	38	45	57	60	71
20	(2,1,1)	9	17	18	31	37	44	56	59	70
21	(3,1,1)	8	16	18	30	36	43	55	65	69
22	(4,1,1)	7	15	18	29	35	49	54	64	68
23	(5,1,1)	6	14	18	28	41	48	53	63	67
24	(6.1,1)	5	13	18	27	40	47	52	62	73
25	(7,1,1)	4	12	18	33	39	46	51	61	72
26	(0,2,1)	1	66	67	68	69	70	71	72	73
27	(1,2,1)	4	11	24	30	37	49	52	63	66
28	(2,2,1)	3	17	23	29	36	48	51	62	66
29	(3,2,1)	9	16	22	28	35	47	57	61	66
30	(4,2,1)	8	15	21	27	41	46	56	60	66
31	(5,2,1)	7	14	20	33	40	45	55	59	66
32	(6,2,1)	6	13	19	32	39	44	54	65	66
33	(7,2,1)	5	12	25	31	38	43	53	64	66
34	(0,3,1)	1	58	59	60	61	62	63	64	65
35	(1,3,1)	5	11	22	29	41	44	55	58	72
36	(2,3,1)	4	17	21	28	40	43	54	58	71
37	(3,3,1)	3	16	20	27	39	49	53	58	70
38	(4,3,1)	9	15	19	33	38	48	52	58	69
39	(5,3,1)	8	14	25	32	37	47	51	58	68
40	(6,3,1)	7	13	24	31	36	46	57	58	67
41	(7,3,1)	6	12	23	30	35	45	56	58	73
42	(0,4,1)	1	50	51	52	53	54	55	56	57
43	(1,4,1)	6	11	21	33	36	47	50	64	70
44	(2,4,1)	5	17	20	32	35	46	50	63	69
45	(3,4,1)	4	16	19	31	41	45	50	62	68
46	(4,4,1)	3	15	25	30	40	44	50	61	67
47	(5,4,1)	9	14	24	29	39	43	50	60	73
48	(6,4,1)	8	13	23	28	38	49	50	59	72
49	(7,4,1)	7	12	22	27	37	48	50	65	71
50	(0,5,1)	1	42	43	44	45	46	47	48	49
51	(1,5,1)	7	11	25	28	39	42	56	62	69
52	(2,5,1)	6	17	24	27	38	42	55	61	68
53	(3,5,1)	5	16	23	33	37	42	54	60	67
54	(4,5,1)	4	15	22	32	36	42	53	59	73
55	(5,5,1)	3	14	21	31	35	42	52	65	72
56	(6,5,1)	9	13	20	30	41	42	51	64	71
57	(7,5,1)	8	12	19	29	40	42	57	63	70
58	(0,6,1)	1	34	35	36	37	38	39	40	41
59	(1,6,1)	8	11	20	31	34	48	54	61	73
60	(2,6,1)	7	17	19	30	34	47	53	60	72
61	(3,6,1)	6	16	25	29	34	46	52	59	71
62	(4,6,1)	5	15	24	28	34	45	51	65	70
63	(5,6,1)	4	14	23	27	34	44	57	64	69
64	(6,6,1)	3	13	22	33	34	43	56	63	68
65	(7,6,1)	9	12	21	32	34	49	55	62	67
66	(0,7,1)	1	26	27	28	29	30	31	32	33
67	(1,7,1)	9	11	23	26	40	46	53	65	68
68	(2,7,1)	8	17	22	26	39	45	52	64	67

69	(3,7,1)	7	16	21	26	38	44	51	63	73
70	(4,7,1)	6	15	20	26	37	43	57	62	72
71	(5,7,1)	5	14	19	26	36	49	56	61	71
72	(6,7,1)	4	13	25	26	35	48	55	60	70
73	(7,7,1)	3	12	24	26	41	47	54	59	69

Table(2)

i	Ni	Li								
1	(1,0,0)	2	10	18	26	34	42	50	58	66
2	(0,1,0)	1	10	11	12	13	14	15	16	17
3	(1,1.0)	3	10	19	28	37	46	55	64	73
4	(2,1,0)	9	10	25	27	36	45	54	63	72
5	(3,1,0)	8	10	24	33	35	44	53	62	71
6	(4,1,0)	7	10	23	32	41	43	52	61	70
7	(5,1,0)	6	10	22	31	40	49	51	60	69
8	(6,1,0)	5	10	21	30	39	48	57	59	68
9	(7,1,0)	4	10	20	29	38	47	56	65	67
10	(0,0,1)	1	2	3	4	5	6	7	8	9
11	(1,0,1)	2	11	19	27	35	43	51	59	67
12	(2,0,1)	2	17	25	33	41	49	57	65	73
13	(3,0,1)	2	16	24	32	40	48	56	64	72
14	(4,0,1)	2	15	23	31	39	47	55	63	71
15	(5,0,1)	2	14	22	30	38	46	54	62	70
16	(6,0,1)	2	13	21	29	37	45	53	61	69
17	(7,0,1)	2	12	20	28	36	44	52	60	68
18	(0,1,1)	1	18	19	20	21	22	23	24	25
19	(1,1,1)	3	11	18	32	38	45	57	60	71
20	(2,1,1)	9	17	18	31	37	44	56	59	70
21	(3,1,1)	8	16	18	30	36	43	55	65	69
22	(4,1,1)	7	15	18	29	35	49	54	64	68
23	(5,1,1)	6	14	18	28	41	48	53	63	67
24	(6.1,1)	5	13	18	27	40	47	52	62	73
25	(7,1,1)	4	12	18	33	39	46	51	61	72
26	(0,2,1)	1	66	67	68	69	70	71	72	73
27	(1,2,1)	4	11	24	30	37	49	52	63	66
28	(2,2,1)	3	17	23	29	36	48	51	62	66
29	(3,2,1)	9	16	22	28	35	47	57	61	66
30	(4,2,1)	8	15	21	27	41	46	56	60	66
31	(5,2,1)	7	14	20	33	40	45	55	59	66
32	(6,2,1)	6	13	19	32	39	44	54	65	66
33	(7,2,1)	5	12	25	31	38	43	53	64	66
34	(0,3,1)	1	58	59	60	61	62	63	64	65
35	(1,3,1)	5	11	22	29	41	44	55	58	72
36	(2,3,1)	4	17	21	28	40	43	54	58	71
37	(3,3,1)	3	16	20	27	39	49	53	58	70
38	(4,3,1)	9	15	19	33	38	48	52	58	69
39	(5,3,1)	8	14	25	32	37	47	51	58	68
40	(6,3,1)	7	13	24	31	36	46	57	58	67
41	(7,3,1)	6	12	23	30	35	45	56	58	73
42	(0,4,1)	1	50	51	52	53	54	55	56	57
43	(1,4,1)	6	11	21	33	36	47	50	64	70
44	(2,4,1)	5	17	20	32	35	46	50	63	69

45	(3,4,1)	4	16	19	31	41	45	50	62	68
46	(4,4,1)	3	15	25	30	40	44	50	61	67
47	(5,4,1)	9	14	24	29	39	43	50	60	73
48	(6,4,1)	8	13	23	28	38	49	50	59	72
49	(7,4,1)	7	12	22	27	37	48	50	65	71
50	(0,5,1)	1	42	43	44	45	46	47	48	49
51	(1,5,1)	7	11	25	28	39	42	56	62	69
52	(2,5,1)	6	17	24	27	38	42	55	61	68
53	(3,5,1)	5	16	23	33	37	42	54	60	67
54	(4,5,1)	4	15	22	32	36	42	53	59	73
55	(5,5,1)	3	14	21	31	35	42	52	65	72
56	(6,5,1)	9	13	20	30	41	42	51	64	71
57	(7,5,1)	8	12	19	29	40	42	57	63	70
58	(0,6,1)	1	34	35	36	37	38	39	40	41
59	(1,6,1)	8	11	20	31	34	48	54	61	73
60	(2,6,1)	7	17	19	30	34	47	53	60	72
61	(3,6,1)	6	16	25	29	34	46	52	59	71
62	(4,6,1)	5	15	24	28	34	45	51	65	70
63	(5,6,1)	4	14	23	27	34	44	57	64	69
64	(6,6,1)	3	13	22	33	34	43	56	63	68
65	(7,6,1)	9	12	21	32	34	49	55	62	67
66	(0,7,1)	1	26	27	28	29	30	31	32	33
67	(1,7,1)	9	11	23	26	40	46	53	65	68
68	(2,7,1)	8	17	22	26	39	45	52	64	67
69	(3,7,1)	7	16	21	26	38	44	51	63	73
70	(4,7,1)	6	15	20	26	37	43	57	62	72
71	(5,7,1)	5	14	19	26	36	49	56	61	71
72	(6,7,1)	4	13	25	26	35	48	55	60	70
73	(7,7,1)	3	12	24	26	41	47	54	59	69

Conclusions:

Form the above results, the complete (k,n)-arcs in PG(2,q) where q=2,4,8 as follows: Table(3)

Notation: -

A (l, t)- blocking set S in PG(2, q) is a set of L points such that every line of PG(2, q) intersects S in at least n points, and there is a line intersecting S in exactly n points. Note that a (k, r)-arc is the complement of a (q^2+q+1-k, q+1− r)-blocking set in a projective plane and conversely. A linear code C of length n and dimension k over GF(q) is a k-dimensional subspace of V (n ,q). Such a code is called [n, k, d ;p]- code if its minimum Hamming distance is d. There is exists a relationship between (k, r)-arc in PG(2,q) and [n,3,d]q codes ,given by the following theorem .

Theorem [6]

There exists a projective [k,3,d]q code if and only if there exists an (n, n−d)-arc in PG(2, q).

Table (3)T he relation between(k,n)- arcs and{l,t }-blocking set and linear codes in the projective planes over Galois field(q) for PG(2,q),q=2,,4,8

P	Arcs	Blocking set	Linear codes
2	(7,3)–arc	………	$[7,3,4]_2$
	(4,2)–arc	(3,1)–blocking set	$[4,3,2]_2$

4	(21,5)–arc	……..	$[21,3,16]_4$
	(16,4)–arc	(5,1)–blocking set	$[16,3,12]_4$
	(9,3)–arc	(12,2)–blocking set	$[9,3,6]_4$
	(4,2)–arc	(17,3)–blocking set	$[4,3,2]_4$
8	(73,9)–arc	…………………..	$[73,3,64]_8$
	(64,8)–arc	(9,1})–blocking set	$[64,3,56]_8$
	(49,7)–arc	(24,2)–blocking set	$[49,3,42]_8$
	(41,6)–arc	(32,3)–blocking set	$[41,3,35]_8$
	(32,5)–arc	(41,4)–blocking set	$[32,3,27]_8$
	(23,4)–arc	(50,5)–blocking set	$[23,3,19]_8$
	(14,3)–arc	(59,6)–blocking set	$[14,3,11]_8$
	(7,2)–arc	(66,7)–blocking set	$[7,3,5]_8$

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On Generalized (α, β) Derivation on Prime Semirings

Maryam K. Rasheed [1], Fatin A. Hameed [2], Abdulrahman. H. Majeed [3]
Mathematics Department, University of Baghdad
maryamkhdhayer@yahoo.com [1]
Mathematics Department, University of Dyala
tonaabd90@gmail.com [2]
Mathematics Department, University of Baghdad
ahmajeed6@yahoo.com [3]
Mathematics Department, University of Baghdad

Abstract: in this paper we introduce generalized (α, β) derivation on Semirings and extend some results of Oznur Golbasi on prime Semiring. Also, we present some results of commutativity of prime Semiring with these derivation.

1. Introduction

Semirings was first introduced in 1934 by vandiver [1]. In 1992 Golan discuss Semirings and their applications and mentioned about the derivation on Semirings [2]. Thereafter, many researchers interested in derivations on Semirings and generalized it in different directions.

Chandramouleeswarn and Thiruveni studied derivations on Semirings, and introduced the notion of (α, β) derivations on semirings, see [3] and [4].

A Semiring is a nonempty set S together with two binary operations (usually denoted by $+$ and $\cdot$) such that $(S, +)$ is commutative Semigroup, $(S, \cdot)$ Semigroup and addition distributive with respect to multiplication on S, we say S is commutative Semiring if and only if $x.y = y.x$ for all $x, y \in S$ [2]. A Semiring S is called additively cancellative if $x + y = x + z$ implies $y = z$ for all $x, y, z \in S$, and it is called multiplicatively cancellative if $x.y = x.z$ implies $y = z$ for all $x, y, z \in S$, so S is called cancellative Semiring if and only if it is both additively and multiplicatively cancellative [5]. Moreover, S is called prime if whenever $x\,S\,y = 0$ implies either $x = 0$ or $y = 0$ for all $x, y \in S$.

Let S be any Semiring, an additive map $d: S \longrightarrow S$ is called derivation on S if $d\,(x\,y) = d\,(x)\,y + x\,d\,(y)$ holds for all $x, y \in S$ [6]. Now, if we suppose that α and β are two nonzero automorphisms on S and d is a derivation on S, then d is said to be (α, β) derivation on S if $d\,(x\,y) = \alpha(x)\,d(y) + d(x)\,\beta(y)$ holds for all $x, y \in S$ [6].

In this paper we introduce the notion of generalized (α, β) derivation on Semirings and extend some important results of Oznur Golbasi [7] on prime Semirings and when these Semirings become commutative.

2. Results

Definition 2.1: - Let S be a Semiring and α, β are two automorphisms on S. An additive map $F: S \longrightarrow S$ is called left generalized (α, β) derivation if there exist nonzero left (α, β) derivation $d: S \longrightarrow S$ such that $F(x\,y) = \alpha\,(x)\,F\,(y) + d\,(x)\,\beta\,(y)$ for all $x, y \in S$, and is called right generalized (α, β) derivation if there exist nonzero right (α, β) derivation $d: S \longrightarrow S$ such that $F(x\,y) = \alpha\,(x)\,d\,(y) + F\,(x)\,\beta\,(y)$ for all $x, y \in S$.

If F is both left and right generalized (α, β) derivation then it is called generalized (α, β) derivation that is $F\,(x\,y) = \alpha\,(x)\,F\,(y) + d\,(x)\,\beta\,(y) = \alpha\,(x)\,d\,(y) + F\,(x)\,\beta\,(y)$ for all x, y ∈ S.

Lemma 2.2: - Let S be a prime Semiring and I be a nonzero ideal of S. If $x\,I\,y = 0$ for all $x, y \in S$, then either $x = 0$ or $y = 0$.

Proof: - Let $x\,I\,y = 0$ for all $x, y \in S$, hence $x\,S\,I\,y = 0$ for all $x, y \in S$.

By primness of S we have either $x = 0$ or $I\,y = 0$.

Now, either $x = 0$ or $I\,S\,y = 0$. By primness of S and since $I \neq 0$, we get $y = 0$.

Theorem 2.3: - Let S be a prime Semiring and I be a nonzero ideal of S. Suppose that $F : S \longrightarrow S$ is a generalized (α, β) derivation on S with $\beta(I) = I$. If $F(I) \subseteq Z(S)$ then S is commutative.

Proof: - Let $F(I) \subseteq Z(S)$, then $F(u) \in Z(S)$ for all $u \in I$.

Replace u in above relation by s u, where $s \in S$, we get:

$$F(s\,u) = \alpha(s)\,F(u) + d(s)\,\beta(u) \in Z(S).$$

Then,

$$[\alpha(s)\,F(u) + d(s)\,\beta(u), \alpha(s)] = 0.$$

$$[\alpha(s)\,F(u), \alpha(s)] + [d(s)\,\beta(u), \alpha(s)] = 0.$$

$$\alpha(s)[F(u), \alpha(s)] + [\alpha(s), \alpha(s)]F(u) + d(s)[\beta(u), \alpha(s)] + [d(s), \alpha(s)]\,\beta(u) = 0$$

Hence,

$$d(s)\,[\beta(u), \alpha(s)] + [d(s), \alpha(s)]\,\beta(u) = 0$$

$$d(s)\,\beta(u)\,\alpha(s) - d(s)\,\alpha(s)\,\beta(u) + d(s)\,\alpha(s)\,\beta(u) - \alpha(s)\,d(s)\,\beta(u) = 0$$

$$d(s)\,\beta(u)\,\alpha(s) - \alpha(s)\,d(s)\,\beta(u) = 0 \qquad \dots (1)$$

Replace u by u v in (1), where $v \in I$. We obtain,

$$d(s)\,\beta(u\,v)\,\alpha(s) - \alpha(s)\,d(s)\,\beta(u\,v) = 0$$

$$d(s)\,\beta(u)\,\beta(v)\,\alpha(s) - \alpha(s)\,d(s)\,\beta(u)\,\beta(v) = 0 \qquad \dots (2)$$

By using (1) we get,

$$d(s)\,\beta(u)\,\beta(v)\,\alpha(s) - d(s)\,\beta(u)\,\alpha(s)\,\beta(v) = 0.$$

Then, for all $u \in I$ implies,

$$d(s)\,\beta(u)\,[\beta(v), \alpha(s)] = 0$$

$$d(s)\,I\,[\beta(v), \alpha(s)] = 0.$$

By Lemma 2.2 and since $d \neq 0$ then for all $v \in I$ we get,

$$[\beta(v), \alpha(s)] = 0.$$

$$[I, \alpha(s)] = 0.$$

Then, $I \subseteq Z(S)$, by [8, Lemma 2.22] we get S is commutative.

Lemma2.4: - Let S be a prime semiring and I be a nonzero ideal of S. Suppose that $F: S \longrightarrow S$ is a nonzero generalized (α, β) derivation and let x ∈ S:

1- If $x. F(u) = 0$ for all u ∈ I then $x = 0$.
2- If $F(u). x = 0$ for all u ∈ I then $x = 0$.

Proof: 1- Let $x. F(u) = 0$ for all u ∈ I.

Replace u in above equation by s u, where s ∈ S. Then for all s ∈ S we have,

$$x. F(s\,u) = 0.$$

$$x. (\alpha(s)\, d(u) + F(s)\, \beta(u)) = x. \alpha(s)\, d(u) + x. F(s)\, \beta(u) = 0$$

Hence,

$$x. \alpha(s)\, d(u) = 0$$

$$\alpha^{-1}(x)\, I\, \alpha^{-1}\big(d(S)\big) = 0.$$

By Lemma 2.2 and since d ≠ 0 we have, $\alpha^{-1}(x) = 0$. Then, x = 0.

Similarly we can prove (2).

Remark 2.5: - Let S be a semiring and α is an automorphism on S. If $\alpha = 0$ on I then α = 0 on S

Proof: - Obvious.

Lemma 2.6: - Let S be a prime semiring and I be a nonzero ideal of S. Suppose that $F: S \longrightarrow S$ is a nonzero generalized (α, β) derivation with nonzero automorphisms α and β. If $F = 0$ on I then $d = 0$ on S.

Proof: - Let $F(u) = 0$ for all u ∈ I. Take s ∈ S then,

$$F(u\,s) = \alpha(u)\, d(s) + F(u)\, \beta(s) = 0.$$

Hence,

$$\alpha(u)\, d(s) = 0.$$

By [8, Lemma 2.27] and since α ≠ 0 then d = 0 on S.

Lemma 2.7: - Let S be a semiring and I be a nonzero ideal of S. Suppose that $F: S \longrightarrow S$ is a generalized (α, β) derivation with nonzero automorphisms α and β. If $F = 0$ on I then $F = 0$ on S.

Proof: - Let $F(u) = 0$ for all u ∈ I. Take s ∈ S then,

$$F(u\,s) = \alpha(u)\, F(s) + d(u)\, \beta(s) = 0.$$

By Lemma 2.6 we get α (u) F (s) = 0.

Now, replace u in above equation by u r, where r ∈ S we get,

$$\alpha(u\,r)\, F(s) = \alpha(u)\, \alpha(r)\, F(s) = 0.$$

Since α is automorphism (onto) Hence, $\alpha (u)\, S\, F(s) = 0$.

By primness and since $\alpha \neq 0$ on S then $F = 0$ on S.

Lemma 2.8: - Let S be a prime semiring and $F: S \longrightarrow S$ be a generalized (α, β) derivation. Suppose that I is an ideal of S. If $0 \neq r \in S$ with $r \,.\, F(x) = 0$ for all $x \in S$, then $F = 0$ on S.

Proof: - Let $r \,.\, F(x) = 0$ for all $x \in S$. Put $x = x\, y$, where $y \in I$ we get,

$$r \,.\, F(x\, y) = 0$$

$$r \,.\, \alpha (x)\, d(y) + r.\, F(x)\, \beta (y) = 0$$

Then,

$$r \,.\, \alpha (x)\, d(y) = 0$$

$$r.\, S\, d(y) = 0.$$

So, by primness of S and since $r \neq 0$ hence, $d(y) = 0$ for all $y \in I$.

That means, $d = 0$ on I. So,

$$F(y\, x) = \alpha (y)\, F(x) + d(y)\, \beta (x) = \alpha (y)\, F(x).$$

Now, $r.\, F(y\, x) = r\, \alpha (y)\, F(x)$ Implies:

$$r.\, \alpha (y)\, F(x) = 0$$

$$r.\, S\, F(x) = 0.$$

By primness of S and since $r \neq 0$ we get, $F = 0$ on S.

Theorem 2.9: - Let S be a prime semiring and I be a nonzero ideal of S. Suppose that $F: S \longrightarrow S$ is a nonzero generalized (α, β) derivation such that $d\, F = F\, d$ and $\alpha\, F = F\, \alpha$. If $[F(u), F(v)] = 0$ for all $u, v \in I$, then S is commutative.

Proof: - Let $[F(u), F(v)] = 0$ for all $u, v \in I$.

Replace v in above equation by v s, where $s \in S$ we get,

$$[F(u), F(v\, s)] = [F(u), \alpha (v)\, d(s) + F(v)\, \beta (s)] = 0$$

$$[F(u), \alpha (v)\, d(s)] + [F(u), F(v)\, \beta (s)] = 0$$

$$\alpha (v)\, [F(u), d(s)] + [F(u), \alpha (v)]\, d(s) + F(v)\, [F(u), \beta (s)] + [F(u), F(v)]\, \beta (s) = 0$$

Hence for all $u, v \in I$ we have,

$$F(v)\, [F(u), \beta (s)] = 0$$

By Lemma 2.8 and since $F \neq 0$ on I (Lemma 2.7). So, for all $u \in I$ implies,

$$[F(u), \beta (s)] = 0$$

Therefore, $F(I) \subseteq Z(S)$, and by Theorem 2.3 we have S is commutative.

Theorem 2.10: - Let S be a cancellative prime semiring and I be a nonzero ideal of S. Suppose that F: S⟶S is a generalized (α, β) derivation with nonzero automorphisms α and β. If F acts as homomorphism on S then d = 0 on S.

Proof: - Since F acts as homomorphism on S then for all x, y ∈ S,

$$F(xy) = F(x)F(y) \quad \text{... (1)}$$

Since F is generalized (α, β) derivation then for all x, y ∈ S,

$$F(xy) = \alpha(x)F(y) + d(x)\beta(y) \quad \text{... (2)}$$

From (1) and (2) we get,

$$F(x)F(y) = \alpha(x)F(y) + d(x)\beta(y) \quad \text{... (3)}$$

Replace y by y s in (3), where s ∈ S we obtain,

$$\alpha(x)F(ys) + d(x)\beta(ys) = F(x)F(ys).$$

$$\alpha(x)F(y)F(s) + d(x)\beta(y)\beta(s) = F(x)F(y)F(s).$$

$$= F(xy)F(s)$$

$$= \alpha(x)F(y)F(s) + d(x)\beta(y)F(s)$$

Since S is cancellative we get, $\beta(s) = F(s)$ for all s ∈ S.

Now, replace s by r s in the above equation, where r ∈ S, w obtain,

$$F(rs) = \beta(rs)$$

$$\alpha(r)d(s) + F(r)\beta(s) = \beta(r)\beta(s)$$

$$= F(r)\beta(s).$$

Since S is cancellative we get, $\alpha(r)d(s) = 0$ for all r, s ∈ S.

By [8, Lemma 2.27] and Since $\alpha \neq 0$ on S then $d = 0$ on S.

Theorem 2.11: - Let S be a cancellative prime semiring and I nonzero ideal of S. Suppose that F: S⟶S is a generalized (α, β) derivation with nonzero automorphisms α and β such that $dF = Fd$ and $\alpha F = F\alpha$. If F acts as anti-homomorphism on S then d = 0 on S.

Proof: - Since F acts as homomorphism on S then for all x, y ∈ S,

$$F(xy) = F(y)F(x) \quad \text{... (1)}$$

Since F is generalized (α, β) derivation then,

$$F(xy) = \alpha(x)F(y) + d(x)\beta(y) \quad \text{... (2)}$$

From (1) and (2) we get,

$$F(y)F(x) = \alpha(x)F(y) + d(x)\beta(y) \quad \text{... (3)}$$

Replace y by y s in (3), where s $\in$ S, we obtain

$$\alpha(x) F(y s) + d(x) \beta(y s) = F(x) F(y s).$$

$$\alpha(x) F(s) F(y) + d(x) \beta(y) \beta(s) = F(s) F(y) F(x).$$

$$= F(s) F(x y)$$

$$= F(s) \alpha(x) F(y) + F(s) d(x) \beta(y).$$

Since $\alpha F = F \alpha$ and S is cancellative we have,

$$d(x) \beta(y) \beta(s) = F(s) d(x) \beta(y)$$

Now, since $d F = F d$ and S is cancellative we have,

$$\beta(s) = F(s) \text{ for all } s \in S.$$

Replace s by r s in the above equation, where r $\in$ S, we get

$$F(r s) = \beta(r s)$$

$$\alpha(r) d(s) + F(r) \beta(s) = \beta(r) \beta(s)$$

$$= F(r) \beta(s)$$

Since S cancellative then, $\alpha(r) d(s) = 0$ for all r, s $\in$ S.

By [8, Lemma 2.27] and Since $\alpha \neq 0$ on S then d = 0 on S.

Theorem 2.12: - Let S be a cancellative prime semiring and I be a nonzero ideal of S. Suppose that $F: S \longrightarrow S$ is a generalized (α, β) derivation with nonzero automorphisms α and β. If F acts as homomorphism on I then $d = 0$ on S.

Proof: - Since F acts as homomorphism on I. Then for all u, v $\in$ I,

$$F(u v) = F(u) F(v) \qquad \ldots (1)$$

Since F is generalized (α, β) derivation then,

$$F(u v) = \alpha(u) F(v) + d(u) \beta(v) \qquad \ldots (2)$$

From (1) and (2) we get,

$$F(u) F(v) = \alpha(u) F(v) + d(u) \beta(v) \qquad \ldots (3)$$

Replace v by v s in (3), where s $\in$ S, we obtain

$$\alpha(u) F(v s) + d(u) \beta(v s) = F(u) F(v s).$$

$$\alpha(u) F(v) F(s) + d(u) \beta(v) \beta(s) = F(u) F(v) F(s)$$

$$= F(u v) F(s)$$

$$= \alpha(u) F(v) F(s) + d(u) \beta(v) F(s).$$

Since S is cancellative we have, $\beta(s) = F(s)$ for all s $\in$ S.

Now, replace s by r s in the above equation, where r ∈ S we get,

$$F\,(r\,s) = \beta\,(r\,s)$$

$$\alpha\,(r)\,d\,(s) + F\,(r)\,\beta\,(s) = \beta\,(r)\,\beta\,(s)$$

$$= F\,(r)\beta\,(s).$$

Since S is cancellative we get, $\alpha\,(r)\,d\,(s) = 0$ for all r, s ∈ S.

By [8, Lemma 2.27] and Since α ≠ 0 on S then d = 0 on S.

Notation: - Throughout the following Theorem we use alpha-beta commutator such that $[\,x, y\,]_{\alpha,\beta} = \alpha\,(x)\,y - y\,\beta\,(x)$.

Theorem 2.13: - Let S be a prime semiring, I nonzero ideal of S and F: S⟶S generalized (α, β) derivation. If α and β commute with d and $F\,(u\,v) = F\,(v\,u)$ for all u, v ∈ I, then S is commutative.

Proof: - Let u, v ∈ I such that $[u, v]$ is constant element say c with $F\,(c) = 0$ and $d\,(c) \neq 0$.

Let z ∈ I hence,

$$F(c\,z) = \alpha\,(c)\,d\,(z) + F\,(c)\,\beta\,(z)$$

$$= \alpha\,(z)F\,(c) + d\,(z)\beta\,(c) = F\,(z\,c).$$

That gives, $\alpha\,(c)\,d\,(z) = d\,(z)\,\beta\,(c)$ for all z ∈ I.

Since α and β are commute with d then for all z ∈ I yields that,

$$[\,d(z), c\,]_{\alpha,\beta} = 0$$

Replace z in the above equation by w z, where w ∈ I we get,

$$[\,d(w\,z), c]_{\alpha,\beta} = [\,d\,(w)\alpha\,(z) + \beta\,(w)d(z), c]_{\alpha,\beta}$$

$$= [\,d\,(w)\alpha\,(z), c)]_{\alpha,\beta} + [\,\beta\,(w)d(z), c]_{\alpha,\beta}$$

$$= 0$$

Now, by add and subtract the terms: d (w) α (z) α (c) and β (w) β (c) d (z) we obtain,

$$d\,(w)\,\alpha\,(z)\,\alpha\,(c) - d\,(w)\,\alpha\,(z)\,\alpha\,(c) + d\,(w)\,\alpha\,(z)\,\alpha\,(c) - d\,(w)\,\beta\,(c\,)\,\alpha\,(z) + d\,(w)\,\alpha\,(c)\,\alpha\,(z) - \beta\,(c)\,d\,(w)\,\alpha\,(z) + \beta\,(w)\,d\,(z)\,\alpha\,(c) - \beta\,(w)\,\beta\,(c)\,d\,(z) + \beta\,(w)\,\beta\,(c)\,d\,(z) - \beta\,(w)\,\beta\,(c)\,d\,(z) + \beta\,(w)\,\alpha\,(c)\,d\,(z) - \beta\,(c)\,\beta\,(w)\,d\,(z) = 0.$$

Hence for all w, z ∈ I,

$$d\,(w)\,\alpha\,[z, c] + [\,c, d(w\,z)]_{\alpha,\beta} + \beta\,(w)\,[\,d(z), c\,]_{\alpha,\beta} + \beta\,[w, c]\,d\,(z) = d\,(w)\,\alpha\,[z, c] + \beta\,[w, c]\,d\,(z) = 0.$$

Replace z in above equation by c then for all w ∈ I we get,

$$\beta\,[w, c]\,d\,(z) = 0$$

$$[w, c]\,\beta^{-1}\,(d(c)) = 0$$

Replace w in above equation by s w, where s ∈ S we obtain,

$$\beta\,[s\,w, c]\; d\,(z) \;=0.$$

Thus, $[s, c]\; w\; \beta^{-1}\;(d(c)) \;=\; 0$ for all w ∈ I. Now, by lemma 2.2 and since d (c) ≠ 0 we get,

$$[s, c] \;=\; 0 \text{ for all } s \in S.$$

Then, I is commutative and by [8, Lemma 2.22] implies S is commutative.

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The M- Projective Tensor of G_1-Manifold

Nawaf Jaber Mohammed[1]

1.Department of Mathematics, Ministry of Education, Basra Education Directorate, Basra, Iraq.

Abstract. This article is devoted to the study of the geometric properties of curvature tensor on certain classes of almost Hermitian manifolds. In particular, we studied the platitude property of M-projective on G_1-manifold and found a link between G_1-manifold, $\mathcal{H}$-manifold and $\mathcal{NK}$-manifold.

1. Introduction

Almost Hermition manifold classifies it as one of the most important topics in differential geometry, which made it one of the most prominent topics addressed by the researchers. This subject is categorized into different composites in an attempt to assign its specifications and characteristics accurately. The first practical study was conducted by Koto 1960 [17]. In 1980 [6], a new study on almost hermit collection types was conducted by Gray and Hervella. In-depth studies have been conducted of these types. In 2010[2], Abood and Mohammed proved that if M is a locally compliant multiple Kahler of pointwise holomorphic sectional curvature and flat projective compliance plan with J-invariant Richi tensor, then M is a manifold Einstein. In 2016 [1], Abood and Abd Ali are given application about the projective-recurrent of Viasman Gray manifold. In 2017 [8], Ignatochkina and Abood investigated the geometrical meaning of flat conharmonicly tensor of VaismanGray manifold. In 2018 [18], Mohammed and Abood are found the necessary and adequate conditions that a projective tensor is vanishes. The $\boldsymbol{G_1}$- manifold that will be addressed in this study is one of the sixteen classes of almost Hermitian manifold. In 1976 [7], Hervella and Vidal studied the geometry of $\boldsymbol{G_1}$-manifold. The aforementioned manifold designated by $\boldsymbol{W_1}\oplus\boldsymbol{W_3}\oplus\boldsymbol{W_4}$, where $\boldsymbol{W_1}, \boldsymbol{W_3}$ and $\boldsymbol{W_4}$ respectively denote the nearly Kahler manifold ($\boldsymbol{\mathcal{NK}}$-manifold), the simi Hermitian manifold ($\boldsymbol{\mathcal{SH}}$-manifold) and locally Kahler manifold ($\boldsymbol{\mathcal{LCK}}$-manifold). In 2000 [13], Kirichenko and Tretiakova proved that the $\boldsymbol{G_1}$-manifold of zero constant type coincides with the class of **6**-dimensional $\boldsymbol{G_1}$-manifold of nonintegrable structure. By using the adjoined $\boldsymbol{G}$- structure space method, we were able to study the geometry properties of one types of *AH*- manifold called *M*-Projective tensor. Before us, the researchers interested in studying this type. In 1971[22], Pokhariyal and Mishra have interested in the study of Riemannian manifold and they also have identified a tensor of type (4.0) as *M*-projective. In 1975 and 1986 [19] [20], Ojha identified the properties of *M*-projective tensor in Sasakian and Kahler manifolds. In 2014[4], De and Mallick In are studied *M*-projective curvature tensor on an *N(k)*-quasi-Einstein manifold. In 2015[5], Devi and Singh are proved that globally ϕ-*M*- projectively symmetric Kenmotsu manifold to be an Einstein manifold. In 2016 [9], Jaiswal and Yadav are found the adequate condition for generalized *M*-projective ϕ-recurrent trans- Sasakian manifold to be an Einstein.

2. Preliminaries

Suppose that *X(M)* is module of vector field. $C^{\infty}(M)$ be the set of smooth function. An almost Hermition manifold (*AH*-manifold) is the treble $\{M, J, g = \langle\cdot,\cdot\rangle\}$, where M is even dimensional greater than 1; smooth manifold ; J is an endomorphism of $T_p(M)$ where $(J_p)^2 = -id,$ and $g =<.,.>$ is Riemann metric on M such that $\langle JX, JY\rangle = \langle X, Y\rangle; X, Y \in X(M)$ [16]. The $T_p(M)$ at the point $p \in M$ has a basis defined by $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ and is called a real adept basis or *RA*-basis. The image of *RA*-basis is construct a new basises $\{\varepsilon_1, \dots, \varepsilon_2, \bar{\varepsilon}_1, \dots, \bar{\varepsilon}_n\}$ on $T_p^c(M)$ which called A- basis[22]. We will use the indexes as following i, j, k, l in the range $1,2,\dots,2n$ and the indices a, b, c, d, e, f in the range $1,2,\dots,n$ and $\hat{a} = a + n$

The matrices of the J, g in a frame are given as follows[12]:

$$(J_j^i) = \begin{pmatrix} \sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n \end{pmatrix}, (g_{ij}) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

Definition 2.1 [3]. According to Banaru's classification, an $AH-manifold$ in the adjoined G-structure space, is called:

1) G_1-manifold if $B^{abc} = B^{[bac]}$.

2) Nearly Kahler manifold ($\mathcal{NK}$- manifold) if $B^{abc} = -B^{bac}$ and $B^{ab}{}_c = 0$;

3) Hermition manifold ($\mathcal{H}$-manifold) if $B^{abc} = 0$;

4) quasi manifold (QK-manifold) if $B^{ab}{}_c = 0$,

Where $B^{abc} = \frac{\sqrt{-1}}{2} J^a_{[\hat{b},\hat{c}]}$ $B^{ab}{}_c = -\frac{\sqrt{-1}}{2} J^a_{\hat{b},c}$ and $X, Y \in X(M)$ and the bracket [ ] denote to the Lie bracket.

Theorem2.1 [14]: The family of the equations of G_1- manifold in the a djoined G- structure space, given by the following forms:

1) $d\omega^a = \omega^a_b \Lambda \omega^b + B^{ab}{}_c \omega^c \Lambda \omega_b + B^{abc} \omega_b \Lambda \omega_c$;

2) $d\omega_a = -\omega^b_a \Lambda \omega_b + B_{ab}{}^c \omega_c \Lambda \omega^b + B_{abc} \omega^b \Lambda \omega^c$;

3) $d\omega^a_b = \omega^a_c \Lambda \omega^c_b + \left(2B^{adh} B_{hbc} + A^{ad}_{bc}\right) \omega^c \Lambda \omega_d (B^{ah}{}_{[c} B_{d]bh}$
$+A^a_{bcd})\, \omega^c \Lambda \omega^d + \left(-B_{bh}{}^{[c} B^{d]ah} + A^{acd}_b\right) \omega_c \Lambda \omega_d$,

where $\{\omega^i\}$ and $\{\omega^i_j\}$ are the components of the differential form and the components of the Riemannian metric g respectively, $\{A^{ad}_{bc}\}$ the components of holomorphic sectional tensor, $\{A^a_{bcd}, A^{acd}_b\}$ are some tensors on adjoined G-structure space.

Definition 2.2[15]: A Riemannian curvature tensor $\mathfrak{R}$ is a tensor of type (4,0) $\mathfrak{R}: T_p(M) \times T_p(M) \times T_p(M) \times T_p(M) \to \mathbb{R}$ which is defined as: $\mathfrak{R}(X, Y, Z, W) = g(\mathfrak{R}(Z, W)Y, X)$, where $\mathfrak{R}(X, Y)Z = \left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right)Z$; $X, Y, Z, W \in T_p(M)$ and satisfies the next properties:

1)$\mathfrak{R}(X, Y, Z, W) = -\mathfrak{R}(Y, X, Z, W)$;

2)$\mathfrak{R}(X, Y, Z, W) = -\mathfrak{R}(X, Y, W, Z)$;

3)$\mathfrak{R}(X, Y, Z, W) + \mathfrak{R}(X, Z, W, Y) + \mathfrak{R}(X, W, Y, Z) = 0$;

4)$\mathfrak{R}(X, Y, Z, W) = \mathfrak{R}(Z, W, X, Y)$.

The following theorem establishes the expression for the components of the Riemannian tensor of G_1- manifold in the adjoined G- structure space.

Theorem 2.2 [14]: The components of the Riemannian curvature tensor of G_1- manifold are given by the following forms:

1) $\mathfrak{R}_{abcd} = 2(B_{ab[cd]} + B^h_{ab} B_{hcd})$;

2) $\mathfrak{R}_{\hat{a}bcd} = 2A^a_{bcd}$;

3)$\mathfrak{R}_{\hat{a}\hat{b}cd} = -2(B^{abh}B_{hcd} + B^{ab}_{[cd]})$;

4) $\mathfrak{R}_{\hat{a}bc\hat{d}} = A^{ad}_{bc} + B^{adh}B_{hbc} - B^{ah}{}_{c}B_{hb}{}^{d}$,

The remaining components of $\mathfrak{R}$ are conjugate or vanishing.

Definition 2.3 [23]. A Richi tensor is a tensor (2, 0) which is defined as $r_{ij=} R^{k}_{ijk} = g^{kl}R_{kijl}$.

Definition 3.2.8 [11]: *AH*- manifold has $J-$invariant Richi tensor when $J \circ r = r \circ J$.

Lemma 3.2.9 [11] : The necessary and adequate conditions *AH*-manifold has $J-$invariant Richi tensor in the adjoined *G*- structure space is $r^{\hat{a}}_{b} = 0$.

Theorem 2.3[10] The components of the Richi tensor of G_1- manifold are given by the following forms:

i) $r_{ab} = 4A^{c}_{(ab)c}$;

ii) $r_{a\hat{b}} = -3B^{hbc}B_{cah} - 2B^{bc}_{[ab]} - A^{bc}_{ca} + B^{hb}_{a}B^{c}_{ch}$,

3. The main results.

The main idea in this paper, is to study the various geometric properties of the *M*-projective of G_1- manifold. The necessary and adequate condition for the G_1-manifold to be an Einstein manifold have been found.

Definition 3.1[22] The *M*-projective tensor is a tensor field of type (4.0), which is define onRiemann manifold by the form:

$$P^{*}(X,Y)Z = \mathfrak{R}(X,Y)Z - \frac{1}{2(n-1)}(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY)),$$

where $\mathfrak{R}$ is Riemannian curvature tensor, S is Richi tensor, Q is Richi operator and g is Riemannian metric.

Now, we can redefine the *M*-projective tensor on *AH*- manifold by the components form as follows:

$$Mp_{ijkl} = \mathfrak{R}_{ijkl} - \frac{1}{2(n-1)}\left(r_{jk}g_{il} - r_{ik}g_{jl} + r_{il}g_{jk} - r_{jl}g_{ik}\right).$$

(3.1)

Let us start with the following theorem, which determined the components of the *M*-projective of G_1- manifold.

Theorem 3.1: The components of the *M*-projective tensor of G_1- manifold are given by the following forms:

1) $Mp_{abcd} = 2\left(B_{ab[cd]} + B_{ab}^{h} B_{hcd}\right);$

2) $Mp_{\hat{a}bcd} = 2A_{bcd}^{a} - \frac{1}{n-1}(2A_{(bc)d}^{d}\delta_{d}^{a} - 2A_{(bd)h}^{h}\delta_{c}^{a});$

3) $Mp_{\hat{a}\hat{b}cd} = -2(B^{abh}\hat{A}_{hcd} + B_{[cd]}^{ab}) - \frac{1}{2(n-1)}(r_{c}^{b}\delta_{d}^{a} - r_{c}^{a}\delta_{d}^{b} + r_{d}^{a}\delta_{c}^{b} - r_{d}^{b}\delta_{c}^{a});$

4) $Mp_{\hat{a}bc\hat{d}} = A_{bc}^{ad} + B^{adh}B_{hbc} - B^{ah}{}_{c}B_{hb}{}^{d} + \frac{1}{2(n-1)}(r_{b}^{d}\delta_{c}^{a} + r_{c}^{a}\ddot{a}_{b}^{d}),$

Proof:

1) put $i = a,\ j = b,\ k = c$ and $l = d$, we get

$$Mp_{abcd} = \Re_{abcd} - \frac{1}{2(n-1)}(r_{bc}g_{ad} - r_{ac}g_{bd} + r_{ad}g_{bc} - r_{bd}g_{ac})$$

Making use of the equation (3.1) , we obtain

$$Mp_{abcd} = 2\left(B_{ab[cd]} + B_{ab}^{h}B_{hcd}\right)$$

2) put $i = \hat{a},\ j = b,\ k = c$ and $l = d$, we have

$$Mp_{\hat{a}bcd} = \Re_{\hat{a}bcd} - \frac{1}{2(n-1)}(r_{bc}g_{\hat{a}d} - r_{\hat{a}c}g_{bd} + r_{\hat{a}d}g_{bc} - r_{bd}g_{\hat{a}c})$$

$$Mp_{\hat{a}bcd} = 2A_{bcd}^{a} - \frac{1}{n-1}(2A_{(bc)d}^{d}\delta_{d}^{a} - 2A_{(bd)h}^{h}\delta_{c}^{a})$$

3) put $i = \hat{a},\ j = \hat{b},\ k = c$ and $l = d$, we obtain

$$Mp_{\hat{a}\hat{b}cd} = \Re_{\hat{a}\hat{b}cd} - \frac{1}{2(n-1)}(r_{\hat{b}c}g_{\hat{a}d} - r_{\hat{a}c}g_{\hat{b}d} + r_{\hat{a}d}g_{\hat{b}c} - r_{\hat{b}d}g_{\hat{a}c})$$

$$Mp_{\hat{a}\hat{b}cd} = -2\left(B^{abh}B_{hcd} + B_{[cd]}^{ab}\right) - \frac{1}{2(n-1)}\left(r_{c}^{b}\delta_{d}^{a} - r_{c}^{a}\delta_{d}^{b} + r_{d}^{a}\delta_{c}^{b} - r_{d}^{b}\delta_{c}^{a}\right)$$

4) put $i = \hat{a},\ j = b,\ k = c$ and $l = \hat{d}$, it follows that

$$Mp_{\hat{a}bc\hat{d}} = \Re_{\hat{a}bc\hat{d}} - \frac{1}{2(n-1)}(r_{bc}g_{\hat{a}\hat{d}} - r_{\hat{a}c}g_{b\hat{d}} + r_{\hat{a}\hat{d}}g_{bc} - r_{b\hat{d}}g_{\hat{a}c})$$

$$Mp_{\hat{a}bc\hat{d}} = A_{bc}^{ad} + B^{adh}B_{hbc} - B^{ah}{}_{c}B_{hb}{}^{d} + \frac{1}{2(n-1)}(r_{b}^{d}\delta_{c}^{a} + r_{c}^{a}\delta_{b}^{d}). \qquad \square$$

In the following theorem, we found the relationship between G_1-manifold and QK-manifold.

Theorem 3.2: Let M be G_1- manifold with vanishing M- projective tensor, then M is QK- manifold if M has vanishing Richi tensor.

Proof:

Suppose that M is G_1- manifold with vanishing M- projective curvature tensor.

Taking into account the Theorem 3.2, we get

$$A_{bc}^{ad} + B^{adh}B_{hbc} - B^{ah}{}_{c}B_{hb}{}^{d} + \frac{1}{2(n-1)}\left(r_b^d\delta_c^a + r_c^a\delta_b^d\right) = 0$$

Since the Richi tensor is vanishing, then we obtain

$$A_{bc}^{ad} + B^{adh}B_{hbc} - B^{ah}{}_{c}B_{hb}{}^{d} = 0$$

From the symmetrizing and anti-symmetrizing the indices (b, c), we obtain

$$-B^{ah}{}_{c}B_{hb}{}^{d} = 0$$

Contracting by the indices (c, d) and (a, b), consequently we get

$B^{ah}{}_{c}B_{ha}{}^{c} = 0$, which implies that $B^{ah}{}_{c}\bar{B}^{ah}{}_{c} = 0 \Leftrightarrow \sum_{a,h,d}\left|B^{ah}{}_{c}\right|^2 = 0 \Leftrightarrow B^{ah}{}_{c} = 0$

Therefore, by the Banaru's classification, we get that M is QK-manifold. □

The next theorem gives the necessary and adequate condition for the G_1- manifold to be the holomorphic sectional tensor identical equal to zero.

Theorem 3.3: Suppose that M ia a G_1- manifold with vanishing M-projective curvature tensor, then M has vanishing holomorphic sectional tensor if, and only if, M has vanishing Richi curvature tensor.

Proof:

Suppose that M is G_1- manifold with vanishing M- projective.

Taking into account Theorem 3.1, we obtain

$$A_{bc}^{ad} + B^{adh}B_{hbc} - B^{ah}{}_{c}B_{hb}{}^{d} + \frac{1}{2(n-1)}\left(r_b^d\delta_c^a + r_c^a\delta_b^d\right) = 0$$

(3.2)

Suppose that M has vanishing holomorphic sectional curvature tensor, consequently we have

$$B^{adh}B_{hbc} - B^{ah}{}_{c}B_{hb}{}^{d} + \frac{1}{2(n-1)}\left(r_b^d\delta_c^a + r_c^a\delta_b^d\right) = 0 \quad (3.3)$$

Symmetrizing and anti-symmetrizing by the indices (h, b), it follows that

$$\frac{1}{2(n-1)}\left(r_b^d\delta_c^a + r_c^a\delta_b^d\right) = 0 \quad (3.4)$$

Contracting by the indices (c, d), so the equation (3.4) becomes

$$\frac{1}{(n-1)} r_b^a = 0$$

Hence, the Richi curvature tensor vanishes.

Conversely, Let the Richi curvature tensor vanishes

By using the equation (3.2), we obtain

$$A_{bc}^{ad} + B^{adh} B_{hbc} - B^{ah}{}_c B_{hb}{}^d = 0$$

Symmetrizing and anti-symmetrizing by the indices (h, b), immediately we have

$$A_{bc}^{ad} = 0.$$

Therefore, M has vanishing holomorphic sectional curvature tensor. □

Definition 3.2 [21]: An Riemannian manifold is called an Einstein manifold if Richi tensor meets the equation $r_{ij} = e g_{ij}$,

where e is cosmological constant.

The necessary and adequate condition for the G_1- manifold to be an Einstein manifold is given in the next theorem.

Theorem 3.4: If M is G_1- manifold with vanishing M- projective tensor and J- invariant Richi tensor, then The necessary and adequate condition for the M to be an Einstein manifold M is $A_{ac}^{ad} = \frac{-e}{(n-1)} \delta_c^d$, where e is cosmological constant.

Proof:

Suppose that M is G_1- manifold with vanishing M- projective tensor. Then by using the Theorem 3.1, we get

$$A_{bc}^{ad} + B^{adh} B_{hbc} - \hat{A}^{ah}{}_c B_{hb}{}^d + \frac{1}{2(n-1)} \left(r_b^d \delta_c^a + r_c^a \delta_b^d \right) = 0$$

Symmetrizing and anti-symmetrizing by the indices (h, b), we get

$$A_{bc}^{ad} + \frac{1}{2(n-1)} \left(r_b^d \delta_c^a + r_c^a \delta_b^d \right) = 0 \qquad (3.5)$$

Since M is Einstein manifold , consequently, we get

$$A_{bc}^{ad} + \frac{e}{2(n-1)} (\delta_c^a \delta_b^d + \delta_b^d \delta_c^a) = 0$$

Contracting by the indices (a, b), we obtain

$$A_{ac}^{ad} = \frac{-e}{(n-1)}\delta_c^d.$$

Conversely, by using the equation (3.5), we have

$$A_{bc}^{ad} + \frac{1}{2(n-1)}(r_c^a\delta_b^d + r_b^d\delta_c^a) = 0$$

Contracting by the indices (a, b), we obtain

$$A_{ac}^{ad} + \frac{1}{(n-1)}r_c^d = 0$$

Substituting A_{ac}^{ad} in the equation (3.5), we get

$$\frac{-e}{(n-1)}\delta_c^d + \frac{1}{(n-1)}r_c^d = 0$$

$$r_c^d = e\delta_c^d$$

Since M has J- invariant Richi tensor, it follows that M is Einstein manifold. □

Theorem 3.5: Let M be a G_1- manifold of vanishing M-projective tensor and J- invariant Richi tenor, if M is Einstein manifold, then M is $\mathcal{NK}$- manifold.

Proof: Suppose that M is a G_1- manifold of vanishing M- projective tensor.

Taking into account the Theorem 3.1, we have

$$A_{bc}^{ad} + B^{adh}B_{hbc} - B^{ah}{}_cB_{hb}{}^d + \frac{1}{2(n-1)}\left(r_b^d\delta_c^a + r_c^a\delta_b^d\right) = 0$$

Contracting by the indices (a, b), we get

$$A_{ac}^{ad} + B^{adh}B_{hac} - B_c^{ah}B_{ha}^d + \frac{e}{(n-1)}\delta_c^d = 0$$

By making use of the Theorem 3.4, we obtain

$$B^{adh}B_{hac} - B_c^{ah}B_{ha}^d = 0$$

Symmetrizing by the indices (a, d), it follows that

$$B_c^{ah}B_{ha}^d = 0$$

Contracting by the indices (d, c), we deduce

$$B^{ah}{}_dB_{ha}{}^d = 0,$$

which implies that: $\bar{B}^{ah}{}_dB^{ah}{}_d = 0 \Longrightarrow \sum_{a,h,d}\left|B^{ah}{}_d\right|^2 = 0 \Leftrightarrow B^{ah}{}_d = 0$

Therefore, M is $\mathcal{NK}$-manifold.

Finally, we were able to find a link between G_1-manifold, $\mathcal{H}$-manifold and $\mathcal{NK}$- manifold.

Theorem 3.6: Suppose that M is G_1- manifold with vanishing M- projective curvature tensor and vanishing Richi curvature tensor, then M is $\mathcal{H}$-manifold if, and only if, M is $\mathcal{NK}$- manifold.

Proof:

Suppose that M is G_1-manifold with vanishing M-projective curvature tensor and vanishing Richi curvature tensor, so according to the Theorems 3.1and 3.3 we obtain

$$B^{adh}B_{hbc} - B^{ah}_{c}B_{hb}^{d} = 0 \tag{3.6}$$

Since M is G_1-manifold, so the equation (3.6) becomes

$$-B^{ahd}B_{hbc} - B^{ah}_{c}B_{hb}^{d} = 0 \tag{3.7}$$

Let M be $\mathcal{H}$-manifold, we deduce

$$B^{ah}_{c}B_{hb}^{d} = 0$$

Contracting by the indices (d, c) and (a, b), we obtain

$$B^{ah}_{d}B^{d}_{ha} = 0 \Leftrightarrow B^{ah}_{d}\bar{B}^{ah}_{d} = 0 \Leftrightarrow \sum_{a,h,d}\left|B^{ah}_{d}\right|^2 = 0 \Leftrightarrow B^{ah}_{d} = 0$$

Hence, M is $\mathcal{NK}$- manifold.

Conversely, suppose that M is $\mathcal{NK}$- manifold, so the equation (3.7), becomes

$$-B^{ahd}B_{hbc} = 0$$

Contracting by the indices (a, c) and (b, d), we have

$$B^{ahb}B_{hba} = 0$$

$$B^{abh}B_{abh} = 0,$$

which implies that $B^{abh}\bar{B}^{abh} = 0 \Rightarrow \sum_{a,b,h}\left|B^{abh}\right|^2 = 0 \Leftrightarrow B^{abh} = 0$

Therefore, according to the Banaru's classification, M is $\mathcal{H}$-manifold. □

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Chaos in Beddington–DeAngelis food chain model with fear

Hiba Abdullah Ibrahim and Raid Kamel Naji

Department of the Mathematics, College of Science, University of Baghdad, Baghdad, IRAQ

haibrahim82@yahoo.com; rknaji@gmail.com

Abstract. In the current paper, the effect of fear in three species Beddington–DeAngelis food chain model is investigated. A three species food chain model incorporating Beddington-DeAngelis functional response is proposed, where the growth rate in the first and second level decreases due to existence of predator in the upper level. The existence, uniqueness and boundedness of the solution of the model are studied. All the possible equilibrium points are determined. The local as well as global stability of the system are investigated. The persistence conditions of the system are established. The local bifurcation analysis of the system is carried out. Finally, numerical simulations are used to investigate the existence of chaos and understand the effect of varying the system parameters. It is observed that the existence of fear up to a critical value has a stabilizing effect on the system; otherwise it works as an extinction factor in the system.

1. Introduction

It is well known that the study of the prey-predator systems is an important subject in ecology and biology, due to the wide existence of such type of interaction in the environment [1-2]. Such prey-predator models have been extensively studied in literatures through previous yeas [3-5]. Most of these studies in literatures mainly concentrated on the local stability as well as persistence [6-7], while recent studies display a direction in exploring dynamical behaviors, for example, local bifurcation and chaos [8-11]. Food chain system is an ecological system that depends completely on the prey-predator interaction in which the energy transfers directly from one level to the higher level.

The effect of predator on the prey population within ecological systems may be direct or indirect or both. In the state of direct effect, the predator preys upon prey through killing them directly [12]. While, in the state of indirect effect, predator motivate fear in prey and change prey's behavior due to decreasing of the prey growth rate [13]. The fear effect is appearance of stress on prey. Recent works presented that the fear is strong enough to affect into the dynamics of ecological systems [14-15]. Many researchers studied the effect of fear in the ecological models. For example, Wang et al [16] have suggested a prey-predator model, where the effect of fear plays important role in the growth of prey. They spotted that the fear can stabilize the system. Zhang et al [17] have investigated the effect of anti-predator behavior that resulting from the fear of predators. They adopted a Holling type-II prey-predator, which incorporating a prey refuge. Pal et al [18] have studied a two species prey-predator model with a functional response of Beddington–DeAngelis type in case of existence of fear. Panday et al [19] investigated the role of fear in a food chain model consisting of three levels with a functional response of Holling type-II, they observed that fear effect can stabilize the system from chaos to stable.

In the present study, we are particularly interested to the dynamics of a food chain model with Beddington–DeAngelis (BD) type of functional response that proposed in [20] in case of existence of fear. It is assumed that the growth rates of prey and middle predator are decreasing as a cost of fear of upper level predator.

In Section (2) the mathematical model is formulated and then all the mathematical properties of the solution of the model are studied. Section (3) studied the stability analysis and determined the conditions of persistent of the model. Local bifurcation near each equilibrium point is discussed in section (4). However, numerical simulation is investigated in section (5). Eventually, in section (6) the discussion and conclusions are carried out from our obtained analytical.

2. **Mathematical Model**

In this section, a BD food chain model with fear is suggested. The mathematical model is formulated according to the following hypotheses:

- Let the densities of prey, middle predator and top predator at time T are given by $X(T)$, $Y(T)$ and $Z(T)$ respectively.
- In the absence of middle predator $Y(T)$, the prey grows according to logistic function with intrinsic growth rate $r > 0$ and carrying capacity $k > 0$. While, the growth rate of prey decreases due to fear from the predation by middle predator with fear rate constant $\alpha > 0$.
- The middle predator $Y(T)$ consumes the prey according to BD functional response with maximum attack rate $a_1 > 0$, the half saturation level $b_1 > 0$ and middle predator's encounters rate $C_1 > 0$. However, The food converted to middle predator $Y(T)$ with conversion rate $0 < e_1 < 1$. It is assumed that, in the absence of the prey, the middle predator decays exponentially with natural death rate $D_1 > 0$. On the other hand, since the middle predator facing predation by top predator $Z(T)$ too, the growth rate of middle predator decreases with fear constant $\beta > 0$.
- The top predator $Z(T)$ consumes the middle predator according to BD functional response with maximum attack rate $a_2 > 0$, the half saturation level $b_2 > 0$, top predator's encounters rate $C_2 > 0$ and then the food consumed by top predator is converted with conversion rate $0 < e_2 < 1$. However, in the absence of middle predator, it is decay exponentially with natural death rate $D_2 > 0$.

According to the above mentioned hypotheses, the dynamics of BD food chain model with fear represented by the following set of differential equations.

$$\begin{aligned}
\frac{dX}{dT} &= \left(\frac{r\,X}{1+\alpha Y}\right)\left(1-\frac{X}{k}\right) - \frac{a_1\,X\,Y}{b_1 Y + X + C_1} \\
\frac{dY}{dT} &= \left(\frac{e_1\,a_1\,X\,Y}{b_1 Y + X + C_1}\right)\left(\frac{1}{1+\beta Z}\right) - \frac{a_2\,Y\,Z}{b_2 Z + Y + C_2} - D_1\,Y \\
\frac{dZ}{dT} &= \frac{e_2\,a_2\,Y\,Z}{b_2 Z + Y + C_2} - D_2\,Z
\end{aligned} \tag{1}$$

where $X(0) \geq 0, Y(0) \geq 0,$ and $Z(0) \geq 0$.

Now, to simplify the model, the following dimensionless variables and parameters are used:

$$\begin{aligned}
&t = rT\,, x = \frac{X}{k}\,, y = \frac{a_1\,Y}{r\,k}\,, z = \frac{a_1 a_2\,Z}{r^2 k}\,, \alpha_1 = \frac{\alpha\,r\,k}{a_1}\,, \beta_1 = \frac{r\,b_1}{a_1}\,, \gamma_1 = \frac{C_1}{k} \\
&\theta_1 = \frac{e_1\,a_1}{r}, \alpha_2 = \frac{r^2 \beta\,k}{a_1 a_2}, \beta_2 = \frac{r\,b_2}{a_2}, \gamma_2 = \frac{C_2\,a_1}{r\,k}\,, d_1 = \frac{D_1}{r}, \theta_2 = \frac{e_2\,a_2}{r}, d_2 = \frac{D_2}{r}
\end{aligned} \tag{2}$$

Therefore, system (1) reduced to:

$$\begin{aligned}
\frac{dx}{dt} &= x\left[\frac{(1-x)}{(1+\alpha_1\,y)} - \frac{y}{\beta_1 y + x + \gamma_1}\right] = x f_1(x,y,z) \\
\frac{dy}{dt} &= y\left[\frac{\theta_1\,x}{\beta_1 y + x + \gamma_1}\left(\frac{1}{1+\alpha_2\,z}\right) - \frac{z}{\beta_2 z + y + \gamma_2} - d_1\right] = y f_2(x,y,z) \\
\frac{dz}{dt} &= z\left[\frac{\theta_2\,y}{\beta_2 z + y + \gamma_2} - d_2\right] = z f_3(x,y,z)
\end{aligned} \tag{3}$$

Theorem 1: System (3) has a uniformly bounded (UB) solutions.

Proof: From the first equation, we get

$$\frac{dx}{dt} \leq x[1-x]$$

By the usual comparison theorem the following is obtained:

$$x(t) \leq \frac{x_0}{x_0 + e^{-t}(1-x_0)}$$

where $x_0 = x(0)$ and then for $t \to \infty$, we get $x(t) \leq 1$.

Now, define the function $\omega(t) = x(t) + y(t) + z(t)$; then the time derivative of $\omega(t)$ is determined by:

$$\frac{d\omega}{dt} = \frac{x(1-x)}{(1+\alpha_1\, y)} - \frac{x\, y}{\beta_1 y + x + \gamma_1}\left(1 - \frac{\theta_1}{1+\alpha_2\, z}\right) - \frac{y\, z(1-\theta_2)}{\beta_2 z + y + \gamma_2} - d_1 y - d_2 z.$$

Therefore, due to the biological meaning of the system's parameters and the bound of $x(t)$, it is obtained that

$$\frac{d\omega}{dt} + \mu\, \omega \leq 2$$

where $\mu = min\{1, d_1, d_2\}$. Hence, due to the Gronwall lemma [21], we obtain $\omega(t) \leq \omega_0 e^{-\mu\, t} + \frac{L}{\mu}(1 - e^{-\mu\, t})$. Thus, for $t \to \infty$, we have that $0 \leq \omega(t) \leq \frac{2}{\mu}$. Hence all solutions of system (3) are UB and the proof is done.

3. The stability analysis

In this section, the existence and stability of the equilibrium points (EPs) are discussed. It's observed that, system (3) has at most four EPs, which can be stated as follows:

1- The trivial equilibrium point $q_0 = (0,0,0)$ always exists.
2- The axial equilibrium point (AEP) that given by $q_1 = (1, 0, 0)$ always exists.
3- The top predator free equilibrium point (TPFEP), which is given by $q_2 = (\bar{x}, \bar{y}, 0)$, where

$$\bar{x} = \frac{d_1\,(\beta_1\, \bar{y} + \gamma_1)}{(\theta_1 - d_1)} \qquad (4a)$$

While, $\bar{y}$ is a unique positive root of the equation:

$$H_1\, y^2 + H_2\, y + H_3 = 0 \qquad (4b)$$

where $H_1 = -(\beta_1^2\, \theta_1\, d_1 + \alpha_1(\theta_1 - d_1)^2) < 0$

$$H_2 = \beta_1\, \theta_1^2 - \beta_1\, \theta_1\, d_1 - 2\, \beta_1\, \theta_1\, \gamma_1\, d_1 - \theta_1^2 + 2\, \theta_1\, d_1 - d_1^2.$$

$$H_3 = \theta_1^2\, \gamma_1 - \theta_1\, \gamma_1\, d_1 - \theta_1\, \gamma_1^2\, d_1.$$

So by *DESCARTES' RULE* of sign [22], equation (4b) has a unique positive root provided that:

$$d_1(1+\gamma_1) < \theta_1$$

(5)

Therefore, q_2 exists uniquily under the above condition.

4- The positive equilibrium point (PEP), that given by $q_3 = (x^*, y^*, z^*)$, where

$$x^* = \frac{-G_2+\sqrt{{G_2}^2-4\,G_3}}{2}\,;\, z^* = \frac{\theta_2\, y^* - d_2\,(y^*+\gamma_2)}{\beta_2 d_2} \qquad (6a)$$

with

$$G_2 = \beta_1\, y^* + \gamma_1 - 1$$

$$G_3 = y^*(1 - \beta_1 + \alpha_1 y^*) - \gamma_1.$$

However, y^* is a positive root of the following equation:

$$K_1\, y^2 + K_2\, y + K_3 = 0 \qquad (6b)$$

here $K_1 = -\beta_1\, d_1(1 + \alpha_2\, z^*) < 0$,

$$K_2 = \theta_1 x^* - (1 + \alpha_2\, z^*)\big(\beta_1 z^*(1 + \beta_2\, d_1) + d_1(\beta_1\, \gamma_1 + x^* + \gamma_1)\big)\,,$$

$$\begin{aligned} K_3 = {} & (\theta_1 - d_1)(\beta_2\, x^* z^* + \gamma_2 x^*) \\ & -(1 + \alpha_2\, z^*)[z^*(x^* + \gamma_1) + d_1\, \gamma_1(\beta_2 z^* + \gamma_2)] \\ & -\alpha_2\, d_1\, x^*\, z^*(\beta_2 z^* + \gamma_2) \end{aligned}$$

So by *DESCARTES' RULE* of sign [22], equation (6b) has a unique positive root provided that:

$$K_3 > 0$$

(7a)

Therefore, the PEP exists uniquely in the $Int.\mathbb{R}^3_+$ provided that in addition to condition (7a) the following conditions hold.

$$y^*(1 + \alpha_1 y^*) < \beta_1\, y^* + \gamma_1 \qquad (7b)$$

$$d_2\,(y^* + \gamma_2) < \theta_2 y^* \qquad (7c)$$

Now the dynamical behavior of system (3) can be studied locally using linearization technique. Observed that it is simple to verify that, the Jacobian matrix (JM) of system (3) at $q_0 = (0,0,0)$ can be written in the form:

$$J(q_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}$$

(8a)

Thus, the eigenvalues of $J(q_0)$ are given by

$\lambda_{01} = 1 > 0, \lambda_{02} = -d_1 < 0, \lambda_{03} = -d_2 < 0.$ (8b)

Therefore, the trivial equilibrium point is a saddle point.

The JM at the (AEP), that is given by $q_1 = (1, 0, 0)$, can be written as:

$$J(q_1) = \begin{bmatrix} -1 & -\left(\frac{1}{1+\gamma_1}\right) & 0 \\ 0 & \frac{\theta_1}{1+\gamma_1} - d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}$$

(9a)

Hence, the eigenvalues of $J(q_1)$ are given by

$\lambda_{11} = -1 < 0, \lambda_{12} = \frac{\theta_1}{1+\gamma_1} - d_1$ and $\lambda_{13} = -d_2 < 0.$ (9b)

Clearly, the AEP is locally asymptotically stable (**$\mathcal{LAS}$**) if the following condition holds:

$$\theta_1 < d_1(1 + \gamma_1)$$

(10)

Moreover, it is easy to verify that, the point q_1 is a saddle point if the condition (5) holds.

The JM at the (TPFEP), $q_2 = (\bar{x}, \bar{y}, 0)$, can be written in the form:

$$J(q_2) = \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}$$

(11a)

where $b_{11} = \bar{x}\left(\frac{-1}{1+\alpha_1\bar{y}} + \frac{\bar{y}}{(\beta_1\bar{y}+\bar{x}+\gamma_1)^2}\right)$, $b_{12} = -\left(\frac{\bar{x}\,(1-\bar{x}\,)\alpha_1}{(1+\alpha_1\bar{y})^2} + \frac{\bar{x}\,(\bar{x}\,+\gamma_1)}{(\beta_1\bar{y}+\bar{x}+\gamma_1)^2}\right)$

$b_{21} = \frac{\theta_1\bar{y}\,(\beta_1\,\bar{y}+\gamma_1)}{(\beta_1\bar{y}+\bar{x}+\gamma_1)^2}$, $b_{22} = \frac{-\theta_1\beta_1\,\bar{x}\,\bar{y}}{(\beta_1\bar{y}+\bar{x}+\gamma_1)^2}$, $b_{23} = -\left(\frac{\alpha_2\theta_1\bar{x}\,\bar{y}}{\beta_1\bar{y}+\bar{x}+\gamma_1} + \frac{\bar{y}}{\bar{y}+\gamma_2}\right)$

$b_{33} = \frac{\theta_2\bar{y}}{\bar{y}+\gamma_2} - d_2$.

Then the characteristic equation of $J(q_2)$ can be determined as follows:

$$(\lambda^2 - T_2\lambda + D_2)\,(b_{33} - \lambda) = 0 \quad (11b)$$

where

$$T_2 = b_{11} + b_{22}$$

$$D_2 = b_{11}b_{22} - b_{12}b_{21}$$

Consequently, the eigenvalues are written as:

$$\lambda_{21} = \frac{T_2}{2} - \frac{\sqrt{{T_2}^2-4D_2}}{2}, \qquad \lambda_{22} = \frac{T_2}{2} + \frac{\sqrt{{T_2}^2-4D_2}}{2}, \qquad \lambda_{23} = \frac{\theta_2\bar{y}}{\bar{y}+\gamma_2} - d_2$$

(11c)

Hence the (TPFEP) is **$\mathcal{LAS}$** provided the following conditions hold:

$$\frac{\bar{y}}{(\beta_1\bar{y}+\bar{x}+\gamma_1)^2} < \frac{1}{1+\alpha_1\bar{y}}$$

(12a)

$$\theta_2 \bar{y} < d_2(\bar{y} + \gamma_2) \tag{12b}$$

The JM at the PEP, that given by $q_3 = (x^*, y^*, z^*)$, can be written in the form

$$J(q_3) = [a_{ij}]_{3\times 3} \tag{13a}$$

where

$a_{11} = x^* \left(\frac{-1}{1+\alpha_1 y^*} + \frac{y^*}{(\beta_1 y^* + x^* + \gamma_1)^2}\right)$, $a_{12} = -\left(\frac{x^*(1-x^*)\alpha_1}{(1+\alpha_1 y^*)^2} + \frac{x^*(x^*+\gamma_1)}{(\beta_1 y^* + x^* + \gamma_1)^2}\right)$, $a_{13} = 0$

$a_{21} = \frac{\theta_1 y^*(\beta_1 y^* + \gamma_1)}{(1+\alpha_2 z^*)(\beta_1 y^* + x^* + \gamma_1)^2} > 0$, $a_{22} = \frac{-\theta_1 \beta_1 x^* y^*}{(1+\alpha_2 z^*)(\beta_1 y^* + x^* + \gamma_1)^2} + \frac{y^* z^*}{(\beta_2 z^* + y^* + \gamma_2)^2}$,

$a_{23} = -\left(\frac{\alpha_2 \theta_1 x^* y^*}{(\beta_1 y^* + x^* + \gamma_1)(1+\alpha_2 z^*)^2} + \frac{y^*(y^* + \gamma_2)}{(\beta_2 z^* + y^* + \gamma_2)^2}\right) < 0$,

$a_{31} = 0$, $a_{32} = \frac{\theta_2 z^*(\beta_2 z^* + \gamma_2)}{(\beta_2 z^* + y^* + \gamma_2)^2} > 0$, $a_{33} = \frac{-\theta_2 \beta_2 y^* z^*}{(\beta_2 z^* + y^* + \gamma_2)^2} < 0$.

Then the characteristic equation of $J(q_3)$ is

$$\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0 \tag{13b}$$

where

$$A_1 = -(a_{11} + a_{22} + a_{33})$$

$$A_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{23}a_{32} - a_{12}a_{21}$$

$$A_3 = a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33}$$

while

$$\begin{aligned}\Delta = A_1 A_2 - A_3 &= -(a_{11} + a_{22})[a_{11}a_{22} - a_{12}a_{21}] - 2a_{11}a_{22}a_{33} \\ &\quad -(a_{22} + a_{33})[a_{22}a_{33} - a_{23}a_{32}] - a_{11}a_{33}[a_{11} + a_{33}]\end{aligned}$$

Now, according to the Routh-Hawirtiz criterion [23], the roots of equation (13b) have negative real parts provided that $A_1 > 0$, $A_3 > 0$ and $\Delta > 0$. Direct calculation shows that these conditions hold provided that

$$\frac{y^*}{(\beta_1 y^* + x^* + \gamma_1)^2} < \frac{1}{1+\alpha_1 y^*} \tag{14a}$$

$$\frac{z^*}{(\beta_2 z^* + y^* + \gamma_2)^2} < \frac{\theta_1 \beta_1 x^*}{(1+\alpha_2 z^*)(\beta_1 y^* + x^* + \gamma_1)^2} \tag{14b}$$

Therefore the PEP is $\mathcal{LAS}$ under the conditions (14a)-(14b).

Obviously system (3) has only one possible subsystem lying in the first quadrant of xy −plane. This subsystem can be written as:

$$\frac{dx}{dt} = x\left[\frac{(1-x)}{(1+\alpha_1\, y)} - \frac{y}{\beta_1 y+x+\gamma_1}\right] = g_1(x,y)$$
$$\frac{dy}{dt} = y\left[\frac{\theta_1\, x}{\beta_1 y+x+\gamma_1} - d_1\right] = g_2(x,y) \qquad (15)$$

Now, in order to investigate the existence of periodic dynamics in the interior of the first quadrant of xy −plane, define the Dulac function as $(x,y) = \frac{1}{x\,y}$. Clearly $B(x,y) > 0$ and C^1 function in the $Int.\mathbb{R}_+^2$ of the xy −plane . Further, we have

$$\Delta(x,y) = \frac{\partial(B\, g_1)}{\partial x} + \frac{\partial(B\, g_2)}{\partial y} = -\frac{1}{y\,(1+\alpha_1\, y)} + \frac{1-\theta_1\,\beta_1}{(\beta_1 y+x+\gamma_1)^2}$$

Then $\Delta(x,y)$ does not identically zero in the $Int.\mathbb{R}_+^2$ of the xy −plane and does not change sign under one of the following two conditions:

$$\frac{1-\theta_1\,\beta_1}{(\beta_1 y+x+\gamma_1)^2} < \frac{1}{y\,(1+\alpha_1\, y)}$$

(16a)

or

$$\frac{1-\theta_1\,\beta_1}{(\beta_1 y+x+\gamma_1)^2} > \frac{1}{y\,(1+\alpha_1\, y)}$$

(16b)

Therefore, by using Dulac-Bendixson criterion [24], there is no closed curve lying in the $Int.\mathbb{R}_+^2$ of the xy −plane for all the trajectories satisfying condition (16a) or condition (16b). Hence according to the Poincare-Bendixon theorem [24], the unique equilibrium point in the $Int.\mathbb{R}_+^2$ of the xy −plane that given by q_2 will be a globally asymptotically stable ($\mathcal{GAS}$) whenever it is $\mathcal{LAS}$.

Theorem 2: Assume that either conditions (16a) or (16b) holds and let the following conditions hold then system (3) is uniformly persistent.

$$d_1(1+\gamma_1) < \theta_1 \qquad (17a)$$

$$d_2(\bar{y}+\gamma_2) < \theta_2\,\bar{y} \qquad (17b)$$

Proof: Let us use the average Lyapunov method [25]. Consider the following function $(x,y,z) = x^{p_1}\, y^{p_2}\, z^{p_3}$, where p_j , $\forall j = 1,2,3$ are positive constants. Obviously $\varphi(x,y,z) > 0$ for all $(x,y,z) \in Int.\mathbb{R}_3^+$ and $\varphi(x,y,z) \to 0$ when $x \to 0$ or $y \to 0$ or $z \to 0$. Consequently, we obtain

$$\Omega(x,y,z) = \frac{\varphi'(x,y,z)}{\varphi(x,y,z)} = p_1\left[\frac{(1-x)}{(1+\alpha_1\, y)} - \frac{y}{\beta_1 y+x+\gamma_1}\right]$$
$$+p_2\left[\frac{\theta_1\, x}{\beta_1 y+x+\gamma_1}\left(\frac{1}{1+\alpha_2\, z}\right) - \frac{z}{\beta_2 z+y+\gamma_2} - d_1\right]$$
$$+\,p_3\left[\frac{\theta_2\, y}{\beta_2 z+y+\gamma_2} - d_2\right]$$

Now, according to average Lyapunov method, the proof follows if $\Omega(E) > 0$ for any boundary equilibrium point E, with suitable choice of constants $p_1 > 0$, $p_2 > 0$, and $p_3 > 0$.

$$\Omega(q_1) = p_2\left(\frac{\theta_1}{1+\gamma_1} - d_1\right) + p_3(-d_2)$$

$$\Omega(q_2) = p_3\left(\frac{\theta_2\,\bar{y}}{\bar{y}+\gamma_2} - d_2\right)$$

Clearly, $\Omega(q_1) > 0$ under condition (17a) for appropriate choice of positive constants p_2 and p_3, so that p_2 is large enough with respect to the constant p_3. While, $\Omega(q_2) > 0$ under condition (17b). Hence the proof is complete.

Theorem 3: Assume that the AEP is $\mathcal{LAS}$, then it is a $\mathcal{GAS}$ in the $\text{Int.}\,\mathbb{R}^3_+$ provided that the following condition holds.

$$\frac{1+\theta_1}{\gamma_1} < d_1$$

(18)

Proof: Define the function

$$u(x,y,z) = \int_1^x \frac{m-1}{m}dm + y + \frac{1}{\theta_2}z$$

Clearly the function u is positive definite so that $u(1,0,0) = 0$ and $u(x,y,z) > 0$ for all $(x,y,z) \in \mathbb{R}^3_+$ with $(x,y,z) \neq (1,0,0)$ and $x > 0$.

Now, straightforward calculations give that

$$\frac{du}{dt} \leq -\frac{(x-1)^2}{1+\alpha_1 y} - \left[d_1 - \frac{1}{\beta_1 y+x+\gamma_1} - \frac{\theta_1 x}{(\beta_1 y+x+\gamma_1)(1+\alpha_2 z)}\right]y - \frac{d_2}{\theta_2}z$$

$$\frac{du}{dt} < -\frac{(x-1)^2}{1+\alpha_1 y} - \left[d_1 - \frac{1+\theta_1}{\gamma_1}\right]y - \frac{d_2}{\theta_2}z$$

Hence under condition (18), we obtain that $\frac{du}{dt}$ will be negative definite. Then u is a Lyapunov function ($\mathcal{LF}$). Therefore AEP is a $\mathcal{GAS}$.

Theorem 4: Assume that the PFEP is $\mathcal{LAS}$, then it is a $\mathcal{GAS}$ in the $\text{Int.}\,\mathbb{R}^3_+$ provided that the following conditions hold.

$$R_1 < (1+\alpha_1\bar{y})R_2 \qquad (19a)$$

$$q_{12}{}^2 < 4\, q_{11}q_{22} \qquad (19b)$$

$$\frac{\theta_2\,\bar{y}}{\gamma_2} < d_2 \qquad (19c)$$

where all the symbols are described clearly in the proof.

Proof: Consider the following function

$$V(x,y,z) = \int_{\bar{x}}^x \frac{u-\bar{x}}{u}du + \frac{1}{\theta_2}\int_{\bar{y}}^y \frac{v-\bar{y}}{v}dv + z.$$

Obviously the function $V(x,y,z) > 0$ is a continuously differentiable real valued function for all $(x,y,z) \in \mathbb{R}^3_+$ and $(x,y,z) \neq (\bar{x},\bar{y},0)$ with $x > 0, y > 0$, while $V(\bar{x},\bar{y},0) = 0$.

Now, straightforward calculations give that

$$\frac{dV}{dt} \leq -q_{11}(x-\bar{x})^2 - q_{12}(x-\bar{x})(y-\overline{y}) - q_{22}(y-\overline{y})^2 - z\,R_1R_2\left[d_2 - \frac{\theta_2\,\overline{y}}{\gamma_2}\right]$$

where $q_{11} = (1+\alpha_1\overline{y})R_2 - R_1$,

$q_{12} = (1+\alpha_1\overline{x})R_2 + \gamma_1\left(1+\frac{\theta_1\,\theta_2}{(1+\alpha_2 z)}\right)R_1 + \left(\overline{x} - \frac{\theta_1\,\theta_2\,\beta_1\,\overline{y}}{(1+\alpha_2 z)}\right)R_1$,

$q_{22} = \frac{\theta_1\,\theta_2\,\beta_1\,\bar{x}}{(1+\alpha_2 z)}R_1$.

with $R_1 = (1+\alpha_1 y)(1+\alpha_1\overline{y})$ and $R_2 = (\beta_1\,y + x + \gamma_1)(\beta_1\,\overline{y} + \bar{x} + \gamma_1)$. Accordingly, by using the given conditions (19a)–(19c), we obtain

$$\frac{dV}{dt} \leq -\left[\sqrt{q_{11}}(x-\bar{x}) + \sqrt{q_{22}}(y-\overline{y})\right]^2 - z\,R_1R_2\left[d_2 - \frac{\theta_2\,\overline{y}}{\gamma_2}\right].$$

Therefore, the derivative $\frac{dV}{dt}$ is negative definite and then V is a $\mathcal{LF}$. Thus the PFEP is a $\mathcal{GAS}$.

Theorem 5: Assume that the PEP is $\mathcal{LAS}$ in the Int. $\mathbb{R}^3_+$, then it is a $\mathcal{GAS}$ provided that the following conditions hold:

$$q_{12}{}^2 < 2\,q_{11}q_{22} \tag{20a}$$

$$q_{23}{}^2 < 2\,q_{22}q_{33} \tag{20b}$$

$$\frac{y^*}{R_2} < \frac{(1+\alpha_1 y^*)}{R_1} \tag{20c}$$

$$\frac{z^*}{R_4} < \frac{\theta_1\,\beta_1\,x^*\,(1+\alpha_2 z^*)}{R_2\,R_3} \tag{20d}$$

where all the symbols are described clearly in the proof.

Proof: Consider the positive definite function

$$l(x,y,z) = \int_{x^*}^{x}\frac{u-x^*}{u}du + \int_{y^*}^{y}\frac{v-y^*}{v}dv + \frac{1}{\theta_2}\int_{z^*}^{z}\frac{w-z^*}{w}dw$$

Clearly, the function $l(x,y,z) > 0$ is a continuously differentiable real valued function for all $(x,y,z) \in \mathbb{R}^3_+$ with $(x,y,z) \neq (x^*,y^*,z^*)$ and $x > 0, y > 0, z > 0$, while $l(x^*,y^*,z^*) = 0$.

Now, the derivative of this function with respect to time can be written as

$$\frac{dl}{dt} = -q_{11}(x-x^*)^2 - q_{12}(x-x^*)(y-y^*) - q_{22}(y-y^*)^2 - q_{23}(y-y^*)(z-z^*) - q_{33}(z-z^*)^2$$

Here $q_{11} = \frac{R_2(1+\alpha_1 y^*) - R_1\,y^*}{R_1\,R_2}$, $q_{12} = \frac{\alpha_1(1-x^*)}{R_1} + \frac{\gamma_1+x^*}{R_2} - \frac{\theta_1\,(\beta_1\,y^*+\gamma_1)(1+\alpha_2\,z^*)}{R_2\,R_3}$,

$q_{22} = \frac{\theta_1\,\beta_1\,x^*(1+\alpha_2\,z^*)}{R_2\,R_3} - \frac{z^*}{R_4}$, $q_{23} = \frac{\theta_1\,\alpha_2 x^*(\beta_1\,y+x+\gamma_1)}{R_2\,R_3} + \frac{\gamma_2+y^*}{R_4} - \frac{\theta_2\,(\beta_2\,z^*+\gamma_2)}{R_4}$

and $\quad q_{33} = \frac{\beta_2\, \theta_2 y^*}{R_4}$.

while $\quad R_1 = (1+\alpha_1 y)(1+\alpha_1 y^*), \qquad R_2 = (\beta_1\, y + x + \gamma_1)(\beta_1\, y^* + x^* + \gamma_1),$

$R_3 = (1+\alpha_2 z)(1+\alpha_2 z^*)$ and $R_4 = (\beta_2\, z + y + \gamma_2)(\beta_2\, z^* + y^* + \gamma_2)$.

Accordingly, by using the given conditions (20a)–(20d) we obtain

$$\frac{dl}{dt} \leq -\left[\sqrt{q_{11}}(x-x^*) + \sqrt{\frac{q_{22}}{2}}(y-y^*)\right]^2 - \left[\sqrt{\frac{q_{22}}{2}}(y-y^*) + \sqrt{q_{33}}(z-z^*)\right]^2$$

Therefore, the derivative $\frac{dl}{dt}$ is negative definite and hence l is a $\mathcal{LF}$. Thus, the PEP is a $\mathcal{GAS}$.

4. Local Bifurcation

In this section, the local bifurcation near the possible EPs of system (3) is discussed with the help of Sotomayor's theorem [21]. It is well known that the existence of non-hyperbolic equilibrium point represents a necessary but not sufficient condition for occurrence of bifurcation. Therefore the candidate bifurcation parameter that is make the equilibrium point non-hyperbolic at a specific value of that parameter is selected. Now rewrite system (3) in the form:

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{F}(\boldsymbol{X}) \tag{21}$$

where $\boldsymbol{X} = (x,y,z)^T$ and $\boldsymbol{F} = (xf_1, yf_2, zf_3)^T$ with $f_i; i = 1,2,3$ represent the interaction functions in the right hand side of system (3). Then straightforward computation on the JM of system (3) with any non-zero vector $V = (v_1, v_2, v_3)^T$, gives the following second directional derivative

$$D^2\boldsymbol{F}(x,y,z)(\boldsymbol{V},\boldsymbol{V}) = (c_{ij})_{3\times 1}$$

(22)

where

$$c_{11} = 2\left(\frac{-1}{(1+\alpha_1 y)} + \frac{y(\beta_1 y+\gamma_1)}{(\beta_1 y+x+\gamma_1)^3}\right){v_1}^2 - 2\left(\frac{\alpha_1(1-2x)}{(1+\alpha_1 y)^2} + \frac{2\,\beta_1 xy + \gamma_1(\beta_1 y+x+\gamma_1)}{(\beta_1 y+x+\gamma_1)^3}\right)v_1 v_2$$
$$+2\left(\frac{{\alpha_1}^2 x(1-x)}{(1+\alpha_1 y)^3} + \frac{\beta_1 x(x+\gamma_1)}{(\beta_1 y+x+\gamma_1)^3}\right){v_2}^2$$

$$c_{21} = -2\left(\frac{\theta_1 y\,(\beta_1 y+\gamma_1)}{(1+\alpha_2 z)(\beta_1 y+x+\gamma_1)^3}\right){v_1}^2 + 2\left(\frac{2\theta_1\beta_1 xy + \theta_1\gamma_1(\beta_1 y+x+\gamma_1)}{(1+\alpha_2 z)(\beta_1 y+x+\gamma_1)^3}\right)v_1 v_2$$
$$-2\left(\frac{\theta_1\alpha_2 y\,(\beta_1 y+\gamma_1)}{(1+\alpha_2 z)^2(\beta_1 y+x+\gamma_1)^2}\right)v_1 v_3 + 2\left(\frac{-\,\theta_1\beta_1 x\,(x+\gamma_1)}{(1+\alpha_2 z)(\beta_1 y+x+\gamma_1)^3} + \frac{z(\beta_2 z+\gamma_2)}{(\beta_2 z+y+\gamma_2)^3}\right){v_2}^2$$
$$-2\left(\frac{\theta_1\alpha_2 x(x+\gamma_1)}{(1+\alpha_2 z)^2(\beta_1 y+x+\gamma_1)^2} + \frac{2\beta_2 yz+\gamma_2(\beta_2 z+y+\gamma_2)}{(\beta_2 z+y+\gamma_2)^3}\right)v_2 v_3$$
$$+2\left(\frac{\theta_1{\alpha_2}^2 xy}{(1+\alpha_2 z)^3(\beta_1 y+x+\gamma_1)} + \frac{\beta_2 y(y+\gamma_2)}{(\beta_2 z+y+\gamma_2)^3}\right){v_3}^2$$

$$c_{31} = -2\left(\frac{\theta_2 z(\beta_2 z+\gamma_2)}{(\beta_2 z+y+\gamma_2)^3}\right){v_2}^2 + 2\left(\frac{2\theta_2\beta_2 yz+\gamma_2\theta_2(\beta_2 z+y+\gamma_2)}{(\beta_2 z+y+\gamma_2)^3}\right)v_2 v_3$$
$$-2\left(\frac{\theta_2\beta_2 y(y+\gamma_2)}{(\beta_2 z+y+\gamma_2)^3}\right){v_3}^2$$

Theorem 6: System (3) at AEP undergoes a transcritical bifurcation ($\mathcal{TB}$)but neither saddle node bifurcation ($\mathcal{SNB}$) nor pitchfork bifurcation ($\mathcal{PB}$) can occurs when the parameter θ_1 passes through the value $\theta_1^* = d_1(1+\gamma_1)$.

Proof: According to the JM that given in equation (9a), system (3) at AEP with $\theta_1 = \theta_1^*$ has the following JM, say $J(q_1, \theta_1^*) = J_1$, where

$$J_1 = \begin{bmatrix} -1 & \frac{-1}{1+\gamma_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}$$

Clearly, J_1 has a zero eigenvalue given by $\lambda_{12}^* = 0$ and hence AEP is a nonhyperbolic point.

Now, let $\boldsymbol{U}^{[1]} = \left(u_1^{[1]}, u_2^{[1]}, u_3^{[1]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{12}^* = 0$.

Thus $J_1 \boldsymbol{U}^{[1]} = \boldsymbol{0}$ gives that $\boldsymbol{U}^{[1]} = \left(n\, u_2^{[1]}, u_2^{[1]}, 0\right)^T$, where $n = \frac{-1}{1+\gamma_1} < 0$ and $u_2^{[1]}$ represents any nonzero real number. Also, let $\boldsymbol{\psi}^{[1]} = \left(\psi_1^{[1]}, \psi_2^{[1]}, \psi_3^{[1]}\right)^T$, represents the eigenvector corresponding to the eigenvalue $\lambda_{12}^* = 0$ of ${J_1}^T$.

Hence ${J_1}^T \boldsymbol{\psi}^{[1]} = \boldsymbol{0}$ gives that $\boldsymbol{\psi}^{[1]} = \left(0, \psi_2^{[1]}, 0\right)^T$, where $\psi_2^{[1]}$ stands for any nonzero real number. Now because

$$\frac{\partial \boldsymbol{F}}{\partial \theta_1} = \boldsymbol{F}_{\theta_1}(\boldsymbol{X}, \theta_1) = \left(0, \frac{x\ y}{(1+\alpha_2 z)(\beta_1 y + x + \gamma_1)}, 0\right)^T$$

Thus, $\boldsymbol{F}_{\theta_1}(q_1, \theta_1^*) = (0,0,0)^T$, which gives $\left(\boldsymbol{\psi}^{[1]}\right)^T \boldsymbol{F}_{\theta_1}(q_1, \theta_1^*) = 0$. So according to Sotomayor's theorem for local bifurcation, system (3) has no $\mathcal{SNB}$ at $\theta_1 = \theta_1^*$. Furthermore because we have

$$D\boldsymbol{F}_{\theta_1}(q_1, \theta_1^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{1+\gamma_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then we obtain,

$$\left(\boldsymbol{\psi}^{[1]}\right)^T \left(D\boldsymbol{F}_{\theta_1}(q_1, \theta_1^*) \boldsymbol{U}^{[1]}\right) = \psi_2^{[1]} \frac{u_2^{[1]}}{1+\gamma_1} \neq 0$$

Moreover using equation (22) with q_1, θ_1^* and $\boldsymbol{U}^{[1]}$ gives

$$D^2\boldsymbol{F}(q_1, \theta_1^*)\left(\boldsymbol{U}^{[1]}, \boldsymbol{U}^{[1]}\right)$$
$$= 2\left(u_2^{[1]}\right)^2 \left(-n^2 - n\left(-\alpha_1 + \frac{\gamma_1}{(1+\gamma_1)^2}\right) + \frac{\beta_1}{(1+\gamma_1)^2}, \frac{n\,\gamma_1\,\theta_1^*}{(1+\gamma_1)^2} - \frac{\beta_1\,\theta_1^*}{(1+\gamma_1)^2}, 0\right)^T$$

Hence, it is obtained that

$$\left(\boldsymbol{\psi}^{[1]}\right)^T D^2\boldsymbol{F}(q_1, \theta_1^*)\left(\boldsymbol{U}^{[1]}, \boldsymbol{U}^{[1]}\right) = \frac{2\,d_1}{(1+\gamma_1)}(n\,\gamma_1 - \beta_1)\psi_2^{[1]}\left(u_2^{[1]}\right)^2 \neq 0.$$

Thus, based on Sotomayor's theorem, system (3) at AEP has a $\mathcal{TB}$ as the parameter θ_1 passes through the bifurcation value θ_1^*, while $\mathcal{PB}$ cannot occurs and that complete the proof.

Theorem 7: Assume that condition (12a) holds, then system (3) at TPFEP undergoes a $\mathcal{TB}$ but neither $\mathcal{SNB}$ nor $\mathcal{PB}$ can occurs when the parameter d_2 passes through the value $d_2^* = \frac{\theta_2 \bar{y}}{(\bar{y}+\gamma_2)}$.

Proof: From the JM that given in equation (11a), system (3) at TPFEP with $d_2 = d_2^*$ has the following JM, say $J(q_2, d_2^*) = J_2$, which has zero eigenvalue, say $\lambda_{23}^* = 0$.

$$J_2 = \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

where b_{ij}; $\forall\, ij = 1,2,3$ are given in equation (11a).

Now, let $\boldsymbol{U}^{[2]} = \left(u_1^{[2]}, u_2^{[2]}, u_3^{[2]}\right)^T$ represents the eigenvector corresponding to the eigenvalue $\lambda_{23}^* = 0$.

Therefore, $J_2 \boldsymbol{U}^{[2]} = \boldsymbol{0}$ gives that $\boldsymbol{U}^{[2]} = \left(m_1 u_3^{[2]}, m_2 u_3^{[2]}, u_3^{[2]}\right)^T$, where $m_1 = \frac{b_{12}\, b_{23}}{b_{11} b_{22} - b_{12}\, b_{21}} > 0$, $m_2 = -\frac{b_{11}\, b_{23}}{b_{11} b_{22} - b_{12}\, b_{21}} < 0$ and $u_3^{[2]}$ represents any nonzero real number. Also, let $\boldsymbol{\psi}^{[2]} = \left(\psi_1^{[2]}, \psi_2^{[2]}, \psi_3^{[2]}\right)^T$ represents the eigenvector corresponding to the eigenvalue $\lambda_{23}^* = 0$ of $J_2{}^T$.

Hence $J_2{}^T \boldsymbol{\psi}^{[2]} = \boldsymbol{0}$ gives that $\boldsymbol{\psi}^{[2]} = \left(0,0,\psi_3^{[2]}\right)^T$, where $\psi_3^{[2]}$ stands for any nonzero real number. Now because we have

$$\frac{\partial \boldsymbol{F}}{\partial d_2} = \boldsymbol{F}_{d_2}(\boldsymbol{X}, d_2) = (0,0,-z)^T$$

Thus $\boldsymbol{F}_{d_2}(q_2, d_2^*) = (0,0,0)^T$, which gives $\left(\boldsymbol{\psi}^{[2]}\right)^T \boldsymbol{F}_{d_2}(q_2, d_2^*) = 0$. So according to Sotomayor's theorem for local bifurcation, system (3) has no $\mathcal{SNB}$ at $d_2 = d_2^*$. Furthermore because we have

$$D\boldsymbol{F}_{d_2}(q_2, d_2^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

We can show that

$$\left(\boldsymbol{\psi}^{[2]}\right)^T \left(D\boldsymbol{F}_{d_2}(q_2, d_2^*) \boldsymbol{U}^{[2]}\right) = \left(0,0,\psi_3^{[2]}\right)\left(0,0,-u_3^{[2]}\right)^T = -\psi_3^{[2]} u_3^{[2]} \neq 0$$

Moreover, using equation (22) with q_2, d_2^* and $\boldsymbol{U}^{[2]}$ gives

$$D^2\boldsymbol{F}(q_2, d_2^*)\left(\boldsymbol{U}^{[2]}, \boldsymbol{U}^{[2]}\right) = 2\left(u_3^{[2]}\right)^2 \left(c_{ij}^{[2]}\right)_{3\times 1}$$

Where

$$c_{11}^{[2]} = \left(\frac{-1}{(1+\alpha_1 \bar{y})} + \frac{\bar{y}(\beta_1\bar{y}+\gamma_1)}{(\beta_1\bar{y}+\bar{x}+\gamma_1)^3}\right) {m_1}^2 - \left(\frac{\alpha_1(1-2\bar{x})}{(1+\alpha_1\bar{y})^2} + \frac{2\,\beta_1\bar{x}\bar{y}\,+\gamma_1(\beta_1\bar{y}+\bar{x}+\gamma_1)}{(\beta_1\bar{y}+\bar{x}+\gamma_1)^3}\right) m_1 m_2 + \left(\frac{{\alpha_1}^2\bar{x}(1-\bar{x})}{(1+\alpha_1\bar{y})^3} + \frac{\beta_1\bar{x}(\bar{x}+\gamma_1)}{(\beta_1\bar{y}+\bar{x}+\gamma_1)^3}\right) {m_2}^2$$

$$c_{21}^{[2]} = -\left(\frac{\theta_1\bar{y}\,(\beta_1\bar{y}+\gamma_1)}{(\beta_1\bar{y}+\bar{x}+\gamma_1)^3}\right) {m_1}^2 + \left(\frac{2\,\theta_1\beta_1\bar{x}\bar{y}\,+\theta_1\gamma_1(\beta_1\bar{y}+\bar{x}+\gamma_1)}{(\beta_1 y+x+\gamma_1)^3}\right) m_1 m_2 - \left(\frac{\theta_1\alpha_2\bar{y}\,(\beta_1\bar{y}+\gamma_1)}{(\beta_1\bar{y}+\bar{x}+\gamma_1)^2}\right) m_1 - \left(\frac{\theta_1\beta_1\bar{x}\,(\bar{x}+\gamma_1)}{(\beta_1\bar{y}+\bar{x}+\gamma_1)^3}\right) {m_2}^2 - \left(\frac{\theta_1\alpha_2\bar{x}(\bar{x}+\gamma_1)}{(\beta_1\bar{y}+\bar{x}+\gamma_1)^2} + \frac{\gamma_2}{(\bar{y}+\gamma_2)^2}\right) m_2 + \left(\frac{\theta_1{\alpha_2}^2\bar{x}\,\bar{y}}{(\beta_1\bar{y}+\bar{x}+\gamma_1)} + \frac{\beta_2\bar{y}}{(\bar{y}+\gamma_2)^2}\right)$$

$$c_{31}^{[2]} = \left(\frac{\gamma_2\theta_2}{(\bar{y}+\gamma_2)^2}\right) m_2 - \left(\frac{\theta_2\beta_2\bar{y}}{(\bar{y}+\gamma_2)^2}\right)$$

Hence, it is obtained that

$$\left(\boldsymbol{\psi}^{[2]}\right)^T D^2\boldsymbol{F}(q_2, d_2^*)\left(\boldsymbol{U}^{[2]}, \boldsymbol{U}^{[2]}\right) = \frac{2\,\theta_2}{(\bar{y}+\gamma_2)^2}\,(m_2\,\gamma_2 - \beta_2\bar{y})\psi_3^{[2]}\left(u_3^{[2]}\right)^2 \neq 0.$$

Therefore, by Sotomayor's theorem, system (3) at TPFEP has a $\mathcal{TB}$ as the parameter d_2 passes through the bifurcation value d_2^*, while $\mathcal{PB}$ cannot occurs and hence the proof is complete.

Theorem 8: Assume that condition (14a) along with the following sufficient conditions hold

$$\frac{\theta_1\beta_1\,x^*}{{\rho_2}^2\rho_3} < \frac{z^*}{{\rho_4}^2} \qquad (23a)$$

$$\rho_3 < \theta_1\beta_1 \qquad (23b)$$

$$1 < \theta_2\,\beta_2 \qquad (23c)$$

$$\frac{x^*y^*z^*}{\rho_1{\rho_4}^2} < \frac{x^*{y^*}^2z^*}{{\rho_2}^2{\rho_4}^2} + \frac{\theta_1\beta_1{x^*}^2y^*}{\rho_1{\rho_2}^2\rho_3} + \frac{\theta_1\alpha_1x^*y^*(1-x^*)(\beta_1\,y^*+\gamma_1)}{{\rho_1}^2{\rho_2}^2\rho_3} + \frac{\theta_1\gamma_1\,x^*y^*}{{\rho_2}^3\rho_3} \qquad (23d)$$

$$\frac{x^*}{\rho_1} + \frac{x^*y^*}{{\rho_2}^2\rho_3}(\theta_1\beta_1 - \rho_3) < \frac{y^*\,z^*}{{\rho_4}^2} \qquad (23e)$$

$$[A_1({\theta_2}^*)A_2({\theta_2}^*)]' < {A_3}'({\theta_2}^*) \qquad (23f)$$

where $\rho_1 = 1 + \alpha_1 y^*$; $\rho_2 = \beta_1 y^* + x^* + \gamma_1$; $\rho_3 = 1 + \alpha_2 z^*$ and $\rho_4 = \beta_2 z^* + y^* + \gamma_2$; while $A_i; i = 1,2,3$ are given in equation (13b). Then system (3) undergoes a Hopf bifurcation ($\mathcal{HB}$) around the equilibrium point q_3 as the parameter θ_2 passes through the positive value ${\theta_2}^*$, that given in the proof.

Proof: Recall that, according to the $\mathcal{HB}$ theorem [26] for the three dimensional autonomous system, such as system (3), undergoes a $\mathcal{HB}$ as the parameter θ_2 passes through the positive value ${\theta_2}^*$ provided that:

The JM of system (3) at the equilibrium point q_3 has a simple pair of complex eigenvalues, say $\sigma_1(\theta_2) \pm i\sigma_2(\theta_2)$, such that they become purely imaginary at $\theta_2 = {\theta_2}^*$, while the third eigenvalue remain real and negative. Moreover, the transversality condition $\left.\frac{d\sigma_1(\theta_2)}{d\theta_2}\right|_{\theta_2={\theta_2}^*} \neq 0$ should be hold.

Note that the above first condition will be satisfied if the coefficients of the characteristic equation given by (13b) satisfy that $\Delta = A_1\, A_2 - A_3 = 0$. So straightforward computation gives that this is equivalent to

$$r_1{\theta_2}^2 + r_2\theta_2 + r_3 = 0$$

(24a)

Where $$r_1 = -[\beta_2 y^*(a_{11} + a_{22}) + (\beta_2 z^* + \gamma_2)a_{23}]\frac{\beta_2 y^* {z^*}^2}{{\rho_4}^4}$$

$$r_2 = [\beta_2 y^*(a_{11} + a_{22})^2 + (\beta_2 z^* + \gamma_2)a_{22}a_{23}]\frac{z^*}{{\rho_4}^2},$$

$$r_3 = -(a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21})$$

Clearly, according to the signs of JM elements that given by equation (13a) in addition to the sufficient conditions (14a), (23a), (23b), (23d) and (23e) it is easy to verify that $a_{11} < 0$, $a_{22} > 0$, $r_1 > 0$ and $r_3 < 0$, and then equation (24a) has a unique positive root denoted by ${\theta_2}^*$ that satisfies $A_1({\theta_2}^*)A_2({\theta_2}^*) = A_3({\theta_2}^*)$. Consequently, as $\theta_2 = {\theta_2}^*$ the characteristic equation given by (13b) will be

$$(\lambda + A_1)(\lambda^2 + A_2) = 0$$

(24b)

Thus, equation (24b) has the following roots

$$\lambda_1 = -A_1({\theta_2}^*) \text{ and } \lambda_{2,3} = \pm i\sqrt{A_2({\theta_2}^*)} = \pm i\sigma_2({\theta_2}^*).$$

Again, the given conditions (23b)-(23d) with the signs of JM elements guarantee that $A_i > 0$ for all $i = 1,2,3$. Therefore the first condition of the $\mathcal{HB}$ follows.

Now in order to check the occurrence of the transversality condition, substitute $\sigma_1(\theta_2) + i\sigma_2(\theta_2)$, where θ_2 in the neighborhood of ${\theta_2}^*$, in the equation (24b) and then take the derivative with respect to the bifurcation parameter θ_2. Then comparing the two sides of this equation and then equating their real and imaginary parts, we get

$$\begin{aligned} \Psi(\theta_2){\sigma_1}'(\theta_2) - \Phi(\theta_2){\sigma_2}'(\theta_2) &= -\Theta(\theta_2) \\ \Phi(\theta_2){\sigma_1}'(\theta_2) + \Psi(\theta_2){\sigma_2}'(\theta_2) &= -\Gamma(\theta_2) \end{aligned} \qquad (25a)$$

where $\Theta(\theta_2) = {A_1}'(\theta_2)[\sigma_1(\theta_2)]^2 - {A_1}'(\theta_2)[\sigma_2(\theta_2)]^2 + {A_2}'(\theta_2)\sigma_1(\theta_2) + {A_3}'(\theta_2)$

$\Psi(\theta_2) = 3[\sigma_1(\theta_2)]^2 + 2A_1(\theta_2)\sigma_1(\theta_2) - 3[\sigma_2(\theta_2)]^2 + A_2(\theta_2)$

$\Gamma(\theta_2) = 2{A_1}'(\theta_2)\sigma_1(\theta_2)\sigma_2(\theta_2) + {A_2}'(\theta_2)\sigma_2(\theta_2)$

$\Phi(\theta_2) = 6\sigma_1(\theta_2)\sigma_2(\theta_2) + 2A_1(\theta_2)\sigma_2(\theta_2)$

Solving the linear system (25a) by using Cramer's rule for the unknowns $\sigma_1'(\theta_2)$ and $\sigma_2'(\theta_2)$ gives that

$$\sigma_1'(\theta_2) = -\frac{\Theta(\theta_2)\Psi(\theta_2)+\Gamma(\theta_2)\Phi(\theta_2)}{[\Psi(\theta_2)]^2+[\Phi(\theta_2)]^2}, \sigma_2'(\theta_2) = -\frac{\Gamma(\theta_2)\Psi(\theta_2)-\Theta(\theta_2)\Phi(\theta_2)}{[\Psi(\theta_2)]^2+[\Phi(\theta_2)]^2} \quad (25b)$$

Hence the transversality condition is satisfied provided that

$$\Theta(\theta_2{}^*)\Psi(\theta_2{}^*) + \Gamma(\theta_2{}^*)\Phi(\theta_2{}^*) \neq 0$$

Obviously, we have that $\sigma_1(\theta_2{}^*) = 0$ and $\sigma_2(\theta_2{}^*) = \sqrt{A_2(\theta_2{}^*)}$, so we obtain that

$$\Theta(\theta_2{}^*) = -A_1'(\theta_2{}^*)A_2(\theta_2{}^*) + A_3'(\theta_2{}^*)$$

$$\Psi(\theta_2{}^*) = -2A_2(\theta_2{}^*)$$

$$\Gamma(\theta_2{}^*) = A_2'(\theta_2{}^*)\sqrt{A_2(\theta_2{}^*)}$$

$$\Phi(\theta_2{}^*) = 2A_1(\theta_2)\sqrt{A_2(\theta_2{}^*)}$$

Accordingly, we get that

$$\sigma_1'(\theta_2{}^*) = 2A_2(\theta_2{}^*)\frac{\left[A_3'(\theta_2{}^*) - \left(A_1'(\theta_2{}^*)A_2(\theta_2{}^*) + A_1(\theta_2{}^*)A_2'(\theta_2{}^*)\right)\right]}{[\Psi(\theta_2{}^*)]^2 + [\Phi(\theta_2{}^*)]^2}$$

Consequently, $\sigma_1'(\theta_2{}^*) > 0$ under condition (23f), and then the transversality condition hold. Hence $\mathcal{HB}$ occurs at $\theta_2 = \theta_2{}^*$.

Not that, according the above theorem, we have that for $\theta_2 > \theta_2{}^*$ the equilibrium point q_3 of system (3) is stable; when $\theta_2 = \theta_2{}^*$ loses its stability and the $\mathcal{HB}$ occurs at q_3, , and when $\theta_2 < \theta_2{}^*$, the equilibrium point q_3 becomes unstable and a family of periodic solutions bifurcates from q_3.

5. Numerical Simulation

In this section, the global dynamics of system (3) is investigated numerically. The main objectives understand the effect of fear on the dynamics of system (3), specify the set of parameters that control the dynamical behavior of the system (3) and confirm our obtained results. Different tools are used through this investigation such as bifurcation diagram ($\mathcal{BD}$), chaotic attractor, 3D phase plot and time series. Predictor-Corrector method with six-order Range Kutta methods are used for solving the system, while MATLAB version 6 is used to present these numerical results.

The following hypothetical set of parameters is used.

$$\alpha_1 = 0,\ \beta_1 = 0.2,\ \gamma_1 = 0.2,\ \theta_1 = 0.5,\ \alpha_2 = 0,\ \beta_2 = 0.2,$$
$$\gamma_2 = 0.2,\ d_1 = 0.2,\ \theta_2 = 0.3,\ d_2 = 0.1$$

(26)

Clearly, in the above set of data, there is no fear in the system (3). It is observed that system (3) undergoes a chaotic dynamics for the above set of data as shown in the Figure 1.

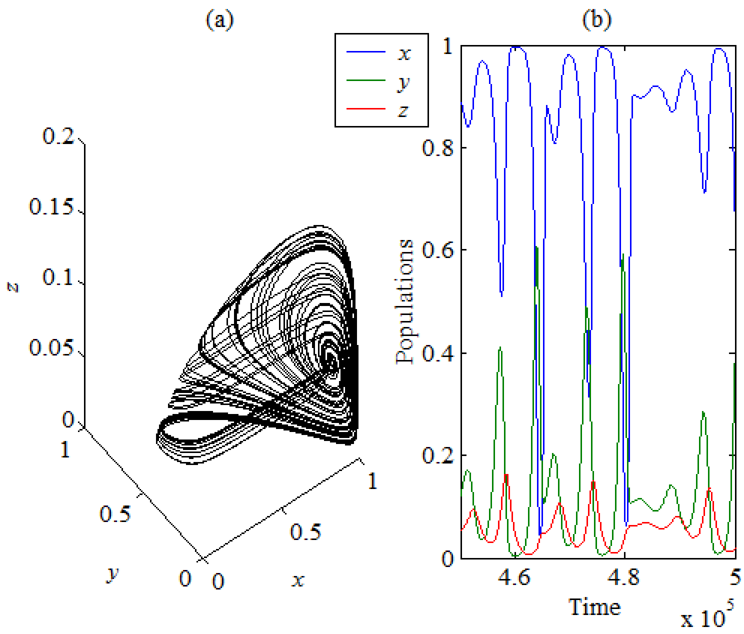


Figure 1. The trajectory of system (3) for the data (26). (a) Chaotic attractor. (b) Time series of the attractor in (a).

Obviously from Figure 1, system (3) without fear has a chaotic dynamics at the data (26), which indicates to existence of complex dynamics. Now, to investigate the impact of varying the parameters θ_1, θ_2 and d_2 on the dynamics of system (3), the $\mathcal{BD}$ for the trajectory of system (3) as a function of each parameter are drawn in Figure 2 – Figure 4 respectively. It is known that, the $\mathcal{BD}$ summarizes the dynamical behavior of the system as a function of a specific. These parameters are selected according to the analytical study in section (4).

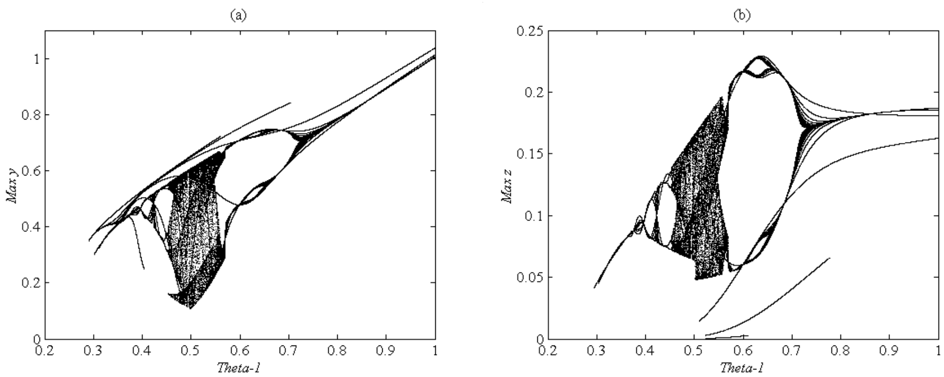


Figure 2. $\mathcal{BD}$ of system (3) as a function of $\theta_1 \in (0.2,1)$ keeping other parameters as in the data (26). (a) Maximum of the trajectory of y versus θ_1. (b) Maximum of the trajectory of z versus θ_1.

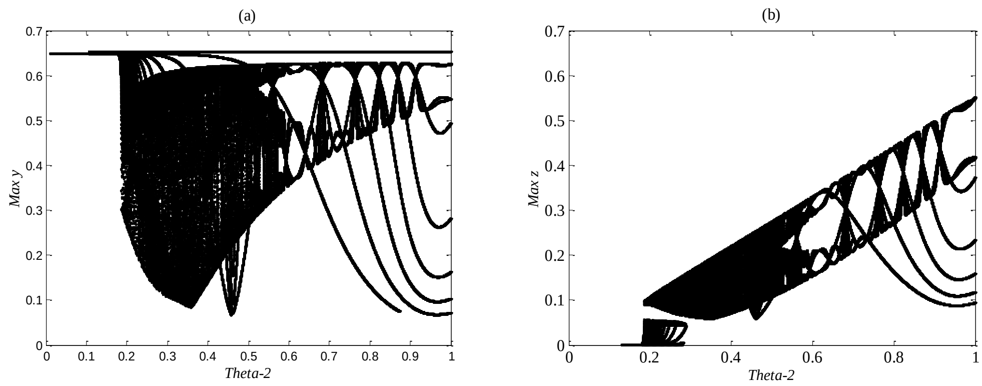


Figure 3. $\mathscr{BD}$ of system (3) as a function of $\theta_2 \in (0,1)$ keeping other parameters as in the data (26). (a) Maximum of the trajectory of y versus θ_2. (b) Maximum of the trajectory of z versus θ_2.

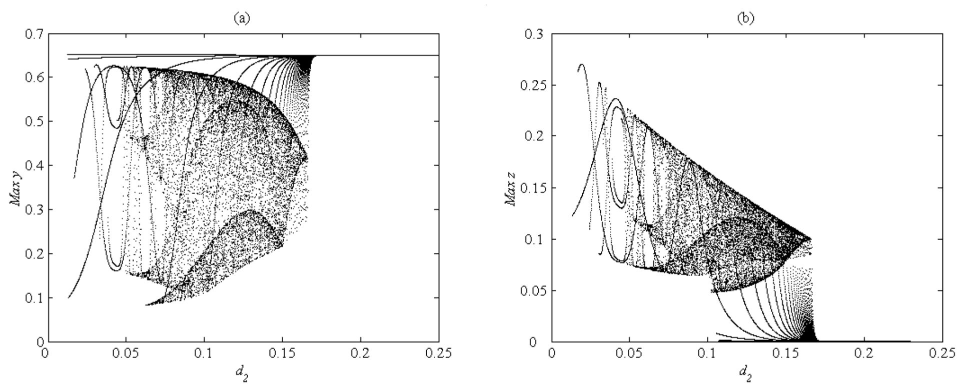
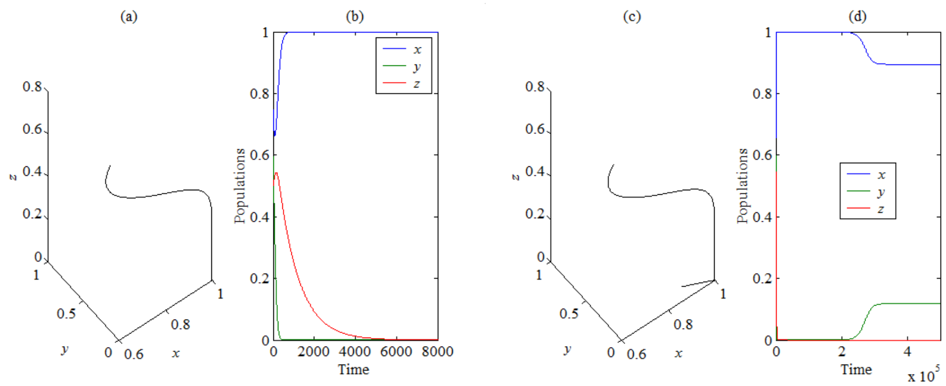


Figure 4. $\mathscr{BD}$ of system (3) as a function of $d_2 \in (0,0.25)$ keeping other parameters as in the data (26). (a) Maximum of the trajectory of y versus d_2. (b) Maximum of the trajectory of z versus d_2.

Clearly, as shown in the above $\mathscr{BD}$, system (3) is very sensitive for varying the parameters θ_1, θ_2 and d_2. Different types of bifurcations are obtained and system (3) enters to chaotic and periodic regions. Furthermore, it is obtained that system (3) approaches asymptotically to AEP for the range $\theta_1 \in (0,0.24)$, which is confirm stability condition (10). It is approaches asymptotically to TPFEP, where $q_2 = (0.89,0.11,0)$, for the range $\theta_1 \in (0.24,0.26)$. While it is approach asymptotically to PEP, with $q_3 = (0.91,0.1,0.005)$, for the range $\theta_1 \in (0.26,0.28)$. Finally, system (3) approaches asymptotically to a periodic dynamics in $Int.\mathbb{R}^3_+$, see Figure 5 for typical values of θ_1 and Table 1 for varying other parameters.



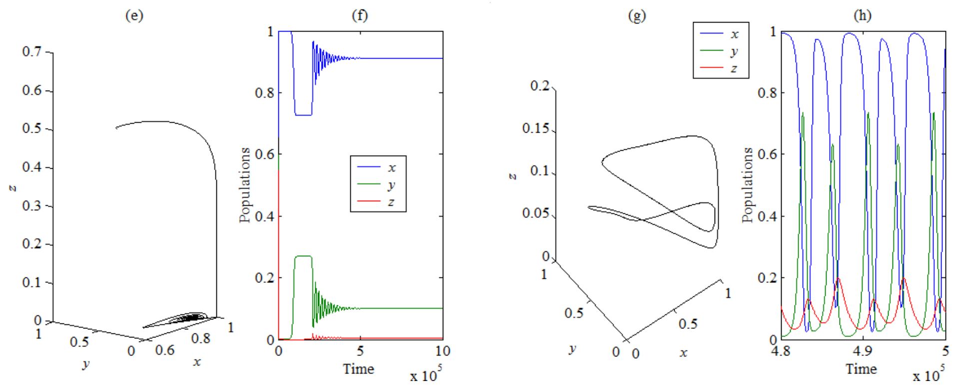


Figure 5. The trajectory of system (3) for the data (26) with different values of θ_1. (a) System (3) approach asymptotically to $q_1 = (1,0,0)$ for $\theta_1 = 0.2$. (b) Time series of the attractor in (a). (c) System (3) approach asymptotically to $q_2 = (0.89,0.11,0)$ for $\theta_1 = 0.25$. (d) Time series of the attractor in (c). (e) System (3) approach asymptotically to $q_3 = (0.91,0.1,0.005)$ for $\theta_1 = 0.27$. (g) System (3) approach asymptotically to period two attractor for $\theta_1 = 0.7$. (h) Time series of the attractor in (g).

Table 1. The dynamical behavior of system (3) using data (26) with varying one parameter each time

The parameter	The range of varying	The dynamical behavior of system (3)
β_1	$0 < \beta_1 < 0.26$ $0.26 \le \beta_1 < 1.5$ $1.5 \le \beta_1$	Complex dynamics involving periodic and chaos Periodic dynamics in $Int.\mathbb{R}^3_+$ Approaches to PEP in $Int.\mathbb{R}^3_+$
γ_1	$0 < \gamma_1 < 0.09$ $0.09 < \gamma_1 < 0.23$ $0.23 \le \gamma_1$	Periodic in the xy −plane Complex dynamics involving periodic and chaos Periodic dynamics in $Int.\mathbb{R}^3_+$
d_1	$0 < d_1 < 0.36$ $0.36 \le d_1 < 0.4$ $0.4 \le d_1 < 0.42$ $0.42 \le d_1 < 1$	Complex dynamics involving periodic and chaos Approaches to PEP in $Int.\mathbb{R}^3_+$ Approaches to TPFEP in xy −plane Approaches to AEP
β_2	$0 < \beta_2 < 0.71$ $0.71 \le \beta_2 < 1$	Complex dynamics involving periodic and chaos Periodic dynamics in $Int.\mathbb{R}^3_+$
γ_2	$0 < \gamma_2 < 0.3$ $0.3 \le \gamma_2 < 0.47$ $0.47 \le \gamma_2 < 0.69$ $0.69 \le \gamma_2 < 1$	Complex dynamics involving periodic and chaos Periodic dynamics in $Int.\mathbb{R}^3_+$ Approaches to PEP in $Int.\mathbb{R}^3_+$ Periodic in the xy −plane
θ_2	$0 < \theta_2 < 0.18$ $0.18 \le \theta_2 < 0.7$ $0.7 \le \theta_2 < 1$	Periodic in the xy −plane Complex dynamics involving periodic and chaos Periodic dynamics in $Int.\mathbb{R}^3_+$

d_2	$0 < d_2 < 0.18$	Complex dynamics involving periodic and chaos
	$0.18 \leq d_2 < 1$	Periodic in the xy −plane

Now, in order to understand the effects of varying the fear rates on the dynamics of system (3) using the data (26), the system is solved numerically with different values of prey's fear rate α_1 and different values of intermediate predator's fear rate α_2 as shown in Figure 6 and Figure 7 respectively.

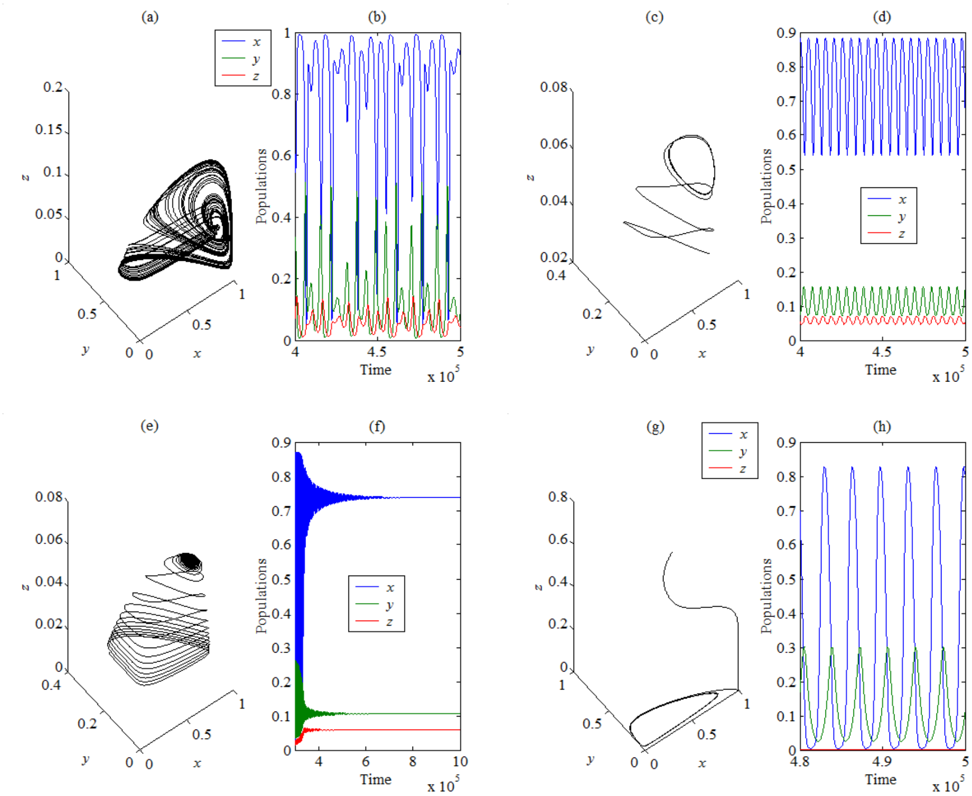
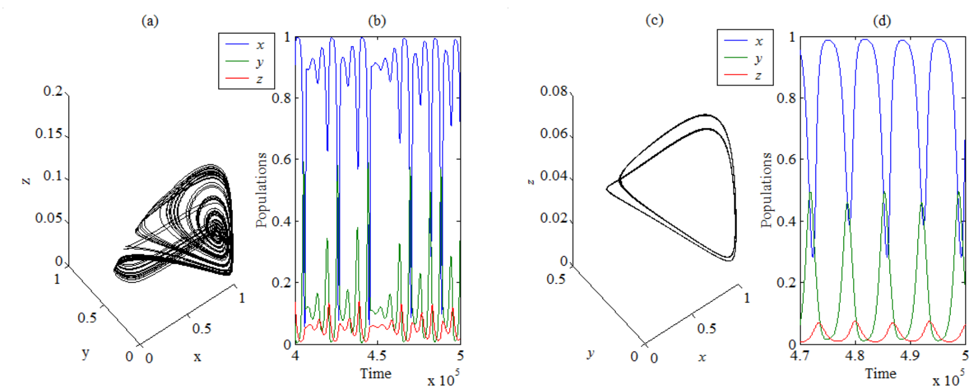


Figure 6. The trajectory of system (3) for the data (26) with different values of α_1. (a) System (3) approach asymptotically to chaotic attractor for $\alpha_1 = 0.5$. (b) Time series of the attractor in (a). (c) System (3) approach asymptotically to periodic attractor in $Int.\mathbb{R}^3_+$ for $\alpha_1 = 10$. (d) Time series of the attractor in (c). (e) System (3) approach asymptotically to $q_3 = (0.73,0.1,0.05)$ for $\alpha_1 = 13$. (f) Time series of the attractor in (e). (g) System (3) approach asymptotically periodic dynamics in the xy −plane for $\alpha_1 = 15$. (h) Time series of the attractor in (g).



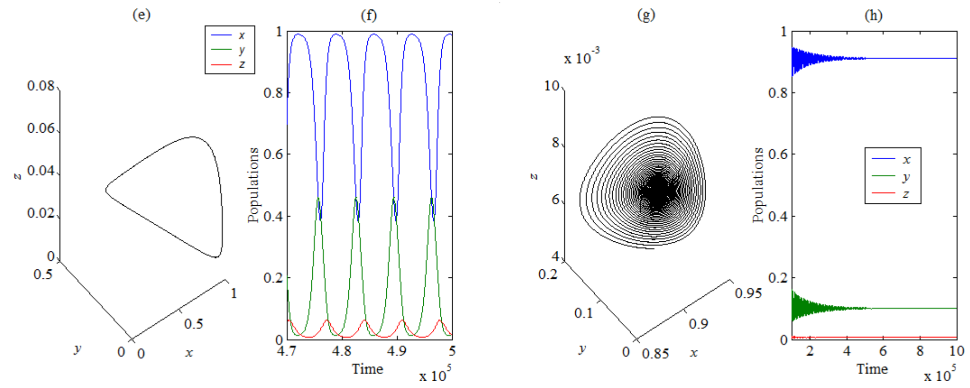


Figure 7. The trajectory of system (3) for the data (26) with different values of α_2. (a) System (3) approach asymptotically to chaotic attractor for $\alpha_2 = 1$. (b) Time series of the attractor in (a). (c) System (3) approach asymptotically to period two attractor in $Int.\mathbb{R}^3_+$ for $\alpha_2 = 10$. (d) Time series of the attractor in (c). (e) System (3) approach asymptotically to periodic attractor for $\alpha_2 = 15$. (f) Time series of the attractor in (e). (g) System (3) approach asymptotically to $q_3 = (0.91,0.1,0.006)$ for $\alpha_2 = 125$. (h) Time series of the attractor in (g).

However, for the data set given by (26) with $\alpha_1 = 10$ and $\alpha_2 = 15$, it is observed that the trajectory of system (3) approaches asymptotically to PEP represented by $q_3 = (0.79,0.1,0.02)$ as shown in Figure 8.

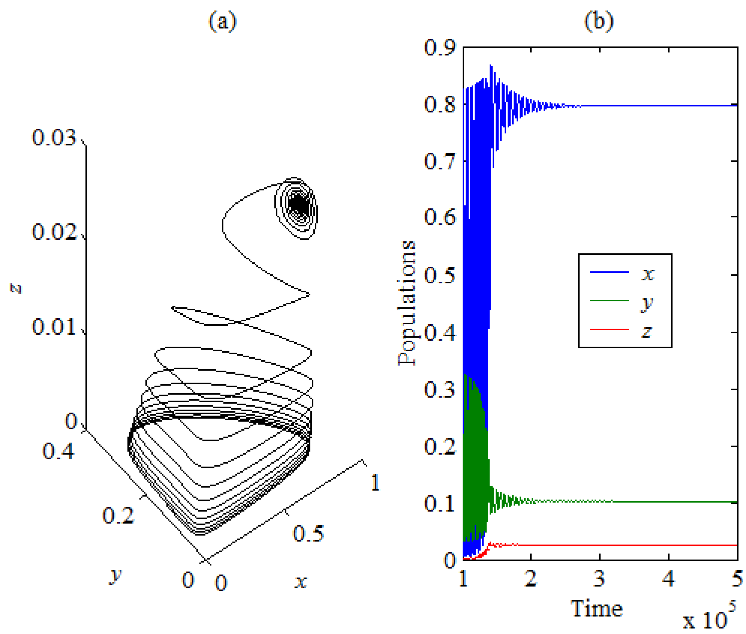


Figure 8. The trajectory of system (3) for the data (26) with $\alpha_1 = 10$ and $\alpha_2 = 15$. (a) System (3) approaches asymptotically to $q_3 = (0.79,0.1,0.02)$. (b) Time series of the attractor in (a).

Keeping the obtained results in view, the effect of varying the parameters of system (3) on the dynamical behavior of the system in case of having asymptotically stable PEP using the data given by equation (26) with $\alpha_1 = 10$ and $\alpha_2 = 15$ is also studied numerically and obtained results are summarized in Table 2.

Table 2. The dynamical behavior of system (3) using data (26) with $\alpha_1 = 10$ and $\alpha_2 = 15$ in case of varying one parameter each time

The parameter	The range of varying	The dynamical behavior of system (3)
β_1	$0 < \beta_1 < 0.09$	Periodic in the xy −plane
	$0.09 \le \beta_1$	Approaches to PEP in $Int.\mathbb{R}^3_+$
γ_1	$0 < \gamma_1 < 0.18$	Periodic in the xy −plane
	$0.18 \le \gamma_1 \le 1$	Approaches to PEP in $Int.\mathbb{R}^3_+$
θ_1	$0 < \theta_1 < 0.25$	Approaches to AEP
	$0.25 \le \theta_1 < 0.28$	Approaches to TPFEP in xy −plane
	$0.28 \le \theta_1 < 0.6$	Approaches to PEP in $Int.\mathbb{R}^3_+$
	$0.6 \le \theta_1 < 0.8$	Periodic dynamics in $Int.\mathbb{R}^3_+$
	$0.8 \le \theta_1 < 1$	Periodic in the xy −plane
d_1	$0 < d_1 < 0.13$	Periodic in the xy −plane
	$0.13 \le d_1 < 0.18$	Periodic dynamics in $Int.\mathbb{R}^3_+$
	$0.18 \le d_1 < 0.37$	Approaches to PEP in $Int.\mathbb{R}^3_+$
	$0.37 \le d_1 < 0.41$	Approaches to TPFEP in xy −plane
	$0.41 \le d_1 < 1$	Approaches to AEP
β_2	$0 < \beta_2 < 1$	Approaches to PEP in $Int.\mathbb{R}^3_+$
γ_2	$0 < \gamma_2 < 0.19$	Periodic dynamics in $Int.\mathbb{R}^3_+$
	$0.19 \le \gamma_2 < 0.23$	Approaches to PEP in $Int.\mathbb{R}^3_+$
	$0.23 \le \gamma_2 < 1$	Periodic in the xy −plane
θ_2	$0 < \theta_2 < 0.29$	Periodic in the xy −plane
	$0.29 \le \theta_2 < 0.32$	Approaches to PEP in $Int.\mathbb{R}^3_+$
	$0.32 \le \theta_2 < 1$	Periodic dynamics in $Int.\mathbb{R}^3_+$
d_2	$0 < d_2 < 0.1$	Periodic dynamics in $Int.\mathbb{R}^3_+$
	$0.1 \le d_2 < 0.11$	Approaches to PEP in $Int.\mathbb{R}^3_+$
	$0.11 \le d_2 < 1$	Approaches to PEP in $Int.\mathbb{R}^3_+$

6. Conclusion and discussion

In this paper, a BD food chain model incorporating fear factors in the first and second traffic levels of the chain is proposed and studied. The objective is to investigating the role of fear on the dynamical behavior of the system. The boundedness of the solution is proved. All the EPs are determined and their stability analyses are investigated locally as well as globally. The persistence conditions of the system are established. The occurrence of local bifurcation around the equilibrium points is investigated too. Finally, the numerical simulation of the system in case of nonexistence and existence of fear is carried out. It is observed that using the hypothetical set of data given by equation (26) the food chain without fear has a complex dynamics involving chaos that is very sensitive for varying of most the parameters. Furthermore, it is clear that the existence of fear has a stabilizing effect, through removing the complex dynamics of the system. Now, according the numerical simulation results using the hypothetical set of data (26) the following observations are obtained.

1. System (3) without fear has complex dynamics including chaos and periodic.

2. Increasing the fear in the first level up to a specific value removes the chaotic dynamics and the trajectory of system (3) approaches asymptotically to stable PEP. However, further increasing the fear at the first level more than a critical value makes the system losing the persistence and then the trajectory approaches asymptotically to a periodic dynamics in the xy −plane.
3. Increasing the fear rate in the second level up to a specific value removes the chaos too and the trajectory of system (3) approaches asymptotically to periodic attractor in $Int.\mathbb{R}^3_+$. Moreover, increasing the fear rate further above a critical value stabilizes the system and the trajectory approaches asymptotically to PEP.
4. The ***BDs*** show that the system is very sensitive to varying in the conversion rates θ_1, θ_2 and the top predator death rate d_2. Different points of bifurcation have been obtained. In fact, decreasing the value of the conversion rates θ_1, θ_2 or increasing the value of predators death rates d_1, d_2 causes extinction in top predator and the system loses their persistence.
5. Similar observation has been obtained regarding increasing the values of top predator half saturation constant γ_2 as that obtained in case of increasing the predators death rate.
6. In case of existence of constant values of fear rates $\alpha_1 = 10$ and $\alpha_2 = 15$ with rest of parameters as given in equation (26), it is observed that the system persists at the PEP. While decreasing the value of encounters between the intermediate predator individuals or the intermediate predator half saturation constant causes extinction in top predator and system (3) approaches asymptotically top periodic dynamics in the xy −plane.
7. Decreasing (increasing) the conversion rate of the intermediate predator θ_1 (death rate of intermediate predator d_1) below (above) a specific value causes extinction in top predator and the solution of system (3) approaches asymptotically to TPFEP in the xy −plane. Further decreasing (increasing) in these parameters leads to extinction in intermediate predator too and the system approaches asymptotically to AEP. On the other hand, increasing θ_1 (decreasing d_1) above (below) a specific value leads to extinction in top predator and the solution approaches asymptotically to TPFEP in xy −plane.
8. Increasing the half saturation constant γ_2 or the death rate d_2 of top predator above a specific value causes losing the persistence of the system and the solution approaches asymptotically to periodic dynamics in xy −plane. However, decreasing these rates leads to losing the stability of the PEP and the system still persist at periodic attractor in $Int.\mathbb{R}^3_+$.
9. Finally, decreasing the top predator conversion rate θ_2 below a specific value causes losing the persistence of the system and the solution approaches asymptotically to periodic dynamics in xy −plane. However, increasing this rate leads to losing the stability of the PEP and the system still persist at periodic attractor in $Int.\mathbb{R}^3_+$.

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Focal Function in i-Topological Spaces via Proximity Spaces

Yiezi Kadham Mahdi Altalkany and Luay A. A. Al Swidi

[1]Department of Mathematics, College of education for Girl, University of Kufa , Najaf

[2]Department of Mathematics, College of Education for Pure Science, Babylon

Email[1]: yiezik.altalkany@uokufa.edu.iq

Email[2]: drluayha11@yahoo.com

Abstract: Through the characteristic and properties of ideal we were able to give a new definition to neighborhood of a certain point, but these neighborhoods do not necessarily contain that point. we also introduced a new definition to the local function by using both proximity relation and the idea of the neighborhoods that were indicated ,finally we presented most important results and their properties.

1. Introduction: This research is based on the concept of proximity relation which was known by the Riecs [10] in his theory of enchainment in 1909 and in 1952, Efremovic [5] rediscover the concept of proximity spaces.

Later, many researchers and mathematicians presented several studies on this topic like A. Kandil, O.A. Tantawy, S.A. El-Sheikh, A. Zakaria [1] in 2014.

As well as they were used to define a new type of set introduced by Luay Al Swidi and Dhearrgham Ali [2] in 2020 and they are named it the centre set.

The second crutch of this research, it is the concept of ideals that has been defined by the Kuratowski [4] in 1933 , this concept has evolved and developed In topological spaces to be the triple (X, T,I), which is named the ideal Topological space and more of mathematician like T. R. Hamlett D. Jankovi´c[13,14] in 1990 and 1992 and R. Vaidyanathaswamy [11] in 1960 studied on this concept .

In 2006 the Irina Zevina [3] developed a new definition of some topological spaces using the ideal tools and it was named i-topological spaces.

As for the third crutch it is the local function that was defined. By K. Kuratowski [4] in 1933 and this function was studied by Many of mathematician in various forms, some of them studied by the fuzzy set such as Mohammed Majid Najm, Luay A. A. AL-Swidi[7]in 2020 and Reghad almohammed ,luay al swidi [12]in 2020 and some of them studied by the soft set, such as manash Jyoti and bipan Hazarika [8] in 2019 and Luay A. Al-Swidi and Sameer A. Al-Fathly[6]in 2017.

In this work all the crutch and concepts were invested to provide a new definition of the sets we named it the focal set and we studied the most important properties, and we also presented a new type of closure using the focal set, and finally the focal function was defined with the confirmation of some facts about it.

2. Fundamentals:

We will begin with some of basic concepts that we are needed in our work and we will mean by a space is i-topological space.

2.1 Definition

an ideal is a family I of X satisfy the following [4 ,11]

$1) X \notin I,\quad 2) \mathfrak{W}, \mathcal{K} \in I\ implies\ \mathfrak{W} \cup \mathcal{K} \in I, and \quad 3) \mathfrak{W} \in I\ and\ \mathcal{K} \subseteq \mathfrak{W}\ imply\ \mathcal{K} \in I. the\ following\ relations\ \propto\ and\ \approx\ on\ P(X)\ defined\ as\ as\ follows$:

(1) $\mathfrak{W} \propto \mathcal{K}\ iff\ \mathfrak{W} \cap \mathcal{K}^c \in I,$ [3]

(2) $\mathfrak{W} \approx \mathcal{K}\ mod\ I\ iff\ (\mathfrak{W} \cap \mathcal{K}^c) \cup (\mathcal{K} \cap \mathfrak{W}^c) \in I$ [5]

2.2 Definition [3]

Let I is ideal defined on X an i-topology on X is a family T of X that check conditions:

1. $\emptyset, X \in T$
2. $for\ any\ U \subseteq T\ there\ exists\ U \in T\ such\ that\ \bigcup U \approx U$
3. $for\ any\ V, W \in T\ there\ exists\ U \in T\ such\ that\ V \cap W \approx U$
4. $T \cap I = \{\emptyset\}$.

Then (X, T , I) is named an i-topological space and an item of T are named i-open sets, and $T(x) = \{ U \in T \mid x \in U \}$ for any x∈ X.

2.3Example

Let X be any set then (X, T_D, I) where T_D is the discrete topology and $I = \{\emptyset, \{x\}\}$is not space

2.4Example:

If (X, T) is the indiscrete topological space and I is any ideal on X then (X, T, I) is i-topological space.

2.5 Definition [1,2]:

A proximity space $(X, \boldsymbol{\delta})$ is a set X with relation **δ** between subsets of X satisfying the following properties:

For all subsets $\mathfrak{W}$, $\mathcal{K}$ and C of X

1. $\mathfrak{W}\, \boldsymbol{\delta}\, \mathcal{K} \Rightarrow \mathcal{K}\, \boldsymbol{\delta}\, \mathfrak{W}$
2. $\mathfrak{W}\, \boldsymbol{\delta}\, \mathcal{K} \Rightarrow \mathfrak{W} \neq \emptyset$
3. $\mathfrak{W} \cap \mathcal{K} \neq \emptyset \Rightarrow \mathfrak{W}\, \boldsymbol{\delta}\, B$
4. $\mathfrak{W}\, \boldsymbol{\delta}\, (\mathcal{K} \cup C) \Leftrightarrow (\mathfrak{W}\, \boldsymbol{\delta}\, \mathcal{K}\ or\ \mathfrak{W}\, \boldsymbol{\delta}\, C)$
5. $(\forall E,\ \mathfrak{W}\, \boldsymbol{\delta}\, E\ or\ \mathcal{K}\, \boldsymbol{\delta}\, (X - E)) \Rightarrow \mathfrak{W}\, \boldsymbol{\delta}\, \mathcal{K}$
6. If $\mathfrak{W}$ **δ** B we say $\mathfrak{W}$ is near $\mathcal{K}$ or $\mathfrak{W}$ and $\mathcal{K}$ are proximal; otherwise we say $\mathfrak{W}$ and $\mathcal{K}$ are apart.

2.6 Example Let X={a,b,c} and δ defined as follow , AδB iff A∩B≠φ then δ is a proximity relation on X .

2.7 Proposition [1,2] : let (X, δ) be a A proximity space then for all subsets A, B and C of X:

1. $If\ A \supseteq B\delta C, then\ A\delta C$
2. $If\ B\delta C \subseteq D, then\ B\delta D$
3. $It\ is\ false\ that\ \emptyset\delta A$
4. $It\ is\ false\ that\ A\delta\emptyset$
5. $If\ A \subseteq B\bar{\delta}C, then\ A\bar{\delta}C$
6. $If\ B\bar{\delta}C \supseteq D, then\ B\bar{\delta}D$
7. $If\ A\bar{\delta}B\ and\ A\bar{\delta}\ C\ iff\ A\bar{\delta}B \cap C$

3. In this section we will define the focal set with some of facts about it

The following proposition will prove some of important properties of the relation $\propto$

3.1. Proposition:

Let I be an ideal on the space X and A, B, C, D are subsets of X, then :

1. $A \propto X$, $foe\ each\ subset\ Aof\ X$
2. $If\ A \in I\ \ then\ A \propto \varphi$
3. $If\ A \in I\ then\ A \propto B\ \ for\ each\ subset\ B\ of\ X$
4. $If\ \ C \subseteq A\ \ \ such\ that\ A \propto B\ then\ C \propto B$
5. $If\ \ B \subseteq D\ \ such\ that\ A \propto B\ then\ A \propto D$
6. $If\ A \propto B1\ and\ A \propto B2\ then$

i- $A \propto B1\ \cap B2$ $ii-\ A \propto\ B1 \cup B2$

7. $If\ A \propto B\lambda\ \ for\ each\ \lambda\ \in\ \Lambda\ \ , wher\ \Lambda\ is\ any\ index\ \ then$

$i - A \propto \bigcap_{i=1}^{n} B_I$ $ii-\ A \propto\ \cup B\lambda$

8. $If\ A\lambda \propto B\ \ for\ each\ \lambda\ \in\ \Lambda\ \ , wher\ \Lambda\ is\ any\ index\ \ \ then$

i- $\cap A\lambda \propto B$ $ii-\ \bigcup_{I=1}^{n} A_i \propto\ B$

9. $A \propto A\ for\ each\ subset\ A\ of\ X$
10. $If\ A \propto B\ and\ B \propto\ C\ then\ A \propto C$

Proof: we will prove (7) and (8) and the other cases exist by definition (2-1)

7) i-

1- By a summation we have that $A \cap B_1^C \in I$ and $A \cap B_2^C \in I$ then

$A \cap B_1^C \cup A \cap B_2^C = A \cap (B_1^C \cup B_2^C) = A \cap (\mathrm{B1}\ \cap \mathrm{B2})^c \in I$ Thus $A \propto B_1 \cap B_2$

ii- by (6) the result exists

8) i- by using part (7) and by induction the result exist

ii-By (6) and since $A \propto B_\lambda \subseteq \cup B_\lambda$ then the result exists

9)Is a similar proof of (8)

10) Obvious

11) Obvious

3.2 Corollary:

Let (X,T,I) is i-topology if $A_\lambda \propto B_\lambda$ for $each\ \lambda\ then$

1. $\bigcap_{\lambda} A_\lambda \propto \bigcup_{\lambda} B_\lambda$
2. $\bigcap_{\lambda} A_\lambda \propto \bigcap_{\lambda} B_\lambda$

Proof: obvious

Now we will define the focal set and give some properties about it

3.3 Definition :

Let (X, T, I) be a space and x∈X , a subset A of X is named a focal set if then we have U∈T(x) such that U∝A , the system of all focal sets of a point x denoted by $I_\oint(x)$.

3.4 Example :

In the space $(X, T_D, \{\varphi\})$, $I_\oint(x)$=T(x) for each x in X when $T(x) = \{U: x\ \in X\}$.

Now useful facts about the focal set are introduce in the following proposition .

3.5 Proposition :

Let (X,T,I) be a space and $\mathfrak{W}, \mathcal{K}$, are subsets of X then the following propositions are holds :

1. If $\mathfrak{W} \in I_{\notin}(x)$ and $\mathfrak{W} \subseteq \mathcal{K}$ then $\mathcal{K} \in I_{\notin}(x)$
2. If $\mathfrak{W}, \mathcal{K} \in I_{\notin}(x)$ then $\mathfrak{W} \cap \mathcal{K} \in I_{\notin}(x)$
3. If For each $\mathfrak{W} \in I_{\notin}(x)$ then we have B such that $\mathfrak{W} \propto \mathcal{K}$ and $\mathcal{K} \in I_{\notin}(y)$ for each y in $\mathcal{K}$
4. If For each $\mathfrak{W} \in I$ and each $x \in \mathfrak{W}$ then $\mathfrak{W} \notin I_{\notin}(x)$
5. If For each $\mathfrak{W} \in T(x)$ then $\mathfrak{W} \in I_{\notin}(x)$
6. If $\mathfrak{W} \in I_{\notin}(x)$ then $\mathfrak{W}^c \notin I_{\notin}(x)$
7. If $\mathfrak{W} \in I$ then $\mathfrak{W}^c \in I_{\notin}(x)$
8. If $\mathfrak{W}, \mathcal{K} \in I_{\notin}(x)$ then $\mathfrak{W} \cup \mathcal{K} \in I_{\notin}(x)$

Proof:

1) the proof is obvious

2) let U_1 , $U_2 \in T(x)$ such that $U_1 \propto \mathfrak{W}, U_2 \propto \mathcal{K}$, since $U_1, U_2 \in T$ then then we have $w \in T$ such that $U_1 \cap U_2 \approx w$, which imply $U_1 \cap U_2 \propto w$ and $w \propto U_1 \cap U_2$, Since $U_1 \propto \mathfrak{W}, U_2 \propto \mathcal{K}$ by corollary (3-2) (2) we have $U_1 \cap U_2 \propto \mathfrak{W} \cap \mathcal{K}$ hence

We get $w \propto U_1 \cap U_2 \propto \mathfrak{W} \cap \mathcal{K}$ so $w \propto \mathfrak{W} \cap \mathcal{K}$, Now, to prove that $x \in w$, if bearable that $x \notin w$ imply that $x \in w^c$ thus $x \in (U_1 \cap U_2 \cap w^c) \in I$, this is mean $\{x\} \in I$ for each $x \in X$, from that we get $I = P(x)$ and this is contradiction ,thus $x \in w$ and $w \in T(x)$, Hence $\mathfrak{W} \cap \mathcal{K} \in I_{\notin}(x)$

3)let $\mathfrak{W} \in I_{\notin}(x)$, then then we have $\mathcal{K} \in T(x)$ such that $\mathcal{K} \propto \mathfrak{W}$,therefor for each $y \in \mathcal{K}$, $\mathcal{K} \in T(y)$ but $\mathcal{K} \propto \mathcal{K}$, hence $\mathcal{K} \in I_{\notin}(y)$

4) Suppose that $\mathfrak{W} \in I_{\notin}(x)$ so then we have $U \in T(x)$ such that $U \propto \mathfrak{W}$ But $\mathfrak{W} \in I$ then $(U \cap \mathfrak{W})^c \cup \mathfrak{W} \in I$, from that we get $U \subseteq U \cup \mathfrak{W} \in I$ and this is contradiction, then $\mathfrak{W} \notin I_{\notin}(x)$

5) For any $\mathfrak{W} \in T(x)$ by proposition (3-1) (10) , $\mathfrak{W} \propto \mathfrak{W}$ for every $\mathfrak{W}$ in X then $\mathfrak{W} \in I_{\notin}(x)$.

6) Let $\mathfrak{W} \in I_{\notin}(x)$ and suppose that $\mathfrak{W}^c \in I_{\notin}(x)$ by (2) $\varphi = \mathfrak{W} \cap \mathfrak{W}^c \in I_{\notin}(x)$ and this is contradiction, thus $\mathfrak{W}^c \notin I_{\notin}(x)$

7) let $\mathfrak{W} \in I$, if bearable that $\mathfrak{W}^c \notin I_{\notin}(x)$, then for every $U \in T(x), U \cap (\mathfrak{W}^c)^c \notin I$

Hence $U \cap \mathfrak{W} \notin I$, But $\mathfrak{W} \in I$ and this contradiction and therefore $\mathfrak{W}^c \in I_{\notin}(x)$

8) The prove is similar to (2)

The following proposition discuss the relation of the focal set of two i-topological spaces with respect to
the same i-topology T of X

3.6. Proposition:

Let (X,T,I_1) and (X,T,I_2) be spaces such that $I_1 \subseteq I_2$ then $I_{1\notin}(x) \subseteq I_{2\notin}(x)$

Proof:

Let $A \in I_{1\notin}(x)$, then we have $U \in T(x)$ such that $U \cap A^c \in I_1$ and then belongs to I_2 so $A \in I_{2\notin}(x)$.

As a consequently with the above proposition $(I_1 \cap I_2) \subseteq I_1$ and I_2.

3.7 Proposition :

Let (X, T, I) be a space if U is i-open set then U is focal set for each of its points

Proof : let U is i-open set and x be any point of X such that $x \in U$, then by proposition (3-1)(5) U is focal set of x .

The antagonistic is not true as we see bellow

3.8. Example :

In the space (X, Ti, I) where T_i is the indiscrete topology and $I = \{\emptyset , \{a\}, \{b\}, \{a, b\}\}$ then

$I_\notin$(a)= $I_\notin$(b)= $I_\notin$(c)={X,{c},{a,c},{b,c}} so clearly that {c} is a focal set of c but it is not i-open set .

3.9.Remark :

1. From the above example we can see that if A∈ $I_\notin$(x) then it is not necessary that x∈A like {c} is a focal set of a but not containing a .
2. ∅ is not focal set for each x in X
3. X is focal set for each x in X

3.10. Proposition :

Let (X, T, I) be space then the system of focal set constructed a filter for each x in X .

Proof:

By proposition (3-1) (1,2) and remark (3-9)(2 and 3) the result exist .

3.11. Definition :

Let (X,T,I) be a space and $A \subset X, x \subset X$, then x is named a $\mathcal{F}$-limit point of A iff for each $U \in$ $I_\notin$(x) such that $x \in U$ then $(Ux \cap A) - \{x\} \neq \emptyset$ and the set of all a limit point of A is named the focal derived set and denoted by $\mathcal{F}d$(A), and $\mathcal{F}$cl(A)=A∪ $\mathcal{F}d$A) and is named the focal closure of the set A

3.12. Definition :

Let (X,T,I) be a space and $A \subset X$, then the intersection of all i-closed supersets of A is named the i-closure of A and is denoted by i −cl(A) , i.e,

$i - cl(A) = \cap\{H \subseteq X:\ A \subset H, , H \text{ is } i - closed\ set\ for\ each\ i\}$

3.13. Proposition :

Let (X,T,I) be a space if a subset A of X is i-closed set then $i - cl(A) = A$

3.14. Proposition :

Let (X,T,I) be a space then if $a \in i - cl(A)$ then $U \cap A \neq \emptyset$ for each $U \in T(a)$.

Proof : let $a \in i - cl(A)$ and suppose that $U \cap A = \varphi$ then A$\subset X - U$ and since $i - cl(A)$ is the intersection of all i-closed set containing A hence $i - cl(A) \subset X - U$ and this is contradiction so $U \cap A \neq \emptyset$ for each $U \in T(a)$

3.15. Proposition :

Let (X, T, I) be a space then $\mathcal{F}cl$ (A)$\subseteq i - cl(A)$ for each subset A of X

Proof:

let p∈ $\mathcal{F}cl(A)$ then p∈ A or p∈$\mathcal{F}d$ (A) hence if $p \in A \subset i - cl(A)$ the result exist and if $p \in \mathcal{F}d(A)$

then p is $\mathcal{F} -$ limit point and for each $U \in$ $I_\notin$(x) such that $x \in U$ then $(Ux \cap A) - \{x\} \neq \emptyset$

Now if bearable that p$\notin i - cl(A)$ then by proposition (3-14) then we have $U \in T(p)$ such that $U \cap A = \varphi$ and this is contradiction so p∈ A $\subset i - cl(A)$ hence $\mathcal{F}$ cl(A)$\subseteq i - cl(A)$

3.16 Proposition

Let (X,T,I) is space then if $a \in \mathcal{F}cl(A)$ then $U \cap A \neq \emptyset$ for each U∈T(a) .

3.17. Remark:

$i - cl(A)$ and $\mathcal{F}$cl(A) is not necessarily i-closed set and $Fcl(A) \neq i - cl\{A)$ as we show in the following example

3.18. Example:

Let X={a,b,c} , $T = \{ X , \emptyset, \{a\}, \{b\} \}$,$I = \{\emptyset, \{c\}\}$ then if A={c} then $i - cl(A)$={c} which is not i-closed set

3.19. Example:

$X = \{a, b, c\}$ $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$ and $I = \{\emptyset, \{c\}\}$ then if A={b,c} then $\mathcal{F}$ d(A)= φ then A∪$\mathcal{F}$ d(A)={b,c}≠$i - cl(A)$ =X and {b,c} is not i-closed set

4. in this section we will define the focal function with some results about it

4.1.Definition

Let (X,T,I) is space and (X,δ) is a proximity space and $\mathcal{B}$ is a subset of X then a point x∈X is named occlusion point of $\mathcal{B}$ if for each U∈ $I_{\oint}$(x) , $x \in U$, U$\delta\mathcal{B}$. The set of all occlution points of $\mathcal{B}$ is denoted by $\oint(\mathcal{B})$,also we will call that occlusion set $\oint(\mathcal{B})$ is a focal function

4.2 Example

Let X={a,b,c} and $T = \{X, \emptyset, \{a, b\}, \{a, c\}\}$ and $I = \{\emptyset, \{c\}\}$ then (X,T,I) is space we define δ as a proximity relation as follow

AδB iff A∩B≠φ then $\oint$ ({a})=$\oint$ ({a,b})=$\oint$ ({a,c})=$\oint$ ({X})=X and

$\oint(\{b\}) = \oint(\{b, c\}) = \{b\}$ and $\oint(\{c\}) = \oint(\{\emptyset\}) = \emptyset$

Some of properties of focal function introduce in the following proposition

4.3 Proposition :

Let (X, T, I) is space and A,B,C are subsets of X then :

1. If $\mathfrak{W} \subseteq \mathcal{K}$ $then$ $\oint(\mathfrak{W}) \subseteq \oint(\mathcal{K})$
2. $\oint(\mathfrak{W} \cap \mathcal{K}) \subseteq \oint(\mathfrak{W}) \cap \oint(\mathcal{K})$
3. $\oint(\mathfrak{W} \cup \mathcal{K}) = \oint(\mathfrak{W}) \cup \oint(\mathcal{K})$
4. $\oint(\oint(\mathfrak{W})) = \oint(\mathfrak{W})$
5. If $\mathfrak{W} \in I$ $then$ $\oint(\mathfrak{W}) = \emptyset$
6. If $\mathfrak{W} \in I$ $then$ $\oint(\mathfrak{W} \cup \mathcal{K}) = \oint(\mathcal{K}) = \oint(\mathcal{K} - \mathfrak{W})$
7. $\oint(\mathfrak{W}) = i - cl(\oint(\mathfrak{W}))$
8. If $x \in X$ and for $each$ U in I $where$ $I \neq \varphi$ $then$ $U \cap \oint(\mathfrak{W}) \subset \oint(U \cap \mathfrak{W})$

Proof:

1) Let x ∈ $\oint(\mathfrak{W})$ then for each U ∈ $I_{\oint}$(x) , U$\delta\mathfrak{W}$,but $\mathfrak{W} \subseteq \mathcal{K}$ then U$\delta\mathcal{K}$ and hence x ∈ $\oint(\mathfrak{W})$

2) since $(\mathfrak{W} \cap \mathcal{K}) \subseteq \mathfrak{W}$ and $(\mathfrak{W} \cap \mathcal{K}) \subseteq \mathcal{K}$ then by (1) $\oint(\mathfrak{W} \cap \mathcal{K}) \subset \oint(\mathfrak{W})$

And $\oint(\mathfrak{W} \cap \mathcal{K}) \subseteq \oint(\mathcal{K})$ and hence $\oint(\mathfrak{W} \cap \mathcal{K}) \subseteq \oint(\mathfrak{W}) \cap \oint(\mathcal{K})$.

3) $\oint(\mathfrak{W} \cup \mathcal{K}) = \{x \in X: \text{for each } U \in I_{\oint}(x), U\delta(\mathfrak{W} \cup \mathcal{K})\}$

$$= \{x \in X: \forall u \in I_{\oint}(x), U\delta\mathfrak{W} \text{ or } U\delta\mathcal{K}\}$$
$$= \{x \in X: \text{for each } U \in I_{\oint}(x), U\delta\mathfrak{W}\} \text{ or } \{x \in X: \text{for each } U \in I_{\oint}(x), U\delta\mathcal{K}\}$$
$$= \oint(\mathfrak{W}) \cup \oint(\mathcal{K})$$

4) Let $x \in \oint(\mathfrak{W})$ then for each $U \in I_{\oint}(x), U\delta\mathfrak{W}$,if bearable that $x \notin \oint(\oint(\mathfrak{W}))$ then we have $w \in I_{\oint}(x)$ such that $w\bar{\delta}\oint(\mathfrak{W})$, but $w \in I_{\oint}(x)$, Then $x \in w \cap \oint(\mathfrak{W})$and this contradiction so $\oint(\mathfrak{W}) \subset \oint(\oint(\mathfrak{W}))$, now let $x \in \oint(\oint(\mathfrak{W}))$ then for each $U \in I_{\oint}(x)$, $U\delta\oint(\mathfrak{W})$ and this imply that $U \cap \oint(\mathfrak{W}) \neq \emptyset$, hence then we have $z \in U \cap \oint(\mathfrak{W})$ then $z \in \oint(\mathfrak{W})$ this mean that for each $w \in I_{\oint}(z)$, $w\delta\mathfrak{W}$,But $z \in U$,by proposition (3-5) , $U \in I_{\oint}(z)$, then $U\delta\mathfrak{W}$ for each $U \in I_{\oint}(x)$ Therefore $x \in \oint(\mathfrak{W})$.

5) if bearable that $\oint(\mathfrak{W}) \neq \varphi$ then then we have $x \in \oint(\mathfrak{W})$ Such that for each U is focal set of x such that $x \in U$, $U\delta\mathfrak{W}$, but $\mathfrak{W}^c$ is also focal set and this is contradiction hence $\oint(\mathfrak{W}) = \emptyset$

6) by using part (3)and (5)we get that $\oint(\mathfrak{W} \cup \mathcal{K}) = \oint(\mathcal{K})$

Now to prove $\oint(\mathcal{K} - \mathfrak{W}) = \oint(B)$, since $\mathcal{K} \cap \mathfrak{W}^c \subset B$ by (1) then $\oint(\mathcal{K} \cap \mathfrak{W}^c) \subset \oint(\mathcal{K})$

Now, let $x \in \oint(\mathcal{K})$ then, for each $U \in I_{\oint}(x)$, $x \in U$, $U\delta\mathcal{K}$ if bearable that $x \notin \oint(\mathcal{K} - \mathfrak{W})$ then then we have $V \in I_{\oint}(x)$ such that $V\bar{\delta}(\mathcal{K} - \mathfrak{W})$ iff $V\bar{\delta}B$ and $V\bar{\delta}\,\mathfrak{W}^c$ then $x \notin \oint(\mathcal{K})$ and this is contradiction , Hence $x \in \oint(\mathcal{K} - \mathfrak{W})$ so $\oint(\mathcal{K} - \mathfrak{W}) = \oint(\mathcal{K})$.

7) Let $x \in i - cl(\oint\mathfrak{W})$ then for each $U \in T(x)$, $U \cap \oint(\mathfrak{W}) \neq \varphi$, then then we have $y \in U$ and $y \in \oint(\mathfrak{W})$ and this imply for each $W \in I_{\oint}(y)$, $W\delta\mathfrak{W}$ but $U \in T(y)$ and by proposition (3-5)(6) , $U \in I_{\oint}(y)$ so $U\delta\mathfrak{W}$ hence $x \in \oint(\mathfrak{W})$ and since $\oint(\mathfrak{W}) \subset i - cl(\oint\mathfrak{W})$ we get that $i - cl(\oint(\mathfrak{W})) = \oint(\mathfrak{W})$

8) let $x \in w \cap \oint(\mathfrak{W})$ then $x \in \oint(\mathfrak{W})$, if bearable that $x \notin \oint(w \cap \mathfrak{W})$ then then we have $U \in I_{\oint}(x)$ such that $U\bar{\delta}w \cap \mathfrak{W}$ iff $U\bar{\delta}w$ and $U\bar{\delta}\mathfrak{W}$ then then we have $U \in I_{\oint}(x)$ such that $U\bar{\delta}\mathfrak{W}$ Then $x \notin \oint(\mathfrak{W})$and this contradiction so $w \cap \oint(\mathfrak{W}) \subset \oint(w \cap \mathfrak{W})$,

The antagonistic of (2) in the above proposition is not true as in the following example

4.4 Example :

By example (4-2) clearly that if A={a,c} and B={ b,c} then A∩B={c}

Hence $\oint(A \cap B) = (\oint\{c\}) = \emptyset \not\supseteq \oint(A) \cap \oint(B) = X \cap \{b\} = \{b\}$

The following proposition explain the relation of focal function of two spaces defined on the same family T.

4.5 Proposition :

Let (X,T,I_1) and (X,T,I_2) be i-topological spaces such that $I_1 \subseteq I_2$ then

$\oint(B(I_{2\oint}(x))) \subseteq \oint(B(I_{1\oint}(x)))$

Proof : exist by proposition (3-6) and (4-3)(1)

5. Conclusion:

Through our study of the subject, we found that the definition of the focal set does not achieve some of the attributes that the set of neighbors achieve in the usual topological spaces, and that the nature of the definition of this set affected some definitions and theories such as closure and its theories in addition to the definition of the local function

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Improved Alternating Direction Implicit Method

Esraa Abbas Al-taai[1] **Adel Rashed A. Ali**[2*]

[1,2]Department of Mathematics, College of Education for Pure Science (Ibn al-Haitham), University of Baghdad, Baghdad, Iraq

[*]Corresponding author: adil.r.a@ihcoedu.uobaghdad.edu.iq

Abstract. The alternating direction implicit method (ADI) is a common classical numerical method that was first introduced to solve the heat equation in two or more spatial dimensions and can also be used to solve parabolic and elliptic partial differential equations as well. In this paper, We introduce an improvement to the alternating direction implicit (ADI) method to get an equivalent scheme to Crank-Nicolson differences scheme in two dimensions with the main feature of ADI method. The new scheme can be solved by similar ADI algorithm with some modifications. A numerical example was provided to support the theoretical results in the research.

1. Introduction:

The alternating direction implicit (ADI) method was first proposed in the first place for partial differential parabolic equations in two spatial dimensions by D. Peaceman and H. Rachford in 1955 [1], they produce the ADI method to solve multidimensional petroleum simulators reservoir, which is between the multi-scale many types of systems which that require implicit discretization. For solving the problem of any useful size, memory-efficient, fast converging methods are needed to solve the large linear equations that arise at each time step [2]. Although computers at that time were of limited capacity, they were able to use this method to solve the problem of heat diffusion in two spatial dimensions. Later ADI method developed to solve other problems and became a significant approach in numerical methods to solve different type of partial differential equations in two or more dimensions [3, 4, 5].

Consider the two-dimensional heat equation:

$$\frac{\partial u(x,y,t)}{\partial t} = \sigma\left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right) \qquad (1)$$

$$u(x,y,0) = \varphi(x,y) \qquad (x,y) \in \Omega \cup \partial\Omega$$

$$u(x,y,t) = \phi(x,y) \qquad (x,y) \in \partial\Omega$$

Where $(x,y) \in \Omega$, $\Omega = \{(x,y)|\ 0 < x < 1, 0 < y < 1\}$, σ is a positive constant. We will consider the rectangle domain $0 < x < 1$, $0 < y < 1$ with Dirichlet boundary conditions, so that $u(x,y,t)$ is given at all rectangular boundary points, for all $t > 0$ and an initial condition $u(x,y,0)$ is given. The region is covered with a uniform rectangular grid of points, with a spacing $h = \Delta x$ in the x-axis and $k = \Delta y$ in the y–axis , where $h = \frac{1}{N_x}$, $k = \frac{1}{N_y}$, $\Delta t = \tau$, N_x, N_y are positive integer numbers, Which denote the approximated solution is then the finite difference $u_{l,m}^n = u(lh, mk, n\tau) = u(x_l, y_m, t_n)$, $l = 0,1,2, \ldots, N_x$, $m = 0,1,2, \ldots, N_y$, for simplicity suppose $(N_x = N_y = N)$.

The explicit finite difference schemes are used for solving such problems but these schemes are conditionally stable so the time the step must take a small value to achieve the stability conditions, while the implicit finite difference schemes are unconditionally stable but these schemes lead to a linear large system of equations must be solved, solving $(N-1)^2$ linear equations.

The mentioned method to solve the heat conduction equation is the Crank-Nicolson method which it like implicit method need the same number of equations in implicit method to solve at every time step. But with the ADI method we need to solve $(N-1)$ systems of linear equations and every system consist of $(N-1)$ of linear equations. After fifty years of their pioneering work on alternating direction implicit methods, D. Peaceman and H. Rachford attended a conference organized to honor them and celebrate a legacy that continues to grow [2].

2. Theory and Calculations:

To explain the advantages of the ADI method for the parabolic equation we will consider the explicit scheme, the implicit and the Crank Nicolson finite difference Scheme for equation (1) with their basic properties.

2.1. Explicit Finite Difference Scheme:

The simplest difference analog to equation (1) is the explicit finite difference scheme which can be found by replacing the time derivative $\frac{\partial u}{\partial t}$ use the difference forward approximation at the point (x_l, y_m, t_n) and the space derivatives $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ with the central difference approximation at the same grid point, then the explicit finite difference scheme has the following difference equation

$$\frac{u_{l,m}^{n+1}-u_{l,m}^{n}}{\tau} = \sigma\left(\frac{1}{h^2}\delta_x^2 u_{l,m}^n + \frac{1}{k^2}\delta_y^2 u_{l,m}^n\right) \tag{2}$$

where $\delta_x^2 u_{l,m}^n = u_{l-1,m}^n - 2u_{l,m}^n + u_{l+1,m}^n$, $\delta_y^2 u_{l,m}^n = u_{l,m-1}^n - 2u_{l,m}^n + u_{l,m+1}^n$ and can be rewritten as:

$$u_{l,m}^{n+1} = (1 + r_x\delta_x^2 + r_y\delta_y^2)u_{l,m}^n \tag{3}$$

where $r_x = \frac{\sigma\tau}{h^2}$ and $r_y = \frac{\sigma\tau}{k^2}$. This is the explicit scheme which can be solved explicitly for $u_{l,m}^{n+1}$, it is stable with conditions and the stability condition is

$$r_x + r_y \le \frac{1}{2} \tag{4}$$

For the case $h = k$ the condition of stability becomes

$$r_x = \frac{\sigma\tau}{h^2} \le \frac{1}{4} \tag{5}$$

That it is as restrictive twice as the one dimensional case [6, 7].

2.2 Implicit Finite Difference Scheme:

The implicit finite difference scheme can be obtained by replacing the derivative time $\frac{\partial u}{\partial t}$ using forward difference approximation at the point (x_l, y_m, t_n) and the space derivatives $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ with the central difference approximation at the grid point (x_l, y_m, t_{n+1}), then the implicit finite difference scheme has the following difference scheme

$$u_{l,m}^n = u_{l,m}^{n+1} - r_x\delta_x^2 u_{l,m}^{n+1} - r_y\delta_y^2 u_{l,m}^{n+1} \tag{6}$$

or

$$u_{l,m}^n = (1 - r_x\delta_x^2 - r_y\delta_y^2)u_{l,m}^{n+1} \tag{7}$$

The implicit scheme is unconditionally stable [6, 7], but leads to large number of linear equations which are more difficult to solve than the explicit scheme.

2.3 Crank Nicolson Finite Difference Scheme:

It is another implicit difference scheme and can be found by replacing the time derivative $\frac{\partial u}{\partial t}$ using forward difference approximation at the point (x_l, y_m, t_n) and the space derivatives $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ with the central difference approximation at the two grid points (x_l, y_m, t_n) and (x_l, y_m, t_{n+1}) and take the average then the Crank Nicolson difference scheme given by:

$$u_{l,m}^{n+1} - u_{l,m}^n = \frac{1}{2}\left[\left(r_x\delta_x^2 + r_y\delta_y^2\right)u_{l,m}^n + \left(r_x\delta_x^2 + r_y\delta_y^2\right)u_{l,m}^{n+1}\right] \tag{8}$$

which can be rewritten as

$$\left[1-\frac{1}{2}\left(r_x\delta_x^2+r_y\delta_y^2\right)\right]u_{l,m}^{n+1}=\left[1+\frac{1}{2}\left(r_x\delta_x^2+r_y\delta_y^2\right)\right]u_{l,m}^{n} \tag{9}$$

3. ADI method:

The ADI method is a finite difference method for two dimensional (or more) heat flow and diffusion problems. The main idea of the ADI method is to divide the scheme from t to $t+\Delta t$ into two steps, in the first half step, from t to $t+\frac{\Delta t}{2}$, treating one of the spatial derivatives implicitly (say $\frac{\partial^2 u}{\partial x^2}$) and treating the other derivative (say u_{yy}) explicitly, this lead to the difference equation:

$$u_{l,m}^{n+\frac{1}{2}}-u_{l,m}^{n}=\frac{r_x}{2}\delta_x^2u_{l,m}^{n+\frac{1}{2}}+\frac{r_y}{2}\delta_y^2u_{l,m}^{n} \tag{10}$$

The matrix of the unknowns $u_{l,m}^{n+\frac{1}{2}}$ will appearing in (10) as a block tridiagonal linear algebraic system of equations and that can solved by the algorithm of tridiagonal linear system. For the second step reverse the treating of the spatial derivatives, i. e. from $t+\frac{\Delta t}{2}$ to $t+1$, treating $\frac{\partial^2 u}{\partial x^2}$ explicitly and treating $\frac{\partial^2 u}{\partial y^2}$ implicitly and this lead to the second difference equation:

$$u_{l,m}^{n+1}-u_{l,m}^{n+\frac{1}{2}}=\frac{r_x}{2}\delta_x^2u_{l,m}^{n+\frac{1}{2}}+\frac{r_y}{2}\delta_y^2u_{l,m}^{n+1} \tag{11}$$

The unknowns $u_{l,m}^{n+1}$ in (11) will appearing like equation (10) as a block tridiagonal linear system of algebraic equations and can be solved by the same algorithm.
The two equations (10) and (11) consist the ADI scheme. The ADI Scheme is unconditional stability with simplicity in calculation. Nowadays there are many versions of the method, with applications to elliptic and hyperbolic problems as well as to systems of parabolic equations.
For comparing ADI with the Crank Nicolson scheme consider equation (10), that rewritten as:

$$\left(1-\frac{r_x}{2}\delta_x^2\right)u_{l,m}^{n+\frac{1}{2}}=\left(1+\frac{r_y}{2}\delta_y^2\right)u_{l,m}^{n} \tag{12}$$

or

$$u_{l,m}^{n+\frac{1}{2}}=\left(1-\frac{r_x}{2}\delta_x^2\right)^{-1}\left(1+\frac{r_y}{2}\delta_y^2\right)u_{l,m}^{n} \tag{13}$$

similarly, equation (11) can be rewritten as:

$$u_{l,m}^{n+1}=\left(1+\frac{r_x}{2}\delta_x^2\right)u_{l,m}^{n+\frac{1}{2}}+\frac{r_y}{2}\delta_y^2u_{l,m}^{n+1} \tag{14}$$

from equation (13) and equation (14) we can get

$$u_{l,m}^{n+1}=\left(1+\frac{r_x}{2}\delta_x^2\right)\left(1-\frac{r_x}{2}\delta_x^2\right)^{-1}\left(1+\frac{r_y}{2}\delta_y^2\right)u_{l,m}^{n}+\frac{r_y}{2}\delta_y^2u_{l,m}^{n+1} \tag{15}$$

By simplifying the equation we get

$$\left(1-\frac{r_y}{2}\delta_y^2\right)u_{l,m}^{n+1}=\left(1+\frac{r_x}{2}\delta_x^2\right)\left(1-\frac{r_x}{2}\delta_x^2\right)^{-1}\times\left(1+\frac{r_y}{2}\delta_y^2\right)u_{l,m}^{n} \tag{16}$$

or

$$\left(1-\frac{r_y}{2}\delta_y^2\right)\left(1-\frac{r_x}{2}\delta_x^2\right)u_{l,m}^{n+1}=\left(1+\frac{r_x}{2}\delta_x^2\right)\times\left(1+\frac{r_y}{2}\delta_y^2\right)u_{l,m}^{n} \tag{17}$$

with more simplifying we get

$$\left[1-\frac{1}{2}\left(r_x\delta_x^2+r_y\delta_y^2\right)+\frac{1}{4}r_xr_y\delta_x^2\delta_y^2\right]u_{l,m}^{n+1}=\left[1+\frac{1}{2}\left(r_x\delta_x^2+r_y\delta_y^2\right)+\frac{1}{4}r_xr_y\delta_x^2\delta_y^2\right]u_{l,m}^{n} \tag{18}$$

Now by comparing equation (18) with the Crank Nicolson scheme, equation (9) we can notice that ADI consider

$$\left(\frac{1}{4}r_x r_y \delta_x^2 \delta_y^2\right) u_{l,m}^{n+1} = \left(\frac{1}{4}r_x r_y \delta_x^2 \delta_y^2\right) u_{l,m}^{n} \quad (19)$$

in two time steps, n and $n+1$.

4. Improved ADI

ADI semi-implicit method is because it expresses one spatial derivative in an explicit difference scheme and the other spatial derivative in implicit difference scheme, if we have a three-dimensional problem then the ADI method will be a one-third implicit method and so on.

In this work, we introduce an improvement to the ADI method to get a finite difference scheme similar to the Crank-Nicolson scheme as follows:

$$u_{l,m}^{\star n+1} - u_{l,m}^{n} = r_x \delta_x^2 u_{l,m}^{\star n+1} + r_y \delta_y^2 u_{l,m}^{n} \quad (20)$$

$$u_{l,m}^{\star\star n+1} - u_{l,m}^{n} = r_x \delta_x^2 u_{l,m}^{n} + r_y \delta_y^2 u_{l,m}^{\star\star n+1} \quad (21)$$

Then by addition and divided by 2 for equations (20) and (21) we get the average:

$$u_{l,m}^{n+1} = \frac{1}{2}\left[u_{l,m}^{\star n+1} + u_{l,m}^{\star\star n+1}\right] \quad (22)$$

The three equations (20), (21) and (22) are consist the Improved ADI scheme. The main idea in improved ADI method is of only one of the 2nd-order replaced derivatives, like ADI method, using implicit finite difference approximation in terms of unknown values of u from $(n+1)$th level time, and the other 2nd-order derivative being replaced by an explicit finite difference approximation then solve the resulting system to get the first solution. And repeat this for the second derivative to the same time level and get the second solution, then gather the two solutions and divide them by two. With this technique, we will get a scheme similar to the Crank Nicolson scheme.

The equations (20) and (21) can be solved by tridiagonal matrix algorithm. Both $u_{l,m}^{\star n+1}$ and $u_{l,m}^{\star\star n+1}$ represent a solution of equation (1) so replaced by an $u_{l,m}^{n+1}$, and the equations (20) and (21) can be rewritten as:

$$(1 - r_x \delta_x^2) u_{l,m}^{n+1} = \left(1 + r_y \delta_y^2\right) u_{l,m}^{n} \quad (23)$$

$$\left(1 - r_y \delta_y^2\right) u_{l,m}^{n+1} = (1 + r_x \delta_x^2) u_{l,m}^{n} \quad (24)$$

Add (23) with (24) to get

$$\left[(1 - r_x \delta_x^2) + \left(1 - r_y \delta_y^2\right)\right] u_{l,m}^{n+1} = \left[(1 + r_x \delta_x^2) + \left(1 + r_y \delta_y^2\right)\right] u_{l,m}^{n} \quad (25)$$

By simplifying the equation, we can get

$$2u_{l,m}^{n+1} - r_x \delta_x^2 u_{l,m}^{n+1} - r_y \delta_y^2 u_{l,m}^{n+1} = 2u_{l,m}^{n} + r_x \delta_x^2 u_{l,m}^{n} + r_y \delta_y^2 u_{l,m}^{n} \quad (26)$$

or

$$2\left(u_{l,m}^{n+1} - u_{l,m}^{n}\right) = r_x \delta_x^2 \left(u_{l,m}^{n+1} + u_{l,m}^{n}\right) + r_y \delta_y^2 \left(u_{l,m}^{n+1} + u_{l,m}^{n}\right) \quad (27)$$

$$u_{l,m}^{n+1} - u_{l,m}^{n} = \frac{1}{2} r_x \delta_x^2 \left(u_{l,m}^{n+1} + u_{l,m}^{n}\right) + \frac{1}{2} r_y \delta_y^2 \left(u_{l,m}^{n+1} + u_{l,m}^{n}\right) \quad (28)$$

and this led to:

$$\left[1 - \frac{1}{2}\left(r_x \delta_x^2 + r_y \delta_y^2\right)\right] u_{l,m}^{n+1} = \left[1 + \frac{1}{2}\left(r_x \delta_x^2 + r_y \delta_y^2\right)\right] u_{l,m}^{n} \quad (29)$$

We can notice that this is the same Crank Nicolson finite difference scheme (9). And so the improved ADI method has the same accuracy as Crank-Nicolson method.

4.1 Stability Analysis of Improved ADI method:

We will use the Von Neumann method to find the stability condition for the improved ADI finite difference scheme [8]. It is common to write

$$u_{l,m}^{n} = \xi^{n} e^{i\beta lh} e^{i\gamma mk} \qquad (30)$$

where $i = \sqrt{-1}$, β ,γ are real spatial wave numbers and ξ is the amplification factor.

Theorem 1: The improved ADI finite difference scheme is unconditionally stable.

Proof: The finite difference scheme (20) and (21) and (22) can be rewritten as

$$u_{l,m}^{n+1} = \frac{1}{2}\left[u_{l,m}^{\star n+1} + u_{l,m}^{\star\star n+1}\right]$$

$$= \tfrac{1}{2}[(u_{l,m}^{n} + r_x\delta_x^2 u_{l,m}^{\star n+1} + r_y\delta_y^2 u_{l,m}^{n}) + (u_{l,m}^{n} + r_x\delta_x^2 u_{l,m}^{n} + r_y\delta_y^2 u_{l,m}^{\star\star n+1})]$$

$$= \tfrac{1}{2}[u_{l,m}^{n} + r_x(u_{l-1,m}^{\star n+1} - 2u_{l,m}^{\star n+1} + u_{l+1,m}^{\star n+1}) + r_y(u_{l,m-1}^{n} - 2u_{l,m}^{n} + u_{l,m+1}^{n})]$$

$$+ \tfrac{1}{2}[u_{l,m}^{n} + r_x(u_{l-1,m}^{n} - 2u_{l,m}^{n} + u_{l+1,m}^{n}) + r_y(u_{l,m-1}^{\star\star n+1} - 2u_{l,m}^{\star\star n+1} + u_{l,m+1}^{\star\star n+1})]$$

use the expression (30) to get we obtain

$$2\xi^{n+1}e^{i\beta lh}e^{i\gamma mk} = 2\xi^{n}e^{i\beta lh}e^{i\gamma mk} + r_x(\xi^{\star n+1}e^{i\beta(l-1)h}e^{i\gamma mk} - 2\xi^{\star n+1}e^{i\beta lh}e^{i\gamma mk}$$
$$+\xi^{\star n+1}e^{i\beta(l+1)h}e^{i\gamma mk}) + r_y(\xi^{n}e^{i\beta lh}e^{i\gamma(m-1)k} - 2\xi^{n}e^{i\beta lh}e^{i\gamma mk}$$
$$+\xi^{n}e^{i\beta lh}e^{i\gamma(m+1)k}) + r_x(\xi^{n}e^{i\beta(l-1)h}e^{i\gamma mk} - 2\xi^{n}e^{i\beta lh}e^{i\gamma mk}$$
$$+\xi^{n}e^{i\beta(l+1)h}e^{i\gamma mk}) + r_y(\xi^{\star\star n+1}e^{i\beta lh}e^{i\gamma(m-1)k}$$
$$-2\xi^{\star\star n+1}e^{i\beta lh}e^{i\gamma mk} + \xi^{\star\star n+1}e^{i\beta lh}e^{i\gamma(m+1)k})$$

We consider both $u_{l,m}^{\star n+1}$ and $u_{l,m}^{\star\star n+1}$ as a solution of equation (1) in the level $n+1$ so we can consider $\xi^{\star\star n+1}, \xi^{\star n+1}$ as ξ^{n+1}. Divided the above equation by $\xi^{n}e^{i\beta lh}e^{i\gamma mk}$ to get

$$2\xi = 2 + r_x(\xi e^{-i\beta h} - 2\xi + \xi e^{i\beta h}) + r_y(e^{-i\gamma k} - 2 + e^{i\gamma k})$$
$$+r_x(e^{-i\beta h} - 2 + e^{i\beta h}) + r_y(\xi e^{-i\gamma k} - 2\xi + \xi e^{i\gamma})$$

using the formula ($e^{i\theta} - 2 + e^{-i\theta} = -4\sin^2\frac{\theta}{2}$) to get

$$2\xi = 2 + r_x\xi\left(-4\sin^2\frac{\beta h}{2}\right) + r_y\left(-4\sin^2\frac{\gamma k}{2}\right) + r_x\left(-4\sin^2\frac{\beta h}{2}\right) + r_y\xi\left(-4\sin^2\frac{\gamma k}{2}\right)$$

rearrange the equation and divided by two to get

$$\xi + r_x\xi\left(2\sin^2\frac{\beta h}{2}\right) + r_y\xi\left(2\sin^2\frac{\gamma k}{2}\right) = 1 + r_y\left(-2\sin^2\frac{\gamma k}{2}\right) + r_x\left(-2\sin^2\frac{\beta h}{2}\right)$$

this lead to

$$\xi = \frac{1 - 2r_y\sin^2\frac{\gamma k}{2} - 2r_x\sin^2\frac{\beta h}{2}}{1 + 2r_x\sin^2\frac{\beta h}{2} + 2r_y\sin^2\frac{\gamma k}{2}} \qquad (31)$$

For stability we require $|\xi| \leq 1$, and from equation (31) for all values of r_x, r_y, β, γ This ratio has an absolute value less than or equal to one.

Table 1: Results of the example when $\Delta x = \Delta y = \Delta t$ with $T = 0.5$.

	$N \times M$	ADI Method		Improve ADI	
		Average Error	**Max Error**	**Average Error**	**Max Error**

1	10×10	0.003791562244134	0.013263668239204	0.001359974472108	0.004383685897438
2	15×15	0.001990674075311	0.006252252554629	0.000279746819906	0.000995690451892
3	20×20	0.001145347137462	0.003448770747655	0.000576055621512	0.001399504353320
4	25×25	0.000777710885193	0.002261185531372	0.000144191125440	0.000730750281131
5	30×30	0.000541626617549	0.001554638768872	0.000084594696706	0.000651841988994
6	35×35	0.000411224398929	0.001161040587889	0.000054859072335	0.000824484762950
7	40×40	0.000314098601357	0.000875586629557	0.000064951419127	0.000604186421875

Numerical Example: *For comparison between the improved ADI and ADI, we will consider the following diffusion equation in two dimensions*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y, t)$$

where $f(x, y, t) = (2\pi^2 - 1)u$ on the domain $0 < x < 1$, $0 < y < 1$ with $0 \leq t$ and the initial condition $u(x, y, 0) = \cos(\pi x)\sin(\pi y)$ and Dirichlet boundary conditions on the rectangle in the form $u(0, y, t) = -u(1, y, t) = e^{-t}\sin(\pi y)$, $u(x, 0, t) = u(x, 1, t) = 0$. The exact solution is given by $u(x, y, t) = e^{-t}\cos(\pi x)\sin(\pi y)$.

The table (1) represents the results of the example with different values of $N \times M$ with two error measures, the average error and the maximum of errors.

5. Results and Conclusion:

The Improved ADI is stable with out condition and consistent with a local truncation error $O((\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2)$ as Crank Nicolson method. Then by Lax's Equivalence Theorem [8], the converge conditions are satisfied. The ADI method has a local truncation error $O((\Delta t) + (\Delta x)^2 + (\Delta y)^2)$.

To solve the two dimensional diffusion equation by Crank Nicolson method we need to solve a linear system of $(N-1)^2$ equation in every time step, but The ADI techniques reduce the Number of arithmetic operation, we need to solve $(n-1)$ linear system and every system have $(N-1)$ linear equations at every half time step. But with improved ADI method we need to solve $(n-1)$ linear system and every system have $(N-1)$ linear equations two times at every time step.

The numerical examples show that the improved ADI method have a good agreement with the theoretical findings. in this paper we consider the diffusion equation in two dimensions, it can be possibly generalized and extended to elliptic and hyperbolic problems and for more than two dimensions.

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A Multilevel Approach for Stability Conditions in Fractional Time Diffusion Problems

I. K. Youssef [1] and Adel Rashed A. Ali[2*]

[1] Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt. E-mail (kaoud22@hotmail.com).

[2] Department of Mathematics, College of Education for Pure Science (Ibn al-Haitham), University of Baghdad, Baghdad, Iraq.

[*]Corresponding author: adil.r.a@ihcoedu.uobaghdad.edu.iq

Abstract. The Caputo definition of fractional derivatives introduces solution to the difficulties appears in the numerical treatment of differential equations due its consistency in differentiating constant functions. In the same time the memory and hereditary behaviors of the time fractional order derivatives (TFODE) still common in all definitions of fractional derivatives. The use of properties of companion matrices appears in reformulating multilevel schemes as generalized two level schemes is employed with the Gerschgorin disc theorems to prove stability condition. Caputo fractional derivatives with finite difference representations is considered. Moreover the effect of using the inverse operator which transmit the memory and hereditary effects to other terms is examined. The theoretical results is applied to a numerical example. The calculated solution has a good agreement with the exact solution.

1. Introduction

The numerical treatment of the standard parabolic equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \qquad (1)$$

with initial condition $u(x,0) = g(x)$ and Dirichlet boundary conditions of the form $u(0,t) = u(1,t) = 0$, defined on the domain $0 < x < 1$, $0 < t < T$, is the cornerstone in the numerical treatment of PDE's in general. Most of the characteristics as well as the difficulties of finite difference method and its common properties appear in this simple form.

The basic idea of the finite difference method depends on the replacement of the derivatives by functional values at different arguments. Accordingly, replacing the functional differential equations by an algebraic relation. The accuracy of the solutions obtained by the use of the finite difference method depends on the convergence, consistency and stability requirements of the corresponding discrete problem. Studying the stability of implicit as well as explicit schemes for equation (1) was the main topic in many publications. Lax equivalence theorem states that satisfaction of only two among the convergence, the consistency and the stability will guarantee the satisfaction of the third. In this work we focus on studying the stability. The importance of proving stability conditions appears in many scientific and economic situations rather than the reliability of solutions. Choosing large steps within the admissible range well reduce the storage requirements as well as the running time. There are different methods used in the stability treatment, Von Neumann, energy and matrix methods are standard techniques [1, 2].

Our main task is to obtain with simple straightforward, easy and realistic method the stability conditions of the explicit scheme of the fractional time counter part equation (1)

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),\ \ 0 < \alpha < 1 \qquad (2)$$

with initial condition $u(x,0) = u_0(x)$ and Dirichlet boundary conditions of the form $u(0,t) = u(1,t) = 0$, defined on the domain $0 < x < 1$, $0 < t < T$, where the fractional order time derivative is understood in the Caputo sense.
The correspondence with the classical multilevel schemes treated in Richtmyer and Morton [2] with the relations on the norm of Frobenius matrices (appears in the reformulation of multilevel schemes as block two level schemes) and moreover the well-known Gerschgorin disc theorems have been reemployed to introduce systematic treatment.
Definition 1.1 *The Caputo time fractional derivative of order $\alpha > 0$ of the function $u(x,t)$ is defined by [3, 4]:*

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n}{ds^n} u(x,s) ds \tag{3}$$

where $\alpha \in (n-1, n)$, $n \in N$. If $\alpha \in N$ then this will coincide with the classical partial derivative.
Equation (2) have appeared in many applications in physics, continuum mechanics, signal processing, and electromagnetic. Also, many publications have mentioned in biology, chemistry and biochemistry, hydrology, medicine, and finance [3, 4]. The fractional order partial differential equations (FOPDEs) are used to model anomalous diffusion and Hamiltonian Chaos. These equations describe the asymptotic behavior of continuous time random walks. Stochastic solutions to the simplest governing equations are Levy motions, generalizing the Brownian motion solution to the classical diffusion equation. Fractional kinetic equations have proved particularly useful in the context of anomalous subdiffusion [5, 6].
The fractional derivative considers the memory and hereditary effects which is not the case of the classical integer derivative which considers only the local behavior. In this work we are interested in this point and its effects on the stability conditions of the explicit schemes. Moreover, the corresponding between the treatment in the stability of multilevel schemes in the integer case and the explicit schemes in the fractional order case have been considered.
Models described in the form of FOPDEs, tend to be more appropriate for the description of memorial and hereditary properties of various materials and processes than the traditional integer order models [7].
It is interesting to note that the FOPDEs is a generalization of the classical partial differential equations and the limiting prosses as the fractional order approaches the classical integer order must introduce the classical case $0 < \alpha < 1$, [8].
It is well known that there is no analytical method that can be considered as a master method for solving PDEs the situation in FOPDEs is more complicated. Laplace and Fourier transform methods [9] have their limitation. Semianalytic methods like the series solution method, the Adomian decomposition method [10] suffer from the complicated integrations. Numerical methods became the most reliable treatment in solving many problems in PDEs due to the development in computer devices. The finite difference method is considered as one of the simplest numerical methods that can treat many different problems [1, 11].
A number of numerical methods have been developed to solve the time fractional diffusion equation with Dirichlet boundary conditions. Yuste and Acedo [12] proposed a procedure with a new Von Neumann-type stability analysis in one dimension using Grünwald approximation for time fractional derivative. Liu et al [8] proposed another stability analysis procedure using discrete non-Markovian random walk approximation for time fractional derivative. LI and XU propose a spectral method in both temporal and spatial discretization [13]. Meerschaert et al. [14] use finite difference approximations for fractional advection-dispersion flow equations and other numerical methods with finite difference approximation to fractional derivative [15, 16, 17, 18] with Von Neumann and matrix methods to study the stability analysis and convergence of the methods.
In the finite difference method, the continuous domain is replaced by a discrete grid superimpose the domain under consideration and the derivatives are replaced by the corresponding differences of functional values obtaining algebraic equation at each grid point. Solutions obtained by the finite difference method must satisfy some tests of consistency, stability and convergence to be reliable.
Some authors prefer to write the time fractional diffusion equation in the form [5, 12]:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \left(\frac{\partial^2 u(x,t)}{\partial x^2} \right) + \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} f(x,t) \tag{4}$$

This form appears to have many difficulties in the finite difference approximation because it includes the time derivatives in both sides even the derivatives in the left-hand side is of integer order.
Any algorithm using a finite difference discretization of the time fractional derivative has to take into account its nonlocal structure, i. e. the computation of the solution at a time level requires information about the solution at all previous time levels, which means high storage requirement.
To deal with this issue, Ford and Simpson [19] and Diethelm and Freed [20], developed a numerical technique to reduce the computational cost of the solution using the so called "fixed memory principle" as described in Podlubny [4]. We will discuss and compare between equation (2) and equation (4) with discretization of time fractional derivative by Caputo definition, formula (7), for both equations with use the Multilevel method to derive the stability conditions.

2. The Finite Difference Method

In the Finite difference method (FDM) every differential equation is approximated by a corresponding finite differences scheme. The domain $[0,1] \times [0,T]$ of the given parabolic equation is superimposed with a grid. The interval $[a,b]$ is divided into J subintervals with length $\triangle x = h = \frac{1}{J}$, $x_j = jh$, for $j = 0,1,2,\cdots,J$ and define the time step $\triangle t = \tau$ and $t_n = n\tau$.
The explicit scheme corresponding to equation (1) can be written in the form [2, 23]

$$u_j^{n+1} = ru_{j-1}^n + (1-2r)u_j^n + ru_{j+1}^n \tag{5}$$

this scheme is consistent and stable for $r = \frac{\tau}{h^2} \leq \frac{1}{2}$. To obtain the corresponding scheme for the fractional order equation (2) one must use the discretization of fractional order derivative, the inverse operator form equation (4) is also considered.

2.1. Discretization of Caputo Fractional derivatives

The time fractional derivative replaced by Caputo fractional derivative of order α, definition 1.1, and we use the following formulation [8]

$$\frac{\partial^\alpha u(x_j,t_{n+1})}{\partial t^\alpha} = w_\alpha^\tau \sum_{k=0}^{n} b_k^\alpha (u_j^{n-k+1} - u_j^{n-k}) \tag{6}$$

where $w_\alpha^\tau = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}$ and $b_k^\alpha = (k+1)^{1-\alpha} - k^{1-\alpha}$, for $k = 0,1,2,\cdots,n$, which can be rearranged in the form

$$\frac{\partial^\alpha u(x_j,t_{n+1})}{\partial t^\alpha} = w_\alpha^\tau \left[u_j^{n+1} - \sum_{k=1}^{n} c_k^\alpha u_j^{n-k+1} - b_n^\alpha u_j^0\right] \tag{7}$$

with $c_k^\alpha = b_{k-1}^\alpha - b_k^\alpha$.

Properties 1: *the coefficients b_k^α and c_k^α having the following properties:*
- $c_k^\alpha = 2k^{1-\alpha} - (k-1)^{1-\alpha} - (k+1)^{1-\alpha}$, $k = 1,2,3,\cdots$. And $\sum_{k=1}^{\infty} c_k^\alpha = 1$.
- $1 > 2 - 2^{1-\alpha} = c_1^\alpha > c_2^\alpha > c_3^\alpha > \cdots$, with $\lim_{k\to\infty} c_k^\alpha = 0$.
- $1 = b_0^\alpha > b_1^\alpha > b_2^\alpha > b_3^\alpha > \cdots$, with $\lim_{k\to\infty} b_k^\alpha = 0$.

Replacing the time derivative using equation (7) at the grid point (x_l, t_n) and the space derivatives with the central difference approximation at the same grid point (x_l, t_n), then the explicit scheme for the solution of equation (2) have the following difference equation

$$u_j^{n+1} = b_n^\alpha u_j^0 + \sum_{k=1}^{n} c_k^\alpha u_j^{n+1-k} + r_\alpha \Gamma(2-\alpha)\delta_x^2 u_j^n + \frac{1}{w_\alpha^\tau} f_j^n \tag{8}$$

3. Stability in Multilevel schemes

The Von Neumann technique for stability analysis uses for a two-time level finite difference scheme but for more than two-time level schemes we need to use the multilevel technique to check the stability conditions, for more details about this technique see [21, 22].

4. Discretization of Time Fractional Derivatives

Replacing the derivatives appears in differential equation by their finite difference approximations one obtains a corresponding scheme. The scheme properties (consistency, stability and convergence) should be examined to obtain reliable results. The same approach is used in case of fractional derivatives. We consider the fractional time derivative in Caputo definition and study its finite difference approximations, also we use this approximation in the diffusion like equations (2) and (4).
The amplification matrix described above can be obtained with the Von Neumann method and Multilevel finite difference technique to study the stability conditions of the fractional time finite difference scheme. Putting

$$ {}_j^n = \xi^n e^{i\beta jh} \tag{9}$$

where $i = \sqrt{-1}$ and β is a real spatial wave number.
The explicit scheme (8) is conditionally stable and the stability condition is $r_\alpha \le \frac{1-2^{-\alpha}}{\Gamma(2-\alpha)}$, Liu et al [8].
We use the multilevel approach and obtain the same stability condition in the next theorem 4.1. The condition is depending on α, figure 1 (a).

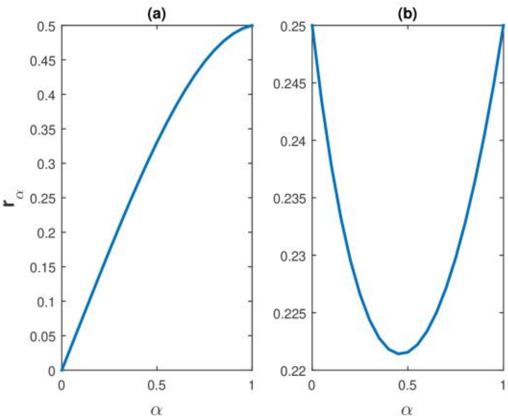

Figure 1: the stability condition on r_α, left (a) for equation (8) and right (b) for equation (22)

Theorem 4.1: *The fractional explicit scheme (8) is conditionally stable and the stability condition is* $r_\alpha \le \frac{1-2^{-\alpha}}{\Gamma(2-\alpha)}$.

Proof: The scheme (8) is a multilevel scheme and can be rewritten

$$u_j^{n+1} = [c_1^\alpha + r_\alpha \Gamma(2-\alpha)\delta_x^2]u_j^n + \sum_{k=2}^{n} c_k^\alpha u_j^{n+1-k} + b_n^\alpha u_j^0 + \frac{1}{w_\alpha^\tau} f_j^n \tag{10}$$

then the multilevel amplification matrix C can be defined by square block matrix of order $(n+1)$ and every element of C is of order $J-1$:

$$\mathbf{C} = \begin{bmatrix} (c_1^\alpha + r_\alpha\Gamma(2-\alpha)\delta_x^2)I & (c_2^\alpha)I & (c_3^\alpha)I & \dots & (c_n^\alpha)I & (b_n^\alpha)I \\ I & O & O & \dots & O & O \\ O & I & O & \dots & O & O \\ \vdots & \vdots & \vdots & \dots & O & \vdots \\ O & O & O & \dots & I & O \end{bmatrix} \tag{11}$$

for the amplification matrix Insert expression (9) in equation (7) then we get

$$\xi^{n+1}e^{i\beta jh} = \xi^0 b_n^\alpha e^{i\beta jh} + \sum_{k=1}^{n} c_k^\alpha \xi^{n+1-k} e^{i\beta jh} + r_\alpha \Gamma(2-\alpha)\xi^n \left[e^{i\beta(j+1)h} - 2e^{i\beta jh} + e^{i\beta(j-1)h}\right] \tag{12}$$

Divided by $e^{i\beta jh}$ and using the formula $\left(e^{i\theta} - 2 + e^{-i\theta}\right) = -4\sin^2\frac{\theta}{2}$ to get

$$\xi^{n+1} = \xi^0 b_n^\alpha + \sum_{k=1}^{n} c_k^\alpha \xi^{n+1-k} + r_\alpha \Gamma(2-\alpha)\xi^n \left[-4\sin^2\frac{\beta h}{2}\right] \tag{13}$$

can be rewritten

$$\xi^{n+1} = \xi^{n}\left[c_1^{\alpha} - 4r_{\alpha}\Gamma(2-\alpha)\sin^2\frac{\beta h}{2}\right] + \sum_{k=2}^{n} c_k^{\alpha}\xi^{n+1-k} + \xi^{0} b_n^{\alpha} \tag{14}$$

then the amplification matrix M can be defined by square block matrix of order $(n+1)$ and every element of M is of order $(J-1)$:

$$M = \begin{bmatrix} (c_1^{\alpha} - 4r_{\alpha}\Gamma(2-\alpha)\sigma)I & (c_2^{\alpha})I & (c_3^{\alpha})I & \dots & (c_n^{\alpha})I & (b_n^{\alpha})I \\ I & O & O & \dots & O & O \\ O & I & O & \dots & O & O \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ O & O & O & \dots & I & O \end{bmatrix} \tag{15}$$

From Gerschgorin theorem for estimating the eigne values of any matrix [24], all rows of matrix M gives eigne values lies in the union of unit discs centered at (0,0) in the complex plan except those corresponding to the first block. For each row of the first block one can see that the corresponding eigenvalue satisfies

$$|\lambda| \leq \left|c_1^{\alpha} - 4r_{\alpha}\Gamma(2-\alpha)\sin^2\frac{\beta h}{2}\right| + \sum_{k=2}^{n} |c_k^{\alpha}| + |b_n^{\alpha}| \tag{16}$$

by the properties1 we have $c_k^{\alpha} > 0$, $b_n^{\alpha} > 0$, and $\sum_{k=2}^{n} c_k^{\alpha} = 2^{1-\alpha} - 1 - b_n^{\alpha}$, this lead to

$$|\lambda| \leq \left|2 - 2^{1-\alpha} - 4r_{\alpha}\Gamma(2-\alpha)\sin^2\frac{\beta h}{2}\right| + 2^{1-\alpha} - 1 \tag{17}$$

if the right-hand inequality is less than or equal to one then $|\lambda| \leq 1$, then we have

$$-(2 - 2^{1-\alpha}) \leq \left(2 - 2^{1-\alpha} - 4r_{\alpha}\Gamma(2-\alpha)\sin^2\frac{\beta h}{2}\right) \leq 2 - 2^{1-\alpha} \tag{18}$$

the right-hand inequality is satisfied and we need to calculate the condition on r_{α} to make the left-hand inequality satisfied, this lead to

$$4r_{\alpha}\Gamma(2-\alpha)\sin^2\frac{\beta h}{2} \leq 4 - 2^{2-\alpha} \tag{19}$$

then the stability condition is

$$r_{\alpha} \leq \frac{1-2^{-\alpha}}{\Gamma(2-\alpha)} \tag{20}$$

For equation (4) the time derivatives appears in both sides makes the finite difference representation is implicit and to obtain the explicit scheme and moreover the implicit, we introduce the weighted average approach to the time derivatives in the right hand side i.e we replace the term $\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)$ by its weighted approximation at the preceding time levels.

$$\theta\left[\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}\left(\frac{\partial^2 u(x,t_n)}{\partial x^2}\right)\right] + (1-\theta)\left[\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}\left(\frac{\partial^2 u(x,t_{n-1})}{\partial x^2}\right)\right] \tag{21}$$

Thus for $\theta = 0$, one obtains the explicit scheme obtained for equation (4) in the form

$$\frac{u_j^{n+1} - u_j^{n}}{\tau} = \frac{1}{h^2}\frac{\tau^{-(1-\alpha)}}{\Gamma(2-(1-\alpha))}\left[\delta_x^2 u_j^n - \sum_{k=1}^{n-1} c_k^{1-\alpha}\delta_x^2 u_j^{n-k} - b_{n-1}^{1-\alpha}\delta_x^2 u_j^0\right]$$
$$+\frac{\tau^{-(1-\alpha)}}{\Gamma(2-(1-\alpha))}\sum_{k=0}^{n-1} b_k^{1-\alpha}\left[f(x_j, t_{n-k}) - f(x_j, t_{n-1-k})\right] \tag{22}$$

This fractional explicit scheme is conditionally stable and the stability condition is $r_{\alpha} \leq \frac{\Gamma(1+\alpha)}{4}$, theorem 4.2. It is apparent that the condition is depend on α the fractional order of the time derivative as shown in figure 1 (b), this is more convenient and includes the integer case.

Theorem 4.2: *The fractional explicit scheme (22) is conditionally stable and the stability condition is* $r_\alpha \leq \frac{\Gamma(1+\alpha)}{4}$.

Proof. The scheme (22) is a multilevel scheme and can be rewritten

$$u_j^{n+1} = u_j^n + \mu\left[\delta_x^2 u_j^n - \sum_{k=1}^{n-1} c_k^\gamma \delta_x^2 u_j^{n-k} - b_{n-1}^\gamma \delta_x^2 u_j^0\right] + \frac{\tau^\alpha}{\Gamma(2-\gamma)}\left[f_j^n - \sum_{k=1}^{n-1} c_k^\gamma f_j^{n-k} - b_{n-1}^\gamma f_j^0\right] \quad (23)$$

where $\gamma = 1-\alpha$, and $\mu = \frac{r_\alpha}{\Gamma(1+\alpha)}$, then the multilevel amplification matrix C can be defined by square block matrix of order $(n+1)$ and every element of C is of order $J-1$:

$$\mathbf{C} = \begin{bmatrix} (1+\mu\ \delta_x^2)I & (-\mu\ c_1^\gamma\ \delta_x^2)I & (-\mu\ c_2^\gamma\ \delta_x^2)I & \dots & (-\mu\ c_{n-1}^\gamma\ \delta_x^2)I & (b_{n-1}^\gamma\ \delta_x^2)I \\ I & O & O & \dots & O & O \\ O & I & O & \dots & O & O \\ \vdots & \vdots & \vdots & \dots & O & \vdots \\ O & O & O & \dots & I & O \end{bmatrix} \quad (24)$$

for the amplification matrix insert expression (9) in equation (23) and divide by $e^{i\beta lh}$ to get

$$\xi^{n+1} = \xi^n + \mu\left[(-4\sin^2\frac{\beta h}{2})\xi^n + 4\sin^2\frac{\beta h}{2}\sum_{k=1}^{n-1} c_k^\gamma \xi^{n-k} + 4b_{n-1}^\gamma \sin^2\frac{\beta h}{2}\xi^0\right] \quad (25)$$

can be rewritten

$$\xi^{n+1} = (1-4\mu\sigma)\xi^n + 4\mu\sigma\sum_{k=1}^{n-1} c_k^\gamma \xi^{n-k} + 4\mu\sigma b_{n-1}^\gamma \xi^0 \quad (26)$$

then the amplification matrix M can be defined by square block matrix of order $(n+1)$ and every element of M is of order $(J-1)$:

$$M = \begin{bmatrix} (1-4\mu\sigma)I & (4\mu\sigma c_1^\gamma)I & (4\mu\sigma c_2^\gamma)I & \dots & (4\mu\sigma c_{n-1}^\gamma)I & (4\mu\sigma b_{n-1}^\gamma)I \\ & O & O & \dots & O & O \\ O & I & O & \dots & O & O \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ O & O & O & \dots & I & O \end{bmatrix} \quad (27)$$

Employing the same procedure of using Gerschgorin theorem [24], as used in theorem 4.1 we find

$$|\lambda| \leq |1-4\mu\sigma| + \sum_{k=1}^{n-1}\left|4\mu\sigma\ c_k^\gamma\right| + \left|4\mu\sigma b_{n-1}^\gamma\right| \quad (28)$$

by the properties 1 we have $c_k^\gamma > 0$, $b_{n-1}^\gamma > 0$, and $\sum_{k=1}^{n-1} c_k^\gamma = 1 + (n-1)^{1-\gamma} - n^{1-\gamma} = 1 - b_{n-1}$, this lead to

$$|\lambda| \leq |1-4\mu\sigma| + 4\mu\sigma(1-b_{n-1}^\gamma) + 4\mu\sigma b_{n-1}^\gamma \quad (29)$$

if the right hand inequality is less than or equal to one then $|\lambda| \leq 1$, then we have

$$|1-4\ \mu\ \sigma| + 4\ \mu\ \sigma \leq 1 \quad (30)$$

then one can write

$$-(1-4\ \mu\ \sigma) \leq 1-4\ \mu\ \sigma \leq 1-4\ \mu\ \sigma \quad (31)$$

the right hand inequality is satisfied and we need to calculate the condition on r_α to make the left hand inequality satisfied, this lead to

$$8\mu\ \sigma \leq 2 \quad (32)$$

then the stability condition is

$$r_\alpha \leq \frac{\Gamma(1+\alpha)}{4} \quad (33)$$

5. Consistency of Time Fractional Finite Difference Schemes

The Caputo fractional derivative of $O(\tau)$ [8], and from the Taylor's expansion, we have

$$\frac{1}{h^2}\delta_x^2 u_j^n = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{2}h^2\frac{\partial^4 u(x,t)}{\partial x^4} + O(h^4) \tag{34}$$

Therefore, the difference schemes (8) and (22) for TFODE are consistent. The truncation error can be calculated and it is of the form $[O(\tau) + O(h^2)]$.

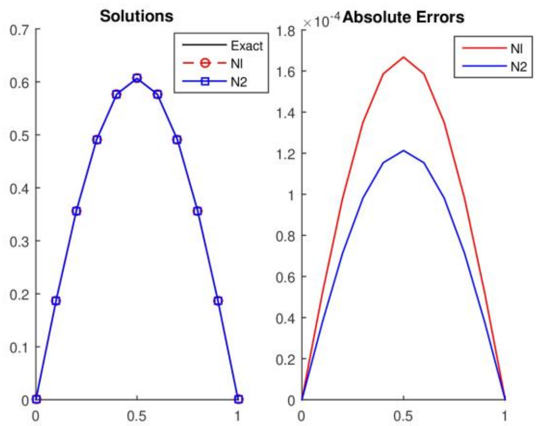

Figure 2: Comparison between the two schemes with the exact solution where $\alpha = 0.9$, $\Delta x = 0.1$, $\Delta t = 0.00125$ and $T = 0.025$. The absolute Errors in the right and the solutions in the left where $N1$ and $N2$ are the numerical solutions by schemes (8) and (22) respectively.

Example 5.1 *To test the two explicit formulas (8) and (22) consider equation (2) with* $f(x,t) = [(\frac{\partial^\alpha}{\partial t^\alpha}e^{-t}) + \pi^2 e^{-t}]sin(\pi x)$, *with initial condition* $u(x,0) = sin(\pi x)$ *and Dirichlet boundary conditions in the form* $u(0,t) = u(1,t) = 0$ *the exact solution is* $u(x,t) = e^{-t}sin(\pi x)$.

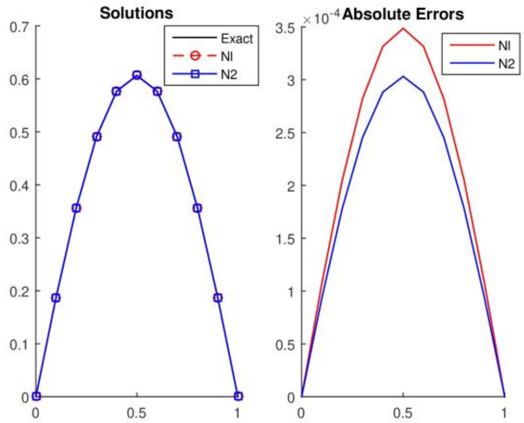

Figure 3: Comparison between the two schemes with the exact solution where $\alpha = 0.8$, $\Delta x = 0.1$, $\Delta t = 0.0005$ and $T = 0.01$. The absolute Errors in the right and the solutions in the left where $N1$ and $N2$ are the numerical solutions by schemes (8) and (22) respectively.

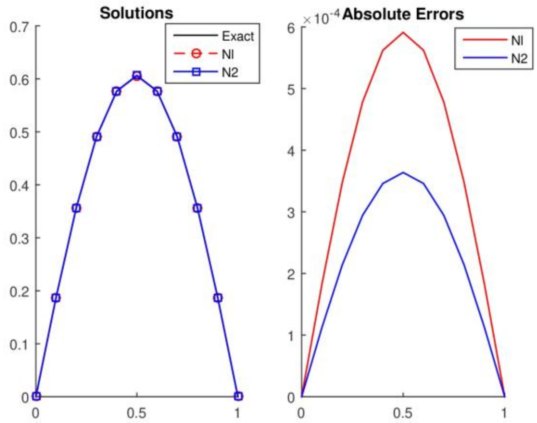

Figure 4: Comparison between the two schemes with the exact solution where $\alpha = 0.7$, $\Delta x = 0.1$, $\Delta t = 0.00025$ and $T = 0.005$. The absolute Errors in the right and the solutions in the left where $N1$ and $N2$ are the numerical solutions by schemes (8) and (22) respectively.

6. Discussion and Conclusion

The implicit schemes are generally unconditional stable and the explicit schemes are conditionally stable and. In explicit schemes one obtains the solutions easily but the conditions on time steps restrict and increase the computational work. In the implicit schemes one has to solve coupled large algebraic systems in each time level. There are many methods to establish stability conditions among them the Von Neumann and the matrix methods are easily used. Consistency is a simple property and its prove is a reversible process to see the original differential equation from its finite difference representation. There are many problems in describing and establishing the properties of the finite difference schemes in the fractional order cases in comparison with the classical integer cases some of them due to the memory and hereditary effects. Simple stability proves through using the techniques of classical multilevel schemes were introduced. The theorems of Gerschgorin's discs are applied to the amplification matrices. We have used the technique of multilevel in proving the condition of stability for two schemes for the time fractional diffusion equation. The method of prove is straightforward and more convenient and contains memory effects implicitly. we examined the conditions on numerical example.

In conclusion the explicit schemes still require small time steps in comparison with implicit schemes. The use of inverse operator has improved the calculated solutions and this is acceptable as illustrated because it extended the memory effects to the spatial terms.

It should be pointed that, the suggested methods can be possibly extended to finite difference schemes for variable order TFODE [25], anomalous order TFODE [26] and fractional advection diffusion equations [27].

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A Modified Generalization of Fractional Calculus Operators in A Complex Domain

Zainab E. Abdulnaby

Department of Mathematics, College of Science, Mustansiriyah University,

Baghdad, Iraq

zainabesa@uomustansiriyah.edu.iq

Abstract: This investigation deals with a new generalization for fractional calculus operators in a complex domain based on the well-known hypergeometric function. Conditions are forced for these generalized operators such as the upper bounds. Other properties for the above operators are also presented. Besides, the employment of these operators is proposed in the
geometric function theory.
Keywords: Fractional Integral operator, Fractional differential operators,Univalent function, Convex function, Hypergeometric function, Bessel function, Wright functions.

1. Introduction

Fractional Calculus is a powerful tool that has been recently applied to complex mathematical with linear operators. Despite its complicated mathematical background, fractional calculus came to open a new window of opportunity to mathematical and real-world, which has appeared many new problems and acceptable results. For instance, the concepts of fractional calculus operators and their generalizations of analytic and univalent functions have been successfully obtained to determine the basic geometric properties such as the coefficient estimates and distortion inequalities for numerous subclasses of analytic functions, adding to that studied some their topological properties in a complex plane (see [1-3]).

In [4] introduced an approach of the fractional integral operator defined for $|z| < 1$ and real numbers $\rho, \mu \in R$, $\mathcal{R}(\omega) > 0$ by

$$\wp_{0,z}^{\omega,\mu,\rho}\psi(z) := \frac{z^{-(\omega+\mu)}}{\Gamma(\omega)} \int_0^z (z-\zeta)^{\omega-1}\psi(\zeta)\ {}_2F_1\left(\omega+\mu, -\rho, \mu; 1-\frac{\zeta}{z}\right) d\zeta, \tag{1}$$

where the function $\psi(z)$ is analytic in a simply-connected region of the $z-$ plane containing the origin, with the order $\psi(z) = O(|z|^{\epsilon})$, $(z \to 0)$, for $\epsilon > \max\{0, \mu-\rho\} - 1$, and the multiplicty of $(z-\zeta)^{\omega-1}$ is removed by requiring $log(z-\zeta)$ to be real when $z-\zeta > 0$. Here, $\Gamma(.)$ is the Gamma function and ${}_2F_1(a,b,c;z)$ is the absolutely convergent Gauss hypergeometric function given for $a,b,c \in C, c > 0$ by the power series [5]:

$${}_2F_1(a,b,c;z) = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} \frac{z^m}{m!}, \qquad |z| < 1$$

where

$$(\gamma)_m = \frac{\Gamma(m+\gamma)}{\Gamma(\gamma)} = \begin{cases} 1, & \text{if } m = 0 \\ \gamma(\gamma+1)\ldots(\gamma+m-1), & \forall\, m \in N \end{cases}$$

is the Pochhammer symbol defined in terms of Gamma function.

Recently, [6] defined a modification of the fractional integral $\Phi_z^{\alpha,\beta}$ and differential $T_z^{\alpha,\beta}$ operators of order two parameters $0 < \alpha \leq 1$ and $0 < \beta \leq 1$ such that $0 \leq \alpha - \beta < 1$, respectively, are presented as follows:

$$\Phi_z^{\alpha,\beta}\psi(z) := \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\, z^{1-\alpha} \int_0^z (z-\zeta)^{\alpha-\beta-1}\zeta^{\beta-1}\psi(\zeta)d\zeta \qquad (2)$$

and

$$T_z^{\alpha,\beta}\psi(z) := \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)}\, z^{1-\beta} \int_0^z (z-\zeta)^{\beta-\alpha}\zeta^{\alpha-1}\psi(\zeta)d\zeta \qquad (3)$$

where the function $\psi(z)$ is analytic in a simply-connected region of the $z-$ plane containing the origin, both of the multiplicity of $(z-\zeta)^{\alpha-\beta-1}$ and $(z-\zeta)^{\beta-\alpha}$ are respectively removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

In this study, we shall restrict our attention to define new fractional calculus operators in the complex plane. The upper bounds for these operators given in terms of the univalent and convex functions. Some geometric applications associated with the Bessel function of the first kind are presented by the generalized Wright functions in the sense of generalization.

2. New classes of generalized fractional calculus operators

In this section, we proposed to define generalized fractional integral and differential operators in the classical definitions, where the order of the fractional integral and fractional differential operators must be positive real numbers. Our definition has been based on important remarks concerning in equations (2) and (3).

Now, we employ equation (1) in (2) to introduce a new generalized fractional integral operator $\mathcal{M}_z^{\alpha,\beta,\mu,\rho}$ as follows:

Definition 1. Let $\mu > 0$ and $\rho > 0$ be real numbers and $0 < \alpha \leq 1$, $0 < \beta \leq 1$ such that

$0 < \alpha - \beta \leq 1$. Then the fractional integral operator $\mathcal{M}_z^{\alpha,\beta,\mu,\rho}$ is defined by

$$\mathcal{M}_z^{\alpha,\beta,\mu,\rho}\psi(z) := \frac{\Gamma(\alpha)z^{1-2\alpha-\mu+\beta}}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^z (z-\zeta)^{\alpha-\beta-1}\zeta^{\beta-1}\psi(\zeta)\,{}_2F_1\left(\alpha-\beta+\mu, -\rho, \alpha-\beta; 1-\frac{\zeta}{z}\right)d\zeta$$
(4)

where the function $\psi(z)$ is analytic in a simply-connected region of the $z-$plane containing the origin with the order $\psi(z) = O(|z|^{\epsilon})$, $(z \to 0)$, for $\epsilon > \max\{0, \mu-\rho\} - 1$ and the multiplicity of $(z-\zeta)^{\alpha-\beta-1}$ is removed as in equations (2) .

Remarke 1. By setting $\mu = \beta - \alpha$ in (4), we have

$$\mathcal{M}_z^{\alpha,\beta,\beta-\alpha,\rho}\psi(z) = \Phi_z^{\alpha,\beta}\psi(z)\,.$$

Next, we applying equation (1) in (3) to define a new generalized fractional differential operator $\aleph_z^{\alpha,\beta,\mu,\rho}$ by the following formula.

Definition 2. Let $\mu > 0$ and $\rho > 0$ be real numbers and $0 < \alpha \leq 1$, $0 < \beta \leq 1$ such that $0 \leq \alpha - \beta < 1$. The generalized fractional differential operator $\aleph_z^{\alpha,\beta,\mu,\rho}$ is defined by:

$$\aleph_z^{\alpha,\beta,\mu,\rho}\psi(z) := \frac{\Gamma(\beta)z^{1-\beta}}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)}\frac{d}{dz}\{z^{\alpha-\beta-\mu}\int_0^z \zeta^{\alpha-1}(z-\zeta)^{\beta-\alpha}\ \psi(\zeta)\ {}_2F_1\left(\beta-\alpha+\mu, 1-\rho, 1-\alpha+\beta; 1-\frac{\zeta}{z}\right)d\zeta \qquad (5)$$

where the function $\psi(z)$ is analytic in a simply-connected region of the $z-$plane containing the origin with order as given by (3).

Remark 2. By setting $\mu = \alpha - \beta$ in (5), then we obtain the following closed results:

$$\aleph_z^{\alpha,\beta,\alpha-\beta,\rho}\psi(z) = T_z^{\alpha,\beta}\psi(z).$$

We shall need the following Definition to present the next outcomes in our investigation.

Definition 3. [5] For the real numbers $c > 0$ and $\sigma > 0$, the hypergeometric function ${}_2F_1$ in the integral terms is shown as follows:

$${}_2F_1(a,b,c;z) := \int_0^1 \gamma(s)\ {}_2F_1(a,b,\sigma;zs)ds$$

where

$$\gamma(s) = \frac{\Gamma(c)}{\Gamma(\sigma)\Gamma(c-\sigma)}s^{\sigma-1}(1-s)^{c-\sigma-1}.$$

Also, we use the familiar Gauss equation

$${}_2F_1(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \mathcal{R}(c-a-b) > 0.$$

The next results are based on two formulas of generalized fractional integral (4) and generalized fractional differential (5) with a power function.

Lemma 1. If $0 < \alpha \leq 1$, $0 < \beta \leq 1$ such that $0 < \alpha - \beta \leq 1$ and $\nu > \mu - \rho - 1$, then

$$\mathcal{M}_z^{\alpha,\beta,\mu,\rho}z^\nu = \frac{\Gamma(\alpha)\Gamma(\nu+\beta)\Gamma(\nu+\beta+\rho-\mu)}{\Gamma(\beta)\Gamma(\nu+\beta-\mu)\Gamma(\nu+\alpha+\rho)}z^{\beta-(\alpha+\mu)+\nu}, \qquad |z| < 1 \qquad (6)$$

in particular,

$$\mathcal{M}_z^{\alpha,\beta,\beta-\alpha,\rho}z^\nu = \Phi_z^{\alpha,\beta}z^\nu.$$

Proof. By using equation (4) and applying Definition 3, we get

$$\mathcal{M}_z^{\alpha,\beta,\mu,\rho}z^\nu = \frac{\Gamma(\alpha)z^{1-2\alpha-\mu+\beta}}{\Gamma(\beta)\Gamma(\alpha-\beta)}\int_0^z (z-\zeta)^{\alpha-\beta-1}\zeta^{\nu+\beta-1}{}_2F_1\left(\alpha-\beta+\mu, -\rho, \alpha-\beta; 1-\frac{\zeta}{z}\right)d\zeta$$

$$= \frac{\Gamma(\alpha) z^{\beta-(\alpha+\mu)+\nu}}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^1 t^{\alpha-\beta-1}(1-t)^{\nu+\beta-1}{}_2F_1(\alpha-\beta+\mu, -\rho, \alpha-\beta; t)dt$$

$$= \frac{\Gamma(\alpha)\Gamma(\nu+\beta)}{\Gamma(\beta)\Gamma(\nu+\alpha)} z^{\beta-(\alpha+\mu)+\nu}\ {}_2F_1(\alpha-\beta+\mu, -\rho, \nu+\alpha; 1)$$

$$= \frac{\Gamma(\alpha)\Gamma(\nu+\beta)\Gamma(\nu+\beta+\rho-\mu)}{\Gamma(\beta)\Gamma(\nu+\beta-\mu)\Gamma(\nu+\alpha+\rho)} z^{\beta-(\alpha+\mu)+\nu}.$$

Similarly to the proof of Lemma 1, it is proved the association of the generalized fractional differential operator (5) with a power function.

Lemma 2. If $0 < \alpha \leq 1$, $0 < \beta \leq 1$ such that $0 \leq \alpha - \beta < 1$ and $\nu > \mu - \rho - 1$, then

$$\aleph_z^{\alpha,\beta,\mu,\rho} z^\nu = \frac{\Gamma(\beta)\Gamma(\nu+\alpha)\Gamma(\nu+\alpha-\mu+\rho)}{\Gamma(\alpha)\Gamma(\nu+\alpha-\mu)\Gamma(\nu+\beta+\rho)} z^{\nu+\alpha-\beta-\mu}, \quad |z| < 1 \qquad (7)$$

in particular,

$$\aleph_z^{\alpha,\beta,\alpha-\beta,\rho} z^\nu = T_z^{\alpha,\beta} z^\nu.$$

Proof. By using equation (5) to the function z^ν, we have

$$\aleph_z^{\alpha,\beta,\mu,\rho} z^\nu = \frac{\Gamma(\beta) z^{1-\beta}}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} \frac{d}{dz}\left\{ z^{\alpha-\beta-\mu} \int_0^z \zeta^{\nu+\alpha-1}(z - \zeta)^{\beta-\alpha}{}_2F_1\left(\beta-\alpha+\mu, , 1-\rho, 1-\alpha+\beta; 1-\frac{\zeta}{z}\right) d\zeta \right\}$$

$$= \frac{\Gamma(\beta) z^{1-\beta}}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} \frac{d}{dz}\left\{ z^{-\mu} \int_0^z \zeta^{\nu+\alpha-1}(1-\frac{\zeta}{z})^{\beta-\alpha}{}_2F_1\left(\beta-\alpha+\mu, , 1-\rho, 1-\alpha+\beta; 1-\frac{\zeta}{z}\right) d\zeta \right\},$$

by employing Definition 1 in the above expression, we get

$$\aleph_z^{\alpha,\beta,\mu,\rho} z^\nu = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} \left\{ z^{1-\beta} \frac{d}{dz} z^{-\mu+\nu+\alpha} \right\} \int_0^1 (1 - t)^{\nu+\alpha-1} t^{\beta-\alpha}{}_2F_1(\beta-\alpha+\mu, , 1-\rho, 1-\alpha+\beta; t)dt,$$

$$= \frac{(\nu+\alpha-\mu)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} z^{\nu+\alpha-\beta-\mu} \int_0^1 (1-t)^{\nu+\alpha-1} t^{\beta-\alpha}{}_2F_1(\beta-\alpha+\mu, , 1-\rho, 1-\alpha+\beta; t)dt.$$

Thus, we obtain

$$\aleph_z^{\alpha,\beta,\mu,\rho} z^\nu = \frac{\Gamma(\beta)\Gamma(\nu+\alpha)}{\Gamma(\alpha)\Gamma(\nu+\beta)} z^{\nu+\alpha-\beta-\mu}{}_2F_1(\beta-\alpha+\mu, , 1-\rho, \nu+\beta+1; 1),$$

$$= \frac{\Gamma(\beta)\Gamma(\nu+\alpha)\Gamma(\nu+\alpha-\mu+\rho)}{\Gamma(\alpha)\Gamma(\nu+\alpha-\mu)\Gamma(\nu+\beta+\rho)} z^{\nu+\alpha-\beta-\mu}.$$

Hence, we arrive at the desired results.

3. Upper Bounds

In this section, we deal with some applications of the new generalizations of fractional operators (4) and (5) in view of the univalent and convex functions in the open unit disk

$$U = \{z : |z| < 1\}.$$

Let A denote the class of all normlaized functions f of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m , \quad m \in N \setminus \{0\} , \tag{8}$$

which are analytic in U of the complex plane C. A function f is called *univalent* and denoted by $f \in S = \{f \in A \mid f \text{ one} - \text{to} - \text{one in } U\}$. A function $f \in A$ that maps U onto a convex domain is called a *convex* function. Let denote K the class of all functions $f \in A$ that are convex. Further, the convolution product for two analytic functions is given by

$$(f * g)(z) = z + \sum_{m=2}^{\infty} a_m \omega_m z^m,$$

where $g(z) = z + \sum_{m=2}^{\infty} \omega_m z^m$ and $z \in U$.

Lemma 3. [7] Let S and K be subclasses of A. If f defined by (8) is in the class S, then $|a_m| \leq m$ for all $m \in N \setminus \{1\}$ and for $z \in U$ the equality holds for the Koebe function defined by

$$f(z) = \frac{z}{(1-z)^2}.$$

Adding to that, if the function f presented by (8) is in the class K, then $|a_m| \leq 1$ and for $z \in U$ the equality holds for

$$f(z) = \frac{z}{(1-z)}.$$

Theorem 1. For $|z| < r,\ r < 1$, let $f \in S$ then

$$\left|\mathcal{M}_z^{\alpha,\beta,\mu,\rho} f(z)\right| \leq r^{1+\beta-(\alpha+\mu)} \sum_{m=0}^{\infty} \frac{(2)_m\, B(m+1+\rho,\alpha) B(m+1,\rho)}{B(m+1+\beta-\mu,\rho) B(m+1,\beta)} \frac{r^m}{m!},$$

the equality holds for the Koebe function.

Proof. Let the function $f(z) \in S$. Then, by utilizing Lemma 1, we have

$$\mathcal{M}_z^{\alpha,\beta,\mu,\rho} f(z) = \sum_{m=1}^{\infty} a_m \frac{\Gamma(\alpha)\Gamma(m+\beta)\Gamma(m+\beta+\rho-\mu)}{\Gamma(\beta)\Gamma(m+\beta-\mu)\Gamma(m+\alpha+\rho)} z^{\beta-(\alpha+\mu)+m}.$$

Thus by using the fact that $|a_m| \leq m$ in Lemma 3, we obtain

$$\left|\mathcal{M}_z^{\alpha,\beta,\mu,\rho} f(z)\right| \leq \sum_{m=1}^{\infty} |a_m| \frac{\Gamma(\alpha)\Gamma(m+\beta)\Gamma(m+\beta+\rho-\mu)}{\Gamma(\beta)\Gamma(m+\beta-\mu)\Gamma(m+\alpha+\rho)} |z|^{\beta-(\alpha+\mu)+m}$$

$$\leq r^{\beta-(\alpha+\mu)} \sum_{m=1}^{\infty} m \frac{\Gamma(\alpha)\Gamma(m+\beta)\Gamma(m+\beta+\rho-\mu)}{\Gamma(\beta)\Gamma(m+\beta-\mu)\Gamma(m+\alpha+\rho)} r^m$$

$$= r^{1+\beta-(\alpha+\mu)} \sum_{m=0}^{\infty} (m+1) \frac{\Gamma(\alpha)\Gamma(m+\beta+1)\Gamma(m+\beta+\rho-\mu+1)}{\Gamma(\beta)\Gamma(m+\beta-\mu+1)\Gamma(m+\alpha+\rho+1)} r^m$$

$$= r^{1+\beta-(\alpha+\mu)} \sum_{m=0}^{\infty} \frac{(2)_m \, B(m+1+\rho,\alpha)B(m+1,\rho)}{B(m+1+\beta-\mu,\rho)B(m+1,\beta)} \frac{r^m}{m!},$$

where $B(t_1, t_2)$ represnts the Beta function in terms of Gamma function given by [5]

$$B(t_1, t_2) = \frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma(t_1+t_2)}.$$

This completes the proof.

Theorem 2. For $|z| < r,\ r < 1$, let $f \in K$ then

$$\left|\mathcal{M}_z^{\alpha,\beta,\mu,\rho} f(z)\right| \leq r^{1+\beta-(\alpha+\mu)} \sum_{m=0}^{\infty} \frac{(1)_m \, B(m+1+\rho,\alpha)B(m+1,\rho)}{B(m+1+\beta-\mu,\rho)B(m+1,\beta)} \frac{r^m}{m!},$$

the equality holds for the Koebe function.

Proof. Let the function $f(z) \in K$. Then, by applying Lemma 1 and Lemma 3, we have

$$\left|\mathcal{M}_z^{\alpha,\beta,\mu,\rho} f(z)\right| \leq \sum_{m=1}^{\infty} |a_m| \frac{\Gamma(\alpha)\Gamma(m+\beta)\Gamma(m+\beta+\rho-\mu)}{\Gamma(\beta)\Gamma(m+\beta-\mu)\Gamma(m+\alpha+\rho)} |z|^{\beta-(\alpha+\mu)+m}, \qquad |a_m| \leq 1$$

$$\leq r^{\beta-(\alpha+\mu)+1} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha)\Gamma(m+\beta+1)\Gamma(m+\beta+\rho-\mu+1)}{\Gamma(\beta)\Gamma(m+\beta-\mu+1)\Gamma(m+\alpha+\rho+1)} r^m$$

$$= r^{1+\beta-(\alpha+\mu)} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha)\Gamma(m+\beta+1)\Gamma(m+\beta+\rho-\mu+1)}{\Gamma(\beta)\Gamma(m+\beta-\mu+1)\Gamma(m+\alpha+\rho+1)} r^m$$

$$= r^{1+\beta-(\alpha+\mu)} \sum_{m=0}^{\infty} \frac{(1)_m \, B(m+1+\rho,\alpha)B(m+1,\rho)}{B(m+1+\beta-\mu,\rho)B(m+1,\beta)} \frac{r^m}{m!}.$$

This completes the proof.

Finally, we introduced some observations concerning the operator $\aleph_z^{\alpha,\beta,\mu,\rho}$ of (5) and by considering a similar manner of Theorem 1 and Theorem 2, respectively, we obtain the upper bounds of the above operator in classes of the univalent and convex functions.

Theorem 3. For $|z| < r,\ r < 1$, let $f \in S$ then

$$\left|\aleph_z^{\alpha,\beta,\mu,\rho} f(z)\right| \leq r^{1+\alpha-(\beta+\mu)} \sum_{m=0}^{\infty} \frac{(2)_m \, B(m+1+\rho,\beta)B(m+1,\rho)}{B(m+1+\alpha-\mu,\rho)B(m+1,\alpha)} \frac{r^m}{m!}$$

the equality holds for the Koebe function.

Proof. Let the function $f(z) \in S$ and $|a_m| \le m$. Then, by applying Lemma 2, we obtain

$$\left|\aleph_z^{\alpha,\beta,\mu,\rho} f(z)\right| \le \sum_{m=1}^{\infty} |a_m| \frac{\Gamma(\beta)\Gamma(m+\alpha)\Gamma(m+\alpha-\mu+\rho)}{\Gamma(\alpha)\Gamma(m+\alpha-\mu)\Gamma(m+\beta+\rho)} |z|^{m+\alpha-(\beta+\mu)}, \quad |a_1| \le 1$$

$$\le r^{\alpha-(\beta+\mu)+1} \sum_{m=0}^{\infty} (m+1) \frac{\Gamma(\beta)\Gamma(m+\alpha+1)\Gamma(m+\alpha-\mu+\rho+1)}{\Gamma(\alpha)\Gamma(m+\alpha-\mu+1)\Gamma(m+\beta+\rho+1)} r^m$$

$$= r^{1+\alpha-(\beta+\mu)} \sum_{m=0}^{\infty} \frac{(2)_m\, B(m+1+\rho,\beta) B(m+1,\rho)}{B(m+\alpha+1-\mu,\rho) B(m+1,\alpha)} \frac{r^m}{m!}.$$

Theorem 4. For $|z| < r,\ r < 1$, let $f \in K$ then

$$\left|\aleph_z^{\alpha,\beta,\mu,\rho} f(z)\right| \le r^{1+\alpha-(\beta+\mu)} \sum_{m=0}^{\infty} \frac{(1)_m\, B(m+1+\rho,\beta) B(m+1,\rho)}{B(m+1+\alpha-\mu,\rho) B(m+1,\alpha)} \frac{r^m}{m!}$$

the equality holds for the Koebe function.

Proof. By supposing $f(z) \in K$, such that $|a_m| \le 1$ and applying Lemma 2. Then, we conclude the proof.

4. Applications in terms of generalized Wright functions

In view of definitions of the fractional integral operator (4) and fractional differential operator (5), we investigate to present some generalized properties associated with the Bessel function of the first kind $\mathcal{J}_\upsilon(z)$ formulated for $z, \upsilon \in C$ such that $z \ne 0$ and $\mathcal{R}(\upsilon) > -1$ by [8]:

$$\mathcal{J}_\upsilon(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{z}{2})^{\upsilon+2m}}{m!\ \Gamma(\upsilon+m+1)}, \quad \upsilon \ne -1, -2, \dots . \qquad (9)$$

We demonstrate that such associated are expanded in terms of the generalized Wright function ${}_q\Psi_p(z)$ which is given by the following formula:

$${}_q\Psi_p(z) = {}_q\Psi_p(z)\left[\begin{matrix}(a_i, \vartheta_i)_{1,p} \\ (b_j, \omega_j)_{1,q}\end{matrix}\middle| z\right] = \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \vartheta_i\, m)}{\prod_{j=1}^{q} \Gamma(b_j + \omega_j\, m)} \frac{z^m}{m!}, \qquad (10)$$

where $z, a_i, b_j \in C$ and ϑ_i, ω_j real numbers in R $(i = 1,2,\dots,p;\ j = 1,2,\dots,q)$, under the condition $\sum_{j=1}^{q} \omega_j - \sum_{i=1}^{p} \vartheta_i > 1$.

In the following, we provide the generalized fractional integral operator (4) associated with the Bessel functions (9).

Theorem 5. Let $\mu,\ \rho$ be positive non-zero numbers, $\nu > -1$ and $0 < \alpha \le 1,\ 0 < \beta \le 1$ such that $0 < \alpha - \beta \le 1$. Then

$$\mathcal{M}_z^{\alpha,\beta,\mu,\rho}\ (\mathcal{J}_\upsilon)(z) = \frac{z^{\beta-(\alpha+\mu)+\nu}\Gamma(\alpha)}{2^\nu \Gamma(\beta)}\,{}_2\Psi_3(z)\left[\begin{matrix}(\nu+\beta,2),(\nu+\beta+\rho-\mu,2)\\(\nu+\beta-\mu,2),(\nu+\alpha+\rho,2),(\upsilon+1,1)\end{matrix}\middle| -\frac{z^2}{4}\right].$$

Proof. Utilizing equation (4) and equation (9), we obtain

$$\mathcal{M}_z^{\alpha,\beta,\mu,\rho}\ (\mathcal{J}_\upsilon)(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}\right)^{\upsilon+2m}}{m!\ \Gamma(\upsilon+m+1)}\left(\mathcal{M}_z^{\alpha,\beta,\mu,\rho} z^{\upsilon+2m}\right).$$

Using Lemma 1, we obtain

$$\mathcal{M}_z^{\alpha,\beta,\mu,\rho}\ (\mathcal{J}_\upsilon)(z) = \frac{z^{\beta-(\alpha+\mu)+\nu}\Gamma(\alpha)}{2^\nu \Gamma(\beta)}\sum_{m=0}^{\infty}\frac{\Gamma(\nu+\beta+2m)\Gamma(\nu+\beta+\rho-\mu+2m)}{\Gamma(\nu+\beta-\mu+2m)+\Gamma(\nu+\alpha+\rho+2m)\Gamma(\upsilon+m+1)}\frac{(-z^2)^m}{4^m m!}.$$

By applying Equation (10), we have

$$\mathcal{M}_z^{\alpha,\beta,\mu,\rho}\ (\mathcal{J}_\upsilon)(z) = \frac{z^{\beta-(\alpha+\mu)+\nu}\Gamma(\alpha)}{2^\nu \Gamma(\beta)}\,{}_2\Psi_3(z)\left[\begin{matrix}(\nu+\beta,2),(\nu+\beta+\rho-\mu,2)\\(\nu+\beta-\mu,2),(\nu+\alpha+\rho,2),(\upsilon+1,1)\end{matrix}\middle| -\frac{z^2}{4}\right].$$

Corollary 1. Let $\mu,\ \rho,\nu \in C$ be such that $\nu > -1$, and $0 < \alpha \leq 1,\ 0 < \beta \leq 1$ with $0 < \alpha - \beta \leq 1$. Then

$$\Phi_z^{\alpha,\beta}\ (\mathcal{J}_\upsilon)(z) = \frac{z^{\nu}\Gamma(\alpha)}{2^\nu \Gamma(\beta)}\,{}_2\Psi_3(z)\left[\begin{matrix}(\nu+\beta,2),(\nu+\alpha+\rho,2)\\(\nu+\alpha,2),(\nu+\alpha+\rho,2),(\upsilon+1,1)\end{matrix}\middle| -\frac{z^2}{4}\right].$$

Corollary 1 achieves from Theorem 5 in respective cases $\mu = \beta - \alpha$.

The following Theorem 6 introduces the generalized fractional differential operator (5) of the Bessel function (9).

Theorem 6. Let $\mu,\ \rho$ be positive non-zero numbers, $\nu > -1$ and $0 < \alpha \leq 1,\ 0 < \beta \leq 1$ be such that $0 < \alpha - \beta \leq 1$. Then

$$\aleph_z^{\alpha,\beta,\mu,\rho}\ (\mathcal{J}_\upsilon)(z) = \frac{z^{\nu+\alpha-(\beta+\mu)}\Gamma(\beta)}{2^\nu \Gamma(\alpha)}\,{}_2\Psi_3(z)\left[\begin{matrix}(\upsilon+\alpha,2),(\upsilon+\alpha-\mu+\rho,2)\\(\upsilon+\alpha-\mu,2),(\upsilon+\beta+\rho,2),(\upsilon+1,1)\end{matrix}\middle| -\frac{z^2}{4}\right].$$

Proof. Applying Equation (5) and Equation (9), we have

$$\aleph_z^{\alpha,\beta,\mu,\rho}\ (\mathcal{J}_\upsilon)(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}\right)^{\upsilon+2m}}{m!\ \Gamma(\upsilon+m+1)}\left(\aleph_z^{\alpha,\beta,\mu,\rho} z^{\upsilon+2m}\right).$$

By using Lemma 2

$$\aleph_z^{\alpha,\beta,\mu,\rho}\ (\mathcal{J}_\upsilon)(z) = \frac{z^{\upsilon+\alpha-\beta-\mu}\Gamma(\beta)}{2^\nu \Gamma(\alpha)}\sum_{m=0}^{\infty}\frac{\Gamma(\upsilon+\alpha+2m)\Gamma(\upsilon+\alpha-\mu+\rho+2m)}{\Gamma(\upsilon+\alpha-\mu+2m)\Gamma(\upsilon+\beta+\rho+2m)\Gamma(\upsilon+m+1)}\frac{(-z^2)^m}{4^m m!}.$$

By Equation (10)

$$\aleph_z^{\alpha,\beta,\mu,\rho}\,(\mathcal{J}_v)(z)=\frac{z^{v+\alpha-(\beta+\mu)}\Gamma(\beta)}{2^v\Gamma(\alpha)}\,{}_2\Psi_3(z)\left[\begin{matrix}(v+\alpha,2),(v+\alpha-\mu+\rho,2)\\(v+\alpha-\mu,2),(v+\beta+\rho,2),(v+1,1)\end{matrix}\;\middle|\;-\frac{z^2}{4}\right].$$

Corollary 2. Let $\mu,\ \rho,\ \nu\in C$ be such that $\nu>-1$, and $0<\alpha\le 1,\ 0<\beta\le 1$ with $0<\alpha-\beta\le 1$. Then

$$T_z^{\alpha,\beta}\,(\mathcal{J}_v)(z)=\frac{z^v\Gamma(\beta)}{2^v\Gamma(\alpha)}\,{}_2\Psi_3(z)\left[\begin{matrix}(v+\alpha,2),(v+\beta+\rho,2)\\(v+\beta,2),(v+\beta+\rho,2),(v+1,1)\end{matrix}\;\middle|\;-\frac{z^2}{4}\right].$$

Corollary 2 achieves from Theorem 6 in particular cases $\mu=\alpha-\beta$.

5. Conclusion

Conditions for the new fractional calculus operators are obtained. Also, some characteristics for these operators are delivered. Some geometric applications are studied in the sense of generalization.

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A New Iterative Methods For a Family of Asymptotically Severe Mappings

Z H Maibed

Department of Mathematics, College of Education for Pure Science Ibn Al-Haithem University of Baghdad

mrs_ zena.hussein@yahoo.com

Abstract. The aim of this paper is to introduce the concepts of asymptotically p-contractive and asymptotically severe accretive mappings. Also, we give an iterative methods(two step-three step) for finite family of asymptotically p-contractive and asymptotically severe accretive mappings to solve types of equations .

h) Introduction

W Consider the real $\mathcal{B}$anach space $\mathcal{B}$ and dual space $\mathcal{B}^*$. The mapping $\mathfrak{J}: \mathcal{B} \rightarrow 2^{\mathcal{B}^*}$ such that $\mathfrak{J}(a) = \{\mathcal{F} \in \mathcal{B}^* : < a_1, \mathcal{F} > = \| a \| . \| \mathcal{F} \| ; \| a \| = \| \mathcal{F} \| \}$ for all $a \in \mathcal{B}$ is called normalized duality mapping .When $\mathcal{B}$ is a uniformly smooth $\mathcal{B}$anach space, we get $\mathfrak{J}$ is singlevalued and uniformly continuous on every bounded subset of $\mathcal{B}$. Lin, Kang andiShim [1], are introduced the following algorithm:

1.1Definition:

Let $\mathbb{C}$ be a convex nonempty subset of $\mathcal{B}, \mathcal{G}: \mathbb{C} \rightarrow \mathbb{C}$ be a map and $p_0 \in \mathbb{C}$ .Define the algorithm iteration $< p_n >$ as

$$p_{n+1} = (1 - a_n)p_n + a_n \mathcal{G}\, q_n$$

$$q_n = (1 - d_n)p_n + d_n\, \mathcal{G} p_n \quad , n \geq 0$$

This algorithm iteration called Ishikawa, when $< a_n >, < d_n >$ any sequences in [0,1]. If $d_n = 0$ for all $\forall\, n \geq 0$, then the algorithm iteration $< x_n >$ is called Mann iteration. Now, let $\mathcal{G}_1 , \mathcal{G}_2$ are two mappings, the algorithm iteration

$$p_{n+1} = a_n p_n + d_n \mathcal{G} q_n + c_n r_n$$

$$q_n = \acute{a}_n\, p_n + \acute{d}_n \mathcal{G} q_n + \acute{c}_n s_n \quad , n \geq 0$$

This algorithm iteration called Ishikawa with error. If $\acute{d}_n = \acute{c}_n = 0$ for all $n \geq 0$, then the algorithm iteration is called Mann with error. The convergence of the iterative algorithms are studied by many researchers,see([1]-[14])

1.2 Lemma: [2]

If $\mathcal{B}$ real $\mathcal{B}$anach space and $\mathfrak{J}: \mathcal{B} \rightarrow 2^{\mathcal{B}^*}$ be a normalized duality mapping. Then, for any $\mathfrak{r}, \mathfrak{s} \in \mathcal{B}$

$$\| \mathfrak{r} - \mathfrak{s} \|^2 \leq \| \mathfrak{r} \|^2 + 2 < \mathfrak{s}, \mathfrak{J}(\mathfrak{r} + \mathfrak{s}) >, \forall\, \mathfrak{J}(\mathfrak{r} + \mathfrak{s}) \in J(\mathfrak{r} + \mathfrak{s})$$

1.3 Lemma: [3]

The nonnegative sequence $< a_n >$ satisfied the following inequality

$$a_{n+1} \leq (1 - d_n)a_n + c_n$$

where $c_n \in (0,1)\, , \forall n \in N, \sum_{n=1}^{\infty} c_n = \infty$ and $d_n = 0(c_n)$.Then $\lim_{n \to \infty} a_n = 0$.

In this article, we analyze the convergence of a new algorithm for asymptotically p-contractive and asymptotically severe mappings.

i) Main Results

Now, we introduce asymptotically p-contractive and asymptotically severe mappings as follows:

2.1 Definition: Any map $\mathcal{G}$ with domain $\mathcal{D}(\mathcal{G})$ and rang $\mathcal{R}(\mathcal{G})$ in $\mathcal{B}$ is called

i) asymptotically p-contractive if $\forall p \in \mathcal{D}(\mathcal{G}), \exists$ appositive sequence $\langle \mathcal{L}_n \rangle \in (1, \infty), n \in N$ such that for all $q \in \mathcal{D}(\mathcal{G})$ and $\ell > 0$

$$\| p - q \| \leq \| (1+\ell)(p-q) - \ell\, \mathcal{L}_n \left(\mathcal{G}^n{}_{p} - \mathcal{G}^n{}_{p}\right) \|$$

ii) asymptotically severe accretive if $\forall p \in \mathcal{D}(\mathcal{G})$, $\exists$ positive sequence $\langle k_n \rangle \in (0,1)$ such that for each $q \in \mathcal{D}(\mathcal{G})$, there is

$j(p-q) \in J(p-q)$ satisfying $< \mathcal{G}^n{}_{p} - \mathcal{G}^n{}_{q}, j(p-q) > \geq k_n \| p - q \|^2$.

2.2 Remark: 1.The mapping $\mathcal{G}$ is asymptotically p-contractive mapping if and only if $(I - \mathcal{G}^n)$ is asymptotically severe accretive mapping and $k_n = 1 - \frac{1}{\mathcal{L}_n}$.

2. 3. If $\mathcal{G}$ is asymptotically severe accretive mapping then$(\mathcal{T}^n - k_n I)$ accretive mapping.

It is our aims in this paper to study the convergence of the modified 3-step algorithm with error 3_ asymptotically p-contractive and asymptotically severe accretive mappings .

2.3 Theorem: Let $\mathcal{G}_1, \mathcal{G}_2 : \mathbb{C} \to \mathbb{C}$ are asymptotically severe accretive mappings assume that the equations $\mathcal{G}^n{}_i x = \mathcal{F}$ $(i = 1,2)$, has a solution for some $\mathcal{F} \in \mathbb{C}$. Define the bounded mapping $\mathcal{K}_i : \mathbb{C} \to \mathbb{C}$ such that $\mathcal{K}^n{}_i x = \mathcal{F} + x - \mathcal{G}^n{}_i x$. Consider $x_o \in \mathbb{C}$, the algorithm iteration $< x_n >$ is defined by:

$$x_{n+1} = a_n x_n + d_n \mathcal{K}^n{}_1 y_n + c_n u_n \qquad (1)$$

$$y_n = \acute{a}_n x_n + \acute{d}_n \mathcal{K}^n{}_2 x_n + \acute{c}_n v_n \qquad (2)$$

where $< u_n >$ and $< v_n >$ are two bounded sequences in $\mathcal{B}$ and $< a_n >, < d_n >, < \acute{d}_n >, < c_n >$ and$< \acute{c}_n >$ are real sequences in [0,1] such that

$a_n + d_n + c_n = \acute{a}_n + \acute{d}_n + \acute{c}_n = 1$ satisfying the conditions:

i) $\lim_{n\to\infty} d_n = \lim_{n\to\infty} \acute{d}_n = 0$

ii) $\sum_{n=0}^{\infty} d_n = \infty$,

iii) $c_n \leq d_n$, $\acute{c}_n \leq \acute{d}_n$.

Then the algorithm $< x_n >$ converges strongly to a fixed point of $\mathcal{G}^n{}_i x = \mathcal{F}$.

Proof: Suppose that $\mathcal{G}^n{}_i w = \mathcal{F} \implies w \in F(\mathcal{K}^n{}_i)$, $w \in \mathbb{C}$

Since $\mathcal{G}_i$ is asymptotically severe accretive mapping $\Longrightarrow$

$$< \mathcal{K}^n{}_i x - \mathcal{K}^n{}_i y , j(x-y) > \leq \left(\| x - y \|^2_{\|x-y\|^2}\right) \qquad (3)$$

$y = w$, we have

$$< \mathcal{K}^n{}_i x - \mathcal{K}^n{}_i w , j(x-y) > \leq \| x\_y \|^2 \_ \| x\_w \|^2 \qquad (4)$$

If $\mathfrak{h} \in F(\mathcal{K}_i)$, we have (4) with $x = \mathfrak{h} \implies w = \mathfrak{h}$, we prove that $< x_n >$ *and* $< y_n >$ are bounded sequences.

Let sup $\{\| \mathcal{K}^n{}_i x - \mathcal{K}^n{}_i w \| + \| H^n{}_i y - w \| : n \geq 0\} + \| x_0 - w \| = \mathcal{N}_i$

sup $\{\| u_n \| + \| v_n \| : n \geq 0\} = \mathcal{N}$, $M_i = \mathcal{N}_i + \mathcal{N}$ for all $i = 1,2$ and $M = \sup\{M_1 , M_2\}$. By (1) and (iii) we get

$$\| x_{n+1} - w \| \leq a_n \| x_n - w \| + d_n \| \mathcal{K}^n{}_1 y_n - w \| + c_n \| u_n \|$$

$$\leq a_n \| x_n - w \| + d_n \mathcal{K}_1 + d_n \mathcal{N}$$

$$\leq a_n \| x_n - w \| + d_n M_1$$

$$\leq a_n \| x_n - w \| + d_n M$$

Now, from (2) and(iii) , we get

$$\| y_n - w \| \leq \acute{a}_n \| x_n - w \| + \acute{d}_n \| \mathcal{K}^n{}_2 x_n - w \| + \acute{c}_n \| v_n \|$$

$$\leq \acute{a}_n \| x_n - w \| + \acute{d}_n \mathcal{N}_2 + \acute{d}_n \mathcal{N}$$

$$\| y_n - w \| \leq \acute{a}_n \| x_n - w \| + \acute{d}_n M_2$$

$$\leq \acute{a}_n \| x_n - w \| + \acute{d}_n M \qquad (6)$$

Now , we show by induction that $\| x_n - w \| \leq M$ (7)

For $n = 0$, we have $\| x_o - w \| \leq \mathcal{N}_i \leq M_i \leq M$

Suppose that $\| x_n - w \| \leq M$, then by(5) we get

$$\| x_{n+1} - w \| \leq a_n \| x_n - w \| + d_n M$$

$$\leq (a_n + d_n) M = (1 - c_n) M \leq M$$

Therefore, the inequality (7) holds

Substituting (7) into (6), we get $\| y_n - w \| \leq M$ (8)

From (2.6) , we have

$$\| y_n - w \|^2 \leq \acute{a}^2{}_n \| x_n - w \|^2 + 2 \acute{a}_n \acute{d}_n M \| x_n - w \| + \acute{d}^2{}_n M^2$$

Since $\acute{a}_n \leq 1$ and $\| x_n - w \| \leq M$, we get

$$\| y_n - w \|^2 \leq \| x_n - w \|^2 + 2 \acute{d}_n M^2 + \acute{d}_n M^2$$

$$= \| x_n - w \|^2 + 3 \acute{d}_n M^2 \qquad (9)$$

Using Lemma(2) , we get

$$\| x_{n+1} - w \|^2 \leq \| a_n (x_n - w) + c_n u_n + d_n (\mathcal{K}^n{}_1 y_n - w) \|^2$$

$$\leq \| a_n (x_n - w) + c_n u_n \|^2 + 2 d_n < \mathcal{K}^n{}_1 y_n - w, j(x_{n+1} - w) >$$

$$\leq a_n^2 \| x_n - w \|^2 + 2 a_n c_n \| u_n \| \| x_n - w \| + c^2{}_n \| u_n \|^2 + 2 d_n < \mathcal{K}^n{}_1 y_n - w, j(y_n - w) > + 2 d_n < \mathcal{K}^n{}_1 y_n - w, j(x_{n+1} - w) - j(y_n - w) >$$

$\| x_{n+1} - w \|^2 \le \| x_n - w \|^2 - 2d_n \| x_n - w \|^2 + d_n{}^2 \| x_n - w \|^2 + 2a_n c_n M^2 + c_n^2 M^2 + 2d_n \| y_n - w \|^2 - 2d_n k_n \| y_n - w \|^2 + 2d_n e_n$

where $e_n =< \mathcal{K}^n{}_1 y_n - w, j(x_{n+1} - w) - j(y_n - w) >$ (10)

By (7) and (9) , $c_n \le d_n$ and $-2a_n c_n + c_n{}^2 \le 0$, we obtain

$\| x_{n+1} - w \|^2 \le \| x_n - w \|^2 - 2d_n M^2 + d_n{}^2 M^2 + 2d_n c_n M^2 - c_n{}^2 M^2 + 2c_n{}^2 M + 2d_n\big(M^2 + 2\acute{d}_n M^2\big) - 2d_n k_n \| y_n - w \|^2 + 2d_n e_n$

$\le \| x_n - w \|^2 - d_n M^2 + d_n{}^2 M^2 - (-2a_n c_n - c_n{}^2)M^2 + 2c_n - d_n M^2 + 2d_n M^2 + 4d_n . \acute{d}_n M^2 - 2d_n k_n \| y_n - w \|^2 + 2d_n e_n$

$= \| x_n - w \|^2 - 2d_n k_n \| y_n - w \|^2 + d_n \lambda_n$ (11)

where, $\lambda_n = \big(d_n + 2c_n + 4\acute{d}_n\big)M^2 + 2e_n$

First, we show that $c_n \to 0$ as $n \to \infty$.From (1) and (2) we get

$\| x_{n+1} - y_n \| \le \| (a_n - \acute{a}_n)(x_n - w) + d_n(\mathcal{K}^n{}_1 y_n - w) - \acute{d}_n(\mathcal{K}^n{}_2 x_n - w) + c_n u_n - \acute{c}_n \acute{v}_n$

$\le (a_n - \acute{a}_n) \| x_n - w \| + d_n \| \mathcal{K}^n{}_1 y_n - w \| + \acute{d}_n \| H^n{}_2 x_n - w \|$
$+ c_n \| u_n \| + \acute{c}_n \| v_n \|$

$\le \big(1 - d_n - c_n - 1 + \acute{d}_n + \acute{c}_n\big) \| x_n - w \| + d_n \| \mathcal{K}^n{}_1 y_n - w \| + \acute{d}_n$
$\| \mathcal{K}^n{}_2 x_n - w \| + d_n \| u_n \| + \acute{d}_n \| v_n \|$

By (7) and definition of M. we get

$$\| x_{n+1} - y_n \| \le 2\big(d_n + \acute{d}_n\big)M + \big(d_n + \acute{d}_n\big)M + \big(d_n + \acute{d}_n\big)M$$

i.e., $\| x_{n+1} - y_n \| \le 4\big(d_n - \acute{d}_n\big)M$ (12)

Therefore, $\| x_{n+1} - w - (y_n - w) \| \to 0$ as $n \to \infty$.

Since $< x_{n+1} - y_n >, < y_n - w >$ and $< \mathcal{K}^n{}_1 y_n - w >$ are bounded and j is uniformly continuous on any bounded subset of X we have

$j(x_{n+1} - w) - j(y_n - w) \to 0$ and $< e_n > \to 0$ as $n \to \infty$.

Thus , $\lim_{n\to\infty} \lambda_n = 0$

Let $\vartheta = \inf\{\| y_n - w \|^2 \colon n \ge 0$

To prove that $\vartheta = 0$.Assume the contrary, i.e., $\vartheta > 0$,

Then $\| x_n - w \|^2 \ge \vartheta > 0$ foriall ≥ 0 , hence

$k_n(\| y_n - w \|^2) \ge k_n(\vartheta) > 0$ where $k_n \in (0,1)$

Thus from(11), $\| x_{n+1} - w \|^2 \le \| x_n - w \|^2 - d_n k_n(\vartheta) - b_n[k_x(\vartheta) - \lambda_n] \ldots..(13)$

Since $\lim_{n\to\infty} \lambda_n = 0$, there exists a positive integer no such that $\lambda_n \leq k_n(\vartheta)$ for all $n \geq n_o \Longrightarrow$ From(13) , we have

$\| x_{n+1} - w \|^2 \leq \| x_n - w \|^2 - d_n k_n(\vartheta)$ or

$d_n k_n(\vartheta) \leq \| x_n - w \|^2 - \| x_{n+1} - w \|^2$ for all $n \geq 0$

Hence, $k_n(\vartheta) \sum_{j=n_o}^{n} dj = \| x_{n_0} - w \|^2 + \| x_{n+1} - w \|^2 \leq \| x_{n_0} - w \|^2$.

$\Longrightarrow \sum_{n=0}^{\infty} d_n < \infty$, contradicting(ii)$\Longrightarrow \vartheta = 0$ and there exists$<\| y_i - w \|>$ subsequence of $<\| y_n - w \|>$ s.t $\lim_{n\to\infty} \| y_i - w \| = 0$ (14)

From(2),we have

$$\| x_n - w \|^2 \leq \| y_n - w + \left(\acute{d}_n + \acute{c}_n \right)(x_n - w) - \acute{d}_n(\mathcal{K}^n{}_2 y_n - w) - \acute{c}_n v_n \|$$

$$\leq \| y_n - w \| + \left(\acute{d}_n + \acute{c}_n \right) \| (x_n - w) \| - \acute{d}_n \| (\mathcal{K}^n{}_2 x_n - w) \|$$

$$+ \acute{c}_n \| v_n \|.$$

Since $\acute{c}_n \leq \acute{d}_n$ and by definition $\mathcal{N}_i$, $\mathcal{N}$ and , we get

$\| x_n - w \|^2 \leq \| y_n - w \| + 2\acute{d}_n M$ for all $n \geq 0$ (15)

Thus, $\lim_{j\to\infty} \| x_j - w \| = 0$ (16)

Now, let $\in > 0$ be arbitrary, since $\lim_{n\to\infty} d_n = \lim_{n\to\infty} \acute{d}_n = 0$ and $\lim_{n\to\infty} \lambda_n = 0$, $\exists N_0 \in N$ such that

$d_n \leq \frac{\in}{12M}$, $\acute{d}_n \leq \frac{\in}{12M}$, $\lambda_n \leq k_n\left(\frac{\in}{3}\right)$ for all $n \geq N_0$

From(16), $\exists k \geq N_0$ such that

$\| x_k - w \| < \in$ (17)

Now,we prove that $\| x_{k+n} - w \| < \in$ (18)

Assume that (18) holds and for $n = 0 \Longrightarrow$ the inequality(18) holds by (17).Now, if

$\| x_{k+n+1} - w \| \geq \in$. then by(12) , we get

$$\in \leq \| x_{k+n+1} - w \| = \| y_{k+n} - w + x_{k+n+1} - y_{k+n} \|$$

$$\leq \| y_{k+n} - w \| + \| x_{k+n+1} - y_{k+n} \|$$

$$\leq \| y_{k+n} - w \| + 3\left(d_{k+n} + \acute{d}_{k+n}\right)M$$

$$\leq \| y_{k+n} - w \| + \frac{\in}{2}$$

Hence, $\| y_{k+n} - w \| \geq \frac{\in}{2}$

From(11), we get

$\in^2 \leq \| x_{k+n+1} - w \|^2 \leq \| x_{k+n} - w \|^2 - 2d_{k+n}\, k_n \left(\frac{\in}{3}\right) + d_{k+n}\, k_n$

$\leq \| x_{k+n} - w \|^2 < \in^2$, which is a contradiction

Thus we proved (18) , hence, $\lim_{n\to\infty} \| x_n - w \| = 0$.

2.4 Theorem: Let $\mathbb{C}$ be a nonempty bounded closed subset of $\mathcal{B}$ and $\mathcal{G}_1, \mathcal{G}_2 : \mathbb{C} \to \mathbb{C}$ are asymptotically pseudo-contractive mappings. Let w be a fixed point of $\mathcal{G}_1, \mathcal{G}_2$, and let $x_0 \in \mathbb{C}$ and $< x_n >$ defined as

$x_{n+1} = a_n x_n + d_n \mathcal{G}^n{}_1 y_n + c_n u_n$

$y_n = \acute{a}_n x_n + \acute{d}_n \mathcal{G}^n{}_2 y_n + \acute{c}_n v_n \quad , n \geq 0$

where $< u_n >$ and $< v_n > \subset \mathbb{C}$, $< b_n >, < \acute{b}_n >, < c_n >, < \acute{c}_n >$ are sequences as in theorem (2.3). then$< x_n >$ converges strongly to the unique fixed point of $\mathcal{G}^n$.

Proof: The sequences $< x_n >$ and $< y_n >$ are both contained in $\mathbb{C}$ and therefore, bounded sequences. Since $\mathcal{G}_i$ are asymptotically p-contractive if and only if $(I - \mathcal{G}^n{}_i)$ are asymptotically severe accretive and $k_n = 1 - \frac{1}{\mathcal{L}_n}$, for all $(i = 1,2)$ put $y = w$ and $(\mathcal{G}^n = \mathcal{K}_i)$, we get the result

2.5 Theorem: Let $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$, are asymptotically p-contractive self mappings on $\mathbb{C}$ and $\bigcap_{i=1}^{3} F\left(\mathcal{G}^n{}_i\right) \neq \emptyset$. Define the algorithm iteration $< x_n >$ as,

For $x_1 \in \mathbb{C}$

$x_{n+1} = a_n x_n + d_n \mathcal{G}^n{}_1 y_n + c_n u_n$

$y_n = \acute{a}_n x_n + \acute{d}_n \mathcal{G}^n{}_2 \mathcal{Z}_n + \acute{c}_n v_n$

$\mathcal{Z}_n = \acute{\acute{a}}_n x_n + \acute{\acute{d}}_n \mathcal{G}^n{}_3 y_n + \acute{\acute{c}}_n w_n$

where $< u_n >$, $< v_n >$ and $< w_n >$ are bounded sequences in $\mathbb{C}$and $< a_n >$, $< \acute{a}_n >, < \acute{\acute{a}}_n >$, $< d_n >$, $< \acute{d}_n >$, $< \acute{\acute{d}}_n >$, $< c_n >$, $< \acute{c}_n >$ and $< \acute{\acute{c}}_n >$ are real sequence

in [0.1] such that

$a_n + d_n + c_n = \acute{a}_n + \acute{d}_n + \acute{c}_n = \acute{\acute{a}}_n + \acute{\acute{d}}_n + \acute{\acute{c}}_n = 1$ and satisfying the following:

i) $\sum d_n = \infty$

ii) $\lim_{n\to\infty} d_n = \lim_{n\to\infty} \acute{d}_n = \lim_{n\to\infty} \acute{\acute{d}}_n = 0$

iii) $\alpha_n = d_n + c_n$, $\beta_n = \acute{d}_n + \acute{c}_n$, $\gamma_n = \acute{\acute{d}}_n + \acute{c}_n$and $\lim_{n\to\infty} \frac{1}{1 + k_n \alpha_n} = 0$,for any sequence $k_n \in (0,1)$. Then the scheme $< x_n >$ converges strongly to the unique fixed point of $\mathcal{G}^n{}_i$, for all $n \in N$.

Proof: Since for all $n \in N$ then $\bigcap_{i=1}^{3} F\left(\mathcal{G}^n{}_i\right) \neq \emptyset$, it follows from(1.8) that $\bigcap_{i=1}^{3} F\left(\mathcal{G}^n{}_i\right)$ is singleton say p .The mappings $\mathcal{T}_i$ is asymptotically p -contractive if and only if $(I - \mathcal{G}^n{}_i)$ is asymptotically severe accretive and $k_n = 1 - \frac{1}{\mathcal{L}_n}$, and therefore $(I - \mathcal{G}^n{}_i) - k_n I = I - \mathcal{G}^n{}_i - k_n I$ $(i = 1,2,3)$ is accretive. Hence , for all $r > 0$ and $k_n \in (0,1)$ we have, $\| x - y \| \leq \| x - y + r\left[\left(I - \mathcal{G}^n{}_i - k_n I\right)x - \left(I - \mathcal{G}^n{}_i - k_n I\right)y\right]$

From our hypothesis,

$$x_n = x_{n+1} + \alpha_n x_n - d_n \mathcal{G}^n{}_1 y_n - c_n u_n$$

$$x_n = x_{n+1} + \alpha_n x_{n+1} - \alpha_n x_{n+1} + \alpha_n x_n - d_n \mathcal{G}^n{}_1 y_n - c_n u_n$$

$$=(1 + \alpha_n)x_{n+1} + \alpha_n\left(I - \mathcal{G}^n{}_i - k_n I\right)x_{n+1} - \alpha_n\left(I - \mathcal{G}^n{}_i - k_n I\right)x_{n+1}$$

$$-\alpha_n(x_{n+1} - x_n) - d_n \mathcal{G}^n{}_1 y_n - c_n u_n \qquad (19)$$

$$=(1 + \alpha_n)x_{n+1} + \alpha_n\left(I - \mathcal{G}^n{}_i - k_n I\right)x_{n+1} - \alpha_n(I - k_n I)x_{n+1}$$

$$+\alpha_n \mathcal{G}_i + \alpha_n(x_n - x_{n+1}) - d_n \mathcal{G}^n{}_1 y_n - c_n u_n$$

Since p is a fixed point of $\mathcal{G}_i$, then

$$p = (1 + \alpha_n)p + \alpha_n\left(I - \mathcal{G}^n{}_i - k_n I\right)p - \alpha_n(I - k_n)p \qquad (20)$$

Subtracting (20) from(19) we obtain

$$x_n - p = (1 + \alpha_n)\left[(x_{n+1} - p) + \frac{\alpha_n}{1+\alpha_n}\left\{\left(I - \mathcal{G}^n{}_i - k_n\right)x_{n+1} - \left(I - \mathcal{G}^n{}_i - k_n I\right)p\right\}\right] - \alpha_n(I - k_n)(x_{n+1} - p) + [\,\alpha_n\left(\mathcal{G}^n{}_i - I\right)x_{n+1}\text{-}\, d_n \mathcal{G}^n{}_1 y_n] + [\alpha_n x_n - c_n u_n]$$

$$\| x_n - p \| = \| (1 + \alpha_n)\left[(x_{n+1} - p) + \frac{\alpha_n}{1+\alpha_n}\left\{\left(I - \mathcal{G}^n{}_i - k_n\right)x_{n+1} - \left(I - \mathcal{G}^n{}_i - k_n I\right)p\right\}\right] - \alpha_n(I - k_n)(x_{n+1} - p) + [\,\alpha_n\left(\mathcal{G}^n{}_i - I\right)x_{n+1}\text{-}b_n \mathcal{G}^n{}_1 y_n] + [\alpha_n x_n - c_n u_n\,] \|$$

$$\| x_n - p \| = (1 + \alpha_n)\left[(x_{n+1} - p) + \frac{\alpha_n}{1 + \alpha_n}\left\{\left(I - \mathcal{G}^n{}_i - k_n I\right)x_{n+1} - \left(I - \mathcal{G}^n{}_i - k_n I\right)p\right\}\right]$$
$$\| -\alpha_n(1 - k_n) - \| (x_{n+1} - p) \| - \| \alpha_n\left(\mathcal{G}^n{}_i - I\right)x_{n+1} - c_n u_n \| -$$
$$\| \alpha_n x_n - b_n \mathcal{G}^n{}_1 y_n \| \qquad (21)$$

Since $\mathcal{G}_i$ asymptotically p -contractive, then (21) yields

$$\| x_n - p \| \geq (1 + \alpha_n) \| x_{n+1} - p \| - \alpha_n(I - k_n) \| \alpha_{n+1} - p \| - \| \alpha_n\left(\mathcal{G}^n{}_i - I\right)x_{n+1}\text{-}c_n u_n \| - \| \alpha_n x_n - d_n \mathcal{G}^n{}_1 y_n \|.$$

$$= (1 - k_n\alpha_n) \| x_{n+1} - p \| - \| \alpha_n\left(\mathcal{G}^n{}_i - I\right)x_{n+1} - c_n u_n \| - \| \alpha_n x_n - d_n \mathcal{G}_1 y_n \|.$$

$$\| x_n - p \| \leq \frac{1}{1+k_n\alpha_n}[\| x_n - p \| + \| \alpha_n\left(\mathcal{G}^n{}_i - I\right)x_{n+1}\text{-}c_n u_n \| + \| \alpha_n x_n - d_n \mathcal{G}^n{}_1 y_n \|]$$

Now,

$$\| (x_{n+1} - p) \| \leq \frac{1}{1+\alpha_n k_n} \| x_n - p \| + \frac{1}{1+\alpha_n k_n} M \qquad (22)$$

Now, put $\delta_n = \frac{1}{1+\alpha_n k_n}$, $\sigma_n = \delta_n M$ and $p_n = \| x_n - p \|$

Thus (22) reduces to $p_{n+1} \leq \delta_n p_n + \sigma_n$

Since $0 \leq \delta_n \leq 1$, $\lim_{n\to\infty} \delta_n = 0$ and $\lim_{n\to\infty} \sigma_n = 0$. Therefore by Lemma (1.3) , we have $\lim_{n\to\infty} p_n = 0 \Longrightarrow < x_n >$ converge strongly to p.

2.6 Theorem: Let Let $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$, are uniformly continuous asymptotically severe self mappings of $\mathbb{C}$ and $\bigcap_{i=1}^{3} F\left(\mathcal{G}^n{}_i\right) \neq \emptyset$. Define a mapping $\mathcal{R}_i: \mathcal{C} \to \mathcal{C}$ by $\mathcal{R}^n{}_i x = x - \mathcal{G}^n{}_i x + \mathcal{F}$, for some $\mathcal{F} \in \mathcal{B}$, consider the following algorithm iteration :

For arbitrary $x_1 \epsilon\, \mathbb{C}$,

$$x_{n+1} = a_n x_n + d_n \mathcal{R}^n{}_1 y_n + c_n u_n$$

$$y_n = \acute{a}_n x_n + \acute{d}_n \mathcal{R}^n{}_2 z_n + \acute{c}_n v_n$$

$$z_n = \acute{\acute{a}}_n x_n + \acute{\acute{d}}_n \mathcal{R}^n{}_3 x_n + \acute{\acute{c}}_n w_n$$

Where $< u_n \geq < v_n >$ and $< w_n >$ are bounded sequences in $\mathbb{C}$ and $< a_n >, < \acute{a}_n >, < \acute{\acute{a}}_n >, < d_n >, < \acute{d}_n >, < \acute{\acute{d}}_n > < c_n >, < \acute{c}_n >$ and $< \acute{\acute{c}}_n >$ are real sequences in [0,1] such that $a_n + d_n + c_n = \acute{a}_n + \acute{d}_n + \acute{c}_n = \acute{\acute{a}}_n + \acute{\acute{d}}_n + \acute{\acute{c}}_n = 1$ and satisfying the conditions:

i) $\sum d_n = \infty$

ii) $\lim_{n\to\infty} d_n = \lim_{n\to\infty} \acute{d}_n = \lim_{n\to\infty} \acute{\acute{d}}_n = 0$

iii) $\alpha_n = d_n + c_n$, $\beta_n = \acute{d}_n + \acute{c}_n$, $\gamma_n = \acute{\acute{d}}_n + \acute{\acute{c}}_n$ and $\lim_{n\to\infty} \frac{1}{1+k_n\alpha_n} = 0$.For all $k_n \in (0,1)$.Then the scheme $< x_n >$ converges strongly to a solution of $\mathcal{G}^n{}_i x = \mathcal{F}$.

Proof: Form definition of asymptotically severe map, that $\forall\, x \in \mathbb{C}, \exists\, \langle k_n \rangle \in (0,1)$ such that

$< \mathcal{G}^n{}_i x - \mathcal{G}^n{}_i y\ , j(x-y) > \geq k_n \| x - y \|^2$ for all $y \in \mathbb{C}$, we observe that $\mathcal{R}_i$, $\mathcal{G}_i$ are uniformly continuous and for any given $\mathcal{F} \in \mathbb{C}$.

$$(I - \mathcal{R}^n{}_i)x = x - \mathcal{F} + \mathcal{G}^n{}_i\, x - x = \mathcal{G}^n{}_i x - \mathcal{F}$$

Which implies that

$$< (I - \mathcal{R}^n{}_i)x - (I - \mathcal{R}^n{}_i)y\ , j(x-y) > \ \geq k_n \| x - y \|^2$$

That is $(I - \mathcal{R}^n{}_i)$ is asymptotically severe . Thus $R^n{}_i$ is asymptotically p-contractive. Thus the results follows .

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On Third-Order Sandwich Results for Analytic Functions Defined by Differential Operator

[1]Waggas Galib Atshan, [2]Ali Hussein Battor and [3]Abeer Farhan Abaas

[1]Mathematics Department, University of Al-Qadisiyah

waggas.galib@qu.edu.iq

waggas_hnd@yahoo.com

waggashnd@gmail.com

[2]Mathematics Department, University of Kufa

alih.battoor@uokufa.edu.iq

[2]Mathematics Department, University of Kufa

abeerfarhan688@gmail.com

Abstract

In this paper, by making use the differential operator, suitable classes of admissible functions are investigated and the properties of third-order sandwich theorems for multivalent analytic function are obtained.

Keywords: analytic function, multivalent function, differential subordination, differential Superordination, sandwich theorem, differential operator.

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1. Introduction

Let $\mathcal{g}(U)$ be the class of functions which are analytic in the open unit disk

$$U = \{z: z \in \mathbb{C} \quad \text{and } |z| < 1\}.$$

Also let

$$\mathcal{g}[a, n] \quad ; \quad (n \in N := \{1,2,3, \dots\}; a \in \mathbb{C})$$

be the subclass of the analytic function class $\mathcal{g}$ consisting of functions of the following form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \qquad (z \in \mathbb{C})$$

Let T be a subclass of $\mathcal{g}$ which are analytic in U have the normalized Taylor-Maclaurin series of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in N = \{1,2,3, \dots\}, z \in U). \tag{1}$$

Suppose that f and g are in $\mathcal{g}$. We say that f is subordinate to g, written as follows:

$$f \prec g \text{ in } U \text{ or } f(z) \prec g(z), \quad (z \in U)$$

if there exists a Schwarz function $\omega \in g$,which is analytic in U with $\omega(0) = 0$ and$|\omega(z)| < 1$ $(z \in U)$, such that $f(z) = g(\omega(z))$, $(z \in U)$. Indeed, it is known that

$$f(z) \prec g(z) \Rightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Furthermore, if the function g is univalent in U, we have the following equivalence relationship ([10])

$$g(z) \prec f(z) \Leftrightarrow g(0) = f(0) \text{ and } g(U) \subset f(U), \qquad (z \in U).$$

The concept of differential subordination is a generalization of various inequalities involving complex variables. We recall here some more definitions and terminologies from the theory of differential subordinations and differential superordination.

Definition (1). (see [1]): Let $\chi: \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ and suppose that the function $h(z)$ is univalent in U. If the function $p(z)$ is analytic in U and satisfies the following third-order differential subordination:

$$\chi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z), \tag{2}$$

then $p(z)$ is called a solution of the differential subordination (2). Furthermore, a given univalent function $q(z)$ is called a dominant of the solutions of (2) or, more simply, a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2). A dominant $\check{q}(z)$ that satisfies $\check{q}(z) \prec q(z)$ for all dominants $q(z)$ of (2) is said to be the best dominant.

Definition (2)[15]: Let $\chi: \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ and let the function $h(z)$ be univalent in U, if the functions $p(z)$ and $\chi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$ are univalent in U and satisfy the following third-order differential superordination:

$$h(z) \prec \chi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z), \tag{3}$$

then $p(z)$ is called a solution of the differential superordination given by (3) or more simply a simply a subordinant, if $q(z) \prec p(z)$ for all $p(z)$ satisfying (3). A univalent subordinant $\tilde{q}(z)$ that satisfies

$q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (3) is said to be the best subordinant we note both the best dominant and the best subordinant are unique up to rotation of U.
The monograph by Miller and Mocanu [9] and the more recent book of Bulboacã [5] provide detailed expositions on the theory of differential subordination and differential superordination.

Definition (3). [3]: For $f \in T, m \in N, \lambda \in N/\{1\}$. We define the differential operator

$$\mathcal{L}_\lambda^m : T \longrightarrow T$$

$$L_\lambda^0 f(z) = f(z),$$

$$L_\lambda^1 f(z) = \frac{z^{1-\lambda}}{\lambda + p}\left[z^\lambda L^0 f(z)\right]_z' ,$$

$$L_\lambda^2 f(z) = \frac{z^{1-\lambda}}{\lambda + p}\left[z^\lambda L^1 f(z)\right]_z' , \ldots$$

$$L_\lambda^m f(z) = \frac{z^{1-\lambda}}{\lambda + p}\left[z^\lambda L^{m-1} f(z)\right]_z' = z^p + \sum_{k=p+1}^{\infty}\left(\frac{\lambda + k}{\lambda + p}\right)^m z^k. \tag{4}$$

By simple calculation and using

$$z\left(L_\lambda^m f(z)\right)_z' = (\lambda + p)L_\lambda^{m+1} f(z) - \lambda L_\lambda^m f(z). \tag{5}$$

Definition (4). [1]: Let $\mathbb{Q}$ be the set of all functions q that are analytic and univalent on $\overline{U}/E(q)$,where

$$E(q) = \left\{\xi : \xi \in \partial U : \lim_{z \to \xi}\{q(z)\} = \infty\right\}, \tag{6}$$

and are such that $\min|q'(\xi)| = p > 0$ for $\xi \in \partial U/E(q)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by Q(a) with

$$\mathbb{Q}(0) = \mathbb{Q}_0 \text{ and } \mathbb{Q}(1) = \mathbb{Q}_1. \tag{7}$$

The subordination methodology is applied to appropriate classes of admissible functions. The following class of admissible functions is given by Antonino and Miller[1]

Definition (5).[1]: Let Ω be a set in $\mathbb{C}$.Also let $q \in \mathbb{Q}$ and $n \in N/\{1\}, N$ being the set of positive integers. The class $\Psi_n[\Omega, q]$ of admissible functions consists of those functions $\chi : \mathbb{C}^4 \times U \to \mathbb{C}$, which satisfy the following admissibility conditions:

$$\chi(r, s, t, u : z) \notin \Omega,$$

whenever

$$r = q(\xi), s = k\xi q'(\xi), \Re\left(\frac{t}{s} + 1\right) \geqq k\Re\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right)$$

and

$$\mathcal{R}\left(\frac{u}{s}\right) \geqq k^2 \mathcal{R}\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$$

where $z \in U\ , \xi \in \partial U/E(q)\ and\ k \geqq n$.

Lemma (1) below is the foundation result in the theory of third-order differential subordination.

Lemma (1). [1]: Let $p \in \mathcal{G}[a, n]\ with\ n \geqq 2\ and\ q \in \mathbb{Q}(a)$ satisfying the following conditions:

$$\mathcal{R}\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geqq 0 \qquad and \qquad \left|\frac{zp'(z)}{q'(\xi)}\right| \leqq k,$$

where $z \in U\ , \xi \in \partial U/E(q)\ and\ k \geqq n$. If Ω is a set in $\mathbb{C}$, $\chi \in \Psi_n[\Omega, q]$ and

$$\chi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z) : z) \in \Omega\ ,$$

then

$$p(z) \prec q(z) \qquad (z \in U).$$

Definition (6). [15]: Let Ω be a set in $\mathbb{C}$. Also let $q \in g[a, n]$ and $q'(z) \neq 0$. The class $\Psi'_n[\Omega, q]$ of admissible functions consists of those functions $\chi : \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$ that satisfy the following admissibility conditions:

$$\chi(r, s, t, u; \xi) \in \Omega,$$

whenever

$$r = q(z), \qquad s = \frac{zq'(z)}{m}\ , \Re\left(\frac{t}{s} + 1\right) \leqq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right)\ ,$$

and

$$\mathcal{R}\left(\frac{u}{s}\right) \leqq \frac{1}{m^2}\mathcal{R}\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z \in U$, $\xi \in \partial U$ and $m \geqq n \geqq 2$.

Lemma (2). [15]: Let $p \in \mathcal{g}[a,n]$ with $\chi \in \Psi'_n[\Omega,q]$. If the functions

$$\chi(p(z),zp'(z),z^2p''(z),z^3p'''(z):z) \subset \Omega ,$$

is univalent in U and $p \in \mathbb{Q}(a)$ satisfying the following conditions:

$$\mathcal{R}\left(\frac{zq''(z)}{q'(z)}\right) \geqq 0 \qquad and \qquad \left|\frac{zp'(z)}{q'(z)}\right| \leqq m,$$

where $z \in U$, $\xi \in \partial U$ and $m \geqq n \geqq 2$,then

$\Omega \subset \{\chi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z\ ):z \in U\}$,

implies that

$$p(z) \prec q(z) \qquad (z \in U).$$

The notion of the third-order differential subordination can be found in the work of Ponnusamy and Juneja [11]. The recent work by Tang et al . (see,for example, [14] and [15]; see also [5]) on the differential subordination attracted many researchers in this field . For example, see[2,4,6,7,8,10,11,12,13,14,15].

In the present paper, we investigate suitable classes of admissible functions associated with the differential operator and drive sufficient conditions on the normalized analytic function to satisfy:

$$q_1(z) \prec \vartheta(z) \prec q_2(z) \qquad (z \in U),$$

where q_1, q_2 are univalent in U and ϑ is suitable operator.

2. Third-Order differential subordination results

In this section, we start with a given set Ω and a function q and determine a set of admissible operator χ when (2) holds true . For this purpose, new class of admissible functions was introduced that will be required to prove the main third- order differential subordination theorems for the operator L_λ^m defined by (3).

Definition (7): Let Ω be a set in $\mathbb{C}$ and $q \in \mathbb{Q}_0 \cap \mathcal{g}_0$. The class $\aleph_L[\Omega,q]$ of admissible function consists of those function $\phi:\mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the following admissibility conditions:

$$\phi(\alpha,\beta,\gamma,\delta;z) \notin \Omega,$$

whenever

$$\alpha = q(\xi)\ , \qquad \beta = \frac{k\xi q'(\xi) + \lambda q(\xi)}{\lambda + p}$$

$$\mathcal{R}\left(\frac{\gamma(\lambda+p)^2 - \alpha\lambda^2}{(\beta(\lambda+p) - \alpha\lambda)} - 2\lambda\right) \geqq k\mathcal{R}\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right)$$

and

$$\mathcal{R}\left(\frac{\delta(\lambda+p)^3 - \gamma(\lambda+p)^2\big(3(\lambda+p)\big) + \lambda^2\alpha(3+2\lambda)}{\lambda(\beta-\alpha)+\beta p} + 3\lambda^2 + 6\lambda + 2\right) \geqq k^2\mathcal{R}\left(\frac{\xi^2q'''(\xi)}{q'(\xi)}\right),$$

where $z \in U$, $\xi \in \partial U/E(q)$ and $k \geqq 2$

Theorem (1): Let $\phi \in \aleph_l[\Omega,q]$ .If the function $f \in T$ and $q \in \mathbb{Q}_0$ satisfy the following conditions:

$$\mathcal{R}\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geqq 0 \qquad \left|\frac{L_\lambda^m f(z)}{q'(\xi)}\right| \leqq k, \tag{8}$$

and

$$\{\phi(L_\lambda^m f(z), L_\lambda^m f(z)\ , L_\lambda^m f(z), L_\lambda^m f(z); z\ ): z \in U\} \subset \Omega, \tag{9}$$

then

$$L_\lambda^m f(z) \prec q(z) \qquad (z \in \mathrm{U}).$$

Proof: Define the analytic function $p(z)$ in U by

$$p(z) = L_\lambda^m f(z). \tag{10}$$

From equation (5) and (10), we have

$$L_\lambda^{m+1} f(z) = \frac{zp'(z) + \lambda p(z)}{\lambda + p}. \tag{11}$$

By a similar argument, we get

$$L_\lambda^{m+2} f(z) = \frac{z^2p''(z) + (2\lambda+1)zp'(z) + \lambda^2 p(z)}{(\lambda+p)^2} \quad (12)$$

and

$$L_\lambda^{m+3} f(z) = \frac{z^3p'''(z) + 3(\lambda+1)z^2p''(z) + (3\lambda^2+3\lambda+1)zp'(z) + \lambda^3 p(z)}{(\lambda+p)^3}. \quad (13)$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\alpha(r,s,t,u) = r, \quad \beta(r,s,t,u) = \frac{s+\lambda r}{\lambda+p}, \qquad \gamma(r,s,t,u) = \frac{t+s(2\lambda+1)+\lambda^2 r}{(\lambda+p)^2}, \quad (14)$$

and

$$\delta(r,s,t,u) = \frac{u+3t(\lambda+1)+s(3(\lambda^2+\lambda)+1)+\lambda^3 r}{(\lambda+p)^3}. \quad (15)$$

Let

$\chi(r,s,t,u) = \phi(\alpha,\beta,\gamma,\delta;z) =$

$$= \phi\left(r, \frac{s+\lambda r}{\lambda+p}, \frac{t+s(2\lambda+1)+\lambda^2 r}{(\lambda+p)^2}, \frac{u+3t(\lambda+1)+s(3\lambda^2+3\lambda+1)+\lambda^3 r}{(\lambda+p)^3}\right) \quad (16)$$

The proof will make use of Lemma(1). Using the equations (10) to (13), and from the equations (16), we have

$$\chi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) = \phi\left(L_\lambda^m f(z), L_\lambda^{m+1} f(z), L_\lambda^{m+2} f(z), L_\lambda^{m+3} f(z); z\right). \quad (17)$$

Hence , clearly , (9) becomes

$$\chi(p(z), zp'(z), z^2p''(z), z^3p'''(z) : z) \in \Omega ,$$

We note that

$$\frac{t}{s} + 1 = \frac{\gamma(\lambda+p)^2 - \lambda^2\alpha}{\beta(\lambda+p) - \lambda\alpha} - 2\lambda$$

and

$$\frac{u}{s} = \frac{\delta(\lambda+p)^3 - \gamma(\lambda+p)^2(3\lambda+3) + \lambda^2\alpha(3+2\lambda)}{\lambda(\beta-\alpha) + p\beta} + 3\lambda^2 + 6\lambda + 2.$$

Thus clearly, the admissibility condition for $\phi \in \aleph_l[\Omega, q]$ in Definition (7) is equivalent to admissibility condition for $\chi \in \Psi_2[\Omega, q]$ as given in Definition (5) with $n = 2$.

Therefore, by using (8) and Lemma (1), we have

$L_\lambda^m f(z) p(z) \prec q(z)$.

The proof is complete.

Our next result is a consequences of Theorem (1) for the case when the behavior of $q(z)$ *on* ∂U is un known.

Corollary (1): Let $\Omega \subset \mathbb{C}$ and let function q be univalent in U *with* $q(0) = 0$. Let $\phi \in \aleph_l[\Omega, q_\rho]$ for some $\in (0,1)$, where $q_\rho(z) = q(\rho z)$ .If the function $f \in T$ and q_ρ satisfies the following conditions:

$$\mathcal{R}\left(\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right) \geqq 0 \qquad \left|\frac{L_\lambda^m f(z)}{q_\rho'(\xi)}\right| \leqq k, \qquad (z \in U; k \geqq 2; \xi \in \partial U / E(q_\rho))$$

and

$$\phi\left(L_\lambda^m f(z), L_\lambda^{m+1} f(z), L_\lambda^{m+2} f(z), L_\lambda^{m+3} f(z); z\right) \in \Omega,$$

then

$$L_\lambda^m f(z) \prec q(z) \qquad (z \in U).$$

Proof: By applying Theorem (1), we get

$$L_\lambda^m f(z) \prec q_\rho(z) \qquad (z \in U).$$

The result asserted by Corollary (1) is now deduced from following subordination property

$$q_\rho(z) \prec q(z) \qquad (z \in U).$$

The proof is complete.

If $\Omega \neq \mathbb{C}$ is simply-connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z) of\ U$ onto Ω In this case, the class $\aleph_l[h(U), q]$ is written as $\aleph_l[\Omega, q]$. This leads to the following immediate consequence of Theorem (1).

Theorem (2): Let $\in \aleph_l[h, q]$. If the function $f \in T$ and $q \in \mathbb{Q}_0$ satisfy the following conditions:

$$\mathcal{R}\left(\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right) \geqq 0 \qquad \left|\frac{L_\lambda^{m+1} f(z)}{q_\rho'(\xi)}\right| \leqq k, \tag{18}$$

and

$$\phi\left(L_\lambda^m f(z), L_\lambda^{m+1} f(z), L_\lambda^{m+2} f(z), L_\lambda^{m+3} f(z); z\right) \prec h(z), \tag{19}$$

then

$$L_\lambda^m f(z) \prec q(z) \qquad (z \in U).$$

The next result is an immediate consequence of Corollary (1).

Corollary (2): Let $\Omega \subset \mathbb{C}$ and let function q be univalent in $U\, with\, q(0) = 0$. Also Let $\phi \in \aleph_l[\Omega, q]$ for some $\in (0,1)$, where $q_\rho(z) = q(\rho z)$ .If the function $f \in T$ and q_ρ Satisfies the following conditions:

$$\mathcal{R}\left(\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right) \geqq 0 \quad , \qquad \left|\frac{L_\lambda^{m+1} f(z)}{q_\rho'(\xi)}\right| \leqq k, \qquad \left(z \in U; k \geqq 2; \xi \in \partial U / E(q_\rho)\right)$$

and

$$\phi\left(L_\lambda^m f(z), L_\lambda^{m+1} f(z), L_\lambda^{m+2} f(z), L_\lambda^{m+3} f(z); z\right) \prec h(z)$$

Then

$$L_\lambda^m f(z) \prec f(z) \qquad (z \in U).$$

The following result yield the best dominant of differential subordination (19).

Theorem (3): Let the function h be univalent in U. Also let $\phi: \mathbb{C}^4 \times U \to \mathbb{C}$ and χ be given by (16). Suppose that following differential equation:

$$\chi(q(z), zq'(z), z^2 q''(z), z^3 q'''(z); z) = h(z), \tag{20}$$

has a solution $q(z)$ with $q(0) = 0$, which satisfies the condition (8). If $f \in T$ satisfies the condition (19) and if

$$\phi\left(L_\lambda^m f(z), L_\lambda^{m+1} f(z), L_\lambda^{m+2} f(z), L_\lambda^{m+3} f(z); z\right),$$

is analytic in U, then

$$L_\lambda^m f(z) \prec q(z) \qquad (z \in U)$$

and $q(z)$ is the best dominant.

Proof: From Theorem (1), we see that q is a dominant of (19). Since q satisfies (20), it is also a solution of (19). Therefore, q will be dominated by all dominants. Hence q is the best dominant. This completes the proof of Theorem (3).

In view of Definition (7), and in special case when $q(z) = Mz\ (M > 0)$,the class $\aleph_l[\Omega, q]$ of admissible functions, denoted by $\aleph_l[\Omega, M]$ is expressed follows.

Definition (8): Let Ω be set in $\mathbb{C}$ and > 0 . The class $\aleph_l[\Omega, M\,]$ of admissible functions consists of those function $\phi: \mathbb{C}^4 \times U \to \mathbb{C}$ such that

$$\phi\begin{pmatrix} Me^{i\theta}, \left(\dfrac{k+\lambda}{\lambda+p}\right) Me^{i\theta}\ , \dfrac{L + [(2\lambda+1)k + \lambda^2] Me^{i\theta}}{(\lambda+p)^2}, \\ \dfrac{N + 3(\lambda+1)L + [(3\lambda^2 + 3\lambda + 1)k + \lambda^3] Me^{i\theta}}{(\lambda+p)^3}; z \end{pmatrix} \notin \Omega, \tag{21}$$

whenever $z \in U$,

$$\mathcal{R}(Le^{-i\theta}) \geqq (k-1)kM$$

and

$$\Re\left(Ne^{-i\theta}\right) \geqq 0 \qquad \forall \theta \in \mathbb{R}; k \geqq 2$$

Corollary (3): Let $\phi \in \aleph_l[\Omega, q]$ .If the function $f \in T$ satisfies the following conditions:

$$\left|L_\lambda^{m+1} f(z)\right| \leqq kM \qquad (z \in U; k \geqq 2; M > 0)$$

and

$$\left(L_\lambda^m f(z), L_\lambda^{m+1} f(z), L_\lambda^{m+2} f(z), L_\lambda^{m+3} f(z); z\right) \in \Omega,$$

then

$$\left|L_\lambda^m f(z)\right| < M.$$

In the special case when $\Omega = q(U) = \{w: |w| < M\}$, the class $\aleph_L[\Omega, q]$ is simply denoted by $\aleph_L[M]$. Corollary (3) can now be rewritten in the following form.

Corollary (4): Let $\phi \in \aleph_l[M]$. If the function $f \in T$ satisfies the following conditions:

$$|L_\lambda^m f(z)| \leqq kM \qquad (z \in U; k \geqq 2; M > 0)$$

and

$$\left|\left(L_\lambda^m f(z), L_\lambda^{m+1} f(z), L_\lambda^{m+2} f(z), L_\lambda^{m+3} f(z); z\right)\right| < M,$$

then

$$|L_\lambda^m f(z)| < M.$$

Corollary (5): Let $k \geqq 2, 0 \neq q \in \mathbb{C}$ *and* $M > 0$. If the function $f \in T$ satisfies the following conditions:

$$|L_\lambda^m f(z)| \leqq kM$$

and

$$|L_\lambda^m f(z) - L_\lambda^m f(z)| < \frac{M}{|\lambda + p|},$$

then

$$|L_\lambda^m f(z)| < M.$$

Proof: let $\phi(\alpha, \beta, \gamma, \delta: z) = \beta - \alpha$ and $\Omega = h(U)$,
where

$$h(z) = \frac{Mz}{|\lambda + p|} \qquad\qquad (M > 0)$$

use Corollary (3), we need to show that $\phi \in \aleph_l[\Omega, q]$, That is that the admissibility condition (21) is satisfied. This follows readily, since it is seen that

$$|\phi(\alpha, \beta, \gamma, \delta; z| = \left|\frac{(k-1)Me^{i\theta}}{\lambda + p}\right| \geqq \frac{M}{|\lambda + p|},$$

whenever $z \in U, \theta \in \mathbb{R}$ *and* $k \geqq 2$. The requird result now follows from Corollary (3). This completes the proof of corollary (5).

Definition (9): Let Ω be a set in $\mathbb{C}$ and $q \in \mathbb{Q}_1 \cap \mathcal{g}_1$. The class $\aleph_{l,1}[\Omega, q]$ of admissible functions consists of those functions $\phi: \mathbb{C}^4 \times U \to \mathbb{C}$, which satisfy the following admissibility conditions:
$\phi(\alpha, \beta, \gamma, \delta; z) \notin \Omega$,
whenever

$$\alpha = q(\xi), \qquad \beta = \frac{k\xi q'(\xi) + (\lambda + 1)q(\xi)}{\lambda + p},$$

$$\Re\left(\frac{\gamma(\lambda + p)^2 - \alpha(\lambda + 1)^2}{\beta(\lambda + p) - \alpha(\lambda + 1)} - 2(1 + \lambda)\right) \geqq k\Re\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right)$$

and

$$\mathcal{R}\left(\frac{\delta(\lambda + P)^3 + 3\gamma(\lambda + 2)(\lambda + p)^2 + 3\alpha(\lambda + 2)(\lambda + 1)^2 - (1 + \lambda)^3\alpha}{\beta(\lambda + p) - \alpha(\lambda + 1)} + 3\lambda^2 + 12\lambda + 11\right)$$
$$\geqq k^2\mathcal{R}\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$$

where $z \in U$ $, \xi \in \partial U / E(q)$ *and* $k \geqq n$.

Theorem (4): Let $\in \aleph_{l,1}[\Omega, q]$. If the function $f \in T$ and $q \in \mathbb{Q}_1$ satisfy the following conditions:

$$\mathcal{R}\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geqq 0 \qquad\qquad \left|\frac{L_\lambda^{m+1} f'(z)}{zq'^{(\xi)}}\right| \leqq k, \tag{22}$$

and

$$\left\{\phi\left(\frac{L_\lambda^m f(z)}{z}, \frac{L_\lambda^{m+1} f(z)}{z}, \frac{L_\lambda^{m+2} f(z)}{z}, \frac{L_\lambda^{m+3} f(z)}{z}; z\right); z \in U\right\} \subset \Omega, \tag{23}$$

then

$$\frac{L_\lambda^m f(z)}{z} \prec q(z) \qquad (z \in U).$$

Proof: Define the analytic function $p(z)$ in U by

$$p(z) = \frac{L_\lambda^m f(z)}{z}. \tag{24}$$

From equation (5) and (24), we have

$$\frac{L_\lambda^{m+1} f(z)}{z} = \frac{zp'(z) + (\lambda+1)p(z)}{\lambda+p}. \tag{25}$$

By a similar argument, we get

$$\frac{L_\lambda^{m+2} f(z)}{z} = \frac{z^2p''(z) + (2\lambda+3)zp'(z) + (\lambda+1)^2p(z)}{(\lambda+p)^2} \tag{26}$$

and

$$\frac{L_\lambda^{m+3} f(z)}{z} = \frac{z^3p'''(z) + 3(\lambda+2)z^2p''(z) + (3\lambda^2+9\lambda+7)zp'(z) + (\lambda+1)^3p(z)}{(\lambda+p)^3}. \tag{27}$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\alpha(r,s,t,u) = r, \quad \beta(r,s,t,u) = \frac{s + r(\lambda+1)}{\lambda+p},$$

$$\gamma(r,s,t,u) = \frac{t + s(2\lambda+3) + (\lambda+1)^2r}{(\lambda+p)^2}, \tag{28}$$

and

$$\delta(r,s,t,u) = \frac{u + 3t(\lambda+2) + s(3\lambda^2+9\lambda+7) + (\lambda+1)^3r}{(\lambda+p)^3}. \tag{29}$$

Let

$\chi(r,s,t,u) = \phi(\alpha,\beta,\gamma,\delta;z) =$

$$= \phi\left(\begin{array}{c} r, \dfrac{s+(\lambda+1)r}{\lambda+p}, \quad \dfrac{t+s(2\lambda+3)+(\lambda+1)^2r}{(\lambda+p)^2}, \\ \dfrac{u+3t(\lambda+2)+s(3\lambda^2+9\lambda+7)+(\lambda+1)^3r}{(\lambda+p)^3}; z \end{array}\right). \tag{30}$$

The proof will make use of Lemma (1). Using the equations (24) to (26), and from the equations (30), we have

$$\chi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) = \phi\left(\frac{L_\lambda^m f(z)}{z}, \frac{L_\lambda^{m+1} f(z)}{z}, \frac{L_\lambda^{m+2} f(z)}{z}, \frac{L_\lambda^{m+3} f(z)}{z}; z\right). \tag{31}$$

Hence, clearly, (23) becomes

$$\chi(p(z), zp'(z), z^2p''(z), z^3p'''(z): z) \in \Omega \,.$$

We note that

$$\frac{t}{s} + 1 = \frac{\gamma(\lambda+p)^2 - \alpha(\lambda+1)^2}{\beta(\lambda+p) - \alpha(\lambda+1)} - 2(\lambda+1)$$

and

$$\frac{u}{s} = \frac{\delta(\lambda+p)^3 - 3\gamma(\lambda+p)^2(\lambda+2) + 3\alpha(\lambda+2)(\lambda+1)^2 - (\lambda+1)^3}{\beta(\lambda+p) - \alpha(\lambda+1)} + 3\lambda^2 + 12\lambda + 11.$$

Thus clearly, the admissibility condition for $\phi \in \aleph_l[\Omega, q]$ in Definition (9) is equivalent to admissibility condition for $\chi \in \Psi_2[\Omega, q]$ as given in Definition (5) with $n = 2$.

Therefore, by using (22) and Lemma (1), we have

$\frac{L_\lambda^m f(z)}{z} \prec q(z)$.

This completes the proof of Theorem (4).

If $\Omega \neq \mathbb{C}$ is simply-connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z) of\ U$ onto Ω .In this case, the class$\aleph_{l,1}[h(U), q]$ is written as $\aleph_{l,1}[\Omega, q]$. This leads to the following immediate consequence of Theorem (4) is stated below.

Theorem (5): let $\in \aleph_{l,1}[\Omega, q]$. If the function $f \in T$ and $q \in \mathbb{Q}_1$ satisfy the following conditions:

$$\mathcal{R}\left(\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right) \geqq 0 \qquad \left|\frac{L_\lambda^{m+1} f(z)}{zq_\rho'(\xi)}\right| \leqq k, \tag{32}$$

and

$$\phi\left(\frac{L_\lambda^m f(z)}{z},\frac{L_\lambda^{m+1} f(z)}{z},\frac{L_\lambda^{m+2} f(z)}{z},\frac{L_\lambda^{m+3} f(z)}{z};z\right) \prec h(z), \tag{33}$$

then

$$\frac{L_\lambda^m f(z)}{z} \prec q(z) \qquad (z \in U).$$

In view of Definition (10), and in special case when $q(z) = Mz\ (M > 0)$,the class $\aleph_{L,1}[\Omega, q]$ of admissible functions , denoted by $\aleph_{l,1}[\Omega, M]$ is expressed follows.

Definition (10): Let Ω be set in $\mathbb{C}$ and$M > o$. The class $\aleph_{l,1}[\Omega, M]$ of admissible functions consists of those function $\phi: \mathbb{C}^4 \times U \to \mathbb{C}$ such that

$$\phi\left(\begin{matrix} Me^{i\theta}, \dfrac{(k+\lambda+1)Me^{i\theta}}{\lambda+p}, \dfrac{L+[(3+2\lambda)k+(\lambda+1)^2]Me^{i\theta}}{(\lambda+p)^2}, \\ \dfrac{N+3(\lambda+2)L+[k(3\lambda^2+9\lambda+7)+(\lambda+1)^3]Me^{i\theta}}{(\lambda+p)^3}; z \end{matrix}\right) \notin \Omega. \tag{34}$$

Whenever $z \in U$,

$$\mathcal{R}(Le^{-i\theta}) \geqq (k-1)kMm$$

and

$$\Re\left(Ne^{-i\theta}\right) \geqq 0 \qquad \forall \theta \in \mathbb{R}; k \geqq 2.$$

Corollary (6): Let $\phi \in \aleph_{l,1}[\Omega, M]$ .If the function $f \in T$ satisfy the following conditions:

$$\left|\frac{L_\lambda^{m+1} f(z)}{z}\right| \leqq kM \qquad (z \in U; k \geqq 2; M > 0),$$

and

$$\phi\left(\frac{L_\lambda^m f(z)}{z},\frac{L_\lambda^{m+1} f(z)}{z},\frac{L_\lambda^{m+2} f(z)}{z},\frac{L_\lambda^{m+3} f(z)}{z};z\right) \in \Omega,$$

then

$$\left|\frac{L_\lambda^m f(z)}{z}\right| < M.$$

In the special case when $\Omega = q(U) = \{w: |w| < M\}$, the class $\aleph_{l,1}[\Omega, M]$ is simply denoted by $\aleph_{l,1}[M]$. Corollary (6) can now be rewritten in the following form.

Corollary (7): Let $\phi \in \aleph_{l,1}[\Omega, M]$ .If the function $f \in T$ satisfy the following conditions:

$$\left|\frac{L_\lambda^{m+1} f(z)}{z}\right| \leqq kM \qquad (z \in U; k \geqq 2; M > 0),$$

and

$$\left|\phi\left(\frac{L_\lambda^m f(z)}{z},\frac{L_\lambda^{m+1} f(z)}{z},\frac{L_\lambda^{m+2} f(z)}{z},\frac{L_\lambda^{m+3} f(z)}{z};z\right)\right| < M,$$

then

$$\left|\frac{L_\lambda^m f(z)}{z}\right| < M.$$

Definition (11): Let Ω be a set in $\mathbb{C}$.Also let $q \in \mathbb{Q}_1 \cap \mathcal{g}_1$.The class $\aleph_{l,2}[\Omega, q]$ of admissible functions consists of those functions $\phi: \mathbb{C}^4 \times U \to \mathbb{C}$, which satisfy the following admissibility conditions:

$$\phi(\alpha, \beta, \gamma, \delta: z) \notin \Omega,$$

whenever

$$\alpha = q(\xi), \qquad \beta = \frac{1}{\lambda+p}\left(\frac{k\psi\xi q'(\xi)}{q(\xi)} + (\lambda+1)q(\xi)\right),$$

$$\Re\left(\frac{(\lambda+p)(\beta\gamma+2\alpha^2-3\alpha\beta)}{(\beta-\alpha)}\right) \geqq k\Re\left(\frac{\xi q''(\xi)}{q'(\xi)}+1\right)$$

and

$$\mathcal{R}\Big((\delta-\gamma)(\lambda+p)^2\beta\gamma-(\lambda+p)^2(\gamma-\beta)(1-\beta-\gamma+3\alpha)-3\beta(\lambda+p)(\gamma-\beta)+2(\beta-\alpha) + 3\alpha(\lambda+p)(\beta-\alpha)+(\beta-\alpha)^2(\lambda+p)\big((\beta-\alpha)(\lambda+p)-3-4\alpha(\lambda+p)\big) + \alpha^2(\beta-\alpha)(\lambda+p)^2\Big)(\beta-\alpha)^{-1} \geqq k^2\mathcal{R}\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$$

where $z \in U$, $\xi \in \partial U/E(q)$ and $k \geqq n$.

Theorem (6): Let $\in \aleph_{l,2}[\Omega, q]$. If the function f $\in T$ and $q \in \mathbb{Q}_1$ satisfy the following conditions:

$$\mathcal{R}\left(\frac{\xi q''{}_p(\xi)}{q'(\xi)}\right) \geqq 0 \qquad \left|\frac{L_\lambda^{m+2}f(z)}{L_\lambda^{m+1}f(z)q'(\xi)}\right| \leqq k, \tag{35}$$

and

$$\left\{\phi\left(\frac{L_\lambda^{m+1}f(z)}{L_\lambda^{m}f(z)},\frac{L_\lambda^{m+2}f(z)}{L_\lambda^{m+1}f(z)},\frac{L_\lambda^{m+3}f(z)}{L_\lambda^{m+2}f(z)},\frac{L_\lambda^{m+4}f(z)}{L_\lambda^{m+3}f(z)};z\right), z \in U\right\} \subset \Omega, \tag{36}$$

then

$$\frac{L_\lambda^{m+1}f(z)}{L_\lambda^{m}f(z)} \prec q(z) \qquad (z \in U).$$

Proof: Define the analytic function $p(z)$ in U by

$$p(z) = \frac{L_\lambda^{m+1}f(z)}{L_\lambda^{m}f(z)}. \tag{37}$$

From equation (5) and (37), we have

$$\frac{L_\lambda^{m+2}f(z)}{L_\lambda^{m+1}f(z)} = \frac{1}{\lambda+p}\left[\frac{zp'(z)}{p(z)}+(\lambda+p)p(z)\right] = \frac{A}{\lambda+p}. \tag{38}$$

By a similar argument , we get

$$\frac{L_\lambda^{m+3}f(z)}{L_\lambda^{m+2}f(z)} = \frac{B}{\lambda+p} \tag{39}$$

and

$$\frac{L_\lambda^{m+4}f(z)}{L_\lambda^{m+3}f(z)} = \frac{1}{\lambda+p}[B+B^{-1}(C+A^{-1}D-A^{-2}C^2)], \tag{40}$$

where

$$B = (\lambda+p)p(z)+\frac{zp'(z)}{p(z)}+\frac{\frac{z^2p''(z)}{p(z)}+\frac{zp'(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2+(\lambda+p)zp'(z)}{\frac{zp'(z)}{p(z)}+(\lambda+p)p(z)},$$

$$C = \frac{z^2p''(z)}{p(z)}+\frac{zp'(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2+(\lambda+p)zp'(z)$$

and

$$D = \frac{3z^2p''(z)}{p(z)}+\frac{z^3p'''(z)}{p(z)}+\frac{zp'(z)}{p(z)}-3\left(\frac{zp'(z)}{p(z)}\right)^2-\frac{3z^3p'(z)p''(z)}{p^2(z)}+2\left(\frac{zp'(z)}{p(z)}\right)^3+(\lambda+\mathrm{p})zp'(z)+(\lambda+p)z^2p''(z).$$

We now define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\alpha(r,s,t,u) = r, \quad \beta(r,s,t,u) = \frac{1}{\lambda+p}\left[\frac{s}{r}+(\lambda+p)\gamma\right] := \frac{E}{\lambda+p},$$

$$\gamma(r,s,t,u) = \frac{1}{\lambda+p}\left[\frac{s}{r}+(\lambda+p)r+\frac{\frac{t}{r}+\frac{s}{r}-\left(\frac{s}{r}\right)^2+(\lambda+p)s}{\frac{s}{r}+(\lambda+p)r}\right] := \frac{F}{\lambda+p} \tag{41}$$

and

$$\delta(r,s,t,u) = \frac{1}{\lambda+p}[F+F^{-1}(L+E^{-1}H-E^{-2}L^{-2}], \tag{42}$$

where

$$L = s(\lambda+p)+\frac{t}{r}+\frac{s}{r}-\left(\frac{s}{r}\right)^2$$

and

$$H=\frac{3t}{r}+\frac{u}{r}+\frac{s}{r}-3\left(\frac{s}{r}\right)^2-3\left(\frac{st}{r^2}\right)+2\left(\frac{s}{r}\right)^3+(\lambda+p)(s+t).$$

Let

$$\chi(r,s,t,u)=\phi(\alpha,\beta,\gamma,\delta;z)=\phi\left(r,\frac{E}{\lambda+p},\frac{F}{\lambda+p},\frac{1}{\lambda+p}[F+F^{-1}(L+E^{-1}H-E^{-2}L^2]\right). \quad (43)$$

The proof will make use of Lemma(1). Using the equations (37) to (40), and from the equations (43), we have

$$\chi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)$$
$$=\phi\left(\frac{L_\lambda^{m+1}f(z)}{L_\lambda^{m}f(z)},\frac{L_\lambda^{m+2}f(z)}{L_\lambda^{m+1}f(z)},\frac{L_\lambda^{m+3}f(z)}{L_\lambda^{m+2}f(z)},\frac{L_\lambda^{m+4}f(z)}{L_\lambda^{m+3}f(z)};z\right). \qquad (44)$$

Hence, clearly, (35) becomes

$$\chi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)\in\Omega\,.$$

We note that

$$\frac{t}{s}+1=\frac{(\lambda+p)(\beta\gamma+2\alpha^2-3\alpha\beta)}{(\beta-\alpha)}$$

and

$(\beta-\alpha)^{-1}$.

Thus clearly, the admissibility condition for $\phi\in\aleph_{l,2}[\Omega,q]$ in Definition (11) is equivalent to admissibility condition for $\chi\in\Psi_2[\Omega,q]$ as given in Definition (5) with $n=2$.

Therefore, by using (35) and Lemma (1), we have

$\frac{L_\lambda^{m+1}f(z)}{L_\lambda^{m}f(z)}\prec q(z)$ (45) .

This completes the proof of Theorem (6).

If $\Omega\neq\mathbb{C}$ is simply-connected domain, then $\Omega=h(U)$ for some conformal mapping $h(z) of\ U$ onto Ω .In this case , the class $\aleph_{l,2}[h(U),q]$ is written as $\aleph_{l,2}\,[\Omega,q]$. An immediate consequence of Theorem (6) is now stated below without proof.

Theorem (7): Let $\in\aleph_{l,2}[h,q]$. If the function $f\in T$ and $q\in\mathbb{Q}_1$ satisfy the following conditions (37) and

$$\phi\left(\frac{L_\lambda^{m+1}f(z)}{L_\lambda^{m}f(z)},\frac{L_\lambda^{m+2}f(z)}{L_\lambda^{m+1}f(z)},\frac{L_\lambda^{m+3}f(z)}{L_\lambda^{m+2}f(z)},\frac{L_\lambda^{m+4}f(z)}{L_\lambda^{m+3}f(z)};z\right)\prec h(z), \qquad (46)$$

then

$$\frac{L_\lambda^{m+1}f(z)}{L_\lambda^{m}f(z)}\prec q(z) \qquad (z\in U).$$

3. Result Related to the Third-Order Superordination

In this section, we investigate and prove several theorems involving the third-order differential superordination for the operator Defined in (5). For the purpose, we consider the following class of admissible functions.

Definition (12): let Ω be a set in $\mathbb{C}$ and $q\in\mathbb{Q}_1\cap\mathcal{G}_1$. The class $\aleph_{l,2}[\Omega,q]$ of admissible functions consists of those functions $\phi:\mathbb{C}^4\times U\to\mathbb{C}$, which satisfy the following admissibility conditions:

$\phi(\alpha,\beta,\gamma,\delta;z)\notin\Omega$,

whenever

$$\alpha=q(\xi),\qquad \beta=\frac{\xi q'(\xi)+m\lambda q(\xi)}{m(\lambda+p)},$$
$$\Re\left(\frac{\gamma(\lambda+p)^2-\lambda^2\alpha}{\beta(\lambda+p)-\lambda\alpha}-2\lambda\right)\leqq\frac{1}{m}\Re\left(\frac{\xi q''(\xi)}{q'(\xi)}+1\right)$$

and

$$\mathcal{R}\left(\frac{\delta(\lambda+P)^3-\gamma(\lambda+p)^2(3\lambda+3)+\lambda^2\alpha(3+2\lambda)}{\lambda(\beta-\alpha)+\beta p}+3\lambda^2+6\lambda+2\right)\leqq\frac{1}{m^2}\mathcal{R}\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z\in U\ ,\xi\in\partial U\ and\ m\geqq 2$.

Theorem (8): Let $\phi\in\aleph'_L[\Omega,q]$ .If the function $f\in T$, with $L_\lambda^m f(z)\in Q_0$, and if $q\in\mathcal{G}_0$ with $q'(z)=0$, satisfying the following conditions:

$$\mathcal{R}\left(\frac{zq''(z)}{q'(z)}\right) \geqq 0 \qquad\qquad \left|\frac{L_\lambda^{m+1} f(z)}{q'(z)}\right| \leqq m, \tag{47}$$

and the function

$$\phi\left(L_\lambda^m f(z), L_\lambda^{m+1} f(z), L_\lambda^{m+2} f(z), L_\lambda^{m+3} f(z); z\right)$$

Is univalent in U, then

$$\Omega \subset \left\{\phi\left(L_\lambda^m f(z), L_\lambda^{m+1} f(z), L_\lambda^{m+2} f(z), L_\lambda^{m+3} f(z); z\right): z \in U\right\} \tag{48}$$

Implies that

$$q(z) \prec L_\lambda^m f(z) \qquad (z \in U).$$

Proof: Let the function $p(z)$ be defined by (24) and χ by (16). Since $\phi \in N_L{}'[\Omega, q]$, from (17) and (48), we have

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z): z \in U\}.$$

From (14) and (15), we see that the admissibility condition for $\phi \in \aleph_l{}'[\Omega, q]$ in Definition (1) is equivalent to the admissibility for $\chi \in \Psi_2[\Omega, q]$ as given in Definition (6) with $n = 2$. Hence $\chi \in \Psi_2{}'[\Omega, q]$ and , by using (48) and Lemma (2) , we find that

$$q(z) \prec L_\lambda^m f(z) \qquad (z \in U).$$

This completes the proof of Theorem (8).

If $\Omega \neq \mathbb{C}$ is simply-connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z) of\ U$ onto Ω .In this case, the class$\aleph_l{}'[h(U), q]$ is written as $\aleph_l{}'[\Omega, q]$. This leads to the following immediate consequence of Theorem (8).

Theorem (9): Let $\phi \in \mathrm{N}_L'[\Omega, q]$ and let h be analytic in U. If the function $f \in T$ and $L_\lambda^m f(z) \in Q_0$, and if $q \in \mathcal{g}_0$ with $q'(z) \neq 0$, satisfying the conditions (47) and the function

$$\phi\left(L_\lambda^m f(z), L_\lambda^{m+1} f(z), L_\lambda^{m+2} f(z), L_\lambda^{m+3} f(z); z\right),$$

is univalent in U, then

$$h(z) \prec \phi\left(L_\lambda^m f(z), L_\lambda^{m+1} f(z), L_\lambda^{m+2} f(z), L_\lambda^{m+3} f(z); z\right), \tag{49}$$

implies that

$$q(z) \prec L_\lambda^m f(z) \qquad (z \in U).$$

Theorem (8) and (9) can only be used to obtain subordination for the third-order differential superordination of the form (48) or (49). The following theorem gives the existence of the best subordination of (49) for suitable ϕ.

Theorem (10): Let the function h be univalent in U. Also let $\phi: \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$ and χ be given by (16). Suppose that following differential equation:

$$\chi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = h(z), \tag{50}$$

has a solution $q(z) \in Q_0$. If the function $f \in T$, with $L_\lambda^m f(z) \in Q_0$ and if $q \in \mathcal{g}_0$ with $q'(z) \neq 0$, satisfying the condition (47) and

$$\phi\left(L_\lambda^m f(z), L_\lambda^{m+1} f(z), L_\lambda^{m+2} f(z), L_\lambda^{m+3} f(z); z\right),$$

is analytic in U, then

$$h(z) \prec \phi\left(L_\lambda^m f(z), L_\lambda^{m+1} f(z), L_\lambda^{m+2} f(z), L_\lambda^{m+3} f(z); z\right)$$
$$q(z) \prec L_\lambda^m f(z) \qquad (z \in U)$$

and $q(z)$ is the best dominant.

Proof: By applying Theorem (8) and Theorem (9), we deduce that q is a subordination of (49). Since q satisfies (50), it is also a solution of (49) and therefore, q will be subordinated by all subordinates. Hence q is the best subordinate. This completes the proof of Theorem (10).

Definition (13): Let Ω be a set in $\mathbb{C}$ and $q \in g_1$ with $q'(z) \neq 0$. The class $\aleph'_{l,1}[\Omega, q]$ of admissible functions consists of those function $\phi: \mathbb{C}^4 \times \overline{U} \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:
$\phi(\alpha, \beta, \gamma, \delta; \xi) \in \Omega$,
whenever

$$\alpha = q(\xi), \qquad \beta = \frac{\xi q'(\xi) + (\lambda + p)m\lambda q(\xi)}{m(\lambda + p)},$$

$$\Re\left(\frac{(\lambda + p)(\gamma - \alpha)}{\beta - \alpha} - 2(\lambda + p)\right) \leqq \frac{1}{m}\Re\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right)$$

and

$$\mathcal{R}\left(\frac{\delta(\lambda+P)^2-3\gamma(\lambda+2)(\lambda+p)+3\alpha(\lambda+2)(\lambda+p)-\alpha(\lambda+p)^2}{\beta-\alpha}+3\lambda^2+12\lambda+11\right)$$
$$\leqq\frac{1}{m^2}\mathcal{R}\left(\frac{z^2q'''(z)}{q'(z)}\right),$$
where $z\in U\ ,\xi\in\partial U\ and\ \ m\geqq 2$.

Theorem (11): Let $\phi\in\aleph'_{l,2}[\Omega,q]$. If the function $f\in T$ and $\frac{L_\lambda^{m+1}f(z)}{z}\in\mathbb{Q}_1$, and if $q\in\mathcal{g}_1$with $q'(z)\neq 0$, satisfying the following conditions:

$$\mathcal{R}\left(\frac{\xi q''(\xi)}{q'(\xi)}\right)\geqq 0\qquad\qquad\left|\frac{L_\lambda^{m+1}f(\xi)}{zq'(\xi)}\right|\leqq k,\tag{51}$$

and the function

$$\phi\left(\frac{L_\lambda^m f(z)}{z},\frac{L_\lambda^{m+1}f(z)}{z},\frac{L_\lambda^{m+2}\mathrm{f}(z)}{z},\frac{L_\lambda^{m+3}f(z)}{z};z\right)\prec h(z),$$

is univalent in U, then

$$\Omega\subset\left\{\phi\left(\frac{L_\lambda^m f(z)}{z},\frac{L_\lambda^{m+1}f(z)}{z},\frac{L_\lambda^{m+2}f(z)}{z},\frac{L_\lambda^{m+3}f(z)}{z};z\right);z\in U\right\}\tag{52}$$

then

$$q(z)\prec\frac{L_\lambda^m f(z)}{z}\qquad(z\in U).$$

Proof: Let the function $p(z)$ be defined by (24) and χ by (30). Since $\phi\in\aleph'_{J,2}[\Omega,q]$, we find from (31) and (52) that

$$\Omega\subset\left\{\phi\left(\frac{L_\lambda^m f(z)}{z},\frac{L_\lambda^{m+1}f(z)}{z},\frac{L_\lambda^{m+2}f(z)}{z},\frac{L_\lambda^{m+3}f(z)}{z};z\right);z\in U\right\}.$$

Form the equations (28) and (29), we see that the admissible condition for $\phi\in\aleph_{j,2}[\Omega,q]$ in Definition (13) is equivalent to the admissible condition for $\chi\in\Psi_2'[\Omega,q]$ and, by using (51) and Lemma (2), we have

$$q(z)\prec\frac{L_\lambda^m f(z)}{z}\qquad(z\in U).$$

If $\Omega\neq\mathbb{C}$ is simply-connected domain, then $\Omega=h(U)$ for some conformal mapping $h(z)of\ U$ onto Ω .In this case, the class$\aleph_{l,1}{}'[h(U),q]$ is written as $\aleph_{l,1}{}'[\Omega,q]$. This leads to the following immediate consequence of Theorem (11).

Theorem (12): Let $\phi\in\aleph'_{l,1}[\Omega,q]$ and let h be analytic in U .If the function $f\in T,\ with\ q\in\mathcal{g}_1$ and $q'(z)\neq 0$, satisfying the conditions (51) and the function

$$\phi\left(\frac{L_\lambda^m f(z)}{z},\frac{L_\lambda^{m+1}f(z)}{z},\frac{L_\lambda^{m+2}f(z)}{z},\frac{L_\lambda^{m+3}f(z)}{z};z\right)$$

is univalent in U, then

$$h(z)\prec\phi\left(\frac{L_\lambda^m f(z)}{z},\frac{L_\lambda^{m+1}f(z)}{z},\frac{L_\lambda^{m+2}f(z)}{z},\frac{L_\lambda^{m+3}f(z)}{z};z\right),$$

implies that

$$q(z)\prec\frac{L_\lambda^m f(z)}{z}\qquad(z\in U).$$

Definition (14): let Ω be a set in $\mathbb{C}$ and $q\in\mathcal{g}_1$ with $q'(z)\neq 0$.The class $\aleph'_{l,2}[\Omega,q]$ of admissible functions consists of those functions $\phi:\mathbb{C}^4\times\overline{U}\to\mathbb{C}$, which satisfy the following admissibility conditions:

$$\phi(\alpha,\beta,\gamma,\delta;\xi)\in\Omega,$$

whenever

$$\alpha=q(\xi),\qquad\beta=\frac{1}{\lambda+p}\left(\frac{zq'(z)}{mq(z)}+(\lambda+p)q(z)\right),$$
$$\Re\left(\frac{(\lambda+p)(\beta\gamma+2\alpha^2-3\alpha\beta)}{(\beta-\alpha)}\right)\geqq k\Re\left(\frac{\xi q''(z)}{q'(\mathrm{z})}+1\right)$$

and

$$\mathcal{R}\big((\delta-\gamma)(\lambda+p)^2\beta\gamma-(\lambda+p)^2(\gamma-\beta)\beta(1-\beta-\gamma+3\alpha)-3(\lambda+p)(\gamma-\beta)\beta+2(\beta-\alpha) \\ +3(\lambda+p)\alpha(\beta-\alpha)+(\beta-\alpha)^2(\lambda+p)((\beta-\alpha)(\lambda+p)-3-4(\lambda+p)\alpha+\alpha^2 \\ +(\lambda+p)^2(\beta-\alpha)\big)(\beta-\alpha)^{-1}\geqq\frac{1}{m^2}\mathcal{R}\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

Where $z\in U$ $,\xi\in\partial U$ and $m\geqq 2$.

Theorem (13): let $\in\aleph'_{l,2}[\Omega,q]$. If the function $f\in T$, with $\frac{L_\lambda^{m+1}f(z)}{L_\lambda^m f(z)}\in\mathbb{Q}_1$ and if $q\in\mathcal{g}_1$ with $q'(z)\neq 0$ satisfy the following conditions:

$$\mathcal{R}\left(\frac{zq''(z)}{q'(z)}\right)\geqq 0 \qquad \left|\frac{L_\lambda^{m+2}f(z)}{L_\lambda^{m+1}f(z)q'(z)}\right|\leqq m, \tag{53}$$

and the function

$$\phi\left(\frac{L_\lambda^{m+1}f(z)}{L_\lambda^m f(z)},\frac{L_\lambda^{m+2}f(z)}{L_\lambda^{m+1}f(z)},\frac{L_\lambda^{m+3}f(z)}{L_\lambda^{m+2}f(z)},\frac{L_\lambda^{m+4}f(z)}{L_\lambda^{m+3}f(z)};z\right),$$

is univalent in U, then

$$\Omega\subset\left\{\phi\left(\frac{L_\lambda^{m+1}f(z)}{L_\lambda^m f(z)},\frac{L_\lambda^{m+2}f(z)}{L_\lambda^{m+1}f(z)},\frac{L_\lambda^{m+3}f(z)}{L_\lambda^{m+2}f(z)},\frac{L_\lambda^{m+4}f(z)}{L_\lambda^{m+3}f(z)};z\right);z\in U\right\} \tag{54}$$

implies that

$$q(z)\prec\frac{L_\lambda^{m+1}f(z)}{L_\lambda^m f(z)} \qquad (z\in U)$$

Proof: Let the function $p(z)$ be defined by (37) and χ by (43). Since $\phi\in\aleph'_{l,2}[\Omega,q]$, we find from (44) and (54) that

$$\Omega\subset\{\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z):z\in U\}.$$

From the equations (41) and (42), we see that the admissible condition for $\phi\in\aleph'_{l,2}[\Omega,q]$ in Definition (14) is equivalent to the admissible condition for χ as given in Defintion (6) with $n=2$ Hence $\chi\in\Psi'_2[\Omega,q]$ and, by using (53) and Lemma (2), we have

$$q(z)\prec\frac{L_\lambda^{m+1}f(z)}{L_\lambda^m f(z)} \qquad (z\in U).$$

This completes the proof of Theorem (13).

Theorem (14): Let $\in\aleph'_{l,2}[\Omega,q]$. If the function $f\in T$, with $\frac{L_\lambda^{m+1}f(z)}{L_\lambda^m f(z)}\in\mathbb{Q}_1$, with $q\in\mathcal{g}_1$ and $q'(z)\neq 0$ satisfy the conditions (53) and the function

$$\phi\left(\frac{L_\lambda^{m+1}f(z)}{L_\lambda^m f(z)},\frac{L_\lambda^{m+2}f(z)}{L_\lambda^{m+1}f(z)},\frac{L_\lambda^{m+3}f(z)}{L_\lambda^{m+2}f(z)},\frac{L_\lambda^{m+4}f(z)}{L_\lambda^{m+3}f(z)};z\right),$$

is univalent in U, then

$$h(z)\prec\phi\left(\frac{L_\lambda^{m+1}f(z)}{L_\lambda^m f(z)},\frac{L_\lambda^{m+2}f(z)}{L_\lambda^{m+1}f(z)},\frac{L_\lambda^{m+3}f(z)}{L_\lambda^{m+2}f(z)},\frac{L_\lambda^{m+4}f(z)}{L_\lambda^{m+3}f(z)};z\right)$$

implies that

$$q(z)\prec\frac{L_\lambda^{m+1}f(z)}{L_\lambda^m f(z)} \qquad (z\in U)$$

4. A Set of Sandwich-Type Results

By combining Theorem (2) and (9), we obtain the following sandwich -type theorem.

Theorem (15): Let h_1 and q_1 be analytic function in U Also let h_2 be univalent function in U and $q_2\in\mathbb{Q}_0$ with $q_1(0)=q_2(0)=0$ and $\phi\in\aleph_l[h_2,q_2]\cap\aleph'_l[h_1,q_1]$. If the function $f\in T$ with $L_\lambda^m f(z)\in\mathbb{Q}_0\cap\mathcal{g}_0$ and the function

$$\phi\big(L_\lambda^m f(z),L_\lambda^{m+1}f(z),L_\lambda^{m+2}f(z),L_\lambda^{m+3}f(z);z\big),$$

is univalent in U, and if the condition (8) and (47) are satisfied, then

$$h_1(z)\prec\phi\big(L_\lambda^m f(z),L_\lambda^{m+1}f(z),L_\lambda^{m+2}f(z),L_\lambda^{m+3}f(z);z\big)\prec h_2(z)$$

Implies that

$$q_1(z)\prec L_\lambda^m f(z)\prec q_2(z) \qquad (z\in U). \tag{55}$$

If, on the order hand, we combine Theorem (5) and (12), we obtain the following sandwich-type theorem.

Theorem (16): let h_1 and q_1 be analytic function in U Also let h_2 be univalent function in U and $q_2 \in \mathbb{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \aleph_{l,1}[h_2, q_2] \cap \aleph'_{l,1}[h_1, q_1]$. If the function $f \in T$ with $\frac{L_\lambda^m f(z)}{z} \in \mathbb{Q}_1 \cap \mathcal{g}_1$ and the function

$$\phi\left(\frac{L_\lambda^m f(z)}{z}, \frac{L_\lambda^{m+1} f(z)}{z}, \frac{L_\lambda^{m+2} f(z)}{z}, \frac{L_\lambda^{m+3} f(z)}{z}; z\right),$$

is univalent in U, and if the condition (22) and (51) are satisfied, then

$$h_1(z) \prec \phi\left(\frac{L_\lambda^m f(z)}{z}, \frac{L_\lambda^{m+1} f(z)}{z}, \frac{L_\lambda^{m+2} f(z)}{z}, \frac{L_\lambda^{m+3} f(z)}{z}; z\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec \frac{L_\lambda^m f(z)}{z} \prec q_2(z) \qquad (z \in U). \tag{56}$$

Finally, by combining Theorem (7) and (14), we obtain the following sandwich-type theorem.

Theorem (17): let h_1 and q_1 be analytic function in U Also let h_2 be univalent function in U and $q_2 \in \mathbb{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \aleph_{l,2}[h_2, q_2] \cap \aleph'_{l,2}[h_1, q_1]$. If the function $f \in T$ with $\frac{L_\lambda^{m+1} f(z)}{L_\lambda^m f(z)} \in \mathbb{Q}_1 \cap \mathcal{g}_1$ and the function

$$\phi\left(\frac{L_\lambda^{m+1} f(z)}{L_\lambda^m f(z)}, \frac{L_\lambda^{m+2} f(z)}{L_\lambda^{m+1} f(z)}, \frac{L_\lambda^{m+3} f(z)}{L_\lambda^{m+2} f(z)}, \frac{L_\lambda^{m+4} f(z)}{L_\lambda^{m+3} f(z)}; z\right),$$

Is univalent in U, and if the condition (35) and (53) are satisfied, then

$$h_1(z) \prec \phi\left(\frac{L_\lambda^{m+1} f(z)}{L_\lambda^m f(z)}, \frac{L_\lambda^{m+2} f(z)}{L_\lambda^{m+1} f(z)}, \frac{L_\lambda^{m+3} f(z)}{L_\lambda^{m+2} f(z)}, \frac{L_\lambda^{m+4} f(z)}{L_\lambda^{m+3} f(z)}; z\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec \frac{L_\lambda^{m+1} f(z)}{L_\lambda^m f(z)} \prec q_2(z) \qquad (z \in U). \tag{57}$$

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Von Neumann Regular Semiring

Alaa Hussein Ali[(1)] **Asaad Mohammed Ali Alhossaini**[(2)]

College of Education for pure Sciences University of Babylon

alaaasd476@gmail.com asaad_hosain@itnet.uobabylon.edu.iq

Abstract. The aim of this action is a study and investigate "Von Neumann regular" semirings, some related concepts, e.g. reduced semirings; duo semiring, quasi-duo, and weakly duo semirings; regular, weakly regular and strongly regular semirings, also investigated. Some known results related to those concepts in rings were converted to semirings. Another aim of this paper is characterization Von Neumann Regular condition by the principal right ideal generated by an idempotent element.

Key words. Semirings**,** reduced semiring; duo, quasi-duo semiring, weakly duo Semiring; regular, weakly regular, strongly regular**;** Boolean semiring**;** semifield**;** Nilpotent

1. Introduction

The concept of **Von Neumann Regular** introduced in (ring theory) in 1936 by J. Von Neumann [1], also was studied in semirings, through much research [2], [3], [4], [5]. A semiring $\mathfrak{R}$ is referred to as **'simply regular'** or **'Von Neumann regular'** if $\forall a \in \mathfrak{R}\ \exists\ x \in \mathfrak{R} \ni axa = a$ [3]. " A non-empty set $\mathfrak{R}$ with two bilateral operations (+) and($\cdot$) is referred to as a semiring if:

(1) ($\mathfrak{R}$, +) is a commutative monoid with identity element 0;
(2) ($\mathfrak{R}$, $\cdot$) is a monoid with identity element $1 \neq 0$;
(3) Both the distributive laws hold in $\mathfrak{R}$;
(4) $a \cdot 0 = 0 \cdot a = 0$ for all $a \in \mathfrak{R}$ ". [6]

A nonempty subset I of a semiring $\mathfrak{R}$ is called a (left, right) **ideal** if $a, b \in I$ and $r \in \mathfrak{R}$ implies $a + b \in I$ and ($ra \in I, ar \in I$ respectively) [6]. An ideal I from a semiring $\mathfrak{R}$ is called **subtractive** if $a,\ a + b \in I, b \in \mathfrak{R}$ implies $b \in I$ [7]. A semiring $\mathfrak{R}$ is called **yoked** if for each x and y in the semiring $\mathfrak{R}$, $x + h = y$ or $x = y + h$ for some h in the semiring $\mathfrak{R}$ [8]. A semiring $\mathfrak{R}$ is called **cancellative** if for every $a, b, c \in \mathfrak{R}$ such that $a + c = b + c$ then $a = b$.[2]. This paper consisting of three sections. In **section one,** we study semirings which that contain no non-zero nilpotent elements; such semirings are called reduced semiring. We give some of their basic properties and provide some examples. **Section two** is devoted to exhibiting several preliminary results on duo semiring, quasi-duo semirings and weakly duo semiring. In **section three**, the properties and definitions of strongly regular, regular and weakly regular semiring were studied.

2. Reduced semiring

In this section, we study semirings that contain no non-zero nilpotent elements; such semirings are called reduced semirings. We give some of their basic properties and provide some examples.

7 Definition 2.1. (see [6], p. 43)

An element x at a semiring $\mathfrak{R}$ is referred to as "nilpotent" iff there exists a positive integer n satisfying $x^n = 0$. We will denote the set for every nilpotent elements from $\mathfrak{R}$ by N.

Lemma 2.2. (see [6], p. 43, 44)

Let N be the set of all nilpotent elements of $\mathfrak{R}$, then N is an ideal of $\mathfrak{R}$.

Definition 2.3. [9]

A semiring $\mathfrak{R}$ is referred to as "reduced" if $\mathfrak{R}$ contains no non-zero nilpotent elements.

Example 1

The semiring of integers modulo 6, $\mathbb{Z}_6$ is reduced while $\mathbb{Z}_8$ is not reduced, since 2,4,6 are nilpotent elements of $\mathbb{Z}_8$.

Definition 2.4. [10]

"A right annihilator" of a non-zero element a in a semiring $\mathfrak{R}$ is defined by

$r(a) = \{b \in \mathfrak{R} : ab = 0\}$.

A left annihilator $l(a)$ is similarly defined.

Proposition 2.5. [9]

Let $\mathfrak{R}$ be "a reduced semiring" . Then, for every $a \in \mathfrak{R}$

1- $r(a)=l(a)$

2- $r(a)=r(a^2)$

3- $\mathfrak{R}/r(a)$ is reduced

Definition 2.6. [1]

An ideal I from a semiring $\mathfrak{R}$ is referred to as essential if and only if $I \cap H \neq 0$ for every nonzero ideal of $\mathfrak{R}$.

Example 2

1- Let $\mathbb{Z}_8$ be the semiring of (integers modulo 8) and I=(2), J=(4), then I and J are essential ideals in $\mathbb{Z}_8$.

2- Let $\mathbb{Z}_6$ be the semiring of (integers modulo 6), then I=(2) is not essential in $\mathbb{Z}_6$.

3- Let $\mathfrak{R} = (\mathbb{N} \cup \{\infty\}, \min, +)$ be the semiring wherever $\mathbb{N}$ is the natural numbers, thus the ideals from $\mathfrak{R}$ are the form $I = \{n, n+1, \ldots\} \cup \{\infty\}$ or $\{\infty\}$

Since :

(i) I = { n , $n+1$, $n+2$, … } ∪ {∞} closed under addition (min).

(ii) Let r be any element belongs to $\mathfrak{R}$ and a be any element belongs to I, $r+a \geq n$, then $r+a \in I$ this implies closed under multiplication by elements of $\mathfrak{R}$ (+).

Now, suppose J is a non-zero ideal contained in $\mathfrak{R}$. indeed :

(i) $\infty \in \mathfrak{R}$, $a \in J$ then $\infty + a = \infty \in J$

(ii) Let n be the smallest element of J. Then J is an ideal, $1 \in \mathfrak{R}$ implies $1+n \in J$.

Then $\{n, n+1, n+2, \ldots\} \subseteq J \rightarrow J = \{n, n+1, \ldots\} \cup \{\infty\}$.

On the another hand every non-zero ideal from $\mathfrak{R}$ is essential from $\mathfrak{R}$ (the zero element from $\mathfrak{R}$ is ∞), if $J \cap K = \{\infty\} \rightarrow$ either $J = \{\infty\}$ or $K = \{\infty\}$

$J = \{n, n+1, \ldots\} \cup \{\infty\}$, $K = \{m, m+1, \ldots\} \cup \{\infty\}$.

$J \cap K = J$ if $n > m$ or $J \cap K = K$ if $n < m$ → if $J \neq \{\infty\}$ and $K \neq \{\infty\}$. Then $J \cap K \neq \{\infty\}$.

Definition 2.7. [1]

Let *x* an element in a semiring $\mathfrak{R}$. Then *x* is referred to as "**a right singular**" iff *r*(*x*) is essential ideal in $\mathfrak{R}$. The set of all "right singular elements" in $\mathfrak{R}$ is denoted by "$rZ(\mathfrak{R})$".

A left singular ideal, denoted by $lZ(\mathfrak{R})$, is similarly defined.

Example 3

1-Let Z_{12} be the semiring of integers modulo 12. Then *r*(6) and *r*(0) are the only essential ideals in Z_{12}. Therefore, $rZ(\mathfrak{R})=lZ(\mathfrak{R})=\{0,6\}$.

2-By(Example 2(3)), if m ≠ ∞ ,

then $r(m) = \{ k \in \mathbb{N}\cup\{\infty\} \mid m+k = \infty\} = \{\infty\}$ not essential in $\mathfrak{R}$.

$r(\infty) = \{ k \in \mathbb{N}\cup\{\infty\} \mid \infty + k = \infty\} = \mathfrak{R}$ essential in $\mathfrak{R}$. This implies that $rZ(\mathfrak{R}) = \{\infty\}$.

The following result is analogous to one in ring theory **(see [11]),** but we will give another proof.

Proposition 2.8.

If $lZ(\mathfrak{R})$ contains no non-zero nilpotent elements, then $lZ(\mathfrak{R})=0$.

Proof:

Since $lZ(\mathfrak{R}) \neq 0$, then there exists $0 \neq z \ni l(z)$ essential in $\mathfrak{R}$.

Thus $l(z) \cap \mathfrak{R}x \neq 0$ for each $x \in \mathfrak{R}$. In particular when $x=z$, then there exists $rz \in l(z) \cap \mathfrak{R}z$ with $rz \neq 0$. So, $(rz)z = 0$, $(zrz)^2 = (zrz)(zrz) = z(rz^2)rz = 0 \rightarrow zrz \in Z(\mathfrak{R})$ and nilpotent $\rightarrow zrz = 0 \rightarrow (zr) \in l(z)$. Now, $(rz)^2 = (rz)(rz) = r(zrz) = 0 \rightarrow (rz) \in Z(\mathfrak{R})$ and rz is nilpotent $\rightarrow rz = 0$, and this is a contradiction. Implies that $lZ(\mathfrak{R}) = 0$. □

By a similar argument in **[12],** the following result can be proved.

Corollary 2.9.

Let $\mathfrak{R}$ be a reduced semiring. Then $lZ(\mathfrak{R})= rZ(\mathfrak{R})=0$.

Remark

It is clear that, if $\mathfrak{R}$ is "a commutative semiring", and K the set for each "nilpotent elements" of $\mathfrak{R}$. Then $\mathfrak{R}/N$ is "a reduced semiring".

3. Duo and quasi-duo semiring

The present section is devoted to exhibiting several preliminary results on "duo semirings", "quasi-duo semirings", and "weakly duo semirings". We shall begin this section with the following definition.

Definition 3.1. [7]

The semiring $\mathfrak{R}$ is referred to as right (left) duo if every right (left) ideal of $\mathfrak{R}$ is a two-sided ideal.

The following definition is analogous to a similar one in ring theory **(see [13])**

Definition 3.2.

A semirings $\mathfrak{R}$ is referred to as "left (right) quasi-duo" if each maximal (left) right ideal of $\mathfrak{R}$ is a two-sided ideal.

A right (quasi-duo) semiring form a non-trivial generalization of right duo semiring.

Definition 3.3. [14]

An element x of a semiring $\mathfrak{R}$ is (a unit) if and only if there exists (a necessarily unique) element x^{-1} of $\mathfrak{R}$ satisfying $xx^{-1}= 1 = x^{-1}x$.

The following definition is analogous to a similar one in ring theory **(see [13])**

Definition 3.4.

A semiring $\mathfrak{R}$ is referred to as (weakly right (left) duo), if for every $x\in \mathfrak{R}$, there exists a positive integer m such that $x^{m}\mathfrak{R}$ $(\mathfrak{R}x^{m})$ is a two-sided ideal of $\mathfrak{R}$.

Note that, every "weakly right (left) duo semiring" is "right (left) quasi-duo".

Definition 3.5. [10]

"The Jacobson radical" of a semiring $\mathfrak{R}$, denoted by $J(\mathfrak{R})$, is the set

$J(\mathfrak{R})= \cap\{M: M$ is a maximal ideal of $\mathfrak{R}\}$.

Definition 3.6. [7]

A semiring $\mathfrak{R}$ is called semi-simple if $J(\mathfrak{R}) = 0$.

Corollary 3.7. [15]

Any proper ideal of a semiring $\mathfrak{R}$ is a subset of a maximal ideal of $\mathfrak{R}$.

The following result is analogous to a similar one in ring theory **(see [16], p. 109),**

Lemma 3.8.

$J(\mathfrak{R}) = \{a \in \mathfrak{R} \mid \mathfrak{R}a \ll \mathfrak{R}\}$.

Proof :

$(\Longrightarrow)$ $\mathfrak{R}a \ll \mathfrak{R}$, C is a maximal ideal of $\mathfrak{R}$, such that $a \notin C \rightarrow \mathfrak{R}a + C = \mathfrak{R} \rightarrow a\mathfrak{R}$ is not small in $\mathfrak{R}$, a **contradiction**. This implies $a \in \bigcap C$, where C is a maximal ideal of $\mathfrak{R}$.

(⟸) Let $a \in \bigcap C$, where *C* is a maximal ideal of $\mathfrak{R}$. Assume $a\mathfrak{R}+U=\mathfrak{R}$, for some proper ideal *U* of $\mathfrak{R}$. We can assume that *U* is a maximal ideal of $\mathfrak{R}$ by **corollary(3.7.).** But $a \in U \rightarrow \mathfrak{R}a \subseteq U \rightarrow \mathfrak{R}a + U = U \neq \mathfrak{R}$, **a contradiction**. Therefore $\mathfrak{R}a \ll \mathfrak{R}$. □

The following result is analogous to a similar one in ring theory **(see [17])**

Proposition 3.9.

Let $\mathfrak{R}$ be "a right quasi-duo semiring". Then $\mathfrak{R}/J(\mathfrak{R})$ is "a reduced semiring".

Proof :

It is enough to prove that any nilpotent element belongs to $J(\mathfrak{R})$. That is, to prove if $x \in \mathfrak{R}$ and $x^m = 0$ for some $m \in Z^+$, then $x \in J(\mathfrak{R}) = \{x \in \mathfrak{R} | \mathfrak{R}x \ll \mathfrak{R}\}$ by **lemma(3.8.).** Suppose that $\mathfrak{R}a + K = \mathfrak{R}$ where *K* is a left ideal from $\mathfrak{R}$, we want to show that $K=\mathfrak{R}$ which implies $x \in J(\mathfrak{R})$, by **corollary (3.7.),** we can assume that *K* is a maximal ideal of $\mathfrak{R}$, multiplying both side by *x* from right we get $\mathfrak{R}x^2 + Kx = \mathfrak{R}x \rightarrow \mathfrak{R}x^2 + Kx + K = \mathfrak{R}$, continuing in this way, we end up with $Kx^{n-1} + \cdots + Kx^2 + Kx + K = \mathfrak{R}$, $(\mathfrak{R}x^n = 0)$. Since $\mathfrak{R}$ is "left quasi-duo", and *K* is maximal, then hence $K=\mathfrak{R}$. □

4. Regular, Strongly Regular, Weakly Regular

In this section, the definitions, and properties of regular, weakly regular and strongly regular semirings are given.

Definition 4.1. [2]

A semiring $\mathfrak{R}$ is said to be "Von Neumann Regular" if, for any $x \in \mathfrak{R}$, there exists $y \in \mathfrak{R}$ such that $x = xyx$.

The following definition is analogous to a similar one in ring theory **(see [18])**

8 **Definition 4.2.**

A semiring $\mathfrak{R}$ is said to be unit regular if, for every $a \in \mathfrak{R}$, there exists a unit *u* in $\mathfrak{R}$ such that $a=aua$.

Definition 4.3. [19]

"A commutative semiring" $\mathfrak{R}$ is referred to as (a semifield) if each non-zero element in $\mathfrak{R}$ has a (multiplicative) inverse in $\mathfrak{R}$.

Definition 4.4. (see [6], p. 7)

The Boolean semiring is the commutative semiring $B = \{0,1\}$, formed by the two-elements, and defined by $1 + 1 = 1$.

Example 1

1- Every semifield is regular.
2- Every Boolean semiring is regular.

3- Let $\mathbb{R}^+ = \{r \geq 0 | r \in \mathbb{R}\}$ be the semiring and $\mathfrak{R}_1 = \begin{bmatrix} \mathbb{R}^+ & \mathbb{R}^+ \\ 0 & \mathbb{R}^+ \end{bmatrix}$, it's clear that $\mathfrak{R}_1$ is "a non-commutative semiring" with identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, but not "regular" because

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ for all } \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathfrak{R}_1.$$

The following result is analogous to a similar one in ring theory **(see [20])**

9 **Lemma 4.5.**

Let $v \in \mathfrak{R}$, if v is "unit regular", then $v=eu$ for some idempotent element e and some unit element u.

Proof:

Suppose that x is a unit regular. Then there exists a unit $v \in \mathfrak{R}$ such that $xvx=x$. Let $e=xv$. Then $e^2=xvxv=xv=e$, so e is an idempotent element of $\mathfrak{R}$. Let $u=v^{-1}$, then we have $x=eu$.*

The following definition is analogous to a similar one in ring theory **(see [21])**

Definition 4.6.

A semiring $\mathfrak{R}$ is referred to as CI-semiring if each idempotent element from $\mathfrak{R}$ is central, ($a \in \mathfrak{R}$ is central if $ab = ba\ \forall\ b \in \mathfrak{R}$).

The following definition is analogous to a similar one ring theory **(see [22]).**

Definition 4.7.

A semiring $\mathfrak{R}$ is called "strongly regular" if, for each $r \in \mathfrak{R}$, there exists

$s, t \in \mathfrak{R}$ such that $r=r^2s=tr^2$.

Remark 2

Every "strongly regular semiring" is "regular". (clear)

We call $\mathfrak{R}$ π-regular (unit π-regular) semiring if for any $x \in \mathfrak{R}$, there exists a positive integer m and an element y (a unit u) of $\mathfrak{R}$ such that $x^m=x^m y x^m$ ($x^m=x^m u x^m$).

Lemma 4.8.

Let $\mathfrak{R}$ a semiring. Then the following statements equivalent conditions:

1- $z^m \in z^{m+1}\mathfrak{R}$ for some integer $m \geq 1$.

2- $z^m\mathfrak{R} = z^{m+1}\mathfrak{R}$ for some integer $m \geq 1$.

3-The chain $z\mathfrak{R} \supseteq z^2\mathfrak{R} \supseteq$.... terminates.

9.1 Proof:

9.2 (1)→(2), $z^{m} \in z^{m+1}\mathfrak{R} \rightarrow z^{m} = z^{m+1}r$ **for some** $r \in \mathcal{R}$, $z^{m}s = z^{m+1}rs \in z^{m+1}\mathfrak{R} \rightarrow z^{m}\mathfrak{R} \subseteq z^{m+1}\mathfrak{R}$, $z^{m+1}r = z^{m}(zr) \in z^{m}\mathfrak{R} \rightarrow z^{m+1}\mathfrak{R} \subseteq z^{m}\mathfrak{R}$.

9.3 (2)→(3), trivial.

9.4 (3)→(1), trivial. □

Definition 4.9.

9.5 An element z in a semiring $\mathfrak{R}$ is called right π-regular if, it satisfies the equivalent conditions in lemma(4.8.)

Definition 4.10.

An element $k \in \mathfrak{R}$ is referred to as (strongly π-regular) if it is both left and right "π-regular", and $\mathfrak{R}$ is referred to as "a strongly π-regular semiring" if each element is "strongly π-regular".

REMARK 3

9.6 Every strongly π-regular semiring is π-regular .(clear)

Definition 4.11. [3]

A semiring $\mathfrak{R}$ is referred to as right (left) "weakly regular" if H^2 =*H* for each right (left) ideal *H* of $\mathfrak{R}$, equivalently "if $w \in w\mathfrak{R}w\mathfrak{R}$ ($w \in \mathfrak{R}w\mathfrak{R}w$) for every $w \in$ □". $\mathfrak{R}$ is referred as to "weakly regular" if it is both right and left "weakly regular".

Remark 4

Every "regular semiring" is "weakly regular".

In case $\mathfrak{R}$ is commutative semiring then $\mathfrak{R}$ is regular if and only if $\mathfrak{R}$ is weakly regular.**[3]**

The following result is analogous to a similar one ring theory **(see [23])**

Proposition 4.12.

Let $\mathfrak{R}$ be a right weakly regular, cancellative and yoked semiring. Then

$\mathfrak{R}=\mathfrak{R}a\mathfrak{R}$ for any right non-zero divisor element *a* of $\mathfrak{R}$.

Proof :

Let a be a right non-zero divisor element of $\mathfrak{R}$. Then $a\mathfrak{R}=(a\mathfrak{R})^2$ (since $a \in (a\mathfrak{R})^2$). Assume that $rat \in \mathfrak{R}a\mathfrak{R}$, then by yoked property either $1 + h = rat$ or $1 = rat + h \ldots (1) \rightarrow a + ah = arat \; or \; a = arat + ah$, since a and $arat \in (a\mathfrak{R})^2$, then by subtractive, we get $ah \in (a\mathfrak{R})^2 \rightarrow ah = auav$ for some $u, v \in \mathfrak{R}$. Again, by yoked property either $h = s + uav \; or \; h + s = uav \; for \; some \; s \in \mathfrak{R} \rightarrow as + auav = auav \; or \; ah + as = ah.$

By cancellative property, we have $as = 0 \rightarrow s = 0(a\ is\ \text{non} - \text{zero divisor}) \rightarrow h = uav \in \Re ah$, then by(1) 1$\in \Re a\Re \rightarrow \Re a\Re = \Re$. □

The following definition is analogous to a similar one in ring theory **(see [24])**

Definition 4.13.

A semiring $\Re$ is called right (left) "weakly π-regular" if $\forall x\in \Re$ there exists a natural number n such that $x^n \in x^n\Re x^n\Re$ ($x^n \in \Re x^n\Re x^n$), $\Re$ is "weakly π-regular" if it is both right and left "weakly π-regular".

Remark 5

Every "π-regular semiring" is "weakly π-regular".

The following result is analogous to a similar one in ring theory **(see [23])**

Proposition 4.14.

For a semiring $\Re$, the following are equivalent :

(a) $\Re$ is "Von Neumann regular".

(b) For each a in $\Re$, there exists an "idempotent" e in $\Re$ such that $a\Re = e\Re$.

Proof:

(a)$\Longrightarrow$ (b) Since $\Re$ is a "Von Neumann regular semiring", then for every element a in $\Re$ there exists an element b in $\Re$ such that $a=aba$. Now we put $e=ab$ yields $e=e^2$ for some e in $\Re$, $a\Re = e\Re$ (since ; $a\Re = (aba)\Re = ea\Re \subseteq e\Re, e\Re = (ab)\Re \subseteq a\Re$).

(b)$\Longrightarrow$ (a) assume $a\Re=e\Re$ where e is an idempotent element. Then $a=ex$ for some x in $\Re$.

Now, $a=ex=e^2x=ea$. Let $e=ab$(since $e \in a\Re \rightarrow e = ab$) we get $a=aba$. So $\Re$ is Von Neumann regular semiring. □

Definition 4.15. [10]

An ideal I from the semiring $\Re$ is referred to as "direct summand" of $\Re$ if there exists an ideal J of $\Re$ such that $\Re=I+J$ and $I \cap J=0$. We usually write $\Re=I \oplus J$.

The following result is analogous to similar one in ring theory **(see [23])**

Proposition 4.16.

A cancellative and yoked semiring $\Re$ is " Von Neumann regular " if and only if every principal right ideal of $\Re$ is a direct summand.

Proof:

Let $\Re$ be a von Neumann regular semiring, if $0\neq a \in\Re$, then by proposition **(4.14)** $a\Re = e\Re$ for some idempotent element e of $\Re$. To prove $e\Re$ is a direct summand of $\Re$. Assume that e is an idempotent element of $\Re$ and $I = e\Re$. If e is a not zero-divisor, then $f: \Re \rightarrow e\Re$ defined by $r \longmapsto er$

is an isomorphism, so, $e\mathfrak{R}$ is "a direct summand" of $\mathfrak{R}$. If e is a zero-divisor, and $eu = 0$ (for some $u \in \mathfrak{R}$).

Claim: $\mathfrak{R} = e\mathfrak{R} + u\mathfrak{R}$ for some u such that $eu = 0$. We need to consider that $\mathfrak{R}$

is yoked. In this case either $e + u = 1$ or $e = 1+u$ for some $u \in \mathfrak{R}$. If $e + u = 1$,

then $\mathfrak{R} = e\mathfrak{R} + u\mathfrak{R}$ and since $e\,(e+u) = e \rightarrow e^2+eu = e \rightarrow e + eu = e \rightarrow eu = 0$.

$x \in e\mathfrak{R} \cap u\mathfrak{R} \rightarrow x = er = us$ for some $r, s \in \mathfrak{R}$. $x = er \rightarrow ex=er=x$ and $ex=eus=0$, so $x=0$. Then $\mathcal{R} = e\mathfrak{R} \oplus u\mathfrak{R}$. In case $e = 1+u$, also we get $eu = 0$, too, and

$e\mathfrak{R} \cap u\mathfrak{R} = 0$. On the other hand $e = 1+u \rightarrow 0 = eu = u+u^2 \rightarrow e+u+u^2 = 1+u$, by cancellative property then $1 = e + u^2 \rightarrow r = er + u^2r \in e\mathfrak{R} + u\mathfrak{R}$; $\forall r \in \mathfrak{R} \rightarrow \mathfrak{R} = e\mathfrak{R} \oplus u\mathfrak{R}$. Therefore $I = \mathfrak{R}e$ is "a direct summand" of $\mathfrak{R}$.

Conversely, let $\mathfrak{R} = a\mathfrak{R} \oplus K$, for some ideal K of $\mathfrak{R}$. Now $1=ar+k$ for some r in $\mathfrak{R}$ and k in K, and $a=ara+ka$, but $ka \in a\mathfrak{R} \cap K = 0$ implies that $a=ara$ and $\mathfrak{R}$ is "Von Neumann regular" semiring. □

The following result is analogous to similar one in ring theory **(see [17])**

Proposition 4.17.

Let $\mathfrak{R}$ be "a right duo semiring". The following statements are equivalent:

1- $\mathfrak{R}$ is a right "weakly regular semiring" ;

2- $\mathfrak{R}$ is "a strongly regular semiring".

3- $\mathfrak{R}$ is "Von Neumann regular";

Proof :

(1)→(2). By **proposition(4.12.)** $\mathfrak{R}=\mathfrak{R}a\mathfrak{R} \rightarrow 1=rat \rightarrow a=arat \rightarrow a=a(as)t$, for some $s \in \mathfrak{R}$, then $a = a^2st \rightarrow a=a^2b$, where $b=st$.

(2)→(3), $\mathfrak{R}$ is strongly regular, then for each $a \in \mathfrak{R}\ \exists\ b,c$ such that $a=a^2b=ca^2$

$\rightarrow a=aab=aba.(ab = ba$, since $\mathfrak{R}$ is a right duo semiring).

(3)→(1), $\forall\ a \in \mathfrak{R}\ \exists\ b \ni a=aba \rightarrow ar = abar \rightarrow a \in a\mathfrak{R}a\mathfrak{R}$

This implies $\mathfrak{R}$ is a right "weakly regular semiring". □

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A Study of Equicontinuous Maps On Uniform G −Spaces

By

Ansam A A [1,2] Ihsan J K [2,3]

[1] ansan.a.ali1987@gmail.com

[2] University of Al- Qadisiyah, College of Science, Department of Mathematics

[3] Ihsan.kadhim@qu.edu.iq

Abstract. In this paper we shall study some new properties of equicontinuous maps on uniform G −Spaces. Here the phase space consider as a uniform space. Also we show the relationship among the equicontinuous maps with the distal dynamical system and expansive dynamical system.

1.Introduction

One of the most significant in the investigation of the hypothesis of dynamical framework is equicontinuous dynamical framework. Numerous creators have been examined the dynamical ideas in a measurement space or in topological space.

R. Das (2012) [1] characterize and study the mayhem of a grouping of maps in "a metric G−space". Additionally, he [2] characterize while research a idea of G−transitive subset for a ceaseless guide upon the smaller metrical G−space.

R. Das [3] (2013) get enough status beneath that consequence from pair maps, at that singular is "Devaney's" G1-befuddled while else is "Devaney's" G2-scattered, is "Devaney's" G1 × G2- chaotic.

R. Das , T. Das [4] (2012) describe and research the thoughts from determinedly and antagonistically "G-asymptotic" spotlights at a homeomorphism by a "metric G-space". Furthermore, in [5] (2012) they describe and research a possibility of "topological transitivity" of an industrious self- chart during the "metric G−space" named like "topologically G-transitive" guide and secure hers depiction.

P. Das and T. Das [6] (2019) show that the course of action of concentrates dual asymptotic into a dot hold measure zero concerning every expansive outside common mensuration to a bi-quantifiable guide on a discernable "uniform space".

I. J. Kadhim and S. K. Jebur [7] (2017) they study the some acclaimed dynamical thoughts, for instance, tricky transitive mixture while equicontinuous at a general topological.

E. Shah and T. Das [8] (2013) portray while research the idea from inconsequentiality while detail for self a homeomorphism from a "metric G−space X". utilize "G−minimality", they get a category concerning maps that don't contain a "G−shadowing property". Further, get the enough event into "G−expansive homeomorphisms" and "G−shadowing property" to have "G−specification property".

Here, we will concentrate some new properties of equicontinuous maps on uniform G-Spaces. In Sec.2, a few ideas related with the uniform space a few properties of a uniform space that required in our work are state. Sec. 3 comprises of the primary aftereffects of our work.

2. Uniform space

Disregard X a set. mean by Δ_X a corner to corner in (X×X), in order to is a set Δ_X={(x,x):x∈X}. a regressive U^(- 1) of the subcategory U⊂X×X is a subcategory of X×X described via U^(- 1)={(x,y):(y,x)∈U}. On decision in order to U is symmetric if U^(- 1)=U. we get U∩U^(- 1) symmetric into each U⊂X×X. We describe the combined U∘V of pair subsets (U ,V) of X×X by U∘V={(x,y):found z∈X to that a degree, such (x,z)∈U and (z,y)∈V} ⊂X×X.

Definition 2.1[10] suppose X is a set. A "uniform structure" on X be a invalid combination U containing subsets of the Cartesian square (X×X) satisfactory a going with situations:

[UN-1] if $U \in \boldsymbol{U}$, then $\Delta_X \subset U$;

[UN-2] if $U \in \boldsymbol{U}$ and $U \subset V \subset X \times X$, then $V \in \boldsymbol{U}$;

[UN-3] if $U \in \boldsymbol{U}$ and $V \in \boldsymbol{U}$,then $U \cap V \in \boldsymbol{U}$;

[UN-4] if $U \in \boldsymbol{U}$,then $U^{-1} \in \boldsymbol{U}$;

[UN-5] if $U \in \boldsymbol{U}$, then there exists $V \in \boldsymbol{U}$ such that $V \circ V \subset U$.

a segments of U is known as the escort of the "uniform structure" while the set X is known as a " uniform space". The consistency U is called segregating (and X is said to be disconnected) if ∩{U:U∈U}=Δ.

annotation in order to the events [UN-3], [UN-4] and [UN-5] propose that, into each organization U found a symmetric escort V with the ultimate objective in order to V∘V⊂U. Disregard X a set while put U⊂X×X. specified a point x∈X, describe a subset U_([x])⊂X by U_([x])={y∈X:(x,y)∈U}.

In case X is" a uniform space", thither a started topology on X charactrized via a way in order to the regions from an emotional dot x∈X include the sets U_([x]) , wherever U works onto every organizations of X. This topology is "Hausdorff " if while just if the interchange purpose of the impressive number of escorts of X is rduced to one side Δ_X.

If (X,d) be an estimation space, thither a trademark uniform build upon X whom organization are the sets U⊂X×X satisfactory the going with situation: found an authentic numeral ε>0such that U involves every paire (x,y)∈X×X with the ultimate objective in order to d(x,y)<ε. The topology related together and such orderly build is subsequently comparable to the topology started by the estimation.

Theorem 2.2 [10] (a) For every x∈X, the assortment ℵ_x={U_([x]) ∶U∈U} structure a local base at x∈X, making X a topological space. A similar topology is delivered if any base B is utilized instead of U. (b) the topology is Hausdorff if and just if U is isolated.

Theorem 2.3[10] The consistency U is isolated if and just if for each x,y∈X with x≠y, there exists U∈U to such an extent that (x,y)∉U.

Corollary 2.4[10] The topology is Hausdorff if and just if for every x,y∈X with x≠y, found U∈U to this an extent that (x,y)∉U.

Definition 2.5. [10]Let (X,U) while (Y,V) be "uniform spaces". A capacity f:X→Y is told into be uniform persistent if for every V∈V, there is some U∈U with the end goal that (x,y)∈U implis that (f(x),f(y))∈V. On the off chance that f is one-one, onto and both f and f^{-1} are uniform persistent, we consider f a uniform isomorphism (uniform comparability) and state X and Y are consistently isomorphic (consistently proportionate). Each consistently constant capacity is nonstop and thus every uniform isomorphism is "homeomorphism".

Definition 2.6. [10] suppose (X,U) and (Y,V) are duo uniform spaces. A mapping f:X→X is told into be uniform equicontinuous on X if for each company V∈V and for each positive number n, fund an escort U∈U to such an extent that

(x,y)∈U infers (f^n (x),f^n (y))∈V.

Obviously that any self-nonstop guide is uniform equicontinuous however the opposite need not be valid.

Definition 2.7 [10] let (X,U) and (Y,V) are "uniform spaces". By subsequently the consequence from (X,U) and (Y,V) is a "uniform space" (Z,W) together the concealed set Z=X×Y while the consistency W on Z whom basis involves the sets

W_(U,V)={((x,y),(x^',y^'))∈Z×Z:(x,x^')∈U,(y,y^')∈V},

wherever U∈U and V∈V. a consistency W is known as the consequence of U , V and is made as W=U×V.

3- Main Results

Right now idea of equicontinuous, sweeping and distal maps in a uniform G-space are presented and some new properties of such ideas are demonstrated.

Definition 3.1[9] through a "G-space" we purpose a triplex (X,G,θ), wherever X is a "Hausdorff space", G is a topological social occasion and θ:G×X→X is a perpetual movement of G on X.

Definition 3.2 The 4-tuple (X,G,U,θ) is said to be Uniform G-space if (X,G,θ) is G-space and (X,U) is uniform space.

For simplest, we shall indicate for $(X,G,\boldsymbol{U},\theta)$ by .

Definition 3.3 The pair of maps

(μ,ψ):(G_1,X,U,θ_1)→(G_2,Y,V,θ_2)

is said to be uniform homomorphism between the two uniform spaces (G_1,X,U,θ_1) and (G_2,Y,V,θ_2) if

(I) μ:G_1→G_2 is topological gathering homomorphism,

(ii) ψ:X→Y is uniform consistent guide and

(iii) ψ(θ_1 (g,x))=θ_2 (μ(g),ψ(x)).

Definition 3.4 suppose X is a uniform G-space. A uniform ceaseless mapping f:X→X is said to be uniform G-equicontinuos on X if for each escort V∈U and for each positive whole number n, there exists a company U∈U to such an extent that

(x,y)∈U infers (f^n (θ(g,x),f^n (θ(p,y))∈V, g,p∈G.

Remark 3.5 beneath the paltry activity of G on X the thoughts of "uniform equicontinuous" and "uniform G-equicontinuous" are agreed.

Theorem 3.6 Suppose X and Y is a "uniform G-spaces" and h_1:X→X , h_1:Y→Y be "equivariant topologically" conjugate by means of φ:X→Y. In the event that h_1 is "uniform G-equicontinuous", at that point so is h_2.

Proof. let h_1 is "uniform $G-$equicontinuous". Let $V \in \boldsymbol{\mathcal{V}}$. Since φ is uniform isomorphism, so we found $U \in \boldsymbol{\mathcal{U}}$ Like that

$$(x_1, x_2) \in U \quad \text{implies} \quad (\varphi(x_1), \varphi(x_2)) \in V. \qquad (1)$$

while $h_1 : X \to X$ is uniform $G-$equicontinuous, so we found an entourage $\widetilde{U} \in \boldsymbol{\mathcal{U}}$ and $g, p \in G$ such that

$$(\tilde{x}_1, \tilde{x}_2) \in \widetilde{U} \quad \text{implies} \quad (h_1^n(\theta(g, \tilde{x}_1)), h_1^n(\theta(p, \tilde{x}_2))) \in U. \qquad (2)$$

Since $\varphi^{-1} : Y \to X$ is uniform continuous, subsist $\tilde{V} \in \boldsymbol{\mathcal{V}}$ Like that

$$(\tilde{y}_1, \tilde{y}_2) \in \tilde{V} \quad \text{denote}(\varphi^{-1}(\tilde{y}_1), \varphi^{-1}(\tilde{y}_2)) \in \widetilde{U}. \qquad (3)$$

By (2) we have

$$(h_1^n(\theta(g, \varphi^{-1}(\tilde{y}_1))), h_1^n(\theta(p, \varphi^{-1}(\tilde{y}_2)))) \in U.$$

By (1) we have

$$(\varphi h_1^n \left(\theta(g, \varphi^{-1}(\tilde{y}_1))\right), \varphi h_1^n \left(\theta(p, \varphi^{-1}(\tilde{y}_2))\right)) \in U.$$

Since h_1 , h_2 be equivariant topologically conjugate via φ, then

$$h_1^n(\theta(g, \varphi^{-1}(y))) = \varphi^{-1}(\sigma(\tilde{g}, h_2^n(y)), \text{ for every } y \in Y \text{ and } \tilde{g} \in G_1$$

$$= \varphi^{-1}(h_2^n(\sigma(\tilde{g}, y))$$

Thus

$$(\varphi\varphi^{-1}(h_2^n(\sigma(\tilde{g}, \tilde{y}_1)), \varphi\varphi^{-1}(h_2^n(\sigma(\tilde{p}, \tilde{y}_2))) \in U.$$

This means that $(h_2^n(\sigma(\tilde{g}, \tilde{y}_1)), h_2^n(\sigma(\tilde{p}, \tilde{y}_2)) \in U$. Consequently h_1 is uniform $G-$equicontinuous.

Theorem 3.7. suppose X and Y be uniform spaces and h_1:X→X , h_1:Y→Y be equivariant topologically conjugate by means of φ:X→Y. In the event that h_1 is uniform equicontinuous, at that point so is h_2.

Proof. Let h_1 is uniform equicontinuous. Let $V \in \boldsymbol{\mathcal{V}}$. Since φ is uniform isomorphism, at that point there exists an escort U∈U with the end goal that

$$(x_1, x_2) \in U \quad \text{implies} \quad (\varphi(x_1), \varphi(x_2)) \in V. \qquad (1)$$

Since $h_1: X \to X$ is uniform equicontinuous, at that point there exists an escort $\widetilde{U} \in \boldsymbol{\mathcal{U}}$ with the end goal that

$$(\tilde{x}_1, \tilde{x}_2) \in \widetilde{U} \quad \text{implies} \quad (h_1^n(\tilde{x}_1), h_1^n(\tilde{x}_2)) \in U. \qquad (2)$$

Since $\varphi^{-1}: Y \to X$ is uniform continuous, subsist $\tilde{V} \in \boldsymbol{\mathcal{V}}$ same that

$$(\tilde{y}_1, \tilde{y}_2) \in \tilde{V} \quad \text{suggest}(\varphi^{-1}(\tilde{y}_1), \varphi^{-1}(\tilde{y}_2)) \in \widetilde{U}. \qquad (3)$$

By (2) we have

$$(h_1^n(\varphi^{-1}(\tilde{y}_1)), h_1^n(\varphi^{-1}(\tilde{y}_2))) \in U.$$

By (1) we have

$$(\varphi h_1^n(\varphi^{-1}(\tilde{y}_1)), \varphi h_1^n(\varphi^{-1}(\tilde{y}_2))) \in U.$$

Since h_1 , h_2 be equivariant topologically conjugate via φ, then

$$h_1^n\big(\varphi^{-1}(y)\big) = \varphi^{-1}(h_2^n(y)), \text{ for every } y \in Y.$$

Thus

$$(\varphi\varphi^{-1}(h_2^n(\tilde{y}_1)), \varphi\varphi^{-1}(h_2^n(\tilde{y}_2))) \in U.$$

This means that $(h_2^n(\tilde{y}_1), h_2^n(\tilde{y}_2)) \in U$. Consequently h_1 is uniform equicontinuous.

Here the relation between the (G-) equicontinuous and (G-)**expansive** is studied in uniform space. First we shall introduce the concepts of **expansive and G- expansive in uniform space.**

Definition 3.8 In the event that (X,U) is a "uniform space" and h∈H(X) at that point h is called far reaching, on the off chance that there exists an escort U∈U with the end goal that at whatever point x,y∈X,x≠y, at that point found a whole number n fulfilling

(h^n (x),h^n (y))∉U;

U is then named a far reaching escort for h.

Definition 3.9 suppose (X,U) be a uniform space and h∈H(X) at that point h is called uniform G-far reaching, in the event that there exists an escort U∈U with the end goal that at whatever point x,y∈X,G(x)≠G(y) at that point found a number n fulfilling

(h^n (u),h^n (v))∉U, for all u∈G(x) and ∈G(y) .

Theorem 3.10 put (X,U) be a uniform space and f∈H(X). On the off chance that f is equicontinuous map, at that point its sweeping.

Proof. Assume that f is uniform equicontinuous. Let x,y∈X with x≠y. Let V∈U be a non-symmetric escort. By speculation there exists an escort U∈U with the end goal that

$(x, y) \in U$ implies $(f^n(x), f^n(y)) \in V$, for every integer n.

while V is non- symmetric and $V^{-1} \in \boldsymbol{\mathcal{U}}$, then $(f^n(x), f^n(y)) \notin V^{-1}$.

This means that f is expansive.

Theorem 3.11 Leave X alone a uniform G-space and f∈H(X). In the event that f is G-equicontinuous map, at that point its G-extensive.

Proof. Assume that f is uniform G-equicontinuous. Let x,y∈X with G(x)≠G(y). Let V∈U be a non-symmetric company. By theory there exists an escort U∈U and g,p∈G to such an extent that

$(x, y) \in U$ implies $(f^n(\theta(g, x)), f^n(\theta(q, y))) \in V$, for every integer n.

While V is not symmetric and $V^{-1} \in \boldsymbol{\mathcal{U}}$, then for every integer n

$$(f^n(\theta(g, x)), f^n(\theta(q, y))) \notin V^{-1} \qquad (1)$$

Let $u \in G(x)$ and $v \in G(y)$. Then there exist $g, q \in G$ such that $u = \theta(g, x), v = \theta(q, y)$. Thus we have

$$(f^n(u), f^n(v)) \notin V^{-1} \text{, for every integer } n.$$

This means that f is G −expansive.This complete the proof.

Definition 3.12 A uniform $G -$ we can state ***distal*** whether, for every pair space $x, y \in X$ with $x \neq y$, the closure of the set $\{(\theta(g, x), \theta(g, y)): g \in G\}$ is disjoint from the diagonal $\Delta = \{(x, x): x \in X\}$ in $X \times X$.

Theorem 3.13 If (G, X, θ) is equicontinuous, then it is distal.

Proof Let $x, y \in X$ with $x \neq y$. Then found an index β on X with $(x, y) \notin \beta$. By equicontinuity found an index α like that $(u, v) \in \alpha$ implies

$$\big(\theta(h,x), \theta(h,y)\big) \in \beta \text{ for all } h \in G.$$

It pursue that

$$\big(\theta(g,x), \theta(g,y)\big) \notin \alpha \text{ for all } g \in G.$$

Otherwise, we could let $u = \theta(g,x)$, $v = \theta(g,y)$, $h \coloneqq g^{-1}$, and reach a contradiction. Thus $\{\big(\theta(g,x), \theta(g,y)\big): g \in G\}$ is disjoint from the diagonal $\Delta = \{(x,x): x \in X\}$ in $X \times X$. Since $\Delta \subseteq \alpha$ and α is open in the product topology, it follows that (G, X, θ) is distal.■

Theorem 2.9. Let , Y be $G-$ spaces and $f_1: X \to X$, $f_1: Y \to Y$ be maps. Then $f_1 \times f_2: X \times Y \to X \times Y$ is uniform $G_1 \times G_2$ $-$equicontinuous iff f_1 is uniform G_1 $-$equicontionuos and f_2 is uniform G_2 $-$equaicontinuous.

Proof. Assume that f≔f_1×f_2 is a uniform G_1×G_2-equicontinuous on X×Y . We will show that f_1 is uniform G_1-equicontionuos on X and correspondingly we can show that f_2 is G_2-equaicontinuous on Y. Let V∈U_X and n be a positive whole number. Since Y×Y∈U_Y at that point

$$V \times (Y \times Y) = W \in \boldsymbol{\mathcal{U}}_{X\times Y}$$

By hypothesis, found $U \in \boldsymbol{\mathcal{U}}_{X\times Y}$ like that if $(x, y) \in U$, after that

$$\big(f^n(\theta(g,x), f^n(\theta(p,y)\big) \in W = V \times (Y \times Y),\ g, p \in G = G_1 \times G_2$$

Since $(x, y) \in U$, then found $U_1 \in \boldsymbol{\mathcal{U}}_X$ and $U_2 \in \boldsymbol{\mathcal{U}}_X$ like that

$x = (x_1, x_2) \in U_1$ and $y = (y_1, y_2) \in U_2$.

But $f^n(\theta(g,x)) = (f_1^n(\theta_1(g_1, x_1), f_1^n(\theta_1(p_1, x_2)) \in V$

This means that f_1 is uniform G_1 $-$equicontionuos. Conversely, suppose that f_1 is uniform G_1 $-$equicontionuos and f_2 is uniform G_2 $-$equaicontinuous. Let $W \in \boldsymbol{\mathcal{U}}_{X\times Y}$. Then there exist $W_1 \in \boldsymbol{\mathcal{U}}_X$ and $W_2 \in \boldsymbol{\mathcal{U}}_Y$ such that $= W_1 \times W_2$. By hypothesis , there exist $U_1 \in \boldsymbol{\mathcal{U}}_X$ and $U_2 \in \boldsymbol{\mathcal{U}}_Y$ like that if $(x, x') \in U_1$ and $(y, y') \in U_2$, after that

$$(f_1^n(\theta_1(g,x), f_1^n(\theta_1(g',x')) \in W_1$$

and

$$(f_2^n(\theta_2(p,y), f_2^n(\theta_2(p',y')) \in W_2$$

for all $(g, g') \in G_1$ and $(p, p') \in G_2$ Set $U_1 \times U_2 = U$. Then $U \in \boldsymbol{\mathcal{U}}_{X\times Y}$. Thus we have

$$\big(f^n(\theta(g,x), f^n(\theta(p,y)\big) \in W\ \bar{g} \in G_1 \times G_2.$$

This means that $f_1 \times f_2 : X \times Y \rightarrow X \times Y$ is uniform $G_1 \times G_2$ −equicontinuous. This complete the proof.

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intuitionistic fuzzy pseudo ideals in Q-algebra

Habeeb Kareem Abdullah and mortda taeh shadhan
Department of Mathematics
Uinversity of Kufa
habeebk.abdullah@uokufa.edu.iq
Department of Mathematics
Uinversity of Kufa
alnbhanymrtdy05@gmail.com

Abstract
Present several types in this paper of intuitionistic fuzzy ideal in Q-algebra , called (intuitionistic fuzzy pseudo ideal , intuitionistic fuzzy k-pseudo ideal, intuitionistic fuzzy c-psudo ideal , intuitionistic fuzzy complete- k-pseudo ideal). We have introduced and illustrated several ideas that evaluate their relationship in a Q-algebra.

1 Introdction

In 1966 , K.Iseki and Y.Imai([7], [14]) introduced BCK-and BCI-algebras. In 2001 H.S.Kim([6]) introduced a new notion, known as Q-algebra, which is BCH / BCI / BCK-algebra generalization. At the same time, A.Iorgulescu and G.Georgescu ([3]) introduced pseudo BCK-algebras as an exemption from bck-algebras. In2016 , Y.B.jun, H.S.Kim and S.S Ahn([13])introduced pseudo Qalgebra as ageneralization of Q-algebra the concept of fuzzy set was introduced in 1969 by L. A .Zadeh ([10]) .In 2005, J.Meng, X.Guo([5]) studied fuzzy ideals of BCK / BCI-algebras. W.A.Dudek and Y.B.Jun ([15]) in 2008, introduced pseudo-BCI-algebras as a natural generalization of BCIalgebras and pseudo-BCK-algebras. At the same time , K. J .Lee([8]) established the fuzzy ideals in pseudo BCI-algebras.in([4]) H. K .Jawad introduced the notion of fuzzy pseudo Ideals of pseudo Q-algebra. In K. ([9]) Intuitionistic Fuzzy Sets(1986) was introduced by T. Atanassov..in 2012 S.M. Abdelnaby and O.R.Elgendy applied the concept of Intuitionistic fuzzy sets on Q-algebra. In this article, we will describe some of the new types of I F pseudo ideal, called (I F pseudo ideal, I F K-pseudo ideal, I F complete ?k-pseudo ideal). Also, we introduced and illustrated the proposition that defines the relationship among them in Q-algebra.

2 Basic concept and notations

In this section , We define Q-algebra , pseudo Q-algebra ,bounded , involutory , and some properties.

Definition (2.1) [11]

A Q- algebra is a set M with a binary operation * and constant 0 that fulfilled the following axioms:

1. $m * m = 0$ 8m 2 M
2. $m * 0 = m$ 8m 2 M
1
3. $(m * b) * d = (m * d) * b$; 8m; b; d 2 M

Remark (2.2)[11]

In a Q-algebra M , we can define a binary relation ≤ on M by $m \geq b$ if and only if $m * b = 0$ 8m; b 2 M

Definition (2.3) [1]

A Q-algebra $(M; *; 0)$ is called bounded if there is an element e 2 M that satisfies $m \leq e$ 8m 2 M then e is said to be an unit .We denotted e*m by m_*,for each m 2 M in bounded Q-algebra.

Example(2.4)

let M = f0; η; θ; βg be a set with the following table :

Table 1: Example of bounded

*	0	η	θ	β
0	0	0	0	0
η	η	0	0	0
θ	θ	0	0	0
β	β	β	β	0

Thereafter $(M;\ *;\ 0)$ be a Q-algebra . Note that M is bounded by unit β

Remark(2.5) [1]

As stated in the following example , the unit in bounded Q-algebra is not unique in general .

Example(2.6)

A binary operation * with $M = f0;\ \eta;\ \theta g$ can be shown in the table : .

Table 2: The unit in bounded Q-algebra is not uniqe

*	**0**	η	θ
0	**0**	**0**	**0**
η	η	**0**	**0**
θ	θ	**0**	**0**

Note that M is bounded with two units $\eta;\ \theta$

Propoition(2.7) [4]

In a bounded Q-algebra M , for any $m;\ b\ 2\ M;$ the following are hold :

1. $e_{*} = 0;\ 0_{*} = e$
2. $m_{*} * b = b_{*} * m$
3. $0 * b = 0$
4. $e_{*} * m = 0$
5. $m_{**} \leq m$

Definition(2.8) [1]

For a bounded Q-algebra M, If element m of M satisfies $m_{**} = m$,then m is called an involution. If every element of M is an involution,we call M is an involutory Q-algebra.

Example (2.9)

let $M = f0;\ \eta;\ \theta;\ \beta;\ g$, can be shown in table :

2

Table 3: **Example of involutory**

*	**0**	η	θ	β	
0	**0**	**0**	**0**	**0**	**0**
η	η	**0**	η	**0**	**0**
θ	θ	θ	**0**	**0**	**0**
β	β	θ	η	**0**	
0	**0**	**0**	**0**		

subsequently $(M;\ *;\ 0)$ is a bounded Q-algebra with unit β. Note that M is involutory.

Definition(2.10) [9]

An intuitionistic fuzzy set (IFS for short) A in a set M is object having the form $A = f < m;\ \mu_A(m);\ \nu_A(m) >: m\ 2\ Mg$, such that $\mu_A : M \text{-!} [0;\ 1]$ and $\nu_A : M \text{-!} [0;\ 1]$ denoted the dagree of membership (namely $\mu_A(m)$) , and the dagree of non membership (namely $\nu_A(m)$) for any element $m\ 2\ M$ to the set A , and $0 \leq \mu_A(m) + \nu_A(m) \leq 1;\ 8m\ 2\ M$ for the sake of simplicity , we shall use the notation $A = f < m;\ \mu_A(m);\ \nu_A(m) > g$ instead of $A = f < m;\ \mu_A(m);\ \nu_A(m) >: m\ 2\ Mg$

Definition(2.11) [2]

if $A = f < m;\ \mu_A(m);\ \nu_A(m) > jm\ 2\ Mgand\ B = f < m;\ \mu_B(m);\ \nu_B(m) > jm\ 2\ Mg$ be any two IFS of a set M then

1. $A \subseteq B$if and only if for all $m\ 2\ M\ \mu_A(m) \geq \mu_B(m)$ and $\mu_A(m) \geq \mu_B(m)$
2. $A = B$ if and only if for all $m\ 2\ M\ \mu_A(m) = \mu_B(m)$ and $\mu_A(m) = \mu_B(m)$
3. $A \backslash B = f < m;\ (\mu_A \backslash \mu_B)(m);\ ;\ (\nu_A [\nu_B)(m);\ m\ 2\ Mg$ where $;\ (\mu_A \backslash \mu_B)(m) = minf\mu_A(m);\ \mu_B(m)gg$ and $;\ (\nu_A\ [\ \mu_B)(m) = maxf\nu_A(m);\ \nu_B(m)gg$
4. $A[B = f < m;\ (\mu_A[\mu_B)(m);\ (\nu_A \backslash \nu_B)(m);\ m\ 2\ Mg$ where $;\ (\mu_A[\mu_B)(m) = maxf\mu_A(m);\ \mu_B(m)gg$ and $;\ (\nu_A \backslash \mu_B)(m) = minf\nu_A(m);\ \nu_B(m)gg$

Definition(2.12)

An intuitionistic fuzzy set $A = f < m;\ \mu_A(m);\ \nu_A(m) > g$ in a Q-algebra M is called an intuitionistic fuzzy ideal if

1. $\mu_A(0) \geq \mu_A(m)\ 8m\ 2\ M$
2. $\nu_A(0) \leq \nu_A(m)\ 8m\ 2\ M$
3. $\mu_A(m) \geq Minf\mu_A(m * b);\ \mu_A(b)g\ 8b;\ m\ 2\ M$

4. $\nu_A(m) \leq Max f \nu_A(m * b); \nu_A(b) g$ $8m; b\ 2\ M$

Definition(2.13) [13]

A pseudo Q-algebra is non-empty set of M with constant 0 and two binary operations * and # that satisfy the following axioms :

1. $m\#m = m * m = 0$ $8m\ 2\ M$
2. $m\#0 = m * 0 = 0$ $8m\ 2\ M$
3. $(m\#b) * c = (m * c)\#b$ $8m; b; c\ 2\ M$

3

Remark(2.14) [13]

In pseudo Q-algebra M , we can define a binary relation $\leq$ by $m \leq b$ if and only if $m\#b =$
0 & $m * b = 0$ $8m; b\ 2\ M$

Remark(2.15) [13]

That Q-algebra is a pseudo Q-algebra but the converse is not true as shown in the example below

Example(2.16) Let $M = f0; \eta; \theta; \beta g$

Table 4: pseudo Q-algebra but not Q-algebra

*	**0**	η	θ	β
0	**0**	**0**	**0**	**0**
η	η	**0**	**0**	**0**
θ	θ	θ	**0**	η
β	β	β	**0**	**0**

#	**0**	η	θ	β
0	**0**	**0**	**0**	**0**
η	η	**0**	**0**	**0**
θ	θ	β	**0**	β
β	β	β	**0**	**0**

Then $(M; *; 0)$ and $(M; \#; 0)$ are not Q-algebra , since $(\theta * \eta) * \beta = \eta\ 6= 0 = (\theta * \beta) * \eta$ and $(\theta\#\eta)\#\beta = 0\ 6= \beta = (\theta\#\beta)\#\eta$, but $(M; *; \#; 0)$ is pseudo Q-algebra .

Proposition(2.17) [12]

Let $(M; *; \#; 0)$ be a pseudo Q-algebra . Then the following hold :

1. $(m * (m\#b))\#b = (m\#(m * b)) * b = 0$ $8m; b\ 2\ M$
2. $m \leq 0 =) m = 0$ $8m\ 2\ M$.

Definition(2.18) [4]

A pseudo -Q-algebra M it is said to bo bounded if there is an element $n\ 2\ M$ satisfying
$m \leq n$ $8m\ 2\ M$ $i:e$; $m \leq n$, $m * n = 0$ *and* $m\#n = 0$ then n is called pseudo unit of M .

A pseudo-Q-algebra with a pseudo uinit is called bounded.

Proposition (2.19) [4]

Let $(M; *; \#; 0)$ be a bounded pseudo Q-algebra . Then the following hold:

1. $e_* = 0 = e_\#$
2. $m_*\#b = b_\# * m$ $8m; b\ 2\ M$
3. $m_*\#b_* = (b_*)_\# * m$ $8m; b\ 2\ M$
4. $m_\# * b_\# = (b_\#)_*\#m$ $8m; b\ 2\ M$

Defintion(2.20) [13]

Let $(M; *; \#; 0)$ be a bounded pseudo Q-algebra . A subset I of M is called the pseudo -ideal of M if it satisfies :

1. $0\ 2\ I$
2. $m * b; m\#b\ 2\ I$ *and* $b\ 2\ I$ imply $m\ 2\ I$ $8m; b\ 2\ I$ *whenever* $m; b\ 2\ I$

Definition(2.21) [9]

Let $(M; *; \#; 0)$ be a bounded pseudo Q-algebra and let $\varphi\ 6= I \subseteq M:$ I is called a pseudo subalgebra of M if $m * b; m\#b\ 2\ I$ wenever $m; b\ 2\ I$

Definition(2.22) [4]

Let M be a pseudo Q-algebra .A fuzzy set μ in M is called a fuzzy pseudo ideal of M if it satisfies :

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1. $\mu(0) \geq \mu(m); 8m\ 2\ M$

2. $\mu(m) \geq Minf\mu(m * b); \mu(m\#b); \mu(b)g$ 8m; b 2 M

Example(2.23)

In Example (2.17) , define the fuzzy set μ by $\mu(m)$ = (0 0: :8 : 6 : *if m if m* = 0 = $\theta; \beta ; \eta$

Then μ is fuzzy pseudo ideal , since $\mu(0) \geq \mu(m)$; 8m 2 M and

$\mu(m) = 0{:}6 \geq Minf\mu(m * b); \mu(m\#b); \mu(b)g = 0{:}6$ 8m 2 Mnfη; 0g and 8b 2 M

Wihle ’(m) = (

00:: 7 : 5 :	ifm = 0 ; η; θ
if	m=β

is not fuzzy pseudo ideal of M , since ’(β) = 0:5 6$\geq$ Minf’$(\beta * \theta)$; ’$(\beta * \theta)$; ’(θ)g = 0:7

Definition(2.24)[4]

A nonempty subset I of a pseudo Q-algebra $(M; *; \#; 0)$ is called complete pseudo ideal (briefly , c-pseudo ideal) , if

1. 0 2 I
2. $m * b$; $m\#b$ 2 I; 8b 2 I such that b 6= 0 implies m 2 I

Definition(2.25)[4]

A nonempty subset I of a bounded pseudo Q-algebra $(M; *; \#; 0)$ is called complete k-pseudo ideal (briefly ,c-k-pseudo ideal), if

1. 0 2 I
2. $m_* * b$; $b_\# * m$ 2 I (resp.$m_\#\#b$; $b_*\#m$ 2 I), 8b 2 I such that b 6= 0 imply m_* 2 I (resp. $m_\#$ 2 M) , 8m 2 M

Note that in bounded pseudo Q-algebra M there ara trivial c-k-pseudo ideals , f0g **Proposition(2.26) [4]**	and	M

Any c-pseudo ideal from bounded pseudo Q-algebra is c-k-pseudo ideal .

Definition(2.27) [4]

Let M be a bounded pseudo Q-algebra . An elementm 2 M satisfies $m_{**} = m = m_{\#\#}$ then m is called pseudo involution (i. e) m is *-involution and # - involution). If every element m 2 M is pseudo involution , we call M is a pseudo Q-algebra .

Example(2.28)

Let M = f0; η; θ; β; g be a set with tables below

Table 5: Pseudo involutory Q-algebra

*	0	η	θ	β	
0	**0**	**0**	**0**	**0**	**0**
η	η	**0**	**0**	**0**	**0**
θ	θ	**0**	**0**	**0**	
β	β	η	**0**	θ	
0	**0**	**0**	**0**		

#	0	η	θ	β	
0	**0**	**0**	**0**	**0**	**0**
η	η	**0**	**0**	**0**	
θ	θ	**0**	**0**	**0**	**0**
β	β	θ	**0**	η	
0	**0**	**0**	**0**		

Then $(M; *; \#; 0)$ is bounded pseudo Q-algebra with unit β. Notice that M is a pseudo involution .

Proposition(2.29) [4]

If I be a c-k-pseudo-ideal in a pseudo-involutory pseudo-Q-algebra M , then I is c-pseudo-ideal.

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Proposition(2.30) [4]

Let μ be a fuzzy pseudo ideal of a pseudo Q-algebra M if $m \leq b$; then $\mu(m) \geq \mu(b)$; 8m; b 2 M

Definition(2.31)

Let M be a pseudo Q-algebra . A fuzzy set μ in M is called a fuzzy pseudo subalgebra of M if it

satisfies :
1. $\mu(m * b) \geq Minf\mu(m); \mu(b)g\ 8m; b\ 2\ M$
2. $\mu(m\#b) \geq Minf\mu(m); \mu(b)g\ 8m; b\ 2\ M$

3 some types of intuitionistic fuzzy pseudo ideal

In this section , we define IF pseudo ideal and IF complete pseudo ideal , IF k-pseudo ideal , IF c-pseudo ideal and some properties among them .

Definition(3.1)

Let M be a pseudo Q-algebra . An intuitionistic fuzzy set A of M is called an intuitionistic fuzzy pseudo ideal if it satisfies :
1. $\mu_A(0) \geq \mu_A(m)\ 8m\ 2\ M$
2. $\nu_A(0) \leq \nu_A(m)\ 8m\ 2\ M$
3. $\mu_A(m) \geq Minf\mu_A(m * b); \mu_A(m\#b); \mu_A(b)g\ 8m; b\ 2\ M$
4. $\nu_A(m) \leq Maxf\nu_A(m * b); \nu_A(m\#b); \nu_A(b)g\ 8m; b\ 2\ M$

Example(3.2)

In Example (2.23) define the intuitionistic fuzzy set A by
$\mu_A(m) =$ (0 0: :8 : 6 : *if m if m* = 0 = $\theta; \beta ; \eta$ & $\nu_A(m) =$ (0 0: :2 : 4 : *if m if m* = 0 = $\theta; \beta ; \eta$
Then A is intuitionistic fuzzy pseudo ideal since ,
$\mu_A(0) \geq \mu_A(m)$ *and* $\nu_A(0) \leq \nu_A(m)\ 8m\ 2\ M;$
$\mu(b) = 0{:}6 \geq Minf\mu_A(b * m)\mu_A(b\#m); \mu_A(m)g = 0{:}6;$

$\nu_A(b) = 0{:}4 \leq Maxf\nu_A(b * m); \nu_A(b\#m); \nu_A(m)g = 0{:}4\ 8m\ 2\ M$ **Definition(3.3)**	*and* $8b\ 2\ Mnf0; \eta g$

Let I be a c-pseudo ideal of a pseudo Q-algebra $(M; *; \#; 0):$ An intuitionistic fuzzy set A is called intuitionistic fuzzy complete pseudo ideal at I (briefly , IF c-pseudo ideal) , if
1. $\mu_A(0) \geq \mu_A(m)\ 8m\ 2\ M$
2. $\nu_A(0) \leq \nu_A(m)\ 8m\ 2\ M$
3. $\mu_A(m) \geq Minf\mu_A(m * b); \mu_A(m\#b); \mu_A(b)g\ 8m; b\ 2\ M; b\ 2\ I$
4. $\nu_A(m) \leq Maxf\nu_A(m * b); \nu_A(m\#b); \nu_A(b)g\ 8b\ 2\ I; 8m\ 2\ M$

Example(3.4)

Let $M = f0; \eta; \theta; \beta g$ be a set with the tables below
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Table 6: intuitionistic fuzzy c-ideal

*	0	η	θ	β
0	**0**	**0**	**0**	**0**
η	η	**0**	**0**	**0**
θ	θ	**0**	**0**	η
β	β	β	**0**	**0**

#	0	η	θ	β
0	**0**	**0**	**0**	**0**
η	η	**0**	**0**	η
θ	θ	θ	**0**	η
β	β	β	**0**	**0**

Then $(M; *; \#; 0)$ is pseudo Q-algebra , a subset $I = f0; \eta; \theta g$ is a c-pseudo ideal of M . Let A is the intuitionistic fuzzy set defined as the following :
$\mu_A =$ (00::5 : 4 : *if m if m* = 0 = $\theta; \eta; \beta$ & $\nu_A(m) =$ (0 0: :5 : 6 : *if m if m* = 0 = $\theta; \eta; \beta$
Then A is the intuitionistic fuzzy c-ideal at I in M , because
$\mu_A(0) \geq \mu_A(m)$ and $\nu_A(0) \leq \nu_A(m)\ 8m\ 2\ M$,
$\mu_A(\theta) = 0{:}4 \geq Minf\mu_A(\theta * b); \mu_A(\theta\#b); \mu_A(b)g = 0{:}4\ 8b\ 2\ I;$
$\nu_A(\theta) = 0{:}6 \leq Maxf\nu_A(\theta * b); \nu_A(\theta\#b); \mu_A(b)g = 0{:}6\ 8b\ 2\ I:$

Proposition(3.5)

Every intuitionistic fuzzy pseudo ideal of a pseudo Q-algebra is an intuitionistic fuzzy c- pseudo ideal .

Proof

suppose that I be a c-pseudo ideal and A is intuitionistic fuzzy pseudo ideal of a pseudo Q-algebra M then by definitin (2.22) we have ,

1. $\mu_A(0) \geq \mu_A(m)\ 8m\ 2\ M$
2. $\nu_A(0) \leq \nu_A(m)\ 8m\ 2\ M$
3. $\mu_A(m) \geq Minf\mu_A(m * b);\ \mu_A(m\#b);\ \mu_A(b)g\ 8m;\ b\ 2\ M$
4. $\nu_A(m) \leq f\nu_A(m * b);\ \nu_A(m\#b);\ \nu_A(b)g\ 8m;\ b\ 2\ M$

since $I \subseteq M$, then $\mu_A(m) \geq Minf\mu_A(m * b);\ \mu_A(m\#b);\ \mu_A(b)g$ and
$\nu_A(m) \leq Maxf\nu_A(m * b);\ \nu_A(m\#b);\ \nu_A(b)g\ 8b\ 2\ I$

Thus A is intuitionistic fuzzy c-pseudo ideal of M .

Remark(3.6)

The following example shows that the converse of proposition (3.5) is not true in genaral

Example(3.7)

In example (3.2) , notice that A is intuitionistic fuzzy c-pseudo ideal at I in M
(When $I = f0;\ \eta;\ \theta g$), but its not is intuitionistic fuzzy pseudo ideal because
$\mu_A(\theta) = 0{:}4\ 6\geq Minf\mu_A(\theta * \beta);\ \mu_A(\theta\#\beta);\ \mu_A(\beta)g = 0{:}5$

Proposition(3.8)

Let I be a c-pseudo ideal of a pseudo involutory pseudo Q-algebra M. An intuitionistic fuzzy set A is intuitionistic fuzzy c-pseudo ideal if and only if satisfies :

1. $\mu_A(0) \geq \mu_A(m)\ 8m\ 2\ M$
2. $\nu_A(0) \leq \nu_A(m)\ 8m\ 2\ M$
3. $\mu_A(m) \geq Minf\mu_A(m_{**} * b);\ \mu_A(b_\# * m_*);\ \mu_A(b)g$

$= \mu_A(m) \geq Minf\mu_A(b_*\#m_\#);\ \mu_A(m_{\#\#}\#b);\ \mu_A(b)g\ 8m;\ b\ 2\ M$

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4. $\nu_A(m) \leq Maxf\nu_A(m_{**} * b);\ \nu_A(b_\# * m_*);\ \nu_A(b)g$ and

$\nu_A(m_\#) \leq Maxf\nu_A(m_\#\#b);\ \nu_A(b_*\#m);\ \nu_A(b)g{:}\ 8m;\ b\ 2\ M$

Proof

by definitin(2.27) and definition(3.3)

Definition(3.9)

An intuitionistic fuzzy set A in bounded pseudo Q-algebra $(M;\ \#;\ *;\ 0)$ is called intuitionistic fuzzy k-pseudo ideal, if

1. $\mu_A(0) \geq \mu_A(m)\ 8m\ 2\ M$
2. $\nu_A(0) \leq \nu_A(m)\ 8m\ 2\ M$
3. $\mu_A(m_*) \geq Minf\mu_A(m_* * b);\ \mu_A(b_\# * m);\ \mu_A(b)g$ and

$\mu_A(m_\#) \geq Minf\mu_A(m_\#\#b);\ \mu_A(b_*\#m);\ \mu_A(b)g\ 8m;\ b\ 2\ M$

4. $\nu_A(m_*) \leq Maxf\nu_A(m_* * b);\ \nu_A(b_\# * m);\ \nu_A(b)g$ and

$\nu_A(m_\#) \leq Maxf\nu_A(m_\#\#b);\ \nu_A(b_*\#m);\ \nu_A(b)g\ 8m;\ b\ 2\ M$

Example(3.10)

1. Every intuitionistic fuzzy constant in bounded paeudo Q-algebra M is intuitionistic fuzzy k-pseudo ideal .
2. Let $M = f0;\ \eta;\ \theta;\ \beta;\ g$ be a set with the tables below

Table 7: Pseudo involutory Q-algebra

*	**0**	η	θ	β	
0	**0**	**0**	**0**	**0**	**0**
η	η	**0**	**0**	**0**	η
θ	θ	**0**	**0**	θ	θ
β	β	**0**	β	**0**	θ
0	**0**	**0**			

#	**0**	η	θ	β	
0	**0**	**0**	**0**	**0**	**0**
η	η	**0**	**0**	η	η
θ	θ	**0**	**0**	**0**	**0**
β	β	**0**	**0**		
0	**0**	**0**			

then $(M; *; \#; 0)$ is bounded pseudo Q-algebra with uinit η and define an intuitionistic fuzzy A by
$\mu_A(m) = ($ 0 0$: :9 : 3 :$ *if m if m* $= 0 = \theta; \beta; ; \eta$ & $\nu_A(m) = ($ 0 0$: :1 : 7 :$ *if m if m* $= 0 = \theta; \beta; ; \eta$
then A is intuitionistic fuzzyk-pseudo ideal of M , because
$\mu_A(0) \geq \mu_A(m)$ *and* $\nu_A(0) \leq \nu_A(m); 8m\ 2\ M$
$\mu_A(m_*) = 0{:}9 \geq Minf\mu_A(m_* * b); \mu_A(b_\# * m); \mu_A(b)g$ is hold $8m; b\ 2\ M$.
also $\mu_A(m_\#) = 0{:}9 \geq Minf\mu_A(m_\#\#b); \mu_A(b_*\#m); \mu_A(b)g$ is hold $8m; b\ 2\ M$ also
$\nu_A(m_*) = 0{:}1 \leq Maxf\nu_A(m_**b); \nu_A(b_\#*m); \nu_A(b)g$ and $\nu_A(m_\#) = 0{:}1 \leq Maxf\nu_A(m_\#\#b); \nu_A(b_*\#m); \nu_A(b)g$

Proposition(3.11)

Every intuitionistic fuzzy pseudo ideal of a bounded pseudo Q-algebra is an intuitionistic fuzzy k-pseudo ideal

Proof

Let A is an intuitionistic fuzzy pseudo ideal of a bounded pseudo Q-algebra then by definition (3.1) we have

1. $\mu_A(0) \geq \mu_A(m)\ 8m\ 2\ M$

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2. $\nu_A(0) \leq \nu_A(m)\ 8m\ 2\ M$
3. $\mu_A(m) \geq Minf\mu_A(m * b); \mu_A(m\#b); \mu_A(b)g$ then

$\mu_A(m_*) \geq Minf\mu_A(m_* * b); \mu_A(m_*\#b); \mu_A(b)g$
$= Minf\mu_A(m_* * b); \mu_A(b_\# * m); \mu_A(b)g\ 8m; b\ 2\ M$ Also
$\mu_A(m_\#) \geq Minf\mu_A(m_\# * b); \mu_A(m_\#\#b); \mu_A(b)g$
$= Minf\mu_A(m_\#\#b); \mu_A(b_*\#m); \mu_A(b)g\ 8m; b\ 2\ M$

4. $\nu_A(m) \leq Maxf\nu_A(m * b); \nu_A(m\#b); \nu_A(b)g$ then

$\nu_A(m_*) \leq Maxf\nu_A(m_* * b); \nu_A(m_*\#b); \nu_A(b)g$
$= Maxf\nu_A(m_* * b); \nu_A(b_\# * m); \nu_A(b)g\ 8m; b\ 2\ M$ Also
$\nu_A(m_\#) \leq Maxf\nu_A(m_\# * b); \nu_A(m_\#\#b); \nu_A(b)g$
$= Maxf\nu_A(m_\#\#b); \nu_A(b_*\#m); \nu_A(b)g\ 8m; b\ 2\ M$
Thus A is intuitionistic fuzzy K-pseudo ideal of M .

Remark(3.12)

In genaral , the converse of Proposition (3.11) needs not ture as shown in the following example .

Example(3.13)

in Example (3.10 -2) A is intuitionistic fuzzy k-pseudo ideal in M , but not intuitionistic fuzzy pseudo ideal in M , because $\mu_A(\theta) = 0{:}3\ 6\geq Minf\mu_A(\theta * \eta); \mu_A(\theta\#\eta); \mu_A(\eta)g = 0{:}9$

Proposition(3.14)

Every intuitionistic fuzzy k-pseudo ideal in a pseudo involutory pseudo Q-algebra M is intuitionistic fuzzy pseudo ideal .

Proof

Assume that A be an intuitionistic fuzzy k-pseudo ideal of M
since M is pseudo involutory pseudo Q-algebra , then
$\mu_A(m) = \mu_A(m_{**}) \geq Minf\mu_A(m_{**} * b); \mu_A(b_\# * m_*); \mu_A(b)g$
$= Minf\mu_A(m*b); \mu_A(m\#b); \mu_A(b)g$ and $\nu_A(m) = \nu_A(m_{**}) \leq Maxf\nu_A(m_{**}*b); \nu_A(b_\#*m_*); \nu_A(b)g$
$= Maxf\nu_A(m * b); \nu_A(m\#b); \nu_A(b)g\ 8m; b\ 2\ M$

Proposition(3.15)

Let A be intuitionistic fuzzy k-pseudo ideal of a bounded pseudo Q-algebra M , then

1. $\mu_A(m_*) \geq \mu_A(e)$ *and* $\mu_A(m_\#) \geq \mu_A(e)\ 8m\ 2\ M$
2. if $\nu(m_*) \leq \nu_A(e)\ \nu_A(m_\#) \leq \nu(e)\ 8m\ 2\ M$
3. if $m_* \leq b;$ then $\mu_A(b) \geq \mu_A(m_*)$ also $\nu_A(b) \leq \nu_A(m_*)$
4. $m_\# \leq b;$ then $\mu_A(b) \geq \mu_A(m_\#)$ also $\nu_A(b) \leq \nu_A(m_\#)$

Proof

1. Since A is intuitionistic fuzzy k -pseudo ideal, we have

$\mu_A(m_*) \geq Minf\mu_A(m_* * e); \mu_A(e_\# * m); \mu_A(e)g$
$= Minf\mu_A(0); \mu_A(e)g = \mu_A(e)$ *and* $\mu_A(m_\#) \geq Minf\mu_A(m_\#\#e); \mu_A(e_*\#m); \mu_A(e)g$
$= Minf\mu_A(0); \mu_A(e)g = \mu_A(e)\ 8m\ 2\ M$

2. Since A is intuitionistic fuzzy k -pseudo ideal, we have

$\nu_A(m_*) \leq Maxf\nu_A(m_* * e); \nu_A(e_\# * m); \nu_A(e)g$

$= Maxf\nu_A(0); \nu_A(e)g = \nu_A(e)$ *and* $\nu_A(m_{\#}) \leq Maxf\nu_A(m_{\#}\#e); \nu_A(e_{*}\#m); \nu_A(e)g$

$= Maxf\nu_A(0); \nu_A(e)g = \nu_A(e)\ 8m\ 2\ M$

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3. if $m_{*} \leq b$ i.e $m_{*} * b = 0$ and $m_{*}\#b = 0$, then

$\mu_A(m_{*}) \geq Minf\mu_A(m_{*} * b); \mu_A(b_{\#} * m)\mu_A(b)g\ 8m; b\ 2\ M$

(since A is intuitionistic fuzzy k-peudo ideal)

$= Minf\mu_A(0); \mu_A(b)g = \mu_A(b)$ and

$\nu_A(m_{*}) \leq Maxf\nu_A(m_{*} * b); \nu_A(b_{\#} * m)\nu_A(b)g\ 8m; b\ 2\ M$

(since A is intuitionistic fuzzy k-peudo ideal)

$= Maxf\nu_A(0); \nu_A(b)g = \nu_A(b)$

4. is similar to the proof of (3)

Definition (3.16)

Let I be a c-k-pseudo ideal of a bounded pseudo Q-algebra $(M; *; \#; 0):$ An intuitionistic fuzzy set A is called intuitionistic fuzzy complete k-pseudo ideal (briefly , intuitionistic fuzzy c-k-pseudo ideal), if

1. $\mu_A(0) \geq \mu_A(m)\ 8m\ 2\ M$
2. $\nu_A(0) \leq \nu_A(m)\ 8m\ 2\ M$
3. $\mu_A(m_{*}) \geq Minf\mu_A(m_{*} * b); \mu_A(b_{\#} * m); \mu_A(b)g$

and $\mu_A(m_{\#}) \geq Minf\mu_A(m_{\#}\#b); \mu_A(b_{*}\#m); \mu_A(b)g\ 8m; b\ 2\ M; b\ 2\ I$

4. $\nu_A(m_{*}) \leq Maxf\nu_A(m_{*} * b); \nu_A(b_{\#} * m); \nu_A(b)g$

and $\nu_A(m_{\#}) \leq Maxf\nu_A(m_{\#}\#b); \nu_A(b_{*}\#m); \nu_A(b)g\ 8m; b\ 2\ M; b\ 2\ I$

Example(3.17)

In Example (2.16) let A be intuitionistic fuzzy set of M where $I = f0; \eta; \theta g$ is c-k-pseudo ideal defined by

$\mu_A(m) = (0\ 0: :6 : 2 :$ *if m if* $m = 0 = \theta; \eta; \beta$ & $\nu_A(m) = (0\ 0: :4 : 8 :$ *if m if* $m = 0 = \theta; \eta; \beta$

Then A is intuitionistic fuzzy complete k-pseudo ideal of M because

$\mu_A(0) \geq \mu_A(m)$ *and* $\nu_A(0) \leq \nu_A(m)\ 8m\ 2\ M;$

$\mu_A(0_{*}) = 0:2 \geq Minf\mu_A(0_{*} * b); \mu_A(b_{\#} * 0); \mu_A(b)g = 0:2\ 8b\ 2\ I$

$\mu_A(\eta_{*}) = 0:2 \geq Minf\mu_A(\eta_{*} * b); \mu_A(b_{\#} * \eta); \mu_A(b)g = 0:2\ 8b\ 2\ I$

$\mu_A(0_{\#}) = 0:2 \geq Minf\mu_A(0_{\#}\#b); \mu_A(b_{*}\#0); \mu_A(b)g = 0:2\ 8b\ 2\ I$ *and*

$\nu_A(0_{*}) = 0:8 \leq Maxf\nu_A(0_{*} * b); \nu_A(b_{\#} * 0); \nu_A(b)g = 0:8\ 8b\ 2\ I$

$\nu_A(\eta_{*}) = 0:8 \leq Maxf\nu_A(\eta_{*} * b); \nu_A(b_{\#} * \eta); \nu_A(b)g = 0:8\ 8b\ 2\ I$

$\nu_A(0_{\#}) = 0:8 \leq Maxf\nu_A(0_{\#}\#b); \nu_A(b_{*}\#0); \nu_A(b)g = 0:8\ 8b\ 2\ I$

Proposition(3.18)

Every intuitionistic fuzzy k-pseudo ideal of a bounded pseudo Q-algebra is an intuitionistic fuzzy c-k-pseudo ideal

Proof

Let I be a c-k-pseudo ideal in bounded pseudo Q-algebra M and A be an intuitionistic fuzzy kpseudo ideal of M , then

$\mu_A(m_{*}) \geq Minf\mu_A(m_{*} * b); \mu_A(b_{\#} * m); \mu_A(b)g$ *and*

$\nu_A(m_{*}) \leq Maxf\nu_A(m_{*} * b); \nu_A(b_{\#} * m); \nu_A(b)g\ 8m; b\ 2\ M$

Since $I \subseteq M$ *we have*

$\mu_A(m_{*}) \geq Minf\mu_A(m_{*} * b); \mu_A(b_{\#} * m); \mu_A(b)g$ *and*

$\nu_A(m_{*}) \leq Maxf\nu_A(m_{*} * b); \nu_A(b_{\#} * m); \nu_A(b)g\ 8b\ 2\ I$ *Also*

$\mu_A(m_{\#}) \geq Minf\mu_A(m_{\#}\#b); \mu_A(b_{*}\#m); \mu_A(b)g$ *and*

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$\nu_A(m_{\#}) \leq Maxf\nu_A(m_{\#}\#b); \nu_A(b_{*}\#m); \nu_A(b)g\ 8m; b\ 2\ M$

Since $I \subseteq M$ *we have*

$\mu_A(m_{*}) \geq Minf\mu_A(m_{*} * b); \mu_A(b_{\#} * m); \mu_A(b)g$ *and*

$\nu_A(m_{*}) \leq Maxf\nu_A(m_{*} * b); \nu_A(b_{\#} * m); \nu_A(b)g\ 8b\ 2\ I$

Remark(3.19)

The converse of proposition(3.18) may not be true and the following example explainwd that .

Example (3.20)

In example (3.17) A be intuitionistic fuzzy c-k-pseudo ideal in M

(Where $I = f0; \eta; \theta g$ is c-k-pseudo ideal) , but its not intuitionistic fuzzy k-pseudo ideal , since $\mu_A(0_\#) = 0{:}2 \; 6\geq Minf\mu_A(0_\#\#\beta); \mu_A(\beta_*\#0); \mu_A(\beta)g = 0{:}6$

corollary (3.21)

Every intuitionistic fuzzy pseudo ideal of bounded pseudo Q-algebra is intuitionistic fuzzy c-k-psudo ideal

proof

by proposition(3.11) and proposition(3.18).

Proposition(3.22)

Any intuitionistic fuzzy c-pseudo ideal from bounded pseudo Q-algebra is intuitionistic fuzzy c-kpseudo ideal .

Proof

Let A be an intuitionistic fuzzy c-pseudo ideal from bounded pseudo Q-algebra M and I be c-pseudo ideal of M .

then I is c-k-pseudo ideal of M by proposition (2.26)

since A intuitionistic fuzzy c-pseudo ideal of M , from definition(3.3) we have :

1. $\mu_A(0) \geq \mu_A(m) \; 8m \; 2 \; M$
2. $\nu_A(0) \leq \nu_A(m) \; 8m \; 2 \; M$
3. $\mu_A(m) \geq Minf\mu_A(m * b); \mu_A(m\#b); \mu_A(b)g \; 8b \; 2 \; I$ thus

$\mu_A(m_*) \geq Minf\mu_A(m_* * b); \mu_A(m_*\#b); \mu_A(b)g$

$= Minf\mu_A(m_* * b); \mu_A(b_\# * m); \mu_A(b)g$	$8b \; 2 \; I$
also $\mu_A(m_\#) \geq Minf\mu_A(m_\# * b); \mu_A(m_\#\#b); \mu_A(b)g$	
$= Minf\mu_A(m_\#\#b); \mu_A(b_*\#m); \mu_A(b)g$	$8b \; 2 \; I$
4. $\nu_A(m) \leq Maxf\nu_A(m * b); \nu_A(m\#b); \nu_A(b)g \; 8b \; 2 \; I$ thus $\nu_A(m_*) \leq Maxf\nu_A(m_* * b); \nu_A(m_*\#b); \nu_A(b)g$	
$= Maxf\nu_A(m_* * b); \nu_A(b_\# * m); \nu_A(b)g$	$8b \; 2 \; I$
also $\nu_A(m_\#) \leq Maxf\nu_A(m_\# * b); \nu_A(m_\#\#b); \nu_A(b)g$	
$= Maxf\nu_A(m_\#\#b); \nu_A(b_*\#m); \nu_A(b)g$	$8b \; 2 \; I$

Hance A is intuitionistic fuzzy c-pseudo ideal of M

Example(3.23)

In example (3.10) if $I = f0; \beta; g;$ then i is a c-k-pseudo ideal and c-pseudo ideal of a bounded Q-algebra M

define the intuitionistic fuzzy set A by :

$\mu_A(m) = (\; 00{::}9 : 6 : if \; m \; if \; m = 0 = \theta;; \eta; \beta$ & $\nu_A(m) = (\; 00{::}1 : 4 : if \; m \; if \; m = 0 = \theta;; \eta; \beta$

then A is intuitionistic fuzzy c-k-pseudo ideal because

$\mu_A(0) \geq \mu_A(m)$ and $\nu_A(0) \leq \nu_A(m) \; 8m \; 2 \; M$

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Also $\mu_A(m_*) = 0{:}9 \geq Minf\mu_A(m_* * b); \mu_A(b_\# * m); \mu_A(b)g$ is hold $8b \; 2 \; I; \; 8m \; 2 \; M$ and $\mu_A(m_\#) = 0{:}9 \geq Minf\mu_A(m_\#\#b); \mu_A(b_*\#m); \mu_A(b)g$ is hold too $8b \; 2 \; I; \; 8m \; 2 \; M$ and

$\nu_A(m_*) = 0{:}1 \leq Maxf\nu_A(m_* * b); \nu_A(b_\# * m); \nu_A(b)g$ is hold $8b \; 2 \; I; \; 8m \; 2 \; M$ and $\nu_A(m_\#) = 0{:}1 \leq Maxf\nu_A(m_\#\#b); \nu_A(b_*\#m); \nu_A(b)g$ is hold too $8b \; 2 \; I; \; 8m \; 2 \; M$

but A is not intuitionistic fuzzy c-pseudo ideal because

$\mu_A(\;) = 0{:}6 \; 6\geq Minf\mu_A(\; * \beta); \mu_A(\; \#\beta); \mu_A(\beta)g = 0{:}9$

Proposition(3.24)

Every intuitionistic fuzzy c-k-pseudo ideal in a pseudo involutory pseudo Q-algebra M is intuitionistic fuzzy c-pseudo ideal .

Proof

suppose that A is intuitionistic fuzzy c-k-pseudo ideal of M . Then I is c-pseudo ideal of M by (proposition (2.29))

since M is pseudo involutory , then

$\mu_A(m) = \mu_A(m_{\#\#}) \geq Minf\mu_A(m_{\#\#}\#b); \mu_A(b_*\#m_\#); \mu_A(b)g$

= $Minf\mu_A(m\#\#\#b); \quad \mu_A(m\#\# * b); \quad \mu_A(b)g$
= $Minf\mu_A(m * b); \quad \mu_A(m\#b); \quad \mu_A(b)g \quad 8b \quad 2 \quad I$
Thus A is intuitionistic fuzzy c-pseudo ideal of M .
Remark(3.25) :
The following diagram shows the relation among intuitionistic fuzzy pseudo ideal ,intuitionistic fuzzy k-pseudo ideal,intuitionistic fuzzy c-pseudo ideal , intuitionistic fuzzy c-k-pseudo ideal in bounded Q-algebra :
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IF pseudo ideal I F k- pseudo ideal
IF c-pseudo ideal IF c-K-pseudo ideal
In involutory Q-algebra
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INDEPENDENT (NON-ADJACENT VERTICES) TOPOLOGICAL SPACES ASSOCIATED WITH UNDIRECTED GRAPHS, WITH SOME APPLICATIONS IN BIOMATHEMATICES

Asmhan Flieh Hassan[1] , Zainab Irhayyim Abed[2]

[1]Department of Mathematics, faculty of education for girls, University of Kufa, Iraq
asmhanf.alzuhairy@uokufa.edu.iq
ORCID ID: https://orcid.org/0000-0002-0855-1873
[2]Department of Mathematics, faculty of education for girls, University of Kufa, Iraq

ABSTRACT: In this work, we associate a new topology to undirected graph $G = (V, E)$ which may contain one isolated vertex or more and we named it Independent (non-adjacent vertices) Topology. A new sub-basis family to generate the *Independent Topology* is introduced on the set of n vertices V and for every vertex v of V the number of adjacent vertices is not greater than $n - 2$ (In simple graph we can say : for every vertex v of V , $\Delta(G) = n - 2$, where $\Delta(G)$ is the maximum degree of vertices in a graph G). Then we give a fundamental step toward investigation of some properties of undirected graphs by their corresponding *Independent Topology* which we introduce in this work. Furthermore, an application to our new model (*Independent Topology*) are presented, that to carry out a framework in practical life like biomathematics (applied examples in biomathematics).

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1. Introduction

In Mathematics graph theory have a long history, one branch of graph theory is a topological graph theory. The relation between graph theory and topological theory existed before and used many times by researchers to deduce a topology from a given graph. Some of them makes models defined on the set of vertices V of the graph G only and others made it on the set of edges E . They studies graphs as a topologies and have been applied in almost every scientific field. Many excellent basics on the mathematics of graph theory, topological graph theory and some applications may be found in the sources [1-7],

In general graphs divided in two types; directed and undirected graph. To an undirected graph some researchers associate a topological spaces as fellow;

In 2013 [8], Jafarian et al. associate a Graphic Topology with the vertex set of a locally finite graph without isolated vertex, and they defined a sub-basis family for a graphic topology as a sets of all vertices adjacent to the vertex v .

And in 2018 [9], Kilicman and Abdulkalek associate an Incidence Topology with a set of vertices for any simple graph without isolated vertex. where they defined a sub-basis family for an incident topology as a sets of all incident vertices with the edge e.

The previous works of topology on graphs was associated with a set of vertices without isolated vertex. Therefore, these topologies are not appropriate to be associated with graphs that have an isolated vertices.

Our motivation or target is to associate a topology on the vertex set of any undirected graph (not only simple graph or locally finite graph) and which may contain one isolated vertex or more. By introducing a new Sub-basis family defined as a sets of all vertices non-adjacent to the vertex v to induce the new topology (which we named it *Independent Topology*), and we present a fundamental steps toward studying some main properties of undirected graphs by their corresponding topologies.

So, we have two goals for this work: First, we introduce a new model of a topology associated with graph which is most general than the previous works. Second, we apply this new model topology in some main subjects in biomathematics.

In Section 2 of the article we give some fundamental definitions and preliminaries of graph theory and topology, also In Section 3 we define our new topology (*independent topology*) on undirected graphs by introducing a sub-basis family for the new topology. Section 4 is devoted to some preliminaries results of independent topology.
In Section 5 some application in biomathematics of new model (*independent topology*) is discussed. In last Section, conclusions of this new topology on undirected graphs are presented.

2. Preliminaries

In this section we give some fundamental definitions and preliminaries of graph theory and topology. All this definitions are standard, and can be found for example in sources [2] [3] [10].

Usually the graph is a pair $\mathrm{G} = (\mathrm{V}, \mathrm{E})$, for more exactly A graph G consist of a non-empty set V of vertices (or nodes), and a set E of edges (or arcs). If e is an edge in G we can write $e = v\,u$ (e is join each vertex v and u),where v and u are vertices in V, then (v and u) are said adjacent vertices and incident with the edge e.If there is no vertex adjacent with a vertex v, then v is said isolated vertex. the degree of the vertex v denoted by $d(v)$ is the number of the edges where v incident with e, and $\Delta(G)$ is the maximum degree of vertices in G. A vertex of degree 0 is isolated. An independent set in a graph G is a set of pairwise non-adjacent vertices. The graph G is finite if the number of the vertices in G also the number of the edges in G is finite, then; otherwise it is an infinite graph. If any vertex can be reached from any other vertex in G by travelling along the edges, then G is called connected graph and is called disconnected otherwise.
We use notations K_n ,$\mathrm{K}_{m,n}$,P_n and C_n for a complete graph with n vertices, the complete bipartite graph when partite sets have sizes m and n, the path on n vertices and the cycle on n vertices, respectively.

A topology $\mathcal{T}$ on a set $\mathcal{X}$ is a combination of subsets of $\mathcal{X}$, called open, such that the union of the members of any subset of $\mathcal{T}$ is a member of $\mathcal{T}$, the intersection of the members of any finite subset of $\mathcal{T}$ is a member of $\mathcal{T}$, and both empty set and $\mathcal{X}$ are in $\mathcal{T}$. The ordered pair $(\mathcal{X}, \mathcal{T})$ is called a topological space. When the topology $\mathcal{T} = \mathrm{P}(\mathcal{X})$ on $\mathcal{X}$ is called discrete topology while the topology $\mathcal{T} = \{\mathcal{X}, \varphi\}$ on $\mathcal{X}$ is called indiscrete (or trivial) topology. A topology in which arbitrary intersection of open set is open called an Alexandroff space.

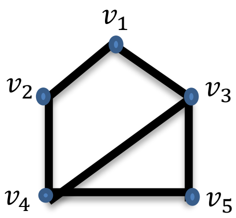
3. *Independent topology* on graphs

Now, we define our new model of topology on undirected graph $G = (\mathrm{V}, \mathrm{E})$ which may contain one isolated vertex or more and we named it Independent (non-adjacent vertices) Topology. A new sub-basis family to generate the *Independent Topology* is introduced on the set of n vertices V and for every vertex v of V the number of adjacent vertices is not greater than $n-2$ (In simple graph we can say : for every vertex v of V, $\Delta(G) = n-2$, where $\Delta(G)$ is the maximum degree of vertices in a graph G).
(i.e. for every vertex $v \in \mathrm{V}$ the number of adjacent vertices is not greater than $n-2$, where n is the number of all vertices in G)
Suppose that I_v is the set of all vertices non-adjacent (independent) to v. It is clear that $v \in I_u$ iff $u \in I_v$ for all $v, u \in V$ and $v \notin I_v$ for all $v \in V$.

Define $\mathcal{S}_{IV}$ as follows $\mathcal{S}_{IV} = \{ I_v \mid v \in V \}$. Since the condition above exist and the graph G can contain one isolated vertex or more, we have $V = \bigcup_{v \in V} I_v$ Hence $\mathcal{S}_{IV}$ forms a sub-basis for a topology $\mathcal{T}_{IV}$ on V, called *Independent topology* of G.
It easy to see that the independent topology of C_n when $n \geq 4$ and the simple graph has $n \geq 2$ isolated vertex are discrete but the independent topology of P_n is not discrete because the set contains just two vertices of degree one is open, the *independent topology* of $\mathrm{k}_{n,m}$ is equal to $\{ \varphi, \mathrm{V}, \mathrm{A}, \mathrm{B}\}$,where A and B are partite sets of $\mathrm{K}_{n,m}$.

Example 3.1. Let $G = (V, E)$ be a simple graph as in Fig.1,clearly G verify the condition (for every vertex $v \in \mathrm{V}$ the number of adjacent vertices is not greater than $n-2$), then:
$V = \{v_1, v_2, v_3, v_4, v_5\}$ We have:

$S_{Iv_1} = \{v_4, v_5\}$, $S_{Iv_2} = \{v_3, v_5\}$, $S_{Iv_3} = \{v_2\}$. $S_{Iv_4} = \{v_1\}$ $S_{Iv_5} = \{ v_1, v_2\}$.



By taking finitely intersection the base obtained,

$\{\{v_5\}, \{v_4, v_5\}, \{v_3, v_5\}, \{v_2\}, \{v_1\}, \{ v_1, v_2\}, \varphi\}$
Then by taking all unions the *Independent topology* can be written as:

Fig. 1

$$\mathcal{T}_{IV} = \{\varphi, V, \{v_1\}, \{v_2\}, \{v_1, v_2\}, \{v_5\}, \{v_4, v_5\}, \{v_3, v_5\}, \{v_2, v_5\}, \{v_1, v_5\}, \{v_1, v_2, v_5\}, \{v_3, v_4, v_5\}, \{v_2, v_4, v_5\}, \{v_1, v_4, v_5\}, \{v_1, v_2, v_4, v_5\}, \{v_1, v_2, v_3, v_5\}, \{v_2, v_3, v_5\}, \{v_1, v_3, v_5\}\}.$$

4. Preliminary result

Proposition 4.1. If $G = (V, E)$ is a graph .then $(V, \mathcal{T}_{IV})$ is an Alexandroff space.
Proof. It is enough to prove that arbitrary intersection of members of $\mathcal{S}_{IV}$ is open. Let $S \subseteq V$. If $v \in \bigcap_{u \in s} I_u$, then $v \in I_u$for each $u \in S$. Hence $u \in I_v$ for each $u \in S$ and so $S \subseteq I_v$, I_v and S are finite sets. This means that if S is infinite, then $\bigcap_{u \in s} I_u$ is empty, but if S is finite, then $\bigcap_{u \in s} I_u$ is the intersection of finitely many open sets and hence $\bigcap_{u \in s} I_u$ is open . □

Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be a graph containing v, for each $v \in V$, the intersection of all open sets containing v is the smallest open set containing v we still call it D_v and the family $B_{IV} = \{D_v \mid v \in \mathrm{V} \}$ is minimal basis for the topological space $(V, \mathcal{T}_{IV})$

Proposition 4.2. Let $\mathrm{G} = (V, E)$ be a graph. Then we have $D_v = \bigcap_{u \in I_v} I_u$ and so D_v is finite for every $v \in \mathrm{V}$.
Proof. Since D_v is the smallest open set containing v and S_{IV} is a sub-basis of $\mathcal{T}_{IV}$ we have $D_v = \bigcap_{w \in s} I_w$ for some subset S of V .This implies that $v \in I_w$ for each $\mathrm{w} \in \mathrm{S}$. Therefore $\mathrm{S} \subseteq I_v$ and so $v \in \bigcap_{w \in I_v} I_w \subseteq D_v$.
Now by definition of D_v, the proof is complete. □

Corollary 4.3. Let $G = (V, E)$ be a graph. Then for every $v, w \in V$ we have $w \in D_v$ if and only if $I_v \subseteq I_w$. Equivalently $D_v = \{w \in V \mid I_v \subseteq I_w\}$.
Proof. By the Proposition above $w \in D_v$ if and only if $w \in I_u$ for each $u \in I_v$ if and only if $u \in I_w$ for each $u \in I_v$.

Remark 4.4. Suppose that $G = (V, E)$ is a graph, then$(V, \mathcal{T}_{IV})$ is a discrete topological space if and only if $I_v \nsubseteq I_u$ and $I_u \nsubseteq I_v$ for every distinct pair of vertices $v, u \in V$.

Remark 4.5. We also know from Remark in [11] that an Alexandroff topological space is T_1 if and only if it is discrete. Now, this implies that the graph $G = (V, E)$ has T_0 *independent topology* $(V, \mathcal{T}_{IV})$ if and only if $I_v \neq I_u$ for every distinct pair of vertices $v, u \in V$. Let $T = (V, E)$ be a tree. Then $(V, \mathcal{T}_{IV})$ is a T_0 space if and only if $I_v \neq I_u$ for every $v, u \in V$ such that $v \neq u$ and $\deg v = \deg u = 1$.

Remark 4.6. Complete graph K_n does not verify the *Independent Topology* but if there exists an one isolated vertex or more in the same graph then it verify the *Independent Topology*

Example 4.7. Let $G = (V, E)$ be a complete graph K_3 as in Fig.2, clearly G satisfy the condition since $n = 4$ and each vertex has not greater than $n - 2$ adjacent vertices such that $V = \{v_1, v_2, v_3, v_4, \}$. We have;

$I_{v_1} = \{v_4\}$, $I_{v_2} = \{v_4\}$, $I_{v_3} = \{v_4\}$, $I_{v_4} = \{v_1, v_2, v_3\}$

Then sub-basis $S_{IV} = \{ \{v_1, v_2, v_3\}, \{v_4\}\}$

By taking finitely intersection the basis obtained

$\{\{v_4\}, \{v_1, v_2, v_3\}, \varphi \}$

Then by taking all unions the *Independent Topology* can be written as:

$\mathcal{T}_{IV} = \{\varphi, V, \{v_4\}, \{v_1, v_2, v_3\}\}$

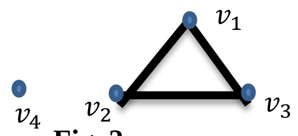


Fig. 2

Definition 4.8. Let $G_1 = (V_1, E_2)$ and $G_2 = (V_2, E_2)$ be two graphs. We call $G_1$and G_2 isomorphic, and write $G_1 \cong G_2$, if there exists a bijection $\xi : V_1 \to V_2$ with $v\,u \in E_1 \Leftrightarrow \xi(x)\xi(y) \in E_2$ for all $v, u \in V_1$, Such a map ξ is called an isomorphism; if $G_1 = G_2$, it is called an automorphism of G_1.

Remark 4.9. It is easy to check, If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic graphs, then topological spaces (V_1, T_{IV_1}) and (V_2, T_{IV_2}) are homeomorphic. The converse is not true, in general. For example C_n when $n \geq 4$ and the simple graph has $n \geq 2$ isolated vertex, are not isomorphic graphs, but their corresponding *independent topologies* are both discrete and hence homeomorphic.

Proposition 4.10. Let $G = (V, E)$ be a graph. Then $(V, \mathcal{T}_{IV})$ is a compact *Independent topological* space if and only if V is finite.

Proof . By Proposition 4.2, D_v is finite for every $v \in V$, hence if V is infinite , then B_{IV} is an open covering of $(V, \mathcal{T}_{IV})$ which has no finite sub cover. □

Definition 4.11. In the graph G if $F \subseteq V(G)$, then we write $G - F$ for the sub graph obtained by deleting the set of vertices F, A cut-vertex of G is a vertex whose deletion increases the number of components of G, i.e. a vertex $v \in V(G)$ such that $G - \{v\}$ has more components than G. A vertex cut of a connected graph G is a set $H \subseteq V(G)$ such that $G - H$ has more than one component. A vertex cut H of G is said to be minimal if every proper subset of H is not a vertex cut.

It is obvious that, if v be a cut vertex in a graph G = (V, E) (not necessarily connected) . Then $\{v\} \notin \mathcal{T}_{IV}$.

Example 4.12. Let $G = (V, E)$ be a graph as in Fig.3 such that $V = \{v_1, v_2, v_3, v_4, v_5\}$. We have;

$I_{v_1} = \{v_4, v_5\}$, $I_{v_2} = \{v_4, v_5\}$, $I_{v_3} = \{v_5\}$, $I_{v_4} = \{v_1, v_2\}$, $I_{v_5} = \{ v_1, v_2, v_3\}$.

Then $S_{IV} = \{\{v_5\}, \{v_1, v_2\} , \{v_4, v_5\} , \{ v_1, v_2, v_3\}\}$

By taking finitely intersection the basis obtained

$\{ \varphi, \{v_5\}, \{v_4, v_5\}, \{v_1, v_2\} , \{ v_1, v_2, v_3\}\}$

Then by taking all unions the *Independent Topology* can be written as:

$\mathcal{T}_{IV} = \{\varphi, V, \{v_5\}, \{v_4, v_5\}, \{v_1, v_2\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4, v_5\}, \{v_1, v_2, v_5\}, \{v_1, v_2, v_3, v_5\}\}$

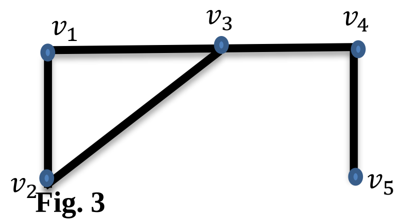


Fig. 3

It is clear in this example $\{v_3\}$ is a cut vertex but $\{v_3\} \notin \mathcal{T}_{IV}$.

Now, the connected graph is a tree if and only if every vertex of degree greater than one is a cut-vertex. Therefore, if $T = (V, E)$ is a tree and $v \in V$ with $deg\, v \geq 2$, then $\{v\} \notin \mathcal{T}_{IV}$.

Example 4.13. Let $T = (V, E)$ be a graph as in Fig. 4 such that $V = \{v_1, v_2, v_3, v_4, v_5\}$ We have;

$I_{v_1}=\{v_3,v_4,v_5\}$, $I_{v_2}=\{v_4,v_5\}$, $I_{v_3}=\{v_1\}$, $I_{v_4}=\{v_1,v_2,v_5\}$, $I_{v_5}=\{v_1,v_2,v_4\}$.
Then $S_{IV}=\{\{v_3,v_4,v_5\},\{v_4,v_5\},\{v_1\},\{v_1,v_2,v_5\},\{v_1,v_2,v_4\}\}$

By taking finitely intersection the basis obtained

$\{\varphi,\{v_3,v_4,v_5\},\{v_4,v_5\},\{v_1\},\{v_1,v_2,v_5\},\{v_1,v_2,v_4\},\{v_1,v_2\},\{v_4\},\{v_5\}\}$

Then by taking all unions the *Independent Topology* can be written as:

$\mathcal{T}_{IV}=\{\varphi,V,\{v_3,v_4,v_5\},\{v_4,v_5\},\{v_1\},\{v_1,v_2,v_5\},\{v_1,v_2,v_4\},\{v_1,v_2\},\{v_4\},\{v_5\}$
$\{v_1,v_3,v_4,v_5\},\{v_1,v_4,v_5\},\{v_1,v_2,v_4,v_5\},\{v_1,v_4\},\{v_1,v_5\}\}$

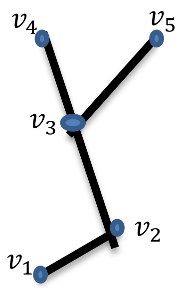


Fig. 4

Clearly, $\{v_2\}$ *and* $\{v_3\}$ are cut vertex in the graph but both of them do not belong to $\mathcal{T}_{IV}$

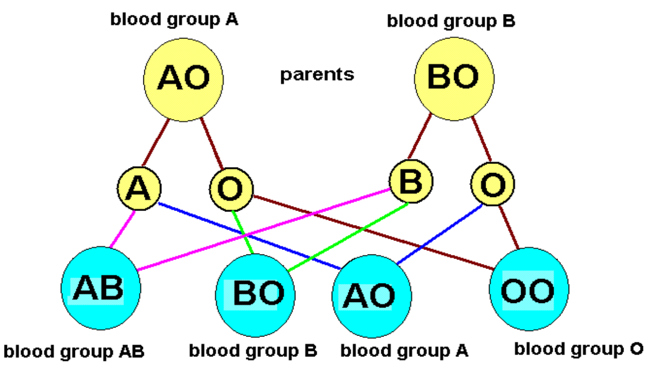
Proposition 4.14. Let $G=(V,E)$ be a connected graph and M is a minimal vertex cut in G. Then $M\in T_{IV}$.
Proof. Suppose that $G-M$ has $k\geq 2$ components, say $G_i=(V_i,E_i)$ for $i=1,2,\ldots,k$. Every vertex $v\in M$ must be adjacent to vertices of at least two different components, say G_1 and G_2, because M is a minimal vertex cut.
Suppose that $\{u_1,\ldots,u_{k1}\}=I_v\cap V_1$ and $\{w_1,\ldots,w_{k2}\}=I_v\cap V_2$, then we have $v\in\cap_{i=1}^{k_1} I_{ui}\subseteq M\cup V_1$ and $v\in\cap_{i=1}^{k_2} I_{wi}\subseteq M\cup V_2$ and so $v\in\left(\cap_{i=1}^{k_1} I_{u_i}\right)\cap\left(\cap_{i=2}^{k_2} I_{w_i}\right)\subseteq M\cup(V_1\cap V_2)=M$ that is v is an interior point of M. □

5. Application of *Independent Topology* in biomathematics.

We apply the above definition on a bio-mathematical applications. We conclude that the undirected graph must be connected for modifying the bio-mathematical state.

5.1. In a possible genetic for the inheritance of blood group.

There are four main blood groups (types of blood) A, B , AB and O, your blood group is determined by the genes you inherit from your parents. Everyone has an (ABO) blood type just like eye or hair color.
Each biological parent donates one of two (ABO) genes to their child, the A and B genes are dominant and the O gene is recessive [12].



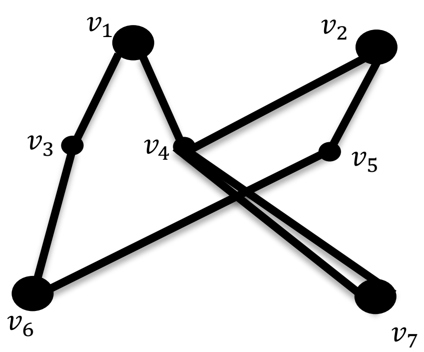


Fig. 5: diagram of a possible genetic for the inheritance of blood group and it is graph.

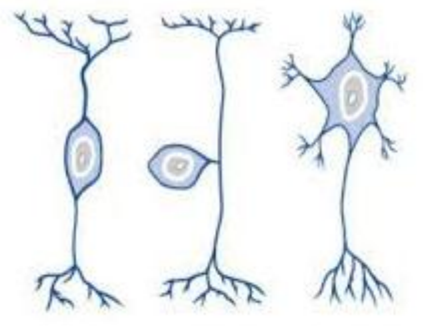
By a graph above, $V = \{v_1, v_2, v_3\ v_4\ v_5\ v_6\ , v_7\}$, where $v_1 = \text{AO}$, $v_2 = \text{BO}$, $v_3 = \text{A}$, $v_4 = \text{O}$, $v_5 = \text{B}$, $v_6 = \text{AB}$, $v_7 = \text{OO}$. We have;
$I_{v_1} = \{v_2, v_5, v_6, v_7\}$, $I_{v_2} = \{v_1, v_3, v_6, v_7\}$, $I_{v_3} = \{v_2, v_4, v_5, v_7\}$, $I_{v_4} = \{v_3, v_5, v_6\}$
$I_{v_5} = \{v_1, v_3, v_4, v_7\}$, $I_{v_6} = \{ v_1, v_2, v_4, v_7\}$, $I_{v_7} = \{v_1, v_2, v_3, v_5, v_6\}$
Then $S_{IV} = \{\{v_2, v_5, v_6, v_7\}, \{v_1, v_3, v_6, v_7\}, \{v_2, v_4, v_5, v_7\}, \{v_3, v_5, v_6\}, \{v_1, v_3, v_4, v_7\}, \{ v_1, v_2, v_4, v_7\}, \{v_1, v_2, v_3, v_5, v_6\}\}$

By taking finitely intersection the basis obtained
$\{ \varphi, \{v_2, v_5, v_6, v_7\}, \{v_1, v_3, v_6, v_7\}, \{v_2, v_4, v_5, v_7\}, \{v_3, v_5, v_6\}, \{v_1, v_3, v_4, v_7\}, \{ v_1, v_2, v_4, v_7\}, \{v_1, v_2, v_3, v_{5,} v_{6,}\}, \{v_6, v_7\}, \{v_2, v_5, v_7\}, \{ v_5, v_6\}, \{v_7\}, \{v_2, v_7\}, \{v_2, v_5, v_6\}, \{v_3, v_6\}, \{v_1, v_3, v_7\}, \{v_1, v_7\}, \{v_1, v_3, v_6\}, \{ v_5\}, \{v_4, v_7\}, \{v_2, v_4, v_7\}, \{v_2, v_5\}, \{v_3\}, \{v_6\}, \{v_1, v_4, v_7\}, \{v_1, v_3\}, \{v_6\}, \{v_1, v_2\}, \{v_2, v_7\}, \{v_2\}, \{v_2, v_5\}, \{v_5, v_6\}, \{v_1, v_7\}\}$

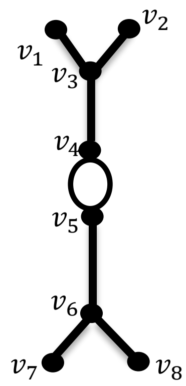
Then by taking all unions the *Independent Topology* can be written as:
$\mathcal{T}_{IV} = \{ \varphi,\ V, \{v_1\}, \{v_2\}, \{v_3\}, \{ v_5\}, \{v_6\}, \{v_7\}, \{v_2, v_5, v_6, v_7\}, \{v_1, v_3, v_6, v_7\}, \{v_3, v_5, v_6\}, \{v_2, v_4, v_5, v_7\}, \{v_1, v_3, v_4, v_7\}, \{ v_1, v_2, v_4, v_7\}, \{v_1, v_2, v_3, v_{5,} v_{6,}\}, \{v_6, v_7\}, \{v_2, v_5, v_7\}, \{ v_5, v_6\}, \{v_2, v_7\}, \{v_2, v_5, v_6\}, \{v_3, v_6\}, \{v_1, v_3, v_7\}, \{v_1, v_7\}, \{v_1, v_3, v_6\}, \{v_4, v_7\}, \{v_2, v_4, v_7\}, \{v_2, v_5\}, \{v_1, v_4, v_7\}, \{v_1, v_3\}, \{v_1, v_2\}, \{v_2, v_7\}, \{v_2, v_5\}, \{v_5, v_6\}, \{v_1, v_7\}, \{v_1, v_5\}, \{v_1, v_6\}, \{v_2, v_3\}\}$

5.2. In general shape of Bipolar neuron.

Neurons are the cells that make up the brain and the nervous system. They are the fundamental units that send and receive signals which allow us to move our muscles, feel the external world, think, form memories and much more.
Just from looking down a microscope, however, it becomes very clear that not all neurons are the same. So just how many types of neurons are there? And how do scientists decide on the categories? For neurons in the brain, at least, this isn't an easy question to answer. For the spinal cord though, we can say that there are three types of neurons: sensory, motor, and interneurons.
Most neurons can be anatomically characterized as: Unipolar, Bipolar, Multipolar.
Bipolar, these neurons have two processes arising from a central cell body, typically one axon and one dendrite. These cells are found in the retina [12].



Bipolar neuron   Unipolar neuron   Multipolar neuron



Graph of Bipolar neuron general shape

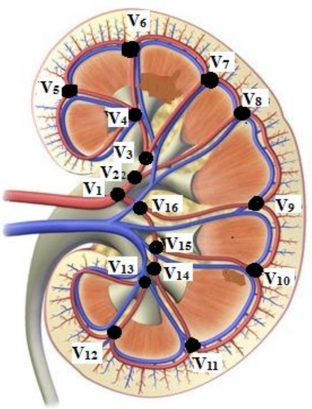
Fig 6: anatomically types of neuron and the graph of Bipolar neuron shape.

By a Graph of Bipolar neuron general shape, $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and;
$I_{v_1}=\{v_2, v_4, v_5, v_6, v_7, v_8\}$, $I_{v_2}=\{v_1, v_4, v_5, v_6, v_7, v_8\}$, $I_{v_3}=\{v_5, v_6, v_7, v_8\}$,
$I_{v_4} = \{v_1, v_2, v_6, v_7, v_8\}$, $I_{v_5} = \{v_1, v_2, v_3, v_7, v_8\}$, $I_{v_6} = \{v_1, v_2, v_3, v_4\}$,
$I_{v_7} = \{v_1, v_2, v_3, v_4, v_5, v_8\}$, $I_{v_8} = \{v_1, v_2, v_3, v_4, v_5, v_7\}$. Then;
$S_{IV} =\{\{v_2, v_4, v_5, v_6, v_7, v_8\}, \{v_1, v_4, v_5, v_6, v_7, v_8\}, \{v_5, v_6, v_7, v_8\}, \{v_1, v_2, v_6, v_7, v_8\}, \{v_1, v_2, v_3, v_7, v_8\}, \{v_1, v_2, v_3, v_4\},\{v_1, v_2, v_3, v_4, v_5, v_8\}, \{v_1, v_2, v_3, v_4, v_5, v_7\}\}$

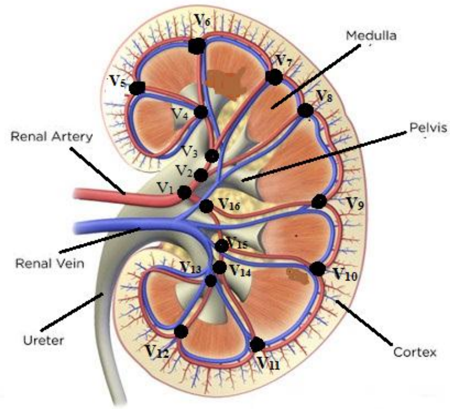
By taking finitely intersection the basis obtained;
$\{ \varphi, \{v_2, v_4, v_5, v_6, v_7, v_8\}, \{v_1, v_4, v_5, v_6, v_7, v_8\}, \{v_5, v_6, v_7, v_8\}, \{v_1, v_2, v_6, v_7, v_8\}, \{v_1, v_8\}, \{v_1, v_2, v_3, v_7, v_8\}, \{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_4, v_5, v_8\}, \{v_1, v_2, v_3, v_4, v_5, v_7\}, \{v_5, v_8\}, \{v_4, v_5, v_6, v_7, v_8\}, \{v_2, v_6, v_7, v_8\}, \{v_2, v_7, v_8\},\{v_2, v_4\}, \{v_2, v_4, v_5, v_8\}, \{v_2, v_4, v_5, v_7\}, \{v_1, v_6, v_7, v_8\}, \{v_1, v_7, v_8\}, \{v_1, v_4\}, \{v_1, v_4, v_5, v_8\}, \{v_1, v_4, v_5, v_7\}, \{v_6, v_7, v_8\}, \{v_7, v_8\}, , \{v_5, v_7\}, \{v_1, v_2, v_7, v_8\}, \{v_1, v_2\}, \{v_1, v_2, v_8\}, \{v_1, v_2, v_7\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_3, v_8\}, \{v_1, v_2, v_3, v_7\}, \{v_1, v_2, v_3, v_4, v_5\} , \{v_4\} , \{v_4, v_5\}, \{v_7\}, \{v_2\}, \{v_8\}, \{v_5\}, \{v_1\}, \{v_2, v_7, v_8\}, \{v_4, v_5, v_7\}\}$

Then by taking all unions the *independent topology* can be written as:
$\mathcal{T}_{IV} = \{\varphi, V, \{v_1\}, \{v_2\}, \{v_4\} , \{v_5\}, \{v_7\}, \{v_8\}, \{v_1, v_2\}\{v_1, v_4\}\{v_1, v_5\}\{v_1, v_7\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_2, v_7\}, \{v_2, v_8\}, \{v_7, v_8\}, \{v_5, v_8\} , \{v_5, v_7\} , \{v_4, v_5\}, \{v_4, v_7\}, \{v_1, v_7, v_8\}, \{v_2, v_7, v_8\} , \{v_6, v_7, v_8\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_5\}, \{v_1, v_2, v_7\}, \{v_1, v_2, v_8\}, \{v_1, v_4, v_5\}, \{v_2, v_4, v_5\} , \{v_1, v_4, v_7\}, \{v_1, v_4, v_8\}, \{v_1, v_5, v_7\}, \{v_1, v_5, v_8\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_7\ \{v_2, v_4, v_5\}, \{v_2, v_4, v_7\},, \{v_2, v_5, v_7\},\{v_2, v_5, v_8\}, \{v_2, v_7, v_8\}, \{v_4, v_7, v_8\}, \{v_4, v_5, v_7\}, \{v_4, v_5, v_8\},\{v_5, v_7, v_8\}, \{v_2, v_4, v_8\} \{v_5, v_6, v_7, v_8\}, \{v_1, v_2, v_3, v_4\},\{v_1, v_2, v_4, v_7\}, \{v_2, v_6, v_7, v_8\} , \{v_2, v_4, v_5, v_8\}, \{v_2, v_4, v_5, v_7\}, \{v_1, v_6, v_7, v_8\}, \{v_2, v_6, v_7, v_8\}, \{v_1, v_4, v_5, v_8\}, \{v_1, v_4, v_5, v_7\}, \{v_1, v_4, v_5, v_7\} \{v_1, v_4, v_7, v_8\}, \{v_1, v_2, v_3, v_8\}, \{v_1, v_2, v_3, v_7\}, \{v_1, v_2, v_6, v_7, v_8\}, \{v_1, v_2, v_3, v_7, v_8\}, \{v_1, v_8\} \{v_1, v_2, v_7, v_8\}, \{v_4, v_5, v_6, v_7, v_8\}, \{v_1, v_2, v_3, v_4, v_5\}, \{v_2, v_4, v_5, v_6, v_7, v_8\}, \{v_1, v_4, v_5, v_6, v_7, v_8\} ,\{v_1, v_2, v_3, v_4, v_5, v_8\}, \{v_1, v_2, v_3, v_4, v_5, v_7\} \}$.

5.3. In connections of the renal artery of human kidney.
The kidneys are a pair of bean-shaped organs on either side of your spine, below your ribs and behind your belly. Each kidney receive blood from the paired renal arteries; blood exits into the paired renal veins. Each kidney is attached to a ureter, a tube that carries excreted urine to the bladder, and has around a million tiny filters called nephrons.
The kidneys' job is to filter your blood. They remove wastes, control the body's fluid balance, and keep the right levels of electrolytes. All of the blood in your body passes through them several times a day [12].



Renal Artery Graph



Anatomy of Human Kidney

Fig. 7: Anatomy of Human Kidney and Renal Artery Graph.

Now, let $G = (V, E)$ be a graph represents the associations (connections) points of the renal artery of human kidney (which is a non-simple graph because it has two multiple edges (v_4, v_5) and (v_{12}, v_{13})) as in Fig.7 such that; $V = \{v_1, v_2, \dots \dots, v_{16}\}$

We have a sub-basis family of the *Independent topology* as fellow:
$I_{v_1} = \{v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$,
$I_{v_2} = \{v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$,
$I_{v_3} = \{v_1, v_5, v_6, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$,
$I_{v_4} = \{v_1, v_2, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$,
$I_{v_5} = \{v_1, v_2, v_3, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$,
$I_{v_6} = \{v_1, v_2, v_3, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$,
$I_{v_7} = \{v_1, v_2, v_4, v_5, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$,
$I_{v_8} = \{v_1, v_3, v_4, v_5, v_6, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$,
$I_{v_9} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$,
$I_{v_{10}} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_{12}, v_{13}, v_{14}, v_{16}\}$,
$I_{v_{11}} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{13}, v_{15}, v_{16}\}$,
$I_{v_{12}} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{14}, v_{15}, v_{16}\}$,
$I_{v_{13}} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{15}, v_{16}\}$,
$I_{v_{14}} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{12}, v_{16}\}$,
$I_{v_{15}} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{11}, v_{12}, v_{13}\}$,
$I_{v_{16}} = \{v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$

Then, by taking finitely intersection we find the basis, and after that find the all unions, the Independent Topology will obtained.

Conclusions :

A synthesis between graph theory and topology has been made. A topology with the set of vertices for any undirected graph has been associated, called idependent topology. The study of some properties of this new model of topology has been presented. It has been shown that this topology is an Alexandroff topology. Useful applications of idependent topology in biomathematics have been introduced. Therefore, this article can be considered as a point of applying another topological concept of graphs in scientific fields, which could lead to another significant applications in the future.

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On $\mathcal{P}$Wπ-regular rings

Raida D. Mahmood / **Khedher J. khider**

Department of Mathematics / College of Computer Science and Mathematics

University of Mosul / Mosul - Iraq

raida.1961@uomosul.edu.iq / khedirjumaa@gmail.com

Abstract.

As a popularization of weakly π-regular rings , we tender the connotation of $\mathcal{P}$Wπ-regular rings , that is if for each $\mathfrak{a} \in ɉ(\mathfrak{N})$, there exist a natural number $\mathfrak{n}$ such that $\mathfrak{a}^{\mathfrak{n}} \in \mathfrak{a}^{\mathfrak{n}}\mathfrak{N}\mathfrak{a}^{\mathfrak{n}}\mathfrak{N}$ ($\mathfrak{a}^{\mathfrak{n}} \in \mathfrak{N}\mathfrak{a}^{\mathfrak{n}}\mathfrak{N}\mathfrak{a}^{\mathfrak{n}}$) . In this treatise , numerous properties of this sort of rings are discussed , some important results are secured . Using the connotation of $\mathcal{P}$Wπ-regular rings . It is show that :

1- Let $\mathfrak{N}$ be a right $\mathcal{P}$Wπ-regular ring and $\aleph$ɉ-rings with $\mathfrak{a}^{\mathfrak{n}}\mathfrak{N} = \mathfrak{N}\mathfrak{a}^{\mathfrak{n}} for\ every$ $\mathfrak{a} \in ɉ(\mathfrak{N})$ and for at least one of a natural number . Then $ɉ(\mathfrak{N}) = \aleph(\mathfrak{N})$.

2- Let $\mathfrak{N}$ a right $\mathcal{P}$Wπ-regular ring and $\mathfrak{a}\mathfrak{N} = \mathfrak{N}\mathfrak{a}$ for each $\in ɉ(\mathfrak{N})$. Then $\mathfrak{N}$ is right $\mathcal{P}.\mathfrak{T}$-ring .

3- Let $\mathfrak{N}$ be a ring with $ɍ(\mathfrak{a}) \subseteq ɭ(\mathfrak{a})$, for each $\in ɉ(\mathfrak{N})$. If any of the next conditions are hold , then $\mathfrak{N}$ is $\mathcal{P}$Wπ-regular rings :

i – Every maximal right ideal of $\mathfrak{N}$ is a right annihilator and right ɉ$\mathcal{PP}$-ring .

ii- any simple singular right $\mathfrak{N}$-module is ɉ-injective and $\mathfrak{N}$ is semi prime .

Keywords : $\mathcal{P}$Wπ-regular ring , ɉ-injective rings , ɉ$\mathcal{PP}$-rings , ɉ-regular ring .

j) Introduction .

Over this treatise, $\mathfrak{N}$ refers to an associative ring with identity and each module is unitary $\mathfrak{N}$-module . We write $ɉ(\mathfrak{N})$, $Y(\mathfrak{N})$, and $\aleph(\mathfrak{N})$ for the Jacobson radical , the right singular ideal and the set of nilpotent elements of $\mathfrak{N}$, respectively . We use the contraction $ɭ(\mathfrak{a})$, $ɍ(\mathfrak{a})$ for the left , right annihilator of $\mathfrak{a}$ in $\mathfrak{N}$.

ɉ-injective rings were defined and discussed [5] , [10]. A ring$\mathfrak{N}$ is define as a right ɉ-injective [10], whether each $\mathfrak{a} \in ɉ(\mathfrak{N})$, $ɭɍ(\mathfrak{a}) = \mathfrak{N}\mathfrak{a}$. Recall that $\mathfrak{N}$ is known as a right (left) weakly π-regular (Wπ-regular) [7] , if every $\mathfrak{a} \in \mathfrak{N}$, there is a natural number $\mathfrak{n}$ such that $\mathfrak{a}^{\mathfrak{n}} \in \mathfrak{a}^{\mathfrak{n}}\mathfrak{N}\mathfrak{a}^{\mathfrak{n}}\mathfrak{N}$ ($\mathfrak{a}^{\mathfrak{n}} \in \mathfrak{N}\mathfrak{a}^{\mathfrak{n}}\mathfrak{N}\mathfrak{a}^{\mathfrak{n}}$). According to [4] $\mathfrak{N}$ is said to be $\mathfrak{n}$-weakly regular ring if for any $\mathfrak{a} \in \aleph(\mathfrak{N})$, $\mathfrak{a} \in \mathfrak{a}\mathfrak{N}\mathfrak{a}$. A ring $\mathfrak{N}$ is said to be reduced if $\aleph(\mathfrak{N})=0$ [3]. $\mathfrak{N}$ is said to be right (left) $\mathcal{SXM}$ if for each $0 \neq \mathfrak{a} \in \mathfrak{N}$, there is a natural number $\mathfrak{n}$ such that $\mathfrak{a}^{\mathfrak{n}} \neq 0$, $ɍ(\mathfrak{a}) = ɍ(\mathfrak{a}^{n})$($ɭ(\mathfrak{a}) = ɭ(\mathfrak{a}^{n})$), every reduced is $\mathcal{SXM}$ but convers is not true [8] . A ring is define is semiprime ring if and only if it contains no non-zero nilpotent ideal [2] .

An element $\mathfrak{a}$ in the ring $\mathfrak{N}$ is said to be right (left) $\mathcal{P}.\mathfrak{T}$- element , if there is an idempotent element $ȩ$ in $\mathfrak{N}$ such that $v = vȩ$ $(v = ȩv)$ and $\mathfrak{r}(v) = \mathfrak{r}(ȩ)(\mathfrak{l}(v) = \mathfrak{l}(ȩ))$. $\mathfrak{N}$ is known as a right (left) $\mathcal{P}.\mathfrak{T}$-ring . whether each element in $\mathfrak{N}$ is right (left) $\mathcal{P}.\mathfrak{T}$-element [1]. For example $\mathcal{Z}_6$ is $\mathcal{P}.\mathfrak{T}$-ring [1] .

In this treatise, we shall popularize the connotation of weakly π-regular rings to $\mathcal{P}$Wπ-regular , numerous properties of this sort of rings are discussed , little conditions under which $\mathcal{P}$Wπ-regular are $\mathcal{P}.\mathfrak{T}$-ring, $\mathfrak{J}$-regular , strongly regular rings will be given .

k) Popularized weakly π-regular rings .

Definition 2.1 : $\mathfrak{N}$ is defined as a right (left) popularized weakly π-regular ($\mathcal{P}$Wπ-regular) if , for each $\mathfrak{a} \in \mathfrak{J}(\mathfrak{N})$, there exist a positive integer $\mathfrak{n}$ such that $\mathfrak{a}^{\mathfrak{n}} \in \mathfrak{a}^{\mathfrak{n}}\mathfrak{N}\mathfrak{a}^{\mathfrak{n}}\mathfrak{N}$ ($\mathfrak{a}^{\mathfrak{n}} \in \mathfrak{N}\mathfrak{a}^{\mathfrak{n}}\mathfrak{N}\mathfrak{a}^{\mathfrak{n}}$) .

Example : Assume that $\mathcal{A}$ is division ring . Then the 2 by 2 upper triangle ring $\mathfrak{N} = \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ 0 & \mathcal{A} \end{bmatrix}$ is $\mathcal{P}$Wπ-regular ring . Clearly $\mathfrak{J}(\mathcal{T}_2(\mathcal{A})) = \begin{bmatrix} 0 & \mathcal{A} \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & \mathcal{A} \\ 0 & 0 \end{bmatrix}^{\mathfrak{n}} \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ 0 & \mathcal{A} \end{bmatrix} \begin{bmatrix} 0 & \mathcal{A} \\ 0 & 0 \end{bmatrix}^{\mathfrak{n}} \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ 0 & \mathcal{A} \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{A} \\ 0 & 0 \end{bmatrix}^{\mathfrak{n}} \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ 0 & \mathcal{A} \end{bmatrix}$.

Remark : Every weakly π-regular ring is $\mathcal{P}$Wπ-regular ring but the converse is not always true : Let $\mathcal{Z}$ be the ring of integer . Then $\mathfrak{J}(\mathcal{Z}) = 0$. Then $\mathcal{Z}$ is $\mathcal{P}$Wπ-regular ring which is not Wπ-regular ring.

Proposition 2.2 : If $\mathfrak{N}$ is right $\mathcal{P}$Wπ-regular ring and $\mathfrak{r}(\mathfrak{a}) = 0$ for all $0 \neq \mathfrak{a} \in \mathfrak{J}(\mathfrak{N})$. Then $\mathfrak{N} = \mathfrak{a}^{\mathfrak{n}}\mathfrak{N}$.

Proof : Let $\mathfrak{N}$ be a right $\mathcal{P}$Wπ-regular . Then for all $0 \neq \mathfrak{a} \in \mathfrak{J}(\mathfrak{N})$, there is a natural number $\mathfrak{n}$ such that $\mathfrak{a}^{\mathfrak{n}}\mathfrak{N} = \mathfrak{a}^{\mathfrak{n}}\mathfrak{N}\mathfrak{a}^{\mathfrak{n}}\mathfrak{N}$, this implies $\sigma^{\mathfrak{n}}(\mathfrak{N} - \mathfrak{N}\mathfrak{a}^{\mathfrak{n}}\mathfrak{N}) = 0$ and hence $(\mathfrak{N} - \mathfrak{N}\mathfrak{a}^{\mathfrak{n}}\mathfrak{N}) \in \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}}) = 0$, hence it follows that $\mathfrak{N} - \mathfrak{N}\mathfrak{a}^{\mathfrak{n}}\mathfrak{N} = 0$. Therefore $= \mathfrak{N}\mathfrak{a}^{\mathfrak{n}}\mathfrak{N}$.

Proposition 2.3 : if $\mathfrak{N}$ is reduced ring . Then it is a right $\mathcal{P}$Wπ-regular iff $\mathfrak{N}$ is left $\mathcal{P}$Wπ-regular .

Proof : suppose that $\mathfrak{R}$ is right $\mathcal{P}$Wπ-regular . Then for each $\mathfrak{a} \in \mathfrak{J}(\mathfrak{R})$ there is a natural number $\mathfrak{n}$ and $\mathfrak{b}, \mathfrak{c} \in \mathfrak{R}$ such that $\mathfrak{a}^{\mathfrak{n}} = \mathfrak{a}^{\mathfrak{n}}\mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c}$. Now $(\mathfrak{a}^{\mathfrak{n}} - \mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c}\mathfrak{a}^{\mathfrak{n}})^2 = 0$. Since $\mathfrak{R}$ is reduced, then $\mathfrak{a}^{\mathfrak{n}} - \mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c}\mathfrak{a}^{\mathfrak{n}} = 0$. Therefore $\mathfrak{a}^{\mathfrak{n}} = \mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c}\mathfrak{a}^{\mathfrak{n}}$. Hence $\mathfrak{R}$ is left $\mathcal{P}$Wπ-regular .The converse is similar .

Following [2], a ring $\mathfrak{R}$ is said to be $\aleph\mathfrak{J}$ if $\aleph(\mathfrak{R}) \subseteq \mathfrak{J}(\mathfrak{R})$.

Theorem 2.4 : Let $\mathfrak{R}$ be a right $\mathcal{P}$Wπ-regular and $\aleph\mathfrak{J}$-ring with $\mathfrak{a}^{\mathfrak{n}}\mathfrak{R} = \mathfrak{R}\mathfrak{a}^{\mathfrak{n}} \textit{for every}\ \mathfrak{a} \in \mathfrak{J}(\mathfrak{R})$ and for some a natural number $\mathfrak{n}$. Then $\mathfrak{J}(\mathfrak{R}) = \aleph(\mathfrak{R})$.

Proof : assume that $0 \neq \mathfrak{a} \in \mathfrak{J}(\mathfrak{R})$ and let $\mathfrak{R}$ be a right $\mathcal{P}$Wπ-regular . Then there is a natural number $\mathfrak{n}$ and $\mathfrak{b}, \mathfrak{c} \in \mathfrak{R}$ such that $\mathfrak{a}^{\mathfrak{n}} = \mathfrak{a}^{\mathfrak{n}}\mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c} = \mathfrak{a}^{\mathfrak{n}}\mathfrak{b}d\mathfrak{a}^{\mathfrak{n}}$, $(\mathfrak{a}^{\mathfrak{n}}\mathfrak{R} = \mathfrak{R}\,\mathfrak{a}^{\mathfrak{n}})$, $d \in \mathfrak{R}$. Then $\mathfrak{a}^{\mathfrak{n}} = \mathfrak{a}^{\mathfrak{n}}h\sigma^{\mathfrak{n}}$, $(h = \mathfrak{b}d)$. implies that $\mathfrak{a}^{\mathfrak{n}}(1 - h\mathfrak{a}^{\mathfrak{n}}) = 0$. Since $\mathfrak{a} \in \mathfrak{J}(\mathfrak{R})$ then $\mathfrak{a}^{\mathfrak{n}} \in \mathfrak{J}(\mathfrak{R})$ gives $(1 - h\mathfrak{a}^{\mathfrak{n}})$ is invertible , so $(1 - h\mathfrak{a}^{\mathfrak{n}})u = 1$ for some $u \in \mathfrak{R}$, implies that $(\mathfrak{a}^{\mathfrak{n}} - \mathfrak{a}^{\mathfrak{n}}h\sigma^{\mathfrak{n}})u = \mathfrak{a}^{\mathfrak{n}} = 0$. Thus $\in \aleph(\mathfrak{R})$, and hence $\mathfrak{J}(\mathfrak{R}) \subseteq \aleph(\mathfrak{R})$. But $\mathfrak{R}$ is $\aleph\mathfrak{J}$, therefore $\aleph(\mathfrak{R}) \subseteq \mathfrak{J}(\mathfrak{R})$ and hence $\mathfrak{J}(\mathfrak{R}) = \aleph(\mathfrak{R})$.

Theorem 2.5: If $\mathfrak{R}$ is $\mathcal{SXM}$ ring and right $\mathcal{P}$Wπ-regular , then $\mathfrak{J}(\mathfrak{R}) \cap \aleph(\mathfrak{R}) = 0$.

Proof : Let $\mathfrak{J}(\mathfrak{R}) \cap \aleph(\mathfrak{R})$ not equal to zero . So there exist $0 \neq \mathfrak{a} \in \mathfrak{J}(\mathfrak{R}) \cap \aleph(\mathfrak{R})$. Since $\mathfrak{R}$ is right $\mathcal{P}$Wπ-regular , so there is a natural number $\mathfrak{n}$ and $\mathfrak{b}, \mathfrak{c} \in \mathfrak{R}$ such that $\mathfrak{a}^{\mathfrak{n}} = \mathfrak{a}^{\mathfrak{n}}\mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c}$. Thus $\mathfrak{a}^{\mathfrak{n}}(1 - \mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c}) = 0$, this implies $(1 - \mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c}) \in \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}}) = \mathfrak{r}(\mathfrak{a})$, implies $\mathfrak{a}(1 - \mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c}) = 0$. Since $\mathfrak{a} \in \mathfrak{J}(\mathfrak{R})$ then $\mathfrak{a}^{\mathfrak{n}} \in \mathfrak{J}(\mathfrak{R})$. So $\mathfrak{a}^{\mathfrak{n}}\mathfrak{c} \in \mathfrak{J}(\mathfrak{R})$, gives $(1 - \mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c})$ is invertible , so $(1 - \mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c})u = 1$ for some $u \in \mathfrak{R}$, implies that$(\mathfrak{a} - \mathfrak{a}\mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c})u = \mathfrak{a} = 0$. This is contradiction. Hence $\mathfrak{J}(\mathfrak{R}) \cap \aleph(\mathfrak{R}) = 0$.

Proposition 2.6 : Let $\mathfrak{R}$ be reduced ring . Then $\mathfrak{R}$ is right $\mathcal{P}$Wπ-regular iff $\mathfrak{R}/_{\mathfrak{r}(\mathfrak{a})}$ is right $\mathcal{P}$Wπ-regular .

Proof : Suppose that $\mathfrak{R}/_{\mathfrak{r}(\mathfrak{a})}$ is right $\mathcal{P}$Wπ-regular , then for every $\mathfrak{a} \in \mathfrak{J}(\mathfrak{R})$ there is a natural number $\mathfrak{n}$ and $\mathfrak{b}, \mathfrak{c} \in \mathfrak{R}$ such that $\big(\mathfrak{a} + \mathfrak{r}(\mathfrak{a})\big)^{\mathfrak{n}} = \big(\mathfrak{a} + \mathfrak{r}(\mathfrak{a})\big)^{\mathfrak{n}}\big(\mathfrak{b} + \mathfrak{r}(\mathfrak{a})\big)\big(\mathfrak{a} + \mathfrak{r}(\mathfrak{a})\big)^{\mathfrak{n}}\big(\mathfrak{c} + \mathfrak{r}(\mathfrak{a})\big)$, implies that $\mathfrak{a}^{\mathfrak{n}} + \mathfrak{r}(\mathfrak{a}) = \mathfrak{a}^{\mathfrak{n}}\mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c} + \mathfrak{r}(\mathfrak{a})$. Therefore $(\mathfrak{a}^{\mathfrak{n}} - \mathfrak{a}^{\mathfrak{n}}\mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c}) \in \mathfrak{r}(\mathfrak{a})$ and so $\mathfrak{a}\mathfrak{a}^{\mathfrak{n}}(1 - \mathfrak{b}\sigma^{\mathfrak{n}}\mathfrak{c}) = 0$,implies that $(1 - \mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c}) \in \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}+1}) = \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}})$ ($\mathfrak{R}$ is reduced) . Therefore $\mathfrak{a}^{\mathfrak{n}}(1 - \mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c}) = 0$, which yields $\mathfrak{a}^{\mathfrak{n}} = \mathfrak{a}^{\mathfrak{n}}\mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c}$. Hence $\mathfrak{R}$ is right $\mathcal{P}$Wπ-regular, The conversely is clear .

l) The relevance among right $\mathcal{P}$Wπ-regular and other rings

Following [10] , $\mathfrak{N}$ is called right $\mathfrak{j}$-regular ring ($\mathfrak{j}$-regular) . whether for each $\mathfrak{a} \in \mathfrak{j}(\mathfrak{N})$, $\mathfrak{a} \in \mathfrak{a}\mathfrak{N}\mathfrak{a}$.

Theorem 3.1 : Assume that $\mathfrak{N}$ is right $\mathcal{P}$Wπ-regular and $\mathfrak{a}^{\mathfrak{n}}\mathfrak{N} = \mathfrak{N}\mathfrak{a}$ for each $\mathfrak{a} \in \mathfrak{j}(\mathfrak{N})$ and a natural number n . Then $\mathfrak{N}$ is $\mathfrak{j}$-regular .

Proof : Assume that $\mathfrak{a} \in \mathfrak{j}(\mathfrak{N})$, and let $\mathfrak{N}$ is right $\mathcal{P}$Wπ-regular , then there is a natural number $\mathfrak{n}$ such that $\mathfrak{a}^{\mathfrak{n}}\mathfrak{N} = \mathfrak{a}^{\mathfrak{n}}\mathfrak{N}\mathfrak{a}^{\mathfrak{n}}\mathfrak{N}$, since $\mathfrak{a}^{\mathfrak{n}}\mathfrak{N} = \mathfrak{N}\mathfrak{a}$, then $\mathfrak{a} \in \mathfrak{N}\mathfrak{a} = \mathfrak{a}^{\mathfrak{n}}\mathfrak{N}\mathfrak{N}\mathfrak{a} = \mathfrak{a}\mathfrak{a}^{\mathfrak{n}-1}\mathfrak{N}\mathfrak{N}\mathfrak{a} = \mathfrak{a}\mathfrak{N}\mathfrak{a}$, implies that $\mathfrak{a} \in \mathfrak{a}\mathfrak{N}\mathfrak{a}$ for every $\mathfrak{a} \in \mathfrak{j}(\mathfrak{N})$. Hence $\mathfrak{N}$ is $\mathfrak{j}$-regular .

Proposition 3.2 : Suppose that $\mathfrak{N}$ is $\aleph\mathfrak{j}$ with $\mathfrak{a}^{\mathfrak{n}}\mathfrak{N} = \mathfrak{N}\mathfrak{a}$ for each $\mathfrak{a} \in \mathfrak{j}(\mathfrak{N})$ and a natural number n . Then $\mathfrak{N}$ is $\mathfrak{n}$-weakly regular ring iff $\mathfrak{N}$ is $\mathcal{P}$Wπ-regular .

Proof : $\mathfrak{j}(\mathfrak{N}) = \aleph(\mathfrak{N})$ (Theorem 2.4) . So $\mathfrak{N}$ is $\mathfrak{n}$-weakly regular iff $\mathfrak{N}$ is $\mathcal{P}$Wπ-regular .

Theorem 3.3 : Suppose that $\mathfrak{N}$ is right $\mathcal{P}$Wπ-regular and $\mathfrak{a}\mathfrak{N} = \mathfrak{N}\mathfrak{a}$ for each $\mathfrak{a} \in \mathfrak{j}(\mathfrak{N})$. Then $\mathfrak{N}$ is right $\mathcal{P}.\mathfrak{T}$-ring .

Proof : Since $\mathfrak{N}$ is right $\mathcal{P}$Wπ-regular ring . Then for any $\mathfrak{a} \in \mathfrak{j}(\mathfrak{N})$, there is a natural number $\mathfrak{n}$ and $\mathfrak{b}, d \in \mathfrak{N}$ such that $\mathfrak{a}^{\mathfrak{n}} = \mathfrak{a}^{\mathfrak{n}}\mathfrak{b}\mathfrak{a}^{\mathfrak{n}}d = \mathfrak{a}^{\mathfrak{n}}\mathfrak{b}\mathfrak{c}\mathfrak{a}^{\mathfrak{n}} = \mathfrak{a}^{\mathfrak{n}}w\mathfrak{a}^{\mathfrak{n}}$, when $w = \mathfrak{b}\mathfrak{c}$, if we take $f = w\mathfrak{a}^{\mathfrak{n}}$, then $f^2 = w\mathfrak{a}^{\mathfrak{n}}w\mathfrak{a}^{\mathfrak{n}} = w\mathfrak{a}^{\mathfrak{n}} = f$, then f is idempotent element and $\mathfrak{a}^{\mathfrak{n}} = \mathfrak{a}^{\mathfrak{n}}f$. Now let $\mathfrak{b} \in \mathfrak{r}(f)$, implies $f\mathfrak{b} = 0$, and $w\mathfrak{a}^{\mathfrak{n}}\mathfrak{b} = 0$, implies that $\mathfrak{a}^{\mathfrak{n}}w\mathfrak{a}^{\mathfrak{n}}\mathfrak{b} = 0$, and hence $\mathfrak{a}^{\mathfrak{n}}\mathfrak{b} = 0$. Therefore $\mathfrak{b} \in \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}})$ and we get $\mathfrak{r}(f) \subseteq \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}}) \ldots (1)$. Now let $\in \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}})$, implies $\mathfrak{a}^{\mathfrak{n}}z = 0$ and $w\mathfrak{a}^{\mathfrak{n}}z = 0$, implies that $fz = 0$. Therefore$z \in \mathfrak{r}(f)$ and we get$\mathfrak{r}(\mathfrak{a}^{\mathfrak{n}}) \subseteq \mathfrak{r}(f) \ldots (2)$. From (1) and (2) we get $\mathfrak{r}(\mathfrak{a}^{\mathfrak{n}}) = \mathfrak{r}(f)$. Hence $\mathfrak{N}$ is $\mathcal{P}.\mathfrak{T}$ -ring .

Following [10] , $\mathfrak{N}$ is said to be right $\mathfrak{j}\mathcal{P}\mathcal{P}$-ring . If $\mathfrak{a}\mathfrak{N}$ is projective for each $\in \mathfrak{J}(\mathfrak{N})$. In [10] we give the following lemma:

Lemma 3.4 : Let $\mathfrak{N}$ be a ring .Then it is right $\mathfrak{j}\mathcal{P}\mathcal{P}$-ring iff $\mathfrak{r}(\mathfrak{a}) = \c{e}\mathfrak{N}$, $\c{e}$ is some idempotent element in $\mathfrak{N}$, $\mathfrak{a} \in \mathfrak{j}(\mathfrak{N})$.

Proposition 3.5 : If $\mathfrak{N}$ is right $\mathfrak{j}\mathcal{P}\mathcal{P}$-ring , then $\Upsilon(\mathfrak{N}) = 0$.

Proof : Assume that $0 \neq \mathfrak{a} \in \Upsilon(\mathfrak{R})$, $\mathfrak{a}^2 = 0$. it is clear that $\mathfrak{a}\mathfrak{R}$ is projective , then $\mathfrak{r}(\mathfrak{a})$ must be direct summand of $\mathfrak{R}$. But $\in \Upsilon(\mathfrak{R})$, $\mathfrak{r}(\mathfrak{a})$ it then essential in $\mathfrak{R}$, but this is contradiction. Therefore $\Upsilon(\mathfrak{R}) = 0$.

Lemma 3.6 : Assume that $\mathfrak{R}$ is right $\mathfrak{f}\mathcal{PP}$-ring , $\mathfrak{r}(\mathfrak{a}) \subseteq \mathfrak{l}(\mathfrak{a})$, for each $\mathfrak{a} \in \mathfrak{f}(\mathfrak{R})$. Then $\mathfrak{R}$ is reduced .

Proof : Trivial .

Theorem 3.7 : Let $\mathfrak{R}$ is right $\mathfrak{f}\mathcal{PP}$-ring , $\mathfrak{r}(\mathfrak{a}) \subseteq \mathfrak{l}(\mathfrak{a})$, for each $\mathfrak{a} \in \mathfrak{f}(\mathfrak{R})$, and any right maximal ideal of $\mathfrak{R}$ is a right annihilator .Then $\mathfrak{R}$ is $\mathcal{P}\mathrm{W}\pi$-regular .

Proof : Suppose that $\mathfrak{a} \in \mathfrak{f}(\mathfrak{R})$, we must show that $\mathfrak{R}\mathfrak{a}^{\mathfrak{n}}\mathfrak{R} + \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}}) = \mathfrak{R}$. If it is not hold, then there is a right maximal ideal $\mathcal{N}$ containing $\mathfrak{R}\mathfrak{a}^{\mathfrak{n}}\mathfrak{R} + \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}})$. If $= \mathfrak{r}(\mathfrak{b})$, for some $0 \neq \mathfrak{b} \in \mathfrak{f}(\mathfrak{R})$, we have $\mathfrak{b} \in \mathfrak{l}\big(\mathfrak{R}\mathfrak{a}^{\mathfrak{n}}\mathfrak{R} + \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}})\big) \subseteq \mathfrak{l}(\mathfrak{a}^{\mathfrak{n}}) = \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}}) \subseteq \mathcal{N} = \mathfrak{r}(\mathfrak{b})$, which implies $\mathfrak{b} \in \mathfrak{r}(\mathfrak{b})$. Then $\mathfrak{b}^2 = 0 , \mathfrak{b} = 0$, a contradiction . Therefore $\mathfrak{a}^{\mathfrak{n}}\mathfrak{R} + \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}}) = \mathfrak{R}$. In particular $x\mathfrak{a}^{\mathfrak{n}}y + d = 1$, with $x, y \in \mathfrak{R}$, and $d \in \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}})$. Hence $\mathfrak{a}^{\mathfrak{n}}x\mathfrak{a}^{\mathfrak{n}}y = \mathfrak{a}^{\mathfrak{n}}$ which proves $\mathfrak{R}$ is right $\mathcal{P}\mathrm{W}\pi$-regular .

Following [3], $\mathfrak{R}$ is called strongly regular ring , if for each $\mathfrak{a} \in \mathfrak{R}$, there is $\mathfrak{b} \in \mathfrak{R}$, $\mathfrak{a} = \mathfrak{a}^2\mathfrak{b}$.

Theorem 3.8 : Assume that $\mathfrak{R}$ is right $\mathcal{P}\mathrm{W}\pi$-regular, $\mathfrak{f}(\mathfrak{R})$ is reduced and $\mathfrak{a}^{n}\mathfrak{R} = \mathfrak{R}\mathfrak{a}$, for each $\mathfrak{a} \in \mathfrak{f}(\mathfrak{R})$. Then $\mathfrak{f}(\mathfrak{R})$ is strongly regular ideal .

Proof : Assume that $\mathfrak{f}(\mathfrak{R})$ be a reduced of $\mathfrak{R}$ and let $\mathfrak{a} \in \mathfrak{f}(\mathfrak{R})$. Since $\mathfrak{R}$ is right $\mathcal{P}\mathrm{W}\pi$-regular , there is a natural number $\mathfrak{n}$ and $\mathfrak{c}, \mathfrak{b} \in \mathfrak{R}$ such that $\mathfrak{a}^{\mathfrak{n}} = \mathfrak{a}^{\mathfrak{n}}\mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c}$, which implies $\mathfrak{a}^{\mathfrak{n}}(1 - \mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c}) = 0$ and $(1 - \mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c}) \in \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}}) = \mathfrak{r}(\mathfrak{a})$, gives $\mathfrak{a} = \mathfrak{a}\mathfrak{b}\mathfrak{a}^{\mathfrak{n}}\mathfrak{c} = \mathfrak{a}h\mathfrak{a}$ ($\mathfrak{a}^{n}\mathfrak{R} = \mathfrak{R}\mathfrak{a}$) . Consider $(\mathfrak{a} - \mathfrak{a}^2h)^2 = \mathfrak{a}^2 - \mathfrak{a}^3h - \mathfrak{a}^2h\mathfrak{a} + \mathfrak{a}^2h\mathfrak{a}^2h = \mathfrak{a}^2 - \mathfrak{a}^3h - \mathfrak{a}(\mathfrak{a}h\mathfrak{a}) + \mathfrak{a}(\mathfrak{a}h\mathfrak{a})\mathfrak{a}h = \mathfrak{a}^2 - \mathfrak{a}^3h - \mathfrak{a}^2 + \mathfrak{a}^3h = 0$. But $\mathfrak{f}(\mathfrak{R})$ is reduced , then $\mathfrak{a} - \mathfrak{a}^2h = 0$, implies that $\mathfrak{a} = \mathfrak{a}^2h$. Hence $\mathfrak{f}(\mathfrak{R})$ is strongly regular ideal .

Theorem 3.9 : Assume that $\mathfrak{R}$ is semi prime and any singular simple right $\mathfrak{R}$-module is $\mathfrak{f}$-injective with $\mathfrak{r}(\mathfrak{a}) \subseteq \mathfrak{l}(\mathfrak{a})$, for each $\mathfrak{a} \in \mathfrak{f}(\mathfrak{R})$. Then $\mathfrak{R}$ is right $\mathcal{P}\mathrm{W}\pi$-regular .

Proof : Assume that $\mathfrak{R}\mathfrak{a}^{\mathfrak{n}}\mathfrak{R} + \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}}) = \mathfrak{R}$, for every $\mathfrak{a} \in \mathfrak{f}(\mathfrak{R})$. If $\mathfrak{a}^{\mathfrak{n}}\mathfrak{R} + \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}}) \neq \mathfrak{R}$, then there is a right maximal ideal $\mathcal{N}$ of $\mathfrak{R}$ such that $\mathfrak{R}\mathfrak{a}^{\mathfrak{n}}\mathfrak{R} + \mathfrak{r}(\mathfrak{a}^{\mathfrak{n}}) \subseteq \mathcal{N}$ and if $\mathcal{N}$ is not essential of $\mathfrak{R}$. Then $\mathcal{N}$ is a direct summand. And then there exists $0 \neq \mathfrak{e} = \mathfrak{e}^2 \in \mathfrak{R}$ such that $= \mathfrak{r}(\mathfrak{e})$.

Now , $\mathfrak{N}a^n ȩ \subseteq \mathfrak{N}a^n\mathfrak{N} \subseteq \mathcal{N} = r(ȩ)$, implies that $ȩ\mathfrak{R}a^n ȩ = 0$ and $(a^n ȩ\mathfrak{R})^2 = a^n ȩ\mathfrak{R}a^n ȩ\mathfrak{R} = 0$. So $a^n ȩ\mathfrak{N} = 0$ ($\mathfrak{N}$ is semi prime) and $a^n ȩ = 0$, $ȩ \in r(a^n) \subseteq \mathcal{N} = r(e)$, and $ȩ^2 = 0$, a contradiction . So $\mathcal{N}$ is maximal essential right ideal of . Since $\mathfrak{N}/_{\mathcal{N}}$ is J-injective , then for any right $\mathfrak{N}$-homomorphism, $f : \sigma^n\mathfrak{N} \to \mathfrak{N}/_{\mathcal{N}}$, known as $f(a^n z) = z + \mathcal{N}$, for every $z \in \mathfrak{R}$. Note f is well define and it will be extended from $\mathfrak{N}$ into $\mathfrak{N}/_{\mathcal{N}}$. So $1 + \mathcal{N} = f(a^n) = ca^n + \mathcal{N}$, where $c \in \mathfrak{N}$, and $(1 - ca^n) \in \mathcal{N}$. Since $a^n \in \mathfrak{R}a^n\mathfrak{R} \subseteq \mathcal{N}$. So that $1 \in \mathcal{N}$, and this is contradiction , hence $\mathfrak{N}a^n\mathfrak{N} + r(a^n) = \mathfrak{N}$. In specific $xa^n y + v = 1$, $a^n x a^n y + a^n v = a^n$. Therefore $a^n x a^n y = a^n$, and $\mathfrak{N}$ is right $\mathcal{P}W\pi$-regular .

From Theorem 3.9 and Lemma 3.6 we get :

Corollary 3.10 : If every simple singular right $\mathfrak{N}$-module is J-injective and $\mathfrak{N}$ is right $J\mathcal{P}\mathcal{P}$-ring , $r(a) \subseteq l(a)$, for each $a \in J(\mathfrak{N})$. Then $\mathfrak{N}$ is $\mathcal{P}W\pi$-regular .

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Separation axioms via $\alpha g_{Ị}$-open set

Sufyan G Saeed[1], R B Esmaeel[2]

[1, 2]*Department of Mathematics, Ibn Al-Haitham*

College of Education University of Baghdad, IRAQ

[1]*Safsaf201030@gmail.com*, [2]*ranamumosa@yahoo.com*

ABSTRACT

The main objective of this paper is to use the concept $\alpha g_{Ị}$-openness to offered new classes of separation axioms in ideal spaces. Those new classes are; $\alpha g_{Ị}$-T_0-space, $\alpha g_{Ị}$-T_1-space, $\alpha g_{Ị}$-T_2-space. Also new type of concepts of convergence in ideal spaces via the $\alpha g_{Ị}$-open set were offered.

Keywords. $\alpha g_{Ị}$-closed set, $\alpha g_{Ị}O$-functions, $\alpha g_{Ị}C$-functions, $\alpha g_{Ị}$-continuous function, ideal, $\alpha g_{Ị}$-T_0-space, $\alpha g_{Ị}$-T_1-space, $\alpha g_{Ị}$-T_2-space, $\alpha g_{Ị}$-convergence.

1- Introduction

An α-open was studied in 1965 by O. Njastad, as a subset Ç is α-open set where Ç ⊆ $int(cl(int(Ç)))$[1,2]. The notion of ideal was studied by Kuratowski[3,4], that Ị is an ideal on X̧, when Ị is a collection of all subsets of X̧ an ideal have two properties (if Ç, Đ ∈ Ị, then Ç ∪ Đ ∈ Ị) and (if Ç ∈ Ị and Đ ⊆ Ç, then Đ ∈ Ị.

There are many types for the ideal[5-8]

i. $Ị_{\{\emptyset\}}$: the trivial ideal where Ị={∅}.
ii. $Ị_n$: the ideal of all nowhere dense sets
 $Ị_n$ ={ Ç ⊆ X̧: $int(cl(Ç))$={∅}}.
iii. $Ị_f$: the ideal of all finite subsets of X̧
 $Ị_f$ = { Ç ⊆ X̧: Ç is a finite set}.

The collection of all α-open sets denoted by" $\tilde{\iota}_\alpha$" and the collection of all α-closed denoted by" ƚ_α".

2- On $\alpha g_{Ị}$-closed set

Definition 2.1. In ideal topological space (Ϫ, ῖ, Ị), Let Ç ⊆ Ϫ. Ç is said Ị-α-g-closed set denoted by "$\alpha g_{Ị}$-closed" ,if Ç-Ơ ∈ Ị then, cl(Ç)-Ơ ∈ Ị where Ơ ⊆ Ϫ and Ơ is an α-open sets.

Now, $Ç^c$ is an Ị-α-g-open sets denoted by "$\alpha g_{Ị}$-open" .The collection of all $\alpha g_{Ị}$-closed sets where $Ç^c$ ∈ Ϫ, denoted by "$\alpha g_{Ị}C$(Ϫ). The collection of all $\alpha g_{Ị}$-open sets "$\alpha g_{Ị}O$(Ϫ)".

Example 2.2. Consider the space (Ϫ, ῖ, Ị) where Ϫ={w̃,v̈}, ῖ={Ϫ,∅,{w̃}} and Ị={∅,{v̈}}.Then $\tilde{\iota}_\alpha$={Ϫ,∅,{w̃}} and ƚ_α={Ϫ,∅,{v̈}}, so $\alpha g_{Ị}C$(Ϫ) = $\alpha g_{Ị}O$(Ϫ) = {Ϫ,∅,{w̃},{v̈}}.

Example 2.3. Consider the space (Ϫ, ῖ, Ị) where Ϫ={w̃,v̈,ʐ}, ῖ={Ϫ,∅,{w̃}} and Ị={∅,{v̈}}. Then $\tilde{\iota}_\alpha$={Ϫ,∅,{w̃},{w̃,v̈},{w̃,ʐ}} ƚ_α={Ϫ,∅,{v̈,ʐ},{ʐ,},{v̈}}, so $\alpha g_{Ị}C$(Ϫ)={Ϫ,∅,{v̈,ʐ},{ʐ},{w̃,ʐ}} $\alpha g_{Ị}O$(Ϫ) = {Ϫ,∅,{w̃},{w̃,v̈},{v̈}}.

Remark 2.4.

i. Each closed set in (Ϫ , ῖ) is an $\alpha g_{Ị}$-closed in (Ϫ, ῖ, Ị).
ii. Each open set in (Ϫ , ῖ) is an $\alpha g_{Ị}$-open in (Ϫ, ῖ, Ị).

Proof:

i. Let Ç is any closed set in (Ϫ, ῖ, Ị) and Ơ be an α-open set such that Ç-Ơ ∈ Ị since cl(Ç) = Ç this implies Ç is an $\alpha g_{Ị}$-closed set.
ii. Let Ơ ∈ Ϫ, then $Ơ^c$ is a closed set this implies $Ơ^c$ is an $\alpha g_{Ị}$-closed set, so Ơ is an $\alpha g_{Ị}$-open set.

The converse of Remark 2.4 is not true in general see Example 2.2. Since {w̃} is closed in (Ϫ, ῖ, Ị), but not closed in (Ϫ, ῖ), and {v̈} is open in (Ϫ, ῖ, Ị), but not open in (Ϫ, ῖ).

2.1 Open function

Definition 2.1.1. The function ʆ: (Ϫ, ῖ, Ị) → (Ƴ, ƚ, ʝ) is called;

i. αg_{I}-open function, denoted by "$\alpha g_{I}O$-function" if ʄ(Ơ) is an αg_{J}-open set in Ƴ. Whenever Ơ is an αg_{I}-open in Ҳ.
ii. αg_{I}^{*}-open function, denoted by "$\alpha g_{I}^{*}O$-function" if ʄ(Ơ) is an αg_{J}-open set in Ƴ. Whenever Ơ ∈ ĩ.
iii. αg_{I}^{**}-open function, denoted by "$\alpha g_{I}^{**}O$-function" if ʄ(Ơ) is an open set in Ƴ. Whenever Ơ is an αg_{I}-open set in Ҳ.

Proposition 2.1.2. Let ʄ: (Ҳ, ĩ, I) → (Ƴ, ɉ, J) is a function;

i. If ʄ is an open function then ʄ is $\alpha g_{I}^{*}O$-function
Proof: Let Ơ ∈ ĩ, since ʄ is an open function then ʄ(Ơ)∈ ɉ, since for each open sets is an αg_{I}-open set then ʄ(Ơ) is an αg_{J}-open set in Ƴ, then ʄ is an $\alpha g_{I}^{*}O$-function.

ii. If ʄ is an $\alpha g_{I}^{**}O$-function then ʄ is an $\alpha g_{I}O$- function.
Proof: Let Ơ is an αg_{I}-open set in Ҳ, since ʄ is an $\alpha g_{I}^{**}O$-function, then ʄ(Ơ)∈ ɉ, since for each open set is an αg_{I}-open set, this implies ʄ(Ơ) is an αg_{J}-open set in Ƴ, then ʄ is an αg_{I}-open function. ■
iii. If ʄ is an $\alpha g_{I}O$-function then ʄ is an $\alpha g_{I}^{*}O$-function.
Proof: Let Ơ ∈ ĩ, since for each open set is an αg_{I}-open set, then ʄ(Ơ) is an αg_{J}-open set in Ƴ, thus ʄ is an $\alpha g_{I}^{*}O$-function.

iv. If ʄ is an $\alpha g_{I}^{**}O$-function then ʄ is an open function.
Proof: Let Ơ ∈ ĩ, since for each open set is an αg_{I}-open set, then Ơ be an αg_{I}-open set in Ҳ, since ʄ is an $\alpha g_{I}^{**}O$-function thus ʄ(Ơ) is an open set in Ƴ, then ʄ is an open function.

v. If ʄ is an $\alpha g_{I}^{**}O$-function then ʄ is an $\alpha g_{I}^{*}O$-function.
Proof: The prove is complete.

The following, examples show that the opposite direction of the above proposition is incorrect.

Example 2.1.3. A function ʄ: (Ҳ, ĩ, I) → (Ҳ, ĩ, J), where Ҳ ={ė₁, ė₂, ė₃} such that ʄ(ė₁) = (ė₂), ʄ(ė₂) = (ė₁), ʄ(ė₃) = (ė₃), ĩ={Ҳ,∅,{ė₁}}, I={∅} and J={∅,{ė₂},{ė₃},{ė₂,ė₃}} then $ĩ_{\alpha}$={Ҳ,∅,{ė₁},{ė₁, ė₂},{ė₁, ė₃}} then $\alpha g_{I}C$(Ҳ) = {Ҳ,∅,{ė₂,ė₃}} and $\alpha g_{I}O$(Ҳ) = {Ҳ,∅,{ė₁}}. So $\alpha g_{J}C$(Ҳ) = Ᵽ(Ҳ) and $\alpha g_{I}O$(Ҳ) = Ᵽ(Ҳ).

Then ʄ is $\alpha g_{I}O$-function and $\alpha g_{I}^{*}O$-function which is not $\alpha g_{I}^{**}O$-function and not open function, since {ė₁} is an open set in Ҳ and αg_{I}-open set, but ʄ(ė₁)=(ė₂) which is not open.

Example 2.1.4. The function ʄ: (Ҳ, ĩ, I) → (Ҳ, ĩ, I); where Ҳ={ė₁, ė₂, ė₃} such that ʄ(ė) = (ė), ∀ ė∈Ҳ, ĩ={Ҳ,∅,{ė₁}}, I={∅,{ė₂},{ė₃},{ė₂, ė₃}} and J={∅}. Then $ĩ_{\alpha}$={Ҳ,∅,{ė₁},{ė₁, ė₂},{ė₁, ė₃}} then $\alpha g_{I}C$(Ҳ) = Ᵽ(Ҳ) and $\alpha g_{I}O$(Ҳ) = Ᵽ(Ҳ). So $\alpha g_{J}C$(Ҳ) = {Ҳ,∅,{ė₂,ė₃}} and $\alpha g_{J}O$(Ҳ) = {Ҳ,∅,{ė₁}}.

It is easy to see that ƒ is open function and $\alpha g_{I}^{*}O$-function but it is not $\alpha g_{I}O$-function and not $\alpha g_{I}^{**}O$-function, since {ė₂}∈ $\alpha g_{I}O$(Ꭓ) but ƒ(ė₂) = (ė₂) which is not open and not αg_{J}-open set.

Definition 2.1.5. The function ƒ: (Ꭓ, ĩ, Ị) → (Ɣ, ɟ, ȷ) is said,

i. αg_{I}-closed function, denoted by "$\alpha g_{I}C$-function" if ƒ(Ơ) is αg_{J}-closed in Ɣ whenever Ơ is on αg_{I}-closed in Ꭓ.
ii. αg_{I}^{*}-closed function, denoted by "$\alpha g_{I}^{*}C$-function", if ƒ(Ơ) is αg_{J}-closed in Ɣ whenever Ơ is an closed in Ꭓ.
iii. αg_{I}^{**}-closed function, denoted by "$\alpha g_{I}^{**}C$-function", if ƒ(Ơ) is closed in Ɣ whenever Ơ is an αg_{I}-closed in Ꭓ.

Proposition 2.1.6. Let ƒ: (Ꭓ, ĩ, Ị) → (Ɣ, ɟ, ȷ) is function,

i. If ƒ is a closed function then ƒ is an $\alpha g_{I}^{*}C$-function.
ii. If ƒ is an $\alpha g_{I}^{**}C$-function then ƒ is an $\alpha g_{I}C$-function.
iii. If ƒ is an $\alpha g_{I}^{**}C$-function then ƒ is a closed function.
iv. If ƒ is an $\alpha g_{I}C$-function then ƒ is an $\alpha g_{I}^{*}C$-function.
v. If ƒ is an $\alpha g_{I}^{**}C$-function then ƒ is an $\alpha g_{I}^{*}C$-function.

Proof: By Remark 2.4 and Definition 2.1.5 the prove is complete.

Example 2.1.3 and 2.1.4 show that the opposite direction of the above proposition is incorrect.

2.2- Near continuous function

Definition 2.2.1. A function ƒ: (Ꭓ, ĩ, Ị) → (Ɣ, ɟ, ȷ) is called;

i. Ị-α-g-continuous function, denoted by "αg_{I}-continuous function", if $ƒ^{-1}$(Ơ) is an αg_{I}-open set in Ꭓ, where Ơ ∈ ɟ.
ii. Strongly Ị-α-g-continuous function, denoted by "Strongly αg_{I}-continuous function" if $ƒ^{-1}$(Ơ) ∈ ĩ, whenever Ơ is an αg_{J}-open set in Ɣ.
iii. Ị-α-g-irresolute function, denoted by "αg_{I}-irresolute function", ", if $ƒ^{-1}$(Ơ) is an αg_{I}-open set in Ꭓ, where Ơ is an αg_{J}-open set in Ɣ.

Proposition 2.2.2. Let ƒ: (Ꭓ, ĩ, Ị) → (Ɣ, ɟ, ȷ) is a function;

i. If ƒ is a continuous function, then ƒ is an αg_{I}-continuous function.
ii. If ƒ is Strongly αg_{I}-continuous function, then ƒ is a continuous function.
iii. If ƒ is an αg_{I}-irresolute function, then ƒ is an αg_{I}-continuous function.
iv. If ƒ is Strongly αg_{I}-continuous function, then ƒ is an αg_{I}-irresolute function.

v. If ƒ is Strongly $\alpha g_{Ị}$-continuous function, then ƒ is an $\alpha g_{Ị}$-continuous function.

Proof:

i. Let Ơ ∈ ɟ. Since ƒ is a continuous function, then $ƒ^{-1}$(Ơ) ∈ ĩ. $ƒ^{-1}$(Ơ) is an $\alpha g_{Ị}$-open set in Ӽ By Remark 2.4. Hence ƒ is an $\alpha g_{Ị}$-continuous function.

ii. Let Ơ ∈ ɟ. By Remark 2.4, Ơ is an $\alpha g_{ʝ}$-open set in Ƴ. Since ƒ is Strongly $\alpha g_{Ị}$-continuous function, then $ƒ^{-1}$(Ơ) ∈ ĩ. Hence ƒ is a continuous function.

iii. Let Ơ ∈ ɟ, this implies to Ơ is $\alpha g_{ʝ}$-open set in Ƴ. Since ƒ is an $\alpha g_{Ị}$-irresolute function then $ƒ^{-1}$(Ơ) is an $\alpha g_{Ị}$-open set in Ӽ. Then ƒ is an $\alpha g_{Ị}$-continuous function.

iv. Let Ơ is an $\alpha g_{ʝ}$-open set in Ӽ. Since ƒ is a Strongly $\alpha g_{Ị}$-continuous function, then $ƒ^{-1}$(Ơ) ∈ ĩ. By Remark 2.4, ƒ(Ơ) is $\alpha g_{Ị}$-open set in Ӽ. This implies ƒ is an $\alpha g_{Ị}$-irresolute function.

v. Let Ơ ∈ ɟ this implies Ơ is an $\alpha g_{ʝ}$-open set and since ƒ is a Strongly $\alpha g_{Ị}$-continuous function, thus $ƒ^{-1}$(Ơ) is open set in Ӽ by Remark 2.4 $ƒ^{-1}$(Ơ) is an $\alpha g_{Ị}$-open set, so ƒ is an $\alpha g_{Ị}$-continuous function.

The following, examples show that the opposite direction of the above proposition is incorrect.

Example 2.2.3. The function ƒ: (Ӽ, ĩ, Ị) → (Ӽ, ĩ, ʝ), where Ӽ={$ė_1, ė_2, ė_3$} such that ƒ($ė_1$) = ($ė_1$), ƒ($ė_2$) = ($ė_2$), ƒ($ė_3$) = ($ė_3$), ĩ = {Ӽ,∅,{$ė_1$}}, Ị={∅} and ʝ={∅,{$ė_2$},{$ė_3$},{$ė_2$,$ė_3$}} then $ĩ_\alpha$={Ӽ,∅,{$ė_1$},{$ė_1, ė_2$},{$ė_1, ė_3$}} then $\alpha g_{Ị}C$(Ӽ) = {Ӽ,∅,{$ė_2$,$ė_3$}} and $\alpha g_{Ị}O$(Ӽ) = {Ӽ,∅,{$ė_1$}}. So $\alpha g_{ʝ}C$(Ӽ) = Ᵽ(Ӽ) and $\alpha g_{ʝ}O$(Ӽ) = Ᵽ(Ӽ).

It is easy to see that ƒ is continuous and $\alpha g_{Ị}$-continuous function but not $\alpha g_{Ị}$-irresolute function since {$ė_3$} is an $\alpha g_{ʝ}$-open set in Ƴ but $ƒ^{-1}$($ė_3$) = $ė_3$ is not an $\alpha g_{Ị}$-open set in Ӽ.

Example 2.2.4. The function ƒ: (Ӽ, ĩ, Ị) → (Ӽ, ĩ, ʝ), where Ӽ={$ė_1, ė_2, ė_3$} such that ƒ($ė_1$) = ($ė_2$), ƒ($ė_2$) = ($ė_1$), ƒ($ė_3$) = ($ė_3$), ĩ = {Ӽ,∅,{$ė_1$}}, ʝ={∅} and Ị={∅,{$ė_2$},{$ė_3$},{$ė_2$,$ė_3$}} then $ĩ_\alpha$={Ӽ,∅,{$ė_1$},{$ė_1, ė_2$},{$ė_1, ė_3$}} then $\alpha g_{ʝ}C$(Ӽ) = {Ӽ,∅,{$ė_2$,$ė_3$}} and $\alpha g_{ʝ}O$(Ӽ) = {Ӽ,∅,{$ė_1$}}. So $\alpha g_{Ị}C$(Ӽ) = Ᵽ(Ӽ) and $\alpha g_{Ị}O$(Ӽ) = Ᵽ(Ӽ).

It is easy to see that ƒ is $\alpha g_{Ị}$-continuous function but not continuous function since {$ė_1$}∈ ĩ but $ƒ^{-1}$($ė_1$) = $ė_2$ is not open in Ӽ, and not Strongly $\alpha g_{Ị}$-continuous function since {$ė_1$} ∈ $\alpha g_{ʝ}O$(Ӽ) but $ƒ^{-1}$($ė_1$) = $ė_2$ is not open in Ӽ.

3-On $\alpha g_{Ị}$-Separation Axioms.

Definition 3.1. A space (Ӽ, ĩ, Ị) is said α-g- Ị-T_0-space denoted by "$\alpha g_{Ị}$-T_0-space" if ∀ $ė_1 \neq ė_2$, ∃ an $\alpha g_{Ị}$-open set contains one of them.

Example3.2. In $(\mathrm{X}, \tilde{\iota}, \mathrm{I})$; where X={$\dot{e}_1, \dot{e}_2, \dot{e}_3$}, $\tilde{\iota}$={X,∅,{$\dot{e}_1$}{$\dot{e}_1, \dot{e}_2$}} I={∅,{$\dot{e}_3$}.
Then $\tilde{\iota}_\alpha$={X,∅,{$\dot{e}_1$},{$\dot{e}_1, \dot{e}_2$},{$\dot{e}_1, \dot{e}_3$}}, $\jmath_\alpha$={X,∅,{$\dot{e}_2$},{$\dot{e}_3$},{$\dot{e}_2, \dot{e}_3$}}, *so* $\alpha g_I C$(X)={X,∅,{$\dot{e}_2$},{$\dot{e}_3$},{$\dot{e}_1, \dot{e}_2$},{$\dot{e}_2, \dot{e}_3$} and $\alpha g_I O$(X)={X,∅,{$\dot{e}_1$},{$\dot{e}_3$},{$\dot{e}_1, \dot{e}_2$},{$\dot{e}_1, \dot{e}_3$}}. Then $(\mathrm{X}, \tilde{\iota}, \mathrm{I})$ is an αg_I-T_0-space.

Theorem 3.3. The space $(\mathrm{X}, \tilde{\iota}, \mathrm{I})$ is an αg_I-T_0-space if and only if, ∀ $\dot{e}_1 \neq \dot{e}_2$, there is an αg_I-closed set contains one of them.

Proof;(→) Let $\dot{e}_1, \dot{e}_2 \in$ X where $\dot{e}_1 \neq \dot{e}_2$.Since X is an αg_I-T_0-space, then there is an αg_I-open set Ơ contains one of them, then (X-Ơ) is an αg_I-closed set contains the other one.

(←) Let $\dot{e}_1, \dot{e}_2 \in$ X where $\dot{e}_1 \neq \dot{e}_2$ and there is an αg_I-closed set ṽ contains one of them (X-ṽ) is an αg_I-open set contains the other one.

Remark 3.4. If $(\mathrm{X}, \tilde{\iota})$ is a T_0-space then $(\mathrm{X}, \tilde{\iota}, \mathrm{I})$ is an αg_I-T_0-space.

Proof: Let $\dot{e}_1, \dot{e}_2 \in$ X where $\dot{e}_1 \neq \dot{e}_2$. Since $(\mathrm{X}, \tilde{\iota})$ is a T_0-space, then there is Ơ contains one of them, where Ơ is an open set. Then Ơ is an αg_I-open set contains one of them, since by Remark (2,4) for each open set in $(\mathrm{X}, \tilde{\iota})$ is an αg_I-open in $(\mathrm{X}, \tilde{\iota}, \mathrm{I})$.

Definition 3.5. A space $(\mathrm{X}, \tilde{\iota}, \mathrm{I})$ is said α-g- I-T_1-space denoted by "αg_I-T_1-space" if ∀ $\dot{e}_1 \neq \dot{e}_2$, there are αg_I-open set $Ơ_1$ and $Ơ_2$, satisfies $\dot{e}_1 \in$ ($Ơ_1$-$Ơ_2$) and $\dot{e}_2 \in$ ($Ơ_2$-$Ơ_1$).

Example 3.6. Let X={$\dot{e}_1, \dot{e}_2, \dot{e}_3$}, $\tilde{\iota}$={X,∅} and I=P(X). $\tilde{\iota}_\alpha$= $\jmath_\alpha$= P(X), $\alpha g_I C$(X)= $\alpha g_I O$(X) = P(X). Then $(\mathrm{X}, \tilde{\iota}, \mathrm{I})$ is an αg_I-T_1-space.

Remark 3.7. If $(\mathrm{X}, \tilde{\iota})$ is a T_1-space then $(\mathrm{X}, \tilde{\iota}, \mathrm{I})$ is an αg_I-T_1-space.

Proof: Let $\dot{e}_1, \dot{e}_2 \in$ X, where $\dot{e}_1 \neq \dot{e}_2$. Since $(\mathrm{X}, \tilde{\iota})$ is a T_1-space, then there are $Ơ_1$,$Ơ_2$ where $Ơ_1$ and $Ơ_2$ are two open set, such that $\dot{e}_1 \in$ ($Ơ_1$-$Ơ_2$) and $\dot{e}_2 \in$ ($Ơ_2$-$Ơ_1$). By Remark 2.4, $Ơ_1$ and $Ơ_2$ are αg_I-open sets whenever $\dot{e}_1 \in$ ($Ơ_1$-$Ơ_2$) and $\dot{e}_2 \in$ ($Ơ_2$-$Ơ_1$).

Proposition 3.8. Every αg_I-T_1-space is an αg_I-T_0-space.

Proof: Let $\dot{e}_1, \dot{e}_2 \in$ X, where $\dot{e}_1 \neq \dot{e}_2$. Since $(\mathrm{X}, \tilde{\iota}, \mathrm{I})$ is an αg_I-T_1-space, then there are αg_I-open sets $Ơ_1$,$Ơ_2$ whenever $\dot{e}_1 \in$ ($Ơ_1$-$Ơ_2$) and $\dot{e}_2 \in$ ($Ơ_2$-$Ơ_1$). Then there is an αg_I-open sets Ơ contains one of them.

The opposite direction of proposition 3.8, is generally incorrect, as the following example.

Example 3.9. A space $(\text{Ẋ}, \tilde{\iota}, \text{Ị})$ is an $\alpha g_{\text{Ị}}$-T_0-space where Ẋ={$\dot{e}_1, \dot{e}_2, \dot{e}_3$}, $\tilde{\iota}$={Ẋ,∅,{$\dot{e}_1$},{$\dot{e}_2$},{$\dot{e}_1, \dot{e}_2$}} and Ị={∅},$\tilde{\iota}_\alpha$={Ẋ,∅,{$\dot{e}_1$},{$\dot{e}_2$},{$\dot{e}_1, \dot{e}_2$},{$\dot{e}_1, \dot{e}_3$},{$\dot{e}_2, \dot{e}_3$}},ƚ_α={Ẋ,∅,{$\dot{e}_1$},{$\dot{e}_2$},{$\dot{e}_3$} ,{$\dot{e}_1, \dot{e}_3$},{$\dot{e}_2, \dot{e}_3$}} $\alpha g_{\text{Ị}}C$(Ẋ)={Ẋ,∅,{$\dot{e}_3$},{$\dot{e}_1, \dot{e}_3$},{$\dot{e}_2, \dot{e}_3$}} $\alpha g_{\text{Ị}}O$(Ẋ)={Ẋ,∅,{$\dot{e}_1$},{$\dot{e}_2$},{$\dot{e}_1, \dot{e}_2$}}. Then the space $(\text{Ẋ}, \tilde{\iota}, \text{Ị})$ is not $\alpha g_{\text{Ị}}$-T_1-space, since the elements $\dot{e}_2 \neq \dot{e}_3$, ∄ $\alpha g_{\text{Ị}}$-open set Ơ contains $\dot{e}_3$ which does not contains $\dot{e}_2$.

Theorem 3.10. For a space $(\text{Ẋ}, \tilde{\iota}, \text{Ị})$: $(\text{Ẋ}, \tilde{\iota}, \text{Ị})$ is an $\alpha g_{\text{Ị}}$-T_1-space if and only if ∀ $\dot{e}_1 \neq \dot{e}_2$, ∃ $\alpha g_{\text{Ị}}$-closed sets Ç$_1$ and Ç$_2$, such that $\dot{e}_1$ ∈ (Ç$_1$- Ç$_2$), $\dot{e}_2$ ∈ (Ç$_2$- Ç$_1$).

Proof:(→) Let $\dot{e}_1, \dot{e}_2$ ∈ Ẋ, where $\dot{e}_1 \neq \dot{e}_2$. Since Ẋ is an $\alpha g_{\text{Ị}}$-T_1-space, then ∃ $\alpha g_{\text{Ị}}$-open sets Ơ$_1$ and Ơ$_2$, such that $\dot{e}_1$ ∈ (Ơ$_1$-Ơ$_2$) and $\dot{e}_2$ ∈ (Ơ$_2$-Ơ$_1$). Then ∃ $\alpha g_{\text{Ị}}$-closed sets Ơ$_1^c$ and Ơ$_2^c$ such that $\dot{e}_1$ ∈ Ơ$_2^c$-Ơ$_1^c$, $\dot{e}_2$ ∈ Ơ$_1^c$-Ơ$_2^c$ where Ơ$_2^c$= Ç$_1$ and Ơ$_1^c$= Ç$_2$. Then ∃ $\alpha g_{\text{Ị}}$-closed sets Ç$_1$ and Ç$_2$ satisfy $\dot{e}_1$ ∈ (Ç$_1$∩ Ç$_2^c$) and $\dot{e}_2$ ∈ (Ç$_2$∩ Ç$_1^c$), therefore $\dot{e}_1$ ∈ (Ç$_1$- Ç$_2$) and $\dot{e}_2$ ∈ (Ç$_2$- Ç$_1$).

(←) Let $\dot{e}_1, \dot{e}_2$ ∈ Ẋ, where $\dot{e}_1 \neq \dot{e}_2$, ∃ $\alpha g_{\text{Ị}}$-closed sets Ç$_1$ and Ç$_2$ satisfy $\dot{e}_1$ ∈ (Ç$_2^c$ ∩ Ç$_1$) and $\dot{e}_2$ ∈ (Ç$_1^c$ ∩ Ç$_2$), then ∃ $\alpha g_{\text{Ị}}$-open sets Ç$_1^c$ and Ç$_2^c$ whenever $\dot{e}_1$ ∈ (Ç$_2^c$-Ç$_1^c$), $\dot{e}_2$ ∈ (Ç$_1^c$-Ç$_2^c$), where Ç$_2^c$= Ơ$_1$, Ç$_1^c$= Ơ$_2$.

Proposition 3.11. A space $(\text{Ẋ}, \tilde{\iota}, \text{Ị})$ is an $\alpha g_{\text{Ị}}$-T_1-space, if {ė} is an $\alpha g_{\text{Ị}}$-closed set for each elements ė in Ẋ.

Proof: Let $\dot{e}_1, \dot{e}_2$ ∈ Ẋ, where $\dot{e}_1 \neq \dot{e}_2$. Since {$\dot{e}_1$}, {$\dot{e}_2$} are $\alpha g_{\text{Ị}}$-closed sets. So (Ẋ-{$\dot{e}_1$}) and (Ẋ-{$\dot{e}_2$}) are $\alpha g_{\text{Ị}}$-open sets. Then ∃ $\alpha g_{\text{Ị}}$-open sets Ơ$_1$ and Ơ$_2$ where Ơ$_1$= (Ẋ-{$\dot{e}_1$}) and Ơ$_2$= (Ẋ-{$\dot{e}_2$}) such that $\dot{e}_1$ ∈ (Ơ$_1$-Ơ$_2$) and $\dot{e}_2$ ∈ (Ơ$_2$-Ơ$_1$).

Definition 3.12. A space $(\text{Ẋ}, \tilde{\iota}, \text{Ị})$ is said α-g- Ị-T_2-space denoted by "$\alpha g_{\text{Ị}}$-T_2-space" if ∀ $\dot{e}_1 \neq \dot{e}_2$, there are $\alpha g_{\text{Ị}}$-open sets Ơ$_1$ and Ơ$_2$, satisfies $\dot{e}_1$ ∈ Ơ$_1$ and $\dot{e}_2$ ∈ Ơ$_2$ and Ơ$_1$∩ Ơ$_2$=∅.

Remark 3.13. If $(\text{Ẋ}, \tilde{\iota})$ is a T_2-space, then $(\text{Ẋ}, \tilde{\iota}, \text{Ị})$ is an $\alpha g_{\text{Ị}}$-T_2-space.

Proof: Let $\dot{e}_1, \dot{e}_2$ ∈ Ẋ, where $\dot{e}_1 \neq \dot{e}_2$**.** Since $(\text{Ẋ}, \tilde{\iota})$ is a T_2-space, then ∃ Ơ$_1$, Ơ$_2$ ∈ $\tilde{\iota}$ satisfy $\dot{e}_1$ ∈ Ơ$_1$ and $\dot{e}_2$ ∈ Ơ$_2$ and Ơ$_1$∩ Ơ$_2$=∅. By Remark 2.4, there are $\alpha g_{\text{Ị}}$-open sets Ơ$_1$ and Ơ$_2$, satisfies $\dot{e}_1$ ∈ Ơ$_1$ and $\dot{e}_2$ ∈ Ơ$_2$ and Ơ$_1$∩ Ơ$_2$=∅.

Proposition 3.14. Every $\alpha g_{\text{Ị}}$-T_2-space is an $\alpha g_{\text{Ị}}$-T_1-space.

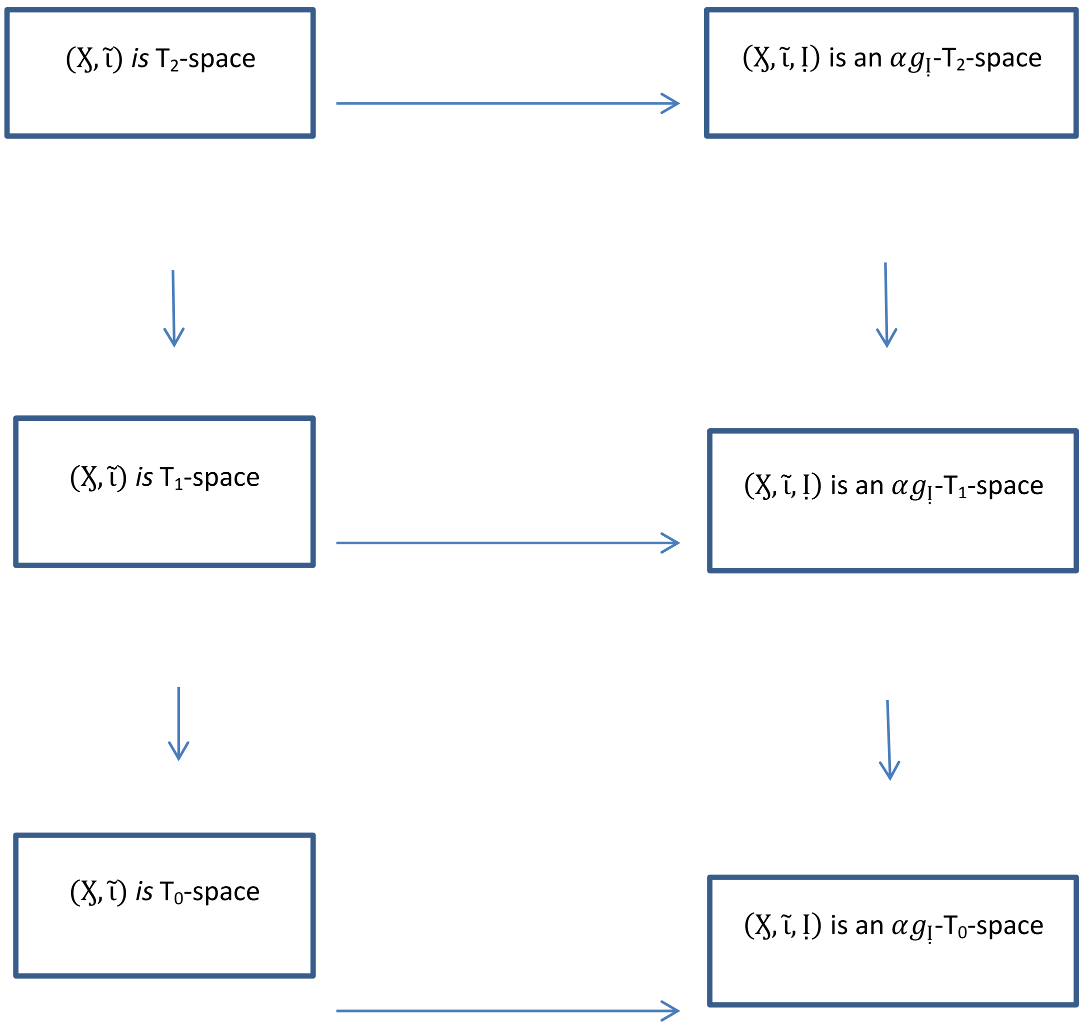
Proof: Let $(\text{Ẋ}, \tilde{\iota}, \text{Ị})$ is an $\alpha g_{\text{Ị}}$-T_2-space and Let $\dot{e}_1, \dot{e}_2$ ∈ Ẋ, where $\dot{e}_1 \neq \dot{e}_2$**.** Since $(\text{Ẋ}, \tilde{\iota}, \text{Ị})$ is a T_2-space, then there are $\alpha g_{\text{Ị}}$-open sets Ơ$_1$ and Ơ$_2$, satisfies $\dot{e}_1$ ∈ Ơ$_1$ and $\dot{e}_2$ ∈ Ơ$_2$ and Ơ$_1$∩ Ơ$_2$=∅. Then there are $\alpha g_{\text{Ị}}$-open sets Ơ$_1$ and Ơ$_2$, such that $\dot{e}_1$ ∈ Ơ$_1$- Ơ$_2$ and $\dot{e}_2$ ∈ Ơ$_2$- Ơ$_1$.

The opposite direction of proposition 3.14, is generally incorrect.

Example 3.15. Let (X̧, ĩ, Į) is a space, such that X̧ = Ŋ, the set of all natural numbers ĩ = ĩ cof, the collection of all complement finite topology and Į = ∅, then $ĩ_{\alpha}$ = ĩ cof, $\alpha g_{Į}C$(X̧) ={ Ơ ⊆ X̧, Ơ is a finite set}∪ X̧, then (X̧, ĩ, Į) is an $\alpha g_{Į}$-T_1-space but not $\alpha g_{Į}$-T_2-space.

Proposition 3.16. If (X̧, ĩ) is a T_i-space i={1,2,3} then the ideal space (X̧, ĩ, Į) is an $\alpha g_{Į}$-T_i-space. But the converse is not true, as shown in the following Arrow chart

Arrow chart (3.1)



Relationships between T_i-space and $\alpha g_{Į}$-T_i-space

The next example shows that the converse of the arrow chart 3.1is incorrect.

Example 3.17. Let $(X,\tilde{\iota},I)$ is an αg_I-T_i-space, where $X=\{\dot{e}_1,\dot{e}_2,\dot{e}_3\}$, $\tilde{\iota}=\{X,\emptyset,\{\dot{e}_2\}\}$ and $I=P(X)$ is not T_i-space, where $i=\{1,2,3\}$, and $\tilde{\iota}_\alpha=\{X,\emptyset,\{\dot{e}_2\},\{\dot{e}_1,\dot{e}_2\},\{\dot{e}_2,\dot{e}_3\}\}$, $\jmath_\alpha=\{X,\emptyset,\{\dot{e}_1\},\{\dot{e}_3\},\{\dot{e}_1,\dot{e}_3\}\}$ and $\alpha g_I C(X)=\alpha g_I O(X)=P(X)$.

3.1- On αg_I-Separtion Axioms via some types of function.

Proposition 3.1.1. If $(X,\tilde{\iota},I)$ is an αg_I-T_i-space whenever (i∈{0,1,2} and $f:(X,\tilde{\iota},I)\to(Y,\jmath,J)$ is a surjective, $\alpha g_I O$-function implies that $(Y,\jmath,J)$ is an αg_J-T_i-space.

Proof: If, $i=0$: Let $\dot{e}_1\neq\dot{e}_2$, where $\dot{e}_1,\dot{e}_2\in Y$. Since f is a surjective function then $f^{-1}(\dot{e}_1)\neq\emptyset$, $f^{-1}(\dot{e}_2)\neq\emptyset$, and $f^{-1}(\dot{e}_1)\neq f^{-1}(\dot{e}_2)$, where $f^{-1}(\dot{e}_1), f^{-1}(\dot{e}_2)\in X$, since X is an αg_I-T_0-space then there is an αg_I-open set O in X contains one of elements $f^{-1}(\dot{e}_1)$ and $f^{-1}(\dot{e}_2)$. Since f is an $\alpha g_I O$-function. Then $f(O)$ is an αg_J-open set contains one of two elements $\dot{e}_1$ and $\dot{e}_2$. Hence Y is an αg_J-T_0-space.

If $i=1$, Let $\dot{e}_1\neq\dot{e}_2$, where $\dot{e}_1,\dot{e}_2\in Y$. Since f is a surjective function then $f^{-1}(\dot{e}_1)\neq\emptyset$, $f^{-1}(\dot{e}_2)\neq\emptyset$, and $f^{-1}(\dot{e}_1)\neq f^{-1}(\dot{e}_2)$, where $f^{-1}(\dot{e}_1), f^{-1}(\dot{e}_2)\in X$, since X is an αg_I-T_1-space then there is an αg_I-open sets O_1 and O_2 in X such that $f^{-1}(\dot{e}_1)\in(O_1-O_2)$ and $f^{-1}(\dot{e}_2)\in(O_2-O_1)$. Since f is an $\alpha g_I O$-function. Then $f(O_1)$ and $f(O_2)$ are αg_J-open set, such that $\dot{e}_1\in(f(O_1)-f(O_2))$ and $\dot{e}_2\in(f(O_2)-f(O_1))$. Hence Y is an αg_J-T_1-space.

If $i=2$, Let $\dot{e}_1\neq\dot{e}_2$, where $\dot{e}_1,\dot{e}_2\in Y$. Since f is a surjective function then $f^{-1}(\dot{e}_1)\neq\emptyset$, $f^{-1}(\dot{e}_2)\neq\emptyset$, and $f^{-1}(\dot{e}_1)\neq f^{-1}(\dot{e}_2)$, where $f^{-1}(\dot{e}_1), f^{-1}(\dot{e}_2)\in X$, since X is an αg_I-T_2-space then there is an αg_I-open sets O_1 and O_2 in X such that $f^{-1}(\dot{e}_1)\in O_1$, $f^{-1}(\dot{e}_2)\in O_2$ and $O_1\cap O_2=\emptyset$. Since f is an $\alpha g_I O$-function. Then $f(O_1)$ and $f(O_2)$ are αg_J-open set, such that $\dot{e}_1\in f(O_1)$ and $\dot{e}_2\in f(O_2)$ and $f(O_1)\cap f(O_2)=f(\emptyset)=\emptyset$. Hence Y is an αg_J-T_2-space.

Proposition 3.1.2. If X is a T_i-space (i∈{0,1,2}) and $f:(X,\tilde{\iota},I)\to(Y,\jmath,J)$ is a surjective αg_I^*o-function then Y is an αg_J-T_i-space.

Proof: Similar to the proof of proposition 3.1.1. Since f is an αg_I^*-open function then $f(O)$ is an αg_J-open set in Y for all open set O in X.

Proposition 3.1.3. If $(X,\tilde{\iota},I)$ is an αg_I-T_i-space whenever (i∈{0,1,2} and f is a surjective αg_I^{**}o-function from $(X,\tilde{\iota},I)$ to $(Y,\jmath,J)$ then $(Y,\jmath)$ is T_i-space.

Proof: Similar to the proof of proposition 3.1.1. Since f is an αg_I^{**}-open function then $f(O)\in Y$, whenever O is an αg_I-open set in X.

Proposition 3.1.4. If Y is a T_i-space (i∈ $\{0,1,2\}$) and ʃ is an injective αg_{I}-continuous function from (Ҳ, ĩ, Ị) to (Y, ɟ, ʝ) then Ҳ is an αg_{I}-T_i-space.

Proof: If $i = 0$: Let $\dot{e}_1 \neq \dot{e}_2$, where $\dot{e}_1, \dot{e}_2 \in$ Ҳ. Since ʃ is a injective function then $ʃ(\dot{e}_1) \neq ʃ(\dot{e}_2)$, where $ʃ(\dot{e}_1)$, $ʃ(\dot{e}_2) \in$ Y. So Y is a T_0-space, then ∃Ơ ∈ Y contains one of the two elements $ʃ(\dot{e}_1)$ or $ʃ(\dot{e}_2)$. Since ʃ is an αg_{I}-continuous, then $ʃ^{-1}$(Ơ) is an αg_{I}-open set contains one of two elements $\dot{e}_1$ or $\dot{e}_2$. Hence Ҳ is an αg_{I}-T_0-space.

If $i = 1$: Let $\dot{e}_1 \neq \dot{e}_2$, where $\dot{e}_1, \dot{e}_2 \in$ Ҳ. Since ʃ is a injective function then $ʃ(\dot{e}_1) \neq ʃ(\dot{e}_2)$, where $ʃ(\dot{e}_1)$, $ʃ(\dot{e}_2) \in$ Y. So Y is a T_1-space, then ∃ $Ơ_1$, $Ơ_2$∈ Y, such that ʃ $(\dot{e}_1) \in$ ($Ơ_1$-$Ơ_2$) and ʃ $(\dot{e}_2) \in$ ($Ơ_2$-$Ơ_1$). Since ʃ is an αg_{I}-continuous, then $ʃ^{-1}$($Ơ_1$) and $ʃ^{-1}$($Ơ_2$) are αg_{I}-open set whenever $\dot{e}_1 \in$ ($ʃ^{-1}$($Ơ_1$)-$ʃ^{-1}$($Ơ_2$)), $\dot{e}_2 \in$ ($ʃ^{-1}$($Ơ_2$)-$ʃ^{-1}$($Ơ_1$)). Hence Ҳ is an αg_{I}-T_1-space.

If $i = 2$: Let $\dot{e}_1 \neq \dot{e}_2$, where $\dot{e}_1, \dot{e}_2 \in$ Ҳ. Since ʃ is a injective function then $ʃ(\dot{e}_1) \neq ʃ(\dot{e}_2)$, where $ʃ(\dot{e}_1)$, $ʃ(\dot{e}_2) \in$ Y. So Y is a T_2-space, then ∃ $Ơ_1$, $Ơ_2$∈ Y, such that ʃ $(\dot{e}_1) \in Ơ_1$ and ʃ $(\dot{e}_2) \in Ơ_2$ and $Ơ_1$ ∩ $Ơ_2$= $\emptyset$. Since ʃ is an αg_{I}-continuous, then $ʃ^{-1}$($Ơ_1$) and $ʃ^{-1}$($Ơ_2$) are αg_{I}-open set whenever $\dot{e}_1 \in ʃ^{-1}$($Ơ_1$), $\dot{e}_2 \in ʃ^{-1}$($Ơ_2$) and $ʃ^{-1}$($Ơ_1$) ∩ $ʃ^{-1}$($Ơ_2$)= $ʃ^{-1}(\emptyset) = \emptyset$. Hence Ҳ is an αg_{I}-T_2-space.

Corollary 3.1.5. If Y is a T_i-space and ʃ is an injective continuous function from (Ҳ, ĩ, Ị) to (Y, ɟ, ʝ) then Ҳ is an αg_{I}-T_i-space whenever (i∈ $\{0,1,2\}$).

Proof: Since, every continuous function is an αg_{I}-continuous function, then by proposition 2.2.2 and by proposition 3.1.4, then Ҳ is an αg_{I}-T_i-space.

Proposition 3.1.6. If Y is an αg_{J}-T_i-space and ʃ is an injective strongly αg_{I}-continuous function from (Ҳ, ĩ, Ị) to (Y, ɟ, ʝ) then Ҳ is a T_i-space whenever (i∈ $\{0,1,2\}$).

Proof: Similar to the proof of proposition 3.1.4.

Proposition 3.1.7. If Y is an αg_{J}-T_i-space and ʃ is an injective αg_{I}-irresolute function from (Ҳ, ĩ, Ị) to (Y, ɟ, ʝ) then Ҳ is an αg_{I}-T_i-space whenever (i∈ $\{0,1,2\}$).

Proof: Similar to the proof of proposition 3.1.5.

4- On αg_{I}-convergence.

Definition 4.1. Let (Ӽ, ĩ, Ị) be an ideal topological space, ė$_0$ ∈ Ӽ and (ʂ$_ŋ$)$_{ŋ∈N}$ be a sequence in Ӽ. Then (ʂ$_ŋ$)$_{ŋ∈N}$ is called $\alpha g_{Ị}$-convergence to ė$_0$ in simple terms ʂ$_ŋ$⟼ ė$_0$ if for every $\alpha g_{Ị}$-open set Ơ contined ė$_0$, $\exists\kappa$ ∈ Ɲ where ʂ$_ŋ$∈ Ơ ∀ŋ $\geq \kappa$.

A sequence (ʂ$_ŋ$)$_{ŋ∈N}$ is called $\alpha g_{Ị}$-divergence if it is not $\alpha g_{Ị}$-convergence.

Theorem 4.2. If (Ӽ, ĩ, Ị) is an $\alpha g_{Ị}$-T$_2$-space then every $\alpha g_{Ị}$-convergence sequence in Ӽ has only one limit point.

Proof: If we consider (ʂ$_ŋ$)$_{ŋ∈N}$ be a sequence in Ӽ and ʂ$_ŋ$⟼ ė$_1$ and ʂ$_ŋ$⟼ ė$_2$, ė$_1$ ≠ ė$_2$ where ė$_1$, ė$_2$ ∈ Ӽ. Since (Ӽ, ĩ, Ị) is an $\alpha g_{Ị}$-T$_2$-space, then there are disjoin $\alpha g_{Ị}$-open set Ơ$_1$ and Ơ$_2$ such that ė$_1$ ∈ Ơ$_1$ and ė$_2$ ∈ Ơ$_2$ since ʂ$_ŋ$⟼ ė$_1$and ė$_1$ ∈ Ơ$_1$ leads to $\exists\kappa_1$∈ Ɲ; ʂ$_ŋ$∈ Ơ$_1$ ∀ŋ $\geq \kappa_1$. So ʂ$_ŋ$⟼ ė$_2$and ė$_2$ ∈ Ơ$_2$ leads to $\exists\kappa_2$∈ Ɲ; ʂ$_ŋ$∈ Ơ$_2$ ∀ŋ $\geq \kappa_2$. Hence Ơ$_1$ ∩ Ơ$_2$≠ ∅, and that a contradiction.

The precondition that a space Ӽ is an $\alpha g_{Ị}$-T$_2$-space is very requisite to make Theorem 4.2 is valid.

Example 4.3. For a space (Ӽ, ĩ, Ị) where Ӽ={ė$_1$, ė$_2$, ė$_3$}, ĩ={Ӽ,∅,{ė$_1$},{ė$_1$, ė$_2$}} and Ị={∅} then ĩ$_\alpha$={Ӽ,∅,{ė$_1$},{ė$_1$, ė$_2$}, {ė$_1$, ė$_3$}}, then $\alpha g_{Ị}C$(Ӽ) = {Ӽ, ∅, {ė$_3$}, {ė$_2$, ė$_3$}} and $\alpha g_{Ị}O$(Ӽ) ={Ӽ, ∅, {ė$_1$}, {ė$_1$, ė$_2$}} .

The sequence (ʂ$_ŋ$)$_{ŋ∈N}$ in Ӽ, where ʂ$_ŋ$ = ė$_3$ ∀ŋ, has one limit point; ʂ$_ŋ$⟼ ė$_3$. But (Ӽ, ĩ, Ị) is not $\alpha g_{Ị}$-T$_2$-space.

The following proposition explains the relationships between convergence and $\alpha g_{Ị}$-convergence to ė$_0$.

Proposition 4.4. If a sequence (ʂ$_ŋ$)$_{ŋ∈N}$ is an $\alpha g_{Ị}$-convergence to ė$_0$ in an ideal space Ӽ, then it is a convergence to ė$_0$.

Proof: Let Ơ is open set in Ӽ contains ė$_0$. By Remark 2.4, Ơ $\alpha g_{Ị}$-open set in Ӽ contains ė$_0$. Since (ʂ$_ŋ$)$_{ŋ∈N}$ is an $\alpha g_{Ị}$-convergence to ė$_0$ then $\exists\kappa$ ∈ Ɲ, where ʂ$_ŋ$∈ Ơ ∀ŋ $\geq \kappa$. Hence (ʂ$_ŋ$)$_{ŋ∈N}$ is a convergence to ė$_0$.

Reverse the proposition 4.4, is incorrect in general.

Example 4.5. For an ideal space $(Ꭓ, ĩ, Ị)$, where $Ꭓ = Ɲ$, the set of all natural numbers ĩ={Ꭓ,∅} and Ị=$\mathrm{P}(Ꭓ)$. Then $ĩ_{\alpha}$=$ɟ_{\alpha}$={Ꭓ,∅}, so $\alpha g_{Ị}C$(Ꭓ)= $\alpha g_{Ị}O$(Ꭓ) = $\mathrm{P}(Ꭓ)$. The sequence $(ʂ_ŋ)_{ŋ\in Ɲ}$, where $ʂ_ŋ = ŋ$, ∀ŋ ∈ Ɲ, is convergent to ŋ=1 which is not $\alpha g_{Ị}$-convergence.

Proposition 4.6. Let $ʄ: (Ꭓ, ĩ, Ị) \rightarrow (ϒ, ɟ, ʝ)$ be an $\alpha g_{Ị}$-irresolute function and $(ʂ_ŋ)_{ŋ\in Ɲ}$ be a sequence in Ꭓ. If $ʂ_ŋ \longmapsto ẻ_0$ in Ꭓ then $ʄ(ʂ_ŋ) \longmapsto ʄ(ẻ_0)$ in ϒ.

Proof: Let Ơ is an $\alpha g_{ʝ}$-open set in ϒ contains $ʄ(ẻ_0)$. Since ʄ be an $\alpha g_{Ị}$-irresolute function, then $ʄ^{-1}$(Ơ) is an $\alpha g_{Ị}$-open set in Ꭓ contains $ẻ_0$. By $(ʂ_ŋ)_{ŋ\in Ɲ}$ is an $\alpha g_{Ị}$-convergence to $ẻ_0$, then $\exists \kappa \in Ɲ$, where $ʂ_ŋ \in ʄ^{-1}$(Ơ) $\forall ŋ \geq \kappa$, implies $\exists \kappa \in Ɲ$, where $ʄ(ʂ_ŋ) \in$ Ơ $\forall ŋ \geq \kappa$. Hence $ʄ(ʂ_ŋ)$ is an $\alpha g_{Ị}$-convergence to $ʄ(ẻ_0)$.

Theorem 4.7. Let $ʄ: (Ꭓ, ĩ, Ị) \rightarrow (ϒ, ɟ, ʝ)$ be an $\alpha g_{Ị}$-continuous function and $(ʂ_ŋ)_{ŋ\in Ɲ}$ be a sequence in Ꭓ. If $ʂ_ŋ \longmapsto ẻ_0$ in Ꭓ then $ʄ(ʂ_ŋ)$ convergent *to* $ʄ(ẻ_0)$ in ϒ.

Proof: Let Ơ is an open set in ϒ contains $ʄ(ẻ_0)$. Since ʄ be an $\alpha g_{Ị}$-continuous function, then $ʄ^{-1}$(Ơ) is an $\alpha g_{Ị}$-open set in Ꭓ contains $ẻ_0$. By $(ʂ_ŋ)_{ŋ\in Ɲ}$ is an $\alpha g_{Ị}$-convergence to $ẻ_0$, then $\exists \kappa \in Ɲ$, where $ʂ_ŋ \in ʄ^{-1}$(Ơ) $\forall ŋ \geq \kappa$, implies $\exists \kappa \in Ɲ$, where $ʄ(ʂ_ŋ) \in$ Ơ $\forall ŋ \geq \kappa$. Hence $ʄ(ʂ_ŋ)$ is an $\alpha g_{Ị}$-convergence to $ʄ(ẻ_0)$.

Proposition 4.8. Let $ʄ: (Ꭓ, ĩ, Ị) \rightarrow (ϒ, ɟ, ʝ)$ be a strongly-$\alpha g_{Ị}$-continuous function and $(ʂ_ŋ)_{ŋ\in Ɲ}$ be a sequence in Ꭓ. Then $ʄ(ʂ_ŋ)$ convergent *to* $ʄ(ẻ_0)$ in ϒ whenever if $ʂ_ŋ \longmapsto ẻ_0$ in Ꭓ.

Proof: Let Ơ is an $\alpha g_{ʝ}$-open set in ϒ contains $ʄ(ẻ_0)$. Since ʄ is a strongly-$\alpha g_{Ị}$-continuous function, then $ʄ^{-1}$(Ơ) is an open set in Ꭓ contains $ẻ_0$. By $(ʂ_ŋ)_{ŋ\in Ɲ}$ is a convergence to $ẻ_0$, then $\exists \kappa \in Ɲ$, where $ʂ_ŋ \in ʄ^{-1}$(Ơ) $\forall ŋ \geq \kappa$, implies $\exists \kappa \in Ɲ$, where $ʄ(ʂ_ŋ) \in$ Ơ $\forall ŋ \geq \kappa$. Hence $ʄ(ʂ_ŋ)$ is an $\alpha g_{Ị}$-convergence to $ʄ(ẻ_0)$.

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HYERS-ULAM STABILITY OF INTEGRAL EQUATIONS WITH TWO VARIABLES

Qusuay H. Alqifiary
Department of Mathematics,College of Science, University of Al-Qadisiyah,
Al-Diwaniya, Iraq
E-mail: qhaq2010@gmail.com ,qusuay.alqifiary@qu.edu.iq

Abstract
We will apply the classical Banach contraction for proving the generalized Hyers-Ulam stability and Hyers-Ulam stability of Volterra integral equations with two variables.

1 Introduction

Volterra integral equations appearance in 1896, therefore it have been extensively studied. Interest in this equation has emerged due to its importance in applications, for instance in chemical reactions,fluid flow, , semiconductors and elasticity, see ([3, 6, 11, 19]).
We say a functional equation is stable when for every approximate solution, there exists near it an exact solution. The concept of stability has been studied for different equations in a quite extensive way, during the last decades.
In 1940,S.M. Ulam [32] posed a famous question concerning the stability of functional equations: \Give conditions in order for a linear function near an approximately linear function to exist." In 1941, a partial answer to the equation of Ulam given by D.H. Hyers [12] for additive functions defined on Banach spaces: Suppose that *X* and *Y* are real Banach spaces and " > 0. Then for every function ^ : *X ! Y* with the property

// ^ (*x* + *y*) - ^(*x*) - ^(*y*)// ≤ "	(*x; y 2 X*);
there exists a unique additive function *T* : *X ! Y* such that the relation below comes	
true // ^ (*x*) - *T*(*x*)// ≤ "	
(*x 2 X*):	

In 1978, Th.M.Rassias in ([26]) considered unbounded right-hand sides in the inequality introducing therefore called the Hyers-Ulam-Rassias stability.
After that, many mathematicians have extended Ulam's problem to other functional equations and generalized Hyers's result in various directions (see [7, 13, 18, 26]).

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A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations (see [2, 20, 21, 30, 31, 15, 17, 16, 28, 29, 23, 24, 25, 14]). M.Gachpazen and O. Baghani ([9]), by successive method proved the HyersUlam stability of a nonlinear integral equation, then in 2011, M.Akkouchi, A. Bounabat and M.H.L.Rhali ([1]) proved Hyers-Ulam-Rassias by the classical Banach contraction.
Despite the large amount of workes on integral equations.(see [5, 10, 22, 4]).
In this paper, we proving Hyers-Ulam-Rassias stability, by using the fixed point alternative theory, for Volterra-type integral equations with two variables .

$u(x; y) = f(x; y) + \int_0^x g(x; y; \xi; u(\xi; y))d\xi + \int_0^x \int_0^y h(x; y; \sigma; \tau; u(\sigma; \tau))d\tau d\sigma$

for *x; y 2* R_+, where *f 2 C*(*E;* R_n),*g 2 C*($E_1 \times R_n$*;* R_n) and *h 2 C*($E_2 \times R_n$*;* R_n) are

functions and u is the unknown function to be found.
In addition, we proving generalize Hyers-Ulam stability for the Volterra-Fredholm -type integral equation in the form
$u(x; y) = h(x; y) + \int_0^x \int_0^y F(x; y; s; t; u(s; t))dtds + \int_0^1 \int_0^1 G(x; y; s; t; u(s; t))dtds;$
for $x; y \in \mathbb{R}_+$, where $h \in C(E; \mathbb{R}^n)$, $F \in C(E_2 \times \mathbb{R}^n; \mathbb{R}^n)$, $G \in C(E_2 \times \mathbb{R}^n; \mathbb{R}^n)$.

2 Preliminaries

Definition 2.1 *For a nonempty set Y, a function $\wedge : Y \times Y \to [0; \infty]$ is called a generalized metric on Y if and only if the function $\wedge$ satisfies :*
(*i*) $\wedge(x_1; x_2) = 0$ *if and only if* $x_1 = x_2$;
(*ii*) $\wedge(x_1; x_2) = \wedge(x_2; x_1)$ *for all* $x_1; x_2 \in Y$;
(*iii*) $\wedge(x_1; x_2) \leq \wedge(x_1; y) + \wedge(y; x_2)$ *for all* $x_1; x_2; y \in Y$.

Theorem 2.1 *(The fixed point alternative) [8] Assume that $(X; d)$ is a generalized complete metric space and $\wedge : X \to X$ is a strictly contractive operator with Lipschitz constant $L < 1$. If there exists a nonnegative integer c such that $d(\wedge^{c+1}x; \wedge^c x) < \infty$ for some $x \in X$, then the followings are true :*
(*a*) *The sequence $\{\wedge^n x\}$ convergens to a fixed point x_* of $\wedge$;*
(*b*) *x_* is the unique fixed point of $\wedge$ in*
$X_* = \{y \in X / d(\wedge^c x; y) < \infty\}$;
(*c*) *If $y \in X_*$, then* $d(y; x_*) \leq \frac{1}{1-L} d(\wedge y; y)$:

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Throughout this paper we will use the notation $E = \mathbb{R}_+ \times \mathbb{R}_+$, $E_0 = I_a \times I_b$, $E_1 = \{(x; y; s) : 0 \leq s \leq x < \infty; y \in \mathbb{R}\}$ and $E_2 = \{(x; y; s; t) \in E_2 : 0 \leq s \leq x < y; 0 \leq t \leq y < \infty\}$.
Let S be the space of all functions $u \in C(E; \mathbb{R}^n)$ which satisfies the condition.
$|u(x; y)| = O(exp(\Lambda(x + y)));$ (2.1)
where $\Lambda > 0$ is a constant.
we define in the space S the norm
$|u|_s = sup_{(x;y) \in E}[|u(x; y)| (exp(-\Lambda(x + y)))];$ (2.2)
such that, we get Banach space from S with norm defined in (2.2). From the condition (2.1) we get there exists a constant $N \geq 0$ such that $|u(x; y)| \leq N(exp(-\Lambda(x + y))$. By using this fact in (2.2), we observe that
$|u(x; y)|_s \leq N$: (2.3)
Now, define a metric function $d_s : S \times S \to [0; \infty]$ such that
$d_s(u(x; y) - v(x; y)) = |u(x; y) - v(x; y)|_s$;
for all $u; v \in S$. We obtain a generalization metric space $(S; d_s)$

Lemma 2.2 ([27]) *Suppose that*
(i) the functions $g; h$ in equation (3.1) satisfy the conditions
$|g(x; y; \Omega; u) - g(x; y; \Omega; u)| \leq a(x; y; \Omega) |u - u|$ (2.4)
$|h(x; y; \sigma; \tau; u) - h(x; y; \sigma; \tau; u)| \leq b(x; y; \sigma; \tau) |u - u|$ (2.5)
where $a \in C(E_1; \mathbb{R}_+)$, $b \in C(E_2; \mathbb{R}_+)$
(ii) for λ as in (2.1),
(*ii$_a$*) *There exists a constant α such that* $0 < \alpha < 1$ *and*
$\int_0^x a(x; y; \Omega)exp(\lambda(\Omega+y))d\Omega+\int_0^x \int_0^y b(x; y; \sigma; \tau)exp(\lambda(\sigma +\tau))d\tau d\sigma \leq \alpha exp(\lambda(x+y))$ (2.6)
(*ii$_b$*) *There exists a constant β such that* $0 < \beta$ *and*
$f(x; y) - \int_0^x g(x; y; \Omega; 0)d\Omega + \int_0^x \int_0^y h(x; y; \sigma; \tau; 0)d\tau d\sigma \leq \beta exp(\lambda(x + y))$
where $f; g; h$ are the functions in equation (3.1). Then the operator T which defined as follows:
$(Tu)(x; y) = f(x; y)+\int_0^x g(x; y; \Omega; u(\Omega; y))d\Omega+\int_0^x \int_0^y h(x; y; \sigma; \tau; u(\sigma; \tau))d\tau d\sigma$ (2.8)
where $u \in S$ is maps S into itself.
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Proof.

We must verify that (2.1) is fulfilled. We have from (2.8) and (2.3)

$j(Tu)(x; y)j \leq f(x; y) + \int_{0}^{x} g(x; y; \Omega; 0)d\Omega + \int_{0}^{x}\int_{0}^{y} h(x; y; \sigma; \tau; 0)d\tau d\sigma$

$+ \int_{0}^{x} jg(x; y; \Omega; u(\Omega; y)) - g(x; y; \Omega; 0)j\ d\Omega + \int_{0}^{x}\int_{0}^{y} jh(x; y; \sigma; \tau; u(\sigma; \tau)) - h(x; y; \sigma; \tau; 0)j\ d\tau d\sigma$

$\leq \beta exp(\lambda(x + y)) + \int_{0}^{x} a(x; y; \Omega)\ ju(\Omega; y)j\ d\Omega + \int_{0}^{x}\int_{0}^{y} b(x; y; \sigma; \tau)\ ju(\sigma; \tau)j\ d\tau d\Omega$

$\leq \beta exp(\lambda(x+y)) + jujs\ [\int_{0}^{x} a(x; y; \Omega)exp(\Omega; y)d\Omega + \int_{0}^{x}\int_{0}^{y} b(x; y; \sigma; \tau)exp(\lambda(\sigma+\Omega))d\tau d\sigma]$

$\leq [\beta + N\alpha]exp(\lambda(x + y))$

Hence, $Tu\ 2\ S$ and mean that is T maps S into itself.

3 Main Results

3.1 Volterra-type integral equation

In this subsection, we consider the integral equation

$u(x; y) = f(x; y) + \int_{0}^{x} g(x; y; \Omega; u(\Omega; y))d\Omega + \int_{0}^{x}\int_{0}^{y} h(x; y; \sigma; \tau; u(\sigma; \tau))d\tau d\sigma$ (3.1)

for $x; y\ 2\ \mathbb{R}_+$, where $f\ 2\ C(E; \mathbb{R}^n)$,$g\ 2\ C(E_1 \times \mathbb{R}^n; \mathbb{R}^n)$ and $h\ 2\ C(E_2 \times \mathbb{R}^n; \mathbb{R}^n)$ are functions and u is the unknown function to be found.

We start with the following theorem which ensures the equation (3.1) has HyersUlam-Rassias stability.

Theorem 3.1 *Under the same conditions in Lemma (2.2), let θ is a continuous function $\theta : E \to \mathbb{R}_+$ and $u\ 2\ S$ is such that*

$u(x; y) - f(x; y) + \int$	$h(x; y; \sigma; \tau; u(\sigma; \tau))d\tau d\sigma \leq \theta(x; y);$

$_{0}^{x} g(x; y; \Omega; u(\Omega; y))d\Omega + \int_{0}^{x}\int_{0}^{y}$ (3.2)

then there is a unique solution $u_0\ 2\ C(E; \mathbb{R}_+)$ of integral equation (3.1) and constant $0 < \alpha < 1$ *such that*

$ju(x; y) - u_0(x; y)j \leq \theta(x; y)$

$1 - \alpha$

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Proof. Let $u; v\ 2\ S$. Using the hypotheses, consider the operator defined in (2.8)

$d_s(Tu; T v) = j(Tu)(x; y) - (T v)(x; y)j \leq \int_{0}^{x} jg(x; y; \Omega; u(\Omega; y)) - g(x; y; \Omega; v(\Omega; y))j\ d\Omega$

$+ \int_{0}^{x}\int_{0}^{y} jh(x; y; \sigma; \tau; u(\sigma; \tau)) - h(x; y; \sigma; \tau; v(\sigma; \tau))j\ d\tau d\sigma$

$\leq \int_{0}^{x} a(x; y)\ ju(\Omega; y) - v(\Omega; y)j\ d\Omega + \int_{0}^{x}\int_{0}^{y} b(x; y; \sigma; \tau)\ ju(\sigma; \tau) - v(\sigma; \tau)j\ d\tau d\sigma$

$\leq ju - vj_s\ [\int_{0}^{x} a(x; y; \Omega)exp(\lambda(\Omega + y))d\Omega + \int_{0}^{x}\int_{0}^{y} b(x; y; \sigma; \tau)exp(\lambda(\sigma + \tau))d\tau d\sigma]$

$\leq \alpha\ ju - vj_s\ exp(\lambda(x + y))$

we get

$d_s(Tu; T v) \leq \alpha d_s(u; v)$ (3.3)

Since $\alpha < 1$, from Banach fixed point theorem, it follows that T has a unique fixed point u_0 in S is however a solution of integral equation (3.1). We can apply again the Banach fixed point theorem, we get

$d_s(u; u_0) \leq 1$

$1 - \alpha$

$d_s(Tu; u)$

$ju(x; y) - u_0(x; y)j \leq \theta(x; y)$

$1 - \alpha$

Corollary 3.2 *Under the same conditions in Lemma (2.2), let $\epsilon > 0$ and $u\ 2\ S$ is such that*

$u(x; y) - f(x; y) + \int$	$h(x; y; \sigma; \tau; u(\sigma; \tau))d\tau d\sigma \leq \epsilon;$

$_{0}^{x} g(x; y; \Omega; u(\Omega; y))d\Omega + \int_{0}^{x}\int_{0}^{y}$ (3.4)

then there is a unique solution $u_0\ 2\ C(E; \mathbb{R}_+)$ of integral equation (3.1) and constant $0 < \alpha < 1$ *such that* $ju(x; y) - u_0(x; y)j \leq {}_{1-\epsilon\ \alpha}$.

This means that, the integral equation (3.1) has the Hyers-Ulam stability.

Theorem 3.3 *Suppose that the functions f; g in equation (3.1) satisfy the conditions (2.4),(2.5) and let*

$\sup_{x;y 2 \mathbb{R}_+} [\int_{0}^{x} a(x; y; \Omega)d\Omega + \int_{0}^{x}\int_{0}^{y} b(x; y; \sigma; \tau)d\tau d\sigma] \leq \alpha < 1$ (3.5)

then the equation (3.1) has Hyers-Ulam-Rassias stability. That means there is a unique

solution u_0 *2 C*(*E;* R+) *of integral equation (3.1) and constant* 0 < *α* < 1 *such that*
ju(*x; y*) - *u*0(*x; y*)*j* ≤ *θ*(*x; y*)
1 - *α*
6
Proof. Consider the space *C*(*E;* R*n*) with a generalization metric defined by

d(*u; v*) =*sup x;y*2R+ *ju*(*x; y*) - *v*(*x; y*)*j ;*	(3.6)
for *u; v 2 C*(*E;* R*n*). From hypotheses, we can prove that, the operator *T* defined by	
(2*:*8) is a contraction. For any *u; v 2 C*(*E;* R*n*) satisfies *d*(*Tu; T v*) ≤ *αd*(*u; v*)	(3.7)

Thus we can apply the Banach fixed point theorem, we get for u_0 *2 C*(*E;* R*n*) which satisfy equation (3.1),
d(*u; u*0) ≤ *αd*(*Tu; u*)
Thus by theorem (2.1)
ju(*x; y*) - *u*0(*x; y*)*j* ≤ *θ*(*x; y*)
1 - *α*

Corollary 3.4 *Under the same conditions in Theorem (3.3), let ϵ* > 0 *and u 2 S is such that*

u(*x; y*) - *f*(*x; y*) + ∫	*h*(*x; y; σ; τ; u*(*σ; τ*))*dτdσ* ≤ *ϵ;*

0*x* *g*(*x; y;* Ω*; u*(Ω*; y*))*d*Ω + ∫0*x* ∫0*y* (3.8)
then there is a unique solution u_0 *2 C*(*E;* R+) *of integral equation (3.1) and constant* 0 < *α* < 1 *such that ju*(*x; y*) - *u*0(*x; y*)*j* ≤ 1-*ϵα*.
This means that, the integral equation (3.1) has the Hyers-Ulam stability.

3.2 Volterra-Fredholm-type integral equation

In this subsection, we consider the Volterra-Fredholm -type integral equation in the form
u(*x; y*) = *h*(*x; y*) + ∫0*x* ∫0*y* *F*(*x; y; ξ; t; u*(*ξ; t*))*dtdξ* + ∫0*1* ∫0*1* *G*(*x; y; ξ; t; u*(*ξ; t*))*dtdξ;*
(3.9)
for *x; y 2* R+, where *h 2 C*(*E;* R*n*),*F 2 C*(*E*2 × R*n;* R*n*) ,*G 2 C*(*E*2 × R*n;* R*n*).
Theorem 3.5 *Assume that*
(*i*) *the functions F; G in equation (3.9) satisfy the conditions*
jF(*x; y; ξ; t; u*) - *F*(*x; y; ξ; t; v*)*j* ≤ *k*(*x; y; ξ; t*) *ju - vj ;* (3.10)
jG(*x; y; ξ; t; u*) - *G*(*x; y; ξ; t; v*)*j* ≤ *r*(*x; y; ξ; t*) *ju - vj ;* (3.11)
7
where k 2 C(*E*2*;* R+), *r 2 C*(*E*2*;* R+).
(*ii*) *for λ as in inequality (2.1),*
(*b*1) *there exist constants* $\alpha_1 \geq 0$ *and* $\alpha_2 \geq 0$ *such that* $\alpha_1 + \alpha_2 < 1$ *and*
∫0*x* ∫0*y* *k*(*x; y; ξ; t*)*exp*(*λ*(*ξ* + *t*))*dtdξ* ≤ *α*1*exp*(*λ*(*x* + *y*)) (3.12)
∫0*1* ∫0*1* *r*(*x; y; ξ; t*)*exp*(*λ*(*ξ* + *t*))*dtdξ* ≤ *α*2*exp*(*λ*(*x* + *y*)) (3.13)
(*b*2) *there exists a constant β* ≥ 0 *such that*
jh(*x; y*)*j* + ∫0*x* ∫0*y* *jF* (*x; y; ξ; t;* 0)*j dtdξ* + ∫0*1* ∫0*1* *jG*(*x; y; ξ; t;* 0)*j dtdξ* ≤ *βexp*(*λ*(*x* + *y*))*;*
(3.14)
where h; F; G are the functions in equation (3.9). Then if u 2 S is such that
u(*x; y*) - *h*(*x; y*) + ∫0*x* ∫0*y* *F* (*x; y; ξ; t; u*(*ξ; t*))*dtdξ* + ∫0*1* ∫0*1* *G*(*x; y; ξ; t; u*(*ξ; t*))*dtdξ* ≤ *φ*(*x; y*)*;*
(3.15)
where φ : *E !* R+ *there is a unique solution* u_0 *2 C*(*E;* R+) *of integral equation (3.1) and constant* 0 < *α* < 1 *such that*
ju(*x; y*) - *u*0(*x; y*)*j* ≤ *φ*(*x; y*)
1 - *α*
Proof. Let *u 2 S* and define the operator *T* by
(*Tu*)(*x; y*) = *h*(*x; y*)+∫0*x* ∫0*y* *F* (*x; y; ξ; t; u*(*ξ; t*))*dtdξ* +∫0*1* ∫0*1* *G*(*x; y; ξ; t; u*(*ξ; t*))*dtdξ;*
(3.16)
for (*x; y*) *2 E:*
Now, we will show that *T* maps *S* into itself.From equation (3.16), we have

$j(Tu)(x; y)j \leq jh(x; y)j + \int_{0x}\int_{0y} jF(x; y; \xi; t; 0)j\ dtd\xi + \int_{01}\int_{01} jG(x; y; \xi; t; 0)j\ dtd\xi +$
$\int_{0x}\int_{0y} jF(x; y; \xi; t; u(\xi; t)) - F(x; y; \xi; t; 0)j\ dtd\xi + \int_{01}\int_{01} jG(x; y; \xi; t; u(\xi; t)) - G(x; y; \xi; t; 0)j\ dtd\xi$
$\leq \beta exp(\lambda(x + y)) + \int_{0x}\int_{0y} k(x; y; \xi; t)\ ju(\xi; t)j\ dtd\xi + \int_{01}\int_{01} r(x; y; \xi; t)\ ju(\xi; t)j\ dtd\xi$
$\leq \beta exp(\lambda(x+y)) + juj_s\ [\int_{0x}\int_{0y} k(x; y; \xi; t)exp(\lambda(\xi+t))dtd\xi + \int_{01}\int_{01} r(x; y; \xi; t)exp(\lambda(\xi+t))dtd\xi]$
$\leq [\beta + N(\alpha_1 + \alpha_2)]exp(\lambda(x + y))$
That means that $Tu\ 2\ S$.
$d_s(Tu\text{-}T v) = j(Tu)(x; y) - (T v)(x; y)j \leq \int_{0x}\int_{0y} jF(x; y; \xi; t; u(\xi; t)) - F(x; y; \xi; t; v(\xi; t))j\ dtd\xi +$
8
$\int_{01}\int_{01} jG(x; y; \xi; t; u(\xi; t)) - G(x; y; \xi; t; v(\xi; t))j\ dtd\xi$
$\leq \int_{0x}\int_{0y} k(x; y; \xi; t)\ ju(\xi; t) - v(\xi; t)j\ dtd\xi + \int_{01}\int_{01} r(x; y; \xi; t)\ ju(\xi; t) - v(\xi; t)j\ dtd\xi$
$\leq ju - vj_s\ [\int_{0x}\int_{0y} k(x; y; \xi; t)exp(\lambda(\xi + t))dtd\xi + \int_{01}\int_{01} r(x; y; \xi; t)exp(\lambda(\xi + t))dtd\xi]$
$\leq (\alpha_1 + \alpha_2)\ ju - vj_s\ exp(\lambda(x + y))$.
We get,
$d_s(Tu; T v) \leq (\alpha_1 + \alpha_2)d_s(u; v)$ (3.17)
Since $\alpha_1 + \alpha_2 < 1$, from Banach fixed point theorem, it follows that T has a unique fixed point u_0 in S is however a solution of integral equation (3.9). We can apply again the Banach fixed point theorem, we get
$d_s(u; u_0) \leq 1$
$1 - (\alpha_1 + \alpha_2)d_s(Tu; u)$
$ju(x; y) - u_0(x; y)j \leq \varphi(x; y)$
$1 - (\alpha_1 + \alpha_2)$

Theorem 3.6 *Suppose that the functions F; G in equation (3.9) satisfy the conditions (3.10),(3.11) and let*
sup
$_{x;y2\mathrm{R}_+}[\int_{0x}\int_{0y} k(x; y; \xi; t)dtd\xi + \int_{01}\int_{01} r(x; y; \xi; t)dtd\xi] \leq \alpha < 1$ (3.18)
then the equation (3.9) has Hyers-Ulam-Rassias stability. That means there is a unique solution $u_0\ 2\ C(E; \mathrm{R}_+)$ *of integral equation (3.9) and constant* $0 < \alpha < 1$ *such that*
$ju(x; y) - u_0(x; y)j \leq \theta(x; y)$
$1 - \alpha$
Proof. Consider the space $C(E; \mathrm{R}_n)$ with a generalization metric defined by
$d(u; v) = {}_{sup\ x;y2\mathrm{R}_+} ju(x; y) - v(x; y)j\ ;$ (3.19)
for $u; v\ 2\ C(E; \mathrm{R}_n)$. We can complete proof, in same way proof of theorem (3.3) .
9

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Some results of Mixed Fuzzy Topological Ring

Munir Abdul Khalik AL-Khafaji
AL-Mustinsiryah University \ College of Education \ Department of Mathematics
Basim Mohammed Melgat
University of Al- Furat Al- Awsat techniques \ College of Health and Medical Technique/ Kuffa

Abstract

The theory of fuzzy topological ring has wide scope of applicability than order topological ring theory. The reason is fuzzy can provide better result. Therefore, fuzzy topological ring has been found in Robotics, computer, artificial intelligent, etc.

In this paper, we continue the study of mixed fuzzy topological ring [13]. We are studying the mixed fuzzy topological subring space, mixed fuzzy quotient topological ring and mixed product fuzzy topological ring.

Keywords: Fuzzy topological ring, Mixed Fuzzy topological ring, mixed fuzzy Quotient topological ring and mixed product fuzzy topological ring

Introduction:

In 1965 [14], Zadeh L. A. gave the definition of fuzziness. After three years C. Chang [2] gave the notion of fuzzy topology. In 1990[1], Ahsanullah and Ganguli, depended on the convergent in fuzzy topological space in the sense of Lowen [7, 8] to introduce the concept of fuzzy nbhd. In 2009, Deb Ray, A. and Chettri, P [3] introduced fuzzy topology on a ring. Also in [4] they introduced fuzzy continuous function and studied left fuzzy topological ring.

In [9,10,11 and 12] we studied the induced fuzzy topological ring space by fuzzy pseudo norm ring space, fuzzy nbhds system fuzzy separation axiom, study fuzzy compactness and Bohr fuzzy compactification of fuzzy topological ring space. In [13] we construct a mixed fuzzy topological ring also we study the relationship between fuzzy continuities of fuzzy homo. with respect to different fuzzy topologies

In this present article we continue the study of mixed fuzzy topological ring and have obtained several significant results of mixed fuzzy topological subring space , mixed fuzzy quotient topological ring and mixed product fuzzy topological ring

For rich the paper, some basic concept of fuzzy set , fuzzy topology and fuzzy topological ring are given below. The symbol I will denote to the closed interval $[0,1]$. Let R be a non-empty set:

Definition [14] 1.1

A fuzzy set in R is a map $\partial: R \to I$ and, that is, belonging to I^R(the set of all fuzzy set of R) . Let $E \in I^R$, for every $r \in R$, we expressed by $E(r)$ of the degree of membership of r in R . If $E(r)$ be an element of $\{0, 1\}$, then E is said a crisp set.

Definition [2] 1.2

A class $\mu \in I^R$ of fuzzy set is called a fuzzy topology for R if the following are satisfied

1) $\emptyset, R \in \mu$
2) $\forall E, H \in \mu \to E \wedge H \in \mu$
3) $\forall (E_j)_{j\in J} \in \mu \to \vee_{j\in J} E_j \in \mu$

(R, μ) is called fuzzy topological space. if $A \in \mu$ Then A is fuzzy open and A^c (complement of A)is a fuzzy closed set.

Definition [1, 3 and 4] 1.3

A pair (R, μ) , where R a ring and μ a fuzzy topology on R, is called fuzzy topological ring if the following functions are fuzzy continuous:

1) $R\times R \to R, (r, k) \to r+k.$
2) $R \to R, \ r \to -r$
3) $R\times R \to R, (r, k) \to r.k$

Definition [4]1.4

A family B of fuzzy nbhds of r_α, for $0 < \alpha \le 1$, is called a fund. system of fuzzy nbhds of r_α iff for any fuzzy nbhd V of r_α, there is $U \in B$ such that $r_\alpha \le U \le V$

Definition [4]1.5

Let R be a ring and μ a FZT on R. Let U and V are fuzzy sets in R. We define $U + V$, $-V$ and $U.V$ as follows

$(U+V)(k) = \sup_{k=k_1+k_2} min\{U(k_1), V(k_2)\}$

$-V(k) = V(-k)$

$(U.V)(k) = \sup_{k=k_1+k_2} min\{U(k_1), V(k_2)\}$

Theorem [4]1.6

If R is a fuzzy topological ring then there is a fundamental system of fuzzy nbhds B of 0 $(0 < \alpha \le 1)$, such that the conditions:

(i) $\forall U \in B$, then $-U \in B$

(ii) $\forall U \in B$, then U is symmetric

(iii) $\forall U, V \in B$, then $U \wedge V \in B$

(iv) $\forall U \in B$, there is $V \in B$ such that $V + V \le U$

(v) $\forall U \in B$, there is $V \in B$ such that $V.V \le U$

(vi) $\forall r \in R, \forall U \in B$, there is $V \in B$ such that a $r.V \le U$ and $V.r \le U$.

Definition [7] 1.7

(R, μ) is fully stratified fuzzy topology on R if the fuzzy topology μ on R contain all constant fuzzy set

Theorem [5]1.8

Let (R, μ) and (R, ρ) be two fuzzy topological spaces and let $\mu(\rho) = \{E \in I^R : \exists\ U \in \rho\ s.t\ cl_\mu(U) \le E\}$. Then $\mu(\rho)$ is a mixed fuzzy topology on R

Theorem 1.9[4]

Every fuzzy subring of fuzzy topological ring is a fuzzy topological ring.

Proposition 1.10[5]

I f (G, ρ) is a fuzzy regular topological space and μ any other topology on E such that $\mu > \rho$, then $\mu(\rho) = \rho$

2.0 Mixed Fuzzy Topological subring

We study a fuzzy subring E of a bi- fuzzy topological ring (R, μ, ρ), we mean the bi-fuzzy topological Subring (E, μ_E, ρ_E), where μ_E and ρ_E are relative fuzzy topologies on E induced by μ and ρ respectively.

Let (R, μ, ρ) be any bi- fuzzy topological ring and E be a fuzzy subring of R. Clearly, the mixed fuzzy topological on E can be constructed in two different methods, the first method by mixing the relative fuzzy topologies μ_E and ρ_E on E and the second method by the mixed fuzzy topological $\mu(\rho)$, of R on E.

Definition 2.1

Let R be any ring equipped with two fuzzy topological ring space μ and ρ. Then the triplet (R, μ, ρ) is defined as a bi- fuzzy topological ring space.

Example 2.2

Let R be any ring with the indiscrete fuzzy topology I and the discrete fuzzy topology D. Then, (R, I, D) is a bi- fuzzy topological ring

Theorem 2.3

L et (R, μ, ρ) be any bi- fuzzy topological ring. I f $\mu < \rho$, then $\mu < \mu(\rho) < \rho$

Proof

Let us consider the identity map

$$i : (R, \rho) \rightarrow (G, \mu(\rho)).$$

For $cl_\mu(E) \in N_{\mu(\rho)}$,

$$i^{-1}\left(cl_\mu(E)\right) = cl_\mu(E) \supseteq E \in \rho$$

So, i is fuzzy continuous and consequently,

$$\rho > \mu(\rho)$$

For the other part, let $N_\mu = \{U\}$ be a fuzzy fundamental system of μ-fuzzy closed fuzzy nbhds of 0 in (R, μ). Since $\mu < \rho$, for each $U \in N_\mu$, there is a $V \in N_\rho$ such that

$$V \subseteq U$$

Therefore,

$$cl_\mu(V) \subseteq cl_\rho(U)$$

Thus, for each $U \in N_\mu$ there exists $Cl_\mu\ (V)\ \in N_{\mu(\rho)})$ such that

$$Cl_\mu\ (V) \subseteq U$$

This implies that

$$\rho < \ \mu(\rho)$$

Combining (1) and (2), the result follows.

Theorem 2.4

In either of the two above cases, every fuzzy subring of mixed fuzzy topological ring is a mixed fuzzy topological ring.

Proof

Let E be a fuzzy subring of a bi- fuzzy topological ring $(R\ ,\mu,\rho)$. By theorem 1.9, μ_E and ρ_E are fuzzy topological ring on E, then $\big(E, \mu_E\ (\ \rho_E)\big)$ where $\mu_E(\ \rho_E) = \{U \in I^E : \exists V \in \rho_E\ s.t\ cl_{\mu_E}(V) \leq U\}$, is fuzzy topological ring space. Also since $(R\ ,\mu(\rho))$ is mixed fuzzy topological ring, then$(\mu(\rho))_E$ is a mixed fuzzy topological ring on E

Theorem 2.5

Let $(R, \mu\ )$ and $(R, \rho\ )$ be two fuzzy topological rings such that $\mu < \rho$. If $(R, \mu\ )$ is fuzzy T_2-space then $(R, \mu(\rho)\ )$is also T_2-space.

Proof

Let $(R, \mu\ )$ is a T_2-space. Let us consider $r, k \in R$ and $r \neq k$. Then there are disjoint μ-fuzzy open sets $U\ , V$ such that

$$(U)(r) > 0 \text{ and } (V)(k) > 0$$

Since $\mu\ < \mu(\rho),\ then\ U\ and\ V$ also $\mu(\rho)$-fuzzy open sets. Thus, given $r, k \in R,\ r \neq k$, we have disjoint $\mu(\rho)$-fuzzy open sets $U\ , V$ such that

$$(U)(r) > 0 \text{ and } (V)(k) > 0$$

. So $(R, \mu(\rho))$ is a T_2-space.

Theorem 2.6

Let E be a fuzzy subring of a bi- fuzzy topological ring $(R\ ,\mu,\rho)$;

(1) If E be μ- fuzzy closed, then $(\mu(\rho))_E \leq \mu_E(\rho_E)$

(2) If (R, ρ) be fuzzy Hausdorff and $\mu > \rho$, then $\mu(\rho)_E = \mu_E(\rho_E)$

Proof

(1) Let $U \in \{V_{(\mu(\rho))_E}(0)\}$ be an element of fuzzy open nbhds of 0 in $(E, (\mu(\rho))_E)$, then there exists $V \in \{V_\rho(0)\}$ with $V(0) > 0$ s.t $cl_\mu(V) \wedge E = U$. Since E is μ- fuzzy closed,

$$cl_{\mu_E}(V \wedge E) = cl_\mu(V \wedge E) \wedge E$$

$$\leq cl_\mu(V) \wedge cl_\mu(E) \wedge E = cl_\mu(V) \wedge E = U$$

Also

$(V \wedge E)(0) = \min\{V(0), E(0)\} > 0$, implies $cl_{\mu_E}(V \wedge E)(0) > 0$

This obtain that there existed a fuzzy element of the fuzzy open nbhd of 0 for the mixed fuzzy topological $\mu_E(\rho_E)$ on E contained in every element of the fuzzy open nbhds of 0 for the mixed fuzzy topological $(\mu(\rho))_E$ on E

Thus

$$(\mu(\rho))_E \leq \mu_E(\rho_E)$$

(2) By Proposition 1.10, we have $\rho = \mu(\rho)$ on R. So,

$$\rho_E = (\mu(\rho))_E, \text{ on } E \qquad (i)$$

Clearly, (E, ρ_E) is Hausdorff and $\mu_E > \rho_E$on E since $\mu > \rho$ on R. By proposition 1.10, $\rho_E = \mu_E(\rho_E)$ on E $\qquad (ii)$

from (i) and (ii), the result follows.

Hence the theorem

Theorem 2.7

Let E be a fuzzy subring of a bi- fuzzy topological ring (R, μ, ρ) such that $\mu < \rho$ and (R, μ) is fuzzy Hausdorff; then

(a) $(E, \mu_E(\rho_E))$ is fuzzy Hausdorff

(b) $(E, \mu(\rho)_E)$ is fuzzy compact if (R, ρ) is fuzzy compact.

Proof

(a)

$(R, \mu)$is fuzzy Hausdorff and $\mu < \rho$, then $\mu < \mu(\rho) < \rho$, and by theorem 2.5,$(R, \mu(\rho))$ is also fuzzy Hausdorff. Then, $(E, \mu(\rho)_E)$ is fuzzy Hausdorff and hence by Theorem 2.6, $(E, \mu_E(\rho_E))$ is also fuzzy Hausdorff.

(b)

By hypothesis $\mu < \rho$, then $\mu < \mu(\rho) < \rho$ on R, and (R, ρ) is fuzzy compact then$(R, \mu(\rho))$ and (R, μ) are fuzzy compact.

Also (R,μ)being fuzzy Hausdorff, $(R,\mu(\rho))$ is also fuzzy Hausdorff [by theorem 2.5]. Then $(E,\mu(\rho)_E)$ is fuzzy compact

Theorem 2.8

Let E be a fuzzy subring of a bi-FZT ring (R,μ,ρ) such that $\mu > \rho$ and (R,ρ) is fuzzy Hausdorff, then

(a) $(E,\mu_E(\rho_E))$ is fuzzy Hausdorff,

(b) $(E,\mu_E(\rho_E))$ is fuzzy compact if (R,ρ) is fuzzy compact and E is ρ- fuzzy closed,

(c) $(E,\mu_E(\rho_E))$ is fuzzy locally compact if $(R,\rho)$is fuzzy locally compact and E is ρ-fuzzy closed,

Proof

Since $(R,\rho)$is fuzzy Hausdorff and $\mu > \rho$, therefore by Prop 1.10, $\rho = \mu(\rho)$ i.e $\mu(\rho)$ is fuzzy Hausdorff, fuzzy compact implies $\mu(\rho)_E$ is fuzzy Hausdorff, fuzzy compact . Also by Theorem 2.6

$$\mu(\rho)_E = \mu_E(\rho_E) \text{ on } E$$

Then we get the results

3. Mixed Fuzzy Quotient Topological Ring

This section deals the fuzzy quotient topology corresponding to the mixed fuzzy topological $\mu(\rho)$ on R is the same as the mixed fuzzy topological of the two fuzzy quotient topologies corresponding to μ and ρ on R.

Theorem 3.1

Let (R,μ,ρ) be any bi- fuzzy topological ring with $\mu(\rho)$ as the mixed fuzzy topological on R. For any subring E of R, let $\mu/E, \rho/E,$ and $\mu(\rho)/E$ be the fuzzy quotient topologies on R/E derived from the fuzzy topological ring(R,μ) , (R,ρ) and $(R,\mu(\rho))$ respectively. Then

$$\mu/E(\rho/E) = \mu(\rho)/E$$

Proof

Clearly, $\mu(\rho)/E$ is the finest fuzzy topological ring on R/E such that the map $f:(R,\mu(\rho)) \to (R/E.\mu(\rho)/E)$, is fuzzy continuous. Assume

$$l:(R,\mu(\rho)) \to (R/E.\mu/E(\rho/E))$$

If U be a fuzzy element of $\mu/E(\rho/E)$)-fuzzy nbhds of the $0+E$ of R/E, then there exists a ρ/E – fuzzy open nbhd. V of $0+E$ of R/E s.t

$$U \geq cl_{\mu/E}\ (V)$$

Now,

$l^{-1}(U) \geq l^{-1}(cl_{\mu/E}\ (V)) = l^{-1}[\wedge(V+H)]$

where $\{H\}$be a fuzzy fundamental μ/E –fuzzy nbhds system of $0+E$

$$= \wedge\big(l^{-1}(V+H)\big) = cl_{\mu(\rho)}(V)$$

Thus, $l: (R, \mu(\rho)) \to (R/E. \mu/E(\rho/E))$ is fuzzy continuous. Since $\mu(\rho)/E$ is the finest fuzzy topological ring on R/E for which l is fuzzy continuous,

$$\mu(\rho)/E \geq \mu/E(\rho/E)$$

For the converse, let us consider the identity map

$$i : (R/E. \mu/E(\rho/E)) \to (R/E. \mu(\rho)/E)$$

Let U be a fuzzy element of the $\mu(\rho)/E$)-fuzzy nbhds of the $0+E$ of R/E, then there exists a $\mu(\rho)$ – fuzzy open nbhd V of 0 s.t

$$U \geq cl_{\mu}\ (V) + E = \wedge(V+H) + E$$

where $\{H\}$be a fuzzy fundamental μ –fuzzy nbhds system of 0

$$= \wedge(V+E) + (H+E) = cl_{\mu/E}(V+E) \in \mu/E(\rho/E)$$

since

$$i^{-1}(U) = U$$

and hence *i* is fuzzy cont. which means that

$$\mu(\rho)/E \geq \mu/E(\rho/E)$$

Thus

$$\mu/E(\rho/E) = \ \mu(\rho)/E$$

Theorem 3.2

Let (R, μ, ρ) be a bi- fuzzy topological ring with $\mu < \rho$ and E an μ- fuzzy closed subring of R. Then $(R/E, \mu/E(\rho/E))$ is compact if (R, ρ) is fuzzy compact.

Proof

Since $\mu < \rho$, we have by Theorem 2.3,

$$\mu < \mu(\rho) < \rho$$

Let R be ρ- fuzzy compact topological ring. In view of the above ordering of the fuzzy topologies μ , $\mu(\rho)$ and ρ, it follows that R is also $\mu(\rho)$- fuzzy compact, Hence by Theorem 3.1, $(R/E, \mu/E(\rho/E))$ is compact.

Theorem 3.3

Let (R, μ, ρ) be a bi- fuzzy topological ring such that (R, ρ) is fuzzy Hausdorff and $\mu > \rho$. Also, let E be a ρ- fuzzy closed subring of R. Then $(R/E, \mu/E(\rho/E))$ is fuzzy compact (fuzzy locally compact) if (R, ρ) is fuzzy compact (fuzzy locally compact),

Proof

Since $\mu > \rho$ then $\mu(\rho) = \rho$ and E is $\mu(\rho)$ fuzzy closed. Now we note that $(R, \mu(\rho))$ is fuzzy compact (fuzzy locally compact) whenever (R, ρ) is fuzzy compact (fuzzy locally compact) and by virtue of Theorem3.1**,** the result follows immediately.

Theorem 3.4

Let I be any index set and $\{(R_i, \mu_i, \rho_i) : i \in J\}$ be a family of bi- fuzzy topological rings. Then

$$\prod_{i\in I} \mu_i\,(\rho_i) = \prod_{i\in I} \mu_i\,(\prod_{i\in I} \rho_i)$$

On the product ring $R = \prod_{i\in I} R_i$.

Proof

Let U be fuzzy open nbhds system of the $0 = (0_1, 0_2, \dots 0_n)$ of $\prod_{i\in I} R_i$ relative to the fuzzy topology $\prod_{i\in I} \mu_i\,(\rho_i)$ so that if V_i is the fuzzy open nbhds system of the identity 0_i of R_i. Then

$$U = \prod_{i\in I} cl_{\mu_i}(V_i)$$

We know that

$$\prod_{i\in I} cl_{\mu_i}(V_i) = cl_{\mu_i}(\prod_{i\in I} V)$$

Hence U is a fuzzy nbhd of $0 = 0_i,\ i \in I$ relative to the mixed fuzzy topology $\prod_{i\in I} \mu_i\,(\prod_{i\in I} \rho_i)$

Thus, every fuzzy nbhd of $0 = 0_i,\ i \in I$ In the fuzzy topology $\prod_{i\in I} \mu_i\,(\rho_i)$ is also a fuzzy nbhd of $0 = 0_i,\ i \in I$ in the fuzzy topology $\prod_{i\in I} \mu_i\,(\prod_{i\in I} \rho_i)$ and vice versa.

Hence follows the result.

Theorem 3.5

Let I be any index set and$\{(R_i, \mu_i, \rho_i) : i \in I\}$ be a bi- fuzzy topological ring for each $i \in I$. If for each $i \in I$, $\mu_i < \rho_i$ then

$$\prod_{i\in I}\mu_i < \prod_{i\in I}\mu_i\,(\prod_{i\in I}\rho_i) < \prod_{i\in I}\rho_i)$$

On the product ring $R = \prod_{i\in I} R_i$.

Proof

It is sufficient to prove that

$$\prod_{i\in I}\mu_i < \prod_{i\in I}\rho_i$$

Because in that case the required result would follow immediately by Theorem 2.3 For each $i \in I$, (R_i, μ_i, ρ_i) is a bi-FZT ring with $\mu_i < \rho_i$. Therefore, there are fundamental systems $\{U_i\}$ and $\{V_i\}$ of fuzzy nbhds of $0_i \in R_i$ in the fuzzy topologies μ_i and ρ_i respectively such that for each $U_i \in \{U_i\}$ there is a $V_i \in \{V_i\}$ with

$$0_i \in V_i \subset\ U_i$$

Thus,

$$0 = (0_i)_{i\in I} \in \prod_{i\in I} V_i \subset \prod_{i\in I} U_i$$

In particular,

$$0 = (0_i)_{i\in I} \in \prod_{i\in I} V_i \subset \prod_{i\in I} U_i$$

for $i = i_m,\ m =\ 1,2,\ldots,n$ (n finite) and for $i \neq i_m$

$$R_i = V_i = U_i \qquad (1)$$

But $\prod_{i\in I} V_i$ and $\prod_{i\in I} U_i$ in (1) above, form the fundamental systems of fuzzy nbhds of $0 = (0_i)_{i\in I} \in \prod_{i\in I} R_i$ in the product fuzzy topological $\prod_{i\in I}\mu_i$ and $\prod_{i\in I}\rho_i$ respectively.

Hence,

$$\prod_{i\in I}\mu_i < \prod_{i\in I}\rho_i$$

from which follows the required result.

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The irreducible modular projective characters of the symmetric groups S_{21} modulo $p = 19$

Jenan Abd Alreda Resen

Jenanabdalreda8@gmail.com

Jenan.resean@uobasrah.edu.iq

Enas Wahab Abood

enaswahab223@gmail.com

enas.abood@uobasrah.edu.iq

Jehan Mohammed Khudhir Al-Ameri

jehankhudhir@yahoo.com

jehan.khudhir@uobasrah.edu.iq

Department of Mathematics,College of Sciences,University of Basrah, Iraq

Abstract In this paper we invention all irreducible modular spin(projective) characters of the symmetric group S_n , when $n = 12$ and the characteristic of the field =19.

Key words spin(projective) characters , modular characters , decomposition matrix , AMS 2010,15C15,15C20,15C25.

1-Introduction

The decomposition matrix for the projective characters is appear the link between the irreducible projective characters and irreducible modular projective characters[1] , then when we find this matrix as amounting to find all irreducible modular projective characters. Characters is known modular or ordinary when the characteristic of the field is prime or zero[2]. Every finite group has covering group[3] ,then S_n has like this group. The characters of the covering group which are identical the characters of S_n are called modular or ordinary characters of S_n ,the rest characters are called projective(spin) of S_n [4]. Finding the decomposition matrix for the projective characters will become more difficult when n increasing ,and there is no general method to find this matrix[4]. Many Mathematicians work in this field like Adul Kareem A.Yaseen, Saeed Abdul-Ameer Taban ,Ahmed Hussein Jassim and Marwa Mohammed Jawad [5],[6],[7],[8]. In this paper decomposition matrices S_{21} modulo $p = 19$ have been calculated by using $(r, \bar{r})$-inducing method, we induce the principal indecomposable characters(P.i.s)of S_{20} (see creek*) to have (P.i.s) or principal characters (P.s) of S_{21}.

2-Rudiments

1-The spin characters of S_n can be written as a linear combination , with non-negative integer coefficients , of the irreducible spin characters[8].

2- Projective characters $\langle \gamma \rangle = \langle \gamma_1, \dots, \gamma_l \rangle$ have degree which is equal $2^{\left[\frac{o-l}{2}\right]} \frac{o!}{\prod_{i=1}^{l}(\gamma_i!)} \prod_{1 \le i < j \le l} \frac{(\gamma_i - \gamma_j)}{(\gamma_i + \gamma_j)}$ [9].

3-The values of associate characters $\langle \gamma \rangle, \langle \gamma \rangle'$ are same on the class except on the class corresponding to γ they have values $\pm i^{\frac{o-l+1}{2}} \sqrt{\left(\frac{\gamma_1 \dots \gamma_l}{2}\right)}$ [9].

4-The inducing from group or restriction from the subgroup of the projective characters are also projective characters [1].

5-If o is odd and $p \nmid (o-1)$,then $\langle o-1,1 \rangle$ and $\langle o-1,1 \rangle'$ are distinct I.m.s of grade $2^{\left[\frac{(o-3)}{2}\right]} \times (o-2)$ which are denoted by $\rho\langle o-1,1 \rangle$ and $\rho\langle o-1,1 \rangle'$ [9].

6- Let p be an odd prime and let σ, γ be a bar partition of o which are not p-bar core. Then $\langle \sigma \rangle$(and $\langle \sigma \rangle'$ if σ is odd) and $\langle \gamma \rangle$(and $\langle \gamma \rangle'$ if γ is odd) are in the same p-block$\leftrightarrow$ $\widetilde{\langle \sigma \rangle} = \widetilde{\langle \gamma \rangle}$.If α be a bar partition of o and $\langle \sigma \rangle$= $\widetilde{\langle \sigma \rangle}$, then $\langle \sigma \rangle$(and $\langle \sigma \rangle'$ if σ is odd) forms a p-block of defect 0 [4].

7- Let p be an odd prime and $\gamma = (\gamma_1, \dots \gamma_l)$ be a bar partition of o ,then all I.m.s in the block B are double(associate) , if $(o-l-m_0)$ is even(odd), where m_0 the number of parts of γ divisible by p [4].

8- If o is even and $p \nmid (o)$,then $\langle o \rangle$ and $\langle o \rangle'$ are distinct I.m.s of grade $2^{\left[\frac{(o-1)}{2}\right]}$ which are denoted by $\gamma\langle o \rangle$ and$\gamma\langle o \rangle'$ [9].

3-The spin block of S_{21}

The matrix required of the projective(spin) characters of $S_{21}, p = 19$ has 115 irreducible spin characters and 113 (19, α)-regular classes [10].

From preliminaries (6) ,there are 105 blocks of $S_{21}, p = 19$, theses blocks are $M_2, \dots, M_{104}$ of defect zero except the block M_1 of defect one.

The blocks of defect zero $M_2, \dots, M_{104}$ includes
$\langle 20,1 \rangle, \langle 20,1 \rangle', \langle 18,3 \rangle, \langle 18,3 \rangle', \langle 17,4 \rangle, \langle 17,4 \rangle', \langle 17,3,1 \rangle^*, \langle 16,5 \rangle, \langle 16,5 \rangle', \langle 16,4,1 \rangle^*,$

$\langle 15,6 \rangle, \langle 15,6 \rangle', \langle 15,5,1 \rangle^*, \langle 15,3,2,1 \rangle, \langle 15,3,2,1 \rangle', \langle 14,7 \rangle, \langle 14,7 \rangle', \langle 14,6,1 \rangle^*, \langle 14,4,3 \rangle^*,$

$\langle 14,4,2,1 \rangle, \langle 14,4,2,1 \rangle', \langle 13,8 \rangle, \langle 13,8 \rangle', \langle 13,7,1 \rangle^*, \langle 13,6,2 \rangle^*, \langle 13,5,3 \rangle^*, \langle 13,5,2,1 \rangle, \langle 13,5,2,1 \rangle',$

$\langle 13,4,3,1 \rangle, \langle 13,4,3,1 \rangle', \langle 12,9 \rangle, \langle 12,9 \rangle', \langle 12,8,1 \rangle^*, \langle 12,6,3 \rangle^*, \langle 12,6,2,1 \rangle, \langle 12,6,2,1 \rangle', \langle 12,5,4 \rangle^*,$

$\langle 12,5,3,1 \rangle, \langle 12,5,3,1 \rangle', \langle 12,4,3,2 \rangle, \langle 12,4,3,2 \rangle', \langle 11,10 \rangle, \langle 11,10 \rangle', \langle 11,9,1 \rangle^*, \langle 11,7,3 \rangle^*,$

$\langle 11,7,2,1 \rangle, \langle 11,7,2,1 \rangle', \langle 11,6,4 \rangle^*, \langle 11,6,3,1 \rangle, \langle 11,6,3,1 \rangle', \langle 11,5,4,1 \rangle, \langle 11,5,4,1 \rangle'$, $\langle 11,5,3,2 \rangle, \langle 11,5,3,2 \rangle',$

$\langle 11,4,3,2,1 \rangle^*, \langle 10,9,2 \rangle^*, \langle 10,8,3 \rangle^*, \langle 10,8,2,1 \rangle, \langle 10,8,2,1 \rangle', \langle 10,7,4 \rangle^*, \langle 10,7,3,1 \rangle, \langle 10,7,3,1 \rangle',$

$\langle 10,6,5 \rangle^*, \langle 10,6,4,1 \rangle, \langle 10,6,4,1 \rangle', \langle 10,6,3,2 \rangle, \langle 10,6,3,2 \rangle'$, $\langle 10,5,4,2 \rangle, \langle 10,5,4,2 \rangle'$, $\langle 10,5,3,2,1 \rangle^*,$

$\langle 9,8,4 \rangle^*, \langle 9,8,3,1 \rangle, \langle 9,8,3,1 \rangle', \langle 9,7,5 \rangle^*, \langle 9,7,4,1 \rangle, \langle 9,7,4,1 \rangle', \langle 9,7,3,2 \rangle, \langle 9,7,3,2 \rangle',$

$\langle 9,6,5,1 \rangle, \langle 9,6,5,1 \rangle', \langle 9,6,4,2 \rangle, \langle 9,6,4,2 \rangle', \langle 9,6,3,2,1 \rangle^*, \langle 9,5,4,3 \rangle, \langle 9,5,4,3 \rangle', \langle 9,5,4,2,1 \rangle^*, \langle 8,7,6 \rangle^*,$

$\langle 8,7,5,1 \rangle, \langle 8,7,5,1 \rangle', \langle 8,7,4,2 \rangle, \langle 8,7,4,2 \rangle', \langle 8,7,3,2,1 \rangle^*, \langle 8,6,5,2 \rangle, \langle 8,6,5,2 \rangle', \langle 8,6,4,3 \rangle, \langle 8,6,4,3 \rangle',$

$\langle 8,6,4,2,1 \rangle^*, \langle 8,5,4,3,1 \rangle^*, \langle 7,6,5,3 \rangle, \langle 7,6,5,3 \rangle', \langle 7,6,5,2,1 \rangle^*$ $\langle 7,6,4,3,1 \rangle, \langle 7,6,4,3,1 \rangle', \langle 7,5,4,3,2 \rangle^*,$

$\langle 6,5,4,3,2,1\rangle, \langle 6,5,4,3,2,1\rangle'$ .respectively ,these characters are irreducible modular spin character (preliminaries 6).The principle block M_1contains the remaining projective characters.

4-The decomposition matrix for the block M_1 of defect one

From preliminaries (7,3) all I.m.s. for the block B_1 are double and $\langle\alpha\rangle = \langle\alpha\rangle'$ on (19, α)-regular classes respectively.

Theorem(4.1):

The matrix required of the projective(spin) characters of S_{21} is

$H_{19,19} = H_{21,19}^{(1)} \oplus \ldots \oplus H_{21,19}^{(104)}$

Proof:

Through technique and the method $(r,\bar{r})$-inducing of P.i.s. of $S_{20}, p = 19$ (see creek *) to S_{21} we have

$z_1 \uparrow^{(1,0)} S_{20} = y_1$, $z_3 \uparrow^{(1,0)} S_{20} = y_2$, $z_5 \uparrow^{(17,3)} S_{20} = y_3$,

$z_7 \uparrow^{(16,4)} S_{20} = y_4$, $z_9 \uparrow^{(1,0)} S_{20} = y_5$, $z_{11} \uparrow^{(1,0)} S_{20} = y_6$,

$z_{13} \uparrow^{(1,0)} S_{20} = y_7$, $z_{15} \uparrow^{(1,0)} S_{20} = y_8$, $z_{17} \uparrow^{(1,0)} S_{20} = y_9$.

So, the matrix required for this block is as given in creek(1).

Creek(1)

The grade of the projective characters	The projective characters	$H_{21,19}^{1}$								
1024	$\langle 21\rangle^*$	1								
193536	$\langle 19,2\rangle$	1	1							
193536	$\langle 19,2\rangle'$	1	1							
487424	$\langle 18,2,1\rangle^*$		1	1						
62899200	$\langle 16,3,2\rangle^*$			1	1					
253338624	$\langle 15,4,2\rangle^*$				1	1				
684343296	$\langle 14,5,2\rangle^*$					1	1			
1316044800	$\langle 13,6,2\rangle^*$						1	1		
1809561600	$\langle 12,7,2\rangle^*$							1	1	
1663334400	$\langle 11,8,2\rangle^*$								1	1
684343296	$\langle 10,9,2\rangle^*$									1
		y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9

Creek(*)

The grade of the projective character	The projective characters	$D_{20,19}^{1}$																	
512	$\langle 20\rangle$	1																	
512	$\langle 20\rangle'$		1																
9216	$\langle 19,1\rangle^{*}$	1	1	1	1														
204800	$\langle 17,2,1\rangle$			1		1													
204800	$\langle 17,2,1\rangle'$				1		1												
1497600	$\langle 16,3,1\rangle$					1		1											
1497600	$\langle 16,3,1\rangle'$						1		1										
6031872	$\langle 15,4,1\rangle$							1		1									
6031872	$\langle 15,4,1\rangle'$								1		1								
16293888	$\langle 14,5,1\rangle$									1		1							
16293888	$\langle 14,5,1\rangle'$										1		1						
31334400	$\langle 13,6,1\rangle$											1		1					
31334400	$\langle 13,6,1\rangle'$												1		1				
43084800	$\langle 12,7,1\rangle$													1		1			
43084800	$\langle 12,7,1\rangle'$														1		1		
39603200	$\langle 11,8,1\rangle$															1		1	
39603200	$\langle 11,8,1\rangle'$																1		1
16293888	$\langle 10,9,1\rangle$																	1	
16293888	$\langle 10,9,1\rangle'$																		1
		z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}	z_{13}	z_{14}	z_{15}	z_{16}	z_{17}	z_{18}

Future work

We can find the irreducible modular projective characters or indecomposable principal characters for the symmetric group S_{21} modulo $p = 19$.

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Numerical Solutions of Boundary Value Problems by using A new Cubic B-spline Method

Bushra A. Taha & Renna D. Abdul-Wahhab

University of Basrah / College of Science/ Dept. of Mathematics

Abstract

In this study, cubic B-spline method is used with a new approximation of the second derivative to find a numerical solution for boundary value problems of the second order. An error analysis was performed for the method and the accuracy of the method was tested via four numerical examples and the results were compared with the exact solution and cubic B-spline method.

Keywords: boundary value problems, error analysis, cubic B-spline, exact solution.

1. Introduction

Splines, especially B-splines, play an important role in the areas of mathematics and engineering today [2],[17]. Splines are popular in computer graphics because of their finesse, flexibility and accuracy. Historically, Isaac Jacob Schoenberg discovered splines in 1946 [6-10], his work motivated other scientists such as Carl de Boor. In the early seventies de Boor [3], [4], [5] discovered a recursive definition for splines. Birkhoff and de Boor (1964) [1] investigated the error bound and convergence of of spline interpolation. Manguia and Bhatta (2015) [18] used cubic B-spline(CBS) functions for solution of second order boundary value problems(BVPs). Reza and Akram [23], applied of cubic B-splines collocation method for solving nonlinear inverse parabolic partial differential equations. Suardi et. al. [26] used the cubic B-spline solution of two-point boundary value problem using HSKSOR iteration and they presented solutions of two-point boundary value problems by using quarter-sweep SOR iteration with cubic B-Spline scheme[27] .

In this study, approximate solutions was found to problems of second order linear arrangement using B-cubes with a new approximation of the second derivative. Lang and X. Xu[16], introduced a new cubic B-spline method for approximating the solution of a class of nonlinear second-order boundary value problem with two dependent variables. His work was a motivation to other mathematicians such Tassaddiq and others [28] to used his method for solve non-linear differential equations arising in visco-elastic flows and hydrodynamic stability problems.

The presented scheme is based on new approximations for the second order derivatives. The approximation for second order derivative is calculated using appropriate linear combinations to approximate the typical B-spline $y''(x)$ at neighbouring values. In the past two decades, several numerical techniques have been used to explore the numerical solution of linear BVP but as far as we know, this new approximation has not been used for this purpose before for solving BVPs. This work is presented as follows. Section 2 is explanation about the cubic B-splines schemes. We presented the new approximation for $y''(x)$ in Section 3.In Section 4, we descripted of the numerical method for new cubic B-spline. The error analysis of the method is described in Section 5. Section 6 tests numerical experiments to demonstrate the feasibility of the proposed method, and this article ends with a conclusion in Section 7.

2. Derivation of the Cubic B-spline Schemes

Let n be a positive integer and $a = x_0 < x_1 < \cdots < x_n = b$ be a uniform partition of $[a,b]$, $x_i = x_o + ih, i \in \mathbb{Z}$ and $h = \frac{b-a}{n}$. The typical third degree B-spline basis functions are defined: [11-14], [24-26]

$$B_i(x)=\frac{1}{6h^3}\begin{cases}(x-x_{i-2})^3 & if \quad x\in[x_{i-2},x_{i-1}]\\ -3(x-x_{i-1})^3+3h(x-x_{i-1})^2+3h^2(x-x_{i-1})+h^3 & if \quad x\in[x_{i-1},x_i]\\ -3(x_{i+1}-x)^3+3h(x_{i+1}-x)^2+3h^2(x_{i+1}-x)+h^3 & if \quad x\in[x_i,x_{i+1}]\\ (x_{i+2}-x)^3 & if \quad x\in[x_{i+1},x_{i+2}]\\ 0 & if \quad otherwise\end{cases} \quad (1)$$

Where $i=-1,2,\ldots,n+1.$ For a sufficiently smooth function $y(x)$ there always exists a unique third degree spline $Y(x)$,

$$Y(x)=\sum_{i=-1}^{n+1}c_iB_i(x)$$

(2)

which satisfies the prescribed interpolating conditions

$Y'(a)=y'(a)$ *and* $Y'(b)=y'(b)$, $i=0,1,\ldots,n$ for all $Y(x_i)=y(x_i)$,

Where c_i's are finite constants yet to be determined.

For simplicity, we express the CBS approximations, $Y(x), Y'(x)$ and $Y''(x)$ by Y_j, t_j and T_j, respectively. The cubic B-spline basis function (1) together with (2) and by using Table (1) gives the following relations,

$$Y_j=\sum_{i=j-1}^{i+1}c_iB_i(x)=\frac{1}{6}(c_{j-1}+4c_j+c_{j+1}), \quad (3)$$

$$t_j=\sum_{i=j-1}^{j+1}c_iB_i'(x)=\frac{1}{2h}(-c_{j-1}+c_{j+1}), \quad (4)$$

$$T_j=\sum_{i=j-1}^{j+1}c_iB_i''(x)=\frac{1}{h^2}(c_{j-1}-2c_j+c_{j+1}).$$

(5)

Moreover ,from (3)-(5) relationships can be created.[7]

$$t_j = y'\left(x_j\right) - \frac{1}{180} h^4 y^{(5)}\left(x_j\right) + \cdots ,$$

(6)

$$T_j = y''\left(x_j\right) - \frac{1}{12} h^2 y^{(4)}\left(x_j\right) + \frac{1}{360} h^4 y^{(6)}\left(x_j\right) + \cdots .$$

(7) , we have. and (7) From (6)

$\left\| T_j - y''(x_j) \right\|_\infty = O(h^2)$. and $\left\| t_j - y'(x_j) \right\|_\infty = O(h^4)$

This gives enough motivation to craft a better approximation to, the $y''(x)$.

Table 1: Coefficients of cubic B-spline and its derivative at nodes x_i .

	x_{i-1}	x_i	x_{i+1}	Else
$B_i(x)$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	0
$B_i^{(1)}(x)$	$-\frac{1}{2h}$	0	$\frac{1}{2h}$	0
$B_i^{(2)}(x)$	$\frac{1}{h^2}$	$-\frac{2}{h^2}$	$\frac{1}{h^2}$	0

3. The New Approximation for $y''(x)$

In order to formulate a new approximation to $y''(x)$, we use (7), to establish the following expression for $\left(T_{j-1}\right)$, in knots, x_j, $j = 1, 2, 3, \cdots, n-1$, [15-16]

$$T_{j-1} = y''\left(x_{j-1}\right) - \frac{1}{12} h^2 y^{(4)}\left(x_{j-1}\right) + \frac{1}{360} h^4 y^{(6)}\left(x_{j-1}\right) + \cdots ,$$

$$= y''\left(x_j\right) - h\, y^{(3)}\left(x_j\right) + \frac{5}{12} h^2 y^{(4)}\left(x_j\right) - \frac{1}{12} h^3 y^{(5)}\left(x_j\right) + \cdots \quad .$$

Similarly,

$$T_{j+1} = y''(x_j) + h\,y^{(3)}(x_j) + \frac{5}{12}h^2 y^{(4)}(x_j) + \frac{1}{12}h^3 y^{(5)}(x_j) + \dots ,$$

be a new approximation to $y''(x_j)$ such that, T_j let

$$\tilde{T}_j = B_1 T_j + B_2 T_{j-1} + B_3 T_{j+1}.$$

(8)

Choosing three parameters B_1, B_2 and B_3 so that the error order of $\tilde{T}_j$ is as high as possible , we obtain

$$B_1 + B_2 + B_3 = 1,$$

$$-B_2 + B_3 = 0,$$

$$-B_1 + 5B_2 + 5B_3 = 0.$$

Hence $B_1 = \frac{5}{6}$, and $B_2 = B_3 = \frac{1}{12}$.

The expression (8) takes the following form,

$$\tilde{T}_j = B_1 T_j + B_2 T_{j-1} + B_3 T_{j+1} = \frac{1}{12h^2}\left(c_{j-2} + 8c_{j-1} - 18c_j + 8c_{j+1} + c_{j+2}\right).$$

(9)

Now we approximate $y''(x)$ at the knot x_0 using four neighboring values, such that.

$$\tilde{T}_0 = B_0 T_0 + B_1 T_1 + B_2 T_2 + B_3 T_3 ,$$

(10)

where.

$$T_1 = y''(x_0) + h\,y^{(3)}(x_0) + \frac{5}{12}h^2 y^{(4)}(x_0) + \frac{1}{12}h^3 y^{(5)}(x_0) + \cdots \;,$$

$$T_2 = y''(x_0) + 2h\,y^{(3)}(x_0) + \frac{23}{12}h^2 y^{(4)} + \frac{7}{6}h^3 y^{(5)}(x_0) + \cdots \;,$$

$$T_3 = y''(x_0) + 3h\,y^{(3)}(x_0) + \frac{53}{12}h^2 y^{(4)}(x_0) + \frac{17}{4}h^3 y^{(5)}(x_0) + \cdots \;.$$

The expression (9) yields the following four equations,

$B_0 + B_1 + B_2 + B_3 = 1,$

$B_1 + 2B_2 + 3B_3 = 0,$

$-B_0 + 5B_1 + 23B_2 + 53B_3 = 0,$

$B_1 + 14B_2 + 51B_3 = 0.$

Hence $B_0 = \frac{7}{6},\ B_1 = -\frac{5}{12},\ B_2 = \frac{1}{3}$ and $B_3 = -\frac{1}{12}.$

Using these values in (10), we have

$$\tilde{T}_0 = \frac{1}{12h^2}\left(14c_{-1} - 33c_0 + 28c_1 - 14c_2 + 6c_3 - c_4\right).$$

(11)

the same style, rounding is presented at node x_n by working in When

(12) $$\tilde{T}_n = \frac{1}{12h^2}\left(-c_{n-4} + 6c_{n-3} - 14c_{n-2} + 28c_{n-1} - 33c_n + 14c_{n+1}\right),$$

4. Description of the Numerical Method.

In this section, consider the boundary value problems,

$$p(x)y''(x) + q(x)y'(x) + r(x)y(x) = f(x)$$

(13)

with boundary conditions

$$y(a)=\alpha,\ y(b)=\beta.$$

Where $p(x)\neq 0, q(x), r(x)$ and $f(x)$ are continuous real-valued functions on the interval $[a,b]$.

Let $Y(x)$ be the cubic B-spline solution to (14) satisfying the interpolating conditions such that

$$Y(x)=\sum_{i=-1}^{n+1} c_i B_i(x).$$

(15)

Discretize Eq.(14) in knots $x_j,\ j=1,2,\cdots,n-1,$ we get,

$$p(x_j)Y''_{k+1}(x_j)+q(x_j)Y'_{k+1}(x_j)+r(x_j)Y_{k+1}(x_j)=f(x_j).$$

(16)

Using Eqs.(3)-(4) and (9) in Eq.(16) ,we have

$$p(x_j)\left(\frac{c_{j-2}+8c_{j-1}-18c_j+8c_{j+1}+c_{j+2}}{12h^2}\right)$$

$$+q(x_j)\left(\frac{-c_{j-1}+c_{j+1}}{2h}\right)+r(x_j)\left(\frac{c_{j-1}+4c_j+c_{j+1}}{6}\right)=f(x_j).$$

(17)

Similarly, at the knots x_0 and x_n , the following equations are obtained

$$p(x_0)\left(\frac{14c_{-1}-33c_0+28c_1-14c_2+6c_3-c_4}{12h^2}\right)$$

$$+q(x_0)\left(\frac{-c_{-1}+c_1}{2h}\right)+r(x_0)\left(\frac{c_{-1}+4c_0+c_1}{6}\right)=f(x_0),$$

(18)

$$p(x_n)\left(\frac{14c_{n-1}-33c_n+28c_{n+1}-14c_{n+2}+6c_{n+3}-c_{n+4}}{12h^2}\right)$$

$$+q(x_n)\left(\frac{-c_{n-1}+c_{n+1}}{2h}\right)+r(x_n)\left(\frac{c_{n-1}+4c_n+c_{n+1}}{6}\right)= f(x_n)\ .$$

(19)

The boundary conditions are giving of the following two equations

$$c_{-1}+4c_0+c_1=6\alpha,$$

(20)

$$c_{n-1}+4c_n+c_{n+1}=6\beta.$$

(21)

In This way they have a system of $(n+3)$ linear equations .Eqs.(17)-(19) which can be written in matrix form as

$$Ac=b.$$

(22)

Where A is the coefficients matrix given by

$$A=\begin{pmatrix} 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & & & \\ o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & & & & & \\ a_1 & b_1 & c_1 & d_1 & e_1 & & & & & & \\ 0 & a_2 & b_2 & c_2 & d_2 & e_2 & & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & a_{n+1} & b_{n+1} & c_{n+1} & d_{n+1} & e_{n+1} & & \\ & & & & & m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \\ & & & & & & & & 1 & 4 & 1 \end{pmatrix}$$

where

$$o_1 = 14p(x_0) - 6hq(x_0) + 2h^2 r(x_0)$$
$$o_2 = -33p(x_0) + 8h^2 r(x_0),$$
$$o_3 = 28p(x_0) + 6hq(x_0) + 2h^2 r(x_0)$$
$$o_4 = 14p(x_0),$$
$$o_5 = 6p(x_0),$$
$$o_6 = -p(x_0),$$

$$where\ i = 1, 2, \ldots, n-1,$$
$$a_i = p(x_i),$$
$$b_i = 8p(x_i) - 6hq(x_i) + 2h^2 r(x_i),$$
$$c_i = -18p(x_i) + 8h^2 r(x_i),$$
$$d_i = 8p(x_i) + 6hq(x_i) + 2h^2 r(x_i),$$
$$e_i = p(x_i)$$

$$m_1 = -p(x_n),$$
$$m_2 = 6p(x_n),$$
$$m_3 = -14p(x_n),$$
$$m_4 = 28p(x_n) - 6hq(x_n) + 2h^2 r(x_n),$$
$$m_5 = -33p(x_n) + 8h^2 r(x_n),$$
$$m_6 = 14p(x_n) + 6hq(x_n) + 2h^2 r(x_n).$$

and $c = [c_{-1}, c_0, c_1, \ldots, c_n, c_{n+1}]^T$,

$$b = [6\alpha, 12h^2 f(x_0), 12h^2 f(x_1), \ldots, 12h^2 f(x_{n-1}), 12h^2 f(x_n), 6\beta]^T,$$

since A is a non-singular matrix, so can solve the system $Ac = b$ for $c_{-1}, c_0, c_1, \ldots c_{n-1}, c_n, c_{n+1}$ substituting these values in Eq. (15), to get the required approximate solution.

Error Analysis 5.

Now, the error analysis is investigated by using the cubic B-spline approximations Eqs.(3)-(5) and Eq.(9) the following relationships can be established

$$h\left[\frac{1}{6}Y'\left(x_{j-1}\right)+\frac{4}{6}Y'\left(x_{j}\right)+\frac{1}{6}Y'\left(x_{j+1}\right)\right]=\frac{1}{2}\left[Y\left(x_{j+1}\right)-Y\left(x_{j-1}\right)\right],$$

(23)

$$h^2Y''\left(x_j\right)=\frac{1}{2}\left(7Y\left(x_{j-1}\right)-8Y\left(x_j\right)+Y\left(x_{j+1}\right)\right)+h\left(Y'\left(x_{j-1}\right)+2Y'\left(x_j\right)\right).$$

(24)

Moreover ,we have

$$h^3Y'''\left(x_j\right)=12\left[Y\left(x_j\right)-Y\left(x_{j+1}\right)\right]+6h\left[Y'\left(x_j\right)+Y'\left(x_{j+1}\right)\right],$$

(25)

$$h^3Y'''\left(x_j\right)=12\left[Y\left(x_{j-1}\right)-Y\left(x_j\right)\right]+6h\left[Y'\left(x_{j-1}\right)+Y'\left(x_j\right)\right].$$

(26)

Where $Y'''\left(x_{j^+}\right) and\, Y'''\left(x_{j^-}\right)$ indicate approximate values of in $Y'''\left(x_j\right)$ in $[x_j, x_{j+1}]$ and $[x_{j-1}, x_j]$ respectively.

$E^{\lambda}\left(Y'\left(x_j\right)\right)=Y'\left(x_{j+\lambda}\right), \lambda\in Z,$ Using the operator notation

Equation (19) can also be written as

$h\left[\frac{1}{6}E^{-1}+\frac{4}{6}+\frac{1}{6}E\right]Y'\left(x_j\right)=\frac{1}{2}\left[E-E^{-1}\right]y\left(x_j\right),$ Hence

$$hS'\left(x_j\right)=3\left(E-E^{-1}\right)\left[E^{-1}+4+E\right]^{-1}s\left(x_j\right),$$

(27)

Using $E=e^{hD},\quad D=\frac{d}{dx},$ we can get it

$$E+E^{-1}=e^{hD}+e^{-hD}=2\left[1+\frac{h^2D^2}{2!}+\frac{h^4D^4}{4!}+\frac{h^6D^6}{6!}+\cdots\right],$$

$$E - E^{-1} = e^{hD} - e^{-hD} = 2\left[hD + \frac{h^3D^3}{3!} + \frac{h^5D^5}{5!} + \frac{h^7D^7}{7!} + \cdots\right].$$

Therefore, Eq. (27) can be expressed as.

$$Y'\left(x_j\right) = \left(D + \frac{h^2D^3}{3!} + \frac{h^4D^5}{5!} + \cdots\right)\left[1 + \left(\frac{h^2D^2}{6} + \frac{h^4D^4}{72} + \frac{h^6D^6}{2160} + \cdots\right)\right]^{-1} y\left(x_j\right),$$

Simplify, we get.

$$Y'\left(x_j\right) = \left(D - \frac{h^4D^5}{180} + \frac{h^6D^7}{1512} - \cdots\right) y\left(x_j\right),$$

Hence

$$Y'\left(x_j\right) = y'\left(x_j\right) - \frac{1}{180}h^4 y^{(5)}\left(x_j\right) + \cdots ,$$

(28)

Similarly, writing Eq. (20) in operator notation we have

$$h^2Y''\left(x_j\right) = \frac{1}{2}\left[7E^{-1} - 8 + E\right]y\left(x_j\right) + h\left[E^{-1} + 2\right]y'\left(x_j\right),$$

$$= \left(-3hD + 2h^2D^2 - \frac{h^3D^3}{2} + \frac{h^4D^4}{6} - \frac{h^5D^5}{40} + \frac{h^6D^6}{180} - \cdots\right) y\left(x_j\right)$$

$$+ \left(3h - h^2D + \frac{h^3D^2}{2} - \frac{h^4D^3}{6} + \frac{h^5D^4}{24} - \frac{h^6D^5}{120} + \cdots\right) y'\left(x_j\right).$$

Simplify the relationship above, we have.

$$Y''\left(x_j\right) = y''\left(x_j\right) + \frac{1}{60}h^3 y^{(5)}\left(x_j\right) - \frac{1}{360}h^4 y^{(6)}\left(x_j\right) + \cdots .$$

(29)

Using the same method in Eq.(21) it can also be written,

$$Y'''(x_j) \cong \frac{1}{2}\left[y'''\left(x_{j^+}\right) + y'''\left(x_{j^-}\right)\right] = y'''(x_j) + \frac{1}{12}h^2 y^{(5)}(x_j) + \ldots \;.$$

(30)

Let us define the term error $e(x) = Y(x) - y(x)$, using relations (24) and (26) in the Taylor series expand $e(x)$ we get

$$e\left(x_j + \theta h\right) = \frac{\theta(5\theta-2)(\theta+1)}{360} h^5 y^{(5)}(x_j) - \frac{\theta^2}{720} h^6 y^{(6)}(x_j) + \ldots \;.$$

(31)

Where $\theta \in [0,1]$, from Eq. (31) The new B-spline approximation is $O\left(h^5\right)$ accurate.

6. Numerical Examples

In this section we illustrate the numerical techniques discussed in the previous sections by the following two boundary value problems of Eqs.(1-2) , in order to illustrate the comparative performance of our method over other existing methods. We now consider four numerical examples to illustrate the comparative performance of our method. All calculations are implemented by Maple.

Example 1: We consider a linear boundary value problem with constant coefficients :[18]

$$y''(x) + y'(x) - 6y(x) = x,$$

with boundary conditions

$$y(0) = 0,\; y(1) = 1,$$

The exact solution to boundary value problem is

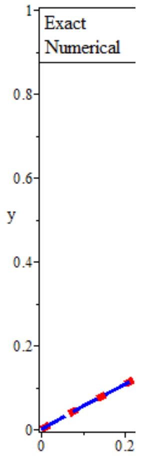
$$y(x) = \frac{(43-e^2)e^{-3x} - (43-e^{-3})e^{2x}}{36(e^{-3}-e^2)} - \frac{1}{6}x - \frac{1}{36}.$$

The numerical result of the example (1) are presented in the Table (2) for with $n = 20$ .In Table 3 the observed maximum absolute errors and compared our result with the results given in cubic b-spline method [18]. Figure 1 shows the comparison of the exact and numerical solutions for $n = 20$.

Table 2: The numerical solutions and exact solution of example (1).

x	New Cubic B-Spline	Cubic B-Spline[18]
0	0	0
0.2	5.59E-8	2.3534E-5
0.3	6.23 E-8	4.41179E-5
0.4	6.06 E-8	6.46773E-5
0.5	5.44 E-8	8.19815E-5
0.6	4.57 E-8	9.30536E-5
0.7	3.59 E-8	9.47169E-5
0.8	2.54 E-8	8.31905E-5
0.9	1.52 E-8	5.36906E-5
1	0	0

Table 3: Comparison of the error proposed method with CBS[18] for example(1) .



x	New Cubic B-Spline	Exact Solution
0	0	0
0.2	0.1074285058	0.1074285617
0.3	0.1636254812	0.1636255435
0.4	0.2267411540	0.2267412146
0.5	0.3006953149	0.3006953693
0.6	0.3896566891	0.3896567348
0.7	0.4982584629	0.4982584988
0.8	0.6318199536	0.6318199790
0.9	0.796586555	0.7965865702
1	1	1

Figure 1: Comparison of the exact and the proposed method of example(1) for n=20

Example 2: We consider a linear boundary value problem with constant coefficients[18],

$$y''(x)+2y'(x)+5y(x)=6\cos(2x)-7\sin(2x),\ \text{for } 0<x<\frac{\pi}{4},$$

with boundary conditions

$$y(0)=4,\ y(\frac{\pi}{4})=1.$$

The exact solution to boundary value problem is

$$y(x)=2(1+e^{-x})\cos(2x)+\sin(2x).$$

The numerical result of the example (2) are presented in the Table 4 compared our result with the exact solution. In Table 5 the observed maximum absolute errors and compared our result with the results given in cubic B-spline method [18]. Figure 2 shows the comparison of the exact and numerical solutions for $n=20$.

Table 4: The numerical solutions and exact solution of example (2) .

x	New Cubic B-	Exact Solution

	Spline	
$\frac{\pi}{80}$	3.989348208	3.9893481701

x	New Cubic B-Spline	Cubic B-Spline[18]
$\frac{3\pi}{80}$	3.906796607	3.9067967056
$\frac{5\pi}{80}$	3.748792026	3.7487922376
$\frac{7\pi}{80}$	3.523205708	3.5232056151
$\frac{9\pi}{80}$	3.238294433	3.2382892895
$\frac{11\pi}{80}$	2.902583355	2.9025837374
$\frac{13\pi}{80}$	2.524830455	2.5248342470
$\frac{15\pi}{80}$	2.113912251	2.1139139602
$\frac{17\pi}{80}$	1.678750121	1.6787494845
$\frac{19\pi}{80}$	1.228243494	1.2282459716

Table 5: Comparison of the error proposed method with CBS[18] for example(2) .

$\frac{\pi}{80}$	3.8E-8	2.0634E-5
$\frac{3\pi}{80}$	9.9 E-8	4.8130E-5
$\frac{5\pi}{80}$	2.12 E-7	6.0894E-5
$\frac{7\pi}{80}$	9.3 E-8	6.2779E-5
$\frac{9\pi}{80}$	5.143E-6	5.70988E-5
$\frac{11\pi}{80}$	3.82 E-7	4.67074E-5
$\frac{13\pi}{80}$	3.792E-6	3.40587E-5
$\frac{15\pi}{80}$	1.709E-6	2.12666E-5
$\frac{17\pi}{80}$	6.37 E-7	1.01538E-5
$\frac{19\pi}{80}$	2.478E-6	2.2885E-5

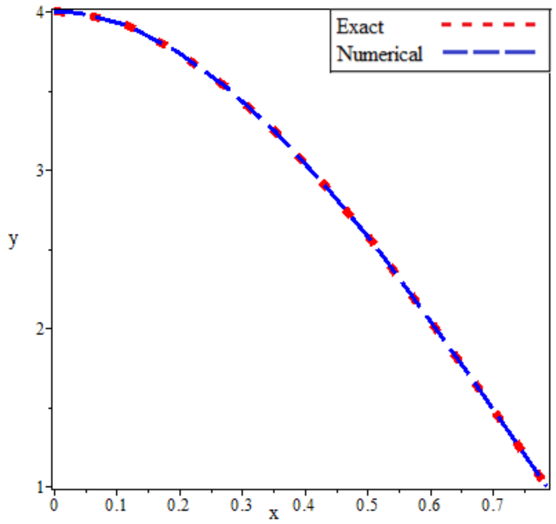


Figure 2 : Comparison of the exact and the proposed method of example(2) for n=20.

Example 3: We consider a linear boundary value problem with constant coefficients[18]

$$x^2 y''(x) + 3xy'(x) + 3y = 0 \quad \text{for} \quad 1 < x < 2,$$

with boundary conditions

$$y(1) = 5, \; y(2) = 0.$$

The exact solution to boundary value problem is

$$y(x) = \frac{5}{x}[\cos(\sqrt{2}\ln x) - \cot(\sqrt{2}\ln 2)\sin(\sqrt{2}\ln x)].$$

The numerical result of the example (3) are presented in the Table (6) for with. In Table 7 the observed maximum absolute errors and compared our result with the results given in cubic B-spline method [18]. Figure 3 shows the comparison of the exact and numerical solutions for $n = 20$.

Table 6: The numerical solutions and exact solution of example (3).

x	New Cubic B-Spline	Exact Solution
1.1	4.094768326	4.0947693502
1.2	3.316711309	3.3167126115
1.3	2.649607254	2.6496084276
1.4	2.077976455	2.0779773959
1.5	1.587980746	1.5879814418
1.6	1.167624994	1.1676254805
1.7	0.806670353	0.8066706529
1.8	0.496442085	0.4964422651
1.9	0.229613526	0.2296136048

Table 7: Comparison of the error proposed method with CBS [18] for example(3) .

x	New Cubic B-Spline	Cubic B-Spline[18]
1.1	1.024E-6	1.609202E-4
1.2	1.303E-6	3.065565E-4
1.3	1.174E-6	3.980724E-4
1.4	9.41E-7	4.327606E-4

1.5	6.96E-7	4.197742E-4
1.6	4.86E-7	3.707492E-4
1.7	2.999E-7	2.964084E-4
1.8	1.801E-7	2.055654E-4
1.9	7.88E-8	1.050575E-4

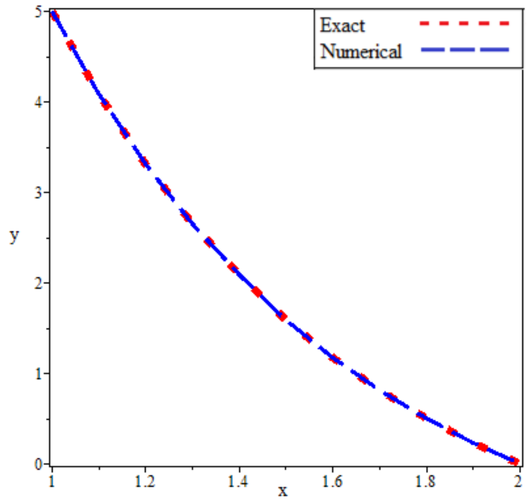


Figure 3 : Comparison of the exact and the proposed method of example(3) for n=20.

Example 4: We consider a linear boundary value problem with constant coefficients,[18]

$xy''(x)+y'(x)=x$ for $1<x<2$,

with boundary conditions

$y(1)=1,\ y(2)=1.$

The exact solution to boundary value problem is

$$y(x)=\frac{x^2}{4}-\frac{3\ln x}{4\ln 2}+\frac{3}{4}.$$

The numerical result of the example (4) are presented in the Table 8 for with .In Table 9 the observed maximum absolute errors and compared our result with the results given in cubic B-spline method [18]. Figure 4 shows the comparison of the exact and numerical solutions for $n = 20$.

Table 8: The numerical solutions and exact solution of example (4).

x	New Cubic B-Spline	Exact Solution
1.1	0.9493723880	0.9493723572
1.2	0.9127242439	0.9127241956
1.3	0.8886163346	0.8886162826
1.4	0.8759299325	0.8759298796
1.5	0.8737781718	0.8737781245
1.6	0.8814461109	0.8814460712
1.7	0.8983489704	0.8983489402
1.8	0.9240023401	0.9240023201
1.9	0.9580004464	0.9580004361

Table 9: Comparison of the error proposed method with CBS [18] for example(4) .

x	New Cubic B-Spline	Cubic B-Spline[18]
1.1	3.08E-8	2.38675E-5
1.2	4.83 E-8	3.66902E-5
1.3	5.20 E-8	4.21471E-5
1.4	5.29 E-8	4.25917E-5
1.5	4.73 E-8	3.95759E-5
1.6	3.97 E-8	3.41494E-5
1.7	3.02 E-8	270371E-5
1.8	2.00 E-8	1.87491E-5
1.9	1.03 E-8	9.6491E-5

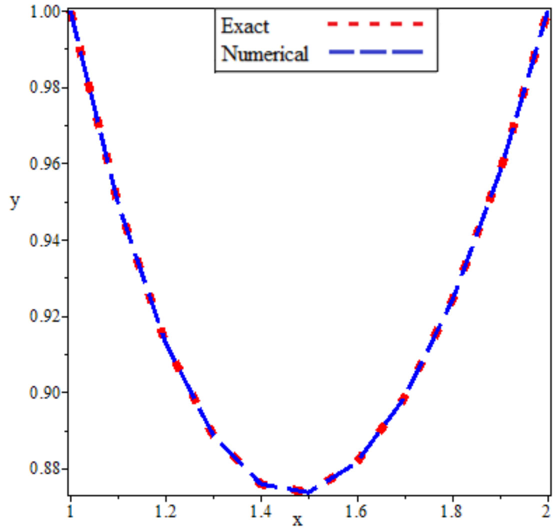


Figure 4 : Comparison of the exact and the proposed method of example(4) for n=20.

7. Conclusion

The cubic B-spline method with a new approximation of the second derivative is developed for the approximate solution of second order two –point BVPs in this paper. Four examples are considered for numerical illustration of the method. Numerical result are presented in Tables (2), (4), (6), and (8) and compared with the exact solutions. We also compared the results with the (CBS) method [18] in Tables (3), (5), (7), and (9) and It can be concluded that this method is quite suitable, accurate.

The obtained numerical results show that the proposed methods maintain a high accuracy which make them are very encouraging for dealing with the solution of this type of two point boundary value problems.

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The Formula for the product of Sines of multiple Arcs

$Dr. Ahmed\ Khalaf\ Radhi$

Department of Mathematics - College of Education

AL-Mustansiriyah University

Abstract

By using the theory of residues of holomorphic functions, one formula is obtained for the product of a finite number of Sines of multiple arcs and one improper integral is computed.

Introduction

As known, the theory of functions of complex variable has various applications in different sections of mathematics . In the book [2], the properties of complex numbers were used to solve exercises and proofs of theorems from elementary geometry . Chapter 7 of the book [1] is devoted to the applications of complex integrations, in particular , to the calculation of real integrals by using the method of transition to complex variables.

In the present paper , using the calculation of the residue of one special function, we have proved the formula (*) for the product of a finite number of Sines of multiple arcs, the proven formula has been verified for several particular values of the parameter which get into the formula. The same function and the residue theorem are used to calculate one improper real integral (**).

1. If $z = a$ be an isolated singular point of the holomorphic function $f(z)$ and γ be a simple closed piecewise smooth curve that, together with the interior, belongs to the domain of holomorphy of the function $f(z)$, except the point $z = a \in Int(\gamma)$, then the residue of $f(z)$ at this point is equal to

$$res_{z=a}f(z) = \frac{1}{2\pi i}\oint_{\gamma} f(z)dz . \qquad (1)$$

If $z = a$ is a simple pole of the holomorphic function $f(z)$, then

$$res_{z=a}f(z) = \lim_{z\to a}(z-a)f(z) . \qquad (2)$$

If $f(z) = P(z) / Q(z)$, where $P(z)$ and $Q(z)$ are holomorphic in a neighborhood of the point $z = a$ and $Q(z)$ has a zero of first order at this point, then

$$res_{z=a} f(z) = \frac{P(a)}{Q'(a)}. \quad (3)$$

These formulas can be found in [1] and [3] .

2. Consider the function $f(z) = \frac{1}{z^n+1}$, where $n \geq 2$ is a natural number .

Let $z = a = e^{i\frac{\pi}{n}}$. This point is a simple pole for the considered function $z^n + 1$ has a zero of first order at this point (the first derivative at this point is not zero). By the formula (3) with $P(z) = 1$ and $Q(z) = z^n + 1$, we obtain

$$A = res_{z=a} f(z) = \frac{P(a)}{Q'(a)} = \frac{1}{ne^{i\pi\frac{n-1}{n}}}. \quad (4)$$

We calculate the residue A by formula (2) , by using the expansion

$$z^n + 1 = \prod_{k=0}^{n-1} \left(z - e^{i\pi\frac{2k+1}{n}} \right).$$

Then we get

$$A = \lim_{z \to e^{i\frac{\pi}{n}}} \frac{z - e^{i\pi\frac{1}{n}}}{\prod_{k=0}^{n-1}\left(z - e^{i\pi\frac{2k+1}{n}}\right)} = \frac{1}{\prod_{k=1}^{n-1}\left(e^{i\frac{\pi}{n}} - e^{i\pi\frac{2k+1}{n}}\right)} = \frac{e^{-i\pi\frac{n-1}{n}}}{\prod_{k=1}^{n-1}\left(1 - e^{ik\frac{2\pi}{n}}\right)}.$$

We transform expression

$$\left(1 - e^{ik\frac{2\pi}{n}}\right)^{-1} = \left(1 - \cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n}\right)^{-1} = \frac{1 - \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}}{4\left(\sin\frac{k\pi}{n}\right)^2} =$$

$$= \frac{i\left(\cos\frac{k\pi}{n} - i\sin\frac{k\pi}{n}\right)}{2\sin\frac{k\pi}{n}} = \frac{e^{i\frac{\pi}{2}} e^{-i\frac{k\pi}{n}}}{2\sin\frac{k\pi}{n}}$$

Substituting the transformed expression into the formula for the residue A , we have

$$A = \frac{e^{-i\pi\frac{n-1}{n}}}{\prod_{k=1}^{n-1} 2\sin\frac{k\pi}{n}} \cdot e^{i(n-1)\frac{\pi}{2}} \cdot \prod_{k=1}^{n-1} e^{-ik\frac{\pi}{n}} = \frac{e^{-i\pi\frac{n-1}{n}}}{\prod_{k=1}^{n-1} 2\sin\frac{k\pi}{n}} \cdot e^{i(n-1)\frac{\pi}{2}} \cdot e^{-i\frac{\pi}{n} \cdot \frac{n(n-1)}{2}}.$$

Summing up the exponents, we find

$$A = \frac{e^{-i\pi\frac{n-1}{n}}}{\prod_{k=1}^{n-1} 2\sin\frac{k\pi}{n}}$$

We equate the resulting expression for the residue A with the previously obtained expression from formula (4**):**

$$\frac{1}{ne^{i\pi\frac{n-1}{n}}} = \frac{e^{-i\pi\frac{n-1}{n}}}{\prod_{k=1}^{n-1} 2\sin\frac{k\pi}{n}} .$$

From this equality we obtain an equation that proves the following theorem

Theorem.

For positive integers $n > 1$ the following formula is true:

$$\prod_{k=1}^{n-1} \sin\frac{k\pi}{n} = \frac{n}{2^{n-1}} . \qquad (*)$$

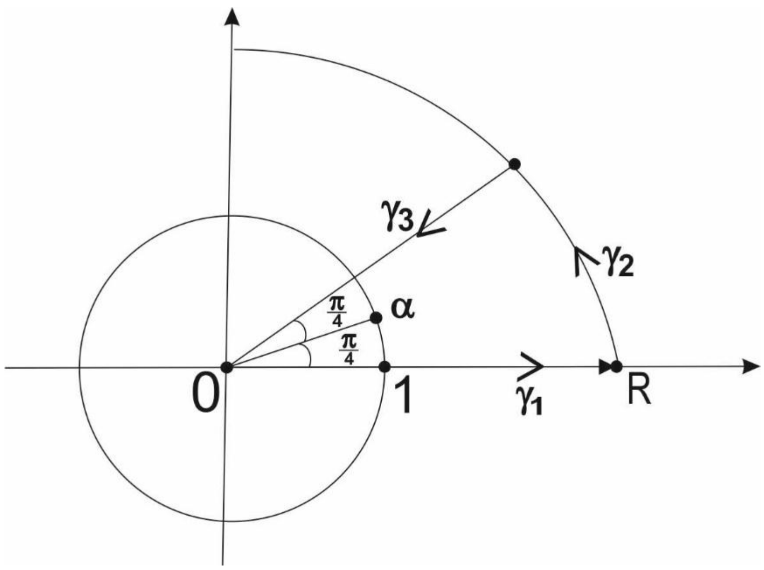
Let us verify the equality for several values of the parameter n:

If *n = 2* we have: $\sin\frac{\pi}{2} = \frac{2}{2}$.

If *n = 3* we have: $\sin\frac{\pi}{3} \cdot \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3}{2^2}$.

If *n = 4* we have: $\sin\frac{\pi}{4} \cdot \sin\frac{2\pi}{4} \cdot \sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2} \cdot 1 \cdot \frac{\sqrt{2}}{2} = \frac{1}{2} = \frac{4}{2^3}$.

3. We now calculate the residue A by the formula (1) and consider the curvilinear triangle (figure)



$\gamma = \gamma(R) = \gamma_1(R) + \gamma_2(R) + \gamma_3(R)$ as the contour γ, where $R > 1$ is a fixed real number and the parametrization of the constituent arcs of a curvilinear triangle is given by:

$\gamma_1(R): z = x \in [0, R]$; $\gamma_2(R): z = Re^{it}, t \in [0, \frac{2\pi}{n}]$; $\gamma_3(R): z = \rho e^{i\frac{2\pi}{n}}, \rho \epsilon [0, R]$.

These arcs are oriented in accordance with the increase of the variable parameter.

$\gamma =$ Note that inside the contour

$\gamma(R)$ there is only one singular point

$z = a = e^{i\frac{\pi}{n}}$ of the function $f(z) = \frac{1}{(z^n + 1)}$, regardless of R.

From formula (1), taking into account the orientation of the contour γ, we obtain

$$\frac{1}{2\pi i}\oint_{\gamma} \frac{dz}{z^n+1} = \frac{1}{2\pi i}\int_{\gamma_1} \frac{dz}{z^n+1} + \frac{1}{2\pi i}\int_{\gamma_2} \frac{dz}{z^n+1} - \frac{1}{2\pi i}\int_{\gamma_3} \frac{dz}{z^n+1}. \quad (5)$$

We calculate the complex integrals on the right-hand side of (5) by reducing them to the Riemann integral and find the limits of these integrals as $R \to \infty$,

we get :

$$\frac{1}{2\pi i}\int_{\gamma_1} \frac{dz}{z^n+1} = \frac{1}{2\pi i}\int_0^R \frac{dx}{x^n+1} \xrightarrow{R\to\infty} \frac{1}{2\pi i} I,$$

where $I = \int_0^\infty \frac{dx}{x^n+1}$. (6)

$$\frac{1}{2\pi i}\int_{\gamma_2} \frac{dz}{z^n+1} = \frac{1}{2\pi i}\int_0^{\frac{2\pi}{n}} \frac{Rie^{it}dt}{\left(Re^{it}\right)^n+1} \xrightarrow{R\to\infty} 0,$$

So

$$\left|\frac{1}{2\pi i}\int_0^{\frac{2\pi}{n}} \frac{Rie^{it}dt}{\left(Re^{it}\right)^n+1}\right| \le \frac{1}{2\pi}\cdot\frac{R\cdot\frac{2\pi}{n}}{R^n-1} \xrightarrow{R\to\infty} 0$$

, when *n> 1*

.

Finally,

$$\frac{1}{2\pi i}\int_{\gamma_3} \frac{dz}{z^n+1} = \frac{1}{2\pi i}\int_0^R \frac{e^{i\frac{2\pi}{n}}d\rho}{\left(\rho e^{i\frac{2\pi}{n}}\right)^n+1} = \frac{1}{2\pi i}\int_0^R \frac{e^{i\frac{2\pi}{n}}d\rho}{(\rho)^n+1} \xrightarrow{R\to\infty} \frac{1}{2\pi i}\cdot e^{i\frac{2\pi}{n}} I .$$

Note that the residue A does not depend on the value

$R > 1$, and (1) holds for all values of R.

Therefore, from equalities (1) and (5), passing to the limit for $R \to\infty$

we obtain the following equality:

$$A = \frac{1}{2\pi i} \cdot I + 0 - \frac{1}{2\pi i} \cdot e^{\frac{i2\pi}{n}} \cdot I.$$

Where we find the expression for the integral *I*:

$$I = \frac{2\pi i \cdot A}{1 - e^{i\frac{2\pi}{n}}}$$

We substitute in the last equality, instead of the residue A, its value $\frac{1}{ne^{i\pi\frac{n-1}{n}}}$

By formula (4) and transform the resulting expression:

$$I = \frac{2\pi}{n} \cdot \frac{i \cdot e^{-i\pi} \cdot e^{i\frac{\pi}{n}}}{1 - \cos\frac{2\pi}{n} - i \cdot \sin\frac{2\pi}{n}} = -\frac{2\pi i}{n} \cdot \frac{e^{i\frac{\pi}{n}}\left(1 - \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)}{\left(1 - \cos\frac{2\pi}{n}\right)^2 + \left(\sin\frac{2\pi}{n}\right)^2} =$$

$$= -\frac{2\pi i}{n} \cdot \frac{\left(\cos\frac{\pi}{n} + i\sin\frac{\pi}{n}\right)\left(1 - \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)}{2 - 2\cos\frac{2\pi}{n}} = \frac{2\pi \cdot 2\sin\frac{\pi}{n}}{4n\left(\sin\frac{\pi}{n}\right)^2} =$$

$$= \frac{\pi}{n \cdot \sin\frac{\pi}{n}}.$$

This proves the equality

$$\int_0^\infty \frac{dx}{x^n + 1} = \frac{\pi}{n \cdot \sin\frac{\pi}{n}} . \qquad (**) \qquad ■$$

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Semi-Analytical Method with Laplace Transform for Certain Types of Nonlinear Problems

Areej Salah Mohammed[1] , Sinan H. Abd Almjeed[1]
[1]Department of mathematics, College of Education for pure Sciences/ Ibn-AL-Haithem, University of Baghdad, Baghdad, Iraq

Email: areej.s.m@ihcoedu.uobaghdad.edu.iq;sinan.h.a@ihcoedu.uobaghdad.edu.iq

Abstract. In this paper, the approximate solution is found for the Fornberg-Whitham equation (F-W) by using two analytical methods which are the Laplace decomposition method (LDM) and modified Laplace decomposition method (MLDM) with comparison between these methods for which gave the best approximate solution near to the exact solution, The analytical results of these methods have been received in terms of convergent series with easily calculable components. The results show that the modified method was found to be efficient, accurate and fast compared to the second method used in this research.

1. Introduction

Many important phenomena can be represented by nonlinear equations, both ordinary and partial, such as population models, chemical kinetics and fluid dynamics. Many efforts have been made to implement either approximate or analytical methods to solve the nonlinear equations such as [1] and [2]. The F-W gave as [3, 4]

$$v_t - v_{bbt} + v_b = vv_{bbb} - vv_b + 3v_b v_{bb} \quad (1.1)$$

It consists of a type of travelling wave solution called a kink-like wave solution and anti-kink-like wave solutions. No such sorts of travel wave solutions have been found for F-W. These days, numerous distinct methods have been presented to solve the F-W such as homotopy analysis method (HAM) [5], variational iteration method (VIM) [6], Daftardar-Jafari iterative method (DJM) and homotopy perturbation transform method (HPTM) [7]. Temimi and Ansari method (TAM) and Banach contraction method (BCM)[8].

In this paper, we implemented the LDM introduced by wazwaz [9] and MLDM introduced by Khuri [10, 11] to solve F-W, and the solution will be compared in both methods, those iterative methods have been successfully used to solve several kinds of problems. For example the linear and nonlinear fractional diffusion–wave equation was solved by applying the LDM [12], MLDM used to solve lane-Emden type differential equations [13]. In the following sections, the LDM and MLDM application are presented to solve the F-W and the validity of these methods to find the appropriate approximate solution.

2. The basic idea of the methods

To illustrate the solution steps for the MLDM, we consider the following nonlinear partial differential problem:

$$Lv(b,t) = Rv(b,t) + Nv(b,t) \quad (2.1)$$

$$v(b,0) = f(b),\ v_t(b,0) = g(b), \quad (2.2)$$

wherein L, is an differential operator ∂/∂t in eq. (2.1), R is another linear differential factor, N is a nonlinear differential factor.

By taking Laplace transform (LT) (indicated by C), we get:

$$C[Lv(b,t)] = C[Rv(b,t)] + C[Nv(b,t)], \quad (2.3)$$

using the differentiation property of LT and initial condition in eq. (2.3)

$$sC[v(b,t)] - f(b) = C[Rv(b,t)] + C[Nv(b,t)], \quad (2.4)$$

$$C[v(b,t)] = \frac{1}{s}f(b) + \frac{1}{s}C[Rv(b,t)] + \frac{1}{s}C[Nv(b,t)], \quad (2.5)$$

Then the solution can be represented as an infinite series mentioned below:

$$v(b,t) = \sum_{i=0}^{\infty} v_i(b,t), \quad (2.6)$$

The nonlinear operator is disintegrating as

$$Nv(b,t) = \sum_{i=0}^{\infty} A_i, \quad (2.7)$$

Where Ai are Adomian polynomials [14] of v1, v2, …, vi and it can be evaluated by the following formula

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[N\sum_{n=0}^{\infty}\lambda^n v_n\right] \quad i = 0,1,2,.... \quad (2.8)$$

By substituted (2.6) and (2.7) in (2.5)

$$C\left[\sum_{i=0}^{\infty} v_i(b,t)\right] = \frac{1}{s}f(b) + \frac{1}{s}C[Rv(b,t)] + \frac{1}{s}C\left[\sum_{i=0}^{\infty} A_i\right], \quad (2.9)$$

As C is the linear operator where

$$\left[\sum_{i=0}^{\infty} C\, v_i(b,t)\right] = \frac{1}{s}f(b) + \frac{1}{s}C[Rv(b,t)] + \frac{1}{s}C\left[\sum_{i=0}^{\infty} A_i\right], \quad (2.10)$$

By correspondence both sides of eq. (2.10) we have the following:

$$Cv_0(b,t) = \frac{1}{s}f(b) = h(b,s), \quad (2.11)$$

$$C\, v_1(b,t) = \frac{1}{s}C[Rv_0(b,t)] + \frac{1}{s}C[A_0], \quad (2.12)$$

$$C\, v_2(b,t) = \frac{1}{s}C[Rv_1(b,t)] + \frac{1}{s}C[A_1], \quad (2.13)$$

$$\vdots$$

$$C\, v_{i+1}(b,t) = \frac{1}{s}C[Rv_i(b,t)] + \frac{1}{s}C[A_i], \quad (2.14)$$

By applying the inverse LT we get:

$$v_0(b,t) = h(b,t), \quad (2.15)$$

$$v_{i+1}(b,t) = C^{-1}\left[\frac{1}{s}C[Rv_i(b,t)] + \frac{1}{s}C[A_i]\right], i \geq 0, \quad (2.16)$$

Wherein h (b, t) depict the term originating from origin term and define initial conditions. Now, first of all, we stratifying LT of the terms on the right-hand facet of Eq. (2.16) then stratifying inverse LT we get the values of v1, v2... vi each in order.

To applied MLDM, we imposed that

$$h(b,t) = h_0(b,t) + h_1(b,t). \quad (2.17)$$

According to this assumption, a small change should be made on the components v_0, v_1. The difference we suggest is that only part $h_0(b,t)$ is set tov_0, at the same time as the ultimate part $h_1(b,t)$ is combined with other terms in eq. (2.16) to fined v_1. Based totally on those suggestions, we formulate the modified iterative algorithm is as follows

$$v_0(b,t) = h_0(b,t), \quad (2.18)$$

$$v_1(b,t) = h_1(b,t) - C^{-1}\left[\frac{1}{s}C[Rv_0(b,t)] + \frac{1}{s}C[A_0]\right], \quad (2.19)$$

$$v_{i+1}(b,t) = -C^{-1}\left[\frac{1}{s}C[Rv_i(b,t)] + \frac{1}{s}C[A_i]\right], i \geq 1. \quad (2.20)$$

The solution using the modified Adomian analysis method in large part relies upon on the choice of $h_0(b,t)$ and $h_1(b,t)$.

3. The application of methods

We will discuss the use of LDM and MLDM for the solution of the F-W in this section.

3.1. Applying the LDM

By considering the F-W (1.1):

With the initial condition

$$v(b,0) = e^{\frac{b}{2}}, \quad (3.1)$$

And the exact solution is

given: $v(b,t) = e^{\frac{b}{2}-\frac{2t}{3}}$, (3.2)

Applying the LT on eq. (1.1) we have

$$Cv_t = -Cv_b + Cv_{bbt} + Cvv_{bbb} - Cvv_b + C3v_bv_{bb}, \quad (3.3)$$

By the differentiation property of LT and initial condition in eq. (3.3), we get:

$$sv(b,s) - v(b,0) = -Cv_b + Cv_{bbt} + Cvv_{bbb} - Cvv_b + C3v_bv_{bb}, \quad (3.4)$$

$$v(b,s) = \frac{1}{s}e^{\frac{b}{2}} - \frac{1}{s}Cv_b + \frac{1}{s}Cv_{bbt} + \frac{1}{s}Cvv_{bbb} - \frac{1}{s}Cvv_b + \frac{1}{s}C3v_bv_{bb}, \quad (3.5)$$

Applying inverse LT

$$v(b,t) = e^{\frac{b}{2}} - C^{-1}\left[\frac{1}{s}Cv_b\right] + C^{-1}\left[\frac{1}{s}Cv_{bbt}\right] + C^{-1}\left[\frac{1}{s}Cvv_{bbb}\right] - C^{-1}\left[\frac{1}{s}Cvv_b\right] + C^{-1}\left[\frac{1}{s}C3v_bv_{bb}\right], \quad (3.6)$$

we represent the solution as an infinite series as follows

$$v(b,t) = \sum_{i=0}^{\infty} v_i(b,t), \quad (3.7)$$

The nonlinear operator is decomposed as

$$vv_{bbb} = \sum_{i=0}^{\infty} A_i, \quad (3.8)$$

$$vv_b = \sum_{i=0}^{\infty} B_i, \quad (3.9)$$

$$v_bv_{bb} = \sum_{i=0}^{\infty} C_i, \quad (3.10)$$

By replacing eq. (3.7), (3.8), (3.9) and (3.10) in eq. (3.6) we get:

$$\sum_{i=0}^{\infty} v_i(b,t) = e^{\frac{b}{2}} - C^{-1}\left[\frac{1}{s}Cv_b\right] + C^{-1}\left[\frac{1}{s}Cv_{bbt}\right] + C^{-1}\left[\frac{1}{s}C\sum_{i=0}^{\infty} A_i\right] - C^{-1}\left[\frac{1}{s}C\sum_{i=0}^{\infty} B_i\right] + C^{-1}\left[\frac{1}{s}C3\sum_{i=0}^{\infty} C_i\right], \quad (3.11)$$

Then we get repetition relation

$$v_0(b,t) = e^{\frac{b}{2}}, \quad (3.12)$$

$$v_1(b,t) = -C^{-1}\left[\frac{1}{s}Cv_{0_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{0_{bbt}}\right] + C^{-1}\left[\frac{1}{s}CA_0\right] - C^{-1}\left[\frac{1}{s}CB_0\right] + C^{-1}\left[\frac{1}{s}C3C_0\right], \quad (3.13)$$

$v_{i+1}(b,t) = -C^{-1}\left[\frac{1}{s}Cv_{i_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{i_{bbt}}\right] + C^{-1}\left[\frac{1}{s}CA_i\right] - C^{-1}\left[\frac{1}{s}CB_i\right] + C^{-1}\left[\frac{1}{s}C3C_i\right], i \geq 1,$ (3.14)

Then other constituents…..

$v_1(b,t) = -C^{-1}\left[\frac{1}{s}Cv_{0_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{0_{bbt}}\right] + C^{-1}\left[\frac{1}{s}Cv_0v_{0_{bbb}}\right] - C^{-1}\left[\frac{1}{s}Cv_0v_{0_b}\right] + C^{-1}\left[\frac{1}{s}C3v_{0_b}v_{0_{bb}}\right],$ (3.15)

$v_1(b,t) = -\frac{1}{4}e^{b/2}t,$ (3.16)

$v_2(b,t) = -C^{-1}\left[\frac{1}{s}Cv_{1_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{1_{bbt}}\right] + C^{-1}\left[\frac{1}{s}CA_1\right] - C^{-1}\left[\frac{1}{s}CB_1\right] + C^{-1}\left[\frac{1}{s}C3C_1\right],$ (3.17)

$v_2(b,t) = -C^{-1}\left[\frac{1}{s}Cv_{1_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{1_{bbt}}\right] + C^{-1}\left[\frac{1}{s}C[v_1v_{0_{bbb}} + v_0v_{1_{bbb}}]\right] - C^{-1}\left[\frac{1}{s}C[v_1v_{0_b} + v_0v_{1_b}]\right] + C^{-1}\left[\frac{1}{s}C3[v_{0_b}v_{1_{bb}} + v_{1_b}v_{0_{bb}}]\right],$ (3.18)

$v_2(b,t) = \frac{1}{16}e^{b/2}(-t + t^2),$ (3.19)

$v_3(b,t) = -C^{-1}\left[\frac{1}{s}Cv_{2_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{2_{bbt}}\right] + C^{-1}\left[\frac{1}{s}CA_2\right] - C^{-1}\left[\frac{1}{s}CB_2\right] + C^{-1}\left[\frac{1}{s}C3C_2\right],$ (3.20)

$v_3(b,t) = -C^{-1}\left[\frac{1}{s}Cv_{2_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{2_{bbt}}\right] + C^{-1}\left[\frac{1}{s}C[v_0v_{2_{bbb}} + v_1v_{1_{bbb}} + v_2v_{0_{bbb}}]\right] - C^{-1}\left[\frac{1}{s}C[v_0v_{2_b} + v_1v_{1_b} + v_2v_{0_b}]\right] + C^{-1}\left[\frac{1}{s}C3[v_{0_b}v_{2_{bb}} + v_{1_b}v_{1_{bb}} + v_{2_b}v_{0_{bb}}]\right],$ (3.21)

$v_3(b,t) = -\frac{1}{192}e^{b/2}(3t - 6t^2 + 2t^3),$ (3.22)

$v_4(b,t) = -C^{-1}\left[\frac{1}{s}Cv_{3_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{3_{bbt}}\right] + C^{-1}\left[\frac{1}{s}CA_3\right] - C^{-1}\left[\frac{1}{s}CB_3\right] + C^{-1}\left[\frac{1}{s}C3C_3\right],$ (3.23)

$v_4(b,t) = -C^{-1}\left[\frac{1}{s}Cv_{3_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{3_{bbt}}\right] + C^{-1}\left[\frac{1}{s}C[v_0v_{3_{bbb}} + v_1v_{2_{bbb}} + v_2v_{1_{bbb}} + v_3v_{0_{bbb}}]\right] - C^{-1}\left[\frac{1}{s}C[v_0v_{3_b} + v_1v_{2_b} + v_2v_{1_b} + v_3v_{0_b}]\right] + C^{-1}\left[\frac{1}{s}C3[v_{0_b}v_{3_{bb}} + v_{1_b}v_{2_{bb}} + v_{2_b}v_{1_{bb}} + v_{3_b}v_{0_{bb}}]\right],$ (3.24)

$v_4(b,t) = \frac{1}{768}e^{b/2}(-3t + 9t^2 - 6t^3 + t^4),$ (3.25)

$v(b,t) = \sum_{i=0}^{\infty} v_i(b,t) = e^{\frac{b}{2}} + \frac{1}{16}e^{b/2}(-t + t^2) - \frac{1}{192}e^{b/2}(3t - 6t^2 + 2t^3) + \frac{1}{768}e^{b/2}(-3t + 9t^2 - 6t^3 + t^4) + \cdots,$ (3.26) *3.2.*

Applying the MLDM

By considering the F-W (1.1) with initial condition (1.2), applying the LT we have

$Cv_t = -Cv_b + Cv_{bbt} + Cvv_{bbb} - Cvv_b + C3v_bv_{bb},$ (3.27)

By the differentiation property of LT and initial condition in eq. (2.3)

$sv(b,s) - v(b,0) = -Cv_b + Cv_{bbt} + Cvv_{bbb} - Cvv_b + C3v_bv_{bb},$ (3.28)

$v(b,s) =$
$\frac{1}{s}e^{\frac{b}{2}} - \frac{1}{s}Cv_b + \frac{1}{s}Cv_{bbt} + \frac{1}{s}Cvv_{bbb} - \frac{1}{s}Cvv_b + \frac{1}{s}C3v_bv_{bb},$ (3.29)

Applying inverse LT

$v(b,t) =$
$\frac{1}{2}e^{\frac{b}{2}} + \frac{1}{2}e^{\frac{b}{2}} - C^{-1}\left[\frac{1}{s}Cv_b\right] + C^{-1}\left[\frac{1}{s}Cv_{bbt}\right] + C^{-1}\left[\frac{1}{s}Cvv_{bbb}\right] - C^{-1}\left[\frac{1}{s}Cvv_b\right] +$
$C^{-1}\left[\frac{1}{s}C3v_bv_{bb}\right],$ (3.30)

we constitute solution as an infinite series as follows

$v(b,t) = \sum_{i=0}^{\infty} v_i(b,t),$ (3.31)

The nonlinear operator is decomposed as

$vv_{bbb} =$
$\sum_{i=0}^{\infty} A_i,$ (3.32)

$vv_b =$
$\sum_{i=0}^{\infty} B_i,$ (3.33)

$v_bv_{bb} = \sum_{i=0}^{\infty} C_i,$ (3.34)

By substituting Eq. (3.31), (3.32), (3.33) and (3.34) in eq. (3.30)

$\sum_{i=0}^{\infty} v_i(b,t) = \frac{1}{2}e^{\frac{b}{2}} + \frac{1}{2}e^{\frac{b}{2}} - C^{-1}\left[\frac{1}{s}Cv_b\right] + C^{-1}\left[\frac{1}{s}Cv_{bbt}\right] + C^{-1}\left[\frac{1}{s}C\sum_{i=0}^{\infty} A_i\right] - C^{-1}\left[\frac{1}{s}C\sum_{i=0}^{\infty} B_i\right] +$
$C^{-1}\left[\frac{1}{s}C3\sum_{i=0}^{\infty} C_i\right],$ (3.35)

Then we have

$v_0(b,t) =$
$\frac{1}{2}e^{\frac{b}{2}},$ (3.36)

$v_1(b,t) =$
$\frac{1}{2}e^{\frac{b}{2}} - C^{-1}\left[\frac{1}{s}Cv_{0_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{0_{bbt}}\right] + C^{-1}\left[\frac{1}{s}CA_0\right] - C^{-1}\left[\frac{1}{s}CB_0\right] + C^{-1}\left[\frac{1}{s}C3C_0\right],$ (3.37)

$v_{i+1}(b,t) = -C^{-1}\left[\frac{1}{s}Cv_{i_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{i_{bbt}}\right] + C^{-1}\left[\frac{1}{s}CA_i\right] - C^{-1}\left[\frac{1}{s}CB_i\right] + C^{-1}\left[\frac{1}{s}C3C_i\right], i \geq$
$1,$ (3.38)

Then

$$v_1(b,t) = \frac{1}{2}e^{\frac{b}{2}} - C^{-1}\left[\frac{1}{s}Cv_{0_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{0_{bbt}}\right] + C^{-1}\left[\frac{1}{s}Cv_0v_{0_{bbb}}\right] - C^{-1}\left[\frac{1}{s}Cv_0v_{0_b}\right] + C^{-1}\left[\frac{1}{s}C3v_{0_b}v_{0_{bb}}\right], \quad (3.39)$$

$v_1(b,t) = \frac{e^{b/2}}{2} - \frac{1}{4}e^{b/2}t,$ (3.40)

$$v_2(b,t) = -C^{-1}\left[\frac{1}{s}Cv_{1_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{1_{bbt}}\right] + C^{-1}\left[\frac{1}{s}C[v_1v_{0_{bbb}} + v_0v_{1_{bbb}}]\right] - C^{-1}\left[\frac{1}{s}C[v_1v_{0_b} + v_0v_{1_b}]\right] + C^{-1}\left[\frac{1}{s}C3[v_{0_b}v_{1_{bb}} + v_{1_b}v_{0_{bb}}]\right], \quad (3.41)$$

$v_2(b,t) = \frac{1}{16}e^{b/2}(-5t + t^2),$ (3.42)

$$v_3(b,t) = -C^{-1}\left[\frac{1}{s}Cv_{2_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{2_{bbt}}\right] + C^{-1}\left[\frac{1}{s}CA_2\right] - C^{-1}\left[\frac{1}{s}CB_2\right] + C^{-1}\left[\frac{1}{s}C3C_2\right], \quad (3.43)$$

$v_3(b,t) =$
$-C^{-1}\left[\frac{1}{s}Cv_{2_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{2_{bbt}}\right] + C^{-1}\left[\frac{1}{s}C[v_0v_{2_{bbb}} + v_1v_{1_{bbb}} + v_2v_{0_{bbb}}]\right] - C^{-1}\left[\frac{1}{s}C[v_0v_{2_b} +$

$v_1 v_{1_b} + v_2 v_{0_b}]\Big] +$
$C^{-1}\left[\frac{1}{s}C3[v_{0_b}v_{2_{bb}} + v_{1_b}v_{1_{bb}} + v_{2_b}v_{0_{bb}}]\right],$ (3.44)

$$v_3(b,t) = -\frac{1}{192}e^{b/2}(15t - 18t^2 + 2t^3), \quad (3.45)$$

$$v_4(b,t) = -C^{-1}\left[\frac{1}{s}Cv_{3_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{3_{bbt}}\right] + C^{-1}\left[\frac{1}{s}CA_3\right] - C^{-1}\left[\frac{1}{s}CB_3\right] + C^{-1}\left[\frac{1}{s}C3C_3\right], \quad (3.46)$$

$v_4(b,t) = -C^{-1}\left[\frac{1}{s}Cv_{3_b}\right] + C^{-1}\left[\frac{1}{s}Cv_{3_{bbt}}\right] + C^{-1}\left[\frac{1}{s}C[v_0 v_{3_{bbb}} + v_1 v_{2_{bbb}} + v_2 v_{1_{bbb}} + v_3 v_{0_{bbb}}]\right] -$
$C^{-1}\left[\frac{1}{s}C[v_0 v_{3_b} + v_1 v_{2_b} + v_2 v_{1_b} + v_3 v_{0_b}]\right] +$
$C^{-1}\left[\frac{1}{s}\mathcal{L}3[v_{0_b}v_{3_{bb}} + v_{1_b}v_{2_{bb}} + v_{2_b}v_{1_{bb}} +$
$v_{3_b}v_{0_{bb}}]\right],$ (3.47)

$$v_4(b,t) = \frac{1}{768}e^{b/2}(-15t + 33t^2 - 14t^3 + t^4), \quad (3.48)$$

$$v(b,t) = \sum_{i=0}^{\infty} v_i(b,t) = \frac{e^{b/2}}{2} - \frac{1}{4}e^{b/2}t + \frac{1}{16}e^{b/2}(-5t + t^2) - \frac{1}{192}e^{b/2}(15t - 18t^2 + 2t^3) + \frac{1}{768}e^{b/2}(-15t + 33t^2 - 14t^3 + t^4), \quad (3.49)$$

4. **Numerical analysis's**

In Table 1, absolute errors are calculated for the differences between the exact solution (3.2) and the approximate solutions (3.26) and (3.49) obtained by LDM and MLDM. Besides, Figure 1, Figure 2 and Figure 3 show the approximate and the exact solutions for the Fornberg-Whitham problem respectively, Figure 4 and Figure 5 show the behaviour of exact and approximate solutions obtained by the LDM and MLDM.

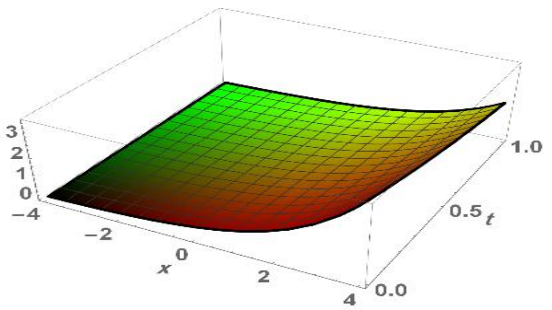

Figure 1. The approximate solution obtained by the LDM of the Fornberg-Whitham problem

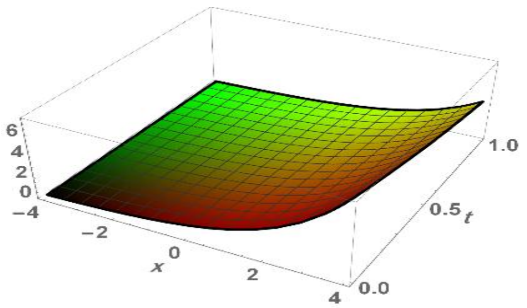

Figure 2. The approximate solution obtained by the MLDM of the Fornberg-Whitham problem

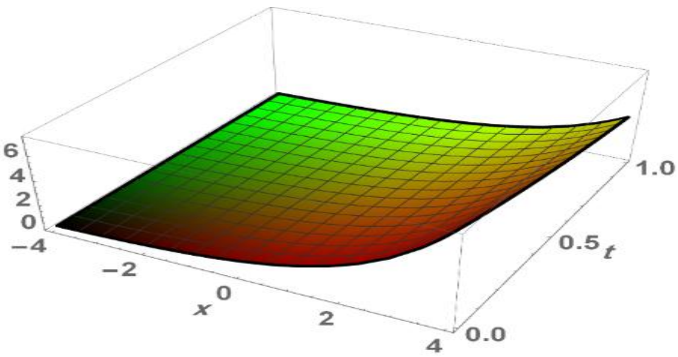

Figure 3. The exact solution of the Fornberg-Whitham problem

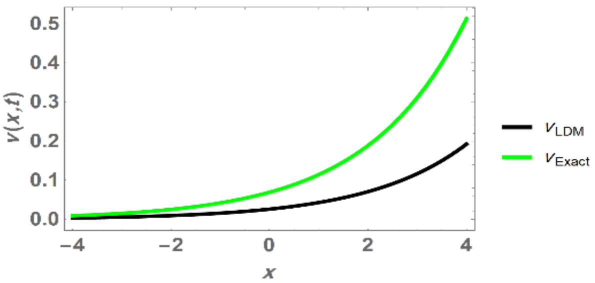

Figure 4. Comparison between the exact solution and approximate solution by LDM.

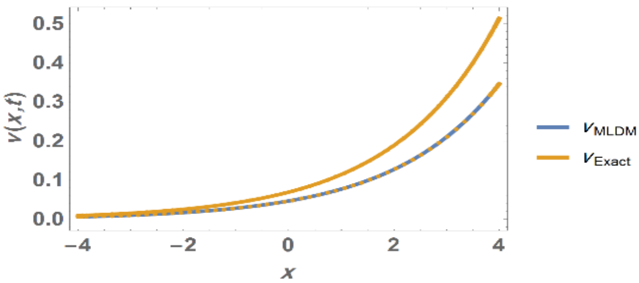

Figure 5. Comparison between the exact solution and approximate solution by MLDM.

Table 1. the numerical values for the exact and the approximate solutions with the absolute errors at t=4

v_{LDM}	v_{MLDM}	v_{EXACT}	$\text{ABSERROR}_{\text{LDM}}$	$\text{ABSERROR}_{\text{MLDM}}$
0.00352435633428678	0.006343841401716	0.0094035625514952	0.0058792062172084	0.003059721149778986
0.0095801937805063	0.017244348804911	0.0255615332065074	0.015981339426001	0.0083171844015960
0.02604166666666666	0.046875	0.0694834512228015	0.0434417845561348	0.0226084512228015
0.0707885892827876	0.127419460709017	0.188875602837561	0.118087013554774	0.06145614212854408
0.19242333590965235	0.34636200463737	0.513417119032592	0.3209937831229397	0.1670551143952178

Conclusion

In this paper, we dealt with analytical solutions include the LDM and the MLDM, which we discussed convergence and compared to the exact solution where we found that the convergence achieved by the modification method is more efficient and accurate than the Laplace decomposition method.

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On the Dirichlet Problem for the Nonlinear Diffusion Equation with Convection and Reaction

Habeeb A. Aal-Rkhais

Department of Mathematics, College of Computer Science and Mathematics,

University of Thi-Qar, Iraq(habeebk@utq.edu.iq)

Abstract: We consider the nonlinear parabolic equations for the nonlinear diffusion-convection-reaction processes applying in many areas of science and engineering, such as filtration of gas or fluid in porous media. The aim of this paper is to concentrate on the existence of the weak solutions and boundary regularity for the Dirichlet problem of the degenerate parabolic equations in irregular domains in some cases where both the convection and reaction terms have the same exponents. The notion of parabolic modulus has a significant role for the boundary continuity of the solutions.

Keywords: Dirichlet problems, degenerate parabolic equations, weak solutions.

1. Introduction

Consider nonlinear degenerate parabolic PDEs

$$(1.1) \qquad \mathcal{L}(u) \equiv u_t - a(u^m)_{xx} + b\big(\ell(u)\big)_x + c\ell(u) = 0$$

with $x \in \mathbb{R}$, $0 < t < \tau$, $a > 0$, $b, c \in \mathbb{R}$, $\ell(u) = u^p$ and p is a positive exponent for both convection and reaction terms. (1.1) is usually called a porous media equation with convection and reaction terms. It has wide applications in chemistry, physics and biology involving diffusion with convection or advection and accompanied with additional source as for instance in modeling filtration in porosity of the medium, flow of a chemical reacting fluid on a flat surface, transport of thermal energy in a plasma, evolution and development of populations. In [12], the mathematical theory of nonlinear implicit degenerate parabolic equations begins and the general theory, which is represented the concepts of the existence, uniqueness and boundary regularity of the boundary problems for classical general nonlinear diffusion equation in special case the equation (1.1) with m > 1, b = 0, and c = 0 have been discussed. A lots of works on this equation introduced by a general list of references [9, 16, 14] etc. General theory for reaction-diffusion in non-cylindrical domains was introduced in a serious works in [1,2,3].

In this paper, we study the existence of weak solutions for DP of nonlinear diffusion-convection-reaction equations in particular case where the convection and reaction terms have the same exponent p. There are in a literature review some papers dealing with the boundary value problems in irregular or noncylindrical domain with nonsmooth boundaries. Let consider the following problem for the heat equation $u = u_{xx} \quad in\ \Omega$ with initial and boundary conditions:

$$(1.2) \qquad u(x,0) = u_0(x), \quad \rho_1(0) \le x \le \rho_2(0)$$

$$(1.3) \qquad u(\rho_i(t), t) = \psi_i(t), \quad 0 \le t \le \tau,$$

where $\Omega = \{(x,t): \rho_1(t) \le x \le \rho_2(t), 0 \le t \le \tau\}$, $0 < \tau \le +\infty$, $\rho_i, \psi_i \in C[0;\tau]$, $i = 1,2$, $\rho_1(t) < \rho_2(t)\ for\ t \in [0;\tau]$, $u_0 \in C([\rho_1(0)\,;\,\rho_2(0)])$ and $u_0(\rho_i(0)) = \psi_i(0)$. Proving the existence of a classical solution to the DP for the heat equation with the conditions (1.2) ,(1.3) was established in [8] if the boundary curves $\rho_i(t)$ satisfy a Hölder condition with Hölder exponent bigger than $1/2$.

2. Statement of Problem

In this paper, we focus on the following problem

Problem(*Dirichlet problem (DP)*): Finding a weak solution of a nonlinear parabolic equation (1.1) in Ω with the conditions (1.3)-(1.4).

Definition 2.1. Let $u(x,t)$ be a function. It a weak solution of the DP in Ω if

(a) u is a nonnegative continuous function in $\bar{\Omega}$, and $u \in L_\infty(\Omega \cap (\tau_1 \ge t))$ for a finite $0 < \tau_1 \le T$.

(b) for any $\beta_i(t), t_0 \le t \le t_1, i = 1,2;$ are $C^\infty(\Omega)$ functions such that $\rho_1(t) < \beta_1(t) < \beta_2(t) < \rho_2(t)$ for $[t_0, t_1] \subset [0, T]$, the integral identity holds

$$J(u, \phi, \Omega_1) = \int_{t_0}^{t_1} \int_{\beta_1(t)}^{\beta_2(t)} (u\,\phi_t + au^m \phi_{xx} - bu^p \phi + cu^p \phi_x)\,dxdt$$

(2.1) $$- \int_{t_0}^{t_1} au^m \phi_x \Big|_{\beta_1(t)}^{\beta_2(t)} dt - \int_{\beta_1(t)}^{\beta_2(t)} u\phi \Big|_{t_2}^{t_1} dx = 0\ ,$$

where $\Omega_1 = \{(x,t): \beta_1(t) \le x \le \beta_2(t),\ t_0 < t < t_1\}$ and $\phi \in C_{x,t}^{2,1}(\bar{\Omega}_1)$ is a function that equals zero when $x = \beta_i(t)$, $t_0 \le t \le t_1, i = 1,2$.

Definition 2.2. A function $u(x,t)$ is said to be a supersolution of the DP in Ω if **(a)** and **(b)** of definition 2.1 are satisfied except for $J(u, \phi, \Omega_1) \le 0$ for any nonnegative function $\phi \in C_{x,t}^{2,1}(\bar{\Omega}_1)$.

Definition 2.3. A function $u(x,t)$ is said to be a subsolution of the DP in Ω if **(a)** and **(b)** of definition 2.1 are satisfied except for $J(u, \phi, \Omega_1) \ge 0$ for any nonnegative function $\phi \in C_{x,t}^{2,1}(\bar{\Omega}_1)$.

Definition 2.4. Let $\rho_i \in C[0;\tau], i = 1,2$ and for any fixed $t_0 > 0$ consider a function

$$\omega_{t_0}^{-}(\rho_1;\delta) = max(\rho_1(t_0) - \rho_2(t): t_0 - \delta \le t \le t_0$$

$$\omega_{t_0}^{+}(\rho_2;\delta) = min(\rho_1(t_0) - \rho_2(t): t_0 - \delta \le t \le t_0$$

with $\delta > 0$ is sufficiently small and this function is well defined and converge to zero as $\delta \to 0^+$. The function $\omega_{t_0}^{-}(\rho_1;.)$ is called the left modulus of lower semi-continuity of the function ρ_1 at the point t_0; and $\omega_{t_0}^{+}(\rho_2;.)$ is called the left modulus of upper semi-continuity of the function ρ_2 at the point t_0.

Assumption($\mathcal{L}$): Let $\mathcal{F}(\delta)$ be a function such that $\mathcal{F}$ is defined for sufficiently small $\delta > 0$; $\mathcal{F}$ is positive and converges to 0 as $\delta \to 0^+$and

(2.2) $$\omega_{t_0}^{-}(\rho_1;\delta) \le \delta^{\frac{1}{2}}\mathcal{F}(\delta).$$

Assumption($\mathcal{R}$): Let $\mathcal{F}(\delta)$ be a function such that $\mathcal{F}$ is defined for sufficiently small $\delta > 0$; $\mathcal{F}$ is positive and converges to 0 as $\delta \to 0^+$and

$$(2.3) \qquad \omega_{t_0}^{+}(\rho_2;\delta) \leq \delta^{\frac{1}{2}}\mathcal{F}(\delta).$$

The sufficient and necessary condition to satisfy regularity of boundary points in initial boundary value problem and in Wiener type [15] and the geometric characterizations for boundary points of any bounded open subset of R^{N+1} for a heat equation have been established in [5, 10]. The sufficient conditions are presented in [7, 18] for regularity of the boundary in the situation of general nonlinear non-degenerate parabolic equations. In [13], the multidimensional Kolmogorov Petrovski test is presented for the boundary regularity for the heat equations. In this paper we are interested in DP to equation (1.1) and the general strategies for the existence results coincide with the classical solution for the DP to Laplace equation [15].

The goal of this paper is to get our attention by studying the existence of a weak solution and boundary regularity of the DP for the nonlinear degenerate parabolic diffusion equation (1.1) with convection and reaction terms in irregular domain or non-smooth boundary curves. The methods that we use are standard parabolic regularizations, construction of barriers and Bernstein method. First, we use an approximation of both the domain Ω and boundary function, as well as standard regularization of (1.1), we also construct a sequence of classical solutions in smooth domains which converges to a solution of (1.1). We then use barriers and a limiting process to prove a boundary regularity. In particular, we study the regularity of the boundary point under the assumptions ($\mathcal{L}$) and ($\mathcal{R}$).

2. Preliminary Results

In this section we use parabolic regularization technique to construct the auxiliary classical problem to prove the preliminary results. Let $\{\epsilon_n\}$ and $\{\tau_n\}$ be a monotonic sequences with $\epsilon_n \to 0^+$. Let $\tau_n \equiv \tau$ if $\tau < +\infty$ and $\{\tau_n\}$ be positive sequence such that $\tau_n \to +\infty$ as $n \to +\infty$ if $\tau = +\infty$. Let $\{\rho_{jn}\}$, $j = 1,2$ be sequences of functions and $\rho_{in} \in C^\infty[0;\ \tau_n]$, $\rho_{1n}(t) < \rho_{2n}(t)$. For $t \in [0;\ \tau_n]$ and

$$\lim_{n\to+\infty} \max_{0\leq t\leq \tau_n} |\rho_{in}(t) - \rho_i(t)| = 0.$$

Suppose that $\rho_1(0) = 0$, $\rho_2(0) = \mathrm{D} > 0$, $\rho_{1n}(0) = \rho_{1n}^0$, $\rho_{2n}(0) = \rho_{2n}^0$. Also, we consider some restrictions on the sequence $\{\rho_{in}^0\}$ will be expressed below. Let γ be any number which satisfies

$$\gamma = 1 \text{ if } c < 0 \text{ and } \gamma > \max(m^{-1}; p^{-1}; 1) \quad \text{if} \quad c \geq 0.$$

Let consider $\epsilon_1^{\gamma} < H$ without loss of generality and sequences of functions $\{u_{0n}\}, \{\psi_{1n}\}, \{\psi_{2n}\}$ and numbers $\{\phi_{1n}^0\}, \{\phi_{2n}^0\}$ such that

(i) $\rho_{1n}^0 \in [0;\ \vartheta/4], \rho_{2n}^0 \in [(3/4)\vartheta;\ \vartheta], lim_{n\to\infty}\, \rho_{1\,n}^0 = 0,\ lim_{n\to\infty}\, \rho_{2\,n}^0 = \vartheta,$

(ii) $u_0(0) - \chi(\epsilon_n)/2 \leq u_0(\rho_{1\,n}^0) \leq (u_0^m(0) + (\chi(\epsilon_n)/2)^m)^{1/m},$

(iii) $u_0(\vartheta) - \chi(\epsilon_n)/2 \leq u_0(\rho_{2\,n}^0) \leq (u_0^m(\vartheta) + (\chi(\epsilon_n)/2)^m)^{1/m},$

(iv) $\epsilon_n^{\gamma} \leq u_{0n}(x),\ \psi_{in}(t) \leq H$ *for* $(x, t) \in [0; \vartheta] \times [0;\ \tau_n]$,

(v) $u_{0n} \in C^{\infty}[0; \vartheta],\ \psi_{in} \in\ C^{\infty}[0; \vartheta], \qquad i = 1,2,$

(vi) $u_{0n}(\rho_{1\,n}^0) =\ \psi_{1n}(0), a\ (u_{0n}^m)''(\rho_{1n}^0) + (\rho_{1n}')u_{0n}'(\rho_{1n}^0)\text{-}b(u_{0n}^p)'(\rho_{1\,n}^0)$

$$-cu_{0n}^p(\rho_{1\,n}^0) + c\theta_c\ \epsilon_n^{p\gamma} =\ \psi_{1\,n}'(0),$$

(vii) $u_{0n}(\rho_{2\,n}^0) =\ \psi_{2n}(0), a\ (u_{0n}^m)''\,(\rho_{2\,n}^0) + (\rho_{2\,n}')\, u_{0n}'(\rho_{2\,n}^0)\text{-}b\ (u_{0n}^p)'(\rho_{2\,n}^0)$

$$-cu_{0n}^p(\rho_{2\,n}^0) + c\theta_c\ \epsilon_n^{p\gamma} =\ \psi_{2\,n}'(0)$$

$(viii)$ $0 \leq u_{0n}(x) - u_0(x) \leq\ \chi(\epsilon_n)$, *for* $0 \leq x \leq \vartheta.$

(ix) $0 \leq\ \psi_{in}^m(t) - \psi_i^m(t) \leq\ \chi^m(\epsilon_n)$, *for* $0 \leq t \leq\ \tau_n, i = 1,2.$

where $\chi(x) = Cx^{\gamma}$ for $x \geq 0$ and $C > 0$. Let assume that $\chi(x)$ is an arbitrary positive monotonic and continuous function with $lim_{x\to 0^+}\, \chi(x) = 0$, if the boundary and initial functions have a positive infimum value. Consider the following auxiliary DP problem

(3.1) $\qquad L_n u \equiv \mathcal{L}(u) -\ c\theta_c \epsilon_n^{p\gamma}\ in\ \Omega_n,$

(3.2) $\qquad u(x, 0) = u_{0n}(x), \quad \rho_{1\,n}^0 \leq x\ \leq \quad \rho_{2\,n}^0\ ,$

(3.3) $\qquad u(p_{in}(t), t) = \psi_{in}(t),\ \ 0 \leq t \leq\ \ \tau_n, \qquad i = 1, 2,$

where $\Omega_n = \{(x, t):\ \rho_{1n}(t) \leq x \leq\ \rho_{2n}(t),\ 0 \leq t \leq\ \tau_n\}.$

Lemma 1. Suppose that the sequences of functions $\{\psi_{in}\}$ and $\{u_{0n}\}$, and sequences of numbers $\{\rho_{in}^0\}$ satisfy the conditions (i)-(ix) then there exists a classical solution $u_n(x, t)$ of the problem (3.1)-(3.3) which satisfy

(3.4) $\quad (u_n)_x \in C_{x,t}^{2+\mu_1,1+\mu_1/2}(\Omega_n)$ for some $\mu_1 > 0$,

(3.5) $\quad \epsilon_n^{\gamma} \le u_n(x,t) \le \psi_{2n}(t)$, for $(x,t) \in \bar{\Omega}_n$.

Proof of Lemma 1. By applying a standard method, and we suppose without loss of generality that the sequences $\{\rho_{1n}\}$, $\{\rho_{2n}\}$ satisfy the conditions (i)-(ix), If we consider a new variable

$$\vartheta(\mathrm{x} - \mathrm{p}_{1\mathrm{n}}(\mathrm{t}))(\mathrm{p}_{2\mathrm{n}}(\mathrm{t}) - \mathrm{p}_{1\mathrm{n}}(\mathrm{t}))^{-1} \to \mathrm{y},$$

Then (3.1)-(3.3) will be changed to the problem

$$v_t = a\vartheta^2 (\rho_{2n}(t) - \rho_{1n}(t))^{-2}(v^m)_{yy} + \left(\vartheta\, \rho'_{1\,n}(t) + \left((\rho'_{2\,n}(t) - \rho'_{1\,n}(t)\right)y\right) \times$$

(3.6) $$(\rho_{2n}(t) - \rho_{1n}(t))^{-1} v_y - b\vartheta\, (\rho_{2n}(t) - \rho_{1n}(t))^{-1}(v^p)_y$$

$$-c(v^p - c\theta_c\ \epsilon_n^{p\gamma}) \quad in \quad \Omega'_n,$$

(3.7) $$v(y,0) = u_{0n}(\rho_{1\,n}^0 + \vartheta^{-1}(\rho_{2\,n}^0 - \rho_{1\,n}^0)y) for\ 0 \le y \le \vartheta,$$

(3.8) $$v(0,t) = \psi_{1n}(t), \quad v(\vartheta,t) = \psi_{2n}(t) \ \ 0 \le t \le \tau_n,$$

where $\Omega'_n = \{(y,t): 0 \le y \le \vartheta \ , 0 \le t \le \tau_n\}$, and $\psi_{2n}(t) = \Psi(t)$ such that

$$\Psi(t) = \begin{cases} [H^{1-P} - c(1-\theta_C)(1-p)t]^{1/(1-p)}, & if\ p \ne t, \\ H\, exp(-c(1-\theta_C)t), & if\ \ p \ne 0 \end{cases}$$

we get from [11] that the problem (3.6)-(3.8) has a unique classical solution $v = v_n(y,t)$ such that $v_n \in C_{x,t}^{2+\mu_1,1+\mu_1/2}(\Omega'_n)$ with some $\mu_1 > 0$. From the maximum principle, we get

(3.9) $$\epsilon_n^{\gamma} \le v_n(y,t) \le \Psi(t) \ \ in \quad \Omega'_n\,.$$

Hence, the functions $u_n(x,t) = v_n\left(\vartheta(x - \rho_{1n}(t))\left(\rho_{2n}(t) - \rho_{1n}(t)\right)^{-1}, t\right)$ is the classical solution of the problem (3.1)-(3.3), then (4.5) follows. Also, from [6], (3.4) holds. .

From the priori estimations in lemma 1, we proved that the classical solution to (3.1)-(3.3) exists. we apply Hölder estimates for the classical solutions in [17], then we have the following lemma.

Lemma 2. Let u_n be a classical solution of (3.1)-(3.3) and a sequences of numbers $\{\rho_{in}^0\}$ and the sequences of functions $\{u_{0n}\}$ and $\{\psi_{in}\}$ satisfy the conditions (i)-(ix) then we get the limit solution

$$(3.10) \qquad u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

Proof of Lemma 2. Since the problem (3.1)-(3.3) has a classical solution by lemma 1, we can consider that a sequence $\{\Omega^k\}$ of compact subsets of Ω such that

$$(3.11) \qquad \Omega^k \subset \Omega^{k+1}, \qquad \bigcup_{k=1}^{\infty} \Omega^k = \Omega.$$

Obviously, for each fixed k, there exists a number $n(k)$ such that $\Omega^k \subset \Omega_n$ for $n(k) \leq k$. The sequence $\{u_n\}$ should be satisfy the following inequality

$$(3.12) \qquad \left|\frac{\partial u_n^m}{\partial x}\right| \leq H_1 \text{ in } \quad \Omega^k,$$

where $H_1 = H_1(k)$ is an arbitrary constant. From [14], by using method of Bernstein then the estimation (3.12) is established. It implies that

$$(3.13) \qquad |u_n(x,t) - u_n(y,t)| \leq H_2|x-y|^\alpha \quad in \quad \Omega^k,$$

where $H_2 = H_2(k)$ and $\alpha = min(m^{-1}; 1)$. From (3.13) that we can easily establish the Hölder estimation with respect to variable of time. Then, we prove the estimation (3.14) in similar way as it is proved in [12],

$$(3.14) \qquad |u_n(x,t) - u_n(y,t)| \leq H_3(|x-y|^\alpha + |t-\tau|^{\frac{\alpha}{1+\alpha}}) \quad in \quad \Omega^k.$$

where $H_3 = H_3(k)$. Therefore for $n(k) \leq n$, the sequence $\{u_n\}$ is uniformly bounded and equicontinuous in the sequence of sets Ω^k. From [4], we get more general results to establish that the sequence $\{u_n\}$ is equicontinuous in Ω^k. From (3.14), (3.11), and by applying an Arzela-Ascoli theorem and a diagonalisation argument, we get a subsequence $\{u_j\}$ such that $u_j \to u'$ as $j \to \infty$, pointwise in Ω, where a limit function u' is continuous. Also, $\{u_j\}$ is uniformly convergent on all compact subsets of Ω. Obviously, $u' \in L_\infty(\Omega)$ if $c \geq 0$ or $c < 0$ and $p > 1$ and $u' \in L_\infty(\Omega \cap (t \leq \tau_1))$ for a finite value $\tau_1 > 0$ if $c < 0$ and $0 < p \leq 1$.

Thus, we consider that $u(x;\ t)$ is a function such that $u(x,t) = u'(x,t)$ for $(\mathrm{x,t}) \in \Omega$; with the initial condition $u(x,0) = u_0(x)$ where $\rho_1(0) \le x < \rho_2(0)$ and the boundary functions $u(\rho_i(0),t) = \psi_i(t)$ where $t \in [0,\ \tau_n]$.

2. Main Results

Depending on the results in sections 2 and 3, we prove the main existence theorem for the DP (1.1), (1.3), (1.4) by using same strategies and methods which presented in [1, 10, 13].

Theorem 4.1. If ρ_1 and ρ_2 satisfy the assumptions $(\mathcal{L})$ and $(\mathcal{R})$, respectively; then a weak solution for the DP (1.1), (1.3), (1.4) in Ω exists.

Proof. From lemma 1 and lemma 2 we proved the existence of classical solution $\mathrm{u_n}$ which satisfies the limit solution (3.10). Also, the integral identity (2.1) is satisfied. Now, according to definition 2.1, we have to get our attention in proving the continuity of function $\mathrm{u(x,t)}$. Obviously, we may be easily establish the continuity of u at along the line $t = 0$. If function $u_0^m(x)$ is locally Lipschitz continuous, then from lemma 2, the estimations (3.13), (3.14) are established at the point $(x_0, 0)$, $x_0 > \rho_1(0)$. Generally, we establish the continuity of u at the point $(x_0, 0)$ by using barriers technique. Next, we will prove the continuity of the function u at the points (x,t), $x = \rho_i(t)$, $t \ge 0$. For that, we consider the following function

$$\upsilon(y,t) = u(\rho_1(t) + \vartheta^{-1}(\rho_2(t) - \rho_1(t))y, t), \quad (y,t) \in \overline{\Omega}'.$$

where $\Omega' = \{(y,t): y \in [0,\vartheta]\ \ , t \in [0,\ \tau_n]\}$. Clearly that

$$\upsilon \in \mathrm{C}(\Omega') \cap \mathrm{L}_\infty(\Omega') \text{ if } p > 1 \text{ and } \mathrm{c} \ge 0 \text{ or } \mathrm{c} < 0,$$

$$\upsilon \in \mathrm{C}(\Omega') \cap \mathrm{L}_\infty(\Omega' \cap (\mathrm{t} \le \tau_1)) \text{ if } \quad 0 < p \le 1 \text{ and } \mathrm{c} < 0,$$

where $\tau_1 \in (0;\tau]$ is a finite number. Then we have a point-wise convergent sequence $\{\upsilon_j\}$ to the function υ as $\mathrm{j} \to +\infty$ in Ω'_n and uniformly convergent on all compact subsets of Ω'. Since there are equivalence between the continuity of u along the points $x = \rho_i(t), i = 1,2$; and the continuity of υ along the points $x = 0$ and $x = \vartheta$. We will divide the proof into two steps:

Step 1. To show v is continuous on $y = 0; t \geq 0$, we shall prove that the following two inequalities are valid for $\tau_0 \geq 0$, $\psi_1(\tau_0) > 0$

(4.1) $$\lim_{(y,t)\to(0,\tau_0)} inf\, v(y,t) \ \geq \ \psi_1(\tau_0) - \epsilon\,,$$

(4.2) $$\lim_{(y,t)\to(0,\tau_0)} sup\, v(y,t) \geq \ \psi_1(\tau_0) - \epsilon\,.$$

Since $\epsilon > 0$ is an arbitrary real number, then v is continuous at the boundary points $(0, \tau_0)$ which comes from (4.1), (4.2). If $\psi_1(\tau_0) = 0$, and since (4.1) with $\epsilon = 0$ in the lower bound inequality directly comes from that v is non-negative in $\overline{\Omega}'$, then it is the same way to prove (4.2). First, we will prove (4.1) if $\epsilon > 0$, $\psi_1(\tau_0) > 0$, so we must estimate the subsolution from the lemma below and thereby complete the first part of the proof.

Lemma 3. Let $\psi_1(\tau_0) > 0$ and $\epsilon \in (0; \psi_1(\tau_0)\,)$ and consider a function $w_n(y,t) = f(\zeta)$ such that $f(\zeta) = H_1(\zeta/h(\mu))^{\alpha}$, $H_1 = \psi_1(\tau_0) - \epsilon$, $\mu > 0$, $h > 0$, where

$$\zeta = h(\mu) - \left(1 - \vartheta^{-1}\rho_{1n}(t)\right)y + \mu(t - t_0) + \rho_{1n}(t_0) - \rho_{1n}(t)$$

then we can choose μ is so large such that

(4.3) $$L_n\omega_n \leq 0 \quad \text{for} \quad 0 \leq \zeta \leq h(\mu)(1+\lambda).$$

Proof of Lemma 3. Let $t_0 > 0$ be fixed where $\psi_1(\tau_0) > 0$ and $\epsilon \in (0; \psi_1(\tau_0)\,)$ then we take $h(\mu) = H_3\mathcal{F}(\mu^{-2})\mu^{-1}$ such that

$$H_3 = 1/\left((H_2H_1^{-1})^{\frac{1}{\alpha}} - 1\right), \qquad H_2 = \psi_1(\tau_0) - \tfrac{\epsilon}{2}, \qquad \mu_0 < \mu;$$

where $\mu_0 = \sqrt{1/\delta_0}$, and we suppose that the boundary curves ρ_1 satisfies (2.2) from assumption $(\mathcal{L})$ at the point $t = \tau_0$. If $\tau_0 = 0$ and $\mu \geq \mu_0 = 1$, then we can choose $h(\mu) = 1/\mu^2$, $\mu \geq \mu_0 = 1$. Let $f(\zeta)$ be a function for all $\zeta \in [0, h(\mu)(1+\lambda)]$, where $\lambda > 0$ such that $\lambda \geq (H_2H_1^{-1})^{\frac{1}{\alpha}} - 1$.

Assume that either $b \geq 0$ or $b < 0$, and if $c > 0$ or $c \leq 0, p \geq 1$, we take two cases as shown in Figure 1:

(A) if $0 < m \leq 1$, then $\alpha > m^{-1}$,
(B) if $m > 1$, then $m^{-1} < \alpha < (m-1)^{-1}$.

If $b < 0$, $c > 0$, $0 < p < 1$, we take four different cases as shown in Figure 1:

(C) if $m > 1,\ 0 < p < 1$, then $m^{-1} < \alpha \le \min\{(m-1)^{-1}; (1-p)^{-1}\}$,

(D) if $0 < p < 1, 1-p < m \le 1$, then $m^{-1} < \alpha \le (1-p)^{-1}$,

(E) if $0 < m \le \frac{1}{2}$, $m \le p \le 1-m$, then $\alpha > m^{-1}$,

(F) if $0 < p \le \frac{1}{2}$, $p \le m \le 1-p$, then $m^{-1} < \alpha \le 2/(m-p)$.

It may be easily checked that

$$L_n\omega_n \equiv \mu f' - a(f^m)'' + b(f^p)' + cf^p - c\theta_c\epsilon_n^{p\gamma}, \tag{4.4}$$

Assume that $b \ge 0$, or $b < 0,\ p \ge 1$ and the cases (A)-(B) are satisfied (see Figure 1), then we have the following estimation from (4.4)

$$L_n\omega_n \le f^{\frac{\alpha-1}{\alpha}}\Big\{\alpha H_1^{\frac{1}{\alpha}}\mu h^{-1}(\mu) - h^{-2}(\mu)aH_1^{\frac{2}{\alpha}}m\alpha(m\alpha-1)H_4^{((m-1)\alpha-1)/\alpha}$$

$$-b(1-\theta_b)H_1^{\frac{1}{\alpha}}\alpha p h^{-1}(\mu)H_4^{p-1} + c\theta_c H_4^{p-1+1/\alpha}\Big\} \tag{4.5}$$

where $H_4 = H_1(1+\lambda)^\alpha$. We take $\mu\, h(\mu) \to 0$ as $\mu \uparrow \infty$, and choose $\mu_1 \ge \mu_0$ is fixed and so large if $\mu \ge \mu_1$, then (4.3) is satisfied.

If however, $b < 0, c > 0, 0 < p < 1$ and m, p are in the regions (C)-(F)(see Figure 1), then we have the following estimation from (4.4)

$$L_n\omega_n \le f^p\Big\{\mu h^{-1}(\mu)\alpha H_1^{\frac{1}{\alpha}}H_4^{1-p-1/\alpha} - h^{-2}(\mu)aH_1^{\frac{2}{\alpha}}m\alpha(m\alpha-1)H_4^{m-p-2/\alpha}$$

$$-b(1-\theta_b)H_1^{\frac{1}{\alpha}}\alpha p h^{-1}(\mu)H_4^{-1/\alpha} + c\Big\} \tag{4.6}$$

As before, from (4.6), we can choose $\mu_1 \ge \mu_0$ is fixed and so large if $\mu \ge \mu_1$, then (4.3) is satisfied. The lemma is proved.

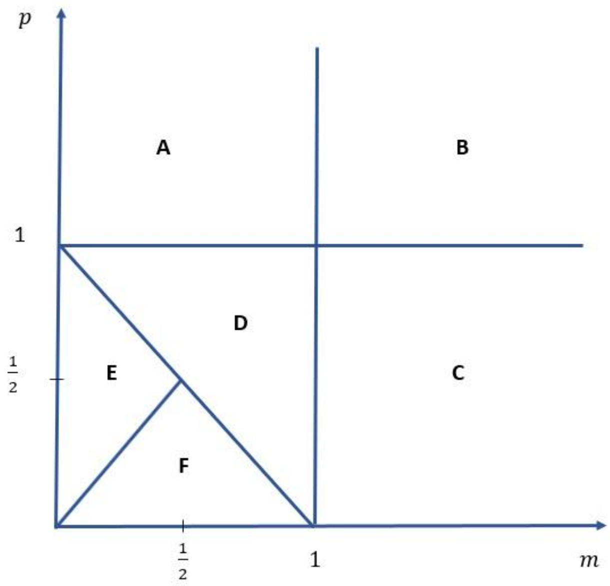

Figure 1.Domain of the boundary regularity barrier in the parameter space (m, p).

We now return to complete proof of the inequality (4.1). Since $\psi_1(t)$ is continuous then we consider the numbers $\mu_1 \leq \mu_2$ such that $\psi_1(\tau_0) - \frac{\epsilon}{2} < \psi_1(t)$ for $\tau_0 - \frac{1}{\mu_0^2} \leq t \leq \tau_0 + \delta_1$, where if $\tau_0 = T < +\infty$ then we choose $\delta_1 > 0$ depends on $\epsilon > 0$ such that $\tau_0 + \delta_1 < T$. If $\tau_0 = T < +\infty$ then we choose $\delta_1 > 0$ depends on $\epsilon > 0$ such that $\psi_1(0) - \frac{\epsilon}{2} < \psi_1(t)$ for $0 \leq t \leq \delta_1$. We will estimate $\omega_n(0, t)$ in the neighborhood of τ_0. Since we have $\omega_n(0, \tau_0) = \psi_1(\tau_0) - \epsilon$ and for a continuous function f and a uniformly convergent sequence $\{\rho_n\}$ to ρ as $n \uparrow +\infty$ for every $\mu \geq \mu_2$ there exists $\delta_2 > 0$ which depends on μ , ϵ, and does not depend on n such that $\delta_2 \leq \delta_1$. Let N_1 be a number which depends on μ , ϵ such that for $n \geq N_1$, $\omega_n(0, t) < \psi_1(\tau_0) - \frac{\epsilon}{2}$ for $\tau_0 \leq t \leq \tau_0 + \delta_2$. We choose $\delta_2 = 0$ if $\tau_0 = T$ and $\delta_2 > 0$ if $\tau_0 < T$. Let suppose that $\tau_0 > 0$ and we consider the function $\omega_n(0, t)$ for $t_0 - \mu^{-2} \leq t \leq \tau_0 + \delta_2, \mu \geq \mu_2$ and $n \geq N_1$. Let consider a uniformly convergent sequence $\{\rho_{1n}\}$ to ρ_1 as $n \uparrow +\infty$, then without loss of generality we suppose that $\omega_{\tau_0}^-(\rho_{1n}; \delta)$ is satisfied (2.2) uniformly for $n \geq N_1, 0 < \delta \leq \delta_0$. If $\tau_0 - \mu^{-2} \leq t \leq \tau_0$, then we have

$$\omega_n(0, t) \leq f\big(h(\mu) + \rho_{1n}(\tau_0) - \rho_{1n}(t)\big) \leq f\big((H_3^{-1} + 1)h(\mu)\big) = \psi_1(\tau_0) - \frac{\epsilon}{2}.$$

If $\tau_0 = 0$ we choose and fix numbers $\mu_2 \geq \mu_1$ and $N_2 \geq N_1$ so large that if $\mu \geq \mu_2$, and $n \geq N_2$, then

$u_0\big((1-\rho_{1n}^0\vartheta^{-1})y+\rho_{1n}^0\big) \geq \psi_1(0)-\frac{\epsilon}{2}$ for $0 \leq y \leq \vartheta(\vartheta-\rho_{1n}^0)^{-1}h(\mu)$.

Now let $N_3 \geq N_2$ be chosen so large that $\epsilon_n^\gamma < \psi_1(\tau_0)-\epsilon$ for $n \geq N_3$. Let $\eta_n = \big(M_1^{-1}\epsilon_n^\gamma\big)^{\frac{1}{\alpha}}h(\mu), n \geq N_3$. Obviously, η_n converges university to 0 as $n \uparrow +\infty$ with respect $\mu \geq \mu_2$. Then we set

$$\Omega_n = \{(y,t):\ 0 < y < \xi_n(t), \qquad \tau_0 - d_{\tau_0}(\mu) < t \leq \tau_0+\delta_2\},$$

$$\Gamma_n = \{(y,t):\ y = \xi_n(t), \qquad \tau_0 - d_{\tau_0}(\mu) < t \leq \tau_0+\delta_2\},$$

$$\xi_n(t) = \vartheta\big(\vartheta-\rho_{1n}(t)\big)^{-1}(h(\mu)+\mu(t-\tau_0)+\rho_{1n}(\tau_0)-\rho_{1n}(t)+\eta_n),$$

where $d_{\tau_0}(\mu) = \begin{cases} 0, & \tau_0=0; \\ \mu^{-1}, & \tau_0>0. \end{cases}$

If $\tau_0 > 0$, then since

(4.7) $$\xi_n(\tau_0-\mu^{-2}) \leq (h(\mu)(1+H_3^{-1})-1/\mu)\vartheta\big(\vartheta-\rho_{1n}(\tau_0-\mu^{-2})\big)^{-1}$$

Let $\mu_3 \geq \mu_2$ so large and for arbitrary $\mu \geq \mu_3$,

(4.8) $$\xi_n(\tau_0-\mu^{-2}) \leq 0 \text{ for } n \geq N_3.$$

In the case, if $\tau = +\infty$ then for $\mu \geq \mu_3$,

(4.9) $$\tau_0+\delta_2 \leq \tau_n \text{ for } \quad n \geq N_3.$$

If we compare $\omega_n(y,t)$ with $v_n(y,t)$ in Ω_n for $\mu \geq \mu_3$, and for $n \geq N_3(\mu,\epsilon)$:

$$\omega_n = f(\eta_n) = \epsilon_n^\gamma \leq v_n \text{ for } (y,t) \in \Gamma_n,$$

$$\omega_n(0,t) \leq \psi_1(\tau_0)-\frac{\epsilon}{2} < \psi_1(t) \leq \psi_{1n}(t) \leq v_n(0,t) \text{ for } \tau_0-d_{\tau_0}(\mu) < t \leq \tau_0+\delta_2.$$

If $\tau_0 = 0$ we also have

$$\omega_n(y,0) \leq f\big(h(\mu)\big) = u_0(0)-\epsilon \leq u_0\big((1-\vartheta^{-1}\rho_{1n}^0)y+\rho_{1n}^0\big)$$

$$\leq v_n(y,0) \text{ for } 0 \leq y \leq \vartheta\big(\vartheta-\rho_{1n}^0(t)\big)^{-1}(h(\mu)-\eta_n).$$

Since the function ω_n is smooth and bounded away from zero in $\overline{\Omega}_n$ by ϵ_n^γ . Suppose that a function $z = v_n - \omega_n$. On a parabolic boundary of Ω_n, we have $z \geq 0$. Then by apply the maximum principle theorem, it follows $z \geq 0$ in $\bar{\Omega}_n$. Let $\mathcal{O} = \{(y,t): 0 < t \leq \tau_0 + \delta_2;\ 0 < y < y_0\}$, where $y_0 \in (0, r_n)$ and $\Omega_n \subset \mathcal{O} \subset \Omega'_n$. Consider

$$\omega'_n(y,t) = \begin{cases} \omega_n(y,t) & in \ \ \bar{\Omega}_n, \\ \epsilon_n^{\gamma} & in \ \ \bar{\mathcal{O}} \backslash \bar{\Omega}_n. \end{cases}$$

Since $v_n \geq \epsilon_n^{\gamma}$ in $\bar{\mathcal{O}}$, we have $\omega'_n(y,t) \leq v_n(y,t)$ in $\bar{\mathcal{O}}$. Then as $n \uparrow +\infty$, we have

(4.10) $$\omega(y,t) \leq v(y,t) \quad \text{in} \quad \bar{\mathcal{O}},$$

where

$$\omega(y,t) = \begin{cases} f\big(h(\mu) - y + \mu(t-\tau_0) + \rho_1(\tau_0) - \rho_1(t)\big), & (y,t) \in \bar{\Omega} \\ 0, & (y,t) \in \bar{\mathcal{O}} \backslash \bar{\Omega} \end{cases}$$

and $\Omega = \big\{(y,t): \tau_0 - d_{\tau_0}(\mu) < t \leq \tau_0 + \delta_2, 0 < y \leq h(\mu) + \mu(t-\tau_0) + \rho_1(\tau_0) - \rho_1(t)\big\}$. Obviously, we have

(4,11) $$\lim_{\substack{(y,t)\to(0,\tau_0) \\ (y,t)\in\bar{\mathcal{O}}}} \omega(y,t) = \lim_{\substack{(y,t)\to(0,\tau_0) \\ (y,t)\in\bar{\Omega}}} \omega(y,t) = \psi_1(\tau_0) - \epsilon.$$

Hence, from (4.10), (4.1) follows.

Let us now prove (4.2) for $\epsilon > 0,\ \psi_1(\tau_0) > 0$. We will estimate the supersolution from the following lemma and thereby complete the proof.

Lemma 4. Let $\psi(\tau_0) > 0$ and $\epsilon \in (0; \psi(\tau_0))$ and $\omega_n(y,t) = f_1(\xi)$ such that

$$f_1(\xi) = [\bar{H}^{\frac{1}{\alpha}} + \xi h^{-1}(\mu)(\bar{H}^{\frac{1}{\alpha}} - H_5^{\frac{1}{\alpha}})]^{\alpha},$$

where $H_5 = \psi_1(\tau_0) + \epsilon,\ \ \mu > 0,\ \ h_1 > 0$ and $0 < \alpha < \min\{m^{-1}; p^{-1}\}$, and

$\xi = h_1(\mu) - \big(1 - \vartheta^{-1}\rho_n(t)\big)y + \mu(t-\tau_0) + \rho_{1n}(\tau_0) - \rho_{1n}(t)$, then we can choose μ is so large such that

(4.12) $$L_n\omega_n > 0 \quad \text{for} \quad 0 \leq \xi \leq (1+\lambda_1)h_1(\mu).$$

Proof of Lemma 4. Let $H' = \Psi(\tau_0 + \delta')$, where $\delta' > 0$ and Ψ is a continuous function at the point $\tau_0 + \delta'$. Let take $\epsilon > 0$ such that $\psi_1(\tau_0) + \epsilon < H'$. If $\tau_0 > 0$ then we choose

$$h_1(\mu) = H_7\mu^{-1}\mathcal{F}(\mu^{-2}), \qquad H_7 = (\bar{H}^{\frac{1}{\alpha}} - H_5^{\frac{1}{\alpha}})(H_5^{\frac{1}{\alpha}} - H_6^{\frac{1}{\alpha}})^{-1},$$

$$H_6 = \psi_1(\tau_0) + \frac{\epsilon}{2}, \quad \mu \geq \mu_0,$$

Where $\mu_0 = \delta_0^{-\frac{1}{2}}$ and let us take that the curve ρ_1 which satisfies the assumption $(\mathcal{L})$ and the condition (2.2) at τ_0 for $\delta \in (0; \delta_0]$. If $\tau_0 = 0$, we choose $h_1(\mu) = \mu^{-2}, \mu \geq \mu_0 = 1$. Consider a function $f_1(\xi)$ for $0 \leq \xi \leq (1 + \lambda_1)h_1(\mu)$, where $\lambda_1 > 0$ such that

$$(\overline{H}^{\frac{1}{\alpha}} - H_5^{\frac{1}{\alpha}})(H_5^{\frac{1}{\alpha}} - H_6^{\frac{1}{\alpha}})^{-1} \leq \lambda_1 \leq (\overline{H}^{\frac{1}{\alpha}} - H_5^{\frac{1}{\alpha}})H_5^{\frac{1}{\alpha}}.$$

Let us estimate

$$L_n\omega_n = \mu f_1' - a(f_1^m)'' + b\left(f_1^p\right)' + cf_1^p - c\theta_c\epsilon_n^{p\gamma}$$

$$\geq \mu h_1^{-1}(\mu)\alpha(H_5^{\frac{1}{\alpha}} - \overline{H}^{\frac{1}{\alpha}})H_{10} + am\alpha(1 - m\alpha)h_1^{-2}(\mu)(\overline{H}^{\frac{1}{\alpha}} - H_5^{\frac{1}{\alpha}})^2 H_9$$

$$+c(1 - \theta_c)\overline{H}^p - c\theta_c\epsilon_n^{p\gamma} + bh_1^{-1}(\mu)\alpha p(\overline{H}^{\frac{1}{\alpha}} - H_5^{\frac{1}{\alpha}})H_{11}^{\frac{\alpha p-1}{\alpha}},$$

Where $H_8 = [H_5^{\frac{1}{\alpha}} - \lambda_1(\overline{H}^{\frac{1}{\alpha}} - H_5^{\frac{1}{\alpha}})]^\alpha > 0, H_9 = \overline{H}^{\frac{m\alpha-2}{\alpha}}, H_{10} = \overline{H}^{\frac{\alpha-1}{\alpha}}$ if $\alpha \geq 1$, or $H_{10} = H_8^{\frac{\alpha-1}{\alpha}}$ if $\alpha < 1$, and $H_9 = \overline{H}^{\frac{p\alpha-2}{\alpha}}$ if $b \geq 0$, or $H_{11} = H_8^{\frac{\alpha p-1}{\alpha}}$ if $b < 0$. Since $h_1\mu \to 0$ as $\mu \uparrow +\infty$ we can choose and fix $\mu \geq \mu_1$ then (3.11) is satisfied. Here we finished proof the lemma.

Let us now return to complete proof of step 1. Since $\psi_1(t)$ is continuous , then for $\mu_2 \geq \mu_1$ and δ_1 such that $\psi_1(t) < \psi_1(\tau_0) + \frac{\epsilon}{2}$ for $\tau_0 - \mu_2^{-2} \leq t \leq \tau_0 + \delta_1$, where if $\tau_0 = \tau < \infty$, we choose $\delta_1 = 0$ and if $\tau_0 < \tau$ then $\delta_1 = \delta_1(\epsilon) \in (0, \delta']$ such that $\tau_0 + \delta < \tau$. If $\tau_0 = 0$, then we choose that $\delta_1 = \delta_1(\epsilon) > 0$ such that $\psi_1(t) < \psi_1(0) + \frac{\epsilon}{2}$ for $0 \leq t \leq \delta_1$. We now estimate $\omega_n(0, t)$ in the neighborhood of τ_0.

we have $\omega_n(0, \tau_0) = \psi_1(\tau_0) - \epsilon$ and for a continuous function f_1 and a uniformly convergent sequence $\{\rho_{1n}\}$ to a continuous function ρ_1 as $n \uparrow +\infty$ for every $\mu \geq \mu_2$ there exists $\delta_2 > 0$ which depends on μ , ϵ, and does not depend on n such that $\delta_2 \leq \delta_1$. Let N_1 be a number which depends on μ , ϵ such that for $n \geq N_1$, $\omega_n(0, t) > \psi_1(\tau_0) + \frac{\epsilon}{2}$ for $\tau_0 \leq t \leq \tau_0 + \delta_2$. We choose $\delta_2 = 0$ if $\tau_0 = T$ and $\delta_2 > 0$ if $\tau_0 < T$. Let suppose that $\tau_0 > 0$ and we consider the function $\omega_n(0, t)$ for $t_0 - \mu^{-2} \leq t \leq \tau_0 + \delta_2, \mu \geq \mu_2$ and $n \geq N_1$. Let consider a uniformly convergent sequence $\{\rho_{1n}\}$ to ρ_1 as $n \uparrow +\infty$, then without loss of generality we suppose that $\omega^-_{\tau_0}(\rho_{1n};\delta)$ is satisfied (2.2) uniformly for $n \geq N_1, 0 < \delta \leq \delta_0$. If $\tau_0 - \mu^{-2} \leq t \leq \tau_0$, then we have

$$\omega_n(0,t) \geq f_1\big(h(\mu) + \rho_{1n}(\tau_0) - \rho_{1n}(t)\big) \geq f_1\big((H_7^{-1}+1)h_1(\mu)\big) = \psi_1(\tau_0) - \frac{\epsilon}{2}.$$

We can choose $N_2 = N_2(\mu, \epsilon) \geq N_1$ is so large , then for $n \geq N_2$,

$$\psi_1(t) \leq \psi_{1n}(t) < \psi_1(0) + \tfrac{\epsilon}{2} \text{ for } \tau_0 - \mu^{-2} \leq t \leq \tau_0 + \delta_2$$

If $\tau_0 = 0$ we choose and fix numbers $\mu_2 \geq \mu_1$ and $N_2 \geq N_1$ so large that if $\mu \geq \mu_2$, and $n \geq N_2$, then

$$u_0\big((1-\rho_{1n}^0\vartheta^{-1})y + \rho_{1n}^0\big) \leq u_0\big((1-\rho_{1n}^0\vartheta^{-1})y + \rho_{1n}^0\big) + K\epsilon_n^\gamma$$

$$< u_0(0) + \epsilon \text{ for } 0 \leq y \leq \vartheta(\vartheta - \rho_{1n}^0)^{-1}h_1(\mu).$$

As before, consider the sets $\Omega_n, \Gamma_n, \xi_n$, then we replace η_n and h with 0 and h_1, respectively. We can derive (4.7)-(4.9), replacing N_3 and H_3 with N_2 and H_7 respectively.

If we compare $\omega_n(y,t)$ with $\upsilon_n(y,t)$ in Ω_n for $\mu \geq \mu_3$, and for $n \geq N_2(\mu,\epsilon)$,

$$\omega_n(0,t) > \upsilon_n(0,t) \text{ for } \tau_0 - d_{\tau_0}(\mu) < t \leq \tau_0 + \delta_2.$$

$$\omega_n = \bar{M} = \Psi(\tau_0 + \delta') \geq \Psi(\tau_0 + \delta_2) \geq \upsilon_n \text{ for } (y,t) \in \bar{\Gamma}_n,$$

If $\tau_0 = 0$ we also have

$$\omega_n(y,0) \leq f\big(h(\mu)\big) = u_0(0) - \epsilon \leq u_0\big((1-\vartheta^{-1}\rho_{1n}^0)y + \rho_{1n}^0\big)$$

$$\leq \upsilon_n(y,0) \text{ for } 0 \leq y \leq \vartheta\big(\vartheta - \rho_{1n}^0(t)\big)^{-1}(h(\mu) - \eta_n).$$

Suppose that a function $z = \upsilon_n - \omega_n$. On a parabolic boundary of Ω_n, we have $z \leq 0$. Then by apply the maximum principle theorem, it follows $z \leq 0$ in $\bar{\Omega}_n$. As before, consider $\mathcal{O}$, where $y_0 \in (0, r_n)$ and $\Omega_n \subset \mathcal{O} \subset \Omega_n'$. Consider

$$\omega_n'(y,t) = \begin{cases} \omega_n(y,t) & in \ \bar{\Omega}_n, \\ \bar{H} & in \ \bar{\mathcal{O}} \backslash \bar{\Omega}_n. \end{cases}$$

Since $\upsilon_n \leq \bar{H}$ in $\bar{\mathcal{O}}$, we have $\omega_n'(y,t) \geq \upsilon_n(y,t)$ in $\bar{\mathcal{O}}$. Then as $n \uparrow +\infty$, we have

(4.13) $$\omega(y,t) \geq \upsilon(y,t) \quad \text{in} \quad \bar{\mathcal{O}},$$

where

$$\omega(y,t)=\begin{cases} f_1\big(h_1(\mu)-y+\mu(t-\tau_0)+\rho_1(\tau_0)-\rho_1(t)\big), & (y,t)\in\bar{\Omega} \\ \bar{H}, & (y,t)\in\bar{O}\backslash\bar{\Omega}\end{cases}$$

and Ω is defined before. Obviously, we have (4.11) is valid. Hence, from (4.13), (4.2) follows. Therefore, the proof of continuity of the function υ at the point $(0,\tau_0)$ is valid.

Step 2. We want to prove that υ is continuous on $y=\vartheta\,;t\geq 0$, we shall prove that the following two inequalities are valid for $\tau_0\geq 0$, $\psi_2(\tau_0)>0$

(4.14) $$\lim_{(y,t)\to(\vartheta,\tau_0)} inf\; \upsilon(y,t)\;\geq\;\psi_2(\tau_0)-\epsilon\,,$$

(4.15) $$\lim_{(y,t)\to(\vartheta,\tau_0)} sup\; \upsilon(y,t)\geq\;\psi_2(\tau_0)-\epsilon\,.$$

Since $\epsilon>0$ is an arbitrary real number, then υ is continuous at the boundary points (ϑ,τ_0) which comes from (4.14), (4.15). If $\psi_2(\tau_0)=0$, and since (4.14) with $\epsilon=0$ in the lower bound inequality directly comes from that υ is non-negative in $\bar{\Omega}'$, then it is the same way to prove (4.15). If $0<\epsilon<\psi_2(\tau_0)$, $\psi_2(\tau_0)\;>0$, we prove that the inequality (4.14) in similar way to (4.1). Let us consider the following function

$$\omega_n(y,t)=f\big(\vartheta^{-1}(\rho_{2n}(t)-\rho_{1n}(t))y+h(\mu)+\mu(t-\tau_0)+\rho_{2n}(t)-\rho_{1n}(\tau_0)\big),$$

where $\quad f(\zeta)=H_1\zeta^{\alpha}h^{-\alpha}(\mu),\quad H_1=\psi_2(\tau_0)-\epsilon,\;\mu>0,\;h>0$

Depending on the appropriate value of α, this case is divided into several cases as in step 1(see, Figure1). Let $H_i;i=1,2,3;$ and h be chosen as in proof the inequality (4.1), (only replace $\psi_1(t)$ by $\psi_2(t)$) and similarly we get the following estimation

$$\omega_n'(y,t)\leq \upsilon_n(y,t)\quad in\;\;\bar{\Omega}_n'$$

Consider

$$\omega_n'(y,t)=\begin{cases}\omega_n(y,t) & in\;\;\bar{\Omega}_n, \\ \epsilon_n^{\gamma} & in\quad \bar{\Omega}_n'\backslash\bar{U}_n.\end{cases}$$

$$\zeta_n(t)=\vartheta\big(\rho_{2n}(t)-\rho_{1n}(t)\big)^{-1}[-h(\mu)-\mu(t-\tau_0)+\eta_n+\rho_{2n}(\tau_0)-\rho_{1n}(t)]$$

And $d_{\tau_0}(\mu),\eta_n$ are defined as before. Since ρ_2 satisfies (2.3), for a fixed number $\mu>0$ which is so large , then $\exists\; N$ depends on μ, such that

$$\zeta_n(\tau_0 - 1/\mu^2) > \vartheta \text{ for } n \geq N.$$

Then as $n \uparrow +\infty$, we have

$$\omega(y,t) \leq \upsilon(y,t) \quad \text{in} \quad \bar{\Omega}, \tag{4.16}$$

where

$$\omega = \begin{cases} f\left(h(\mu) + \vartheta\big(\rho_2(t) - \rho_1(t)\big)^{-1} + \mu(t-\tau_0) + \rho_2(\tau_0) - \rho_1(t)\right), (y,t) \in \bar{\mathcal{U}} \\ 0, \qquad\qquad\qquad\qquad\qquad\qquad (y,t) \in \bar{\Omega}'\backslash\bar{\mathcal{U}} \end{cases}$$

$$\mathcal{U} = \left\{(y,t): \tau_0 - d_{\tau_0}(\mu) < t \leq \tau_0 + \delta_3, \qquad \zeta(t) < y \leq \vartheta\right\}$$

$$\zeta(t) = \vartheta\big(\rho_2(t) - \rho_1(t)\big)^{-1}[-h(\mu) - \mu(t-\tau_0) + \rho_2(\tau_0) - \rho_1(t)]$$

Obviously, we have

$$\lim_{\substack{(y,t)\to(\vartheta,\tau_0) \\ (y,t)\in\overline{\Omega'}}} \omega(y,t) = \lim_{\substack{(y,t)\to(\vartheta,\tau_0) \\ (y,t)\in\bar{\mathcal{U}}}} \omega(y,t) = \psi_2(\tau_0) - \epsilon.$$

Hence, from (4.16), then the estimation (3.7) is valid. The proof of (4.15) is in similar way as we prove (4.14) so, the proof of continuity of the function υ at the boundary point (ρ_2, t) is valid.

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A new conjugate gradient method for solving a large scale systems of monotone equations.

H A Wasi [1] and M A K Shiker [1, 2]

[1]Mathematics Department, College of Education for Pure Sciences, University of Babylo,Iraq.

Email: [1]mathman1972@gmail.com , [2]mmttmmhh@yahoo.com

[2]Corresponding Author

Abstract. The conjugate gradient method (CGM) is one of the best algorithms that used to solve and minimize constrained optimization problems. In this paper we suggest a new algorithm to solve large-scale nonlinear systems of monotone equations. The suggested method has some advantages such as it doesn't need the Jacobian matrix data nor store matrices at each iteration, also this method has the ability to solve large-scale problems with non-smooth property. With standard conditions, we established the global convergence for the proposed method. The numerical experiment shoes that the new method is promised and efficient by comparing with other famous methods.

Keywords: System of monotone equations, Conjugate gradient method and Global convergence.

1. Introduction

CGM is generally used to solve large-scale problems such as image processing, density physics and environmental science [1]. It is distinguished from the other numerical methods using to solve nonlinear systems of equations, because it is quick to calculate, needs very low memory and also does not need the Hessian matrix for objective functions [2, 3, 4, 5].

In general, the conjugate gradient direction d_k has the following formula

$$d_k = \begin{cases} -F_k + \beta_k d_{k-1} \dots \dots if\ k \geq 1 \\ -F_k \dots \dots \dots \dots \dots \dots if\ k = 0 \end{cases} \tag{1.1}$$

Where β_k is a parameter, its value determines the different conjugate gradient Algorithms.

Consider the following system of monotone equations:

$$F(x) = 0 \ , x \in \Omega \tag{1.2}$$

Where $F: R^n \to R^n$ is a continuous monotonic function, i.e.

$$\langle F(x) - F(y), x - y \rangle \geq 0, \quad \forall\ x, y \in R^n \tag{1.3}$$

Many effective methods have been proposed to solve (1.2) [6] based on the following unconstrained optimization problem:

$$min_{x\in R^n} f(x) = \frac{1}{2} \|F(x)\|^2 \quad (1.4)$$

Where $f : R^n \to R^n$ is continuous and differentiable function. The optimization techniques are iterative, that it depends on the current iteration to find the next one. Most of researchers used the line techniques to find the next iteration. In general, the line search principal regularly takes the following iterative scheme:

$$x_{k+1} = x_k + \alpha_k d_k$$

(1.5)

Where x_k is the current iterate point, d_k is the search direction and $\alpha_k > 0$ is the step length along d_k.

The system of monotone equations operates widely in the case of unconstrained equations, i.e. when $R^n = \Omega$, where Ω is a set of possible solutions, it is non-empty closure convex set.

Over the years, this topic of CG has received a lot of interests and has many applications. This interest has increased sharply in recent years. In 2013, Yunhai Xiao and Hong Zhu [2] introduced a conjugate gradient method to solve convex constrained monotone equations with applications in compressive sensing. In 2018, M. A. Shiker and K. Amini introduced a new CG direction and comparing it with a new projection-based algorithm to solve a large-scale nonlinear System of monotone equations [7].

The new CG algorithm for solving (1.2) is established from the famous solver CG descent [8, 3] and the algorithm of Gonglin Y. and Zengxin W. [4]. Other parts of this paper are structured as follow: In Section 2, we build the new algorithm. Section 3 shows the globally converges of the new algorithm. And in section 4, we show the Numerical results. Finally, we clarified the conclusions. Through paper, $\|\cdot\|$ means the normed vector in Euclidean space.

2. The New Algorithm

Here, in this part of the paper we will build our suggestion through the following steps. For solution (1.2) we will suggest the search direction imposing it in the following formula

$$d_k = \begin{cases} (-F_k + \beta_k . w_k - \vartheta_k . y_k)/ \|w_k\|^2 + \beta_k & if\ k \geq 1 \\ -F_k & if\ k = 0 \end{cases},$$

(2.1)

where:

$$F_k = F(x_k),\ y_k = F_{k+1} - F_k$$

(2.2)

$$w_k = x_{k+1} - x_k \quad \text{and} \quad \vartheta_k = \|w_k\|^2 {F_k}^t y_k / \|Fk\|^2$$

$$\beta_k = \frac{F(_{k+1})^T . (F(_{k+1}) - F(_k))}{\|F(_k)\|^2}$$

(2.3)

Note that through the premise that d_k is considered as a descent direction of the function f at the point x_k, this property is very important and necessary for any iterative algorithm to be convergence

[9, 10]. In order to ensure that d_k satisfies our hypothesis in (2.1), it must be satisfy the following property:

2.1. Lemma

Let $\{d_k\}$ be the sequence generated by (2.1) and assume that ($\|w_k\|^2 + \beta_k \neq 0$), then for every $k > 1$, It holds that

$$|F_k d_k| \geq -\frac{5}{8}\|F_k\|^2 \qquad (2.4)$$

Proof

Theorem 1.1 in [11] holds independent of the definition of y_k. The assertion of this lemma is proving directly. □

To demonstrate our algorithm we use the projection operative $P_\Omega[.]$ which is known as a mapping from R^n to Ω , it is defined as follows:

$$P_\Omega[x] = argmin\{\|y - x\| : y \in \Omega\}, \forall x \in R^n$$

And it is satisfy the following inequality:

$$\|P_\Omega[x] - P_\Omega[y]\| \leq \|x - y\|, \forall x, y \in R^n \; .$$

Now we state the new Algorithm.

2.2. Algorithm

Step 1. Select a randomly initial point $x_0 \in \Omega$, and:

$\rho \in (0,1), \sigma \in (0,1), \epsilon = 0.000001$. Set k =0.

Step 2. If $F(x_k) = 0$, Stop, Else, calculate d_k by (2.1).

$$\alpha_k = \mu , \qquad \text{where} \quad \mu = \left|\frac{F(x_k)^T d_k}{{d_k}^T \nabla F(x_k) d_k}\right|.$$

Step 3. find α_k which satisfy:

$$(F(x_k + \alpha_k d_k)^T d_k \leq \sigma \alpha_k \|d_k\|)/10 \qquad (2.5)$$

Set: $x_{k+1} = x_k + \alpha_k d_k$

Step 4. if $F(x_{k+1})$ = 0 then stop.

Otherwise compute $x_{k+1} = x_k + \alpha_k d_k$, where $\alpha_k = \rho \alpha_k$

Step 5. Set $k := k + 1$. Go to Step 2.

3. Convergence Analysis

In this section, we will prove for the globally convergence for Algorithm 2.2, for this purpose we need the following assumptions:

3.1. *Assumptions*

- A1 Let F is Lipchitz continuous on Ω, i.e., $\exists$ a positive number L > 0, such that

$$\|F[x] - F[y]\| \leq L\|x - y\|, \forall x, y \in \Omega \quad (3.1)$$

- A2 The solution set of (1.2) is non-empty, the next result displays that Algorithm 2.2 is well-defined.

3.2. *Lemma.*

Assume that the assumption A1 is hold, there exists appositive scalar α_k that satisfies

$$(F(x_k + \alpha_k d_k)^T d_k \leq \sigma\alpha_k \|d_k\|)/10$$

Proof

Consider the algorithm stop in $F(z_k)$ =0, where z_k , x_k is solutions.

Suppose $F(z_k) \neq 0$ for each k then $d_k \neq 0$.

From (2.5), $z_k = x_k + \alpha_k d_k$ is a solution.

Now if

$$\alpha_k \neq \rho \left|\frac{F(x_k)^T d_k}{{d_k}^T \nabla F(x_k) d_k}\right|$$

Then α_k not satisfy the line search condition i.e.

$$F(z_k)^T d_k \geq \sigma\alpha_k \|d_k\|, \quad (3.2)$$

then

$$2\, F(z_k)^T d_k \geq \alpha_k \sigma_k \|F_k\| \|d_k\|^2.$$

Since

$$\beta_k = \frac{F(_{k+1})^T.(F(_{k+1}) - F(_k))}{\|F(_k)\|^2},$$

and $\rho > 0$, from (3.2) and by above assumption we get $F(z_k) d_k > 0$.

Now, if $2F(z_k)^T d_k = \alpha_k \sigma_k \|F_k\| \|d_k\|^2$, that is mean

$$\alpha_k = min\left\{\rho, \left|\frac{F(x_k)^T d_k}{{d_k}^T \nabla F(x_k) d_k}\right|\right\}.$$

So, the suggested line search is will define. □

3.3. Lemma

Assume that the assumptions A1 and A2 hold, and $\{x_k\}$ be the sequence generated by the Algorithm (2.2**)**, then for any positive M > 0 we get:

$$\|F(x_k)\| \leq M \ .$$

Proof

For all $x \in S$, from the non-expansiveness of the projection hand, satisfy

$$\begin{aligned}
\|x_{k+1} - \bar{x}\|^2 &= \|P_\Omega[x_k - \alpha_k F(z_k)] - \bar{x}\|^2 \\
&\leq \|x_k - \alpha_k F(z_k)\|^2 \\
&= \|x_k - \bar{x}\|^2 - 2\alpha_k \langle F(z_k), x_k - \bar{x}\rangle + {\alpha_k}^2 \ \|F(z_k)\|^2 \\
&\leq \|x_k - \bar{x}\|^2 - 2\alpha_k \langle F(z_k), x_k - z_k\rangle + {\alpha_k}^2 \ \|F(z_k)\|^2 \\
&= \|x_k - \bar{x}\|^2 - \langle F(z_k), x_k - z_k\rangle^2 \ \|F(z_k)\|^2 \\
&\leq \|x_k - \bar{x}\|^2 \ ,
\end{aligned}$$

which implies that $\|x_k - \bar{x}\| \leq \|x_0 - \bar{x}\|$.

From (3.2), for each k, we have

$$\|F(x_k)\| = \|F(x_k) - F(\bar{x})\| \leq L\|x_k - \bar{x}\| \leq L\|x_0 - \bar{x}\| \ .$$

Suppose M = $L\|x_0 - \bar{x}\|$, then (3.3) is proved clearly. □

3.4. Theorem

Consider that the assumptions A1 and A2 are satisfied and the sequence $\{x_k\}$ is generated by Algorithm (2.2) then:

$$lim_{k\to\infty} \, inf \, \|F_k\| = 0 \qquad (3.3)$$

Proof

If (3.3) unrealized, then for $\epsilon > 0$ it satisfies that:

$$\| \ F_k \ \| \geq \epsilon, \ \ \forall \ \ k \geq 0. \qquad (3.4)$$

Let

$$\begin{aligned}
\|d_k\| &= \|d_k - F_k + F_k\| \\
&\geq (\|d_k - F_k\| - \|F_k\|) \\
&\geq -2(\|d_k - F_k\|) - \|F_k\| \\
&\geq \tfrac{5}{4}\|F_k\| - \|F_k\| \\
&\geq \tfrac{1}{4}\|F_k\|
\end{aligned}$$

Hence $\|d_k\| \geq \frac{1}{4}\|F_k\|$, then

$$\|d_k\| \geq \frac{1}{4}\epsilon, \ \forall \, k \geq 0. \quad (3.5)$$

Since

$$\beta_k = \frac{F(_{k+1})^T.(F(_{k+1}) - F(_k))}{\|F(_k)\|^2}$$

Then

$$|\beta_k| = \left|\frac{F(_{k+1})^T.(F(_{k+1}) - F(_k))}{\|F(_k)\|^2}\right|$$

$$\leq |F(_{k+1})^T.(F(_{k+1}) - F(_k))|/\|F_k\|^2$$

$$\leq \|F(_{k+1})^T\| * \|F(_{k+1}) - F(_k)\|/\|F_k\|^2$$

$$\leq \epsilon\|F_k - F_{K1}\|/\|F_k\|^2 \quad (3.6)$$

Now

$$\|d_k\| \leq \|\beta_k\|.\|d_{k-1}\| + \|F_k\| \quad (3.7)$$

From above relations and Lemma 3.2, we get $\| \, d_k \, \| \leq C$, where C is a positive scalar.

By inequalities (3.4), (3.5) and (3.6) there is a contradiction about inequality (3.7). So, (3.3) holds and the theorem is proved. □

4. Numerical Results

In this section, we will compare our algorithm (H_1) with three famous algorithms used to solve large scale systems of monotone equations.

The experiments were run on a PC with CPU 2.20 GHz and 8 GB RAM. The codes were written in MATLAB R2014 a programming environment. For high accuracy to all test problems, the termination condition is $\| \, F(x_k) \, \| \leq 10^{-5}$, or the total number of iterates exceeds 500000. The problems (1- 4) were taken from Qingna L. and Dong H. L. [8], Problems (5- 6) were taken from Wanyou C. [9] and problem (7) were taken from Qin R.Y. et.al [12].

We compare the new Algorithm (H_1) with the following three famous:

QD: This Algorithm introduced by Qingna L. and Dong H. L. [8].

GX: This Algorithm introduced by Gonglin Y. et.al. [4].

ZZ: This Algorithm introduced by Zhen S. Y. and Zhan H. L. [3].

The parameters are definite as follows: $\rho = 0.5, \ \sigma = 0.1, \ \epsilon = 0.000001$. We took $\mu = \left|\frac{F(x_k)^T d_k}{d_k{}^T \nabla F(x_k) d_k}\right|$ as the initial trial parameter.

The results of tested Algorithms are listed in the following tables. Table 1 contains the functions evaluations and iterations that have occurred by each Algorithm to solve each problem. While Table 2 contains the time that each algorithm took to solve each problem.

Table 1: Functions evaluations (f eval) and iterations (iter).

problem	**Dim**	**f eval**				**Iter**			
		(H_1)	**QD**	**GX**	**ZZ**	**(H_1)**	**QD**	**GX**	**ZZ**
P1	500000	168	443	1500009	216	20	110843	500001	24
	500000	778	100589	1500014	292	62	25161	500001	29
	500000	131	2645	1500045	131	10	661	500001	10
	500000	318	2672	1500017	189	50	668	500001	27
P2	500000	76	31	15	204	5	7	4	14
	500000	56	31	38	180	5	7	5	13
	500000	76	25	9	76	5	5	2	5
	500000	76	25	32	76	5	5	3	5
P3	500000	76	31	15	76	5	7	4	5
	500000	57	93	45	57	5	20	6	5
	500000	76	25	9	76	5	5	2	5
	500000	77	74	41	77	5	16	5	5
P4	500000	70	56555	27143	116	3	14133	9040	5
	500000	70	70415	24398	116	3	17598	8125	5
	500000	70	32755	281363	116	3	8183	93780	5
	500000	70	51551	525401	116	3	12882	175126	5
P5	500000	783	150	2158	*	76	29	622	*
	500000	108	253	2216	*	26	48	632	*
	500000	477	89	2137	*	47	19	619	*
	500000	72	108	2214	*	17	23	633	*
P6	500000	327	270	352	959	78	68	114	125
	500000	371	258	105	1014	89	64	32	130
	500000	379	259	136	958	91	64	42	98
	500000	508	258	241	1065	123	64	77	127
P7	500000	88	2277	*	*	8	238	*	*
	500000	58	4186	*	*	5	406	*	*
	500000	57	417	*	*	5	55	*	*
	500000	58	1045	*	*	5	121	*	*

Table 2: CPU-Time (in seconds)

problem	Dim	CPU-Time			
		(H_1)	**QD**	**GX**	**ZZ**
P1	500000	0.781	1132.218	4195.515	0.718
	500000	2.109	263.890	4165.875	0.906
	500000	0.312	6.890	3719.375	0.406
	500000	0.890	6.781	4067.031	0.593
P2	500000	1.750	0.140	0.026	4.234
	500000	1.218	0.156	0.171	3.796
	500000	1.218	0.093	0	1.609
	500000	1.781	0.093	0.078	1.593
P3	500000	1.765	0.171	0.078	1.781
	500000	1.234	0.390	0.171	1.312
	500000	1.796	0.140	0.031	1.781
	500000	1.828	0.265	0.156	1.890
P4	500000	0.265	211.500	100.734	0.375
	500000	0.250	293.343	92.781	0.390
	500000	0.250	142.656	1180.796	0.421
	500000	0.250	234.125	2457.375	0.421
P5	500000	0.609	0.093	1.703	*
	500000	0.109	0.218	1.578	*
	500000	0.359	0.062	1.658	*
	500000	0.031	0.109	1.562	*
P6	500000	0.156	0.125	0.218	0.515
	500000	0.171	0.140	0.062	0.562
	500000	0.187	0.140	0.062	0.515
	500000	0.265	0.125	0.140	0.5310
P7	500000	1.921	15.375	*	*
	500000	1.046	27.593	*	*
	500000	1.296	2.92187	*	*
	500000	1.421	7.171	*	*

The results in above tables show the efficiency of the new Algorithm (H_1) comparing with the three another methods in all areas of comparison (number of functions evaluations, number of iterations and CPU time). Few results may appear to be less efficient of the new algorithm than the other methods, but it is generally better than other methods in a final outcome.

5. Conclusions

In this paper there is a modest contribution for solving constrained convex monotony equations and global convergence has been proved. The new method (H_1) is very suitable for solving such problems as it does not require neither high memory nor Jacobian data and need little storage for matrices from the iterations process.

The preliminary numerical results indicate that the new method is efficient and promised, that is we compare it with three famous algorithms according to the number of function evaluations (f eval), number of iterations (Iter) and CPU time that every algorithm needs to find the solution of the given problems. The results listed in above tables show that the new algorithm, in general, needs less number of (f eval), (Iter) and CPU time comparing with the other three famous methods, which ensure that (H_1) is very efficient and promised to solve large scale systems of monotone equations.

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Jordan Left Derivation and Centralizer on Skew Matrix Gamma Ring

Rajaa .C.Shaheen

Mathematics Department , Education College ,Al-Qadisiyah University

E-mail:rajaa.chaffat@qu.edu.iq

Abstract

We define Skew matrix Gamma ring and describe constitute of Jordan left Centralizers and derivations of a Skew matrix gamma ring $M_2(M,\Gamma;\sigma,q)$ on a Γ -ring M .

Keywords: Skew matrix ring, Gamma ring ,Jordan left centralizer and Jordan left derivation

1-Introduction

A linear mappings $\mathfrak{D}:\mathcal{R}\to\mathcal{R}$ is said to be a left derivation (resp.,Jordan left derivation)if $\mathfrak{D}$ (ab)=a $\mathfrak{D}$ (b)+b $\mathfrak{D}$ (a) $\forall$ a,b$\in$ $\mathcal{R}$ (if $\mathfrak{D}$ $(a^2) = 2a\mathfrak{D}(a)$ $\forall$a$\in$ $\mathcal{R}$. Bresar and Vukman [3] introduced concept of left derivation and Jordan left derivation .We refer the readers to [4,6 ,10,11] for result concerning Jordan left derivations .A linear mappings $\mathcal{T}:\mathcal{R}\to\mathcal{R}$ is called Jordan left centralizers(resp., left centralizers) if $\mathcal{T}(x^2) = \mathcal{T}(x)x$ (resp.,$\mathcal{T}$ $(xy) = \mathcal{T}$ $(x)y$ $\forall$ x,y $\in$ $\mathcal{R}$). A linear maps $\mathcal{T}:\mathcal{R}\to\mathcal{R}$ is called a Jordan centralizers if $\mathcal{T}$ satisfies $\mathcal{T}$ $(x\ y + yx)$= $\mathcal{T}$ $(x)\,y$+ $y\,\mathcal{T}$ (x)= $\mathcal{T}$ $(y)\,x$+ $x\,\mathcal{T}$ (y) $\forall$ x,y $\in$ $\mathcal{R}$. In [12,13,14] some result about left centralizer .In [5] Hamaguchi ,give a sufficient and necessary conditions for J : $M_2(\mathcal{R};\sigma,q) \to M_2(\mathcal{R};\sigma,q)$ being Jordan derivation and prove that there exist many Jordan derivations of it which are not derivations and indicate to the characterization of derivation on $M_2(\mathcal{R};\sigma,q)$,and Jordan derivation of $M_2(\mathcal{R})$ with invariant ideal. Nobusawa [8] introduced the concept of gamma ring which generalized by Barnes [1] as follows

Let M and Γ with + ,abelian groups ,M is called a Γ-ring if for any x,y,z $\in$M and $\alpha,\beta\in\Gamma$, the following satisfied

(1) $x\ \alpha y$ $\in$M

(2) $(x + y)\ \alpha,z$ =$x\ \alpha,z$+$y\ \alpha,z$

$x\ (\alpha+\beta)\ z$ =$x\ \alpha\ z$ +$x\ \beta\ z$

$x\ \alpha(y + z)$=$x\ \alpha\ y$ +$x\ \alpha\ z$

(3) $(x\ \alpha y)\beta\ z$ =$x\ \alpha(y\beta z)$

In[7] Majeed and Shaheen described form of Jordan left derivation and Centralizers on $M_2(\mathcal{R};\sigma,q)$. In this article, we define Skew matrix Gamma ring, describe constitue Jordan left centralizers and derivation of a Skew matrix Gamma ring $M_2(M,\Gamma;\sigma,q)$ on a Γ -ring M .Now, we shall recall some definitions which are basic in this paper .

Definition 1.1 **:-[9]**

Let $\mathcal{R}$ be a ring ,$q \in \mathcal{R}$ and endomorphism $\sigma: \mathcal{R} \to \mathcal{R}$ such that $\sigma(q) = q$ and $\sigma(r)q = qr$ $\forall r \in \mathcal{R}$.

Let $M_2(\mathcal{R}; \sigma, q)$be the set of 2×2 matrices on $\mathcal{R}$ with the following multiplication

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix} = \begin{bmatrix} x_1y_1 + x_2y_3q & x_1y_2 + x_2y_4 \\ x_3\sigma(y_1) + x_4y_3 & x_3\,\sigma(y_2)q + x_4y_4 \end{bmatrix}$$

and usual addition . $M_2(\mathcal{R}; \sigma, q)$ is said a skew matrix ring over $\mathcal{R}$.

In this article ,we define Skew Matrix Gamma ring as follows

Definition 1.2 **:-**Skew Matrix Gamma ring

Let M be a gamma ring ,$q \in M$ and σ :$M \to M$ such that $\sigma(q) = q$ and $\sigma(r)\alpha\, q = q\alpha r$ $\forall r \in M, \alpha \in \Gamma$.Let $M_2(M; \Gamma, \sigma, q)$be the set of 2×2 matrices over M with the following multiplication

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \alpha \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix} = \begin{bmatrix} x_1\alpha y_1 + x_2\alpha y_3\alpha q & x_1\alpha y_2 + x_2\alpha y_4 \\ x_3\alpha\sigma(y_1) + x_4\alpha y_3 & x_3\,\alpha\sigma(y_2)\alpha q + x_4\alpha y_4 \end{bmatrix}$$

and usual addition $M_2(M; \Gamma, \sigma, q)$ is said a skew matrix Γ −ring over M .

Note that the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by $e_{11}\,a + e_{12}b + e_{21}c + e_{22}d$.

2- **on Skew Matrix Gamma ring with Jordan Left Derivation**

We shall describe constitute of Jordan left derivation of skew matrix Gamma ring .

Let D be a Jordan left derivation of $M_2(M; \Gamma, \sigma, q)$. First ,we set

$$D(e_{11}\,a) = \begin{bmatrix} \delta_1(a) & \delta_2(a) \\ \delta_3(a) & \delta_4(a) \end{bmatrix}, D(e_{12}\,b) = \begin{bmatrix} k_1(b) & k_2(b) \\ k_3(b) & k_4(b) \end{bmatrix}$$

$$D(e_{21}c) = \begin{bmatrix} l_1(c) & l_2(c) \\ l_3(c) & l_4(c) \end{bmatrix}, D(e_{22}\,d) = \begin{bmatrix} h_1(d) & h_2(d) \\ h_3(d) & h_4(d) \end{bmatrix}$$

Where $\delta_i, h_i, k_i, l_i: M \to M$ are linear mapping .

Lemma2.1:- For every a∈M, $\alpha \in \Gamma$

$1 - \delta_1, \delta_2$ are Jordan left derivation of M .

$2 - \delta_3(a\alpha a) = 0$

$3 - \delta_4(a\alpha a) = 0$

Proof:-since

$$D(e_{11}a\alpha a) = 2e_{11}a\,\alpha\, D(e_{11}a)$$

$$\begin{bmatrix} \delta_1(a\alpha a) & \delta_2(a\alpha a) \\ \delta_3(a\alpha a) & \delta_4(a\alpha a) \end{bmatrix} = 2\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \alpha \begin{bmatrix} \delta_1(a) & \delta_2(a) \\ \delta_3(a) & \delta_4(a) \end{bmatrix}$$

$$\begin{bmatrix} \delta_1(a\alpha a) & \delta_2(a\alpha a) \\ \delta_3(a\alpha a) & \delta_4(a\alpha a) \end{bmatrix} = \begin{bmatrix} 2a\alpha\delta_1(a) & 2a\alpha\delta_2(a) \\ 0 & 0 \end{bmatrix}$$

Then $\delta_1(a\alpha a) = 2a\alpha\delta_1(a), \delta_2(a\alpha a) = 2a\alpha\delta_2(a), \delta_3(a\alpha a) = 0\ and\ \delta_4(a\alpha a) = 0$.

Lemma2.2 :- For every d∈M, $\alpha \in \Gamma$

1-$h_3, h_4$are Jordan left derivation of M .
2-$h_1(d\alpha d) = 0$
3-$h_2(d\alpha d) = 0$
Proof:-Since

$$D(e_{22}d\alpha d) = 2e_{22}d\,\alpha D(e_{22}d)$$

$$\begin{bmatrix} h_1(d\alpha d) & h_2(d\alpha d) \\ h_3(d\alpha d) & h_4(d\alpha d) \end{bmatrix} = 2\begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \alpha \begin{bmatrix} h_1(d) & h_2(d) \\ h_3(d) & h_4(d) \end{bmatrix}$$

$$\begin{bmatrix} h_1(d\alpha d) & h_2(d\alpha d) \\ h_3(d\alpha d) & h_4(d\alpha d) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2d\alpha h_3(d) & 2d\alpha h_4(d) \end{bmatrix}$$

then $h_3(d\alpha d) = 2d\alpha h_3(d)$, $h_4(d\alpha d) = 2d\alpha h_4(d)$, $h_1(d\alpha d) = 0$ and $h_2(d\alpha d) = 0$..

Lemma 2.3 :- For any a,b∈ M, $\alpha \in \Gamma$

1- $k_1(a\alpha b) = 2a\alpha k_1(b) + 2b\alpha\delta_3(a)\alpha q$
2- $k_2(a\alpha b) = 2a\alpha k_2(b) + 2b\alpha\delta_4(a)$
3- $k_3(a\alpha b) = 0$
4- $k_4(a\alpha b) = 0$

Proof:- Since

$$D(e_{12}a\alpha b) = D(e_{11}a\alpha e_{12}b + e_{12}b\alpha e_{11}a)$$

$$\begin{bmatrix} k_1(a\alpha b) & k_2(a\alpha b) \\ k_3(a\alpha b) & k_4(a\alpha b) \end{bmatrix} = 2e_{11}a\alpha D(e_{12}b) + 2\,e_{12}b\,\alpha\,D(e_{11}a)$$

$$= 2\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \alpha \begin{bmatrix} k_1(b) & k_2(b) \\ k_3(b) & k_4(b) \end{bmatrix} + 2\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \alpha \begin{bmatrix} \delta_1(a) & \delta_2(a) \\ \delta_3(a) & \delta_4(a) \end{bmatrix}$$

$$= \begin{bmatrix} 2a\alpha\, k_1(b) & 2a\,\alpha k_2(b) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2b\alpha\,\delta_3(a)\alpha q & 2b\,\alpha\delta_4(a) \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2a\,\alpha k_1(b) + 2b\,\alpha\delta_3(a)\alpha q & 2a\alpha\, k_2(b) + 2b\,\alpha\delta_4(a) \\ 0 & 0 \end{bmatrix}$$

Lemma 2.4 :-For every c, d∈ M, $\alpha \in \Gamma$

1-$l_1(d\alpha c) = 0$

2-$l_2(d\alpha c) = 0$

3-$l_3(d\alpha c) = 2d\alpha l_3(c) + 2c\alpha\sigma(h_1(d))$

4-$l_4(d\alpha c) = 2d\alpha l_4(c) + 2c\alpha\sigma(h_2(d))\alpha q$

Proof:-Since

$$D(e_{21}\, d\alpha c) = D(e_{22}\, d\alpha e_{21} c + e_{21} c\alpha e_{22}\, d)$$

$$\begin{bmatrix} l_1(d\alpha c) & l_2(d\alpha c) \\ l_3(d\alpha c) & l_4(d\alpha c) \end{bmatrix} = 2e_{22}\, d\alpha D(e_{21}c) + 2e_{21}c\alpha D\,(e_{22}\, d)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 2d \end{bmatrix} \alpha \begin{bmatrix} l_1(c) & l_2(c) \\ l_3(c) & l_4(c) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2c & 0 \end{bmatrix} \alpha \begin{bmatrix} h_1(d) & h_2(d) \\ h_3(d) & h_4(d) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 2d\alpha l_3(c) & 2d\alpha l_4(c) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2c\alpha\sigma(h_1(d)) & 2c\alpha\sigma(h_2(d))\alpha q \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 2d\alpha l_3(c) + 2c\alpha\sigma(h_1(d)) & 2d\alpha l_4(c) + 2c\alpha\sigma(h_2(d))\alpha q \end{bmatrix}$$

Theorem 2.5 :-Let M be a gamma ring and D be a Jordan left derivation of $M_2(M,\Gamma;\sigma,q)$ then $D\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \delta_1(a) + k_1(b) + l_1(c) + h_1(d) & \delta_2(a) + k_2(b) + l_2(c) + h_2(d) \\ \delta_3(a) + k_3(b) + l_3(c) + h_3(d) & \delta_4(a) + k_4(b) + l_4(c) + h_4(d) \end{bmatrix}$

Such that

1-$\delta_3(a\alpha a) = 0$,$\delta_4(a^2) = 0$,δ_1, δ_2 are Jordan left derivation of M.

2-$h_1(d\alpha d) = 0, h_2(d\alpha d) = 0$ h_3 and $h_4$are Jordan left derivation of M.

3-$k_1(a\alpha b) = 2a\,\alpha k_1(b) + 2b\alpha\,\delta_3(a)\alpha q$,$k_2(a\alpha b) = 2a\,\alpha k_2(b) + 2b\alpha\delta_4(a)$

$k_3(a\alpha b) = 0$and $k_4(a\alpha b) = 0$

4-$l_1(d\alpha c) = 0, l_2(d\alpha c) = 0$,$l_3(d\alpha c) = 2d\alpha l_3(c) + 2c\alpha\sigma(h_1(d))$and

$l_4(d\alpha c) = 2d\alpha l_4(c) + 2c\alpha\sigma\big(h_2(d)\big)\alpha q$.

Proof:-Since $D\begin{bmatrix} a & b \\ c & d \end{bmatrix} = D(e_{11}\, a) + D(e_{12}\, b) + D(e_{21}c) + D(e_{22}\, d)$

$$= \begin{bmatrix} \delta_1(a) & \delta_2(a) \\ \delta_3(a) & \delta_4(a) \end{bmatrix} + \begin{bmatrix} k_1(b) & k_2(b) \\ k_3(b) & k_4(b) \end{bmatrix} + \begin{bmatrix} l_1(c) & l_2(c) \\ l_3(c) & l_4(c) \end{bmatrix} + \begin{bmatrix} h_1(d) & h_2(d) \\ h_3(d) & h_4(d) \end{bmatrix}$$

$$= \begin{bmatrix} \delta_1(a) + k_1(b) + l_1(c) + h_1(d) & \delta_2(a) + k_2(b) + l_2(c) + h_2(d) \\ \delta_3(a) + k_3(b) + l_3(c) + h_3(d) & \delta_4(a) + k_4(b) + l_4(c) + h_4(d) \end{bmatrix}$$

By[lemma 2.1], [lemma 2.2] ,[lemma 2.3]and [lemma 2.4] we get the result .

3- On Skew matrix Gamma ring and Jordan left centralizer

Let J be a Jordan Left Centralizer of $M_2(M,\Gamma;\sigma,q)$ .First ,we set

$$J(e_{11}\,a)=\begin{bmatrix}\delta_1(a) & \delta_2(a)\\ \delta_3(a) & \delta_4(a)\end{bmatrix},\ J(e_{12}\,b)=\begin{bmatrix}l_1(b) & l_2(b)\\ l_3(b) & l_4(b)\end{bmatrix}$$

$$J(e_{21}c)=\begin{bmatrix}k_1(c) & k_2(c)\\ k_3(c) & k_4(c)\end{bmatrix},\ J(e_{22}\,d)=\begin{bmatrix}h_1(d) & h_2(d)\\ h_3(d) & h_4(d)\end{bmatrix}$$

Where $\delta_i, k_i, h_i, l_i : M \to M$ are linear mapping.

Lemma 3.1 :- For every a∈M, $\alpha \in \Gamma$

1-$\delta_1$is Jordan left centralizer of M .
2-$\delta_2(a\alpha a) = 0$
3-$\delta_3(a\alpha a) = \delta_3(a)\alpha\sigma(a)$
4-$\delta_4(a\alpha a) = 0$.
Proof:- Since $J(e_{11}a\alpha a) = J(e_{11}a)\alpha e_{11}a$

$$\begin{bmatrix}\delta_1(a\alpha a) & \delta_2(a\alpha a)\\ \delta_3(a\alpha a) & f_4(a\alpha a)\end{bmatrix}=\begin{bmatrix}\delta_1(a) & \delta_2(a)\\ \delta_3(a) & \delta_4(a)\end{bmatrix}\alpha\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}\delta_1(a\alpha a) & \delta_2(a\alpha a)\\ \delta_3(a\alpha a) & \delta_4(a\alpha a)\end{bmatrix}=\begin{bmatrix}\delta_1(a)\alpha a & 0\\ \delta_3(a)\alpha\sigma(a) & 0\end{bmatrix}$$

Then $\delta_1(a\alpha a) = \delta_1(a)\alpha a, \delta_2(a\alpha a) = 0, \delta_3(a\alpha a) = \delta_3(a)\alpha\sigma(a)$and $\delta_4(a\alpha a) = 0$

Lemma3.2 :- For any d∈M, $\alpha \in \Gamma$

1- $h_2, h_4$are Jordan left centralizer of M .
2- $h_1(d\alpha d) = 0$
3- $h_3(d\alpha d) = 0$
Proof:-Since

$$J(e_{22}d\alpha d) = J(e_{22}d)\alpha\, e_{22}d$$

$$\begin{bmatrix}h_1(d\alpha d) & h_2(d\alpha d)\\ h_3(d\alpha d) & h_4(d\alpha d)\end{bmatrix} = \begin{bmatrix}h_1(d) & h_2(d)\\ h_3(d) & h_4(d)\end{bmatrix}\alpha\begin{bmatrix}0 & 0\\ 0 & d\end{bmatrix}$$

$$= \begin{bmatrix}0 & h_2(d)\alpha d\\ 0 & h_4(d)\alpha d\end{bmatrix}$$

Then $h_1(d\alpha d) = 0, h_3(d\alpha d) = 0$&$h_2, h_4$are Jordan left centralizer of M**.**

Lemma3.3 :- For every a, b∈ R

1-$l_1(a\alpha b) = l_1(b)\alpha a$

2-$l_2(a\alpha b) = \delta_1(a)\alpha b$

3-$l_3(a\alpha b) = l_3(b)\alpha\sigma(a)$

4-$l_4(a\alpha b) = \delta_3(a)\alpha\sigma(b)\alpha q$

Proof:-Since $J(e_{12}a\alpha b) = J(e_{11}a\alpha e_{12}b + e_{12}b\alpha e_{11}a)$

$$\begin{bmatrix} l_1(a\alpha b) & l_2(a\alpha b) \\ l_3(a\alpha b) & l_4(a\alpha b) \end{bmatrix} = J(e_{11}a)\alpha e_{12}b + J(\ e_{12}b\ )\ \alpha e_{11}a$$

$$= \begin{bmatrix} \delta_1(a) & \delta_2(a) \\ \delta_3(a) & \delta_4(a) \end{bmatrix} \alpha \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} l_1(b) & l_2(b) \\ l_3(b) & l_4(b) \end{bmatrix} \alpha \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \delta_1(a)\alpha b \\ 0 & \delta_3(a)\alpha\sigma(b)\alpha q \end{bmatrix} + \begin{bmatrix} l_1(b)\alpha a & 0 \\ l_3\ (b)\alpha\sigma(a) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} l_1(b)\alpha a & \delta_1(a)\alpha b \\ l_3\ (b)\alpha\sigma(a) & \delta_3(a)\alpha\sigma(b)\alpha q \end{bmatrix},$$

Then $l_1(a\alpha b) = l_1(b)\alpha a, l_2(a\alpha b) = \delta_1(a)\alpha b, l_3(a\alpha b) = l_3\ (b)\alpha\sigma(a)$ and

$l_4(a\alpha b) = \delta_3(a)\alpha\sigma(b)\alpha q$.

Lemma 3.4 :-For every c ,d∈ M, $\alpha \in \Gamma$

1-$k_1(d\alpha c) = h_2(d)\ \alpha c\alpha q$

2-$k_2(d\alpha c) = k_2(c)\alpha d$

3-$k_3(d\alpha c) = h_4(d)\alpha c$

4-$k_4(d\alpha c) = k_4(c)\alpha d$

Proof:-Since

$$J(e_{21}\ d\alpha c) = J(e_{22}\ d\alpha e_{21}c + e_{21}c\alpha e_{22}\ d)$$

$$\begin{bmatrix} k_1(d\alpha c) & k_2(d\alpha c) \\ k_3(d\alpha c) & k_4(d\alpha c) \end{bmatrix} = J(e_{22}\ d)\alpha e_{21}c + J(e_{21}c)\alpha e_{22}\ d$$

$$= \begin{bmatrix} h_1(d) & h_2(d) \\ h_3(d) & h_4(d) \end{bmatrix} \alpha \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} k_1(c) & k_2(c) \\ k_3(c) & k_4(c) \end{bmatrix} \alpha \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$

$$= \begin{bmatrix} h_2(d)\ \alpha c\alpha q & 0 \\ h_4(d)\alpha\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & k_2(c)\alpha d \\ 0 & k_4(c)\alpha d \end{bmatrix}$$

$$= \begin{bmatrix} h_2(d)\ \alpha c\alpha q & k_2(c)\alpha d \\ h_4(d)\ \alpha c & k_4(c)\alpha d \end{bmatrix}$$

$k_1(d\alpha c) = h_2(d)\alpha c\alpha q$, $k_2(d\alpha c) = k_2(c)\alpha d$

$k_3(d\alpha c) = h_4(d)\ \alpha c$, $k_4(d\alpha c) = k_4(c)\alpha d$

Theorem 3.5:- Let M be a Gamma ring and J be a Jordan left centralizer of $M_2(M, \Gamma; \sigma, q)$ then

$$J \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \delta_1(a) + l_1(b) + k_1(c) + h_1(d) & \delta_2(a) + l_2(b) + k_2(c) + h_2(d) \\ \delta_3(a) + l_3(b) + k_3(c) + h_3(d) & \delta_4(a) + l_4(b) + k_4(c) + h_4(d) \end{bmatrix}$$

Such that

1-$\delta_1$is Jordan left centralizer of M , $\delta_2(a\alpha a) = 0, \delta_3(a\alpha a) = \delta_3(a)\alpha\sigma(a)and\delta_4(a\alpha a)$ =0.
2- $h_2, h_4$are Jordan left centralizer of R , $h_1(d\alpha d) = 0$and $h_3(d\alpha d) = 0$.
3-$l_1(a\alpha b) = l_1(b)\alpha a$,$l_2(a\alpha b) = \delta_1(a)\alpha b$,$l_3(a\alpha b) = l_3(b)\alpha\sigma(a)$ and

$l_4(a\alpha b) = \delta_3(a)\alpha\sigma(b)\alpha q$

4-$k_1(d\alpha c) = h_2(d)\alpha c\alpha q$, $k_2(d\alpha c) = k_2(c)\alpha d$

$k_3(d\alpha c) = h_4(d)\alpha c$, and $k_4(d\alpha c) = k_4(c)\alpha d$

Proof:- Since $J\begin{bmatrix} a & b \\ c & d \end{bmatrix} = J(e_{11}\, a) + J(e_{12}\, b) + J(e_{21} c) + J(e_{22}\, d)$

$$=\begin{bmatrix} \delta_1(a) & \delta_2(a) \\ \delta_3(a) & \delta_4(a) \end{bmatrix} + \begin{bmatrix} l_1(b) & l_2(b) \\ l_3(b) & l_4(b) \end{bmatrix} + \begin{bmatrix} k(c) & k_2(c) \\ k_3(c) & k_4(c) \end{bmatrix} + \begin{bmatrix} h_1(d) & h_2(d) \\ h_3(d) & h_4(d) \end{bmatrix}$$

$$=\begin{bmatrix} \delta_1(a) + l_1(b) + k_1(c) + h_1(d) & \delta_2(a) + l_2(b) + k_2(c) + h_2(d) \\ \delta_3(a) + l_3(b) + k_3(c) + h_3(d) & \delta_4(a) + l_4(b) + k_4(c) + h_4(d) \end{bmatrix}$$

So by [lemmas 3.1, 3.2, 3.3 and 3.4] ,we have the result.

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On Some Generalized Continuous Mappings in Fuzzy Topological Spaces

Rasha A.Isewid[1], and Samer R.Yaseen[2]

[1]Department **of Mathematics, College of Physical Education and sport science Al-Qadisya University, Iraq.**

[2]Department **of Mathematics, College of pure science, Tikrit University, Iraq.**

Abstract: The purpose of this paper is to introduce a new class of intuitionistic fuzzy closed sets called intuitionistic k-continuous k-irresolute functions in intuitionistic fuzzy topological space (for short, fkt) . Finally , we introduce the concepts of k-open (k-closed) mapping with some properties in fuzzy topological spaces.

Keywords: Intuitionistic fuzzy K-closed sets Intuitionistic fuzzy K-connectedness, Intuitionistic fuzzy K-compactness.

1 Introduction

Atanassov [1,2] was introduced the concept of intuitionistic fuzzy sets (IFS) . After that Change [4] introduced the concept of fuzzy topological spaces. Also many fuzzy topological concepts such as fuzzy compactness [5], fuzzy connectedness [12], fuzzy continuity [6,9], fuzzy g-closed sets [3], fuzzy g continuity [11], fuzzy rg-closed sets [7] have been generalized for IF topological spaces. Further in 2012 Vadivel and Sivakumar [13] were introduced the concept g^*-continuous mapping in fuzzy topology.

2. Preliminaries

We give the following definitions which needed in this paper

Definition 2.1 A fuzzy set ρ in (M, τ) is called:
(1) a fuzzy pre-open set [10] if $\rho \leq cl(int(\rho))$ and a fuzzy pre-closed set if $cl(int(\rho)) \leq \rho$,
(2) a fuzzy α-open set [10] if $\rho \leq int(cl(int(\rho)))$ and a fuzzy α-closed set if $\text{cl}(\text{int}(\text{cl}(\rho))) \leq \rho$,
(3) a fuzzy semi-open set [14] if $\rho \leq cl(int(\rho))$ and a fuzzy semi-closed set if $\text{int}(\text{cl}(\rho)) \leq \rho$,
(4) fuzzy regular open set [10] if $\text{int}(\text{cl}\,\rho) = \rho$ and a fuzzy regular closed set if $\text{cl}(\text{int}(\rho)) = \rho$.

Definition 2.2 A fuzzy set ρ in a fts (M, τ) is called: (1) a fuzzy generalized closed set (for short, fg-closed set) [1] if $cl(\rho) \leq \nu$ whenever $\rho \leq \nu$ and ν is fuzzy open in M,

(2) a fuzzy generalized pre-regular closed set [3] (for short, fgpr-closed fuzzy set) if $pcl(\rho) \subseteq \nu$ whenever $\rho \leq \nu$ and ν is fuzzy regular open set in M .

(3) a fuzzy generalized semi-regular closed set [ ] (for short, fgsr-closed fuzzy set) if $scl(\rho) \subseteq \nu$ whenever $\rho \leq \nu$ and ν is fuzzy semi open set in M .

Definition 2.3 Let M, N are two fuzzy topological spaces. A mapping $L : M \to N$ is called:

(1) fuzzy continuous (for short, f-continuous) [9] if $L^{-1}(\rho)$ is f-open set in M, for every f-open set ρ of N ,

(2) fuzzy α-continuous (for short, fα-continuous) [10] if $L^{-1}(\rho)$ is fuzzy α-closed set in M, for every f-closed set ρ of N ,

(3) fuzzy pre-continuous [10] if $L^{-1}(\rho)$ is fuzzy pre-closed set in M, for every f- closed set ρ of N ,

(4) fuzzy gp-continuous (for short ,fgp-continuous) [6] if $L^{-1}(\rho)$ is fuzzy fgp-closed set in M, for every f-closed set ρ of N,

(5) fuzzy generalized semi-irresolute (for short , fgs-irresolute) [14] if $L^{-1}(\rho)$ is fg-closed set in M, for every fg-closed set ρ of N ,

(6) fuzzy α generalized irresolute (for short , fαg-irresolute) [14] if $L^{-1}(\rho)$ is fαg-closed set in M, for every fαg-closed set ρ of N ,

(7) fuzzy perfectly continuous (for short , fp-continuous) [14] if $L^{-1}(\rho)$ is fuzzy open and f-closed set in M, for every f-open set ρ in N .

Definition 2.4 Let M, N are two fts. A mapping $L : M \to N$ is called:

(1) fuzzy open (for short , f-open) [8] if $L(\rho)$ is fuzzy open set in N , for every f-open set of M,

(2) fuzzy g-open (for short, fg-open) [8] iff $L(\rho)$ is fg-open set in N , for every f-open set in M .

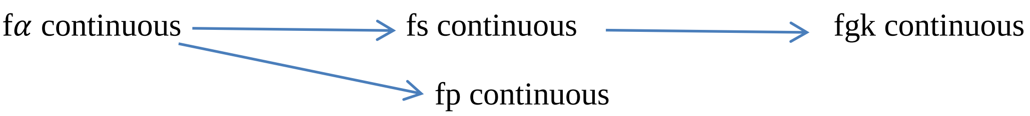
Definition 2.5 Let M, N are two fts. A bijective map $L : M \to N$ is called fuzzy-homeomorphism [14] (for short, f-homeomorphism) if L and L^{-1} are f- continuous.

3. Fuzzy Generalized K-Continuous Mappings

We, start by the following definition:

Definition 3.1 Let M and N be two fts. A function $L : M \to N$ is said to be fuzzy generalized K-continuous (briefly fgk-continuous) if the inverse image of every FOS in N is fgk-open set in M.

Proposition 3.2 Let $L : M \to N$ be a mapping. Then following implications are true :



Proof: fα continuous ⟶ fs continuous : Let ρ be fuzzy α closed set in N . Since L is fα-continuous, L^{-1} is fα closed set in M. so, L^{-1} is fs closed set in M. Thus L is fs continuous.
fα continuous ⟶ fp continuous : Let ρ be fuzzy α closed set in N . Since L is fα-continuous, L^{-1} is fα closed set in M. so, L^{-1} is fp closed set in M. Thus L is fs continuous.

fs continuous ⟶ fgk continuous :it's clear .
The converse of the above proposition need not true in general .The following example show the cases.

Example 3.3.Let $M = N = \{m_1, m_2\}$ and the fuzzy sets δ and ξ defined as following : $\delta = \frac{0.1}{m_1} + \frac{0.5}{m_2}$, $\xi = = \frac{1}{m_1} + \frac{0.9}{m_2}$. Let $\lambda = \{0, 1, \delta\}$ and $\gamma = \{0, 1, \xi\}$. Define $L : (M, \lambda) \rightarrow (N, \gamma)$ by $L(m_1) = m_1$ and $L(m_2) = m_2$. Then L is not fα-continuous because ξ is fα-closed in N and $L^{-1}(\xi) = \xi$ is not fα-closed set in M but L is fs-closed set in M.. Thus L is fs-continuous.Also L is fp-continuous but L is not fα-continuous .

Example 3.4 .Let $M = N = \{m_1, m_2, m_3\}$ and the fuzzy sets δ, ξ and ω defined as following : $\delta = \frac{0}{m_1} + \frac{0.4}{m_2} + \frac{0.2}{m_3}$, $\xi = = \frac{0.6}{m_1} + \frac{0.7}{m_2} + \frac{0.9}{m_3}$, $\omega = = \frac{0.1}{m_1} + \frac{0.2}{m_2} + \frac{0.3}{m_3}$.
Let $\lambda = \{0, 1, \delta\}$ and $\gamma = \{0, 1, \xi\}$. Define $L : (M, \lambda) \rightarrow (N, \gamma)$ by $L(m_1) = m_1$, $L(m_2) = m_2$ and $L(m_3) = m_3$. Then L is not fs-continuous because ξ is fs-closed in N and $L^{-1}(\xi) = \xi$ is not fs-closed set in M but it is fgk-closed set in M . Therefore L is fgk-continuous.

Remark 3.5. the relations between fgk continuous and fp-continuous, also fs continuous and fp-continuous are independent .The following example show the cases.

Example 3.6 .Let $M = N = \{m_1, m_2, m_3\}$ and the fuzzy sets δ, ξ and ω defined as following : $\delta = \frac{0.8}{m_1} + \frac{0.9}{m_2} + \frac{1}{m_3}$, $\xi = = \frac{0.2}{m_1} + \frac{0.6}{m_2} + \frac{0.5}{m_3}$, $\omega = = \frac{0.4}{m_1} + \frac{0.5}{m_2} + \frac{0.6}{m_3}$. Let $\lambda = \{0, 1, \delta\}$ and $\gamma = \{0, 1, \xi\}$. Define $L : (M, \lambda) \rightarrow (N, \gamma)$ by $L(m_1) = m_1$, $L(m_2) = m_2$ and $L(m_3) = m_3$. Then L is not fgk-continuous because δ is fgk-closed in N and $L^{-1}(\delta) = \delta$ is not fgk-closed set in M but it is fgk-closed set in M . Therefore L is fp-continuous.

Example 3.7 .Recall example 3.4 We see that L is not fp-continuous because ξ is fgk-closed in N and $L^{-1}(\xi) = \xi$ is not fp-closed set in M but it is fgk-closed set in M.. Therefore L is fgk-continuous.

Example 3.8 .Let $M = N = \{m_1, m_2, m_3\}$ and the fuzzy sets δ, ξ and ω defined as following : $\delta = \frac{0.1}{m_1} + \frac{0.2}{m_2} + \frac{0.3}{m_3}$, $\xi = = \frac{0.3}{m_1} + \frac{0.4}{m_2} + \frac{0.5}{m_3}$, $\omega = = \frac{0.5}{m_1} + \frac{0.6}{m_2} + \frac{0.7}{m_3}$. Let $\lambda = \{0, 1, \delta\}$ and $\gamma = \{0, 1, \xi\}$. Define $L : (M, \lambda) \rightarrow (N, \gamma)$ by $L(m_1) = m_1$, $L(m_2) = m_2$ and $L(m_3) = m_3$. Then L is not fs-continuous because δ is fs-closed in N and $L^{-1}(\delta) = \delta$ is not fp-closed set in M but it is fs-closed set in M . Therefore L is fs-continuous.

Example 3.9 .Let $M = N = \{m_1, m_2, m_3\}$ and the fuzzy sets δ, ξ and ω defined as following : $\delta = \frac{0}{m_1} + \frac{0.9}{m_2} + \frac{0.2}{m_3}$, $\xi = = \frac{08}{m_1} + \frac{0.7}{m_2} + \frac{1}{m_3}$, $\omega = = \frac{0.3}{m_1} + \frac{0.6}{m_2} + \frac{0.9}{m_3}$. Let $\lambda = \{0, 1, \delta\}$ and $\gamma = \{0, 1, \xi\}$. Define $L : (M, \lambda) \rightarrow (N, \gamma)$ by $L(m_1) = m_1$, $L(m_2) = m_2$ and $L(m_3) = m_3$.

Then L is not fp -continuous because δ is fp-closed in N and $L^{-1}(\xi) = \xi$ is not fs-closed set in M but it is fs-closed set in M . Therefore L is fp-continuous.

Remark 3.10. By transitivity we get : fα continuous ⟶ fgk continuous .

Proposition 3.11. If $L : M \rightarrow N$ is fgk-continuous and $K : N \rightarrow W$ is L-continuous, then $K \circ L : M \rightarrow W$ is fgk-continuous function .

Proof: Let ρ be fuzzy closed set in W. Then $K^{-1}(\rho)$ is fuzzy closed set in N , since K is f-continuous, then $L^{-1}(K^{-1}(\rho))$ is fgk -closed set in M , since L is fgk -continuous. So $(K \circ L)^{-1}(\lambda) = L^{-1}(K^{-1}(\rho))$ is fgk -closed set in M .Therefore $K \circ L : M \rightarrow W$ is fgk-continuous function.

Definition 3.12 A mapping If $L : M \rightarrow N$ is said to be fuzzy generalized k-irresolute (for short fgk -irresolute) if the inverse image of every fgk-closed set in N is fgk -closed fuzzy set in M .

Proposition 3.13 Let $L : M \rightarrow N$ be a mapping .Then every fgk-irresolute function is fgk -continuous.

Proof: Let ρ be a fuzzy closed set in N . So ρ is fgk-closed set in N . Since L is fgk – irresolute, $L^{-1}(\rho)$ is fgk -closed set in M . Thus L is fgk -continuous.

The converse of the above proposition are not true . The following example show the cases .

Example 3.14 . Recall example 3.4 We see that L is fgk-continuous but not fgk-irresolute because the fuzzy closed set ω in N is $L^{-1}(\omega) = \omega$ which is not fgk-closed set in M .

Proposition 3.15. If $L : M \rightarrow N$ and $K : N \rightarrow W$ are two mappings . If L and K are fgk, irresolute mapping ,then $K \circ L : M \rightarrow W$ is fgk- irresolute mapping .

Proof: Let ρ be fuzzy closed set in M. Then $K^{-1}(\rho)$ is fuzzy closed set in N , since K is fgk - irresolute, then $L^{-1}(K^{-1}(\rho))$ is fgk -closed set in M , since L is fgk - irresolute. So $(K \circ L)^{-1}(\lambda) = L^{-1}(K^{-1}(\rho))$ is fgk -closed set in M . Therefore $K \circ L : M \rightarrow W$ is fgk-irresolute function.

Proposition 3.16. Let $L : M \rightarrow N$ and $K : N \rightarrow W$ are two mappings . If L is fgk irresolute and K is fgk-continuous, then $K \circ L : M \rightarrow W$ is fgk-continuous.

Proof : Let ρ be fuzzy closed set in M. Then $K^{-1}(\rho)$ is fgk-closed set in N , since K is fgk -continuous. Since L is fgk-irresolute, $L^{-1}(K^{-1}(\rho)) = (K \circ L)^{-1}(\lambda)$ is fgk -closed set in M . Hence $K \circ L : M \rightarrow W$ is fgk -continuous.

Definition 3.17 A mapping $L : M \rightarrow N$ is called to be fuzzy generalized k-regular open (for short, fgkr-open) if the image of every f-open set in M is fgkr-open set in N .

Definition 3.18 A mapping $L : M \rightarrow N$ is said to be fuzzy generalized k-regular closed (for short, fgkr -closed) if the image of every f-closed set in M is fgkr-closed set in N .

Proposition 3.19 If $L : M \rightarrow N$ is f-closed map and $K : N \rightarrow W$ is fgk-closed maps, then $K \circ L : M \rightarrow W$ is fgk-closed map.

Proof: Let ρ be fuzzy closed set in M . Then L(ρ) is f-closed set in N . Since K is fgk-closed map and since K(L(ρ)) is fgk-closed set in W. So $(K \circ L)(\rho) = K(L(\rho))$ is fgpr-closed set in Z .Therefore $K \circ L : M \rightarrow W$ is fgk-closed map.

Proposition 3.20 Let $L : M \rightarrow N$ and $K : N \rightarrow W$ are two mappings s.t, $K \circ L: M \rightarrow W$ is fgkr -closed map. (1) If L is f-continuous and surjective, then K is fgkr -closed mapping.
(2) If h is fgk$r$ -irresolute and injective, then f is fgkr -closed map.
Proof: (1) Let ρ be fuzzy closed set in N. Then $L^{-1}(\rho)$ is f-closed set in M. Thus $K \circ L$ is fgkr -closed map, $(K \circ L)(L^{-1}(\rho)) = K(\rho)$ is fgkr-closed set in W. Therefore K is fgkr -closed mapping .
(2) Let υ be a f-closed set in N. Then (h ◦ f)(μ) is fgk -closed set in W, so $(K^{-1}(K \circ L)(\upsilon))$ is fgkr -closed set in N. Since K is injective, $L(\upsilon) = K^{-1}(K \circ L)(\upsilon)$ is fgkr-closed set in N . Therefore L is fgkr-closed mapping.

Definition 3.21 Let M and N be two fts . A bijective map $L : M \rightarrow N$ is called fuzzy generalized k-regular homeomorphism (for short, fgkr-homeomorphism) if L and $L^{-1}$are fgkr-continuous.

Proposition 3.22 Every fα-homeomorphism is fgkr-homeomorphism.
Proof: Let $L : M \rightarrow N$ be a fα-homeomorphism. Then Land L^{-1} are f-continuous. Therefore L and $L^{-1}$are fgkr -continuous. So L is fgkr -homeomorphism.
The converse of the above proposition are not true . The following example show the cases .

Example 3.23 Let $M = N = \{m_1, m_2\}$ and the fuzzy sets δ, ξ defined as following :
$\delta = \frac{0.2}{m_1} + \frac{0}{m_2}$, $\xi = = \frac{1}{m_1} + \frac{0.8}{m_2}$. Let $\lambda = \{0, 1, \delta\}$ and $\gamma = \{0, 1, \xi\}$. Define $L : (M, \lambda) \rightarrow (N, \gamma)$ by $L(m_1) = m_1$and $L(m_2) = m_2$. Then L is fgkr-homeomorphism but not fα -homeomorphism because the fuzzy set δ is open in M and its image $L(\delta) = \delta$ is not fα -open set in N , $L^{-1} : M \rightarrow N$ is not fα -continuous.

Definition 3.24. A bijective map $L : M \rightarrow N$ is called fuzzy generalized k-regular-semi-homeomorphism (for short, fgkrs-homeomorphism) if L and L^{-1} are fgkr-irresolute.

Proposition 3.25 Let $L : M \rightarrow N$, $K : N \rightarrow W$ are two fgkrs-homeomorphism , then $K \circ L : M \rightarrow W$ is fgkrs-homeomorphism.

Proof : Let ρ be fgkr-open set in W, and since $K : N \rightarrow W$ is fgkr-irresolute, $K^{-1}(\rho)$ is fgkr -open set in N . Also since $L : M \rightarrow N$ fgkr -irresolute, and $L^{-1}(K^{-1}(\rho)) = (K \circ L)^{-1}(\rho)$ is fgkr-open set in M . So that $K \circ L : M \rightarrow W$ is fgkrs-irresolute .
Now, let ρ be a fgkr-open set in M . Then $(L^{-1})^{-1}(\rho) = L(\rho)$ is fgkr-open set in N . Also $K^{-1} : W \rightarrow N$ is fgkr-irresolute, $(K^{-1})^{-1}(L(\rho)) = K(L(\rho)) = (K \circ L)(\rho)$ is fgpr-open in W. Hence $(K \circ L)^{-1} : W \rightarrow M$ is fgkr-irresolute. Thus $K \circ L$ is fgkrs -homeomorphism.

Proposition 3.26 Let $L : M \rightarrow N$, $K : N \rightarrow W$ are two fgkr-homeomorphism ,then $K \circ L : M \rightarrow W$ is fgkrs-homeomorphism.
Proof :it's obvious .

Conclusion

We studied fgk-continuous, fgk-closed mappings and studied some properties .
It is observed that every fα-continuous and fs-continuous is a fgk-continuous but not conversely. Also every fgk continuous function is a fgkr -continuous function but not conversely. And we get results of composition of fgkr -continuous, fgkr -closed maps, and fgkr-homeomorphisms maps are obtained. Finally f-closed map is fgkr-closed map but not conversely.

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Soft Compact linear operator and soft adjoint linear operator on soft linear spaces

Sabah A. Khaleefah[1], Buthainah A. A. Ahmed[2]

[1] Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.

[2] Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.

E-mail: sab20072008@gmail.com , buthainah41@gmail.com

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Abstract

Day after day, new concepts of soft normed spaces are emerging, which require studying their properties. In our works we have defined the soft compact operator and study some properties of this kind. After that we define soft adjoint operator on soft Banach spaces and study some of its properties. Finally we discuss the relation between the soft compact linear operator and its soft adjoint operator.

Keywords soft compact linear operator, soft adjoint operator

1. INTRODUCTION

Molodtsov (1) in 1999 initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties. He has introduce several applications of this theory in solving many practical problems in engineering, economics, medical science, social science, etc. Maji et al. (3) introduced in 2003 several operations on soft sets and applied soft sets to decision making problems. Ali et al. in (2) defined some new operations on soft sets in 2009. In the line of reduction and addition of parameters of soft sets some works have been done by Chen (15). Aktas and Cagman (16) introduced the notion of soft group and discussed many properties of soft group. Feng et al. (4) worked on soft ideals, soft semiring and idealistic soft semiring. The idea of soft topological spaces was given by M. Shabir, M. Naz, (6). Mappings between soft sets were described by P. Majumdar, S. K. Samanta (17). Feng et al. (2) worked on soft sets combined with fuzzy sets and rough sets. Sujoy Das, S. K. Samanta introduced a notion of soft real sets, soft real numbers, soft complex sets, soft complex numbers and some of their basic properties have been investigated. They present some applications of soft real sets and soft real numbers in real life problems. And later they introduced the concepts of soft metric over an absolute soft set and `soft norm, soft inner product over soft linear spaces. Many properties of soft metric spaces, soft linear spaces, soft normed linear spaces and soft inner product spaces have been investigated with examples and counter examples.

2. PRELIMINARIES

The basic definitions and theorems which were found in earlier study were introduced in the begin :

Definition 2.1 [1] suppose X is a given set and F is a set of parameters. Let $\wp(X)$ symbolize the power set of X and $P \neq \emptyset$ be a subset of F. A pair (H, P) is named a soft set over X, where H is a mapping given by H : P → $\wp(X)$. In a similar term, a soft set over X is a parameterized family of subsets of the universe X. For $w \in$ P, H (w) can be think about as the set of w- approximate elements of the soft set (H, P).

Definition 2.2 [2] For two soft sets (H, P) and (E, D) over a shared universe X, Then (H, P) is a soft subset of (E, D) if:
(1) P ⊆ D and
(2) For all e ∈ P, H (e) ⊆ E (e). We write (H, P) $\tilde{\subseteq}$ (E, D).
(E, D) is called a soft superset of (H, P), We write (H, P) $\tilde{\subseteq}$((E, D) if (H, P) is a soft subset of (E, D).

Definition 2.3 [2] Two soft sets (H, P) and (E, D) over a shared universe X are said to be identical if (H, P) is a soft subset of (E, D) and (E, D) is a soft subset of (H, P).

Definition 2.4 [3] The union of two soft sets (H, P) and (E, D) over the shared universe X is the soft set

(J, Z); where Z = P ∪ D and for all e ∈ Z,

$$J(e) = \begin{cases} H(e) & if\ e \in P - D \\ E(e) & if\ e \in D - P \\ H(e) \cup \mathrm{E(e)} & \text{if e } \in P \cap D \end{cases}$$

We express it as (H, P) $\tilde{\cup}$(E, D) = (J, Z).

Definition 2.5 [4] The intersection of two soft sets (H, P) and (E, D) over the shared universe X is the soft set (L, S), where S = P ∩ D and for all e ∈S, L (e) = H(e) ∩ E(e). We write (H, P) $\widetilde{\cap}$(E , D)= (L, S).

Suppose X be an introductory universal set and P is a set of parameters such that $P \neq \emptyset$. In the upstairs definitions the set of parameters probably different from soft set to another, but we consider, through our work that all soft sets Possess the identical set of the parameters P. Also the upstairs definitions will be useable for these types of soft sets because it'll be a special case of these definitions.

Definition 2.6 [5] The complement of a soft set (F, P) is symbolized by $(F, P)^c$= (F^c, P), where F^c: P→ ℘(X) is a mapping given by $F^c(\lambda)$ = X \ F (λ), for every λ ∈ P

Definition 2.7 [3] A soft set (F, P) over X is said to be an absolute soft set symbolized by $\widetilde{X}$ if F (λ) = X for every λ ∈ P.

Definition 2.8 [3] A soft set (F, P) over X is said to be a null soft set symbolized by $\widetilde{\Phi}$ if for every λ ∈ P, F(λ) = ϕ.

Definition 2.9 [6] The difference (H , P) of two soft sets (F, P)and (E, P)over X, denoted by (F, P)\(E, P), is defined by H (λ) = F(λ)\E(λ) for all λ ∈ P.

Proposition 2.10 [6] Let (M, P) and (N, P) be two soft subsets of $\widetilde{X}$ Then:

(i) $[(M, P)\ \tilde{\cup}\ ((N, P)]^c = (M, P)^c\ \tilde{\cap}\ (N, P)^c$

(ii) $[(M, P)\ \tilde{\cap}\ ((N, P)]^c = (M, P)^c\ \tilde{\cup}\ (N, P)^c$

Definition 2.11 [7] Let X be a non-empty set of elements and $P \neq \emptyset$ is a set of parameter. Then a function ε: P →X is called a soft element of X. A soft element ε of X is belongs to a soft set B of X, which is symbolized by $\varepsilon\ \tilde{\in}$ B, if ε (λ) ∈ B(λ) for every λ ∈ P. Thus for a soft set B of X (with respect to the index set P) we have B (λ) = {$\varepsilon(\lambda)$, ε $\tilde{\in}$B}, λ ∈ P.

It should be mentioned that each singleton soft set (a soft set (H, P) for which H (λ)={x}, x∈X and λ ∈ P) can be assumed as an soft element by replacing the one element set with the element that it contains for all λ ∈ P.

Definition 2.12 [8] Consider 𝔅 (R) the collection of all non-empty bounded subsets of R (R is real number) and P booked as a parameters set. The map H: P → 𝔅 (R) is named a soft real set. It is symbolized by (H, P). If explicitly (H, P) is a singleton soft set, then when detecting (H, P) with the matching soft element, it will be named a soft real number.

The collection of each soft real numbers is symbolized by R(P) while the collection of all non-negative soft real numbers is symbolized by $R(P)^*$.

Definition 2.13 [9] Consider $\wp$ (ℂ) ={ k ; k ⊆ ℂ , k≠ ∅ , k is bounded}. P booked as a parameters set. The map H: P → $\wp$ (ℂ) is named a soft complex set symbolized by (H, P). In particular, if (H, P) is a singleton soft set, then identifying (H, P) with the agreeing soft element, it will be named a soft complex number.

If we take all soft complex numbers as a set, we can call it by ℂ(P).

Definition 2.14 [9] Let (H, P) be a soft complex set. The complex conjugate of (H, P) is symbolized by ($\overline{H}$, P) and is defined by $\overline{H}(\lambda)$ = {$\overline{z}$: z ∈ H (λ)}, for every λ ∈ P. Where $\overline{z}$ is complex conjugate of the ordinary complex number z, The complex conjugate of a soft complex number (H, P) is $\overline{H}(\lambda)$ = $\overline{z}$: z = H(λ) , for every λ ∈ P.

Definition 2.15 [9] Let (F, P), (E, P) $\tilde{\in}$ C (P): Then the sum, difference, product and division are defined by

(F + E) (λ) = z + w, z ∈ F (λ), w ∈ E (λ), for all λ ∈ P.
(F − E) (λ) = z − w; z ∈ F (λ), w ∈ E (λ), for all λ ∈ P.
(FE) (λ) = zw, z ∈ F (λ), w ∈ E (λ), for all λ ∈ P.
(F/E) (λ) = z/w, z ∈ F (λ), w ∈ E (λ), provided E (λ) ≠ ϕ, for all λ ∈ P.

Definition 2.16 [9] Let (F, P) be a soft complex number. Then the modulus of (F, P) is symbolized by ($|F|$, P) and is defined by $|F|(\lambda)$ =$|z|$; z ∈ F (λ), for each λ ∈ P, where z is an ordinary complex number.

Since the modulus of each ordinary complex number and ordinary real number are a non-negative real number and by definition of soft real numbers, we obtained that ($|F|$, P) is a non-negative soft real number for every soft complex number (F, P) or soft real number (F, P) .

Let X is a non-empty set and $\widetilde{X}$ be the absolute soft set i.e., V (λ) = X, for each λ ∈ P .where (V, P) = $\tilde{X}$. Suppose S ($\widetilde{X}$) be the collection of all soft sets (H, P) over X for which H (λ) ≠ ϕ, for all λ ∈ P together with the null soft set $\widetilde{\Phi}$. Let (H, P) (≠ Φ) ∈ S($\widetilde{X}$), then the collection of all soft elements of (H , P) will be denoted by SE (H , P). For a collection 𝔅 of soft elements of $\widetilde{X}$, the soft set generated by 𝔅is denoted by SS(𝔅).

Definition 2.17 [10] Let d: SE($\widetilde{X}$) × SE($\widetilde{X}$) → $R(P)^*$. We called d a soft metric on the soft set $\widetilde{X}$if d Achieves the subsequent conditions:

(1). d ($\tilde{x}$; $\tilde{y}$) $\tilde{\geq}$ $\overline{0}$, for each $\tilde{x}$, $\tilde{y}$ $\tilde{\in}$ $\widetilde{X}$.

(2). d ($\tilde{x}$, $\tilde{y}$) = $\overline{0}$, if and only if $\tilde{x}$ = $\tilde{y}$.

(3). d ($\tilde{x}$, $\tilde{y}$) = d ($\tilde{y}$, $\tilde{x}$) for all $\tilde{x}$, $\tilde{y}$ $\tilde{\in}$ $\widetilde{X}$.

(4). For all $\tilde{x}, \tilde{y}, \tilde{z}$ $\tilde{\in}$ $\widetilde{X}$, d ($\tilde{x}$, $\tilde{z}$) $\tilde{\leq}$ d ($\tilde{x}$, $\tilde{y}$) + d ($\tilde{y}$, $\tilde{z}$)

The soft metric d defined on $\widetilde{X}$ side by side with the soft set $\widetilde{X}$ is called a soft metric space and is symbolized by ($\widetilde{X}$, d, P) or ($\widetilde{X}$, d).

Definition 2.18 [11] Let W be a vector space over Z (which Z is a field) , P is a parameters set. Let H be a soft set on (W, P). If for all λ ∈ P,H (λ) is a vector subspace of W, Then H is called a soft vector space of W over Z.

Definition 2.19 [12] Suppose H is a soft vector space of W over Z. Let L: P → $\wp$ (W) be a soft set over (W, P). If for each $\lambda \in$ P, L(λ) is a vector subspace of W over Z and H (λ) ⊇ L (λ), Then L is called a soft vector subspace of H.

Definition 2.20 [11] Suppose H is a soft vector space of W over Z. Then a soft element of H is called a soft vector of H. In a similar way we can called the soft element of the soft set (Z, P) by soft scalar, where Z is the scalar field.

Definition 2.21 [11] Let $\tilde{x}$, $\tilde{y}$ be soft vectors of G and $\tilde{k}$ be a soft scalar. The addition $\tilde{x}$ + $\tilde{y}$ of $\tilde{x}$, $\tilde{y}$ and scalar multiplication $\tilde{k}. \tilde{x}$ of $\tilde{k}$ and $\tilde{x}$ are defined by ($\tilde{x}$ + $\tilde{y}$) (λ) = $\tilde{x}$ (λ) + $\tilde{y}$ (λ) , $\tilde{k}. \tilde{x}$ (λ) = $\tilde{k}$ (λ) . $\tilde{x}$ (λ) for all $\lambda \in$ A. Obviously, $\tilde{x}$ + $\tilde{y}$, $\tilde{k}$. $\tilde{x}$ are soft vectors of G.

Definition 2.22 [13] Let $\widetilde{X}$ be the absolute soft vector space i.e., $\widetilde{X}$ (λ) = X, for all $\lambda \in$ P. Then a function $\|.\|$: SE($\widetilde{X}$) → $R(P)^*$is called a soft norm on the soft vector space $\widetilde{X}$ if $\|.\|$ satisfies the subsequent situations:

1) $\|.\| \cong \bar{0}$ for every $\tilde{x} \tilde{\in} \widetilde{X}$.

2) $\|\tilde{x}\|$ = $\bar{0}$ if and only if $\tilde{x}$ = Θ.

3) $\|\tilde{\alpha} . \tilde{x}\|$ = $|\tilde{\alpha}|$ $\|\tilde{x}\|$ for each $\tilde{x} \tilde{\in} \widetilde{X}$ as well as for each soft scalar $\tilde{\alpha}$.

4) For each $\tilde{x}$,$\tilde{y} \tilde{\in} \widetilde{X}$, $\|\tilde{x} + \tilde{y}\| \tilde{\leqq} \|\tilde{x}\|$ + $\|\tilde{y}\|$

The soft vector space $\widetilde{X}$ with a soft norm $\|.\|$on $\widetilde{X}$ is called a soft normed linear space and is symbolized by ($\widetilde{X}$,$\|.\|$, P) or ($\widetilde{X}$, $\|.\|$). The exceeding conditions are called soft norm axioms.

Theorem 2.23 [11] Suppose a soft norm $\|.\|$ achieves the situation (N5). For $\xi \in$ X and $\lambda \in$ P the set { $\|\tilde{x}\|(\lambda)$: $\tilde{x}(\lambda)$ = ξ} is a one element set. Then for each $\lambda \in$ P, the mapping $\|.\|_{\lambda}$: X → R^+ defined by $\|\xi\|_{\lambda}$ = $\|\tilde{x}\|(\lambda)$, for all $\xi \in$ X and $\tilde{x} \tilde{\in} \widetilde{X}$. Such that $\tilde{x}$ (λ) = ξ , can be considered as a norm on X.

Definition 2.24 [12] consider ($\widetilde{X}$, $\|.\|$, P) is a soft normed linear space, $\tilde{r} \tilde{\geqq} \bar{0}$ be a soft real number. We define the followings:
B($\tilde{x}$, $\tilde{r}$) = { $\tilde{y} \tilde{\in} \widetilde{X}$:$\|\tilde{x} - \tilde{y}\| \tilde{<} \tilde{r}$ }⊂ SE($\widetilde{X}$)
$\bar{B}$($\tilde{x}$, $\tilde{r}$) = { $\tilde{y} \tilde{\in} \widetilde{X}$:$\|\tilde{x} - \tilde{y}\| \tilde{\leqq} \tilde{r}$ }⊂ SE($\widetilde{X}$)
S($\tilde{x}$, $\tilde{r}$) = { $\tilde{y} \tilde{\in} \widetilde{X}$:$\|\tilde{x} - \tilde{y}\|$ = $\tilde{r}$ }⊂ SE($\widetilde{X}$)
B($\tilde{x}$, $\tilde{r}$) , $\bar{B}$($\tilde{x}$, $\tilde{r}$) , S($\tilde{x}$, $\tilde{r}$) are respectively called an open ball, a closed ball and a sphere with center at $\tilde{x}$ and radius $\tilde{r}$. SS(B($\tilde{x}$, $\tilde{r}$)), SS($\bar{B}$($\tilde{x}$, $\tilde{r}$)) and SS(S($\tilde{x}$, $\tilde{r}$)) are respectively called a soft open ball, a soft closed ball and a soft sphere with center at $\tilde{x}$ and radius $\tilde{r}$.

Definition 2.25 [11] A sequence of soft elements {$\widetilde{x_n}$} in a soft normed space ($\widetilde{X}$,$\|.\|$, P) called convergent sequence if $\|\widetilde{x_n} - \tilde{x}\|$ → $\bar{0}$ as n → ∞, we say the sequence converges to a soft element $\tilde{x}$. In other words for all $\tilde{\epsilon} \tilde{\geqq} \bar{0}$, there exist N∈ ℕ , N = N ($\tilde{\epsilon}$) and $\bar{0} \tilde{\leqq}$ $\|\widetilde{x_n} - \tilde{x}\| \tilde{\leqq} \tilde{\epsilon}$ every time n > N.

i.e., n > N *implies* $\widetilde{x_n}$ ∈ B ($\tilde{x}$, $\tilde{\epsilon}$) . We symbolize this by $\widetilde{x_n}$ → $\tilde{x}$ as n → ∞ or by $\lim_{n\to\infty} \widetilde{x_n} = \tilde{x}$. The soft element $\tilde{x}$ said to be the limit of the sequence $\widetilde{x_n}$ as n → ∞.

Definition 2.26 [11] A sequence {$\widetilde{x_n}$} of soft elements in a soft normed space ($\widetilde{X}$, $\|.\|$, P) is said to be a Cauchy sequence in $\widetilde{X}$ if corresponding to each $\tilde{\epsilon} \tilde{>} \bar{0}$, there exist m > N such that $\|\widetilde{x_i} - \widetilde{x_j}\| \tilde{\leqq} \tilde{\epsilon}$, for all i,j $\geq m$ i.e., $\|\widetilde{x_i} - \widetilde{x_j}\|$ → $\bar{0}$ as i; j → ∞.

Definition 2.27 [11] Let ($\widetilde{X}$, ‖. ‖, P) be a soft normed space. Then $\widetilde{X}$ is called a soft complete if every Cauchy sequence in $\widetilde{X}$ converges to a soft element of $\tilde{X}$. The soft complete normed space is said to be a soft Banach Space.

Theorem 2.28 [11] every Cauchy sequence in R(P), where P is a finite set of parameters, is convergent, i.e., the set of all soft real numbers together with its usual modulus soft norm with respect to finite set of parameters, is a soft Banach space.

Definition 2.29[12] A series $\sum_{k=1}^{\infty} \widetilde{x_k}$ of soft elements called soft convergent if the partial sum of the series $\widetilde{S_n} = \sum_{k=1}^{n} \widetilde{x_k}$ is soft convergent.

Let $\tilde{X}$, $\tilde{Y}$ be the corresponding absolute soft normed spaces i.e., $\tilde{X}(\lambda) = X$, $\tilde{Y}(\lambda) = Y$, for all $\lambda \in P$. We use the notation $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ to represent soft vectors of a soft vector space.

Definition 2.30[5] Let G be a soft vector space of W over Z. Let $\widetilde{\alpha_1}, \widetilde{\alpha_2}$.,……. $\widetilde{\alpha_n} \in G$. A soft vector $\tilde{\beta}$ in G is said to be a linear combination of the soft vectors $\widetilde{\alpha_1}, \widetilde{\alpha_2}$.,……. $\widetilde{\alpha_n}$ if $\tilde{\beta}$ can be expressed as $\tilde{\beta} = \widetilde{c_1}\widetilde{\alpha_1}+\widetilde{c_2}\widetilde{\alpha_2}+\ldots\ldots+\widetilde{c_n}\widetilde{\alpha_n}$, for some soft scalars $\widetilde{c_1}, \widetilde{c_2}, \ldots \ldots \ldots, \widetilde{c_n}$.

Proposition 2.31 [11] A set $S = \{\widetilde{\alpha_1}, \widetilde{\alpha_2}, \ldots \ldots .\widetilde{\alpha_n}\}$ of soft vectors in a soft vector space G over W is linearly independent if and only if the sets $S(\lambda) = \{\widetilde{\alpha_1}(\lambda), \widetilde{\alpha_2}(\lambda), \ldots \ldots .\widetilde{\alpha_n}(\lambda)\}$ are linearly independent in W, for all $\lambda \in P$.

Proposition 2.32 [11] A set $S = \{\widetilde{\alpha_1}, \widetilde{\alpha_2}, \ldots \ldots .\widetilde{\alpha_n}\}$ of soft vectors in a soft vector space G over W is linearly dependent if and only if the sets $S(\lambda) = \{\widetilde{\alpha_1}(\lambda), \widetilde{\alpha_2}(\lambda), \ldots \ldots .\widetilde{\alpha_n}(\lambda)\}$ are linearly dependent in V, for all $\lambda \in P$.

Definition 2.33 [11] A soft linear space $\widetilde{X}$is said to be of finite dimensional if there is a finite set of linearly independent soft vectors in $\widetilde{X}$ which also generates $\tilde{X}$, i.e., any soft element of $\widetilde{X}$ can be stated as a linear combination of those linearly independent soft vectors.

Set of soft vectors which linearly independent is said to be the basis for $\widetilde{X}$ and the number of soft vectors of the basis is called the dimension of $\tilde{X}$.

Definition 2.34[11] Suppose T: SE($\tilde{X}$) → SE($\tilde{Y}$) is an operator. T is called soft linear if
(L1) T ($\widetilde{x_1}$ + $\widetilde{x_2}$) = T ($\widetilde{x_1}$) + T($\widetilde{x_2}$) for all soft elements $\widetilde{x_1}, \widetilde{x_2} \tilde{\in} \tilde{X}$.
(L2) For all soft scalar $\tilde{k}$, T($\tilde{k}$. $\tilde{x}$) = $\tilde{k}$ T ($\tilde{x}$). for all soft element $\tilde{x} \tilde{\in} \tilde{X}$.
The properties (L1) and (L2) can be put in a combined form T ($\widetilde{k_1}$. $\widetilde{x_1}$ + $\widetilde{k_2}$. $\widetilde{x_2}$) = $\widetilde{k_1}$ T ($\widetilde{x_1}$) + $\widetilde{k_2}$ T($\widetilde{x_2}$) for every soft elements $\widetilde{x_1}, \widetilde{x_2} \tilde{\in} \tilde{X}$ and every soft scalars $\widetilde{k_1}$, $\widetilde{k_2}$.

Definition 2.35[11] The operator T : SE($\tilde{X}$) → SE($\tilde{Y}$) is said to be continuous at $\widetilde{x_0} \tilde{\in} \tilde{X}$ if for every sequence $\{\widetilde{x_n}\}$ of soft elements of $\tilde{X}$ with $\widetilde{x_n} \to \widetilde{x_0}$ as n → ∞, we have T($\widetilde{x_n}$) → T($\widetilde{x_0}$) as n → ∞ i.e., $\|\widetilde{x_n} - \widetilde{x_0}\| \to \bar{0}$ as n → ∞ implies $\|T(\widetilde{x_n}) - T(\widetilde{x_0})\| \to \bar{0}$ as n → ∞ . If T is continuous at every soft element of $\widetilde{X}$, then T is called a continuous operator.

Theorem 2.36[11] *Let T: SE*($\widetilde{X}$) →*SE*($\widetilde{Y}$) *be a soft linear operator, where* $\widetilde{X}, \widetilde{Y}$ *are* soft normed linear spaces: If T is continuous at some soft element $\widetilde{x_0} \tilde{\in} \widetilde{X}$ then *T is* continuous at every soft element of $\widetilde{X}$.

Definition 2.37[11] Let T: SE($\tilde{X}$) →SE($\tilde{Y}$) be a soft linear operator, where $\tilde{X}$, $\tilde{Y}$ are soft normed linear spaces .The operator T is said to be bounded if there exists some positive soft real number $\widetilde{M}$ such that for each $\tilde{x} \tilde{\in} \tilde{X}$, $\|T(\tilde{x})\| \lesssim \widetilde{M} \|\tilde{x}\|$.

Theorem 2.38[11] Let T: SE($\tilde{X}$) →SE($\tilde{Y}$) be a soft linear operator, where $\tilde{X}$, $\tilde{Y}$ are soft normed linear spaces. If T is bounded then T is continuous.

Theorem 2.39[11] (Decomposition Theorem) Suppose a soft linear operator T: SE($\widetilde{X}$) → SE($\tilde{Y}$), where $\widetilde{X}$, $\tilde{Y}$ are soft normed spaces, fulfills the situation (L3). For $\xi \in$ X, and $\lambda \in$ P the set {T ($\tilde{x}$) (λ): $\tilde{x} \tilde{\in} \widetilde{X}$ such that $\tilde{x}$ (λ) = ξ} is a one element set. Then for each $\lambda \in$ P, the mapping T_λ : X → Y defined by $T_\lambda(\xi)$ = T ($\tilde{x}$) (λ), for all $\xi \in$ X and $\tilde{x} \tilde{\in} \widetilde{X}$ such that $\tilde{x}$ (λ) = ξ, is a linear operator.

Theorem 2.40[11] Let T_λ : X ⟶Y, $\lambda \in$ P be a family of crisp linear operators from the vector space X to the vector space Y, and $\tilde{X}$, $\tilde{Y}$ be the corresponding absolute soft vector spaces. Then there exists a soft linear operator T : SE($\tilde{X}$) → SE($\tilde{Y}$) defined by T $(\tilde{x})$ (λ) = $T_\lambda(\xi)$ if $\tilde{x}$ (λ) = ξ ¸ $\lambda \in$ P. which satisfies (L3) and T (λ) = T_λ for all $\lambda \in$ P.

Theorem 2.41[11] Let $\tilde{X}$ and $\tilde{Y}$ be soft normed linear spaces which satisfy (N5) and T: SE($\tilde{X}$) → SE($\tilde{X}$) be a soft linear operator satisfying (L3). If T is continuous then T is bounded.

Theorem 2.42[11] Let $\tilde{X}$ and $\tilde{Y}$ be soft normed linear spaces which satisfy (N5) and T: SE($\tilde{X}$) → SE($\tilde{Y}$) be a soft linear operator satisfying (L3). If $\tilde{X}$ is of finite dimension, then T is bounded and hence continuous.

Definition 2.43[11] **(**Let T be a bounded soft linear operator from SE(X̃) into SE(Ỹ). Then the norm of the operator T denoted by ||T||, is a soft real number defined as the following:
For each λ ∈ P $\|T\|(\lambda) = \inf\{t \in R; \|T(\tilde{x})\| \le t. \|\tilde{x}\|(\lambda), for\ each\ \tilde{x} \tilde{\in} \tilde{X}\}$.

Theorem 2.44[11] Let $\tilde{X}, \tilde{Y}$ be soft normed linear spaces which satisfy (N5) and T satisfy (L3). Then for each λ ∈ P, $\|T\|(\lambda) = \|T_\lambda\|_\lambda, where\ \|T_\lambda\|_\lambda$ is the norm of the linear operator T_λ: X → Y.

Theorem 2.45[11] $\|T(\tilde{x})\| \tilde{\leqq} \|T\|\|\tilde{x}\|, for\ all\ \tilde{x}\ \tilde{\in} \tilde{X}.$

Theorem 2.46[11] Let $\tilde{X}$and $\tilde{Y}$ be soft normed linear spaces which satisfy (N5) and T: SE($\tilde{X}$) → SE($\tilde{Y}$) be a soft linear operator satisfying (L3). Then:
(i) $\|T\|(\lambda) = \sup\{\|T(\tilde{x})\|(\lambda): \|\tilde{x}\| \tilde{\leqq} \bar{1}\} = \|T_\lambda\|_\lambda, for\ each\ \lambda \in$ P.
(ii) $\|T\|(\lambda) = \sup\{\|T(\tilde{x})\|(\lambda): \|\tilde{x}\| = \bar{1}\} = \|T_\lambda\|_\lambda, for\ each\ \lambda \in$ P.
(iii) $\|T\|(\lambda) = \sup\left\{\frac{\|T(\tilde{x})\|}{\|\tilde{x}\|}(\lambda): \|\tilde{x}\|(\mu) \neq 0, for\ all\ \mu \in A\right\} = \|T_\lambda\|_\lambda, for\ each\ \lambda \in$ P.

Theorem 2.47[11] Let $\tilde{X}$ and $\tilde{Y}$ be a soft normed linear spaces which satisfy (N5). Let T: SE($\tilde{X}$) → SE($\tilde{Y}$) be a continuous soft linear operator satisfying (L3).Then T_λ is continuous on X for each λ ∈ P.

Theorem 2.48[12] Let $\tilde{X}$and $\tilde{Y}$ be a soft normed linear spaces which satisfy (N5). Let { T_λ; λ ∈ P} be a family of continuous linear operators such that T_λ: X → Y for each λ. Then the soft linear operator T: SE($\tilde{X}$) → SE($\tilde{Y}$) defined by (T(x̃))(λ) = $T_\lambda(\tilde{x}(\lambda)), for\ all$ λ ∈ P is a continuous soft linear operator satisfying (L3).

Definition 2.49[12] (Soft linear space of operators) Let $\tilde{X}, \tilde{Y}$ be soft normed linear spaces satisfying (N5). Consider the set W of all continuous soft linear operators S; T etc. which satisfy (L3) each mapping SE($\tilde{X}$) into SE($\tilde{Y}$) Then using Theorem 2.43, it follows that for each λ ∈ P; S,T, ……. etc. are continuous soft linear operators from X to Y.
Let $W(\lambda) = \{T_\lambda(= T(\lambda)); \mathrm{T} \in \mathrm{W}\}$, for all λ ∈ P. Also using definition 2.43 and Theorem 2.44, it follows that $W(\lambda)$ is the collection of all continuous linear operators from X to Y. By the property of crisp linear operators it follows that $W(\lambda)$ forms a vector space for each λ ∈ P with respect to the usual operations of addition and scalar multiplication of linear operators. It also follows that $W(\lambda)$ is identical with the set of all continuous linear operators from X to Y for all λ ∈ P.Thus the absolute soft set generated by $W(\lambda)$ form an absolute soft vector space. Hence W can be interpreted as to form an absolute soft vector space. We shall denote this absolute soft linear (vector) space by L $(\tilde{X}, \tilde{Y})$.

Proposition 2.50[12] Each element of $SE(L(\tilde{X}, \tilde{Y}))$ can be identified uniquely with a member of W i.e., to a continuous soft linear operator T: SE($\tilde{X}$) → SE($\tilde{Y}$).

Theorem 2.51[12] L $(\tilde{X}, \tilde{Y})$ is a soft normed linear space where for $\hat{f} \in SE(L(\tilde{X}, \tilde{Y})$, we can identify $\hat{f}$ to a unique $T \in W\ and\ \|\hat{f}\|\ is\ defined\ by\ \|\hat{f}\|(\lambda) = \|T\|(\lambda) = \sup\{\|T(\tilde{x})\|(\lambda): \|\tilde{x}\| \tilde{\leqq} \bar{1}\}, for\ each\ \lambda \in$ P.

Definition 2.52[11] Suppose T: SE(X̃) ⟶ SE(Ỹ) be a soft linear operator where X̃, Ỹ are soft normed space. Then T is called injective or one-to-one if T($\tilde{x_1}$)(λ) = T($\tilde{x_2}$)(λ) implise ($\tilde{x_1}$)(λ) = ($\tilde{x_2}$)(λ) ∀ λ ∈ P ,it is called surjective or onto if Rang(T) = SE(Ỹ) ,the operator T is said to be bijective if T is both one-to-one and onto .

Definition2.53 [14]: let $(\widetilde{X}, \tau)$ be a soft topological space, A collection $\{(G_i, P) : i \in I\}$ of soft open sets is called a soft open cover of $(\widetilde{X}, \tau)$ if $\widetilde{X} = \widetilde{\cup}_{i\in I}(G_i, P)$

Definition2.54 [14]: An $(\widetilde{X}, \tau)$ is called soft compact if every soft open cover of $\widetilde{X}$ has a finite soft sub-collection which cover $\widetilde{X}$

Theorem 2.55[12]: Let $\widetilde{X}$ be a real soft normed linear space satisfying (N5). Let f be a continuous soft linear functional on a soft subspace G of $\widetilde{X}$ satisfying (L3). Then there exists a continuous soft linear functional F defined on $\widetilde{X}$ satisfying (L3), such that

(i). $f(\tilde{x}) = F(\tilde{x})$ for all $\tilde{x} \widetilde{\in} G$; and

(ii). $\|f\|_G = \|F\|_{\widetilde{X}} = \|F\|$.

3. SOFT COMPACT LINEAR OPERATOR

Definition 3.1 : A soft normed space $(\widetilde{X}, \|.\|)$ is called a soft compact if every sequence $\{\widetilde{x_n}\}$ of soft vectors in $\widetilde{X}$ has a convergent subsequence , a soft subset (G, P) of $\widetilde{X}$ is called a soft compact soft set if every sequence of soft vectors in (G, P) has a convergent subsequence converges to a soft vector of (G, P).

Proposition 3.2: A soft compact subset (G, P) of soft normed space ($\widetilde{X}$, $\|.\|$) is soft closed and soft bounded.

Proof: for each $\tilde{x} \widetilde{\in} \overline{(G, A)}$ there exist a sequence $\{\widetilde{x_n}\}$ in (G, P) such that $\widetilde{x_n} \to \tilde{x}$, since (G, P) is soft compact then $\tilde{x} \widetilde{\in}$ (G, P). This implies that (G, P) is soft close since $\tilde{x} \widetilde{\in}$ (G, P) was random.

We show that (G, P) is bounded, if (G, P) is not bounded, then it would contain an unbounded sequence $\{\widetilde{x_n}\}$ such that $\|\widetilde{x_n} - \tilde{y}\| \widetilde{>} \widetilde{m}$ where $\tilde{y}$ is a fixed soft vector in (G, P) and $\widetilde{m} \widetilde{\in} R(P)^*$. This sequence could not have a convergent subsequence because a convergent subsequence must be bounded. i.e., (G, P) must be bounded.

The converse of above Proposition is not true in common; it's true only in finite dimension soft normed space.

Lemma 3.3 (Linear combinations). Let $\{\widetilde{x_1}, \widetilde{x_2}, \widetilde{x_3}, \dots \dots \widetilde{x_n}\}$ be a linearly independent set of soft vectors in a soft normed space $\widetilde{X}$ (of any dimension).Then there is a soft real number $\tilde{c} \widetilde{\geq} \overline{0}$ such that for every choice of soft scalars

$\widetilde{\alpha_1}, \widetilde{\alpha_2}, \dots \dots . \widetilde{\alpha_n}$ we have

$\|\widetilde{\alpha_1}\widetilde{x_1} + \widetilde{\alpha_2}\widetilde{x_2} + \dots + \widetilde{\alpha_n}\widetilde{x_n}\| \widetilde{\geq} \tilde{c}(|\widetilde{\alpha_1}| + |\widetilde{\alpha_2}| + \dots + |\widetilde{\alpha_n}|)$.

Prove of lemma was given in (5) briefly, we have proved it in another way with details.

Proof: We write $\tilde{S} = |\widetilde{\alpha_1}| + |\widetilde{\alpha_2}| + \dots + |\widetilde{\alpha_n}|$. if $\tilde{S} = \overline{0}$, all $\widetilde{\alpha_j}$ are $\overline{0}$ and the above statement true for any soft real number $\tilde{c}$. Let $\tilde{S} \widetilde{\geq} \overline{0}$. Dividing both sides by $\tilde{S}$ and writing $\widetilde{\beta_j} = \frac{\widetilde{\alpha_j}}{\tilde{S}}$, that is,

(1)……….. $\|\widetilde{\beta_1}\widetilde{x_1} + \widetilde{\beta_2}\widetilde{x_2} + \dots + \widetilde{\beta_n}\widetilde{x_n}\| \widetilde{\geq} \tilde{c}$ where $\sum_{j=1}^{n}|\widetilde{\beta_j}| = \overline{1}$.

Hence it be enough to prove that there is a $\tilde{c} \widetilde{\geq} \overline{0}$ such that (1) holds for all soft scalars $\widetilde{\beta_j}$ with $\sum_{j=1}^{n}|\widetilde{\beta_j}| = \overline{1}$. Consider that this is not true. Then there is a sequence $\{\widetilde{y_m}\}$ of soft vectors such that

$\widetilde{y_m} = \widetilde{\beta_1^m}\widetilde{x_1} + \widetilde{\beta_2^m}\widetilde{x_2} + \dots + \widetilde{\beta_n^m}\widetilde{x_n}$ $\left(\sum_{j=1}^{n}\left|\widetilde{\beta_j^m}\right| = \overline{1}\right)$ And $\|\widetilde{y_m}\| \to \overline{0}$ as $m \to \infty$.

Since $\sum_{j=1}^{n}\left|\widetilde{\beta_j^m}\right| = \overline{1}$, we have $\left|\widetilde{\beta_j^m}\right| \widetilde{\leq} \overline{1}$. Hence for every static j the sequence $\left\{\widetilde{\beta_j^m}\right\} = \{\widetilde{\beta_j^1}, \widetilde{\beta_j^2}, \dots\}$ is bounded. Consequently, by the Bolzano-Weierstrass theorem $\{\widetilde{\beta_j^m}\}$ has a convergent subsequence. Let $\widetilde{\beta_1}$ represent the limit of that subsequence, and let $\{\widetilde{y_{1,m}}\}$ symbolize the consistent subsequence of $\{\widetilde{y_m}\}$. By the same reason, $\{\widetilde{y_{1,m}}\}$ has a subsequence $\{\widetilde{y_{2,m}}\}$for which the consistent subsequence of soft scalars $\widetilde{\beta_2^m}$ converges; let $\widetilde{\beta_2}$ denote the limit. Ongoing in this way, after n stages we get a subsequence:

$\left\{\widetilde{y_{n,m}}\right\} = \left\{\widetilde{y_{n,1}}, \widetilde{y_{n,2}}, \dots\right\}$of $\{\widetilde{y_m}\}$ Whose terms are of the form $\widetilde{y_{n,m}} = \sum_{j=1}^{n}\widetilde{\gamma_j^m}\widetilde{x_j}$ $\left(\sum_{j=1}^{n}\left|\widetilde{\beta_j^m}\right| = \overline{1}\right)$.

With soft scalars $\widetilde{\gamma_j^m}$ satisfying $\widetilde{\gamma_j^m} \to \widetilde{\beta_j}$ as $m \to \infty$. Hence $\widetilde{y_{n,m}} \to \tilde{y} = \sum_{j=1}^{n}\widetilde{\beta_j}\widetilde{x_j}$.

Where $\sum_{j=1}^{n}|\widetilde{\beta_j}| = \overline{1}$, so that not all $\widetilde{\beta_j}$ can be zero. Since $\{\widetilde{x_1}, \widetilde{x_2}, \widetilde{x_3}, \dots \dots \widetilde{x_{1n}}\}$ is a linearly independent set, we thus have $\tilde{y} \neq \overline{0}$. On the other side, $\widetilde{y_{n,m}} \to \tilde{y}$ implies$\|\widetilde{y_{n,m}}\| \to \|\tilde{y}\|$, by the continuity of the soft norm. Since

$\|\widetilde{y_m}\| \to \bar{0}$ by hypothesis and $\{\widetilde{y_{n,m}}\}$ is a subsequence of $\{\widetilde{y_m}\}$, we obtained that $\|\widetilde{y_{n,m}}\| \to \bar{0}$. Hence, $\|\tilde{y}\| = \bar{0}$, so that $\tilde{y} = \theta$ by (N2). This contradicts $\tilde{y} \neq \theta$, and the result is followed.

Proposition 3.4: in a finite dimension soft normed space $(\tilde{X}, \|.\|)$ any soft subset $M \tilde{\subseteq} \tilde{X}$ is soft compact if and only if M is soft close and soft bounded.

Proof: (if direction)
Let M = (G, P) be a soft closed and soft bounded subset of $\tilde{X}$, let dimension of $\tilde{X}$ = n and let $\{\widetilde{x_1}, \widetilde{x_2}, \dots, \widetilde{x_n}\}$be a base of $\tilde{X}$, we consider any sequence $\{\widetilde{x_m}\}$ in M, then $\widetilde{x_m} = \widetilde{\alpha_{1_m}}\widetilde{x_1} + \widetilde{\alpha_{2_m}}\widetilde{x_2} + \dots + \widetilde{\alpha_{n_m}}\widetilde{x_n}$ for all m.
$\{\widetilde{x_m}\}$ is soft bounded since M is soft bounded, i.e.,$\|\widetilde{x_m}\| \tilde{\leqq} \tilde{k}$ for all m and for $\tilde{k} \tilde{\in} R(P)^*$.
By **lemma 3.3** $\tilde{k} \tilde{\geqq} \|\widetilde{x_m}\| = \left\|\sum_{i=1}^{n} \widetilde{\alpha_{i_m}}\widetilde{x_i}\right\| \tilde{\geqq} \tilde{c}\sum_{i=1}^{n}\left|\alpha_{i_m}\right|$ where $\tilde{c} \tilde{>} \bar{0}$
i.e., $\tilde{k}(\lambda) \tilde{\geqq} [\tilde{c}\sum_{i=1}^{n}\left|\alpha_{i_m}\right|](\lambda)$ for every $\lambda \in$ P, hence the sequence of soft real number $\widetilde{\alpha_{i_m}}(\lambda)$ (i fixed) is soft bounded. Bolzano-weierstrass theorem states that it has a point of accumulation say $\tilde{\alpha}_i$, we determine that $\{\widetilde{x_m}\}$ has a subsequence $\{\widetilde{z_m}\}$ which converge to $\tilde{z} = \sum_{i=1}^{n} \tilde{\alpha}_i\tilde{x}_i$, because M is soft closed, then $\tilde{z} \tilde{\in}$ M .this shows that the sequence $\{\widetilde{x_m}\}$ in M (which is random) has a subsequence which converges in M . Therefor M is soft compact.

Theorem 3.5: Let $\tilde{X}, \tilde{Y}$ be two soft normed spaces. Consider T: SE($\tilde{X}$) ⟶ SE($\tilde{Y}$) to be a soft continuous linear operator. Then the image of any soft compact subset of $\tilde{X}$ under T is soft compact.

proof : let M = (G,A) be a soft compact subset of $\tilde{X}$, it suffices to show that every sequence$\{\tilde{y}\}$in the image $T(M) \tilde{\subseteq} SE(\tilde{Y})$ contain a convergent subsequence such that converges in T(M). Since $\widetilde{y_n} \tilde{\in} T(M)$, then there exist $\widetilde{x_n} \tilde{\in}$ M such that $\widetilde{y_n} = T(\widetilde{x_n})$ for all n ∈ N.
Since M is soft compact, then $\{\widetilde{x_n}\}$ contain subsequence $\{\widetilde{x_{n_k}}\}$ which converge in M .the image of $\widetilde{x_{n_k}}$ is a subsequence of $\{\widetilde{y_n}\}$ which converge in T(M) because T is continuous (if $\widetilde{x_{n_k}} \longrightarrow \widetilde{x_{n_0}}$ then $T\widetilde{x_{n_k}} \longrightarrow T\widetilde{x_{n_0}}$).
So $T\widetilde{x_{n_k}}$ converges. Hence T(M) is soft compact.

Now the definition of soft compact operator is given:
Definition 3.6: (soft compact operator) let $\tilde{X}, \tilde{Y}$ be two soft normed spaces. T: SE($\tilde{X}$) ⟶ SE($\tilde{Y}$) be a soft operator. T is called a soft compact operator if for each bounded soft subset M of $\tilde{X}$, the image T(M) is relatively soft compact i.e., $\overline{T(M)}$ is soft compact .

Proposition 3.7: Let $\tilde{X}, \tilde{Y}$ be two soft normed spaces then every soft compact operator T: SE($\tilde{X}$) ⟶ SE($\tilde{Y}$) is soft bounded , hence soft continuous .
Proof: the set M = $\{ \tilde{x} \tilde{\in} SE(\tilde{X}) : \|\tilde{x}\| = \bar{1}\}$ is soft bounded . Since T is soft compact then $\overline{T(M)}$ is soft compact and $\overline{T(M)}$ is soft bounded, so $\sup\|T\tilde{x}\| \tilde{\leqq} \tilde{k}$ where $\tilde{k} \tilde{\geqq} \bar{0}$, hence T is soft bounded therefore T is soft continuous.
Theorem 3.8 : let $\tilde{X}, \tilde{Y}$ be two soft normed spaces and let T : SE($\tilde{X}$) ⟶ SE($\tilde{Y}$) be soft linear operator then T is soft compact if and only if it maps every soft bounded sequence of soft vectors $\{\widetilde{x_n}\}$ in $\tilde{X}$ onto a sequence $(T\widetilde{x_n})$ in $\tilde{Y}$ which has a convergent subsequence .
Proof: if T is soft compact and $\widetilde{x_n}$ is soft bounded then the closure of $(T\widetilde{x_n})$ in $\tilde{Y}$ is soft compact and by **Definition 3.1** $\{T\widetilde{x_n}\}$ contain a convergent subsequence.

Conversely, assume that every soft bounded sequence $\{\widetilde{x_n}\}$ contain a subsequence $\widetilde{x_{n_k}}$ such that $\{T\widetilde{x_{n_k}}\}$ converge in $\tilde{Y}$. Consider any soft bounded subset $B \tilde{\subseteq} \tilde{X}$ and let $\{\widetilde{y_n}\}$ be random sequence in T(B), then $\widetilde{y_n} = T(\widetilde{x_n})$ for some $\widetilde{x_n} \tilde{\in}$ B, and $\{\widetilde{x_n}\}$ is soft bounded since B is soft bounded.

By assumption {T $\widetilde{x_n}$}contain a convergent subsequence , hence T(B) is soft compact by **Definition 3.1** and T(B) is soft closed by **proposition 3.2** i.e., T(B) = $\overline{T(B)}$ is soft compact .because $\widetilde{y_n}$ in T(B) was arbitrary, hence T is soft compact by **Definition 3.6**

Theorem 3.9 : let $\tilde{X}$, $\tilde{Y}$ be two soft normed spaces which satisfies N_5. Consider T : SE($\tilde{X}$) ⟶ SE($\tilde{Y}$) to be a soft compact linear operator satisfy L_3 then T_λ: X ⟶ Y is a compact linear operator .
Proof: since T satisfy L_3 , then T_λ: X ⟶ Y is a linear operator for all $\lambda \in$ P.
Consider $\widetilde{x_n}$ to be a soft bounded sequence of soft vectors in $\widetilde{X}$, then for fixed λ $\widetilde{x_n}(\lambda) \in$ X and $x_n = \widetilde{x_n}(\lambda)$ for all n ∈N (fixed λ), so x_n is a bounded sequence of crisp element in X .
(In fact if $\|\widetilde{x_n}\| \lesseqq \widetilde{M}$ for all $\widetilde{x_n}$ and $\widetilde{M} \geqq \overline{0}$, then $\|\widetilde{x_n}\|(\lambda) \lesseqq \widetilde{M}(\lambda)$. hence $\|\widetilde{x_n}(\lambda)\| \leq$ M for M = $\widetilde{M}(\lambda)$, M∈ R . and that implies $\|x_n\| \leq$ M , hence $\{x_n\}$ is bounded sequence in X) .

Now, since T is soft compact, then T $\widetilde{x_n}$ having a convergent subsequence says T $\widetilde{x_{n_k}}$. Hence T $\widetilde{x_{n_k}}(\lambda)$ is converge. But T $\widetilde{x_{n_k}}(\lambda)$ = T $x_{n_k} \in$ Y *for all* $n_k \in$ N (fixed λ) ,hence for every bounded sequence x_n in X implies T(x_n) have a convergent subsequence T(x_{n_k}) in Y.

i.e., T_λ: X ⟶ Y is a compact linear operator for all $\lambda \in$ P

Theorem 3.10: let T_λ: X ⟶ Y be a soft linear operator and let $\tilde{X}$, $\tilde{Y}$ be the corresponding absolute soft vector spaces satisfies N_5 , if T_λ is compact for all $\lambda \in$ P , then T : SE($\tilde{X}$) ⟶ SE($\tilde{Y}$) is soft compact linear operator .

Poof: since T_λ is linear for all $\lambda \in$ A, then T: SE($\tilde{X}$) ⟶ SE($\tilde{Y}$) is linear and satisfy $L_3$by (**Theorem 2.36**).
Consider {$\widetilde{x_n}$} to be a soft bounded sequence of soft vectors in $\tilde{X}$ i.e., $\|\widetilde{x_n} - \widetilde{x_m}\| \lesseqq \widetilde{M}$ for each n,m ∈N and $\widetilde{M} \geqq \overline{0}$, then $\|\widetilde{x_n} - \widetilde{x_m}\|(\lambda) \lesseqq \widetilde{M}(\lambda)$ for each $\lambda \in$ P. hence $\|(\widetilde{x_n} - \widetilde{x_m})(\lambda)\| \leq$ M for M = $\widetilde{M}(\lambda)$, M ∈ R . and that implies $\|x_n - x_m\| \leq$ M where $x_n = \widetilde{x_n}(\lambda)$, $x_m = \widetilde{x_m}(\lambda)$ for all n,m ∈N, hence$\{x_n\}$is bounded sequence in X. T_λ is compact for each λ ∈P implies $T_\lambda(\widetilde{x_n}(\lambda)) = T_\lambda(x_n)$ has a convergent subsequence say T($\widetilde{x_{n_k}}$) =$T_\lambda(\widetilde{x_{n_k}}(\lambda))$. Hence T($\widetilde{x_{n_k}}$) is convergent subsequence . i.e., for all {$\widetilde{x_n}$} bounded sequence in $\tilde{X}$, T($\widetilde{x_n}$) has a convergent subsequence . Hence T is soft compact.

□

Theorem 3.11: (soft compactness of product)
Let T: SE($\tilde{X}$) ⟶ SE($\tilde{X}$) be a soft compact operator and S : SE($\tilde{X}$) ⟶ SE($\tilde{X}$) a soft bounded operator . Then ST and TS are soft compact.
Proof : let B $\tilde{\subseteq}$ $\tilde{X}$ be any soft bounded subset , since S is soft bounded , S(B) is a soft bounded set and the soft set T(S(B)) = TS(B) is relatively soft compact since T is soft compact . Hence TS is soft compact.
In the other hand, consider { $\widetilde{x_n}$} to be a soft bounded sequence in $\tilde{X}$. We get T($\widetilde{x_n}$) has a convergent subsequence say {T($\widetilde{x_{n_k}}$)} by definition of soft compact linear operator . S(T($\widetilde{x_{n_k}}$))= ST($\widetilde{x_{n_k}}$) converge since S is soft bounded hence soft continuous . Hence ST is soft compact.

Theorem 3.12: (finite dimensional domain or range)
Let $\tilde{X}$, $\tilde{Y}$ be two soft normed spaces which satisfy N_5 , consider T: SE($\tilde{X}$) ⟶ SE($\tilde{Y}$) to be a soft compact linear operator then :
a) If T is soft bounded and dim T($\tilde{X}$) $< \infty$, then T is soft compact.
b) if dim $\tilde{X} < \infty$, then T is soft compact .
Proof: (a) let {$\widetilde{x_n}$} be any soft bounded sequence in $\tilde{X}$. Since T is soft bounded, then $\|T\widetilde{x_n}\| \leq \|T\| \, \|\widetilde{x_n}\|$.
So {T($\widetilde{x_n}$)} is soft bounded , hence {T($\widetilde{x_n}$)} is soft compact by **proposition (3.4)**.

It follows that $\{T(\widetilde{x_n})\}$ has a convergent subsequence by **definition (3.1).** Because $\{\widetilde{x_n}\}$ was random soft bounded sequence in $\tilde{X}$, then T is soft compact.
(b) Since dim $\tilde{X} < \infty$, we obtained that T is soft bounded. Now, with fact that dim $T(\tilde{X}) \leq$ dim $\tilde{X}$ and from (a) we complete the proof.

4. SOFT ADJOINT OPERATOR

Definition 4.1: Consider T: $SE(\tilde{X}) \rightarrow SE(\tilde{Y})$ to be soft bounded linear operator, where $\tilde{X}$ and $\tilde{Y}$ are soft normed spaces satisfying (N_5). Then the soft adjoint operator T^*: $SE(\tilde{Y}^*) \rightarrow SE(\tilde{X}^*)$ of T is symbolized by :

$T^*g = f$ where $g \tilde{\in} SE(\tilde{Y}^*)$ and $f \tilde{\in} SE(\tilde{X}^*)$

$(T^* g)(\tilde{x}) = g(T(\tilde{x})) = f(\tilde{x})$ $\tilde{x} \tilde{\in} \tilde{X}$ where $\tilde{X}^*$ and $\tilde{Y}^*$ are the dual spaces of $\tilde{X}$ and $\tilde{Y}$, correspondingly.

Theorem 4.2: The soft adjoint operator T^* in previous definition is soft linear and soft bounded, and $\|T^*\| = \|T\|$.
Proof: let $g_1, g_2 \tilde{\in} SE(\tilde{Y}^*)$ and $\tilde{\alpha}, \tilde{\beta}$ be two soft scalar ,

$$[T^*(\tilde{\alpha}g_1 + \tilde{\beta}g_2)](\tilde{x}) = (\tilde{\alpha}g_1 + \tilde{\beta}g_2)T(\tilde{x})$$
$$= \tilde{\alpha}g_1[T(\tilde{x})] + \tilde{\beta}g_2[T(\tilde{x})]$$
$$= \tilde{\alpha}T^*g_1(\tilde{x}) + \tilde{\beta}T^*g_2(\tilde{x})$$
$$= [\tilde{\alpha}T^*g_1 + \tilde{\beta}T^*g_2](\tilde{x})$$

Hence $T^*$is soft linear. Now, $\|T\|(\lambda) = \sup\{\|T(\tilde{x})\|(\lambda) : \|\tilde{x}\| = \bar{1}\}$for all $\lambda \in P$.
Hence $\|T^*\|(\lambda) = \sup\{\|T^*g\|(\lambda) : \|g\| = \bar{1}\} = \sup\{\|g(T(\tilde{x}))\|(\lambda) : \|g\| = \bar{1}\}$
$\leq \sup\{\|g\|\|T(\tilde{x})\|(\lambda) : \|g\| = \bar{1}\} = \sup\{\|T(\tilde{x})\|(\lambda) : \tilde{x} \tilde{\in} \tilde{X}\}$, in particular if $\|\tilde{x}\| = \bar{1}$
$= \|T\|(\lambda)$ for all $\lambda \in P$.
So $\|T^*\| \tilde{\leq} \|T\|$. Hence T^* is soft bounded.

Proposition 4.3: Let T: $SE(\tilde{X}) \rightarrow SE(\tilde{Y})$ be soft bounded linear operator, where $\tilde{X}$ and $\tilde{Y}$ are soft normed spaces satisfying (N_5). Then the soft adjoint operator T^*: $SE(\tilde{Y}^*) \rightarrow SE(\tilde{X}^*)$ of T possess the succeeding possessions :
1) $(R + T)^* = R^* + T^*$
2) $(\tilde{\alpha}T)^* = \tilde{\alpha}T^*$
3) if $\tilde{X}, \tilde{Y}, \tilde{Z}$ are soft normed spaces such that T^*: $SE(\tilde{Y}^*) \rightarrow SE(\tilde{X}^*)$ and R^*: $SE(\tilde{Z}^*) \rightarrow SE(\tilde{Y}^*)$ then :
$$(RT)^* = T^*R^*$$

Proof: (1) $[(R + T)^*g](\tilde{x}) = g[(R + T)(\tilde{x})]$
$$= g[R(\tilde{x}) + T(\tilde{x})]$$
$$= g(R(\tilde{x})) + g(T(\tilde{x}))$$
$$= R^*g(\tilde{x}) + T^*g(\tilde{x})$$
$$= (R^*g + T^*g)(\tilde{x})$$
$$= [(R^* + T^*)g](\tilde{x})$$
Hence $(R + T)^* = R^* + T^*$

(2) $[(\tilde{\alpha}T)^*g](\tilde{x}) = g[\tilde{\alpha}T(\tilde{x})] = \tilde{\alpha}g(T(\tilde{x})) = \tilde{\alpha}T^*g(\tilde{x})$. Hence $(\tilde{\alpha}T)^* = \tilde{\alpha}T^*$.

(3) Let $T^*g = f \text{ and } R^*h = g$ where $g \tilde{\in} \tilde{Y}^*$, $h \tilde{\in} \tilde{Z}^*$

$[(RT)^*h](\tilde{x}) = h(RT)(\tilde{x}) = h[R(T(\tilde{x}))] = R^*h(T(\tilde{x})) = (g(T(\tilde{x})) = T^*g(\tilde{x}) = (T^*R^*)h(\tilde{x})$.
Hence $(RT)^* = T^*R^*$.

We shall consider a soft compact linear operator T: SE($\widetilde{X}$) → SE($\widetilde{X}$) on a soft normed space , the soft adjoint operator T*: SE($\tilde{X}^*$) → SE($\tilde{X}^*$). We try to discovery the result of the equation T($\tilde{x}$)$-\mu\tilde{x} = \tilde{y}$ where $\mu \neq 0$.Firstly we need to prove the following lemma.

Lemma 4.4 :(existence of functional)

Cosider Y to be a proper soft closed subspace of soft normed space $\widetilde{X}$. Let $\tilde{x}_0 \tilde{\in} \widetilde{X} - Y$ be random and the distance from $\tilde{x}_0$ to Y is $\tilde{\delta} = inf_{\tilde{y}\tilde{\in}Y}\|\tilde{y} - \tilde{x}_0\|$. Then we can find f $\tilde{\in}$ $\widetilde{X}^*$ such that $\|f\|$ = $\bar{1}$, f($\tilde{y}$) = $\bar{0}$ for all $\tilde{y} \tilde{\in} Y$, f($\tilde{x}_0$) = $\tilde{\delta}$.

Proof: we consider the subspace Z $\tilde{\subset}$ $\widetilde{X}$ spanned by Y and $\tilde{x}_0$.every $\tilde{z} \tilde{\in} Z$ = span(Y$\tilde{\cup}$ $\tilde{x}_0$) has a unique representation $\tilde{z} = \tilde{y} + \tilde{\alpha}\tilde{x}_0$ where $\tilde{y} \tilde{\in} Y$. Define on Z a soft bounded linear functional f by:
f($\tilde{z}$) = f($\tilde{y} + \tilde{\alpha}\tilde{x}_0$) = $\tilde{\alpha}\tilde{\delta}$. becuase Y is soft closed and $\tilde{\delta} \tilde{\geqq} \bar{0}$, we obtained that f $\tilde{\not\equiv}$ 0.
If $\tilde{\alpha} = \bar{0}$, then f($\tilde{y}$) = $\bar{0}$ with any $\tilde{y} \tilde{\in} Y$. When $\tilde{\alpha} = \bar{1}$ and $\tilde{y} = \theta$ we have f($\tilde{x}_0$) = $\tilde{\delta}$.
We show that f is soft bounded. $\tilde{\alpha} = \bar{0}$ gives f($\tilde{z}$) = θ. Let $\tilde{\alpha} \neq \bar{0}$.
$|f(\tilde{z})|$ = $|\tilde{\alpha}|\tilde{\delta}$ = $|\tilde{\alpha}|inf_{\tilde{y}\tilde{\in}Y}\|\tilde{y} - \tilde{x}_0\| \tilde{\leqq} |\tilde{\alpha}| \left\|-\frac{\bar{1}}{\tilde{\alpha}}\tilde{y} - \tilde{x}_0\right\|$ = $\|\tilde{y} + \tilde{\alpha}\tilde{x}_0\|$ since $-\frac{\bar{1}}{\tilde{\alpha}}\ \tilde{y}\ \tilde{\in} Y$.
That is $|f(\tilde{z})| \tilde{\leqq} \|\tilde{z}\|$. Hence f is soft bounded and $\|f\| \tilde{\leqq} \bar{1}$. We now prove that $\|f\| \tilde{\geqq} \bar{1}$. Use the infimum definition, Y has a sequence $\{\widetilde{y_n}\}$ satisfy $\|\tilde{y} - \tilde{x}_0\| \to \tilde{\delta}$. Let $\widetilde{z_n} = \widetilde{y_n} - \tilde{x}_0$. Then we have f($\widetilde{z_n}$) = $-\tilde{\delta}$ with $\tilde{\alpha} = \overline{(-1)}$. Also $\|f\| = sup_{\substack{\tilde{z}\tilde{\in}Z \\ \tilde{z}\neq\theta}} \frac{|f(\tilde{z})|}{\|\tilde{z}\|} \geq \frac{|f(\widetilde{z_n})|}{\|\widetilde{z_n}\|} = \frac{\tilde{\delta}}{\|\widetilde{z_n}\|} \to \frac{\tilde{\delta}}{\tilde{\delta}} = \bar{1}\ as\ n \to \infty$.
Hence $\|f\| = \bar{1}$. Use the hahan Banach statement for soft normed space; we can enlarge f to all $\widetilde{X}$.

We shall consider a soft compact linear operator T: SE($\tilde{X}$) → SE($\tilde{X}$) on a soft normed space $\widetilde{X}$. The soft adjoint operator T*: SE($\tilde{X}^*$) → SE($\tilde{X}^*$). The equation:
(1) …………T$\tilde{x} - \mu\tilde{x} = \tilde{y}$ where $\tilde{y} \tilde{\in} \widetilde{X}$ given, $\mu \neq 0$.
The corresponding homogeneous equation:
(2) ……………. T$\tilde{x} - \mu\tilde{x} = \theta$
And two similar equation involving the soft adjoint operator,
(3)……………… T*$f - \mu f = g$ where g$\tilde{\in}$ $\widetilde{X}^*$given, $\mu \neq 0$.
And the corresponding homogeneous equation:
(4)……………… T*$f - \mu f = \theta$

Theorem 4.5: Let T: SE($\tilde{X}$) ⟶ SE($\tilde{X}$) be a soft compact linear operator on a soft normed space $\tilde{X}$,and let $\mu \neq 0$. Then (1) has a solution $\tilde{x}$ if and only if $\tilde{y}$ is such that f($\tilde{y}$) = $\bar{0}$ for each f $\tilde{\in}$ $\tilde{X}^*$filling (4). So if (4) has one solution f ≡ 0, then (1) with any assumed $\tilde{y} \tilde{\in} \widetilde{X}$ is solvable.

Proof: suppose (1) has a solution $\tilde{x} = \widetilde{x_0}$, that is $\tilde{y} = T\widetilde{x_0} - \mu\widetilde{x_0} = T_\mu\widetilde{x_0}$.
Let f be any solution for (4). Then we have $f(\tilde{y}) = f(T\widetilde{x_0} - \mu\widetilde{x_0}) = f(T\widetilde{x_0}) - \mu f(\widetilde{x_0})$.
Now, $f(T\widetilde{x_0}) = (\text{T}^*f)(\widetilde{x_0})$ by the definition of the soft adjoint operator.
Hence by (4) $(\tilde{y}) = (\text{T}^*f)(\widetilde{x_0}) - \mu f(\widetilde{x_0}) = \bar{0}$.
Conversely, we assume that $\tilde{y}$ in (1) satisfies $f(\tilde{y}) = \bar{0}$ for every solution of (4) and show that (1) has a solution.
Suppose that (1) has no solution, hence $\tilde{y} = T_\mu\tilde{x}$ for no $\tilde{x}$. Then $\tilde{y} \tilde{\notin} T_\mu(\tilde{X})$.
Since $T_\mu(\tilde{X})$ is soft closed, the distance $\tilde{\delta}$ from $\tilde{y}$ to $T_\mu(\widetilde{X})$ is positive soft scalar. By **lemma 4.4** there existe $f \tilde{\in} \tilde{X}^*$ such that $f(\tilde{y}) = \tilde{\delta}\ and\ f(\tilde{z}) = \bar{0}$ for every $\tilde{z} \tilde{\in} T_\mu(\widetilde{X})$.
Since $\tilde{z} \tilde{\in} T_\mu(\tilde{X})$, we have $\tilde{z} = T_\mu(\tilde{x})$ for some $\tilde{x} \tilde{\in} \tilde{X}$. So that $f(\tilde{z}) = \bar{0}$ becomes:
$f\left(T_\mu(\tilde{x})\right) = f(T\tilde{x}) - \mu f(\tilde{x}) = \text{T}^*f(\tilde{x}) - \mu f(\tilde{x}) = \bar{0}$.
This holds for every $\tilde{x} \tilde{\in} \widetilde{X}$ since $\tilde{z} \tilde{\in} T_\mu(\widetilde{X})$ was arbitrary. Hence f is a solution of (4). By assumption it satisfies $(\tilde{y}) = \bar{0}$. But this contradicts $(\tilde{y}) = \tilde{\delta} \tilde{>} \bar{0}$. Consequently, (1) must have a solution. This proves first part of theorem. The proof of second part follows.

For equation (3) there is an analogue of **Theorem 4.5** which we shall obtain from the following lemma.
Lemma 4.6: Let T: SE($\tilde{X}$) ⟶ SE($\tilde{X}$) be a soft compact linear operator on a soft normed space $\tilde{X}$,and let $\mu \neq 0$ be assumed. We can find a soft real number $\tilde{c} \tilde{\geqq} \bar{0}$ which is free of $\tilde{y}$ in (1) and such that for every $\tilde{y}$ for which (1) has a solution, at least one of these solution call it $\tilde{k}$ satisfies $\|\tilde{k}\| \tilde{\leqq} \tilde{c}\|\tilde{y}\|$ where $\tilde{y} = T_\mu(\tilde{k})$.

Proof: firstly, we show that if (1) with a given $\tilde{y}$ has a solution at all, the set of these solution contains a solution of minimum norm call it $\tilde{k}$.
Let $\widetilde{x_0}$ be a solution of (1). If $\tilde{x}$ is any other solution of (1), the difference $\tilde{z} = \tilde{x} - \widetilde{x_0}$ fulfills (2). Therefore each solution of (1) can be written $\tilde{x} = \widetilde{x_0} + \tilde{z}$ where $\tilde{z} \tilde{\in} \mathcal{N}(T_\mu)$. And, conversely, for every $\tilde{z} \tilde{\in} \mathcal{N}(T_\mu)$ the sum $\widetilde{x_0} + \tilde{z}$ is a solution of (1). For a fixed $\widetilde{x_0}$ the norm of $\tilde{x}$ depends on $\tilde{z}$, we write $p(\tilde{z}) = \|\widetilde{x_0} + \tilde{z}\|$ and $H = inf_{\tilde{z} \tilde{\in} \mathcal{N}(T_\mu)} p(\tilde{z})$. By definition of infimum, $\mathcal{N}(T_\mu)$ contain a sequence $\{\widetilde{z_n}\}$such that $p(\widetilde{z_n}) = \|\widetilde{x_0} + \widetilde{z_n}\| \to H \quad as\, n \to \infty$.
Since $\{p(\widetilde{z_n})\}$ converge, it is bounded. Also $\{\widetilde{z_n}\}$ is bounded because:

$$\|\widetilde{z_n}\| = \|\widetilde{x_0} + \widetilde{z_n} - \widetilde{x_0}\| \tilde{\leq} \|\widetilde{x_0} + \widetilde{z_n}\| + \|\widetilde{x_0}\| = p(\widetilde{z_n}) + \|\widetilde{x_0}\|.$$

Becuase T is soft compact, $\{T(\widetilde{z_n})\}$possess a convergent subsequence. But $\tilde{z} \tilde{\in} \mathcal{N}(T_\mu)$ means that $T_\mu(\widetilde{z_n}) = \theta$, that is, $T\widetilde{z_n} = \mu\widetilde{z_n}$; where $\mu \neq 0$. Hence $\{\widetilde{z_n}\}$ has a convergent subsequence say, $\widetilde{z_{n_j}} \to \widetilde{z_0}$ where $\widetilde{z_0} \tilde{\in} \mathcal{N}(T_\mu)$ since $\mathcal{N}(T_\mu)$ is closed. Also $p(\widetilde{z_{n_j}}) \to p(\widetilde{z_0})$ since p is soft continuous. We thus obtain that $p(\widetilde{z_0}) = \|\widetilde{x_0} + \widetilde{z_0}\| = H$. This mean that if the equation (1) with a assumed $\tilde{y}$ has a solution, then one of these solutions $\tilde{k} = \widetilde{x_0} + \widetilde{z_0}$ has a smallest norm.
Secondly, we have proven that there exist $\tilde{c} \tilde{\geq} \bar{0}$ (independent of $\tilde{y}$) such that $\|\tilde{k}\| \tilde{\leq} \tilde{c}\|\tilde{y}\|$ for a solution $\tilde{k}$ of minimum norm consistent to any $\tilde{y} = T_\mu(\tilde{k})$ wherefore (1) is solvable.
Suppose that is not true. Then there is a sequence $\{\widetilde{y_n}\}$ such that $\frac{\|\widetilde{k_n}\|}{\|\widetilde{y_n}\|} \to \infty \ as\ n \to \infty$. Where $\widetilde{k_n}$ is of least norm and satisfies $\widetilde{y_n} = T_\mu(\widetilde{k_n})$. Multiplication by $\tilde{\alpha}$ shows that to $\tilde{\alpha}\widetilde{y_n}$ there corresponds $\tilde{\alpha}\widetilde{k_n}$ as a solution of least norm. Hence we may accept that $\|\widetilde{k_n}\| = \bar{1}$, without Influence the general meaning.
Then $\frac{\|\widetilde{k_n}\|}{\|\widetilde{y_n}\|} \to \infty\ with\ \|\widetilde{k_n}\| = \bar{1} implies\ \|\widetilde{y_n}\| \to \bar{0}$. Since T is soft compact and $\{\widetilde{k_n}\}$ is soft bounded, $\{T(\widetilde{k_n})\}$ has a convergent subsequence say, $T\widetilde{k_{n_j}} \to T\widetilde{k_0}$. We can write for convenience $T\widetilde{k_{n_j}} \to \mu\widetilde{k_0}\ as\ j \to \infty$.
Since $\widetilde{y_n} = T_\mu(\widetilde{k_n}) = T(\widetilde{k_n}) - \mu\widetilde{k_n}$, we have $\mu\widetilde{k_n} = T(\widetilde{k_n}) - \widetilde{y_n}$. Thus we obtain:

$$T\widetilde{k_{n_j}} = \frac{1}{\mu}(T\left(\widetilde{k_{n_j}}\right) - \widetilde{y_{n_j}}) \to \widetilde{k_0}.$$

Since T is soft continuous, we have $T(\widetilde{k_{n_j}}) \to T(\widetilde{k_0})$. Hence $T(\widetilde{k_0}) = \mu\widetilde{k_0}$ because $T(\widetilde{k_n}) \to \mu\widetilde{k_0}$. Also we see that $\tilde{x} = \widetilde{k_n} - \widetilde{k_0}$ satisfies $\widetilde{y_n} = T(\widetilde{k_n})$.
Since $\widetilde{k_n}$is of minimum norm, $\|\tilde{x}\| = \|\widetilde{k_n} - \widetilde{k_0}\| \tilde{\geq} \|\widetilde{k_n}\| = \bar{1}$. But this contradicts the convergence in, $T\widetilde{k_{n_j}} = \frac{1}{\mu}(T\left(\widetilde{k_{n_j}}\right) - \widetilde{y_{n_j}}) \to \widetilde{k_0}$. Hence $\frac{\|\widetilde{k_n}\|}{\|\widetilde{y_n}\|} \to \infty$ cannot hold. But the sequence of quotients must be soft bounded; that is, we must have $\tilde{c} = sup_{\tilde{y} \tilde{\in} T_\mu(\tilde{X})} \frac{\|\tilde{k}\|}{\|\tilde{y}\|} \tilde{\leq} \tilde{M}\ where\ \tilde{y} = T_\mu(\tilde{x})$. This implies $\|\tilde{k}\| \tilde{\leq} \tilde{c}\|\tilde{y}\|$.

Using this lemma, we *can* now give a characterization of the solvability of (3) similar to that for (1) given in **Theorem 3.5**:

Theorem 4.7: (solution of (3))
Let T: SE($\tilde{X}$) ⟶ SE($\tilde{X}$) be a soft compact linear operator on a soft normed space $\tilde{X}$,and let $\mu \neq 0$ be assumed. Then (3) has a solution f if and only if g is such that $g(\tilde{x}) = \bar{0}$ for all $\tilde{x} \tilde{\in} \tilde{X}$ which satisfy (2). Hence if (2) has the petty solution $\tilde{x} = \theta$ only, then (3) with any $g \tilde{\in} \tilde{X}^*$ is solvable.

Proof: (a) if (3) has a solution f and $\tilde{x}$ satisfies (2), then

$$g(\tilde{x}) = (T^*f)(\tilde{x}) - \mu f(\tilde{x}) = f(T\tilde{x} - \mu\tilde{x}) = f(\theta) = \bar{0}.$$

(b) Conversely, assume that g satisfies $g(\tilde{x}) = \bar{0}$ for every solution $\tilde{x}$ of (2). Consider any $\tilde{x} \tilde{\in} \tilde{X}$ and set $\tilde{y} = T_\mu(\tilde{x})$. Then $\tilde{y} \tilde{\in} T_\mu(\tilde{X})$. We may define a functional f_0 on $T_\mu(\tilde{X})$ by:
$f_0(\tilde{y}) = f_0(T_\mu(\tilde{x}) = g(\tilde{x})$. This definition is unambiguous because if $T_\mu(\widetilde{x_1}) = T_\mu(\widetilde{x_2})$, then $T_\mu(\widetilde{x_1} - \widetilde{x_2}) = \theta$. So that $\widetilde{x_1} - \widetilde{x_2}$ is a solution of (2); hence $g(\widetilde{x_1} - \widetilde{x_2}) = \bar{0}$ by assumption, that is $g(\widetilde{x_1}) = g(\widetilde{x_2})$. f_0 is linear since T_μ and g are linear. **Lemma 4.6** implies that for every $\tilde{y} \tilde{\in} T_\mu(\tilde{X})$, at least one of the corresponding $\tilde{x}'$ssatisfy $\|\tilde{x}\| \tilde{\leq} \tilde{c}\|\tilde{y}\|$ where $\tilde{y} = T_\mu(\tilde{x})$ and $\tilde{c}$ does not depend on $\tilde{y}$. So we have $|f_0(\tilde{y})| = |g(\tilde{x})| \tilde{\leq} \|g\|\|\tilde{x}\| \tilde{\leq} \tilde{c}\|g\|\|\tilde{y}\|$. Hence $\|f_0\| \tilde{\leq} \tilde{c}\|g\|$.
So f_0 is soft bounded. Useing the Hahn-banach statement show that the functional f_0 has an expanding f on $\tilde{X}$ which is a soft bounded linear functional defined on all $\tilde{X}$. By the definition of f_0 ,
$f(T(\tilde{x}) - \mu\tilde{x}) = f(T_\mu(\tilde{x}) = f_0(T_\mu(\tilde{x}) = g(\tilde{x})$.

Definition of the soft adjoint operator show that we have for all $\tilde{x} \tilde{\in} \tilde{X}$:
$f(T(\tilde{x}) - \mu\tilde{x}) = f(T(\tilde{x}) - \mu f\mu(\tilde{x}) = (T^{*}f)(\tilde{x}) - \mu f(\tilde{x})$. From this we conclude that f is a solution of (3) and first Demands of theorem is proves. Consequently, the second Demands follow freely.

Theorem 4.8: Let T: $SE(\tilde{X}) \longrightarrow SE(\tilde{X})$ be a soft compact linear operator on a soft normed space $\tilde{X}$, And let $\mu \neq 0$ then:
(a) Equation (1) has a solution $\tilde{x}$ for every $\tilde{y} \tilde{\in} \tilde{X}$ if and only if the homogeneous equation (2) has only the petty solution $\tilde{x} = \theta$. In this situation the equation (1) has a unique solution, and T_{μ} has a soft bounded inverse.
(b) Equation (3) has a solution f for all $g \tilde{\in} \tilde{X}^{*}$ if and only if (4) has only the petty solution $f \equiv 0$. In this situation the equation (3) has a unique solution.

Proof: Let for each $\tilde{y} \tilde{\in} \tilde{X}$ the equation (1) is solvable. Suppose that $\tilde{x} = \theta$ is not the single solution of (2). Then there exist a solution $\widetilde{x_1} \neq \theta$. Because (1) for any $\tilde{y}$ is solvable, $T_{\mu}(\tilde{x}) = \tilde{y} = \widetilde{x_1}$ has a solution $\widetilde{x_2}$. That is $T_{\mu}(\widetilde{x_2}) = \widetilde{x_1}$. For the same reason there exist $\widetilde{x_3}$ such that $T_{\mu}(\widetilde{x_3}) = \widetilde{x_2}$, etc. thus for each k = 2,3,…….
$\theta \neq \widetilde{x_1} = T_{\mu}(\widetilde{x_2}) = {T_{\mu}}^{2}(\widetilde{x_3}) = \cdots\cdots = {T_{\mu}}^{k-1}(\widetilde{x_k})$ and $\theta = T_{\mu}(\widetilde{x_1}) = {T_{\mu}}^{k}(\widetilde{x_k})$.
Hence $\widetilde{x_k} \tilde{\in} \mathcal{N}({T_{\mu}}^{k})$ but $\widetilde{x_k} \tilde{\notin} \mathcal{N}({T_{\mu}}^{k-1})$. This means that $\mathcal{N}({T_{\mu}}^{k-1})$ is a proper subspace of $\mathcal{N}({T_{\mu}}^{k})$ for all k. but this contradiction. Hence $\tilde{x} = \theta$ must be the unique solution for (2).
On the other hand, suppose that $\tilde{x} = \theta$ is the only solution of (2). Then equation (3) for every g is solvable by (**Theorem 4.7**)
Now, since T^{*} is soft compact, So that by first part of the proof and replace by T^{*}, we conclude that $f \equiv 0$ should be the only solution of (4). Solvability of (1) with any $\tilde{y}$ now follows from **Theorem 4.5**.
Uniqueness of the solution comes from the fact that the difference of two solutions of (1) is a solution of (2). Clearly, such a unique solution $\tilde{x} = {T_{\mu}}^{-1}(\tilde{y})$ is the solution of least norm. And the boundedness of ${T_{\mu}}^{-1}$ follows by **Lemma 4.6**. i.e., $\|\tilde{x}\| = \left\|{T_{\mu}}^{-1}(\tilde{y})\right\| \lesseqqgtr \tilde{c}\|\tilde{y}\|$.
(b) Is a consequence of (a) and note that T^{*} is compact.

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On the soft stability of soft Picard and soft Mann iteration processes

Sabah A. Khaleefah
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.
Email:sab20072008@gmail.com

Buthainah A. A. Ahmed
Assistant Professor, Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.
Email:buthainah41@gmail.com

ABSTRACT.
In this paper, we define the Soft Contraction Operator, soft Picard and soft Mann iteration processes. After that we establish some stability results for the soft Picard and soft Mann iteration processes considered in soft normed spaces.

1. INTRODUCTION

Let $(\tilde{X}, \|.\|)$ be a complete soft normed space and let T : SE($\tilde{X}$) → SE($\tilde{X}$) be a self-map of $\tilde{X}$. Consider F(T) ={$\tilde{p} \,\tilde{\in}\, \tilde{X}$: T$\tilde{p} = \tilde{p}$} denote the set of fixed points of T. Let $\{\tilde{x}_n\}_{n=0}^{\infty}$ be the sequence generated by an iteration procedure involving the operator T,

That is $\tilde{x}_{n+1} = f(T, \tilde{x}_n)$, n = 0, 1, 2, ... (1)

Consider $\tilde{x}_0 \,\tilde{\in}\, \tilde{X}$ is the initial approximation and f is some function. Suppose $\{\tilde{x}_n\}_{n=0}^{\infty}$ converges to a fixed point $\tilde{p}$ of T. Let $\{\tilde{y}_n\}_{n=0}^{\infty} \,\tilde{\subset}\, \tilde{X}$ and set $\epsilon_n = \|\tilde{y}_{n+1} - f(T, \tilde{y}_n)\|$,n = 0, 1, 2, ... Then, the iteration procedure (1) is said to be T-stable or stable with respect to T if and only if $\lim_{n\to\infty} \epsilon_n = \bar{0}$ implies $\lim_{n\to\infty} \tilde{y}_n = \tilde{p}$. Using this concept, we proved some stability results under soft contraction conditions.

2. PRELIMINARIES

The basic definitions and theorems were introduced in this section that may found in earlier studies.

Definition 2.1 [1] Suppose X is a universe set; T is a set of parameters. Consider ℘(X) is the set of all subsets of X and B ≠ ∅ is a subset of T. An ordered pair (H, B) is named a soft set over X, where H is a mapping given by H: B → ℘(X). We can say that a soft set over X is parameterized kindred of subsets of the universe X. H (w) can consider like a set of w-approximate elements of (H, B) for all w ∈ B.

Definition 2.2 [2] Supposes (H, B) and (J, D) are two soft sets over a shared set X, then (H, B) is a soft subset of (J, D) if:
(1) B ⊆ D.
(2) For all w ∈ B, H (w) ⊆ J (w). We write (H, B) $\tilde{\subset}$ (J, D).
(J, D) is said to be a soft superset of (H, B), We write (H, B) $\tilde{\subset}$((J, D) if (H, B) is a soft subset of (J, D).

Definition 2.3 [3] Two soft sets (H, B) and (J, D) over a shared set X are called identical, if (H, B) and (J, D) are soft subset of each other.

Definition 2.4 [3] Let (H, B), (J, D) be two soft sets over the shared set X. The union of (H, B) and (J, D) is the soft set (L, M); Assuming M = B ∪ D and for all $w \in$ M,

$$L(w) = \begin{cases} H(w) & if\ w \in B - D \\ J(w) & if\ w \in D - B \\ H(w) \cup G(w) & if\ w \in B \cap D \end{cases}$$

In Mathematical expression (H, B) $\tilde{\cup}$ (J, D) = (L, M).

Definition 2.5 [4] Let (H, B), (J, D) be two soft sets over the shared set X. The intersection of (H, B) and (J, D) is the soft set (K, M); Assuming M = B ⋂ D and for all $w \in$ M, K (w) = H(w) ∩ J(w). In Mathematical expression (H, B) $\widetilde{\cap}$ (J , D) = (K , M).

Suppose X be an initial universal set and B is a non- flatulent set of parameters. In the upstairs definitions the set of parameters may differ from soft set to another, but in our considerations, through this paper all soft sets have the same set of parameters B. The upstairs definitions are also useable for these types of soft sets as a particular case of those definitions.

Definition 2.6 [5] for a soft set (F, B), the complement of (F, B) is symbolized by $(F, B)^c$= (F^c, B), assuming F^c: B→ ℘(X) defined by $F^c(\lambda)$ = X − F (λ), with any $w \in$ B.

Definition 2.7 [3] A soft set (F, B) over X is called an absolute soft set symbolized via $\tilde{X}$ if F (w) = X with every $w \in$ B.

Definition 2.8 [3] A soft set (F, A) over X is called a null soft set symbolized via $\tilde{\Phi}$ if , F(w) = ϕ with every $w \in$ B.

Definition 2.9 [6] Let (H, B), (J, D) be two soft sets over the shared set X. The difference (H, B) of (F, B) and (G, B), symbolized by (F, B) \ (J, B), is defined via H (w) = F(w)\G(w) with any $w \in$ B.

Proposition 2.10 [6] for two soft sets (F, B) and (J, B) we have:

(i) $[(F, B)\ \tilde{\cup}\ ((J, B)]^c = (F, B)^c\ \tilde{\cap}\ (J, B)^c$.

(ii) $[(F, B)\ \tilde{\cap}\ ((J, B)]^c = (F, B)^c\ \tilde{\cup}\ (J, B)^c$.

Definition 2.11 [7] Let X be a non- flatulent set of elements and B ≠ ∅ is a set of parameter. The function ε: B → X is called a soft element of X. A soft element ε of X is belongs to a soft set R of X, which is symbolized by $\varepsilon\ \tilde{\in}$ R, if $\varepsilon(w) \in$ R (w) for every $w \in$ A. consequently, for a soft set R of X we obtained that R (w) = {ε (w), $\varepsilon\ \tilde{\in}$R}, $w \in$ B.

We can recognized each singleton soft set (a soft set (H, B) for which H (w) is a singleton set, for every $w \in$ B) with a soft element by just identifying the one element set with the element that it contains for all $w \in$ B.

Definition 2.12 [8] Let $\mathfrak{B}$ ($\mathbb{R}$) be the collection of all non-empty bounded subsets of $\mathbb{R}$ ($\mathbb{R}$ is real number) and B booked as a parameters set. Then, a mapping H: B $\rightarrow \mathfrak{B}$ ($\mathbb{R}$) is named a soft real set. and symbolized with (H, B). If specifically (H, B) is a singleton soft set, then when identifying (H, B) with the corresponding soft element, it will be named a soft real number.

The collection of each soft real numbers is symbolized by $\mathbb{R}$ (B) while the collection of non-negative only is symbolized by $\mathbb{R}(B)^*$.

Definition 2.13 [9] Let $\wp$ ($\mathbb{C}$) be the collection of all non- flatulent bounded subsets of the set of complex numbers $\mathbb{C}$. B is a set of parameters. Then, a mapping H: B $\rightarrow \wp$ ($\mathbb{C}$) is named a soft complex set symbolized by (H, B). If in particular (H, B) is a singleton soft set, and then identifying (H, B) with the agreeing soft element, it will be named a soft complex number.

The collection of each soft complex numbers is symbolized by $\mathbb{C}$(B).

Definition 2.14 [9] Let (H, B) be a soft complex set. The complex conjugate of (H, B) is symbolized with ($\overline{H}$, B) and is defined by $\overline{H}(w)$ = {$\bar{z}$: z $\in$ H (w)}, for every $w \in$ B, assuming $\bar{z}$ is complex conjugate of the ordinary complex number z. The complex conjugate of a soft complex number (H, B) is $\overline{H}(w)$ = $\bar{z}$: z = H(λ), for every $w \in$ B.

Definition 2.15 [9] Let (L, B), (J, B) $\tilde{\in}$ $\mathbb{C}$(B). Then, the sum, difference, product and division are defined by:

(L + J) (w) = z + p, z $\in$ L (w), p $\in$ J (w), for all $w \in$ B.
(L − J) (w) = z − p; z $\in$ L (w), p $\in$ J (w), for all $w \in$ B.
(LJ) (w) = zp, z $\in$ L (w), p $\in$ J (w), for all $w \in$ B.
(L/J) (w) = z/p, z $\in$ L (w), p $\in$ J (w), on condition that J (w) $\neq 0$, for all $w \in$ B.

Definition 2.16 [9] Let (L, B) be a soft complex number. The modulus of (L, B) is denoted by ($|L|$, B) and is defined by $|L|(w)$ =$|z|$; z $\in$ L (w), for all $w \in$ B, assuming z is an ordinary complex number.

Since the modulus of all ordinary complex number and ordinary real number are a non-negative real number and by definition of soft real numbers it follows that ($|L|$, B) is a non-negative soft real number for every soft complex number (L, B).

Let X be a non-flatulent set and $\tilde{X}$ be the absolute soft set i.e., V (w) = X, for each $w \in$ B, where (V, B) = $\tilde{X}$. Suppose S ($\tilde{X}$) be the collection of all soft sets (H, B) over X with condition H (w) $\neq \phi$, for all $w \in$ B together with the null soft set $\tilde{\Phi}$. Let (H, B) ($\neq \Phi$) $\tilde{\in}$ S($\tilde{X}$), then the collection of all soft elements of (H, B) will be denoted by SE (H, B), For a collection $\mathfrak{B}$ of soft elements of $\tilde{X}$, the soft set generated by $\mathfrak{B}$ is symbolized with SS($\mathfrak{B}$).

Definition 2.17 [10] A mapping M: SE($\tilde{X}$) × SE($\tilde{X}$) → $R(B)^*$, is called a soft metric on the soft set $\tilde{X}$if d fulfills the following situations:

(1) M ($\tilde{x}$; $\tilde{y}$) $\tilde{\geq}$ $\bar{0}$, with any $\tilde{x}, \tilde{y} \tilde{\in} \tilde{X}$.

(2) M ($\tilde{x}$, $\tilde{y}$) = $\bar{0}$, if and only if $\tilde{x}$ = $\tilde{y}$.

(3) M ($\tilde{x}$, $\tilde{y}$) = M ($\tilde{y}$, $\tilde{x}$) with any $\tilde{x}, \tilde{y} \tilde{\in} \tilde{X}$.

(4) With any $\tilde{x}, \tilde{y}, \bar{z}$ $\tilde{\in}$ $\tilde{X}$, M ($\tilde{x}$, $\bar{z}$) $\tilde{\leq}$ M ($\tilde{x}$, $\tilde{y}$) + M ($\tilde{y}$, $\bar{z}$).

The soft set $\tilde{X}$ together with a soft metric M on $\tilde{X}$ is called a soft metric space and is symbolized by ($\tilde{X}$, M, A) or ($\tilde{X}$, M).

Definition 2.18 [11] Let Q is a vector space over a field K and B is a set of parameters. Let L be a soft set over (Q, B). If for all $\mathcal{w}$ ∈ B,L ($\mathcal{w}$) is a vector subspace of Q, Then L is called a soft vector space of Q over K.

Definition 2.19 [12] Suppose L is a soft vector space of Q over K. Let H: B → $\mathcal{p}$ (Q) be a soft set over (Q, B). If for each $\mathcal{w}$ ∈ B, H ($\mathcal{w}$) is a vector subspace of Q over K and L ($\mathcal{w}$) ⊇ H ($\mathcal{w}$), then H is called a soft vector subspace of L.

Definition 2.20 [11] Suppose LL is a soft vector space of Q over a field K, then, a soft element of L is called a soft vector of L. In the same sense a soft element of the soft set (K, B) is called a soft scalar.

Definition 2.21 [11] Let $\tilde{x}$, $\tilde{y}$ be soft vectors of L and $\tilde{k}$ be a soft scalar. The addition $\tilde{x}$ + $\tilde{y}$ of $\tilde{x}$, $\tilde{y}$ and scalar multiplication $\tilde{k}\tilde{x}$ of $\tilde{k}$ and $\tilde{x}$ are defined by ($\tilde{x}$ + $\tilde{y}$) ($\mathcal{w}$) = $\tilde{x}$ ($\mathcal{w}$) + $\tilde{y}$ ($\mathcal{w}$) , $\tilde{k}\tilde{x}$ ($\mathcal{w}$) = $\tilde{k}$ ($\mathcal{w}$) $\tilde{x}$ ($\mathcal{w}$) for all $\mathcal{w}$ ∈ B. Obviously, $\tilde{x}$ + $\tilde{y}$, $\tilde{k}$ $\tilde{x}$ are soft vectors of L.

Definition 2.22 [13] Let $\tilde{X}$ be the absolute soft vector space i.e., $\tilde{X}$ ($\mathcal{w}$) = X, for all $\mathcal{w}$ ∈ B. Then a mapping ||. || : SE($\tilde{X}$) → $R(B)^*$is called a soft norm on the soft vector space $\tilde{X}$ if ||. || fulfills the succeeding situations:

(1). ||. || $\tilde{\geq}$ $\bar{0}$ for every $\tilde{x} \tilde{\in} \tilde{X}$.

(2). $\|\tilde{x}\|$ = $\bar{0}$ if and only if $\tilde{x}$ = Θ.

(3). $\|\tilde{\alpha} . \tilde{x}\| = |\tilde{\alpha}| \, \|\tilde{x}\|$ for each $\tilde{x} \tilde{\in} \tilde{X}$ as well as for each soft scalar $\tilde{\alpha}$.

(4).With any $\tilde{x}, \tilde{y} \tilde{\in} \tilde{X}$, $\|\tilde{x} + \tilde{y}\| \tilde{\leq} \|\tilde{x}\| + \|\tilde{y}\|$

The soft vector space $\tilde{X}$ with a soft norm ||. ||on $\tilde{X}$ is called a soft normed linear space and is symbolized with ($\tilde{X}$,|| . ||, B) or ($\tilde{X}$, ||. ||). The exceeding conditions are called soft norm axioms.

Theorem 2.23 [11] Suppose a soft norm $\|.\|$ achieves the situation (N5). For $\xi \in$ X and $w \in$ B the set $\{\|\tilde{x}\|(w) : \tilde{x}(w) = \xi\}$ is a one element set. Then with any $w \in$ B, the function $\|.\|_{w}$: X → R^{+} defined with $\|\xi\|_{w} = \|\tilde{x}\|(w)$, with any $\xi \in$ X and $\tilde{x} \tilde{\in} \tilde{X}$ such that $\tilde{x}(w) = \xi$, can be considered as a norm on X.

Definition 2.24 [12] Consider $(\tilde{X}, \| . \|, B)$ is a soft normed linear space, $\tilde{r} \tilde{\geq} \bar{0}$ is a soft real number. We realize the following concepts:

$\mathbb{B}(\tilde{x}, \tilde{r}) = \{ \tilde{y} \tilde{\in} \tilde{X} : \|\tilde{x} - \tilde{y}\| \tilde{<} \tilde{r} \} \subset SE(\tilde{X})$,

$\overline{\mathbb{B}}(\tilde{x}, \tilde{r}) = \{ \tilde{y} \tilde{\in} \tilde{X} : \|\tilde{x} - \tilde{y}\| \tilde{\leq} \tilde{r} \} \subset SE(\tilde{X})$,

$S(\tilde{x}, \tilde{r}) = \{ \tilde{y} \tilde{\in} \tilde{X} : \|\tilde{x} - \tilde{y}\| = \tilde{r} \} \subset SE(\tilde{X})$,

$\mathbb{B}(\tilde{x}, \tilde{r})$, $\overline{\mathbb{B}}(\tilde{x}, \tilde{r})$, $S(\tilde{x}, \tilde{r})$ are respectively called an open ball, a closed ball and a sphere with center at $\tilde{x}$ and radius $\tilde{r}$. $SS(\mathbb{B}(\tilde{x}, \tilde{r}))$, $SS(\overline{\mathbb{B}}(\tilde{x}, \tilde{r}))$ and $SS(S(\tilde{x}, \tilde{r}))$ are respectively called a soft open ball, a soft closed ball and a soft sphere with center at $\tilde{x}$ and radius $\tilde{r}$.

Definition 2.25 [11] A sequence of soft elements $\{\widetilde{x_n}\}$ in a soft normed space $(\tilde{X}, \| . \|, B)$ called convergent sequence, if $\|\widetilde{x_n} - \tilde{x}\| \to \bar{0}$ as n → ∞, we say the sequence converges to a soft element $\tilde{x}$. In other words for each $\tilde{\epsilon} \tilde{\geq} \bar{0}$, there exists N∈ ℕ , N = N $(\tilde{\epsilon})$ and $\bar{0} \tilde{\leq} \|\widetilde{x_n} - \tilde{x}\| \tilde{\leq} \tilde{\epsilon}$ whenever n > N.

i.e., n > N *implies* $\widetilde{x_n} \in \mathbb{B}(\tilde{x}, \tilde{\epsilon})$. We symbolize this by $\widetilde{x_n} \to \tilde{x}$ as n → ∞ or by $\lim_{n\to\infty} \widetilde{x_n} = \tilde{x}$. The soft element $\tilde{x}$ called the limit of the sequence $\widetilde{x_n}$ as n goes to ∞.

Definition 2.26 [11] A sequence $\{\widetilde{x_n}\}$ of soft elements in a soft normed space $(\tilde{X}, \|.\|, B)$ is called a soft Cauchy sequence in $\tilde{X}$, if matching to each $\tilde{\epsilon} \tilde{>} \bar{0}$, there exists m ∈ N satisfy :

$$\left\|\tilde{x_i} - \tilde{x_j}\right\| \tilde{\leq} \tilde{\epsilon} \text{, for all } i,j \geq m \text{ i.e., } \left\|\tilde{x_i} - \tilde{x_j}\right\| \to \bar{0} \text{ as i; j goes to } \infty.$$

Definition 2.27 [11] Suppose $(\tilde{X}, \|.\|, B)$ is a soft normed space. Then, $\tilde{X}$ is called soft complete if every soft Cauchy sequence in $\tilde{X}$ converges to a soft element of $\tilde{X}$. The soft complete normed space is called a soft Banach Space.

Theorem 2.28 [11] Every soft Cauchy sequence in R(B) is convergent provided that B is a finite set of parameters, i.e., the set of all soft real numbers together with its usual modulus soft norm is a soft Banach space, provided that the set of parameters is finite

Definition 2.29[12] A series $\sum_{k=1}^{\infty} \widetilde{x_k}$ of soft elements called soft convergent, if the partial sum of the series $\widetilde{S_n} = \sum_{k=1}^{n} \widetilde{x_k}$ is soft convergent.

Let $\tilde{X}$, $\tilde{Y}$ be the corresponding absolute soft normed spaces i.e., $\tilde{X}(w) = X$, $\tilde{Y}(w) = Y$, for all $w \in$ B. We use the notation $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ to represent soft vectors of a soft vector space.

Definition 2.30[11] Suppose T: $SE(\tilde{X}) \to SE(\tilde{Y})$ is an operator. T is called soft linear, if
(L1). T is additive, i.e., $T(\tilde{x_1} + \tilde{x_2}) = T(\tilde{x_1}) + T(\tilde{x_2})$ with any soft elements $\tilde{x_1}, \tilde{x_2} \tilde{\in} \tilde{X}$.
(L2). T is homogeneous, i.e., with any soft scalar $\tilde{k}$, $T(\tilde{k}.\tilde{x}) = \tilde{k}\, T(\tilde{x})$, with any soft element $\tilde{x} \tilde{\in} \tilde{X}$.
The properties (L1) and (L2) can be combined in one condition $T(\widetilde{k_1}.\tilde{x_1} + \widetilde{k_2}.\tilde{x_2}) = \widetilde{k_1}\, T(\tilde{x_1}) + \widetilde{k_2}\, T(\tilde{x_2})$ for every soft elements $\tilde{x_1}, \tilde{x_2} \tilde{\in} \tilde{X}$ and every soft scalars $\widetilde{k_1}$, $\widetilde{k_2}$.

Definition 2.31[11] The operator T : SE($\widetilde{X}$) → SE($\widetilde{Y}$) is called soft continuous at $\widetilde{x_0}$ $\tilde{\in}$ $\widetilde{X}$, if for every soft sequence {$\widetilde{x_n}$} of soft elements of $\widetilde{X}$ with $\widetilde{x_n} \to \widetilde{x_0}$ as n goes to ∞, the image T($\widetilde{x_n}$) → T($\widetilde{x_0}$) as n goes to ∞. i.e., $\|\widetilde{x_n} - \widetilde{x_0}\| \to \bar{0}$ as n goes to ∞ implies $\|T(\widetilde{x_n}) - T(\widetilde{x_0})\| \to \bar{0}$ as n goes to ∞ . If T is soft continuous at every soft element of $\tilde{X}$, then T is called a soft continuous operator.

Theorem 2.32[11] Let $\widetilde{X}$, $\widetilde{Y}$ are two soft normed linear spaces and T: SE($\widetilde{X}$) →SE($\widetilde{Y}$) be a soft linear operator, If T is soft continuous at some soft element $\widetilde{x_0}$ $\tilde{\in}$ $\widetilde{X}$, then T is soft continuous at every soft element of $\widetilde{X}$.

3. *Soft Contraction Operator, soft Picard and soft Mann iteration processes*

Definition 3.1:

Let $\tilde{X}$ be a soft normed space. A soft operator T:SE($\tilde{X}$) ⟶SE($\tilde{X}$) is called a soft contraction operator if there exists a soft real number $\tilde{\alpha}$ such that $\bar{0} \tilde{\leq} \tilde{\alpha} \tilde{<} \bar{1}$ and for every $\tilde{x}, \tilde{y} \tilde{\in} \tilde{X}$ we have: $\|T\tilde{x} - T\tilde{y}\| \tilde{\leq} \tilde{\alpha}\|\tilde{x} - \tilde{y}\|$.

Example 3.2

Let $\tilde{X}$ be a soft vector space where X = $\mathcal{R}^n$ and A={1, 2, ..., n}.

Let T: SE($\tilde{X}$) ⟶SE($\tilde{X}$) be a soft operator on $\tilde{X}$ such that $T(\tilde{x}) = \overline{0.5}\tilde{x} - \bar{1}$

For all $\tilde{x}, \tilde{y} \tilde{\in} \tilde{X}$, we have $\|T\tilde{x} - T\tilde{y}\| = \|\overline{0.5}\tilde{x} - \bar{1} - \overline{0.5}\tilde{y} + \bar{1}\|$

$$= \|\overline{0.5}\tilde{x} - \overline{0.5}\tilde{y}\| = \overline{0.5}\|\tilde{x} - \tilde{y}\|$$

So, we have $\|T\tilde{x} - T\tilde{y}\| \tilde{\leq} \overline{0.6}\|\tilde{x} - \tilde{y}\|$ for all $\tilde{x}, \tilde{y} \tilde{\in} \tilde{X}$. That is T is soft contraction.

Proposition 3.3

Every soft contraction operator is soft continuous operator.

Proof: Let $\tilde{x} \tilde{\in} \tilde{X}$ be arbitrary soft element. For any $\tilde{\epsilon} \tilde{>} \bar{0}$, let $\|\tilde{x} - \tilde{y}\| \tilde{<} \tilde{\delta}$. Choose $\tilde{\delta} \tilde{<} \tilde{\in}$. Since T is soft contraction, then $\|T\tilde{x} - T\tilde{y}\| \tilde{\leq} \tilde{\alpha}\|\tilde{x} - \tilde{y}\| \tilde{<} \tilde{\alpha}\tilde{\delta} \tilde{<} \tilde{\in}$. Hence T is soft continuous.

Definition 3.4

Let T:SE($\tilde{X}$) ⟶SE($\tilde{X}$) where $\tilde{X}$ is a soft normed space. A soft element $\tilde{x}$ called soft fixed element if, $T(\tilde{x}) = \tilde{x}$.

Theorem 3.5

Let $\widetilde{X}$ be a soft Banach space and T: SE($\widetilde{X}$) → SE($\widetilde{X}$). If T is a soft contraction operator, then there exists a unique soft element $\tilde{x} \tilde{\in} \tilde{X}$ such that $T(\tilde{x}) = \tilde{x}$.

Proof: Let $\widetilde{x_0}$ be any soft element in $\tilde{X}$. We set $\widetilde{x_1} = T(\widetilde{x_0})$, $\widetilde{x_2} = T(\widetilde{x_1})$, … , $\tilde{x}_{n+1} = T(\widetilde{x_n})$.

$\|\tilde{x}_{n+1} - \widetilde{x_n}\| = \|T\widetilde{x_n} - T\tilde{x}_{n-1}\| \cong \tilde{\alpha}\|\widetilde{x_n} - \tilde{x}_{n-1}\|$

$= \tilde{\alpha}\|T\tilde{x}_{n-1} - T\tilde{x}_{n-2}\|$

$\cong \tilde{\alpha}^2\|\tilde{x}_{n-1} - \tilde{x}_{n-2}\|$…..$\cong \tilde{\alpha}^n\|\widetilde{x_1} - \widetilde{x_0}\|$.

Therefore, we have $\|\tilde{x}_{n+1} - \tilde{x}_n\| \cong \tilde{\alpha}^n\|\widetilde{x_1} - \widetilde{x_0}\|$.

Now, for $n > m$ we have:

$\|\tilde{x}_n - \tilde{x}_m\| \cong \|\widetilde{x_n} - \tilde{x}_{n-1}\| + \|\tilde{x}_{n-1} - \tilde{x}_{n-2}\| + \cdots + \|\tilde{x}_{m+1} - \tilde{x}_m\|$

$\cong (\tilde{\alpha}^{n-1} + \tilde{\alpha}^{n-2} + \cdots + \tilde{\alpha}^m)\|\widetilde{x_1} - \widetilde{x_0}\|$

$\cong \frac{\tilde{\alpha}^m}{\bar{1}-\tilde{\alpha}} \|\widetilde{x_1} - \widetilde{x_0}\|$

(Since $\frac{\tilde{\alpha}^m}{\bar{1}-\tilde{\alpha}} = (\tilde{\alpha}^{n-1} + \tilde{\alpha}^{n-2} + \cdots + \tilde{\alpha}^m) + \frac{\tilde{\alpha}^m}{\bar{1}-\tilde{\alpha}}$, $then\ \tilde{\alpha}^{n-1} + \tilde{\alpha}^{n-2} + \cdots + \tilde{\alpha}^m \cong \frac{\tilde{\alpha}^m}{\bar{1}-\tilde{\alpha}}$)

When n,m ⟶ ∞, $\|\widetilde{x_n} - \widetilde{x_m}\| \longrightarrow \bar{0}$. This implies that $\{\widetilde{x_n}\}$ is a soft Cauchy sequence. By completeness of $\tilde{X}$, there is a soft element $\tilde{x} \tilde{\in} \tilde{X}$ such that $\widetilde{x_0} \to \tilde{x}\ as\ n \to \infty$. Therefore,

$\|T\tilde{x} - \tilde{x}\| \cong \|T\widetilde{x_n} - T\tilde{x}\| + \|T\widetilde{x_n} - \tilde{x}\| \cong \tilde{\alpha}\|\widetilde{x_n} - \tilde{x}\| + \|\tilde{x}_{n+1} - \tilde{x}\|$.

We obtained that $\|T\tilde{x} - \tilde{x}\| \to \bar{0}\ as\ n \to \infty$ (i.e., T$\tilde{x} = \tilde{x}$).

If $\tilde{y}$ is another soft fixed element of T, then:

$\|\tilde{x} - \tilde{y}\| = \|T\tilde{x} - T\tilde{y}\| \cong \tilde{\alpha}\|\tilde{x} - \tilde{y}\|$.

This implies that $\|\tilde{x} - \tilde{y}\| = \bar{0}$ (since $\tilde{\alpha} \tilde{<} \bar{1}$) and $\tilde{x} = \tilde{y}$. Hence, the soft fixed element of T is unique.

The iteration procedure using in the last theorem called (**soft Picard iteration procedure)**.

<u>Definition 3.6</u> (soft *Mann iteration*)

Let $\tilde{X}$ be a soft normed space and T: SE($\tilde{X}$) ⟶ SE($\tilde{X}$) is a soft operator on. Let $\{\widetilde{\alpha_n}\}$ be a sequence of non-negative soft real number such that $\bar{0} \leq \tilde{\alpha}_n < \bar{1}\ for\ all\ n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \tilde{\alpha}_n$ is diverge.

Define a soft sequence $\{\widetilde{x_n}\}$ in $\tilde{X}$ by $\tilde{x}_0 \tilde{\in} \tilde{X}$ and $\tilde{x}_{n+1} = M(\widetilde{x_n}, \alpha_n, T)\ \ n \in \mathbb{N}$

Where M$(\widetilde{x_n}, \alpha_n, T) = (1 - \tilde{\alpha}_n)\widetilde{x_n} + \tilde{\alpha}_n T\widetilde{x_n}$.

The sequence $\{\widetilde{x_n}\}$ is called the soft Mann iteration.

<u>Theorem 3.7</u>

Let $\tilde{X}$ be a soft Banach space and T: SE($\tilde{X}$) ⟶SE($\tilde{X}$) is a soft continuous operator on $\tilde{X}$. If the soft Mann iteration $\{\widetilde{x_n}\}$ defined in previous definition converges strongly to a soft element $\tilde{p} \tilde{\in} \tilde{X}$, then $\tilde{p}$ is a soft fixed element of T.

Proof: since $\{\widetilde{x_n}\}$ converges to $\tilde{p}$, then $\lim_{n\to\infty} \widetilde{x_n} = \tilde{p}$. We want to prove that $T\tilde{p} = \tilde{p}$.

Suppose not, that is $T\tilde{p} \neq \tilde{p}$, i.e., $\|T\tilde{p} - \tilde{p}\| \tilde{>} \bar{0}$.

We set $\tilde{\in}_n = \widetilde{x_n} - T\widetilde{x_n} - (\tilde{p} - T\tilde{p})$.

Because $\lim_{n\to\infty} \widetilde{x_n} = \tilde{p}$ and T is soft continuous, we obtained that:

$\lim_{n\to\infty} \tilde{\in}_n = lim_{n\to\infty} (\widetilde{x_n} - T\widetilde{x_n} - (\tilde{p} - T\tilde{p})) = \bar{0}$. So, $\|\tilde{\in}_n\| \to \bar{0}$.

Now, since $\|T\tilde{p} - \tilde{p}\| \tilde{>} \bar{0}$, there exists $k \in \mathbb{N}$ *such that* $\|\tilde{\in}_n\| \tilde{<} \|T\tilde{p} - \tilde{p}\| / \bar{3}$.

For every Cauchy sequence in $\tilde{X}$, $\|\widetilde{x_n} - \widetilde{x_m}\| \tilde{<} \|T\tilde{p} - \tilde{p}\|/\bar{3}$ $\quad for\ all\ n, m \geq k.$

Let H be any positive integer such that $\sum_{i=k}^{k+H} \alpha_i \geq 1$.

We have: $\tilde{x}_{i+1} = (1 - \tilde{\alpha}_n)\tilde{x}_i + \tilde{\alpha}_i T\tilde{x}_i$

$$\tilde{x}_{i+1} - \tilde{x}_i = \tilde{\alpha}_i(T\tilde{x}_i - \tilde{x}_i)$$

Therefore, $\|\tilde{x}_k + \tilde{x}_{k+H+1}\| = \left\|\sum_{i=k}^{k+H}(\tilde{x}_i - \tilde{x}_{i+1}\right\|$

$$= \left\|\sum_{i=k}^{k+H} \tilde{\alpha}_i(\tilde{p} - T\tilde{p} + \tilde{\in}_i)\right\|$$

$$\tilde{\geq} \left\|\sum_{i=k}^{k+H} \tilde{\alpha}_i(\tilde{p} - T\tilde{p})\right\| - \left\|\sum_{i=k}^{k+H} \alpha_i\, \tilde{\in}_n\right\|$$

$$\tilde{\geq} \sum_{i=k}^{k+H} \tilde{\alpha}_i[\|T\tilde{p} - \tilde{p}\| - \|T\tilde{p} - \tilde{p}\|/\bar{3}]$$

$$\tilde{\geq} \frac{\bar{2}\|T\tilde{p} - \tilde{p}\|}{\bar{3}}$$

But $\|\tilde{x}_k + \tilde{x}_{k+H+1}\| \tilde{<} \|T\tilde{p} - \tilde{p}\| / \bar{3}$, which is contradiction.

So, $T\tilde{p} = \tilde{p}$. That is $\tilde{p}$ is a soft fixed element.

Example 3.8

Let $\tilde{X}$ be an absolute soft vector space where X = $\mathcal{R}^3$ and A = {1, 2, 3}.

Let T: SE($\tilde{X}$) ⟶SE($\tilde{X}$) be a soft operator on $\tilde{X}$ such that $T(\tilde{x}) = \bar{1} - \tilde{x}$

It is clear that T is continuous. We choose $\tilde{\alpha}_n = \overline{(\frac{1}{n})}$, $n \in \mathbb{N}$ *and* $\bar{0} \leq \tilde{\alpha}_n < \bar{1}$

Let $\widetilde{x_1} = \{(1, (1,1,1)), (2, (2,2,2)), (3, (3,3,3))\}$.

We have $\tilde{x}_{n+1} = (\bar{1} - \tilde{\alpha}_n)\widetilde{x_n} + \tilde{\alpha}_n T\widetilde{x_n}$ $\quad for\ all\ n \in \mathbb{N}$

$= \left(\bar{1} - \overline{(\frac{1}{n})}\right)\widetilde{x_n} + \overline{(\frac{1}{n})}(\bar{1} - \tilde{x})$

$= \left(\bar{1} - \overline{(\frac{2}{n})}\right)\widetilde{x_n} + \overline{(\frac{1}{n})}$

n =				
1	1	2	3	$\widetilde{x_1}$={(1,(1,1,1)),(2,(2,2,2)),(3,(3,3,3))}
2	0	-1	-2	$\widetilde{x_2}$={(1,(0,0,0)),(2,(-1,-1,-1)),(3,(-2,-2,-2))}
3	0.5	0.5	0.5	$\widetilde{x_3}$={(1,(0.5,0.5,0.5)),(2,(0.5,0.5,0.5)),(3,(0.5,0.5,0.5))}
4	0.5	0.5	0.5	$\widetilde{x_4}$={(1,(0.5,0.5,0.5)),(2,(0.5,0.5,0.5)),(3,(0.5,0.5,0.5))}
5	0.5	0.5	0.5	$\widetilde{x_5}$={(1,(0.5,0.5,0.5)),(2,(0.5,0.5,0.5)),(3,(0.5,0.5,0.5))}

It is clear that $\widetilde{x_n} \to \tilde{x} \quad where\ \tilde{x} = \{(1, (0.5,0.5,0.5)), (2, (0.5,0.5,0.5)), (3, (0.5,0.5,0.5))\}$

So by theorem, $\tilde{x}$ is a soft fixed element of T.

Let $\widetilde{x_1} = \{(1, (-1,0,1)), (2, (1,2,0)), (3, (3 - 2,31, -1))\}$.

n=										
1	-1	0	1	1	2	0	-2	1	-1	$\widetilde{x_1}$={(1,(-1,0,1)),(2,(1,2,0)),(3,(-2,1,-1))}
2	2	1	0	0	-1	1	3	0	2	$\widetilde{x_2}$={(1,(2,1,0)),(2,(0,-1,1)),(3,(3,0,2))}
3	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	$\widetilde{x_3}$={(1,(0.5,0.5,0.5)),(2,(0.5,0.5,0.5)),(3,(0.5,0.5,0.5))}
4	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	$\widetilde{x_4}$={(1,(0.5,0.5,0.5)),(2,(0.5,0.5,0.5)),(3,(0.5,0.5,0.5))}
5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	$\widetilde{x_5}$={(1,(0.5,0.5,0.5)),(2,(0.5,0.5,0.5)),(3,(0.5,0.5,0.5))}

It is clear that $\widetilde{x_n} \to \tilde{x} \quad where\ \tilde{x} = \{(1, (0.5,0.5,0.5)), (2, (0.5,0.5,0.5)), (3, (0.5,0.5,0.5))\}$

So by theorem, $\tilde{x}$ is a soft fixed element of T.

Example 3.9

Let $\tilde{X}$ be an absolute soft vector space where X = $\mathcal{R}^3$ and A = {1, 2, 3}.

Let T: SE($\tilde{X}$) ⟶SE($\tilde{X}$) be a soft operator on $\tilde{X}$ such that $T(\tilde{x}) = \bar{2}\tilde{x}$

It is clear that T is continuous. We choose $\tilde{\alpha}_n = \overline{(\frac{1}{n})}, n \in \mathbb{N}\ and\ \bar{0} \le \tilde{\alpha}_n < \bar{1}$

Let $\widetilde{x_1} = \{(1, (1,1,1)), (2, (2,2,2)), (3, (3,3,3))\}$.

We have $\tilde{x}_{n+1} = (\bar{1} - \tilde{\alpha}_n)\widetilde{x_n} + \tilde{\alpha}_n T\widetilde{x_n} \quad for\ all\ n \in \mathbb{N}$

$$= \left(\bar{1} - \frac{\bar{1}}{n}\right)\widetilde{x_n} + \frac{\bar{1}}{n}(\bar{2}\tilde{x})$$

$$= \overline{\left(\frac{n+1}{n}\right)}\widetilde{x_n}$$

n=					
1	1	2	3		$\widetilde{x_1}$={(1,(1,1,1)),(2,(2,2,2)),(3,(3,3,3))}
2	2	4	6		$\widetilde{x_2}$={(1,(2,2,2)),(2,(4,4,4)),(3,(6,6,6))}
3	3	6	9		$\widetilde{x_3}$={(1,(3,3,3)),(2,(6,6,6)),(3,(9,9,9))}
4	4	8	12		$\widetilde{x_4}$={(1,(4,4,4)),(2,(8,8,8)),(3,(12,12,12))}
5	5	10	15		$\widetilde{x_5}$={(1,(5,5,5)),(2,(10,10,10)),(3,(15,15,15))}
6	6	12	18		$\widetilde{x_6}$={(1,(6,6,6)),(2,(12,12,12)),(3,(18,18,18))}
7	7	14	21		$\widetilde{x_7}$={(1,(7,7,7)),(2,(14,14,14)),(3,(21,21,21))}
8	8	16	24		$\widetilde{x_8}$={(1,(8,8,8)),(2,(16,16,16)),(3,(24,24,24))}
9	9	18	27		$\widetilde{x_9}$={(1,(9,9,9)),(2,(18,18,18)),(3,(27,27,27))}
10	10	20	30		$\tilde{x}_{10}$={(1,(10,10,10)),(2,(20,20,20)),(3,(30,30,30))}

Although that T is soft continuous operator, the soft Mann iteration not converges to a soft element in $\tilde{X}$.

Proposition 3.10

Let $\tilde{X}$ be a soft normed space and T: SE($\tilde{X}$) ⟶ SE($\tilde{X}$) is a soft operator on $\tilde{X}$. $\tilde{p}$ is a fixed element of T such that $\|T\tilde{x} - \tilde{p}\| \tilde{\leq} \|\tilde{x} - \tilde{p}\|\ for\ all\ \tilde{x}\ \tilde{\in}\ \tilde{X}$, then for the soft Mann iteration $\tilde{x}_{n+1} = (\bar{1} - \tilde{\alpha}_n)\widetilde{x_n} + \tilde{\alpha}_n T\widetilde{x_n} \quad for\ all\ n \in \mathbb{N}$ the $\lim_{n\to\infty}\|\widetilde{x_n} - \tilde{p}\|\ exists.$

Proof: Because $\|\tilde{x}_{n+1} - \tilde{p}\| = \|(\bar{1} - \tilde{\alpha}_n)\widetilde{x_n} + \tilde{\alpha}_n T\widetilde{x_n} \quad - \tilde{p}\|$

$$= \|(\bar{1} - \tilde{\alpha}_n)\widetilde{x_n} + \tilde{\alpha}_n T\widetilde{x_n} \quad - (1 - \alpha_n)\tilde{p} - \alpha_n\tilde{p}\|$$

$$\tilde{\leq} \|(\bar{1} - \tilde{\alpha}_n)[\widetilde{x_n} - \tilde{p}]\| + \|\tilde{\alpha}_n(T\widetilde{x_n} - \tilde{p})\|$$

$$= (\bar{1} - \tilde{\alpha}_n)\|\widetilde{x_n} = \tilde{p}\| + \tilde{\alpha}_n\|T\widetilde{x_n} - \tilde{p}\|$$

$$\tilde{\leq} (\bar{1} - \tilde{\alpha}_n)\|\widetilde{x_n} = \tilde{p}\| + \tilde{\alpha}_n\|\widetilde{x_n} - \tilde{p}\|$$

$$= \|\widetilde{x_n} = \tilde{p}\| \quad for\ all\ n \in \mathbb{N}$$

It follows that $\lim_{n\to\infty}\|\widetilde{x_n} - \tilde{p}\|\ exists.$

4. Stability of soft iteration processes

Definition 4.1

Let $\tilde{X}$ be a soft normed space and let $T: SE(\tilde{X}) \rightarrow SE(\tilde{X})$ be a soft operator on $\tilde{X}$. Let $F(T) = \{\tilde{p} \tilde{\in} \tilde{X}, T\tilde{p} = \tilde{p}\}$ a set of soft fixed element of T, consider $\tilde{x}_0 \tilde{\in} \tilde{X}$ and $\{\tilde{x}_n\}$ be a soft sequence such that:

$$\tilde{x}_n = f(T, \tilde{x}_n) \ , n = 0,1,2, \ldots \qquad \ldots\ldots (1)$$

Where $\tilde{x}_0 \tilde{\in} \tilde{X}$, is the initial soft element and f is some function. Assume that $\{\tilde{x}_n\}$ converges to a soft fixed element $\tilde{p}$. Let $\{\tilde{y}_n\}$ be another soft sequence in $\tilde{X}$.

We consider $\tilde{\epsilon} = \|\tilde{y}_{n+1} - f(T, \tilde{y}_n)\| \ , n = 0,1,2,3, \ldots$

The soft iteration procedure (1) is called soft T-stable or soft stable with respect to T if and only if $\lim_{n\to\infty} \tilde{\epsilon} = \bar{0}$ implies $\lim_{n\to\infty} \tilde{y}_n = \tilde{p}$.

<u>Lemma4.2</u>

if $\tilde{\delta}$ is a soft real number such that $\bar{0} \tilde{\leqq} \tilde{\delta} \tilde{\lesssim} \bar{1}$ and $\{\tilde{\epsilon}_n\}_{n=0}^{\infty}$ is a soft sequence of positive soft real number with $\lim_{n\to\infty} \tilde{\epsilon} = \bar{0}$, then for all $\{\tilde{u}_n\}_{n=0}^{\infty} \subset R(A)^*$satisfies:

$$\tilde{u}_{n+1} \tilde{\leqq} \tilde{\delta}\tilde{u}_n + \tilde{\epsilon} \quad , n = 0,1,2,3, \ldots\ldots, \text{ we have } \lim_{n\to\infty} \tilde{u}_n = \bar{0}.$$

<u>Proof</u>: if $\tilde{\delta} = \bar{0}$, the statement is true. Assume $\bar{0} \tilde{\lesssim} \tilde{\delta} \tilde{\lesssim} \bar{1}$, we can multiply both side of Inequality by $\frac{\bar{1}}{\tilde{\delta}^{k+1}} = \tilde{\delta}^{-k-1}$, we obtained that:

$$\tilde{u}_{k+1}\tilde{\delta}^{-k-1} \tilde{\leqq} \tilde{\delta}^{-k}\tilde{u}_k + \tilde{\delta}^{-k-1}\tilde{\epsilon}_k \qquad for \quad k = 0,1,2, \ldots\ldots$$

By sum all inequalities for $k = 0,1,2,3, \ldots\ldots n+1$ and after simplify we obtained that:

$$\bar{0} \tilde{\leqq} \tilde{u}_{n+1} \tilde{\leqq} \tilde{\delta}^{n+1}\, \tilde{u}_0 + \sum_{k=0}^{n} \tilde{\delta}^{n-k}\tilde{\epsilon}_k$$

Now, using lemma in ((stability of k-stable fixed point iteration methods for Persic type contraction mapping)) we get;

$$\lim_{n\to\infty}\left[\sum_{k=0}^{n} \tilde{\delta}^{n-k}\tilde{\epsilon}_k\right] = \bar{0}. \text{ Therefore, } \lim_{n\to\infty} \tilde{u}_n = \bar{0}.$$

Stability of soft iteration processes (with contraction operator)

<u>Theorem 4.3</u> (stability of Picard iteration procedure)

Let $\tilde{X}$ be a soft banach space and let $T: SE(\tilde{X}) \rightarrow SE(\tilde{X})$ be a soft operator on $\tilde{X}$ satisfies the condition:

$$\|T\tilde{x} - T\tilde{y}\| \tilde{\leqq} \tilde{k}\|\tilde{x} - \tilde{y}\| \text{ Where, } \quad \bar{0} \tilde{\leqq} \tilde{k} \tilde{\lesssim} \bar{1}.$$

Then, the soft Picard iteration process where $\tilde{x}_0 \tilde{\in} \tilde{X}$ and $\tilde{x}_{n+1} = T\tilde{x}_n \ , n \geq 0$, is soft T-stable.

Proof: by soft contraction theorem, T has unique soft fixed point $\tilde{p}$. Consider $\{\tilde{y}_n\}_{n=0}^{\infty}$ be a soft sequence in $\tilde{X}$ such that $\tilde{y}_{n+1} = T\tilde{y}_n$ and let $\tilde{\epsilon}_n = \|\tilde{y}_{n+1} - T\tilde{y}_n\|$.

Suppose that $\lim_{n\to\infty} \tilde{\epsilon}_n = \bar{0}$ to prove that $\lim_{n\to\infty} \tilde{y}_n = \tilde{p}$.

$$\begin{aligned} \|\tilde{y}_{n+1} - \tilde{p}\| &\,\tilde{\leqq}\, \|\tilde{y}_{n+1} - T\tilde{y}_n\| + \|T\tilde{y}_n - \tilde{p}\| \\ &= \|T\tilde{y}_n - T\tilde{p}\| + \tilde{\epsilon}_n \\ &\,\tilde{\leqq}\, \tilde{k}\|\tilde{y}_n - \tilde{p}\| + \tilde{\epsilon}_n \end{aligned}$$

Since $\bar{0} \,\tilde{\leqq}\, \tilde{k} \,\tilde{<}\, \bar{1}$ and by lemma, we obtained that $\lim_{n\to\infty}\|\tilde{y}_n - \tilde{p}\| = \bar{0}$ that is $\lim_{n\to\infty} \tilde{y}_n = \tilde{p}$.

On the other hand, let $\lim_{n\to\infty} \tilde{y}_n = \tilde{p}$.

$$\begin{aligned} \tilde{\epsilon}_n = \|\tilde{y}_{n+1} - T\tilde{y}_n\| &\,\tilde{\leqq}\, \|\tilde{y}_{n+1} - \tilde{p}\| + \|\tilde{p} - T\tilde{y}_n\| \\ &\,\tilde{\leqq}\, \|\tilde{y}_{n+1} - \tilde{p}\| + \tilde{k}\|\tilde{y}_n - \tilde{p}\| \end{aligned}$$

When $n \to \infty$, we have $\lim_{n\to\infty} \tilde{\epsilon}_n = \bar{0}$.

Therefore, the Picard iteration procedure is T-stable.

Theorem 4.4 : (stability of Mann iteration procedure)

Let $\tilde{X}$ be a soft banach space and let $T: SE(\tilde{X}) \to SE(\tilde{X})$ be a soft operator on $\tilde{X}$ satisfies the condition: $\|T\tilde{x} - T\tilde{y}\| \,\tilde{\leqq}\, \tilde{k}\|\tilde{x} - \tilde{y}\|$ where $\bar{0} \,\tilde{\leqq}\, \tilde{k} \,\tilde{<}\, \bar{1}$.

Then, the soft Mann iteration process where $\tilde{x}_0 \,\tilde{\in}\, \tilde{X}$ and

$\tilde{x}_{n+1} = (\bar{1} - \tilde{\alpha}_n)\tilde{x}_n + \tilde{\alpha}_n T\tilde{x}_n$, $\tilde{\alpha}_0 = 1$, $\bar{0} \,\tilde{\leqq}\, \tilde{\alpha} \,\tilde{<}\, \bar{1}$ $for\ all\ n \geq 1$, is soft T-stable.

Proof: by soft contraction theorem, T has unique soft fixed point $\tilde{p}$. Consider $\{\tilde{y}_n\}_{n=0}^{\infty}$ be a soft sequence in $\tilde{X}$ such that $\tilde{y}_{n+1} = (\bar{1} - \tilde{\alpha}_n)\tilde{y}_n + \tilde{\alpha}_n T\tilde{y}_n$ and let

$$\tilde{\epsilon}_n = \|\tilde{y}_{n+1} - (\bar{1} - \tilde{\alpha}_n)\tilde{y}_n + \tilde{\alpha}_n T\tilde{y}_n\|.$$

Suppose that $\lim_{n\to\infty} \tilde{\epsilon}_n = \bar{0}$ to prove that $\lim_{n\to\infty} \tilde{y}_n = \tilde{p}$.

$$\begin{aligned} \|\tilde{y}_{n+1} - \tilde{p}\| &\,\tilde{\leqq}\, \|\tilde{y}_{n+1} - (\bar{1} - \tilde{\alpha}_n)\tilde{y}_n - \tilde{\alpha}_n T\tilde{y}_n\| + \|(\bar{1} - \tilde{\alpha}_n)\tilde{y}_n + \tilde{\alpha}_n T\tilde{y}_n - \tilde{p}\| \\ &= \|(\bar{1} - \tilde{\alpha}_n)\tilde{y}_n + \tilde{\alpha}_n T\tilde{y}_n - ((\bar{1} - \tilde{\alpha}_n) + \tilde{\alpha}_n)\tilde{p}\| + \tilde{\epsilon}_n \\ &= \|(\bar{1} - \tilde{\alpha}_n)(\tilde{y}_n - \tilde{p}) + \tilde{\alpha}_n(T\tilde{y}_n - \tilde{p})\| + \tilde{\epsilon}_n \\ &\,\tilde{\leqq}\, (\bar{1} - \tilde{\alpha}_n)\|\tilde{y}_n - \tilde{p}\| + \tilde{\alpha}_n\|T\tilde{y}_n - T\tilde{p}\| + \tilde{\epsilon}_n \\ &\,\tilde{\leqq}\, (\bar{1} - \tilde{\alpha}_n)\|\tilde{y}_n - \tilde{p}\| + \tilde{\alpha}_n\tilde{k}\|\tilde{y}_n - \tilde{p}\| + \tilde{\epsilon}_n \\ &= \left(\bar{1} - \tilde{\alpha}_n + \tilde{\alpha}_n\tilde{k}\right)\|\tilde{y}_n - \tilde{p}\| + \tilde{\epsilon}_n \end{aligned}$$

Since $\left(\bar{1} - \tilde{\alpha}_n + \tilde{\alpha}_n \tilde{k}\right) \tilde{<} \bar{1}$ and by lemma, we obtained that $\lim_{n\to\infty} \|\tilde{y}_n - \tilde{p}\| = \bar{0}$ that is $\lim_{n\to\infty} \tilde{y}_n = \tilde{p}$.

On the other hand, let $\lim_{n\to\infty} \tilde{y}_n = \tilde{p}$.

$$\tilde{\epsilon}_n = \|\tilde{y}_{n+1} - (\bar{1} - \tilde{\alpha}_n)\tilde{y}_n - \tilde{\alpha}_n T\tilde{y}_n\|$$

$$\tilde{\leq} \|\tilde{y}_{n+1} - \tilde{p}\| + \|\tilde{p} - (\bar{1} - \tilde{\alpha}_n)\tilde{y}_n - \tilde{\alpha}_n T\tilde{y}_n\|$$

$$\tilde{\leq} \|\tilde{y}_{n+1} - \tilde{p}\| + \|((\bar{1} - \tilde{\alpha}_n) + \tilde{\alpha}_n)\tilde{p} - (\bar{1} - \tilde{\alpha}_n)\tilde{y}_n - \tilde{\alpha}_n T\tilde{y}_n\|$$

$$\tilde{\leq} \|\tilde{y}_{n+1} - \tilde{p}\| + (\bar{1} - \tilde{\alpha}_n)\|\tilde{y}_n - \tilde{p}\| + \tilde{\alpha}_n\|T\tilde{y}_n - \tilde{p}\|$$

$$\tilde{\leq} \|\tilde{y}_{n+1} - \tilde{p}\| + (\bar{1} - \tilde{\alpha}_n)\|\tilde{y}_n - \tilde{p}\| + \tilde{\alpha}_n\tilde{k}\|\tilde{y}_n - \tilde{p}\|$$

When $n \to \infty$, the $\lim_{n\to\infty} \tilde{\epsilon}_n = \bar{0}$.

Therefore, the Mann iteration procedure is T-stable.

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BS-algebra and Pseudo Z-algebra

Adel Hashem Nouri
Faculty of Education
University of Kufa , Najaf, Iraq
adilh.alhajjar@ uokufa.edu.iq

Abstract:

This paper introduces a new notion of algebra called BS-algebra and some

of its properties are discussed in detail. Also, we define a S(a,b) of BS-algebra

and discus of properties and theorems of it. Some of theorems of a new ideals of BS-algebra are introduced and proved .Also, we introduced some new types of ideals of Z-algebra and we linked these ideals with some theorems and properties .Also , we define the concept of pseudo Z-algebra and introduced some examples and new concepts in it, including the concept of pseudo b-subalgebra and introduced some theorems and properties in this new concepts.

Keywords : BS-algebra , S(a,b) of BS-algebra , Bs –idel , Z-algebra, associative Z-algebra, pseudo Z-algebra, Z -ideal , pseudo b-subalgebra.

1. Introduction.

Algebras structures have an important role in many applications of science such as computer science, information theory, control engineering, etc. In 1966 the world introduced the concept of BCK –algebra and then emerged other concepts and types of algebra. In 2002 , J. Neggers and H. S. Kim introduced the concept of B-algebra which is the generalization of concepts of some types of algebras .In 2008, C. B. Kim and H. S .Kim introduced BG-algebra which is the generalization of B-algebra. In 2012 introduced the of BO-algebra .In this paper we will introduce a new type of algebra namely BS-algebra and we will try to connected it to other types of algebras. Some theorems and properties .Also , we define a new concept of sub- algebra and introduced some theorems and characteristics Finally ,we define a new type of ideals in this type and we will connected to other types of ideals of BS-algebra. M. Chandramouleeswaran, P. Muralikrishna, K. Sujatha and S. Sabarinathan (2017) introduced the concepts of Z-algebra. They tried to provide some theorems that link this type of algebra to other types. They also introduced some types of ideals and filters in this type of algebra.

In this paper, we introduce the notion of some types of ideas of Z-algebra, and investigate its characterization. We also introduce the concepts of associative Z-algebra, and

investigate related properties. We define a pseudo Z-algebra . Some concepts and theorems are given and proved.

2. Preliminaries.

Definition. 2.1.[3] Let **Ж** be a null set has a constant of " O " with a binary operation " ⟐" satisfying the following axioms :

i. ᵯ ⟐ ᵯ = O, ∀ ᵯ ∈ **Ж**.

ii. ᵯ ⟐ O = ᵯ, ∀ ᵯ ∈ **Ж** **.**

iii. (ᵯ ⟐ ᵰ) ⟐ ƶ = ᵯ ⟐ (ƶ ⟐ (O ⟐ ᵰ)) , ∀ ᵯ, ᵰ , ƶ ∈ **Ж** .

Then **Ж** is called a B -algebra .

Definition.. 2.2.[1] Let **Ж** be a null set has a constant of " O " with a binary operation " ⟐" satisfying the following axioms :

i. ᵯ ⟐ ᵯ = O, ∀ ᵯ ∈ **Ж**.

ii. ᵯ ⟐ O = ᵯ, ∀ ᵯ ∈ **Ж** **.**

iii. (ᵯ ⟐ ᵰ) ⟐ (O ⟐ ᵰ) = ᵯ , ∀ ᵯ, ᵰ , ƶ ∈ **Ж** .

Then **Ж** is called a BG -algebra .

Definition. 2.3.[2] Let **Ж** be a null set has a constant of " O " with a binary operation

" ⟐" satisfying the following axioms :

i. ᵯ ⟐ ᵯ = O, ∀ ᵯ ∈ **Ж**.

ii. ᵯ ⟐ O = ᵯ, ∀ ᵯ ∈ **Ж** **.**

iii. ᵯ ⟐ (ᵰ ⟐ ƶ) = (ᵯ ⟐ ᵰ) ⟐ (O ⟐ ƶ) , ∀ ᵯ, ᵰ , ƶ ∈ **Ж** .

Then Ж is called a BO -algebra .

Definition..2.4. : Let **Ж** be a non-empty set have a constant " O " and a binary operation "⟐" satisfying the following axioms:

i. ᵯ ⟐ O = O , ∀ ᵯ ∈ **Ж**.

ii. O ⟐ ᵯ = ᵯ, ∀ ᵯ ∈ **Ж**.

iii. ᵯ ⟐ ᵯ = **κ** , ∀ ᵯ ∈ **Ж.**

iv. ᵯ ⟐ ᵰ = ᵰ ⟐ ᵯ , ∀ ᵯ, ᵰ ∈ **Ж**.

Then **Ж** is called a Z-algebra.

Definition 2.5. Let **Ж** be a Z-algebra, then **Ɨ** is called an **ideal** of **Ж** if :-

i. O ∈ **Ɨ** .

ii. ᵯ ⟐ ᵰ ∈ **Ɨ** , ᵰ ∈ Ɩ ⇒ ᵯ ∈ **Ɨ** , ∀ ᵯ, ᵰ ∈ **Ж**.

3. BS-algebra

Definition. 3.1. Let **Ж** be a null set has a constant of " O " with a binary operation

" ⟐" satisfying the following axioms :

i. ᵯ ⟐ ᵯ = O, ∀ ᵯ ∈ **Ж**.

ii. ᵯ ⟐ O = ᵯ, ∀ ᵯ ∈ **Ж .**

iii. (ᵯ ⟐ ᵰ) ⟐(ᵰ ⟐ ƶ) = ᵯ , ∀ ᵯ , ᵰ , ƶ ∈ **Ж .**

Then **Ж** is called a BS -algebra

Example .3.2. Let **Ж** = { **O**,1,2,3} be a set with the following Cayley tables:

⟐	O	1	2	3
O	O	1	2	3
1	1	O	1	1
2	2	2	O	2
3	3	3	3	O

Then Ж is a BS-algebra .

Definition. 3.3. Let Ж is a BS –algebra , then Ж is called a commutative if

ɱ ⟐ ɲ = ɲ ⟐ ɱ , ∀ ɱ, ɲ ∈ Ж .

Definition. 3.4.. Let Ж is a BS –algebra , then Ж is called a associative if

ɱ ⟐(ɲ⟐ ƶ) = (ɱ ⟐ ɲ)⟐ ƶ , ∀ ɱ, ɲ , ƶ ∈ Ж .

Proposition.3.5. If O ⟐ ɱ = ɱ , then every B-algebra Ж if and only if a BS-algebra.

Proof : Let Ж be B-algebra satisfies O ⟐ ɱ = ɱ .

Let (ɱ ⟐ ɲ) ⟐ (ɲ ⟐O) = (ɱ ⟐ ɲ) ⟐ ɲ = ɱ ⟐ (ɲ ⟐ (O ⟐ ɲ))

= ɱ ⟐ (ɲ ⟐ ɲ) = ɱ

Hence , Ж is BS-algebra.

Conversely , Let Ж be BS-algebra satisfies O ⟐ ɱ = ɱ .

Let ɱ ⟐ (ɲ ⟐ (O ⟐ ɲ)) = ɱ ⟐ (ɲ ⟐ ɲ) = ɱ ⟐ O

= ɱ = (ɱ ⟐ ɲ) ⟐ (ɲ ⟐O) = (ɱ ⟐ ɲ) ⟐ ɲ .

Hence , Ж is B-algebra.

Proposition.3.6. If O ⟐ ɱ = ɱ , then every BG-algebra Ж if and only if a BS-algebra.

Proof : Let Ж be BG-algebra satisfies O ⟐ ɱ = ɱ .

Let (ɱ ⟐ ɲ) ⟐ (ɲ ⟐O) = (ɱ ⟐ ɲ) ⟐ ɲ = (ɱ ⟐ ɲ)⟐ (O ⟐ ɲ) = ɱ

Hence , Ж is BS-algebra.

Conversely , Let Ж be BS-algebra satisfies O ⟐ ɱ = ɱ .

Let ɱ = (ɱ ⟐ ɲ) ⟐ (ɲ ⟐O) = (ɱ ⟐ ɲ) ⟐ ɲ =(ɱ ⟐ ɲ)⟐ (O ⟐ ɲ)

Hence , Ж is BG-algebra

Proposition.3.7. Every associative BO-algebra Ж satisfies O ⟐ ɱ = ɱ is a BS-algebra.

Proof : Let Ж be BO-algebra satisfies O ◇ ꟺ = ꟺ .

Let ꟺ = ꟺ ◇O = ꟺ ◇(ꞥ ◇ ꞥ) = (ꟺ ◇ ꞥ)◇ (O ◇ ꞥ)

= (ꟺ ◇ ꞥ)◇ ꞥ = **(**ꟺ ◇ ꞥ) ◇ (ꞥ ◇O)

Hence , Ж is BS-algebra

Proposition.3.8. Let Ж be BS-algebra ,then the following results are hold:

1. ꟺ **= (**ꟺ ◇ ꞥ) .
2. **(**ꟺ ◇ ꞥ) ◇ ꞥ = ꟺ .
3. ꟺ ◇ **((**ꟺ ◇ ꞥ) ◇ ꞥ) = O .

Proof :

1. ꟺ = **(**ꟺ ◇ ꞥ) ◇(ꞥ ◇ ꞥ) = **(**ꟺ ◇ ꞥ) ◇ O = ꟺ ◇ ꞥ .
2. **(**ꟺ ◇ ꞥ) ◇ ꞥ = **(**ꟺ ◇ ꞥ) ◇ (ꞥ ◇O) = ꟺ .
3. ꟺ ◇ **((**ꟺ ◇ ꞥ) ◇ ꞥ) = ꟺ ◇ ((ꟺ ◇ ꞥ) ◇ (ꞥ ◇O)) = ꟺ ◇ ꟺ = O .

4. S (a,b) of BS-algebra

Definition. 4.1. Let S be a subset of BS -algebra **Ж .** Then S is called a sub-algebra

of Ж if ꟺ ◇ ꞥ ∈ S , for all ꟺ, ꞥ ∈ S .

Definition. 4.2. Let S be a sub-algebra of BS -algebra **Ж .**Then S is called S (a,b) of **Ж ,** if (a ◇ ꟺ) ◇ (b ◇ ꞥ) ∈ S , for all ꟺ, ꞥ ∈ S and for some a,b ∈ **Ж .**

Proposition.4.3. O ∈ S (a,b) of BS-algebra**.**

Proof : Let S (a,b) of BS-algebra **Ж .**

Let (a ◇ ꟺ) ◇ (b ◇ ꞥ) ∈ S , where a,b ∈ Ж and for all ꟺ, ꞥ ∈ S .

Put a= ꟺ and b= ꞥ , we get : (a ◇ a) ◇ (b ◇ b) ∈ S.

Imply Ơ ⟐ Ơ = Ơ ∈ S (a,b) of Ж **.**

Proposition.4.4. Let Ж be a commutative BS-algebra, then S (a,b)= S (b,a)**.**

Proof : Let S (a,b) of commutative BS-algebra **Ж .**

Let (a ⟐ ꟽ) ⟐ (b ⟐ ꞥ) ∈ S , where a,b ∈ Ж and for all ꟽ, ꞥ ∈ S .

Since Ж is a commutative BH-algebra, then (b ⟐ ꞥ) ⟐ (a ⟐ ꟽ) ∈ S.

By definition of S (b,a) ,we get S (a,b)= S (b,a) **,** where ꟽ ⟐ ꞥ = ꞥ ⟐ ꟽ .

5. Bs –ideal of ƁS-algebra

Definition.5.1 Let Ж be a ƁS-algebra , then Ɨ is called an **ideal** of Ж if :-

iii. Ơ ∈ Ɨ .

iv. ꟽ ⟐ ꞥ ∈ Ɨ and ꞥ ∈ Ɨ ⇒ ꟽ ∈ Ɨ , ∀ ꟽ, ꞥ ∈ Ж.

Definition.5.2. Let Ж be a ƁS-algebra , then Ɨ is called an **Bs-ideal** of Ж if :-

i. Ơ ∈ Ɨ .

ii. ((ꟽ ⟐ ꞥ) ⟐(ꞥ ⟐ ƶ)) ⟐ ꞥ ∈ Ɨ and ꞥ ∈ Ɨ ⇒ ꟽ ∈ Ɨ , ∀ ꟽ, ꞥ ∈ Ж.

Proposition.5.3 Let Ж be a ƁS-algebra , then every ideal of Ж if and only if an Bs-ideal of Ж.

Proof : Suppose that Ɨ is an ideal of Ж , ∀ ꟽ, ꞥ , ƶ ∈ Ж

Let ((ꟽ ⟐ ꞥ) ⟐(ꞥ ⟐ ƶ)) ⟐ ꞥ ∈ Ɨ and ꞥ ∈ Ɨ

Since Ж is a ƁS-algebra , we get (ꟽ ⟐ ꞥ) ⟐(ꞥ ⟐ ƶ) = ꟽ

Imply ꟽ ⟐ ꞥ ∈ Ɨ and ꞥ ∈ Ɨ .

Since Ɨ is an ideal of Ж **,** we get ꟽ ∈ Ɨ

Hence , Ɨ is an Bs- ideal of Ж .

Conversely , Suppose that Ɨ is an Bs-ideal of Ж , ∀ ꟽ, ꞥ , ƶ ∈ Ж

Let ꟽ ⟐ ꞥ ∈ Ɨ and ꞥ ∈ Ɨ **.**

Since Ж is a ƁS-algebra , we get (ᵯ ⟐ ᵰ) ⟐(ᵰ ⟐ ƶ) = ᵯ

Imply ((ᵯ ⟐ ᵰ) ⟐(ᵰ ⟐ ƶ)) ⟐ ᵰ ∈ Ɨ and ᵰ ∈ Ɨ

Since Ɨ is an Bs-ideal of **Ж ,** we get ᵯ ∈ Ɨ

Hence , Ɨ is an ideal of Ж .

6 .Some types of ideals of Z-algebra

Dęfįnįţįoņ.6.1.A nonempty subset Ɨ of a Z-algebra Ж is called **a Z_1- ideal** of Ж if :

(i) 0 ∈ Ɨ .

(ii) ((ᵯ⟐ ƶ)⟐ᵯ)⟐ ᵰ ∈ Ɨ and ᵰ ∈ Ɨ imply ᵯ ∈ Ɨ, ∀ ᵯ,ᵰ,ƶ ∈ Ж.

Example .6.2. Let Ж = (**Z,** ⟐,**O**) be the Z-algebra, where Ж = { O,1,2,3} and ⟐ is given by the table :

⟐	O	1	2	3
O	O	1	2	3
1	O	1	1	1
2	O	1	2	2
3	O	1	2	3

Let Ɨ ={ O,2,3} be a subset of Z-algebra, then Ɨ is **Z_1-** ideal of a Z-algebra.

Dęfįnįţįoņ.6.3. A Z-algebra Ж is called **associative Z-algebra** if:

(ᵯ⟐ᵰ) ⟐ƶ = ᵯ ⟐ (ᵰ ⟐ƶ) , ∀ ᵯ,ᵰ,ƶ ∈ Ж.

Proposition.6.4. Let Ж be a associative Z-algebra. Then the following properties hold for all ᵯ, ᵰ ∈ **Ж.**

1- (ᵯ⟐ᵰ)⟐ ᵯ = ᵰ ⟐ ᵯ .
2- ᵯ ⟐ (ᵰ ⟐ ᵯ) = ᵯ⟐ᵰ .
3- (ᵯ ⟐(ᵰ ⟐ ᵯ)) ⟐ᵰ = ᵰ ⟐ ᵯ.

Proof :

1- (ᵯ⟐ᵰ)⟐ ᵯ= (ᵰ ⟐ ᵯ) ⟐ ᵯ = ᵰ ⟐(ᵯ⟐ ᵯ) = ᵰ ⟐ ᵯ .
2- ᵯ ⟐ (ᵰ ⟐ ᵯ) = ᵯ ⟐ (ᵯ ⟐ ᵰ) = (ᵯ⟐ᵯ) ⟐ᵰ = ᵯ⟐ᵰ .

3- (ᵯ ⟐(ꞥ ⟐ ᵯ)) ⟐ꞥ= ꞥ ⟐(ᵯ ⟐(ꞥ ⟐ ᵯ)) = (ꞥ ⟐ ᵯ) ⟐(ꞥ ⟐ ᵯ) = ꞥ ⟐ ᵯ.

Proposition.6.5. Let **Ж** be a Z-algebra, then every **Z_1-** ideal of **Ж** is an ideal of **Ж.**

Proof : Suppose that Ɨ is a **Z_1**- ideal of **Ж** , ∀ ᵯ,ꞥ,ƶ ∈ **Ж**.

Let ((ᵯ⟐ ƶ)⟐ ᵯ) ⟐ ꞥ ∈ Ɨ and ꞥ ∈ Ɨ .

Put ƶ = Ο , we have :

((ᵯ⟐ Ο)⟐ ᵯ) ⟐ ꞥ ∈ Ɨ and ꞥ ∈ Ɨ ,then

(Ο ⟐ ᵯ) ⟐ ꞥ ∈ Ɨ and ꞥ ∈ Ɨ [**Ж** is a Z-algebra]

ᵯ⟐ꞥ ∈ Ɨ and ꞥ ∈ Ɨ [**Ж** is a Z-algebra,]

Imply ꞥ ∈ Ɨ . [Ɨ is an ideal of **Ж**].

Hence , Ɨ is a **Z_1**- ideal of **Ж**.

Dȩfįnįțįoņ.6.6.A nonempty subset Ɩ of a Z-algebra, **Ж** is called **a Z_2-ideal** of **Ж** if

(i) Ο ∈ Ɨ .

(ii) (ᵯ⟐ ƶ) ⟐(ᵯ⟐ ꞥ) ∈ Ɨ and ꞥ ∈ Ɨ imply ᵯ ∈ Ɨ, ∀ ᵯ,ꞥ,ƶ ∈ **Ж**.

Proposition.6.7. Let **Ж** be a Z-algebra,, then every **Z_2**- ideal of **Ж** is an ideal of **Ж** .

Proof : Suppose that Ɨ is a **Z_2**- ideal of **Ж** , ∀ ᵯ,ꞥ,ƶ ∈ **Ж**.

Let (ᵯ⟐ ƶ)⟐(ᵯ⟐ ꞥ) ∈ Ɨ and ꞥ ∈ Ɨ

Put ƶ = Ο , we have :

Let (ᵯ⟐ Ο)⟐(ᵯ⟐ ꞥ) ∈ Ɨ and ꞥ ∈ Ɨ,then

Ο⟐(ᵯ⟐ ꞥ) ∈ Ɨ and ꞥ ∈ Ɨ [**Ж** is a Z-algebra,]

Then(ᵯ⟐ ꞥ) ∈ Ɨ and ꞥ ∈ Ɨ [**Ж** is a Z-algebra,]

Imply ꞥ ∈ Ɨ . [Ɨ is an ideal of **Ж**].

Hence , Ɨ is a **Z_1**- ideal of **Ж**.

Proposition.6.8. Every **Z_1**- ideal of associative Z-algebra, **Ж** if and only if is **Z_2**- ideal of **Ж**.

Proof : Suppose that Ɨ is a **Z_1**- ideal of **Ж** , ∀ ᵯ,ꞥ,ƶ ∈ **Ж**.

Let (ᵯ⟐ ƶ) ⟐ (ᵯ⟐ꞥ)∈ Ɨ and ꞥ ∈ Ɨ.

Since Ж is an associative Z-algebra,,we get

((ᵯ⟐ ƶ)⟐ᵯ)⟐ᵰ∈ Ɨ and ᵰ ∈ Ɨ .

Imply ᵯ ∈ Ɨ [Ɨ is a **Z**$_1$- ideal of Ж]

Hence , Ɨ is a **Z**$_2$- ideal of Ж .

Similarly, Suppose that Ɨ is a **Z**$_2$- ideal of Ж ,   ∀ ᵯ,ᵰ,ƶ ∈ Ж.

Let ((ᵯ⟐ ƶ))⟐ᵯ)⟐ᵰ∈ Ɨ and ᵰ ∈ Ɨ .

Since Ж is an associative Z-algebra,,we get

(ᵯ⟐ ƶ)⟐ᵯ)⟐ᵰ∈ Ɨ and ᵰ ∈ Ɨ.

Imply ᵯ ∈ Ɨ [Ɨ is a **Z**$_1$- ideal of Ж]

Hence , Ɨ is a **Z**$_2$- ideal of Ж .

7. Pseudo Z-algebra

In this section, we define a new type of algebras .It is called a pseudo Z-algebra, and then we introduced some of the concepts and examples in it. Also , some properties and theorems linking them are studied and proved.

Dęfįnįţįoņ.7.1. Let Ж be a non-empty set   have a constant " O " and a two binary operations "⟐" and"◊" satisfying the following axioms :

i. ᵯ ⟐ O = ᵯ ◊ O = O ,   ∀ ᵯ ∈ Ж.

ii. O ⟐ ᵯ = O ◊ ᵯ = ᵯ, ∀ ᵯ ∈ Ж.

iii. ᵯ ⟐ ᵯ = ᵯ ◊ ᵯ = ᵯ , ∀ ᵯ ∈ Ж

iv. ᵯ ⟐ ᵰ = ᵰ ◊ ᵯ , ∀ ᵯ, ᵰ ∈ Ж.

Then Ж is called a **pseudo Z-algebra**

Example .7.2. Let Ж = { O,1,2} be a set with the following Cayley tables:

⟐	O	1	2	3
O	O	1	2	3
1	O	1	1	2
2	O	2	2	1
3	O	1	1	3

◊	O	1	2	3
O	O	1	2	3
1	O	1	2	1
2	O	1	2	1
3	O	2	1	3

Then (**Ж**, ⟐,◊,**O**) is a pseudo Z-algebra.

Proposition.7.3. Let **Ж** be a pseudo Z-algebra. Then the following results hold for all ꝳ, ꞥ ∈ **Ж.**

1- ꝳ⟐ ꞥ = (O⟐ ꞥ) ◊ (O⟐ ꝳ) ,

ꝳ◊ ꞥ = (O◊ ꞥ) ⟐ (O◊ ꝳ).

2- O ⟐(ꝳ◊ ꞥ)= (O⟐ ꝳ) ◊(O⟐ ꞥ)

O ◊ (ꝳ⟐ ꞥ)= (O◊ ꝳ) ⟐ (O◊ ꞥ)

3- ꝳ ⟐(O◊ ꞥ)=(ꞥ ◊ ꝳ)

ꝳ ◊ (O⟐ ꞥ)=(ꞥ ⟐ ꝳ).

Proof :

1- ꝳ⟐ ꞥ = ꞥ ◊ ꝳ = (O⟐ ꞥ) ◊(O ⟐ ꝳ), similarly ,ꝳ◊ ꞥ = (O◊ ꞥ) ⟐ (O◊ ꝳ).
2- O ⟐(ꝳ◊ ꞥ)= ꝳ◊ ꞥ= (O⟐ ꝳ) ◊(O⟐ ꞥ) , similarly, O ◊ (ꝳ⟐ ꞥ)= (O◊ ꝳ) ⟐ (O◊ ꞥ).
3- ꝳ ⟐(O◊ ꞥ)= ꝳ⟐ ꞥ = ꞥ ◊ ꝳ = (O⟐ ꞥ) ◊(O ⟐ ꝳ)

Similarly, ꝳ ◊ (O⟐ ꞥ)=(ꞥ ⟐ ꝳ).

Dęfįnįtįoņ.7.4. A subset **S** of a pseudo Z-algebra **Ж** is called a **pseudo subalgebra of Ж** , if :

(ꝳ⟐ ꞥ), (ꝳ◊ ꞥ) ∈ **S** , for all ꝳ, ꞥ ∈ **S** .

Dęfįnįtįoņ.7.5.A subset **S** of a pseudo Z-algebra **Ж** is called a pseudo **b- subalgebra** of **Ж** ,if :

b ◊(ꝳ⟐ ꞥ), **b** ⟐ (ꝳ◊ ꞥ) ∈ **S** , for some **b** ∈ **Ж** and for all ꝳ, ꞥ ∈ **S** .

Example .7.6. In above example, let S={1,2}

3⟐(1◊2)= 1 ∈ **S ,** 3⟐(2◊1)= 1 ∈ **S** .

3◊ (1⟐2)= 2 ∈ **S ,** 3◊ (2⟐1)= 1 ∈ **S**

Hence, S is a pseudo 3- subalgebra of **Ж**

Proposition.7.6. Every a pseudo O-subalgebra of a pseudo Z-algebra **Ж** is a pseudo subalgebra of **Ж.**

Proof : Let S be a pseudo O-subalgebra **Ж** . Let **O** ◊(ꝳ⟐ ꞥ), **O** ⟐ (ꝳ◊ ꞥ) ∈ **S** .

Since **Ж** is pseudo Z-algebra, we get

(ꝳ⟐ ꞥ), (ꝳ◊ ꞥ) ∈ **S**.

Hence, S is pseudo subalgebra of **Ж.**

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Related to Non – Vanishing Parts of the Dihedral Set D_3

Zinah Makki Kadhim AL-Shuhaib[1], Mohammed Baqer Hashim al Hakeem[2]

Zeanh73@ gmail.com

mbhmsc2015110@gmail.com

Department of mathematics, Directorate General of Education Karbala , Karbala ,Iraq

Abstract

For $n \in z^{+}$ the dihedral group D3, the part $g \in D3$ is supposed as a D3 non–vanishing when $\chi(g) \neq 0$; for whole $\chi \in Irr(D3)$. It's evaluated that the whole the D3 non–vanishing part locate in its Fitting subset F(D3). In this work, an allied is found to the non–vanishing basics of the dihedral set D3 that holds when D3 is solvable set.

Keywords: *Soluble set, Feature, Fitting subset, Non–vanishing part,*

Dihedralset D3.

1. Introduction

For $n \in z^{+}$, the dihedral set is $D_n = < r,s > =\{s^j r^i;\ 0 \le i \le n\text{-}1 \ \&\ 0 \le j \le 1\}$, where $r^n = 1$, $s^2 = 1$, $srs = r^{-1}$, $(sr^i)^2 = 1$, $sr^i s = r^{-i}$ $\forall\ 0 \le i \le n\text{-}1$, such that

$D_3 = \{1, r, r^2, s, sr, sr^2\}$, where $r^3 = 1$, $s^2 = 1$, and $srs = sr^{-1} = sr^2$.

Geometrically, D_3 consists of certain rigid motiaes of an equilateral triangle.

r is a clock wise rotation about the center with angle $\frac{2\pi}{3}$.

r^2 is a clock wise rotation about the center with angle $\frac{4\pi}{3}$.

S, sr, and sr^2 are reproductions near: shapes L1, L2, and L3 in that order [1]; and carve Irr(D3) for the complete set of (D3) complex irreducible features; if $g \in D3$ satisfies $\chi(g) \neq 0$, then g is considered as D3 non-vanishing parts. In [2], it's estimated that the whole non–fading parts of a limited soluble collection (D3) *locate in its correct subset F(D3) [3]. In the present research, V(D3) is used to represent the subset created by the whole non–vanishing parts of D3, i.e. V(D3) ={g| $\chi(g) \neq 0$; for all $\chi \in Irr$ (D3}, which is named the vigorously (D3) vanishing off subset, and the V(D3) locates in the center Z(F(D3)) of F(D3). Expressed in term of V(D3), this conjecture equally confirms that the inequality V(D3 $\leq$ F(D3) is satisfied for the solvable dihedral set D3. In this work, Isaacs [4] is used as a reference for the standard symbols and outputs from the feature theory.*

2. The preliminaries

The next lemma states few principle things of $V(D_3)$.

Lemma 2.1 [5]:

Suppose that D_3 is a finite solvable set, and $V(D_3)$ is its vigorously vanishing off subset, then

1. $V(D_3)$ is a D_3 feature subset.

2. $V(D_3)$ is a suitable D_3 subset when D_3 is non abelian.

3. When C_n is a regular D_3 cyclic subset, thus $V(D_3/C_n)$ in D_3 includes $V(D_3)$.

Lemma 2.2 [5]:

Let $M \geq N$ is regular D_3 subsets. When $\theta^k = e^l$ for $\theta \in Irr(N)$, $l \in Irr(M)$ and (e) is a positive integer, thus there is $\chi \in Irr(D_3)$, where $\chi(a)=0$ for whole $a \in M–N$.

Lemma 2.3 [6]:

Let $V(D_3)$ is a loyal and totally reducible P–module, where P is a G-equivalent, then P possesses 2 normal orbits on $V(D_3)$ at least.

3. Principal Outputs

Some definitions and propositions of the character table of the dihedral set D_3 will be given in this item, and in this way, we will show that related to the set D_3 non–vanishing part.

Definition 3.1 [1]:

The centralizer of x in G *is a subset of G defined by CG (x) = {a $\in$ G: ax = xa} of course $x \in C_G(x)$.*

Theorem 3.2 [7]:

Let G as a limited set, then the function, φ: $G/C_G(g) \longrightarrow CL(g)$ is given by $(xC_G(g))=xgx^{-1}$ is bijective and also $|CL(g)|=[G: C_G(g)]=|G| / |C_G(g)|$.

Proposition 3.3 [1]:

The characters table of G is an invertible matrix.

Example 3.4:

Consider the dihedral set D3. It has three conjugate classes:

[(1)] = {(1)}, [r] ={r,r2} and [s] = {s, sr, sr2} and it has three non-equivalent irreducible representations, T1 is the principal representation, i.e. T1(g) = [1], $\forall g \in D3$.

$T_2(g) = \begin{cases} [1] & if\ g = r^k \\ [-1] & if\ g = sr^k \end{cases}$, $\forall g \in D_3$ *i.e.* $T_2(1) = T_2(r) = T_2(r^2) = 1$, $T_2(s) = T_2(sr) = T_2(sr^2) = -1$, *and T_3 is defined as follows:*

$T_3(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $T_3(r) = \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}$, $T_3(r^2) = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega \end{bmatrix}$

$T_3(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $T_3(sr) = \begin{bmatrix} 0 & \omega^2 \\ \omega & 0 \end{bmatrix}$, $T_3(sr^2) = \begin{bmatrix} 0 & \omega \\ \omega^2 & 0 \end{bmatrix}$

Where, $\omega = e^{2\pi i/3}$

If χ_1', χ_2' and χ_3' are the features of T_1, T_2 and T_3, respectively since all the parts in the conjugate classes are equivalent, thus they have the same character.

$\equiv D_3=$

CL_α	[1]	[r]	[s]
$\|CL_\alpha\|$	1	2	3
$\|C_G(CL_\alpha)\|$	6	3	2
χ_1'	1	1	1
χ_2'	1	1	-1
χ_3'	2	-1	0

Definition 3.5 [7]:

Let G as a set and x, y $\in$ G, then x^{-1}, y^{-1} and xy are named ***commutates of x and y****. The subset of G created by the whole commutates is named commutates subset or the derived set of G and denoted by$D^{\backslash}$.*

Theorem 3.6:

Let the dihedral group D_3 be a solvable set. Then, $V(D_3) \leq F(D_3)$.

Proof: *By definition 3.5 D_3 has three conjugate classes:*

[(1)] = {(1)}, [r] = {r ,r^2} and [s] = {s , sr, sr^2}, then

$|G\backslash CL_1| = | D_3\backslash CL_1| = |6\backslash 1|=6$ *is the Centralizer of 1 in D_3 then $D_3\backslash C_6=C_1$*

$|G\backslash CL_r| = | D_3\backslash CL_r| = |6\backslash 2| = 3$ *is the Centralizer of r in D_3 then $D_3\backslash C_3 = C_2$ And,*

/G\CLs| =| D3\ CLs|=|6\3|=2 is the centralizer of s in D3 then D3\C2 =C3 such that C n is the normal cyclic subset, using Lemma 2.1, and applying Lemma 2.3, it can be concluded that r and r^2 are G-equivalent in D3 that has a regular orbit in P, since D3 is a limited solvable set, it's evident that D3 $\trianglelefteq$ C2 $\trianglelefteq$ [{1}]. So, Lemma 2.2 gives that each V(D3) - F(D3) part is a vanishing part of few D3 irreducible characters. Thus, V(D3) $\leq$ F(D3). This proof is ended.

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On Fuzzy pα-Separation Axioms

Marwa yasir Mohsen

Directorate-General of Education of Wasit

Marwayasir91@gmail.com

Abstract: in this paper we study a type of fuzzy generalized open sets in fuzzy topological spaces namely pα- open set ,and study all types of fuzzy pα- Separation axioms. Properties and relationship of fuzzy pα- Separation axioms are investigated.

Keywords: fuzzy pα- open set; fuzzy pα- Separation axioms; fuzzy pα-regular space; fuzzy pα-normal space.

1. Introduction:

the concept of fuzzy set was introduced by L.A.Zadeh. The notation of a fuzzy subsets naturally plays a significant role in the study of fuzzy topology was introduced by C.L.Chang [1] in 1968 , On the other hand A.S.Bin Shahna (1991) introduced the concept of fuzzy α-open sets .Sabiha I.Mahmood introduced and developed a new type of generalized open sets in topological space namely, pre-α-open sets ,Rubasri.M[1] and Palanisamy.M[2] , are studied the fuzzy pre-α-open sets[3] (2017), In this paper, we introduce and study a fuzzy pα-Separation Axioms.

2. Preiminaries

Throughout this paper by (X,τ) or simply by X we mean a topological space and $f : X \rightarrow Y$ means a mapping from a fuzzy topological space X to a fuzzy topological space Y. If A is a fuzzy set in X then A° , $\overline{A}$, A^c will denote respectively, the interior of A, the closure of A and complement of A.

Now we recall some of the basic definitions in the fuzzy topological space.

2.1.Definition [1,P.182-190]
Let f : X → Y be a mapping from a set X to another set Y .
(i) If λ is a fuzzy set of X, then f(λ) is a fuzzy set of Y defined as:

$$[f(\lambda)](y) = \begin{cases} \sup \lambda(x)_{x \in f^{-1}_{(y)}} & \text{if } f^{-1}_{(y)} \neq \emptyset \\ 0 & otherwise\ of\ each\ y \in Y \end{cases}$$

(ii) If μ is fuzzy set of Y , then $f^{-1}(\mu)$ is a fuzzy set of X defined as:
$[f^{-1}(\mu)](x) = \mu(f(x))$, for each x ∈ X.

2.2.Definition [1,P. 182-190]
Let X be a non- empty set and τ be a family of fuzzy sets of X. Then τ is called a fuzzy topology on X if it satisfies the following conditions:
(i) 0_X and 1_X belong to τ.
(ii) Any union of members of τ is in τ.

(iii) Any finite intersection of members of τ is in τ.

2.3.Definition [1,P. 182-190]
A fuzzy singleton p in X is a fuzzy set defined by: p(x) = t, for x = x_0 and p(x) = 0 otherwise, where 0 < t ≤ 1. The point p is said to have support x_0 and value t.

2.4.Definition [3,P. 2395-4396]
(i) A fuzzy set A in a fuzzy topological space X is called a fuzzy p-open set if A ≤ $\overline{A}^{\circ}$.
(ii) A fuzzy set A in a fuzzy topological space X is called a fuzzy α-open set if A ≤ $\overline{A^{\circ}}^{\circ}$.
(iii) A fuzzy set A in a fuzzy topological space X is called a fuzzy Pα-open set if A ≤ $\overline{A^{\circ}}^{p^{\circ}}$.

The complement of a fuzzy open sets respectively is defined to be.
(i) a fuzzy p-closed.
(ii) a fuzzy α-closed.
(iii) a fuzzy pα-closed.

2.5.Theorem 2.1:[3,P. 2395-4396]
let (X,τ) be a fuzzy topological space and A,B ≤ X. then
(i) $A^{\circ} \leq A^{\circ p} \leq A$
(ii) $A \leq \overline{A}^{p} \leq \overline{A}$.
(iii) $A^{\circ p c} = \overline{A^{c}}^{p}$.

2.6.Theorem
A subset A of a fuzzy topological space (X,τ) is a fuzzy pα-open set if there is an fuzzy open set U such that U ≤ A ≤ $\overline{U}^{p\,o}$.

Proof: ⇒) Let A be a fuzzy subset of (X,τ) and assume that A is pα-open set in X then A ≤ $\overline{A^{\circ}}^{p\,o}$. since A°=A ⇒ A° ≤ A ≤ $\overline{A^{\circ}}^{p\,o}$. Put U = A°⇒ U ≤ A ≤ $\overline{A^{o}}^{p\,o}$.

⇐) Assume that there is an fuzzy open set U of X such that U ≤ A ≤ $\overline{U}^{p\,o}$
Since U ≤ A ⇒ U≤ A° ⇒ $\overline{U}^{p} \leq \overline{A^{o}}^{p}$ ⇒ $\overline{U}^{p\,o} \leq \overline{A^{o}}^{p\,o}$. But A ≤ $\overline{U}^{p\,o}$
⇒ A ≤ $\overline{A^{o}}^{p\,o}$ it's mean A is pα-open subset of X.

2.7.Remark
if U is a fuzzy open set in (X,τ) then U ∩ $\overline{A}^{p} \leq \overline{U \cap A}^{p}$ for any subset A of X.

2.8.Remark
the family of all fuzzy pα-open subsets of X is denoted by $\tau^{p\alpha}$.

2.9.Theorem [3,P.2395-4396]
the family of all fuzzy pα-open subsets of X ($\tau^{p\alpha}$) in a fuzzy topological space form a fuzzy topology on X.

2.10.Definition
let A be a subset of fuzzy topological space (X,τ) then.
(i) The fuzzy Pα-interior of a fuzzy set A is

$A^{\circ p\alpha} = \vee$ {B : B ≤ A is a fuzzy Pα-open set}.

(ii) The fuzzy Pα-closure of a fuzzy set A is
$\overline{A}^{p\alpha} = \wedge$ {B : B ≥ A is a fuzzy Pα-closed set}.

2.11.Theorem [3,P. 2395-4396]
let A be a subset of a fuzzy topological space (X,τ) then the following statement are equivalent:

(i) A is fuzzy pα-closed.

(ii) $\overline{A}^{op} \leq$ A

(iii) There is a fuzzy closed sub set F of X such that $\overline{F}^{op} \leq$ A.

2.12.Theorem
every fuzzy open sets is pα-open, but the converse is not true.

Proof: let (X,τ) be a fuzzy topological space and A be any fuzzy open set in X it is mean A=$A^{\circ} \Rightarrow \overline{A}^{p} = \overline{A^{\circ}}^{p} \Rightarrow \overline{A}^{po} = \overline{A^{\circ}}^{po}$
But A≤ $\overline{A}^{p} \Rightarrow$A≤ $\overline{A^{\circ}}^{p^{\circ}} \Rightarrow$ A is pα-open set in X.

2.13.Example
the convers is not true in general.
Let X = {a, b}
τ= {0, 1, {$a_{0.1}$, b_0}, {$a_{0.01}$, b_0}, {$a_{0.001}$, b_0}, …}
C(x) = {0, 1, {$a_{0.9}$, b_1}, {$a_{0.99}$, b_1}, {$a_{0.999}$, b_1}, …}
A_{λ} = {0, 1, {a_{λ}, b_0}}; $0 \leq \lambda \leq 0.1$ (A_{λ} is fuzzy p-open set)
${A_{\lambda}}^{c}$= {0, 1, {$a_{1-\lambda}$, b_1}}; $0 \leq \lambda \leq 0.1$
Now take λ=0.2

$A_{o.2}$= {$a_{0.2}$, b_0} $\Rightarrow {A_{o.2}}^{o} = 1 \Rightarrow \overline{{A_{o.2}}^{o}}^{p} = 1 \Rightarrow \overline{{A_{o.2}}^{o}}^{po} = 1 \Rightarrow A_{o.2} \leq \overline{{A_{o.2}}^{o}}^{po}$
∴ $A_{o.2}$ is pα-open set, but not open .

2.14.Theorem
let A and B be a subset of fuzzy topological space (X,τ) then.

(i) $A^{\circ} \leq A^{\circ p\alpha} \leq A$; $A \leq \overline{A}^{p\alpha} \leq \overline{A}$.

(ii) $A^{\circ p\alpha}$ is a fuzzy pα-open set in X.
$\overline{A}^{p\alpha}$ is a fuzzy pα-closed set in X.

(iii) If A ≤ B then $A^{\circ p\alpha} \leq B^{\circ p\alpha}$; and $\overline{A}^{p\alpha} \leq \overline{B}^{p\alpha}$.

(iv) A is a fuzzy pα-open ⇔ $A^{\circ p\alpha} = A$
A is a fuzzy pα-closed ⇔ $\overline{A}^{p\alpha} = A$.

(v) $(A \cap B)^{op\alpha} = A^{\circ p\alpha} \cap B^{\circ p\alpha}$; $\overline{A \cup B}^{P\alpha} = \overline{A}^{p\alpha} \cup \overline{B}^{p\alpha}$.

(vi) $(A^{\circ p\alpha})^{op\alpha} = A^{\circ p\alpha}$; $\overline{\overline{A}^{p\alpha}}^{p\alpha} = \overline{A}^{p\alpha}$.

(vii) $x \in A^{\circ p\alpha}$ if and only if there exist a fuzzy pα-open set U in X such that $x \in U \leq A$.

(viii) $x \in \overline{A}^{p\alpha}$ if for every fuzzy pα-open set U containing x , U∩ $A \neq \emptyset$.

Proof: it is obvious,

2.15.Theorem

every fuzzy pα-open set is a fuzzy α-open.

Proof: let A be any fuzzy pα-open set in X, then $A \leq \overline{A^{\circ}}^{p\,o}$.
Since $\overline{A^{\circ}}^{p} \leq \overline{A^{\circ}}^{o}$, thus $A \leq \overline{A^{\circ}}^{o}$, therefore A is an fuzzy α-open in X.

2.16.Example

Let X={a, b}, τ= {0, 1, {$a_{0.1}$, b_0}, {$a_{0.3}$, b_0}, {$a_{0.5}$, b_0}}
Then (X,τ) is a fuzzy topological space.
C(x) = {0, 1, {$a_{0.9}$, b_1}, {$a_{0.7}$, b_1}, {$a_{0.5}$, b_1}}
A_λ = {0, 1, {a_λ, b_0}}, A_λ is p-open set in X such that $\lambda \in [0.9, 1] \cup [0, 0.5]$.
$A_\lambda{}^c$= {0, 1, {$a_{1-\lambda}$, b_1}}

Now if we take the set $A_{0.2}$ = {$a_{0.2}$, b_0} which is α-open set such that
But $A_{0.2}{}^{o}$ = {$a_{0.1}$, b_0}
$\overline{A_{0.2}{}^{o}}$ = {$a_{0.5}$, b_1}
$\overline{A_{0.2}{}^{o}}^{o}$ = {$a_{0.5}$, b_0} $\Rightarrow A_{0.2} \leq \overline{A_{0.2}{}^{o}}^{o}$.

$A_{0.2}{}^{o}$ = {$a_{0.1}$, b_0}
$\overline{A_{0.2}{}^{o}}^{p}$ = {$a_{0.1}$, b_1}
$\overline{A_{0.2}{}^{o}}^{p\,o}$ = {$a_{0.1}$, b_0}
$A_{0.2} \nleq \overline{A_{0.2}{}^{o}}^{p\,o}$ it's mean $A_{0.2}$ is not pα-open set .

2.17.Proposition

if A is a fuzzy pα-open set in (X,τ) and $A \leq B \leq \overline{A}^{p\,o}$, then B is a fuzzy p$\alpha$-open set in X..

proof: let A be a fuzzy pα-open set in (X,τ), then by theorem (2.2) there exist an open set U of X such that $U \leq A \leq \overline{U}^{p\,o}$. Since $A \leq B \Rightarrow U \leq B$ But $\overline{A}^{p\,o} \leq \overline{U}^{p\,o} \Rightarrow U \leq B \leq \overline{U}^{p\,o}$.
Thus B is a fuzzy pα-open set in X.

2.18.Proposition

if A is a fuzzy pα-closed set in (X,τ) and $\overline{A^{\circ p}} \leq B \leq A$, then B is a fuzzy p$\alpha$-closed set in X.

Proof: Since $A^c \leq B^c \leq \overline{A^{\circ p}}^{c} = (A^{op\,c})^{o} = \overline{A^{c}}^{p\,o}$ then by proposition (2.1) B^c is a fuzzy pα-open set in X it's mean B is a fuzzy pα-closed set in X.

2.19.Proposition[3,P. 2395-4396]

let (X,τ_1) and (Y,τ_2) be a fuzzy topological space. If $A_1 \leq X$, $A_2 \leq Y$, then $A_1 \times A_2$ is a fuzzy pα-0pen set in X×Y if and only if A_1 and A_2 are fuzzy pα-0pen sets in X and Y respectively.

2.20.Definition

let Y be a subset of a fuzzy topological space (X,τ); then (Y,$\tau_y{}^{p\alpha}$) is a fuzzy topological subspace on X, if
$\tau_y{}^{p\alpha} = \{ Y \cap A; A\ is\ p\alpha$-open set in X}.

2.21.Proposition

if $(Y, \tau_y{}^{p\alpha})$ is a fuzzy topological subspace of (X,τ), and A≤Y. then

(i) $A^{\circ p\alpha}$ = Y∩ $A^{\circ}{}_{y}{}^{p\alpha}$.

(ii) $\overline{A_y}^{p\alpha}$ =Y∩ $\overline{A}^{p\alpha}$.

2.22.Definition [2,P. 189-202]

A fuzzy topological space (X,τ) is said to be:

(i) Fuzzy T_0 (FT_0) if for every pair of fuzzy singletons p, q with different supports there exists a fuzzy open set U such that either $p \leq U \leq q^c$ or $q \leq U \leq p^c$.

(ii) Fuzzy T_1 (FT_1) if for every pair of fuzzy singletons p, q with different supports there exist fuzzy open sets U and V such that $p \leq U \leq q^c$ and $q \leq V \leq p^c$.

(iii) Fuzzy stronger T_1 (FT_s) if every fuzzy singleton is a fuzzy closed set.

(iv) Fuzzy housdorff (FT_2) if for every pair of fuzzy singletons p, q with different supports, there exist two fuzzy open sets U and V such that $p \leq U \leq q^c$, $q \leq V \leq p^c$ and $U \leq V^c$.

(v) Fuzzy Uryshon ($FT_{2½}$) if for every pair of fuzzy singletons p, q with different supports, there exist two fuzzy open sets U and V such that $p \leq U \leq q^c$, $q \leq V \leq p^c$ and $\overline{U} \leq (\overline{V})^c$.

(vi) Fuzzy regular space (FR) if for a fuzzy singleton p and a fuzzy closed set V, there exist two fuzzy open sets U_1 and U_2 such that $V \leq U_2$, $p \leq U_1$ and $U_1 \leq U_2^c$.

(vii) Fuzzy T_3 (F ST_3) if it is (FR) as well as (F ST_s).

(viii) Fuzzy normal space (FN) if for every pair of fuzzy closed sets V_1 and V_2 such that $V_1 \leq V^c_2$, there exist two fuzzy open sets U_1 and U_2 such that $V_1 \leq U_1$, $V_2 \leq U_2$ and $U_1 \leq U^c_2$.

(ix) Fuzzy T_4 (FT_4) if it is (FN) as well as (FT_s).

3. pα-separation axioms.

In this section we introduce fuzzy pα-separation axioms (pα-T_λ space for λ= 0, 1, 2,2½, 3,4} as follows:

3.1.Definition

A fuzzy topological space is said to be fuzzy pα-T_0 space if for every pair of fuzzy singletons p, q with different supports there exists a fuzzy pα-open set U such that either,

$p \leq U \leq q^c$ or $q \leq U \leq p^c$.

3.2.Remark

it's clear that fuzzy T_0 implies pα-T_0, but the converse is not true.

3.3.Example

X= {a, b}

$\tau = \{0, 1, \{a_{¾}, b_0\}\}$

$\tau^{p\alpha} = \{0, 1, \{a_\lambda, b_0\}\}$; $\lambda \in (0, 1]$

Now if we take v={a_1 , b_0} is a fuzzy pα- open set in (X,τ) but is not a fuzzy open set then (X,τ) is a fuzzy pα-T_0 but not fuzzy T_0 space

3.4.Theorem

a fuzzy topological *space* (X, τ) is fuzzy pα-T_0 space if and only if any two crisp fuzzy singletons with different supports, have disjoint fuzzy pα-closures.

Proof: Let (X, τ) be fuzzy pα-T_0 space and p, q be two crisp fuzzy singletons with supports x_1, x_2, respectively, where $x_1 \neq x_2$,
Since (X,τ) being fuzzy pα-T_0. There exists a fuzzy pα-open set U such that $p \leq U \leq q^c$. This implies that $q \leq \overline{q}^{p\alpha} \leq U^c$, since $p \nleq U^c$,
$p \nleq \overline{q}^{p\alpha}$. But $p \leq \overline{p}^{p\alpha}$. Therefore $\overline{q}^{p\alpha} \neq \overline{p}^{p\alpha}$.
Conversely, let p, q be any two fuzzy singletons with different supports x_1, x_2, respectively. Let p_1, q_1 be fuzzy singletons such that
$p_1(x_1) = q_1(x_1) = 1$. By hypothesis $\overline{p_1}^{p\alpha} \neq \overline{q_1}^{p\alpha}$ and $p_1 \leq \overline{p_1}^{p\alpha}$ implies
$p^c_1 \geq (\overline{p_1}^{p\alpha})^c$, but $p \leq p_1$ implies that $p^c \geq p^c_1 \geq (\overline{p_1}^{p\alpha})^c$. Thus $(\overline{p_1}^{p\alpha})^c$ is a fuzzy pα-open set such that $q \leq (\overline{p_1}^{p\alpha})^c \leq p^c$.
Hence (X,τ) is fuzzy $p\alpha - T_0$ space ■.

3.5.Definition
A fuzzy topological space (X, τ) is said to be a fuzzy pα-T_1 space if for every pair of fuzzy singletons p, q with different supports there exist a fuzzy pα-open sets U and V such that $p \leq U \leq q^c$ and $q \leq V \leq p^c$.

3.6.Remark
Every fuzzy pα-T_1 space is obviously fuzzy pα -T_0 space. But the converse does not need to be true.

3.7.Example
the space in example (3.3) is a fuzzy pα -T_0 space but not fuzzy pα-T_1 space.

3.8.Theorem
A fuzzy topological space (X, τ) is fuzzy $p\alpha - T_1$ if and only if every crisp fuzzy singleton is a fuzzy pα-closed set.

Proof: Let (X, τ) be fuzzy $p\alpha - T_1$; and p_0 be a crisp fuzzy singleton with support x_0. Now, for any fuzzy singleton p with support x in X such that
$x \neq x_0$ there exist fuzzy pα-open sets U and V such that
$p \leq U \leq q^c$ and $q \leq V \leq p^c$.
Since, every fuzzy set is considered as the union of fuzzy singletons it contains; we obtain in particular $p^c_0 = \vee\{p: p \leq p_0^c\}$ from $p^c_0(x_0) = 1 - p_0(x_0) = 0$.
We deduce that $p^c_0 = \vee\{V: p \leq p^c_0\}$ and thus p^c_0 is a fuzzy pα-open set
$\Rightarrow$ p_0 is a fuzzy pα-closed set.
Conversely, let p_1 and p_2 be a fuzzy singleton with different supports x_1, x_2. Also let q_1 and q_2 be fuzzy singletons with different supports x_1, x_2, respectively and such that $q_1(x_1) = q_2(x_2) = 1$. The fuzzy sets q^c_1 and q^c_2 are fuzzy pα-open sets and satisfy the conditions:
$p_1 \leq q^c_2 \leq p^c_2$ and $p_2 \leq q^c_1 \leq p^c_1$. Hence the space (X, τ) is fuzzy $p\alpha - T_1$ space .

3.9.Definition
A fuzzy topological space (X, τ) is said to be a fuzzy pα-stronger-T_1 space.($p\alpha - T_s$) if every fuzzy singleton is a fuzzy pα-closed set.

3.10.Remark
Every fuzzy $p\alpha - T_s$ space is a fuzzy $p\alpha - T_1$, but the converse need not be true.

3.11.Example
let $X = \{a, b\}$

$\tau = \{ 0, 1, \{ a_\lambda , b_0 \}, \{ a_0 , b_r\},\{ a_\lambda , b_r \} \}$; $\lambda, r \in [½,1]$.

$\tau^{p\alpha} = \tau \Rightarrow$ every fuzzy crisp singleton is pα-closed set.

Then $(X, \tau^{p\alpha})$ is fuzzy $p\alpha-T_1$ but not every fuzzy singleton pα-closed set.

3.12.Definition

A fuzzy topological space (X, τ) is said to be fuzzy pα-Housdorff ($p\alpha-T_2$) if for every pair of fuzzy singletons p, q with different supports, there exist two fuzzy pα-open sets U and V such that $p \le U \le q^c$, $q \le V \le p^c$ and $U \le V^c$.

3.13.Remark

every fuzzy pα-Housdorff ($p\alpha-T_2$) is a fuzzy $p\alpha-T_1$, but the converse not needs to be true.

3.14.Definition

A fuzzy topological space (X, τ) is said to be fuzzy pα-Uryshon ($p\alpha-T_{2½}$) if for every pair of fuzzy singletons p, q with different Supports, there exist two fuzzy pα-open sets U and V such that $p \le U \le q^c$, $q \le V \le p^c$ and $\overline{U}^{p\alpha} \le \overline{V}^{p\alpha}$.

3.15.Remark

it's easy to show that if (X,τ) is fuzzy $p\alpha-T_{2½}$ space then (X, τ) is $p\alpha-T_2$ space.

3.16.Definition

A fuzzy topological space (X, τ) is said to be fuzzy pα-regular space (pα-R) if for a fuzzy singleton p and a fuzzy closed set V,
There exist two fuzzy pα-open sets U_1 and U_2 such that $V \le U_2$, $p \le U_1$ and $U_1 \le U^c_2$.

3.17.Theorem

A fuzzy topological space (X, τ) is said to be fuzzy pα-regular space (pα-R) if for every fuzzy singleton p and a fuzzy open subset U of X, with $p \in U$, there exist $V \in \tau^{p\alpha}$ such that
$p \in V \le \overline{V} \le U$.

Proof:⇒)
let (X, τ) be a fuzzy pα-regular space (pα-R) ,and let p any fuzzy singleton in X and U fuzzy open subset of X , with $p \in U$;
It's mean U^c is a fuzzy closed set, since (X, τ) a fuzzy pα-regular space and $p \notin U^c$, then there exist $V_1, V_2 \in \tau^{p\alpha}$; such that $p \in V_1$, $U^c \le V_2$ and
$V_1 \le V_2^c$; since $U^c \le V_2 \Rightarrow V_2^c \le U$ and $V_1 \le V_2^c \Rightarrow \overline{V_1} \le \overline{V_2{}^c}$ But V_2^c is closed $\Rightarrow \overline{V_1} \le V_2^c$ we obtained $p \in V_1 \le \overline{V_1} \le U$.

⇐) let p be any fuzzy singleton and suppose that F be a fuzzy closed set in X such that $p \notin F \Rightarrow P \in F^c \in \tau$
Then there exist there exist $V \in \tau^{p\alpha}$ such that $p \in V \le \overline{V} \le F^c \Rightarrow F \le \overline{V}^c$ and since $V \le (\overline{V}^c)^c$
$\Rightarrow (X, \tau)$ is fuzzy pα-regular space.

3.18.Theorem

let (X, τ) be a fuzzy pα-regular space (pα-R), then for any fuzzy closed subset F of X and a fuzzy singleton p where $p \in F^c$ there exist $U, V \in \tau^{p\alpha}$ such that $p \in U$, $F \le W$ and
$\overline{U} \le (\overline{W})^c$.

Proof: let F be any fuzzy closed subset of X, then F^c is fuzzy closed subset of X then by theorem (3.17), there exist $V \in \tau^{p\alpha}$ such that $p \in V \le \overline{V} \le U = F^c$

Take V= ($\overline{F^c}$)^c then $\overline{V} \le (\overline{U})^c$.

3.19.Theorem
if (X, τ) fuzzy pα − T_0 space and pα − R space then it's pα − $T_{2½}$ space.

Proof: let (X, τ) fuzzy pα − T_0 space and pα − R space, and let p, q be two fuzzy singleton with different supports, since(X, τ) fuzzy pα − T_0 space then there exist $U \in \tau^{p\alpha}$ such that
$p \in U \le q^c$, take $F = U^c \Rightarrow F^c = U$ which is pα-open set and $p \in F^c$ now by theorem(3.18) since F is pα-closed subset of a fuzzy pα − R space then there exist ; V, W $\in \tau^{p\alpha}$, such that $p \in V$ and $F \le W$ with $\overline{V} \le (\overline{W})^c$, but $q \in U^c = F \le W$. we obtained $p \in V$ and $q \in W$ and $\overline{V} \le (\overline{W})^c$ then
(X, τ) pα − $T_{2½}$ space.

3.20.Corollary
if (X, τ) fuzzy pα − T_0 space and pα − R space then it's pα − T_2 space.
Proof: it is obvious.

3.21.Definition
A fuzzy topological space (X, τ) is said to be fuzzy pα-T_3 space (pα− T_3) if it is (pα-R) as well as (pα− T_s) space.

3.22.Definition
A fuzzy topological space (X, τ) is said to be fuzzy pα-normal space (pα-N) if for every pair of fuzzy closed sets V_1 and V_2 such that $V_1 \le V^c_2$, there exist two fuzzy pα-open sets U_1 and U_2 such that, $V_1 \le U_1$, $V_2 \le U_2$ and $U_1 \le U^c_2$.

3.23.Definition
A fuzzy topological space (X, τ) is said to be fuzzy pα-T_4(pα− T_4) if it is (pα-N) as well as (pα− T_s) space.

3.24.Theorem
a fuzzy closed subset of a fuzzy pα-normal space (pα-N) is fuzzy pα-normal.

Proof: let (X, τ) be a fuzzy pα-normal space (pα-N) and let B be a closed subset of X, then (B,${\tau_B}^{p\alpha}$) is a subspace.
Take F_1, F_2 any two fuzzy closed subsets of B with $F_1 \le F_2^c$ in B ,
Since B is fuzzy closed subset of X ⇒ $F_1 \le F_2^c$ in X but (X,τ) a fuzzy pα-normal space, then there exist U, V∈ $\tau^{p\alpha}$ such that $F_1 \le U$, $F_2 \le V$ and
$U \le V^c$. now B∧U and B∧V are two fuzzy pα-open subsets of ${\tau_B}^{p\alpha}$such that
$F_1 \le B \wedge U$, $F_2 \le B \wedge V$ and $B \wedge U \le (B \wedge V)^c$ then B a fuzzy pα-normal .

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Classification of $(k;4)$-arcs up to projective inequivalence, for $k < 10$

Z.S. Hamed[1] and **J.W.P. Hirschfeld**[2]

[1]Department of Mathematics, College of Science, Mustansiriah University, Iraq

[2]Department of Mathematics, University of Sussex, United Kingdom

Abstract

In this paper, the classification of $(k; 4)$-arcs up to projective inequivalence for $k < 10$ in PG(2, 13) is introduced in details according to their inequivalent number, stabilisers, the action of each stabiliser on the associated arc, and the inequivalent classes N_c of secant distributions of arcs. Here, the strategy is to start from the projective line PG(1,13) where there are three projectively inequivalent tetrads.

1 Basic concepts

1.1 Finite fields

A field **F** is a set of elements with two operations, addition (+) and multiplication ($\times$), satisfying the following properties:

(a) $(\mathbf{F}, +)$ is an abelian group with identity 0;

(b) $(\mathbf{F} \setminus \{0\}, \times)$ is an abelian group with identity 1;

(c) $x(y + z) = xy + xz$, for all $x, y, z \in \mathbf{F}$.

1.2 Note

A finite field is defined up to an isomorphism by the number q of its elements. So, q must be an integer power p^h of a prime p. Here, p is the *characteristic* of the finite field. Then, every element $x \in \mathbf{F}_q$ satisfies $x^q - x = 0$. When $q = p$, then $\mathbf{F}_p = \{0, 1, \ldots, p-1\}$; when $q = p^h$, $h > 1$, then $\mathbf{F}_q = \{0, 1, \alpha, \alpha^2, \ldots, \alpha^{q-2} \mid \alpha^{q-1} = 1\}$ for some $\alpha \in \mathbf{F}_q$. The non-zero elements of $\mathbf{F}_q$ form a group $\mathbf{F}_q^*$ of order $q - 1$ such that $\mathbf{F}_q^* \cong Z_{q-1}$.

1.3 Finite groups

Definition .1 *A group is an ordered pair* $(G, *)$, *where* G *is a non-empty set and* $*$ *is a binary operation on* G *such that the following properties hold.*

(1) *For all* $a, b, c \in G$, $a * (b * c) = (a * b) * c$.

(2) *There exists* $e \in G$ *such that for all* $a \in G$, $a * e = a = e * a$.

(3) *For all* $a \in G$, *there exists* $b \in G$ *such that* $a * b = e = b * a$.

1.4 Group action on a set

Let G be a group acts on a set X if for each $g \in G$ and $x \in X$ an element $gx \in X$ is defined, such that $g_2(g_1x) = (g_2g_1)x$ and $ex = x$ for all $x \in X$, $g_1, g_2 \in G$.

The set

$$Orb(x) = \{gx \mid g \in G\},$$

is called the orbit of the element x. The stabilizer of an element x of X is the subgroup

$$S = \{g \in G \mid gx = x\}.$$

The fixed points set of an element g of G is the set defined as follows:

$$Fix(g) = \{x \in X \mid gx = x\}.$$

2 The projective plane PG(2, q)

The projective plane PG(2, q) over $\mathbf{F}_q$ contains q^2+q+1 points and lines. There are $q+1$ points on each line and $q + 1$ lines passing through each point. The value of q that has been used in this work is $q = 13$. Therefore the projective plane PG(2, 13) has 183 points and lines, with 14 points on each line and 14 lines passing through each point. The point $\mathbf{P}(x_0, x_1, x_2)$ in the projective plane, PG(2, q), can be represented as a vector of three coordinates over $\mathbf{F}_q$ as shown in Table 1.

Table 1: **The points in** PG(2, q)

Point format	Number of points
$\mathbf{P}(x_0, x_1, 1)$	q^2
$\mathbf{P}(x_0, 1, 0)$	q
$\mathbf{P}(1, 0, 0)$	1

A line in PG(2, q) is a set of points $\mathbf{P}(x_0, x_1, x_2)$ satisfying the homogeneous linear equation

$$ax_0 + bx_1 + cx_2 = 0,$$

with $a, b, c \in \mathbf{F}_q$ not all zero; it is denoted by $\mathbf{L}(a, b, c)$. Thus, a projective plane is an incidence structure of points and lines with the following properties:

(i) every two points are incident with a unique line;

(ii) every two lines are incident with a unique point;

(iii) there are four points, no three collinear.

3 General linear group of a vector space

Let $\mathbf{F}_q$ is a finite field and let $V(n,q)$ is a vector space of dimension n over $\mathbf{F}_q$, then the linear map $V(n,q) \longrightarrow V(n,q)$, such that $x \longrightarrow xA$, for $x \in V$ a row vector and A a non-singular $n \times n$ matrix over $\mathbf{F}_q$. The group consisting of all linear maps of $V(n,q)$, that is, the group consisting of all

non-singular $n \times n$ matrices over $\mathbf{F}_q$, is called the general linear group and is denoted by GL(n, q). The order of GL(n, q) is as follows:

$$|\mathrm{GL}(n, q)| = (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1}).$$

In addition, the subgroup SL(n,q) consisting of all matrices with determinant 1, and it is called the special linear group of degree n over $\mathbf{F}_q$. The group SL(n, q) contains a subgroup UT(n, q) consisting of those matrices with all entries below the main diagonal zero, and with the entries on the main diagonal equal to the identity. This subgroup is called the unitriangular group of degree n over $\mathbf{F}_q$.

3.1 The fundamental theorem in PG(2, q)

If $\varphi : P \longrightarrow P^J$ is a bijective mapping from one projective plane, PG(2, q), to another, then there is a unique projectivity shifting any quadrangle, that is, a set of four points no three collinear, to

another quadrangle.

11.1.1 Definition .2

A (k; n)-arc K *in* PG(2, q) *is a set of* k *points such that no* $n + 1$ *of them are collinear but some* n *are collinear.*

3.2 Lexicographically least set

Given the sets $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_r\}$ of integers, with $a_1 < a_2 < \cdots < a_r$ and $b_1 < b_2 < \cdots < b_r$. Then $A \leq B$ lexicographically if either $A = B$ or if, for some i with $1 \leq i < r$, we have $a_1 = b_1, \ldots, a_i = b_i$, but $a_{i+1} < b_{i+1}$.

4 Classification of (k;4)-arcs up to projective inequivalence, for $k < 10$

The number of projectively inequivalent (k;4)-arcs for $k < 10$ is given in the following subsections.

4.1 Projectively inequivalent (4; 4)-arcs

In this classification, the number of tetrads is constructed by fixing a triad, $U_1 = \{1, 2, 9\}$. There are eleven tetrads containing U_1. The lexicographically least sets in the G-orbits of tetrads, where $G = PGL(2, 13)$ took 2104 msec. Then among these canonical sets there are three projectively

inequivalent tetrads; this took 1699 msec. Also, the three tetrads have sd-equivalent secant distri- bution. It took 1734. The statistics are shown in Table 2.

Table 2: **Projectively inequivalent tetrads**

Number	Tetrad	$\{t_4, t_3, t_2, t_1, t_0\}$
1	$\{1, 2, 9, 21\}$	$\{1, 0, 0, 52, 130\}$
2	$\{1, 2, 9, 83\}$	$\{1, 0, 0, 52, 130\}$
3	$\{1, 2, 9, 115\}$	$\{1, 0, 0, 52, 130\}$

Theorem .3 *In* PG(1, 13), *there are exactly three projectively inequivalent tetrads.*

4.2 Projectively inequivalent (5; 4)-arcs

The (5; 4)-arcs are constructed by adding all the points from the plane, PG(2, 13), which are not on the line to each inequivalent tetrad given in Table 2. So, the constructed number of (5; 4)-arcs is 507. The lexicographically least set images of the 507 (5;4)-arcs are computed. This shows that

the number Φ_4 of projectively inequivalent (5; 4)-arcs is three. The three (5; 4)-arcs all have the same secant distribution, that is, $\{1, 0, 4, 58, 120\}$. In addition, the stabiliser of each of the three

projectively inequivalent (5; 4)-arcs is $Z_3 \times ((Z_4 \times Z_4) \text{ w } Z_2)$, $Z_3 \times (Z_8 \text{ w } Z_2)$, $Z_3 \times (\text{SL}(2, 3) \text{ w } Z_2)$.

The statistics are given in the following tables:

Table 3: **Projectively inequivalent (5; 4)-arcs**

Number	Φ_4	Stabiliser	$\{t_4, t_3, t_2, t_1, t_0\}$
1	$\{1, 2, 9, 83, 3\}$	$Z_3 \times ((Z_4 \times Z_4) \text{ w } Z_2)$	$\{1, 0, 4, 58, 120\}$
2	$\{1, 2, 9, 21, 3\}$	$Z_3 \times (Z_8 \text{ w } Z_2)$	$\{1, 0, 4, 58, 120\}$
3	$\{1, 2, 9, 115, 3\}$	$Z_3 \times (\text{SL}(2, 3) \text{ w } Z_2)$	$\{1, 0, 4, 58, 120\}$

Theorem .4 *In* PG(2, 13), *there are exactly three projectively inequivalent* (5; 4)-*arcs.*

Table 4: **Points added**

Tetrad	Points added
1	3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 84, 85, 86, 87, 88, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 116, 117, 118, 120, 121, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 183
2	3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 84, 85, 86, 87, 88, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 116, 117, 118, 120, 121, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 162, 163, 164, 165, 166, 167,168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 183
3	3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39,40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 84, 85, 86, 87, 88,90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 116, 117, 118, 120, 121, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 162, 163, 164, 165, 166, 167,168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 183

11.2 Remark

The stabiliser groups in Table 3 split the associated projectively inequivalent (5; 4)-arcs into 2 orbits. They are given as follows.

(1) The group $Z_3 \times ((Z_4 \times Z_4)\,\mathrm{w}\,Z_2)$ partitions the (5;4)-arc $\{1,2,9,83,3\}$ into 2 orbits $\{1,9,2,83\}$,

$\{3\}$.

(2) The group $Z_3 \times (Z_8\ \mathrm{w}\ Z_2)$ splits the (5; 4)-arc $\{1, 2, 9, 21, 3\}$ into 2 orbits $\{1, 9, 2, 21\}$, $\{3\}$.

(3) The group $Z_3 \times (\mathrm{SL}(2,3)\,\mathrm{w}\,Z_2)$ divides the (5;4)-arc $\{1,2,9,115,3\}$ into 2 orbits $\{1,2,115,9\}$,

$\{3\}$.

4.3 Projectively inequivalent (6; 4)-arcs

In Table 3, for each projectively inequivalent (5; 4)-arc the points from the plane which are not on any 4-secant are added to construct the (6;4)-arcs. Therefore, the number of (6;4)-arcs that constructed is 504. Among the 504 (6;4)-arcs the lexicographically least set image and the stabiliser are calculated. So, the number Φ_4 of projectively inequivalent (6; 4)-arcs is 10. Also, the secant

distribution $\{t_4, t_3, t_2, t_1, t_0\}$ for each of the 10 projectively inequivalent (6;4)-arcs is computed. It shows that there are only two *sd*-inequivalent classes N_c of secant distributions. The statistics of the 10 projectively inequivalent (6; 4)-arcs are given in the following tables:

Table 5: **Projectively inequivalent** (6; 4)**-arcs**

Number	Φ_4	Stabiliser	Orbits
1	$\{1,2,9,83,3,4\}$	$Z_2 \times Z_2$	$\{1\},\{2\},\{3,4\},\{9,83\}$
2	$\{1,2,9,21,3,4\}$	Z_2	$\{1\},\{2\},\{3,4\},\{9\},\{21\}$
3	$\{1,2,9,115,3,4\}$	Z_6	$\{1\},\{2,115,9\},\{3,4\}$
4	$\{1,2,9,83,3,8\}$	$Z_4 \times Z_2$	$\{1,9,2,83\},\{3,8\}$
5	$\{1,2,9,21,3,5\}$	Z_2	$\{1\},\{2\},\{3,5\},\{9\},\{21\}$
6	$\{1,2,9,21,3,12\}$	Z_2	$\{1\},\{2\},\{3,12\},\{9\},\{21\}$
7	$\{1,2,9,21,3,14\}$	$Z_2 \times Z_2$	$\{1,2\},\{3,14\},\{9,21\}$

8	$\{1, 2, 9, 83, 3, 5\}$	Z_2	$\{1\}, \{2\}, \{3, 5\}, \{9\}, \{83\}$
9	$\{1, 2, 9, 115, 3, 7\}$	$Z_2 \times Z_2$	$\{1, 115\}, \{2, 9\}, \{3, 7\}$
10	$\{1, 2, 9, 115, 3, 5\}$	Z_6	$\{1\}, \{2, 115, 9\}, \{3, 5\}$

Table 6: N_c **of** t_4, t_3, t_2, t_1, t_0

Number	N_c	Number of N_c
1	$\{1, 1, 6, 65, 110\}$	3
2	$\{1, 0, 9, 62, 111\}$	7

Theorem .5 *In* PG(2, 13), *there are exactly ten projectively inequivalent* (6; 4)-*arcs.*

Projectively inequivalent (7; 4)-arcs

In this process, the constructed number of (7;4)-arcs is 1670. According to their lexicographically least set images, the number of projectively inequivalent (7;4)-arcs is 207. Among the 207 arcs,

there are eleven types of the stabiliser groups. In addition, the secant distribution $\{t_4, t_3, t_2, t_1, t_0\}$ of each of the (7; 4)-arcs is also computed. It shows that there are five *sd*-inequivalent classes of secant distributions. The statistics are given in Tables 7 and 8.

Table 7: **Projectively inequivalent** (7; 4)**-arcs**

Number	Φ_4	Stabiliser
1	{1, 2, 9, 83, 3, 4, 57}	I
2	{1, 2, 9, 83, 3, 4, 5}	I
3	{1, 2, 9, 21, 3, 4, 20}	I
4	{1, 2, 9, 21, 3, 4, 5}	I
5	{1,2,9,21,3,4,22}	I
6	{1,2,9,21,3,4,32}	I
7	{1,2,9,21,3,4,37}	I
8	{1,2,9,21,3,4,58}	I
9	{1, 2, 9, 115, 3, 4, 22}	Z_2
10	{1,2,9,115,3,4,5}	I
11	{1,2,9,83,3,4,51}	I
12	{1, 2, 9, 83, 3, 4, 6}	I
13	{1,2,9,83,3,4,19}	I
14	{1,2,9,21,3,4,13}	I
15	{1,2,9,21,3,4,19}	I
16	{1,2,9,21,3,4,96}	I
17	{1,2,9,21,3,4,27}	I
18	{1,2,9,21,3,4,28}	I
19	{1,2,9,21,3,4,56}	I
20	{1,2,9,21,3,4,149}	I
21	{1,2,9,21,3,4,122}	I
22	{1, 2, 9, 21, 3, 4, 6}	I
23	{1, 2, 9, 83, 3, 4, 30}	I
24	{1,2,9,115,3,4,50}	I
25	{1,2,9,115,3,4,15}	I
26	{1,2,9,83,3,4,27}	I
27	{1,2,9,83,3,4,33}	I
28	{1,2,9,115,3,4,10}	I
29	{1,2,9,115,3,4,30}	I
30	{1,2,9,21,3,4,101}	I
31	{1,2,9,21,3,4,30}	I
32	{1,2,9,83,3,4,47}	I
33	{1,2,9,21,3,4,40}	I
34	{1,2,9,21,3,4,75}	I

35	$\{1,2,9,21,3,4,127\}$	I	
36	$\{1,2,9,21,3,4,100\}$	I	
37	$\{1, 2, 9, 21, 3, 4, 14\}$	I	
38	$\{1, 2, 9, 21, 3, 4, 111\}$	I	
39	$\{1,2,9,21,3,4,12\}$	I	
40	$\{1,2,9,83,3,4,15\}$	I	
41	$\{1,2,9,83,3,4,16\}$	I	
42	$\{1, 2, 9, 115, 3, 4, 103\}$	I	
43	$\{1, 2, 9, 115, 3, 4, 6\}$	I	
44	$\{1, 2, 9, 83, 3, 4, 8\}$	I	
45	$\{1,2,9,83,3,4,43\}$	I	
46	$\{1,2,9,115,3,4,7\}$	I	
47	$\{1, 2, 9, 115, 3, 4, 20\}$	I	
48	$\{1,2,9,21,3,4,18\}$	I	
49	$\{1,2,9,21,3,4,65\}$	I	
50	$\{1,2,9,83,3,4,20\}$	I	
51	$\{1,2,9,83,3,4,92\}$	I	
52	$\{1, 2, 9, 21, 3, 4, 136\}$		Z_2
53	$\{1,2,9,21,3,4,95\}$	I	
54	$\{1,2,9,21,3,4,49\}$	I	
55	$\{1,2,9,21,3,4,44\}$	I	
56	$\{1, 2, 9, 115, 3, 4, 18\}$	I	
57	$\{1,2,9,83,3,4,11\}$		D_4
58	$\{1,2,9,83,3,4,31\}$	I	
59	$\{1,2,9,83,3,4,10\}$		Z_2
60	$\{1,2,9,83,3,4,17\}$	I	
61	$\{1,2,9,83,3,4,49\}$	I	
62	$\{1,2,9,83,3,4,23\}$		Z_2
63	$\{1,2,9,83,3,4,28\}$	I	
64	$\{1,2,9,83,3,4,54\}$	I	
65	$\{1,2,9,83,3,4,13\}$	I	
66	$\{1,2,9,83,3,4,37\}$		Z_2
67	$\{1,2,9,83,3,4,40\}$	I	
68	$\{1,2,9,83,3,4,26\}$	I	
69	$\{1,2,9,83,3,4,76\}$		Z_2
70	$\{1,2,9,83,3,4,25\}$	I	
71	$\{1, 2, 9, 83, 3, 4, 7\}$	I	
72	$\{1,2,9,83,3,4,82\}$		Z_2
73	$\{1,2,9,83,3,4,71\}$	I	
74	$\{1,2,9,83,3,4,108\}$		Z_2
75			Z_2

76	$\{1,2,9,83,3,4,126\}$ $\{1, 2, 9, 83, 3, 4, 14\}$	Z_2
77	$\{1,2,9,83,3,4,130\}$	Z_6
78	$\{1,2,9,83,3,4,100\}$	I
79	$\{1, 2, 9, 21, 3, 4, 35\}$	I
80	$\{1, 2, 9, 115, 3, 4, 24\}$	Z_2
81	$\{1,2,9,21,3,4,16\}$	I
82	$\{1,2,9,21,3,4,43\}$	I
83	$\{1,2,9,21,3,4,46\}$	I
84	$\{1,2,9,21,3,4,51\}$	I
85	$\{1,2,9,115,3,4,8\}$	I
86	$\{1, 2, 9, 115, 3, 4, 16\}$	I
87	$\{1,2,9,21,3,4,50\}$	I
88	$\{1,2,9,21,3,4,82\}$	I
89	$\{1,2,9,21,3,4,17\}$	I
90	$\{1, 2, 9, 21, 3, 4, 152\}$	I
91	$\{1,2,9,21,3,4,76\}$	I
92	$\{1,2,9,21,3,4,55\}$	I
93	$\{1,2,9,21,3,4,94\}$	I
94	$\{1,2,9,115,3,4,17\}$	I
95	$\{1,2,9,115,3,4,33\}$	I
96	$\{1,2,9,115,3,4,34\}$	I
97	$\{1, 2, 9, 21, 3, 4, 8\}$	I
98	$\{1, 2, 9, 21, 3, 4, 57\}$	I
99	$\{1, 2, 9, 21, 3, 4, 103\}$	I
100	$\{1,2,9,21,3,4,47\}$	I
101	$\{1,2,9,21,3,4,48\}$	I
102	$\{1,2,9,21,3,4,34\}$	I
103	$\{1,2,9,21,3,4,26\}$	I
104	$\{1, 2, 9, 21, 3, 4, 108\}$	I
105	$\{1, 2, 9, 21, 3, 4, 25\}$	I
106	$\{1,2,9,115,3,4,74\}$	I
107	$\{1,2,9,115,3,4,26\}$	I
108	$\{1, 2, 9, 21, 3, 4, 15\}$	I
109	$\{1, 2, 9, 21, 3, 4, 7\}$	I
110	$\{1,2,9,21,3,4,118\}$	Z_2
111	$\{1,2,9,115,3,4,35\}$	Z_3
112	$\{1,2,9,21,3,4,23\}$	Z_2
113	$\{1,2,9,21,3,4,71\}$	I
114		I

115	$\{1, 2, 9, 21, 3, 4, 110\}$	I
116	$\{1,2,9,21,3,4,74\}$	I
117	$\{1,2,9,21,3,4,54\}$	I
118	$\{1,2,9,21,3,4,31\}$	I
	$\{1,2,9,21,3,4,33\}$	
119	$\{1, 2, 9, 21, 3, 4, 66\}$	I
120	$\{1, 2, 9, 21, 3, 4, 130\}$	Z_3
121	$\{1, 2, 9, 21, 3, 4, 77\}$	I
122	$\{1,2,9,115,3,4,32\}$	I
123	$\{1,2,9,115,3,4,14\}$	I
124	$\{1, 2, 9, 115, 3, 4, 130\}$	$Z_3 \times S_3$
125	$\{1,2,9,83,3,8,17\}$	Z_2
126	$\{1,2,9,21,3,5,56\}$	I
127	$\{1, 2, 9, 21, 3, 12, 18\}$	I
128	$\{1, 2, 9, 83, 3, 5, 7\}$	I
129	$\{1, 2, 9, 115, 3, 7, 12\}$	Z_2
130	$\{1,2,9,115,3,7,6\}$	Z_2
131	$\{1,2,9,21,3,5,31\}$	I
132	$\{1,2,9,83,3,8,60\}$	$Z_4 \times Z_2$
133	$\{1, 2, 9, 83, 3, 8, 7\}$	I
134	$\{1,2,9,83,3,8,18\}$	I
135	$\{1,2,9,83,3,8,57\}$	Z_{12}
136	$\{1,2,9,83,3,8,40\}$	Z_2
137	$\{1,2,9,83,3,8,24\}$	I
138	$\{1,2,9,83,3,8,62\}$	I
139	$\{1,2,9,83,3,8,26\}$	I
140	$\{1, 2, 9, 83, 3, 8, 5\}$	I
141	$\{1,2,9,83,3,8,19\}$	Z_4
142	$\{1,2,9,83,3,8,12\}$	I
143	$\{1, 2, 9, 21, 3, 12, 17\}$	I
144	$\{1, 2, 9, 21, 3, 5, 13\}$	Z_2
145	$\{1, 2, 9, 115, 3, 7, 16\}$	Z_2
146	$\{1, 2, 9, 83, 3, 5, 6\}$	I
147	$\{1, 2, 9, 21, 3, 12, 68\}$	Z_2
148	$\{1, 2, 9, 21, 3, 5, 6\}$	I
149	$\{1, 2, 9, 21, 3, 14, 52\}$	$Z_2 \times Z_2$
150	$\{1, 2, 9, 83, 3, 5, 13\}$	Z_2
151	$\{1,2,9,115,3,7,49\}$	$Z_2 \times Z_2$
152	$\{1,2,9,115,3,5,13\}$	Z_6
153	$\{1,2,9,21,3,5,111\}$	I
154		I

155	$\{1,2,9,21,3,5,79\}$	I	
156	$\{1,2,9,21,3,5,50\}$	I	
157	$\{1, 2, 9, 21, 3, 12, 30\}$	I	
158	$\{1, 2, 9, 83, 3, 5, 44\}$		Z_2
159	$\{1,2,9,115,3,7,19\}$	I	
160	$\{1,2,9,115,3,7,41\}$	I	
	$\{1, 2, 9, 21, 3, 5, 76\}$		
161	$\{1, 2, 9, 21, 3, 5, 106\}$	I	
162	$\{1, 2, 9, 21, 3, 5, 95\}$	I	
163	$\{1, 2, 9, 21, 3, 12, 58\}$	I	
164	$\{1,2,9,83,3,5,17\}$	I	
165	$\{1,2,9,21,3,5,65\}$	I	
166	$\{1,2,9,21,3,5,66\}$	I	
167	$\{1,2,9,21,3,5,99\}$	I	
168	$\{1,2,9,21,3,5,45\}$	I	
169	$\{1,2,9,21,3,5,42\}$	I	
170	$\{1,2,9,83,3,5,16\}$	I	
171	$\{1,2,9,83,3,5,32\}$	I	
172	$\{1,2,9,21,3,5,40\}$		Z_3
173	$\{1,2,9,21,3,14,55\}$		Z_2
174	$\{1,2,9,21,3,12,66\}$		Z_3
175	$\{1, 2, 9, 115, 3, 5, 6\}$		Z_3
176	$\{1, 2, 9, 21, 3, 14, 31\}$		Z_6
177	$\{1, 2, 9, 83, 3, 5, 40\}$		Z_3
178	$\{1,2,9,115,3,7,92\}$		Z_6
179	$\{1,2,9,115,3,5,40\}$		$Z_3 \times Z_3$
180	$\{1,2,9,21,3,5,28\}$	I	
181	$\{1,2,9,21,3,5,26\}$	I	
182	$\{1, 2, 9, 21, 3, 5, 8\}$		Z_2
183	$\{1,2,9,21,3,5,41\}$	I	
184	$\{1,2,9,21,3,5,27\}$	I	
185	$\{1,2,9,21,3,5,20\}$	I	
186	$\{1, 2, 9, 21, 3, 5, 126\}$	I	
187	$\{1, 2, 9, 21, 3, 5, 17\}$	I	
188	$\{1, 2, 9, 21, 3, 5, 100\}$	I	
189	$\{1,2,9,21,3,5,29\}$		Z_2
190	$\{1,2,9,21,3,5,43\}$	I	
191	$\{1, 2, 9, 21, 3, 5, 167\}$	I	
192	$\{1,2,9,21,3,5,15\}$	I	
193	$\{1,2,9,115,3,7,8\}$	I	
194		I	

195	$\{1,2,9,115,3,7,26\}$	I
196	$\{1,2,9,115,3,7,15\}$	I
197	$\{1, 2, 9, 83, 3, 5, 42\}$	Z_3
198	$\{1,2,9,115,3,5,42\}$	I
199	$\{1,2,9,115,3,7,13\}$	Z_2
200	$\{1,2,9,21,3,12,14\}$	Z_3
201	$\{1,2,9,115,3,7,45\}$	I
202	$\{1, 2, 9, 83, 3, 5, 27\}$	I
	$\{1, 2, 9, 115, 3, 7, 25\}$	
203	$\{1, 2, 9, 115, 3, 7, 5\}$	I
204	$\{1,2,9,115,3,7,20\}$	Z_3
205	$\{1,2,9,115,3,7,52\}$	Z_2
206	$\{1,2,9,21,3,12,96\}$	Z_2
207	$\{1,2,9,21,3,12,15\}$	I

Table 8: N_c **of** t_4, t_3, t_2, t_1, t_0

Number	N_c	Number of N_c
1	$\{ 1, 0, 15, 64, 103 \}$	62
2	$\{ 1, 1, 12, 67, 102 \}$	106
3	$\{ 1, 2, 9, 70, 101 \}$	30
4	$\{ 1, 3, 6, 73, 100 \}$	3
5	$\{ 2, 0, 9, 72, 100 \}$	6

Theorem .6 *In* PG(2, 13)*, there are exactly* 207 *projectively inequivalent* (7; 4)*-arcs.*

11.3 Remark

In Table 7, there are 11 types of the stabiliser groups as follows:

$$I, Z_2, Z_3, Z_4, Z_6, D_4, Z_3 \times S_3, Z_4 \times Z_2, Z_{12}, Z_2 \times Z_2, Z_3 \times Z_3.$$

These stabiliser groups of order at least two divide their corresponding projectively inequivalent (7; 4)-arcs into a number of orbits. All orbits of these groups are listed in Table 9.

Table 9: **Group orbits of projectively inequivalent** (7; 4)**-arcs**

Φ_4	Stabiliser	Orbits
{1, 2, 9, 115, 3, 4, 22}	Z_2	{1, 2},{3},{4, 22}, {9, 115}
{1, 2, 9, 21, 3, 4, 136}	Z_2	{1, 2},{3, 136}, {4},{9, 21}
{1, 2, 9, 83, 3, 4, 11}	D_4	{1},{2, 3},{4, 11, 83, 9}
{1, 2, 9, 83, 3, 4, 10}	Z_2	{1, 2},{3},{4, 10}, {9, 83}
{1, 2, 9, 83, 3, 4, 23}	Z_2	{1},{2},{3},{4},{9, 83}, {23}
{1, 2, 9, 83, 3, 4, 37}	Z_2	{1, 2},{3},{4, 37}, {9},{83}
{1, 2, 9, 83, 3, 4, 76}	Z_2	{1},{2},{3, 4},{9, 83}, {76}
{1, 2, 9, 83, 3, 4, 82}	Z_2	{1},{2},{3, 4},{9, 83}, {82}
{1, 2, 9, 83, 3, 4, 108}	Z_2	{1},{2},{3, 4},{9, 83}, {108}
{1, 2, 9, 83, 3, 4, 126}	Z_2	{1},{2},{3, 4},{9, 83}, {126}
{1, 2, 9, 83, 3, 4, 14}	Z_2	{1},{2},{3, 4},{9, 83}, {14}
{1, 2, 9, 83, 3, 4, 130}	Z_6	{1},{2},{3, 4, 130}, {9, 83}
{1, 2, 9, 115, 3, 4, 24}	Z_2	{1, 2},{3, 24}, {4},{9, 115}

{1, 2, 9, 21, 3, 4, 118}	Z_2	{1, 2},{3, 118}, {4},{9, 21}
{1, 2, 9, 115, 3, 4, 35}	Z_3	{1, 9, 115}, {2},{3, 35, 4}
{1, 2, 9, 21, 3, 4, 23}	Z_2	{1},{2, 3},{4, 21}, {9, 23}
{1, 2, 9, 21, 3, 4, 130}	Z_3	{1},{2},{3, 4, 130}, {9},{21}
{1, 2, 9, 115, 3, 4, 130}	$Z_3 \times S_3$	{1},{2, 3, 9, 115, 130, 4}
{1, 2, 9, 83, 3, 8, 17}	Z_2	{1, 2},{3, 8},{9, 83}, {17}
{1, 2, 9, 115, 3, 7, 12}	Z_2	{1, 9},{2, 115}, {3, 12}, {7}
{1, 2, 9, 115, 3, 7, 6}	Z_2	{1, 115}, {2, 9},{3, 7},{6}
{1,2,9,83,3,8,60}	$Z_4 \times Z_2$	{1, 9, 2, 83}, {3, 8},{60}
{1,2,9,83,3,8,57}	Z_{12}	{1, 9, 2, 83}, {3, 8, 57}
{1, 2, 9, 83, 3, 8, 40}	Z_2	{1, 2},{3, 8},{9, 83}, {40}
{1, 2, 9, 83, 3, 8, 19}	Z_4	{1, 9, 2, 83}, {3},{8},{19}
{1, 2, 9, 21, 3, 5, 13}	Z_2	{1},{2},{3, 13}, {5},{9},{21}
{1, 2, 9, 115, 3, 7, 16}	Z_2	{1, 115}, {2, 9},{3},{7},{16}
{1, 2, 9, 21, 3, 12, 68}	Z_2	{1},{2},{3},{9},{12, 68}, {21}
{1, 2, 9, 21, 3, 14, 52}	$Z_2 \times Z_2$	{1, 2},{3},{9, 21}, {14, 52}
{1, 2, 9, 83, 3, 5, 13}	Z_2	{1},{2},{3, 13}, {5},{9},{83}
{1,2,9,115,3,7,49}	$Z_2 \times Z_2$	{1, 115}, {2, 9},{3},{7, 49}
{1,2,9,115,3,5,13}	Z_6	{1},{2,115,9},{3,13},{5}
{1, 2, 9, 115, 3, 7, 19}	Z_2	{1, 115}, {2, 9},{3, 7},{19}
{1, 2, 9, 21, 3, 5, 40}	Z_3	{1},{2},{3, 5, 40}, {9},{21}
{1, 2, 9, 21, 3, 14, 55}	Z_2	{1, 2},{3},{9, 21}, {14}, {55}
{1, 2, 9, 21, 3, 12, 66}	Z_3	{1},{2},{3, 12, 66}, {9},{21}
{1, 2, 9, 115, 3, 5, 6}	Z_3	{1},{2, 9, 115}, {3},{5},{6}
{1, 2, 9, 21, 3, 14, 31}	Z_6	{1, 2},{3, 14, 31}, {9, 21}
{1, 2, 9, 83, 3, 5, 40}	Z_3	{1},{2},{3, 5, 40}, {9},{83}
{1, 2, 9, 115, 3, 7, 92}	Z_6	{1, 115}, {2, 9},{3, 7, 92}
{1, 2, 9, 115, 3, 5, 40}	$Z_3 \times Z_3$	{1},{2,9,115},{3,5,40}
{1, 2, 9, 21, 3, 5, 8}	Z_2	{1,2},{3},{5, 8},{9, 21}
{1, 2, 9, 21, 3, 5, 29}	Z_2	{1, 2},{3, 29}, {5},{9, 21}
{1, 2, 9, 115, 3, 5, 42}	Z_3	{1, 9, 115}, {2},{3, 5, 42}
{1, 2, 9, 21, 3, 12, 14}	Z_2	{1, 2},{3, 14}, {9, 21}, {12}
{1, 2, 9, 115, 3, 7, 45}	Z_3	{1, 9, 115}, {2},{3, 45, 7}
{1, 2, 9, 115, 3, 7, 20}	Z_3	{1, 2, 9},{3, 7, 20}, {115}
{1, 2, 9, 115, 3, 7, 52}	Z_2	{1, 115}, {2, 9},{3, 7},{52}
{1, 2, 9, 21, 3, 12, 96}	Z_2	{1, 2},{3, 96}, {9, 21}, {12}

4.4 Projectively inequivalent (8; 4)-arcs

In PG(2,13), the number of projectively inequivalent (8;4)-arcs is 7399. The stabliser groups of 7399 projectively inequivalent (8; 4)-arcs are as follows:

$$I, Z_2, Z_3, Z_4, Z_6, Z_{12}, Z_2 \times Z_2, Z_4 \times Z_2, (Z_4 \times Z_4) \text{ w } Z_2, Z_3 \times S_3, D_4.$$

The number of these groups is listed in Table 10. Also, the 7399 projectively inequivalent (8;4)-arcs have eleven *sd*-inequivalent classes of secant distributions as shown in Table 11.

Table 10: **Group statistics of the projectively inequivalent (8; 4)-arcs**

Number	Stabiliser	Number of stabiliser
1	I	6895
2	Z_2	443
3	Z_3	12
4	Z_4	15
5	Z_6	4
6	Z_{12}	1
7	$Z_2 \times Z_2$	18
8	$Z_4 \times Z_2$	2
9	$(Z_4 \times Z_4) \text{ w } Z_2$	1
10	$Z_3 \times S_3$	1
11	D_4	7

Note that the groups of order at least eight are as follows:

$$Z_4 \times Z_2, Z_{12}, (Z_4 \times Z_4) \text{ w } Z_2, Z_3 \times S_3.$$

These groups partition the associated projectively inequivalent (8;4)-arcs into a number of orbits as shown below.

(1) The group Z_{12} splits the (8;4)-arc $\{1, 2, 9, 83, 3, 8, 57, 19\}$ into 3 orbits of sizes 4, 3, 1. They are

$$\{1, 9, 2, 83\}, \{3, 8, 57\}, \{19\}.$$

(2) The group $Z_4 \times Z_2$ partitions the (8;4)-arcs $\{1, 2, 9, 83, 3, 8, 60, 19\}$ and $\{1, 2, 9, 83, 3, 8, 57, 59\}$ into 3 orbits. They are $\{\{1, 9, 2, 83\}, \{3, 60\}, \{8, 19\}\}$ and $\{\{1, 9, 2, 83\}, \{3, 59\}, \{8, 57\}\}$.

(3) The group $(Z_4 \times Z_4) \text{ w } Z_2$ divides the (8;4)-arc $\{1, 2, 9, 83, 3, 8, 19, 59\}$ into one orbit, that is,

$\{1, 2, 3, 19, 8, 83, 59, 9\}$.

(4) The group $Z_3 \times S_3$ separates the (8;4)-arc $\{1, 2, 9, 115, 3, 5, 6, 132\}$ into two orbits of sizes 2, 6. Thay are $\{\{1, 5\}, \{2, 6, 9, 115, 132, 3\}\}$.

Table 11: N_c of

t_4, t_3, t_2, t_1, t_0

Number	N_c	Number of N_c
1	{ 1, 0, 22, 64, 96 }	534
2	{ 1, 1, 19, 67, 95 }	2272
3	{ 1, 2, 16, 70, 94 }	2905
4	{ 1, 3, 13, 73, 93 }	1188
5	{ 2, 0, 16, 72, 93 }	146
6	{ 1, 4, 10, 76, 92 }	182
7	{ 2, 1, 13, 75, 92 }	128
8	{ 1, 5, 7, 79, 91 }	10
9	{ 2, 2, 10, 78, 91 }	30
10	{ 1, 6, 4, 82, 90 }	1
11	{ 2, 3, 7, 81, 90 }	3

er Stabiliser Number of stabiliser 1 I

Theorem .7 *In* PG(2, 13)*, there are exactly* 7399 *projectively inequivalent* (8; 4)-*arcs.*

4.5 Projectively inequivalent (9; 4)-arcs

In PG(2,13), the number of projectively inequivalent (9; 4)-arcs is 222536 according to the inequi- valent lexicographically least set in the *G*-orbit of each (9;4)-arc. These arcs have one of the groups

$I, Z_2, Z_3, Z_4, Z_6, Z_2 \times Z_2, Z_4 \times Z_2, D_4, S_3, S_4, A_4$. In addition, the secant distribution of each

of the 222536 projectively inequivalent arcs is calculated. There are 21 *sd*-inequivalent classes of secant distributions of the projectively inequivalent (9; 4)-arcs. The statistics are given in Tables 12, 13, and 14.

Table 12: **Group statistics of the projectively inequivalent (9; 4)-arcs**

Num

b

2 2 0 7 1 9		
2	Z_2	1702
3	Z_3	59
4	Z_4	14
5	Z_6	8
6	$Z_2 \times Z_2$	22
7	$Z_4 \times Z_2$	1
8	S_3	4
9	S_4	2
10	D_4	3
11	A_4	2

{1, 2, 9, 83, 3, 5, 13, 49, 101}	Z_4
{1, 2, 9, 21, 3, 12, 68, 56, 151}	Z_4
{1, 2, 9, 83, 3, 5, 13, 16, 33}	Z_4
{1, 2, 9, 83, 3, 5, 13, 58, 97}	Z_4
{1, 2, 9, 83, 3, 8, 17, 32, 61}	Z_4
{1, 2, 9, 83, 3, 8, 17, 79, 147}	Z_4
{1, 2, 9, 115, 3, 7, 6, 154, 160}	Z_4
{1, 2, 9, 21, 3, 14, 31, 8, 74}	Z_4
{1, 2, 9, 115, 3, 4, 5, 25, 148}	Z_6
{1, 2, 9, 115, 3, 4, 30, 43, 59}	Z_6
{1, 2, 9, 115, 3, 4, 18, 35, 39}	Z_6
{1, 2, 9, 115, 3, 4, 8, 51, 130}	Z_6
{1, 2, 9, 115, 3, 4, 16, 37, 145}	Z_6
{1, 2, 9, 115, 3, 4, 32, 31, 130}	Z_6
{1, 2, 9, 115, 3, 4, 32, 29, 130}	Z_6
{1, 2, 9, 115, 3, 4, 32, 130, 149}	Z_6
{1, 2, 9, 83, 3, 4, 57, 60, 147}	$Z_4 \times Z_2$
{1, 2, 9, 21, 3, 4, 58, 7, 80}	S_3
{1, 2, 9, 115, 3, 4, 5, 130, 131}	S_3
{1, 2, 9, 21, 3, 4, 96, 163, 166}	S_3
{1, 2, 9, 115, 3, 4, 15, 130, 45}	S_3
{1, 2, 9, 83, 3, 4, 11, 10, 84}	S_4
{1, 2, 9, 83, 3, 4, 11, 37, 129}	S_4
{1, 2, 9, 83, 3, 4, 10, 82, 86}	D_4
{1, 2, 9, 115, 3, 7, 12, 77, 76}	D_4
{1, 2, 9, 115, 3, 7, 12, 70, 177}	D_4
{1, 2, 9, 83, 3, 4, 5, 12, 135}	A_4
{1, 2, 9, 83, 3, 4, 92, 135, 164}	A_4

11.3.1 Remark

In Table 12, the large groups of order at least 4 are Z_4, Z_6, $Z_4 \times Z_2$, S_3, S_4, D_4, A_4. The action of these groups is shown in the following table:

Table 13: **Group orbits of projectively inequivalent** (9; 4)**-arcs**

Φ_4	Stabiliser	Orbits
{1, 2, 9, 83, 3, 4, 57, 99, 105}	Z_4	{1, 9, 2, 83}, {3, 105, 57, 4},{99}
{1, 2, 9, 83, 3, 4, 5, 24, 135}	Z_4	{1},{2, 4},{3, 83, 135, 9},{5, 24}
{1, 2, 9, 21, 3, 4, 22, 24, 108}	Z_4	{1, 2},{3, 22, 24, 4},{9, 21}, {108}
{1, 2, 9, 115, 3, 4, 18, 151, 159}	Z_4	{1, 115}, {2, 9},{3, 18, 159, 4},{151}
{1, 2, 9, 83, 3, 4, 30, 84, 124}	Z_4	{1, 9, 2, 83}, {3},{4, 124, 84, 30}
{1, 2, 9, 83, 3, 4, 92, 135, 118}	Z_4	{1},{2, 4},{3, 83, 135, 9},{92, 118}

Table 14: N_c **of**

t_4, t_3, t_2, t_1, t_0

Number	N_c	Number of N_c
1	{ 1, 0, 30, 62, 90 }	1199
2	{ 1, 1, 27, 65, 89 }	13688
3	{ 1, 2, 24, 68, 88 }	50341
4	{ 1, 3, 21, 71, 87 }	74174
5	{ 2, 0, 24, 70, 87 }	1776
6	{ 1, 4, 18, 74, 86 }	47139
7	{ 2, 1, 21, 73, 86 }	7227
8	{ 2, 2, 18, 76, 85 }	8259
9	{ 1, 5, 15, 77, 85 }	12848
10	{ 1, 6, 12, 80, 84 }	1487
11	{ 2, 3, 15, 79, 84 }	3388
12	{ 3, 0, 18, 78, 84 }	182
13	{ 1, 7, 9, 83, 83 }	68
14	{ 2, 4, 12, 82, 83 }	518
15	{ 3, 1, 15, 81, 83 }	151
16	{ 1, 8, 6, 86, 82 }	2
17	{ 2, 5, 9, 85, 82 }	39
18	{ 3, 2, 12, 84, 82 }	42
19	{ 2, 6, 6, 88, 81 }	2
20	{ 3, 3, 9, 87, 81 }	5
21	{ 3, 4, 6, 90, 80 }	1

Theorem .8 *In* PG(2, 13)*, there are exactly* 222536 *projectively inequivalent* (9; 4)*-arcs.*

4.6 Projectively inequivalent (10; 4)-arcs

The number of (10;4)-arcs is paralleled into 5 processes; each took 6 : 22 : 54 : 11*,* 4 : 15 : 36 : 77*,* 5 :

09 : 28 : 12*,* 5 : 12 : 40 : 46*,* 3 : 21 : 52 : 13 of CPU time respectively for the construction. Then according to the canonical images of the (10; 4)-arcs found from 4 processes, there are at least 5268378 projectively inequivalent (10; 4)-arcs. This took 2403232618 msc. The 5268378 arcs have 36 *sd*-inequivalent classes N_c of *i*-secant distributions as listed in Table 15. The total time is 1726578 msc where it was computed in six processes. Then according to the number of N_c there

are 36 *sd*-inequivalent (10; 4)-arcs, which have five types of stabilisers I, $Z_2 \times Z_2$, Z_2, S_3, $Z_3 \times S_3$.

The timing of these groups was 3633 msec. The statistics of the *sd*-inequivalent (10; 4)-arcs are given in Table 16.

Table 15: N_c **of** t_4, t_3, t_2, t_1, t_0

Number	N_c	Number of N_c
1	$\{1, 7, 18, 79, 78\}$	192599
2	$\{1, 0, 39, 58, 85\}$	661
3	$\{1, 1, 36, 61, 84\}$	15664
4	$\{1, 2, 33, 64, 83\}$	145027
5	$\{1, 3, 30, 67, 82\}$	592731
6	$\{1, 4, 27, 70, 81\}$	1227187
7	$\{1, 5, 24, 73, 80\}$	1322219
8	$\{1, 6, 21, 76, 79\}$	719144
9	$\{1, 8, 15, 82, 77\}$	24434
10	$\{1, 9, 12, 85, 76\}$	1399
11	$\{1, 10, 9, 88, 75\}$	31
12	$\{1, 11, 6, 91, 74\}$	3
13	$\{2, 0, 33, 66, 82\}$	4572
14	$\{2, 1, 30, 69, 81\}$	52934
15	$\{2, 2, 27, 72, 80\}$	207496
16	$\{2, 3, 24, 75, 79\}$	344994
17	$\{2, 4, 21, 78, 78\}$	255989
18	$\{2, 5, 18, 81, 77\}$	87359
19	$\{2, 6, 15, 84, 76\}$	13784
20	$\{2, 7, 12, 87, 75\}$	954
21	$\{2, 8, 9, 90, 74\}$	38
22	$\{2, 10, 3, 96, 72\}$	1
23	$\{3, 0, 27, 74, 79\}$	3944
24	$\{3, 1, 24, 77, 78\}$	17244
25	$\{3, 2, 21, 80, 77\}$	22990
26	$\{3, 3, 18, 83, 76\}$	11598
27	$\{3, 4, 15, 86, 75\}$	2477
28	$\{3, 5, 12, 89, 74\}$	257
29	$\{3, 6, 9, 92, 73\}$	12
30	$\{3, 7, 6, 95, 72\}$	2
31	$\{4, 0, 21, 82, 76\}$	222
32	$\{4, 1, 18, 85, 75\}$	297
33	$\{4, 2, 15, 88, 74\}$	97
34	$\{4, 3, 12, 91, 73\}$	13
35	$\{4, 4, 9, 94, 72\}$	2
36	$\{5, 0, 15, 90, 73\}$	3

Theorem .9 *In* PG(2, 13), *there are at least* 5268378 *projectively inequivalent* (10; 4)-*arcs.*

Table 16: *sd*-**inequivalent** (10; 4)-**arcs**

Symbol	(10; 4)-arc	$\{t_4, t_3, t_2, t_1, t_0\}$	Stabiliser
K_1^J	$\{1,2,9,83,3,4,57,6,166,8\}$	$\{1,7,18,79,78\}$	I
K_2^J	$\{1,2,9,83,3,8,17,40,72,78\}$	$\{1,0,39,58,85\}$	I
K_3^J	$\{1,2,9,83,3,4,6,50,67,63\}$	$\{1,1,36,61,84\}$	I
K_4^J	$\{1,2,9,83,3,4,57,166,99,40\}$	$\{1,2,33,64,83\}$	I
K_5^J	$\{1,2,9,83,3,4,57,6,107,18\}$	$\{1,3,30,67,82\}$	I
K_6^J	$\{1,2,9,83,3,4,57,6,166,33\}$	$\{1,4,27,70,81\}$	I
K_7^J	$\{1,2,9,83,3,4,57,6,166,16\}$	$\{1,5,24,73,80\}$	I
K_8^J	$\{1, 2, 9, 115, 3, 4, 5, 6, 7, 8\}$	$\{1,6,21,76,79\}$	I
K_9^J	$\{1,2,9,115,3,4,5,6,7,90\}$	$\{1,8,15,82,77\}$	I
K_{10}^J	$\{1,2,9,83,3,4,5,129,137,178\}$	$\{1,9,12,85,76\}$	I
K_{11}^J	$\{1,2,9,83,3,4,5,129,178,104\}$	$\{1,10,9,88,75\}$	I
K_{12}^J	$\{1,2,9,83,3,4,5,30,37,51\}$	$\{1,11,6,91,74\}$	$Z_2 \times Z_2$
K_{13}^J	$\{1,2,9,83,3,4,6,11,167,33\}$	$\{2,0,33,66,82\}$	I
K_{14}^J	$\{1,2,9,83,3,4,57,166,11,33\}$	$\{2,1,30,69,81\}$	I
K_{15}^J	$\{1,2,9,83,3,4,57,6,11,18\}$	$\{2,2,27,72,80\}$	I
K_{16}^J	$\{1,2,9,83,3,4,57,6,166,7\}$	$\{2,3,24,75,79\}$	I
K_{17}^J	$\{1,2,9,83,3,4,57,6,166,17\}$	$\{2,4,21,78,78\}$	I
K_{18}^J	$\{1,2,9,83,3,4,57,6,166,87\}$	$\{2,5,18,81,77\}$	I
K_{19}^J	$\{1,2,9,83,3,4,57,6,166,163\}$	$\{2,6,15,84,76\}$	I
K_{20}^J	$\{1,2,9,83,3,4,5,129,137,37\}$	$\{2,7,12,87,75\}$	I
K_{21}^J	$\{1,2,9,83,3,4,5,129,68,11\}$	$\{2,8,9,90,74\}$	Z_2
K_{22}^J	$\{1,2,9,115,3,4,18,183,35,39\}$	$\{2,10,3,96,72\}$	$Z_3 \times S_3$
K_{23}^J	$\{1,2,9,83,3,4,57,166,11,51\}$	$\{3,0,27,74,79\}$	I
K_{24}^J	$\{1,2,9,83,3,4,57,6,11,17\}$	$\{3,1,24,77,78\}$	Z_2
K_{25}^J	$\{1,2,9,83,3,4,57,6,113,77\}$	$\{3,2,21,80,77\}$	I
K_{26}^J	$\{1,2,9,83,3,4,5,129,137,87\}$	$\{3,3,18,83,76\}$	I
K_{27}^J	$\{1,2,9,83,3,4,57,142,131,163\}$	$\{3,4,15,86,75\}$	I
K_{28}^J	$\{1,2,9,83,3,4,57,6,95,163\}$	$\{3,5,12,89,74\}$	I
K_{29}^J	$\{1,2,9,83,3,4,5,129,51,37\}$	$\{3,6,9,92,73\}$	I
K_{30}^J	$\{1,2,9,83,3,4,5,51,37,122\}$	$\{3,7,6,95,72\}$	S_3
K_{31}^J	$\{1,2,9,83,3,4,57,166,38,160\}$	$\{4,0,21,82,76\}$	I
K_{32}^J	$\{1,2,9,83,3,4,57,6,153,91\}$	$\{4,1,18,85,75\}$	I
K_{33}^J	$\{1,2,9,83,3,4,57,142,163,96\}$	$\{4,2,15,88,74\}$	I
K_{34}^J	$\{1,2,9,83,3,4,5,129,112,39\}$	$\{4,3,12,91,73\}$	I
K_{35}^J	$\{1,2,9,83,3,4,5,129,37,11\}$	$\{4,4,9,94,72\}$	Z_2
K_{36}^J	$\{1,2,9,21,3,4,37,91,90,178\}$	$\{5,0,15,90,73\}$	Z_2

11.3.2 Remark

In Table 17, The classification timings of the projectively inquivalent $(k;4)$-arcs for $k=5,\ldots,9$ are given.

Table 17: **Timing (msec) of projectively inequivalent $(k;4)$-arcs for $k = 5, \ldots, 9$**

$(k;4)$-arcs	Construction	Lexicographically least sets	$\{t_4, t_3, t_2, t_1, t_0\}$	Stabilisers
(5;4)-arcs	2011	2134	2193	2181
(6;4)-arcs	2138	2168	2329	2230
(7;4)-arcs	2516	2201	2999	3615
(8;4)-arcs	26606	711630	19554	80338
(9;4)-arcs	22729912	32126643	176130	3848131

5 Complete (38;4)-arcs from the *sd*-inequivalent (10;4)-arcs

In Table 16, there are 36 *sd*-inequivalent (10; 4)-arcs together with the corresponding *sd*-inequivalent classes of the *i*-secant distributions. Therefore, at this stage of the classification the 36-arcs of Table 16 have been extended. The aim of this process is to discover the largest complete $(k;4)$-arc in

PG(2, 13) that can be established. The result of this method is a complete (38; 4)-arc K^J. This complete arc is comes from the *sd*-inequivalent (10; 4)-arc K_8^J. The complete (38; 4)-arc is as follows: $K^J = \{1, 2, 9, 115, 3, 4, 5, 6, 7, 8, 10, 19, 25, 60, 74, 98, 107, 78, 130, 27, 106, 69, 116, 46, 63, 126, 99, 51, 81, 65, 52, 176, 88, 92, 53, 181, 169, 178\}$. The properties of K^J are given in Table 18.

Table 18: **Complete** (38; 4)**-arc in** PG(2, 13)

Symbol	Complete (38;4)-arc	Stabiliser	$\{t_4, t_3, t_2, t_1, t_0\}$
K^J	$\{1,2,9,115,3,4,5,6,7,8,10,19,25,60,74,$ 98, 107, 78, 130, 27, 106, 69, 116, 46, 63, 126, 99, 51, 81, 65, 52, 176, 88, 92, 53, 181, 169, 178$\}$	D_{12}	$\{102,24,19,14,24\}$

11.3.3 Remark

In Table 19, The classification timing of the projectively inquivalent $(k;4)$-arcs for $k=5,\ldots,9$ are given.

Table 19: **Timing (msec) of projectively inequivalent $(k;4)$-arcs for $k = 5, \ldots, 9$**

$(k;4)$-arcs	Construction	Lexicographically least sets	$\{t_4, t_3, t_2, t_1, t_0\}$	Stabilisers
(5;4)-arcs	2011	2134	2193	2181
(6;4)-arcs	2138	2168	2329	2230
(7;4)-arcs	2516	2201	2999	3615
(8;4)-arcs	26606	711630	19554	80338
(9;4)-arcs	22729912	32126643	176130	3848131

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Best multiplier Approximation in $L_{p,\emptyset_n}(X)$ By two dimensions De La Vallee- Poussin Operator

Saheb K. AL-Saidy[1], Naseif J. AL-Jawari[2] and Ali H.Zaboon[3]

[1] Department of Mathematics, College of Science University of Mustansiriyah, IRAQ
[2] Department of Mathematics, College of Science University of Mustansiriyah, IRAQ
[3] Department of Mathematics, College of Science University of Mustansiriyah, IRAQ
*Correspondent author Email[1]: sahebalsaidy@gmail.com
*Correspondent author Email[2]: nsaif642014@yahoo.com
*Correspondent author Email[3]: ali.zaboon1@gmail.com

Abstract
The purpose of this paper is to find best multiplier approximation of unbounded functions in $L_{p,\emptyset_n}$ –space by using Trigonometric polynomials and by two dimensions de la Vallee- Poussin operator for $f \in L_{p,\emptyset_n}(X),\ X = [-\pi,\pi] \times [-\pi,\pi]$, in terms of the modulus of smoothness of order k and the average modulus.

Keywords: multiplier convergence, multiplier Integral.

الخلاصة
الغرض من هذا البحث هو ايجاد افضل تقريب مضاعف للدوال غير المقيدة في الفضاء $L_{p,\emptyset_n}$ باستخدام الحدوديات المثلثية و متعددات حدود دي لا فالية- بواسون المضاعف ذات البعدين للدوال الدورية ذات المتغييرين باستعمال مقاسات النعومة ذوات الرتبة k وكذلك باستعمال نماذج المعدل

1. Introduction and Results

In 1949, **[1]** G. Hardy defined the multiplier sequence for a converge of the series as.

A series $\sum_{n=0}^{\infty} a_n$ is called a multiplier convergent if there is convergent sequence of real numbers$\{\emptyset_n\}_{n=0}^{\infty}$,such that $\sum_{n=0}^{\infty} a_n\ \emptyset_n < \infty$ where, $\{\emptyset_n\}_{n=0}^{\infty}$ is called a multiplier for the convergence, for example.

The series $\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent series and the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ convergent sequence. Since $\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{1}{n} = \sum_{n=1}^{\infty}\frac{1}{n^2}$ which is convergent series then the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is a multiplier convergent.

And from above we have

If $\sum_{n=1}^{\infty} a_n$ is convergent series then it is multiplier convergent, since the sequence $\{\emptyset_n\}_{n=0}^{\infty} = \{1\}_{n=0}^{\infty}$ may be taken. But the convers is not true in general.

Similar to the above we provide the following definition

For any real valued function f defined on $B=[a,b]$, f is called multiplier integral if there is a sequence $\{\emptyset_n\}_{n=0}^{\infty}$ of real numbers such that $\int_B f\emptyset_n(x) < \infty$, as $n\to\infty$ where $\{\emptyset_n\}_{n=0}^{\infty}$ is called a multiplier for the integral.

Let $L_p(B)$ be the space of all bounded measurable functions defined on $B=[a,b]$ with the norm

$$\|f(.)\|_{Lp}=\|f(.)\|_p=(\int_B |f(x)|^p)^{\frac{1}{p}}<\infty \quad ,1\leq p<\infty.$$

Now for any real valued function f the multiplier integral norm can be defined as follows,

$$\|f(.)\|_{L_{p,\emptyset_n}}=\left\{(\int_B |f\emptyset_n(x)|^p\,dx)^{\frac{1}{p}}: x\in B\right\} ,$$

where $\emptyset_n$ is the multiplier for the integral

Let us define the norm $\|f\|_{L_{p,\emptyset_n}}$ by $\|f\|_{p,\emptyset_n}$

Let $L_{p,\emptyset_n}(B)$, be the space of all real valued unbounded functions f

such that $\int_B |f\emptyset_n(x)|^p\,dx<\infty$ with the norm

$\|f(.)\|_{P,\emptyset_n}=\left\{(\int_B |f\emptyset_n(x)|^p\,dx)^{\frac{1}{p}} : x\in B\right\}$, where $\emptyset_n$ is the multiplier for the integral, $\|f\emptyset_n(.)\|_p=\|f(.)\|_{p,\emptyset_n}$ and $B=[-\pi,\pi]$

Now, before we give some examples for the define of $L_{p,\emptyset_n}(B)-space$, we present the following theorem, (Lebesgue Dominated Convergence Theorem).

Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions defined on a Lebesgue measurable set E such that

$\{f_n(x)\}_{n=1}^{\infty}$ Converges pointwise almost everywhere to $f(x)$, then

$$\lim_{n\to\infty}\int_E f_n(x)\,dx=\int_E \lim_{n\to\infty} f_n(x)\,dx=\int_E f(x)dx$$

Example 1:

Let $f(x)=\csc x$ with $x\in X=(0,\pi)$ which is unbounded function, $\emptyset_n=\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$ be a sequence. Then we have

$f\emptyset_n(x)=f_n(x)=\frac{\csc x}{n^2}$ is a sequence of Lebesgue measurable functions defined on a Lebesgue measurable set $=(0,\pi)$, since

$f\emptyset_n(x)=f_n(x)=\frac{\csc x}{n^2}$ Converges pointwise almost everywhere to $f(x)=0$, then by using the above theorem we get the following

$\int_{X} f\emptyset_n(x)dx = \int_{X} f_n(x)dx = \int_X f(x)dx < \infty$ **,** as $n \to \infty$,which means that $\emptyset_n = \left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$ is a multiplier for the Integral **.**

Example 2:

Let $f: B \to \mathbb{R}$ be a function defined as follows

$f(x) = \frac{\boldsymbol{\pi^2 - x^2}}{\boldsymbol{x}}$ for $x \in B = [-\pi, 0) \cup (0, \pi]$**.**

And let $\emptyset_n = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ be the multiplier for integral.

Thus $f \in L_{p,\emptyset_n}(B)$ where$1 \le p < \infty$.

Now, suppose that $x = \frac{\boldsymbol{1}}{\boldsymbol{m}}$ where m be a positive real numbers.

Thus $x = \frac{1}{m} \to 0$ as $m \to \infty$ and for $n \ge m$ we get the following inequality

$$f\emptyset_n(x) = \left(\frac{(\pi^2 - x^2)}{x}\right)\frac{1}{n} = \frac{(\pi^2 - x^2)\frac{1}{n}}{x} = (\pi^2 - x^2)\frac{m}{n} \le f^*(x) = \pi^2 - x^2$$

Thus $f\emptyset_n(x) \le f^*(x)\ \forall\, n \ge m$ that $f\emptyset_n(x) \le f^*(x)\ \forall\ \frac{1}{n} \le \frac{1}{m} = x$

i.e., $f\emptyset_n(x) = \frac{(\pi^2 - x^2)\frac{1}{n}}{x} \le f^*(x) = \pi^2 - x^2 \quad \forall\, x \ge \frac{1}{n}$

Therefor if we take $n \to \infty$ then $\frac{1}{n} \to 0$ and we get that

$f\emptyset_n(x) = \frac{(\pi^2 - x^2)\frac{1}{n}}{x} \le f^*(x) = \pi^2 - x^2$ for all $x \in (\frac{1}{n}, \pi]$

This means that

$\int_B f\emptyset_n(x)dx \le \int_B f^*(x)\, dx\ \ \forall\, x \in \left(\frac{1}{n}, \pi\right]$. From all above we have

$\boldsymbol{L_p(B) \subseteq L_{p,\emptyset_n}(B)}$.

Many researchers presented research in studying the approximation of unbounded periodic functions using multiple types of modulus of smoothness in one dimension **[2, 3]**.

In this paper we approximate the function f which is unbounded function lies in $L_{p,\emptyset_n}(X), X = [-\pi, \pi] \times [-\pi, \pi]$ by two dimensions de la Vallee Poussin sums for periodic functions of two variables.

First for $X = [-\pi,\pi] \times [-\pi,\pi]$, $1 \le p < \infty$ and for any real valued function f of two variables we can define the multiplier integral norm as follows

$$\|f(.,.)\|_{L_{p,\emptyset_n}} = \|f(.,.)\|_{p,\emptyset_n} = \left\{\left(\iint_X |f\emptyset_n(x,y)|^p dx\, dy\right)^{\frac{1}{p}}\right\}$$

$1 \le p < \infty$. Where $\emptyset_n$ is called the multiplier for integral. Also

Let $L_{p,\emptyset_n}(X), X = [-\pi,\pi] \times [-\pi,\pi]$, $1 \le p < \infty$ be the space of all real valued unbounded functions f such that $\iint_X f\emptyset_n(x,y)\, dxdy < \infty$,

with the following norm.

$$\|f(.,.)\|_{L_{p,\emptyset_n}} = \|f(.,.)\|_{p,\emptyset_n} = \left\{\left(\iint_X |f\emptyset_n(x,y)|^p dx\, dy\right)^{\frac{1}{p}}\right\},$$

where $\emptyset_n$ is the multiplier for the integral **,** $f(x\,,y)$ is called multiplier integral

Now let $f \in L_p(B)$, $B = [-\pi,\pi]$. The Fourier series of f is given by, **[4]**

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n(f)\cos nx + b_n(f)\sin nx), \qquad \cdots\cdots\cdots (1.1)$$

The n th partial sums of (1.1) is given by

$$S_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k(f)\cos kx + b_k(f)\sin kx)\ ,$$

and the de la Vallee-Poussin partial sum of (1.1) is defined by

$$V_{n,m}(f,x) = \frac{1}{m+1}\sum_{k=n}^{n+m} S_k\,(f,x) \qquad n\,,m = 0,1,2\cdots$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)dt$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)\cos kt\ dt\ \ ,\ \ b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)\sin kt\ dt$$

Let $L_{p,\emptyset_n}(X)$ be the class of real valued functions of two variables that are continuous unbounded on $X = [-\pi,\pi] \times [-\pi,\pi]$ and 2π −periodic in each variable separately.

For $f \in L_{p,\emptyset_n}(X)$, $X = [-\pi,\pi] \times [-\pi,\pi]$,$1 \le p < \infty$ the multiplier Fourier series of f is given by, **[5]**

$$S_{n.m}(f;x,y) \cong \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\beta_{n,m}(\,a_{n,m}\cos nx\ \cos my + b_{n,m}\sin nx\cos my + c_{n,m}\cos nx\sin my + d_{n,m}\sin nx\sin my)$$. Where

$$\beta_{n,m} = \begin{cases} \frac{1}{4} & \text{if } n=m=0 \\ \frac{1}{2} & \text{if } n\geq 1, m=0 \text{ or } n=0, m\geq 1 \\ 1 & \text{if } n \geq 1, m \geq 1 \end{cases}$$

$$a_{n,m} = \frac{1}{\pi^2}\iint_X f\emptyset_n(u,v)\cos nu\cos mv\ dudv\ ,$$

$$b_{n,m} = \frac{1}{\pi^2}\iint_X f\emptyset_n(u,v)\sin nu\cos mv\, dudv,$$

$$c_{n,m} = \frac{1}{\pi^2}\iint_X f\emptyset_n(u,v)\cos nu\sin mv\, dudv$$

$$d_{n,m} = \frac{1}{\pi^2}\iint_X f\emptyset_n(u,v)\sin nu\ \sin mv\, dudv,$$

and the partial sum of multiplier Fourier series is given by

$$S_{n.m}(f;x,y) = \sum_{k_1=0}^{n}\sum_{k_2=0}^{m}\beta_{k_1,k_2}(a_{k_1,k_2}\cos k_1x\cos k_2y + b_{k_1,k_2}\sin k_1x\cos k_2y + c_{k_1,k_2}\cos k_1x\sin k_2y + d_{k_1,k_2}\sin k_1x\sin k_2y\ ).$$

Also the partial sum of multiplier Fejer series is given by

$\delta_{n,m}(f;x,y) = \frac{1}{(n+1)(m+1)}\sum_{k_1=0}^{n}\sum_{k_2=0}^{m} S_{k_1.k_2}(f;x,y)$, and the partial sum of multiplier de la Vallee-Poussin is given by

$$V_{n,p_1}^{m,p_2}(f;x,y) = \frac{1}{(p_1+1)}\frac{1}{(p_2+1)}\sum_{k_1=n}^{n+p_1}\sum_{k_2=m}^{m+p_2} S_{k_1.k_2}(f;x,y)\ ,$$

$(p_1 \geq 0,\ \ p_2 \geq 0)$.

Denote by $E_n(f)_{p,\emptyset_n}$ the degree of best multiplier approximation of a function by trigonometric polynomials of order not exceeding n, i.e.

$E_n(f)_{p,\emptyset_n} = inf_{g_n\in\mathbb{T}_n}\{\|f-g_n\|_{p,\emptyset_n}, g_n \in \mathbb{T}_n\}$, where $\mathbb{T}_n$ is the set of all trigonometric polynomials

In two dimensions we present this definition,

for $f \in L_{p,\emptyset_n}(X)$ $X = [-\pi,\pi]\times[-\pi,\pi]$,$1 \leq p < \infty$ the degree of best multiplier approximation of a function f with respect to the set of trigonometric polynomial $T_{n,m} \in \mathbb{T}_{n,m}$ is given by

$E_{n,m}(f)_{p,\emptyset_n} = \underbrace{inf}_{T_{n,m}\in\mathbb{T}_{n,m}} \left\{\|f - T_{n,m}\|_{p,\emptyset_n}\right\}$, where $\mathbb{T}_{n,m}$ be the set of all trigonometric polynomials of two variables x, y with order $\leq n$ in x and order $\leq m$ in y.

Now to convert the formulas$S_{n,m}(f,x,y)$, $\delta_{n,m}(f,x,y)$and $V_{n,p_1}^{m,p_2}(f,x,y)$ from the sum formula to the integration formula take the following results.

Proposition 1.1:

Let $f \in L_{p,\emptyset_n}(X)$ $X = [-\pi,\pi] \times [-\pi,\pi]$, we have

$S_{n,m}(f,x,y) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x+u, y+v) D_{n,m}(u,v)dudv$, where

The Dirichlet kernel $D_n(t)$ is given by

$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin(2n+1)\frac{t}{2}}{2\sin\frac{t}{2}}$, and

$D_{n,m}(u,v) = \frac{\sin(2n+1)\frac{u}{2}\ \sin(2m+1)\frac{v}{2}}{4\sin\frac{u}{2}\ \sin\frac{v}{2}}$

Proof:

$S_{n.m}(f,x,y)$

$= \sum_{k_1=0}^{n} \sum_{k_2=0}^{m} \beta_{k_1,k_2}(a_{k_1,k_2} \cos k_1x \cos k_2y + b_{k_1,k_2} \sin k_1x \cos k_2y$

$+ c_{k_1,k_2} \cos k_1x \sin k_2y + d_{k_1,k_2} \sin k_1x \sin k_2y\ )$

$=$

$\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x,y)[\sum_{k_1=0}^{n} \sum_{k_2=0}^{m} \beta_{k_1,k_2}(\cos k_1u \cos k_2v\ \cos k_1x \cos k_2y +$
$\sin k_1u \cos k_2v \sin k_1x \cos k_2y\ + \cos k_1u \sin k_2v \cos k_1x \sin k_2y +$
$\sin k_1u \sin k_2v \sin k_1x \sin k_2y\ )]dudv$

$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x,y)[\sum_{k_1=0}^{n} \sum_{k_2=0}^{m} \beta_{k_1,k_2}(\cos k_1u \cos k_1x(\cos k_2v\ \cos k_2y + \sin k_2v\ \sin k_2y) +$
$\sin k_1u \sin k_1x\ (\sin k_2v \sin k_2y + \cos k_2v \cos k_2y\ ))]dudv =$

$\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x,y)[\sum_{k_1=0}^{n} \sum_{k_2=0}^{m} \beta_{k_1,k_2}(\cos k_1u \cos k_1x\ +$
$\sin k_1u \sin k_1x)\ (\cos k_2v\ \cos k_2y + \sin k_2v \sin k_2y\ )\ ]dudv$

$= \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f \emptyset_{\mathrm{n}}(x, y).$

$\Big[(\frac{1}{2} +$

$\sum_{k_1=0}^{n} (\cos k_1 u \cos k_1 x \ + \sin k_1 u \sin k_1 x)) (\frac{1}{2} + \sum_{k_2=0}^{m} \cos k_2 v \ \cos k_2 y + \sin k_2 v \sin k_2 y \ )) \Big] dudv$

$= \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f \emptyset_n(x, y) \left[(\frac{1}{2} + \sum_{k_1=1}^{n} \cos k_1 (u + x)) \ (\frac{1}{2} + \sum_{k_2=1}^{m} \cos k_2 (v + y)) \right] \ dudv$

$= \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f \emptyset_n(x, y) D_n(u + x) \ D_m(v + y) dudv$. Then

$\mathrm{S}_{n,m}(f, x, y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f \emptyset_n(x + u, y + v) D_{n,m}(u, v) \ dudv$, where

$\mathrm{D}_{n,m}(\mathrm{u}, \mathrm{v}) = \frac{\sin(2n+1)\frac{u}{2} \ \sin(2m+1)\frac{v}{2}}{4 \sin\frac{u}{2} \ \sin\frac{v}{2}}$ ■

Proposition 1.2:

Let $f \in L_{p,\emptyset_n}(X) \ \ X = [-\pi, \pi] \times [-\pi, \pi]$, we have

$\delta_{n,m}(f, x, y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f \emptyset_n(x + u, y + v) F_n(u) F_m(v) dudv$

$= \ \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f \emptyset_n(x + u, y + v) F_{n,m}(u, v) dudv$, where

$F_n(u) = \frac{1}{n+1} [D_0(u) + D_1(u) + \cdots D_n(u)] = \frac{\sin^2 \frac{nu}{2}}{\sin^2 \frac{u}{2}}$ and

$F_{n,m}(u, v) = \frac{\sin^2 \frac{nu}{2} . \sin^2 \frac{mv}{2}}{\sin^2 \frac{u}{2} \ . \sin^2 \frac{v}{2}}$

Proof:

Since $\mathrm{S}_{n,m}(f, \mathrm{x}, \mathrm{y}) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f \emptyset_n(x, y) D_n(u + x) \ D_m(v + y) dudv$

And $\delta_{n,m}(f, x, y) = \frac{1}{(n+1)(m+1)} \sum_{k_1=0}^{n} \sum_{k_2=0}^{m} S_{k_1 . k_2}(f, x, y)$, we have

$\delta_{n,m}(f, x, y)$

$= \frac{1}{(n+1)(m+1)} \sum_{k_1=0}^{n} \sum_{k_2=0}^{m} \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f \emptyset_n(x, y) D_{k_1}(u + x) D_{k_2}(v + y) dudv$

$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x,y)\frac{1}{n+1}\sum_{k_1=0}^{n} D_{k_1}(u+x)\frac{1}{m+1}\sum_{k_2=0}^{m} D_{k_2}(v+y)\ dudv$

$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x,y)F_n(u+x)F_m(v+y)\ dudv$

$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x+u,y+v)F_n(u)F_m(v)\ dudv$

$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x+u,y+v)F_{n,m}(u,v)dudv$, where

$F_{n,m}(u,v) = F_n(u).F_m(v) = \frac{\sin^2\frac{nu}{2}}{\sin^2\frac{u}{2}}.\frac{\sin^2\frac{mv}{2}}{\sin^2\frac{v}{2}} = \frac{\sin^2\frac{nu}{2}.\sin^2\frac{mv}{2}}{\sin^2\frac{u}{2}.\sin^2\frac{v}{2}}$ ■

Proposition 1.3:

Let $f \in L_{p,\emptyset_n}(X)$ $X = [-\pi,\pi]\times[-\pi,\pi]$,we have

$V_{n,p_1}^{m,p_2}(f,x,y) = \frac{1}{\pi^2(p_1+1)(p_2+1)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x+u,y+v)F_{n+p_1}^{m+p_2}(u,v)dudv$, where

$F_{n+p_1}^{m+p_2}(u,v) = \frac{\sin\frac{2n+p_1+1}{2}u\ \ \sin\frac{p_1+1}{2}u}{2\sin^2\frac{u}{2}}\frac{\sin\frac{2m+p_2+1}{2}v\ \ \sin\frac{p_2+1}{2}v}{2\sin^2\frac{v}{2}}$

Proof:

Since $V_{n,p_1}^{m,p_2}(f,x,y) = \frac{1}{(p_1+1)}\frac{1}{(p_2+1)}\sum_{k_1=n}^{n+p_1}\sum_{k_2=m}^{m+p_2} S_{k_1.k_2}(f,x,y)$. And

$S_{k_1,k_2}(f,x,y) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x+u,y+v)D_{k_1}(u)D_{k_2}(v)\ dudv$. Then

$V_{n,p_1}^{m,p_2}(f,x,y) = \frac{1}{(p_1+1)}\frac{1}{(p_2+1)}\sum_{k_1=n}^{n+p_1}\sum_{k_2=m}^{m+p_2}\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x+u,y+v)D_{k_1}(u)D_{k_2}(v)\ dudv$

$V_{n,p_1}^{m,p_2}(f,x,y) = \frac{1}{\pi^2(p_1+1)(p_2+1)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x+u,y+v)\sum_{k_1=n}^{n+p_1} D_{k_1}(u)\sum_{k_2=m}^{m+p_2} D_{k_2}(v)\ dudv$

But $\sum_{k_1=n}^{n+p_1} D_{k_1}(t) = \left[D_n(t) + D_{n+1}(t)\cdots D_{n+p_1}\right]$

$= \frac{\sin\frac{2n+p_1+1}{2}t\ \ \sin\frac{p_1+1}{2}t}{2\sin^2\frac{t}{2}}$ .Thus we get

$\sum_{k_1=n}^{n+p_1} D_{k_1}(u).\sum_{k_2=m}^{m+p_2} D_{k_2}(v) = \frac{\sin\frac{2n+p_1+1}{2}u\ \ \sin\frac{p_1+1}{2}u}{2\sin^2\frac{u}{2}}.\frac{\sin\frac{2m+p_2+1}{2}v\ \ \sin\frac{p_2+1}{2}v}{2\sin^2\frac{v}{2}} = F_{n+p_1}^{m+p_2}(u,v)$

Therefore

$$V_{n,p_1}^{m,p_2}(f,x,y)=\frac{1}{\pi^2(p_1+1)(p_2+1)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x+u,y+v)F_{n+p_1}^{m+p_2}(u,v)dudv \qquad \blacksquare$$

Before presenting the modulus of smoothness of f in two dimensions that we will use in the Main Results, we introduce the following concept, **[6]**

$\omega^k(f,\delta)_{P,\emptyset_n}=sup_{|h|<\delta}\left\|\Delta_h^k f(\cdot)\right\|_{P,\emptyset_n}$, the multiplier modulus of smoothness of the function f of order k where

$\Delta_h^k(f,x)=\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}f\left(x-\frac{kh}{2}+ih\right), x\mp\frac{kh}{2}\in X$ the k^{th}symmetric difference of the function f .

The averaged modulus of smoothness of order k, (τ-modulus) of the function f is the following function of $\delta\in\left[0,\frac{2\pi}{k}\right]$

$$\tau^k(f,\delta)_{P,\emptyset_n}=\|\omega^k(f,.;\delta)\|_{P,\emptyset_n}=\left\{\int_{-\pi}^{\pi}(\omega^k(f\emptyset_n(x);\delta))^p dx)^{\frac{1}{p}}\right\}.$$

Let $X=[-\pi,\pi]\times[-\pi,\pi]$ and $\Delta=\{V_i \; i\in A\}$ be collection pairwise disjoint set with A is index set from $\boldsymbol{Z}_+^2$ such that $X\subseteq\cup_{i\in A} V_i$, where

$V_i=\prod_{k=1}^{2}\left[x_k^{(i_k)},x_k^{(i_k+1)}\right]=\left[x_1^{i_1},x_1^{i_1+1}\right]\times\left[x_2^{i_2},x_2^{i_2+1}\right]$ form a partition of X and . $\boldsymbol{Z}_+$ be positive integer numbers

Now for $k\in Z_+^2$, $h>0$ (i.e. $h_1,h_2>0$), $\delta=(\delta_1,\delta_2)$, $\delta_1,\delta_2>0$ and

$x=(x_1,x_2)\in X$ we define the following

$\Delta_h^k(f,x)=\Delta_{h_1}^{k_1}.\Delta_{h_2}^{k_2}(f(x_1,x_2)$ is the k −th symmetric difference of f where $\Delta_{h_i}^{k_i}(f,x)=\sum_{l=0}^{k_i}\binom{k_i}{l}(-1)^{k_i-l}f\left(x-\frac{k_ih_i}{2}+lh_i\right)$,

is the k_i −th difference of step length h_i with respect to x_i and $i=1,2$ such that

$\Delta_h^1(f(x))=\Delta_{h_1}^1.\Delta_{h_2}^1(f(x))=\Delta_{h_1}^1(\Delta_{h_2}^1(f(x,y)))=f(x+h_1,y+h_2)-f(x+h_1,y)-f(x,y+h_2)+f(x,y)$ and

$\Delta_h^0(f,x)=\Delta_{h_1}^0.\Delta_{h_2}^0(f(x,y))=f(x,y)$.

Let $X(K,h)=\{$ x: $(x_i+s_i.k_i)_{i=1}^2\in X$ for all $s_i\le h_i$, $s_i\in\boldsymbol{R}_+^1$ $\}$, $i=1,2$

Then we define:

$$\omega^k(f,\delta)_{p,\emptyset_n} = \omega^k(f,\delta,\mathrm{x})_{L_{p(X(K,h)),\emptyset_n}} = sup_{0\le h\le\delta}\left\|\Delta_h^k(f,\mathrm{x})\right\|_{p(X(K,h)),\emptyset_n}.$$

Before we present the modulus of smoothness of f with respect to the derivative, we present the following concept

Let n be a positive integer. A vector of n-tuples $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_n)$, where α_i , $i = 1,2, \cdots n$ are non-negative integers, is called multi-index of dimension n . The number $|\alpha| = \sum_{i=1}^{n} \alpha_i$ is called the length of the multi-index, for α, β , we have

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots \alpha_n + \beta_n)$$

We say that multi-index α, β are related by $\alpha \le \beta$ if $\alpha_i \le \beta_i$ for all i

Now for the function $f = f(\mathrm{x})$, $\mathrm{x} = (x_1, x_2, \cdots x_n)$ let

$D^\alpha(f) = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}(f)$ will be called derivative of a function f of order$|\alpha|$.

For special case if $n = 2$,$\alpha = (\alpha_1,\alpha_2) = (1,1)$ we have

$D^\alpha(f) = D^{(1,1)}(f) = \frac{\partial^2}{\partial_{x_1}^1 \ \partial_{x_2}^1} f$. From above let

$W_{p,\emptyset_n}^\alpha(X) = \left\{f \in L_{p,\emptyset_n}(X) : D^\alpha f \in L_{p,\emptyset_n}(X)\right\}$, be multiplier Sebolov space.

Here: Since convexity in two dimensions is very important in approximate theory we introduce the following concept, **[7]**

Let $\mathrm{X} = [-\pi,\pi] \times [-\pi,\pi]$ then X satisfies the following

[if $\mathrm{x} \in X$ and $\mathrm{x} + \mathrm{h}e_i \in X$ for some $\mathrm{h} > 0$, $\mathrm{i} = 1,2$ then $\mathrm{x} + \mathrm{t}e_i \in X$ for all $0 \le \mathrm{t} \le \mathrm{h}$,where $e_1 = (1,0)$, $e_2 = (0,1)$ and $X \subseteq \cup_{i=1}^{m} V_i$ where V_i is a rectangle and for each i there is rectangle R_i with one of its vertices is $0 = (0,0)$ such that if $\mathrm{x} \in V_i \cap X$ then $\mathrm{x} + R_i \subset X$]$\cdots\cdots\cdots$ (1.2)

2. The Main Results

Before we state our main results, we need the following Lemmas and notes

Lemma 2.1[5]:

For the kernel $\frac{1}{\pi(p_1+1)}\int_{-\pi}^{\pi} F_{n,p_1}(u)$, where $F_{n,p_1}(u) = \frac{\sin\frac{2n+p_1+1}{2}u \quad \sin\frac{p_1+1}{2}u}{2\sin^2\frac{u}{2}}$, with $u \neq 0$ we have

$$L_{n,p_1} = \left|\frac{1}{\pi(p_1+1)}\int_{-\pi}^{\pi} F_{n,p_1}(u)du\right| = \frac{1}{\pi(p_1+1)}\int_{-\pi}^{\pi} \frac{\left|\sin\frac{2n+p_1+1}{2}u \quad \sin\frac{p_1+1}{2}u\right|}{2\sin^2\frac{u}{2}}\, du$$

$$= \frac{4}{\pi^2}\ln\frac{2n+p_1+1}{p_1+1} + O\,(1)$$

Note 2.2:[8]

For f and g are two functions we have

$f(x) = O\{\,g(x)\}$ if $\lim_{x\to 0}\frac{f(x)}{g(x)} = A$, A is a constant and $g(x) \neq 0$. In particular, O (1) means bounded function.

Note 2.3[5]:

For the kernel $\frac{1}{\pi^2(p_1+1)(p_2+1)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_{n+p_1}^{m+p_2}(u,v)dudv$, where

$$F_{n+p_1}^{m+p_2}(u,v) = \frac{\sin\frac{2n+p_1+1}{2}u \quad \sin\frac{p_1+1}{2}u}{2\sin^2\frac{u}{2}} \cdot \frac{\sin\frac{2m+p_2+1}{2}v \quad \sin\frac{p_2+1}{2}v}{2\sin^2\frac{v}{2}},$$ with $u \neq 0, v \neq 0$ we have

$$L_{n,p_1}^{m,p_2} = \left|\frac{1}{\pi^2(p_1+1)(p_2+1)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_{n+p_1}^{m+p_2}(u,v)dudv\right|$$

$$= \frac{1}{\pi^2(p_1+1)(p_2+1)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|\sin\frac{2n+p_1+1}{2}u \quad \sin\frac{p_1+1}{2}u \ \sin\frac{2m+p_2+1}{2}v \quad \sin\frac{p_2+1}{2}v\right|}{4\sin^2\frac{u}{2}\ \sin^2\frac{v}{2}}$$

$$= \frac{16}{\pi^4}\ln\frac{2n+p_1+1}{p_1+1}.\ln\frac{2m+p_2+1}{p_2+1} + O\,(1)$$

Lemma 2.4:

Let $f \in L_{p,\emptyset_n}(X)$ $X = [-\pi,\pi]\times[-\pi,\pi]$,$1 \leq p < \infty$ we have

$\|\delta_{n,m}(f,.,.)\|_{p,\emptyset_n} \leq c(p)\|f\|_{p,\emptyset_n}$, where $c(p)$ a constant depends on p

Proof:

Since $\delta_{n,m}(f,x,y) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x+u,y+v)F_n(u)F_m(v)dudv$

And by using Jensen's Inequality we have

$$\left\|\delta_{n,m}(f,.,.)\right\|_{p,\emptyset_n}$$

$$=\left\{\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x+u,y+v)F_n(u)F_m(v)dudv\right|^p dxdy\right)^{\frac{1}{p}}\right\}$$

$$=\left\{\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|f\emptyset_n(x+u,y+v)\frac{1}{\pi}\frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\mathrm{F}_n(\mathrm{u})F_m(\mathrm{v})\mathrm{dudv}\right|^p dxdy\right)^{\frac{1}{p}}\right\}$$

$$\leq\left\{\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f\emptyset_n(x+u,y+v)|^p\, dxdy\ \frac{1}{\pi}\int_{-\pi}^{\pi}F_n(u)du\frac{1}{\pi}\int_{-\pi}^{\pi}F_m(\mathrm{v})\mathrm{dv}\right)^{\frac{1}{p}}\right\}$$

$\leq\left\{\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f\emptyset_n(x+u,y+v)|^p\, dxdy\right)^{\frac{1}{p}}\right\}c(p) = c(p)\|f\|_{p,\emptyset_n}$ Thus

$$\left\|\delta_{n,m}(f,.,.)\right\|_{p,\emptyset_n} \leq c(p)\|f\|_{p,\emptyset_n}$$

Lemma 2.5:

Let $f \in L_{p,\emptyset_n}(X)$ $X = [-\pi,\pi]\times[-\pi,\pi]$,$1 \leq p < \infty$ we have

$\left\|V_{n,p_1}^{m,p_2}(f,.,.)\right\|_{p,\emptyset_n} \leq L_{n,p_1}^{m,p_2}\|f\|_{p,\emptyset_n}$, where

$L_{n,p_1}^{m,p_2} = \frac{16}{\pi^4}\ln\frac{2n+p_1+1}{p_1+1}.\ln\frac{2m+p_2+1}{p_2+1} + O(1)$

Proof:

$$\left\|V_{n,p_1}^{m,p_2}(f,.,.)\right\|_{p,\emptyset_n}$$

$$=\left\{\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\frac{1}{\pi^2(p_1+1)(p_2+1)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x+u,y+v)F_{n+p_1}^{m+p_2}(u,v)dudv\right|^p dxdy\right)^{\frac{1}{p}}\right\}$$

$$=\left\{\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|f\emptyset_n(x+u,y+v)\frac{1}{\pi^2(p_1+1)(p_2+1)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}F_{n+p_1}^{m+p_2}(u,v)dudv\right|^p dxdy\right)^{\frac{1}{p}}\right\}$$

$$\leq \left\{\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f\emptyset_n(x+u,y+v)|^p\,dxdy\right)^{\frac{1}{p}}\left|\frac{1}{\pi^2(p_1+1)(p_2+1)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}F_{n+p_1}^{m+p_2}(u,v)dudv\right|\right\}$$

$$\leq \left\{\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f\emptyset_n(x+u,y+v)|^p\,dxdy\right)^{\frac{1}{p}}\right\}\ L_{n,p_1}^{m,p_2} = L_{n,p_1}^{m,p_2}\ \|f\|_{p,\emptyset_n}\,.$$

Where $L_{n,p_1}^{m,p_2} = \frac{16}{\pi^4}\ln\frac{2n+p_1+1}{p_1+1}.\ln\frac{2m+p_2+1}{p_2+1}$ +O(1). Thus $\left\|V_{n,p_1}^{m,p_2}(f,.,.)\right\|_{p,\emptyset_n}$

$\leq L_{n,p_1}^{m,p_2}\ \|f\|_{p,\emptyset_n}$ ■

Lemma 2.6[7]:

If X satisfy (1.2) and $X \subset Q$ is rectangle with side length vector δ then for each $f \in L_{p(X)}$ there exists $g \in W_p^{\alpha}(X)$ such that

$$\|f-g\|_{p(X)} + \delta^{\alpha}\|D^{\alpha}g\|_{p(X)} \leq c\omega_k(f,\delta,\mathrm{x})_{p(X)}\,.$$

Lemma 2.7:

For each $f \in L_{p,\emptyset_n}(X)$, $X = [-\pi,\pi]^2$ there exists $T_{n,m} \in W_{p,\emptyset_n}^{\alpha}(X)$

Such that $\left\|f-T_{n,m}\right\|_{p,\emptyset_n} \leq c\omega^k(f,\delta)_{p,\emptyset_n}$.

Proof:

Since $f\emptyset_n(x,y)$ is bounded such that $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x,y) < \infty$ and

$T_{n,m} \in W_{p,\emptyset_n}^{\alpha}(X)$ for each $n,m \in \mathbb{N}$, then by using Lemma 2.6 we have

$$\left\|f-T_{n,m}\right\|_{p(X),\emptyset_n} + \delta^{\alpha}\left\|D^{\alpha}T_{n,m}\right\|_{p(X),\emptyset_n}$$

$$= \left\|(f-T_{n,m})\emptyset_n\right\|_{p(X)} + \delta^{\alpha}\left\|D^{\alpha}T_{n,m}\emptyset_n\right\|_{p(X)} = \left\|f\emptyset_n - T_{n,m}\emptyset_n\right\|_{p(X)} + \delta^{\alpha}\left\|D^{\alpha}T_{n,m}\emptyset_n\right\|_{p(X)} \leq c\omega_k(f\emptyset_n,\delta,\mathrm{x})_{p(X)} = c\omega^k(f,\delta)_{p,\emptyset_n}$$, thus

$$\left\|f-T_{n,m}\right\|_{p,\emptyset_n} + \delta^{\alpha}\left\|D^{\alpha}T_{n,m}\right\|_{p(X),\emptyset_n} \leq c\omega^k(f,\delta)_{p,\emptyset_n}$$

But $\delta^{\alpha}\left\|D^{\alpha}T_{n,m}\right\|_{p(X),\emptyset_n} \geq 0$.

Then for each $f \in L_{p,\emptyset_n}(X)$ there is $T_{n,m} \in W_{p,\emptyset_n}^{\alpha}(X)$ such that

$\left\|f - T_{n,m}\right\|_{p,\emptyset_n} \leq c\omega^k(f,\delta)_{p,\emptyset_n}$ ■

Lemma 2.8:

Let $f \in L_{p,\emptyset_n}(X)$ $X = [-\pi,\pi] \times [-\pi,\pi]$,$1 \leq p < \infty$ we have

$\tau^1(f,\delta)_{p,\emptyset_n} \leq \delta\|f'\|_{p,\emptyset_n}$, where $f' = D^{(1,1)}(f) = \frac{\partial^2}{\partial^1 x_1\, \partial^1 x_2}(f)$

is the second derivative of function f and $\delta = (\delta_1, \delta_2)$, $\delta_1, \delta_2 > 0$

Proof:

$\omega^1(f,\delta)_{p,\emptyset_n} = sup_{0\leq h\leq\delta}\|\Delta_h^1(f)\|_{p,\emptyset_n}$

$$= sup_{0\leq h\leq\delta}\left\{\left\{\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|(\Delta_h^1(f\emptyset_n, x, y))|^p\, dxdy\right)^{\frac{1}{p}}\right\}\right\}$$

$$= sup_{0\leq h\leq\delta}\left\{\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\left(\int_0^{h_2}\int_0^{h_1}(f\emptyset_n)'(x+u, y+v)\, dudv\right)\right|^p dxdy\right)^{\frac{1}{p}}\right\}$$

$$= sup_{0\leq h\leq\delta}\left\{\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|(f\emptyset_n)'(x+u, y+v).\int_0^{h_1} du\int_o^{h_2} dv\right|^p dxdy\right)^{\frac{1}{p}}\right\}$$

$$= sup_{0\leq h\leq\delta}\left\{\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|(f\emptyset_n)'(x+u, y+v)|^p\, dxdy\right)^{\frac{1}{p}}.h_1.h_2\right\}$$

$$= sup_{0\leq h\leq\delta}\left\{\|f'\|_{p,\emptyset_n}\right\}.h \leq \delta\|f'\|_{p,\emptyset_n}$$

Thus $\omega^1(f,\delta)_{p,\emptyset_n} \leq \delta\|f'\|_{p,\emptyset_n}$ ■

Lemma 2.9:

Let $f \in L_{p,\emptyset_n}(X)$ $X = [-\pi,\pi] \times [-\pi,\pi]$,$1 \leq p < \infty$ we have

$\omega^k(f,.,\delta)_{p,\emptyset_n} \leq \delta\omega^{k-1}(f',.,\delta)_{p,\emptyset_n}$

Proof:

$\omega^k(f,\delta)_{p,\emptyset_n} = sup_{0\leq h\leq\delta}\left\|\Delta_h^k(f)\right\|_{p,\emptyset_n}$

$$= sup_{0\le h\le\delta}\left\{\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|(\Delta_h^k(f\emptyset_n(x,y))\right|^p dxdy\right)^{\frac{1}{p}}\right\}$$

$$= sup_{0\le h\le\delta}\left\{\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|(\Delta_h^{k-1}\int_0^{h_1}\int_0^{h_2}(f\emptyset_n)'\,(x+u,y+v))dudv)\right|^p dxdy\right)^{\frac{1}{p}}\right\}$$

$$= sup_{0\le h\le\delta}\left\{\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|(\Delta_h^{k-1}(f\emptyset_n)'(x+u,y+v)).\int_0^{h_1}du\int_0^{h_2}dv\right|^p dxdy\right)^{\frac{1}{p}}\right\}$$

$$= sup_{0\le h\le\delta}\left\{\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\Delta_h^{k-1}(f\emptyset_n)'(x+u,y+v)\right|^p dxdy\right)^{\frac{1}{p}}h_1.h_2\right\}$$

$$= sup_{0\le h\le\delta}\left\{\left\|\Delta_h^{k-1}(f')\right\|_{p,\emptyset_n}.h\right\}$$

$$= \omega^{k-1}(f',.,\delta).h \le \delta\omega^{k-1}(f',.,\delta)_{p,\emptyset_n}$$

. Thus $\omega^k(f,.,\delta)_{p,\emptyset_n} \le \delta\omega^{k-1}(f',.,\delta)_{p,\emptyset_n}$ ■

Lemma 2.10:

For $f \in L_{p,\emptyset_n}(X)$, $X = [-\pi,\pi]\times[-\pi,\pi]$, $1 \le p < \infty$ we get

$\omega^k(f,\delta)_{p,\emptyset_n} \le \delta^k\left\|f^{(k)}\right\|_{p,\emptyset_n}$.

Where $f^{(k)} = D^k(f) = D^{(k_1,k_2)}(f) = \frac{\partial^{k_1+k_2}}{\partial^{k_1}x_1\,\partial^{k_2}x_2}(f)$

Proof:

From Lemma 2.9 we have

$$\omega^k(f,\delta)_{p,\emptyset_n} \le \delta\omega^{k-1}(f',\delta)_{p,\emptyset_n} \le \delta\delta\omega^{k-2}(f'',\delta)_{p,\emptyset_n}$$

$$\le \cdots \underbrace{\delta\delta\cdots\delta}_{k-1\ time}\,\omega^1\left(f^{(k-1)},\delta\right)_{p,\emptyset_n}.$$

Then using Lemma 2.8 we get

$$\omega^k(f,\delta)_{p,\emptyset_n} \le \underbrace{\delta\delta\cdots\delta}_{k-1\ time}\,\omega^1\left(f^{(k-1)},\delta\right)_{p,\emptyset_n}$$

$$\le \underbrace{\delta\delta\cdots\delta}_{k-1\ time}.\delta\left\|f^{((k-1)+1)}\right\|_{p,\emptyset_n} = \delta^k\left\|f^{(k)}\right\|_{p,\emptyset_n}$$, thus

$\omega^k(f,\delta)_{p,\emptyset_n} \leq \delta^k \|f^{(k)}\|_{p,\emptyset_n}$ ■

Lemma 2.11:

For $f \in L_{p,\emptyset_n}(X)$, $X = [-\pi,\pi] \times [-\pi,\pi]$, $1 \leq p < \infty$, $x = (x_1, x_2)$,

$h = (h_1, h_2)$ and $k = (k_1, k_2)$, we get $\omega^k(f,\delta)_{p,\emptyset_n} \leq \tau^k(f,\delta)_{p,\emptyset_n}$

Proof:

$\omega^k(f(x_1,x_2),\delta)_{p,\emptyset_n}$

$$= sup_{|h|<\delta}\left\{\left(\int_{-\pi}^{\pi-k_2.h_2}\int_{-\pi}^{\pi-k_1h_1}\left|(\Delta_h^k(f\emptyset_n(x_1,x_2)))\right|^p dx_1dx_2\right)^{\frac{1}{p}}\right\}$$

$$\leq sup_{|h|<\delta}\left\{\left(\int_{-\pi}^{\pi-k_2.h_2}\int_{-\pi}^{\pi-k_1.h_1}\left|(\omega^k(f\emptyset_n(x_1+\tfrac{k_1h_1}{2},x_2+\tfrac{k_2h_2}{2});\delta)\right|^p dx_1dx_2\right)^{\frac{1}{p}}\right\}$$

$$= sup_{|h|<\delta}\left\{\left(\int_{-\pi+\frac{k_2h_2}{2}}^{\pi-\frac{k_2h_2}{2}}\int_{-\pi+\frac{k_1h_1}{2}}^{\pi-\frac{k_1h_1}{2}}|(\omega^k(f\emptyset_n(x_1,x_2);\delta)|^p dx_1dx_2\right)^{\frac{1}{p}}\right\}$$

$\leq sup_{|h|<\delta}\tau^k(f,\delta)_{p,\emptyset_n} = \tau^k(f,\delta)_{p,\emptyset_n}$. Thus

$\omega^k(f,\delta)_{p,\emptyset_n} \leq \tau^k(f,\delta)_{p,\emptyset_n}$ ■

In this paper we prove the following results

Theorem 2.12:

Suppose that $f \in L_{p,\emptyset_n}(X)$, $X = [-\pi,\pi] \times [-\pi,\pi]$, $1 \leq p < \infty$. Then

$E_{n,m}(f)_{P,\emptyset_n} \leq c\omega^k(f,\delta)_{p,\emptyset_n}$ where $x = (x_1,x_2)$, $h = (h_1,h_2)$ *and* $k = (k_1,k_2)$.

Proof:

Using Lemma 2.7 there exists $T_{n,m} \in W^{\alpha}_{p,\emptyset_n}(X)$ such that

$$E_{n,m}(f)_{P,\emptyset_n} = \underbrace{\inf}_{T_{n,m}\in\mathbb{T}_{n,m}}\left\{\|f-T_{n,m}\|_{P,\emptyset_n}\right\} \leq \|f-T_{n,m}\|_{P,\emptyset_n}$$

$\leq c\omega^k(f,\delta)_{p,\emptyset_n}$.Then from Lemma 2.11 we get

$E_{n,m}(f)_{P,\emptyset_n} \leq c\omega^k(f,\delta)_{p,\emptyset_n} \leq c\tau^k(f,\delta)_{p,\emptyset_n}$ ■

Theorem 2.13:

Let $f \in L_{p,\emptyset_n}(X)$, $X = [-\pi,\pi] \times [-\pi,\pi]$, $1 \le p < \infty$ **.** Then

$\|f(.,.) - \delta_{n,m}(f)\|_{p,\emptyset_n} \le (1 + c(p))c\tau^k(f,\delta)_{p,\emptyset_n}$, where $c(p)$a constant

Proof:

Let $T^*_{n,m}(x,y)$ be the best multiplier approximation of function $f(x,y)$

Using linearity and bounded of $\delta_{n,m}$ and $\delta_{n,m}(T^*_{n,m}) = T^*_{n,m}$, Lemma 2.4 and Theorem 2.12 we get

$$\|f(.,.) - \delta_{n,m}(f)\|_{p,\emptyset_n} = \|f(.,.) - T^*_{n,m}(.,.) + T^*_{n,m}(.,.) - \delta_{n,m}(f)\|_{p,\emptyset_n}$$

$$\le \|f(.,.) - T^*_{n,m}(.,.)\|_{p,\emptyset_n} + \|T^*_{n,m}(.,.) - \delta_{n,m}(f)\|_{p,\emptyset_n}$$

$$= \|f(.,.) - T^*_{n,m}(.,.)\|_{p,\emptyset_n} + \|\delta_{n,m}(T^*_{n,m}) - \delta_{n,m}(f)\|_{p,\emptyset_n}$$

$$\le \|f - T^*_{n,m}\|_{p,\emptyset_n} + \|\delta_{n,m}(T^*_{n,m} - f)\|_{p,\emptyset_n}$$

$$\le E_{n,m}(f)_{P,\emptyset_n} + c(p)\|T^*_{n,m} - f\|_{p,\emptyset_n} \le E_{n,m}(f)_{P,\emptyset_n} + c(p)E_{n,m}(f)_{P,\emptyset_n}$$

$$= (1 + c(p))E_{n,m}(f)_{P,\emptyset_n} \le (1 + c(p))\, C\omega^k(f,\delta)_{p,\emptyset_n} \le (1 + c(p))\, c\tau^k(f,\delta)_{p,\emptyset_n}$$ Thus

$$\|f(.,.) - \delta_{n,m}(f)\|_{p,\emptyset_n} \le (1 + c(p))c\tau^k(f,\delta)_{p,\emptyset_n} \quad \blacksquare$$

Corollary 2.14:

If $f \in L_{p,\emptyset_n}(X)$, $X = [-\pi,\pi] \times [-\pi,\pi]$, $1 \le p < \infty$ and from Theorem 2.13 we have

$$\|f(.,.) - \delta_{n,m}(f)\|_{p,\emptyset_n} \le (1 + c(p))\, c\omega^k(f,\delta)_{p,\emptyset_n}$$

Then by using Lemma 2.10 we have that

$$\|f(.,.) - \delta_{n,m}(f)\|_{p,\emptyset_n} \le (1 + c(p))c\delta^k\|f^{(k)}\|_{p,\emptyset_n} \quad \blacksquare$$

Theorem 2.15:

If $f \in L_{p,\emptyset_n}(X)$, $X = [-\pi,\pi] \times [-\pi,\pi]$, $1 \le p < \infty$,we get

$\left\| f(.,.) - V_{n,p_1}^{m,p_2}(f,.,.) \right\|_{p,\emptyset_n} \leq \left(1 + L_{n,p_1}^{m,p_2}\right) c\tau^k(f,\delta)_{p,\emptyset_n}$, where

$L_{n,p_1}^{m,p_2} = \frac{16}{\pi^4} \ln \frac{2n+p_1+1}{p_1+1} . \ln \frac{2m+p_2+1}{p_2+1} + O(1)$

Proof:

Let $T_{n,m}^*(x,y)$ be the best multiplier approximation of function $f(x,y)$.

Using linearity and bounded of V_{n,p_1}^{m,p_2} and $V_{n,p_1}^{m,p_2}\left(T_{n,m}^*\right) = T_{n,m}^*$,

Lemma 2.5, Theorem 2.12 and Lemma 2.11 we get

$\left\| f(.,.) - V_{n,p_1}^{m,p_2}(f,.,.) \right\|_{p,\emptyset_n}$

$= \left\| f(.,.) - T_{n,m}^*(.,.) + T_{n,m}^*(.,.) - V_{n,p_1}^{m,p_2}(f,.,.) \right\|_{p,\emptyset_n}$

$\leq \left\| f(.,.) - T_{n,m}^*(.,.) \right\|_{p,\emptyset_n} + \left\| T_{n,m}^*(.,.) - V_{n,p_1}^{m,p_2}(f,.,.) \right\|_{p,\emptyset_n}$

$= \left\| f(.,.) - T_{n,m}^*(.,.) \right\|_{p,\emptyset_n} + \left\| V_{n,p_1}^{m,p_2}(T_{n,m}^*) - V_{n,p_1}^{m,p_2}(f,.,.) \right\|_{p,\emptyset_n}$

$\leq \left\| f - T_{n,m}^* \right\|_{p,\emptyset_n} + \left\| V_{n,p_1}^{m,p_2}\left(T_{n,m}^* - f\right) \right\|_{p,\emptyset_n} \leq E_{n,m}(f)_{P,\emptyset_n} + L_{n,p_1}^{m,p_2} \left\| T_{n,m}^* - f \right\|_{p,\emptyset_n}$

$\leq E_{n,m}(f)_{P,\emptyset_n} + L_{n,p_1}^{m,p_2}\, E_{n,m}(f)_{P,\emptyset_n}$

$= \left(1 + L_{n,p_1}^{m,p_2}\right) E_{n,m}(f)_{P,\emptyset_n} \leq \left(1 + L_{n,p_1}^{m,p_2}\right) c\omega^k(f,\delta)_{p,\emptyset_n}$

$\leq (1 + L_{n,p_1}^{m,p_2})\, c\tau^k(f,\delta)_{p,\emptyset_n}$, where

$L_{n,p_1}^{m,p_2} = \frac{16}{\pi^4} \ln \frac{2n+p_1+1}{p_1+1} . \ln \frac{2m+p_2+1}{p_2+1} + O(1)$. Thus

$$\left\| f(.,.) - V_{n,p_1}^{m,p_2}(f) \right\|_{p,\emptyset_n} \leq \left(1 + L_{n,p_1}^{m,p_2}\right) c\tau^k(f,\delta)_{p,\emptyset_n}$$

Corollary 2.16:

For $f \in L_{p,\emptyset_n}(X)$, $X = [-\pi,\pi] \times [-\pi,\pi]$, $1 \leq p < \infty$ **,** and from Theorem 2.15 we get
$\left\| f(.,.) - V_{n,p_1}^{m,p_2}(f) \right\|_{p,\emptyset_n} \leq \left(1 + L_{n,p_1}^{m,p_2}\right) c\omega^k(f,\delta)_{p,\emptyset_n}$

Then by using Lemma 2.10 we have
$\left\| f(.,.) - V_{n,p_1}^{m,p_2}(f) \right\|_{p,\emptyset_n} \leq \left(1 + L_{n,p_1}^{m,p_2}\right) c\delta^k \left\| f^{(k)} \right\|_{p,\emptyset_n}$ ■

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Presentation of the subgroups of Mathieu Group using Groupoid

13 **Nisreen Alokbi**[1]**, Faik Mayah**[2]

[1]College of computer science and IT, University of Wasit.

[2] College of science, University of Wasit

E-mail: e-mail [1]nisreen.alokbi@gmail.com, [2]faik.mayah@gmail.com.

Abstract. Mathieu groups are one type of the sporadic simple groups, they turn out not to be isomorphic to any member of the infinite families of finite simple groups. Study these groups is interesting since their orders are very high. Groupoid can be used to find the presentation of the subgroups of the Mathieu groups. The idea is creating a groupoid by acting the Mathieu group on a subset of this group and then calculating the presentation of the vertex group of the groupoid which represents the presentation of the subgroup as the vertex groups are isomorphic.

1. Introduction

Émile Léonard Mathieu (1861, 1873) introduced a special type of groups, they are multiply transitive permutation groups on n objects ($n \in \{11, 12, 22, 23, 24\}$). The Mathieu groups were the first five sporadic simple groups to be discovered and they are denoted by M_{11}, M_{12}, M_{22}, M_{23} and M_{24} [7].

Groups that act on sets of 9, 10, 20, and 21 points, respectively are denoted by M_9, M_{10}, M_{20} and M_{21}. These group are not sporadic simple groups but they are subgroups of the larger groups and can be used to construct the larger ones. One can extend this sequence up to obtain the Mathieu groupoid M_{13} acting on 13 points. Also M_{21} which is simple group, but is not a sporadic group, being isomorphic to projective special linear group PSL(3,4) [5].

Table 1 is showing the orders of the Mathieu group.

Table 1. Order table of the Mathieu groups

Mathieu Group	Order
11	7920
12	95040
22	443520
23	10200960
24	244823040

M_{12} has a maximal simple subgroup of order 660 which is isomorphic to PSL2(F_{11}) over the field of 11 elements. M_{11} is the stabilizer of a point in M_{12}.M_{10}, the stabilizer of two points, is not sporadic, but is an almost simple group whose commutator subgroup is the alternating group A_6. The stabilizer of 3 points is the projective special unitary group PSU(3,22). The stabilizer of 4 points is the quaternion group. Also, M_{24} has a simple subgroup of order 6072 which is a maximal subgroup and it is isomorphic to PSL2(F_{23}). The stabilizers of 1 and 2 points, M_{23} and M_{22} also becomes sporadic simple groups. The stabilizer of 3 points is simple and isomorphic to PSL3(4) (the projective special linear group) [4, 8].

We will try to find a presentation of a subgroup of Mathieu group by construction first a finitely presented groupoid by acting the Mathieu group on the set that generate the subgroup of the Mathieu group and then finding the presentation of the vertex group of the groupoid.

The groupoid is an algebraic structure which is a generalization of the group. It is a category in which all arrows are isomorphisms. So a group is a groupoid with one object and arrows the elements of the group.

In the context of topology, the best example of groupoid is the fundamental groupoid of a topological space in which the objects set is a set of point taken from the space and an arrow from point a to point b to be equivalence classes of paths from a to b [3]. This is generalisation of the idea of the fundamental group.

In this paper, we construct a groupoid whose objects set is the left cosets

$$gH = \{mh \mid h \text{ an element } H\}$$

and m is an element in M (Mathieu group) and H is a subgroup of M . The morphism of the groupoid is induced by the group action, more details later.

2. Groupoids and vertex group

2.1. Groupoids, free groupoids and finitely presented groupoids

A *groupoid* is a special type of category which is a generalization of a group.

Deftnition 2.1. *[6] A groupoid $\mathcal{G}$ is a category in which for each morphism (arrow) $f : A \to B$ there is a morphism (arrow) $f^{-1} : B \to A$ such that $f \circ f^{-1} = 1_B$, $f^{-1} \circ f = 1_A$. The morphism f^{-1} is called the inverse of f.*

A groupoid is *connected* if for each pair of objects A and $B \in Obj(\mathcal{G})$ there is at least one arrow $w \in Arr(\mathcal{G})$ with the property $source(w) = A$ and $target(w) = B$.

The notion "free groupoid" is the corner stone of this work. Since for any free groupoid there is an underlying graph (directed graph). So let us recall the definition and required mathematical fact that help to construct such free groupoid.

Deftnition 2.2. *A directed graph $\Gamma = (V, E, s, t)$ consists of a set V called the set of vertices, a set E called the set of edges of Γ and two functions $s, t : E \to V$. The vertex $s(e)$ is the source of an edge $e \in E$. The vertex $t(e)$ is the target of an edge $e \in E$.*

A map of directed graphs $(V,E,s,t) \rightarrowtail (V',E',s',t')$ consists of functions $f_1 : V \to V'$, f_2 : $E \to E'$ such that $s(f_2(e)) = f_1(s(e))$ and $t(f_2(e)) = f_1(t(e))$ for all $e \in E$.

Deftnition 2.3. *The disjoint union $\Gamma = \Gamma_1 \sqcup \Gamma_2$ of directed graphs Γ_1 and Γ_2 with disjoint vertex sets $V(\Gamma_1)$ and $V(\Gamma_2)$ and edge sets $E(\Gamma_1)$ and $E(\Gamma_2)$ is the directed graph with $V(\Gamma) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma) = E(\Gamma_1) \cup E(\Gamma_2)$.*

Deftnition 2.4. *A maximal tree T of a directed graph Γ is a subgraph which includes every vertex of Γ and contains no cycle.*

Let *Graphs* denote the category whose objects are directed graphs and whose morphisms are maps of directed graphs. Let *Groupoids* denote the category whose objects are groupoids and whose morphisms are functors between groupoids. There is a functor

$$U : Groupoids \to Graphs \qquad (1)$$

which simply forgets the partial composition on a groupoid. If G is a groupoid, then the vertices of $U(G)$ are precisely the objects of G. The directed edges of $U(G)$ are the arrows of G.

There is a functor

$$F : Graphs \to Groupoids \qquad (2)$$

where for a directed graph Γ, the groupoid $F(\Gamma)$ is characterized, up to isomorphism, by the following universal property.

Universal property of a free groupoid on Γ. There is a map of directed graphs $\iota : \Gamma \to U(F(\Gamma))$. For any groupoid G and any map of directed graphs $f : \Gamma \to U(G)$ there exists a unique groupoid morphism $\bar{f} : F(\Gamma) \to G$ for which the following diagram commutes in the category of directed graphs.

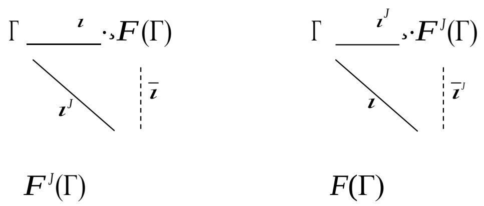
$$\begin{array}{ccc} \Gamma & \xrightarrow{\iota} & U(F(\Gamma)) \\ & \searrow^{f} & \downarrow{\scriptstyle U(\bar{f})} \\ & & U(G) \end{array}$$

We call $F(\Gamma)$ the *free groupoid* on Γ. The existence of $F(\Gamma)$ is established by an explicit construction in terms of words $x_1^{s_1}x_2^{s_2}\dots x_n^{s_n}$ where $s = \pm 1, x_i \in E(\Gamma)$, and $s(x_i^{s_i}) = t(x_{i+1}^{s_{i+1}})$. When the directed graph Γ has just a single vertex we say that $F(\Gamma)$ is the free group on the set $E(\Gamma)$.

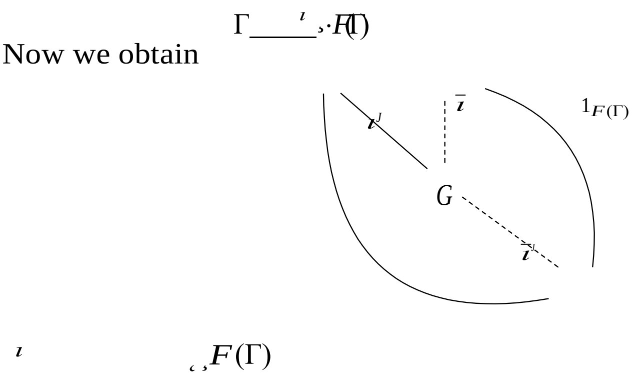
Proposition 2.1. *$F(\Gamma)$ is unique up to isomorphism of groupoids. Proof.*

For simplicity we denote $U(G)$ by G for any groupoid G.

Let Γ be a directed graph, and let $F(\Gamma)$ and $F'(\Gamma)$ be free groupoids on Γ. Let $\iota : \Gamma \to F(\Gamma)$ be a map, and another map $\iota' = \Gamma \to F'(\Gamma)$. By the universal property of free groupoid there is a unique groupoid morphism $\bar{\iota} = F(\Gamma) \to F'(\Gamma)$ such that the following digram



commutes. Now we obtain



By uniqueness, $\overline{\iota'} \circ \overline{\iota} = 1_{F(\Gamma)}$. Similarly, $\overline{\iota} \circ \overline{\iota'} = 1_{F'(\Gamma)}$. Therefore, $F(\Gamma)$ is isomorphic to $F'(\Gamma)$. □

Let G be a groupoid with object set $Obj(G) = V$. Let N be a discrete subgroupoid of G with the same object set $Obj(N) = V$. Thus every arrow of N is an arrow of G and N is closed under groupoid composition. The collection of groups $\{G(v, v) \mid v \in V\}$ is an example of a discrete subgroupoid of G. We say that a discrete subgroupoid N is *normal* in G if $N(v, v)$ is a normal subgroup of $G(v, v)$ for each $v \in V$. Given a discrete normal subgroupoid N in G we can form the quotient groupoid G/N which is characterized up to groupoid isomorphism by the following universal property.

Universal property of a quotient groupoid. There is a morphism of groupoids $\varphi : G \to G/N$. For any groupoid Q with object set $Obj(Q) = V$, and for any morphism $\psi : G \to Q$ that is the identity on V and that sends each element of N to an identity element, there exists a unique morphism of groupoids $\psi' : G/N \to Q$ such that the following diagram in the category of groupoids commutes.

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G/N \\ & \searrow^{\psi} & \downarrow^{\psi'} \\ & & Q \end{array}$$

Proposition 2.2. *For discrete N, G/N is unique up to isomorphism of groupoids.*

Proof. Similar to the proof of the proposition 2.1. □

Deftnition 2.5. *We say that a set $\underline{r}$ of arrows in a discrete subgroupoid N normally generates N if any normal discrete subgroupoid of G containing $\underline{r}$ also contains the subgroupoid N.*

Let G be a groupoid with vertex set $V = Obj(G)$, and let $F(\Gamma)$ be a free groupoid on a directed graph $\Gamma = (V, \underline{x}, s, t)$, and suppose that there is a morphism of groupoids

$$\varphi : F(\Gamma) \to G \qquad (3)$$

that is the identity on objects and that is surjective on arrows. By ker φ we mean the groupoid with vertex set V and with arrows those elements r in $F(\Gamma)$ mapping to an identity arrow $1_{s(r)}$ in G. The groupoid ker φ is a discrete normal subgroupoid and $F(\underline{x})/\ker\varphi$ is isomorphic to G. Let $\underline{r}$ be a set of elements in ker φ that normally generates ker φ. The data $(\underline{x} \mid \underline{r})$ is called a *free presentation* of the groupoid G.

2.2. Vertex group

Let G be a groupoid with object set $Obj(G) = V$. For each object (vertex) $v \in V$ we let $G(v, v)$ denote the group of arrows with source and target equal to v. We refer to $G(v, v)$ as the *vertex group* or *isotropy group* or *object group* at v. The vertex group $G(v, v)$ actually is a subgroupoid consisting of one object v and all arrows of the form $v \to v$.

Let G be a connected groupoid, we can define a homomorphism

$$\theta : G \to G(v,v) \qquad (4)$$

in the following sense.

Let Γ be the generating graph of G, (i.e. $F(\Gamma) = G$), and let T be a maximal tree in Γ. The tree T generates a subgroupoid H of G, which called a *tree of groupoid*. The map θ is defined as

$$\theta(a) = v \qquad a \in Obj(G)$$

$$\theta(w) = xwy, \quad w \in Arr(G), x, y \in H \qquad (5)$$

such that $t(y) = s(w)$, $s(x) = t(w)$ and $s(y) = t(x) = v$.

For $c, d \in H$ (such that $s(c) = t(d) = u$ and $t(c) = s(d) = v$), the product $dc = 1_u$. Its obvious that the map θ maps the whole H onto 1_v.

Proposition 2.3. *The vertex groups of a connected groupoid are all isomorphic.*

Proof. Let G be a groupoid with $Obj(G) = V$. Let $v \in V$ and $G(v, v)$ is the vertex group on v. To prove that all vertex groups are isomorphic to $G(v, v)$, let us choose any object $w \in V$, and any arrow x such that $s(x) = v$ and $t(x) = w$. The map $h \mapsto xhx^{-1}$ is an isomorphism from the vertex group at $G(v, v)$ to the vertex group at $G(w, w)$.

Theorem 2.1. *Let $G = (\underline{x} \mid \underline{r})$ be a finitely presented connected groupoid. If $G(v, v)$ is the vertex group at $v \in Obj(G)$, then $G(v, v) = (\underline{x}' \mid \underline{r}' \cup t)$, where $\underline{x}' = \{\theta(x) : x \in \underline{x}\}$ and $\underline{r}' = \{\theta(r) : r \in \underline{r}$ with expressing $\theta(r)$ as a word $x_1^{s_1} x_2^{s_2} \dots x_k^{s_k}$, $x_i \in \underline{x}', s_i \in \pm 1\}$ and $t = \{t : t$ edge in a maximal tree of $G\}$.*

Proof. Let $\underline{x} = (V, E, s, t)$ be a connected directed graph. Let $F(\underline{x})$ denote the free groupoid on $\underline{x}$. An arrow $r \in Arr(F(\underline{x}))$ is said to be a *loop* if $s(r) = t(r)$. Let $\underline{r}$ denote a set of loops in the groupoid $F(\underline{x})$. Let R denote the normal subgroupoid of $F(\underline{x})$ generated by $\underline{x}$.

The data $(\underline{x} \mid \underline{r})$ is a presentation for the quotient groupoid

$$G = F(\underline{x})/R.$$

Let t denote a maximal tree in the graph $\underline{x}$. Fix some vertex $v \in V$. Then each vertex $w \in V$ determines a unique simple path $p(w)$ in the tree t with $s(p(w)) = w$ and $t(p(w)) = v$. In other words, $p(w)$ is a path in t from w to v.

For each arrow a in the groupoid $F(\underline{x})$ let us set

$$\theta(a) = p(s(a))^{-1} * a * p(t(a)).$$

Thus $\theta(a)$ is a loop in the groupoid $F(\underline{x})$ with source and target equal to v.
Now define

$$\begin{aligned} \underline{x}' &= \{\theta(a) : a \text{ is a dircted edge in } \underline{x} \text{ and } a \notin t\}, \underline{r}' \\ &= \{\theta(a) : a \text{ is an arrow in } \underline{r}\}. \end{aligned}$$

Note that $\underline{x}'$ is a free generating set for the free group $\mathcal{F}(v,v)$. here we are writing $\mathcal{F} = F(\underline{x})$ and letting $\mathcal{F}(v,v)$ denote the vertex group at v.
Note that $\underline{r}'$ is a subset of $\mathcal{F}(v,v)$. Let $\mathcal{R}(v,v)$ denote the normal subgroup of $\mathcal{F}(v,v)$ normally generated by $\underline{r}'$.

We can now regard $(\underline{x}' \mid \underline{r}')$ as a free presentation for the finitely presented group

$$F(v, v)/R(v, v).$$

To prove the theorem we need to see that $\mathcal{F}(v, v)/\mathcal{R}(v, v)$ is isomorphic to the vertex group $\mathcal{G}(v,v)$ in $\mathcal{G}$.

There is a canonical set theoretic function $\lambda' : \underline{x} \to \mathcal{G}$. This function induces a group homomorphism

$$\lambda : F(v, v) \to G(v, v)$$

The kernel of λ, by definition, consists of all loops in $\mathcal{F}(\underline{x})$ at v that can be written as a product of conjugates of loops in $\underline{r}$. So clearly the kernel of λ is normally generated by $\underline{r}'$ and the proof is complete.

The theorem and propositions above are implemented in GAP as a part of the package FpGd [2] available in GitHub website [1].

3. Group actions produce a groupoid

Proposition 3.1. *Suppose that a group tt acts on a set S and that $\underline{x}$ is a set of generators for tt. Then the groupoid Gpd(tt, S) is generated by the collection of arrows $\underline{x} \times S = \{(x, s) : x \in \underline{x}, s \in S\}$.*

Proof. An arbitrary arrow (g, s) in $Gpd(tt, S)$ can be expressed as

$$(x_1^{s_1}, x_2^{s_2}...x_n^{s_n} s)(x_2^{s_2}, x_3^{s_3}...x_n^{s_n} s)...(x_{n-1}^{s_{n-1}}, x_n^{s_n} s)(x_n^{s_n}, s)$$

where

$$g = x_1^{s_1} x_2^{s_2} ... x_n^{s_n}$$

and $x_i \in \underline{x}, s_i = \pm 1$. If $s_i = -1$ then

$$(x_i^{-1}, s) = (x_i, {}^{x_i^{-1}} s)^{-1}.$$

Each arrow in $Gpd(tt, S)$ is a sequence of arrows in $\underline{x} \times S$. □

Let tt be a group with subgroup U. Let $tt/U = \{gU : g \in tt\}$ denote the collection of left cosets $gU = \{{}^g u : u \in U\}$. There is an action of tt on the set $X = tt/U$ given by $(g, hU) \to ghU$ for $g, h \in tt$. This action gives rise to a groupoid $Gpd(tt, U)$.

Proposition 3.2. *For a group tt and subgroup U the groupoid Gpd(tt, U) is connected and all vertex groups are isomorphic to U.*

Proof. The object set of the groupoid $Gpd(tt, U)$ is

$$\{U, U_1, ..., U_n\}, \qquad \text{where } n = \text{Index}(U) - 1.$$

Since any coset $U_i = y_i U$ for some $y_i \in tt$, the groupoid is connected.

To prove that all vertex groups are isomorphic to U, let us choose any object U_i and any element $x = x_1^{s_1}...x_n^{s_n}$ such that $x_2^{s_2}...x_n^{s_n} U = U_i$. The map $h \mapsto x^{-1}hx$ is an isomorphism between the vertex group at U to the vertex group at U_i. □

Proposition 3.3. *Let $tt = \underline{x}\ \underline{r}$ be a finitely presented group with finite index subgroup U. Then the groupoid G = Gpd(tt, U) is finitely presented as follows. The objects of G are the left cosets gU. The generators of G are the arrows (x, gU) for $x \in \underline{x}$. Each relator $r = x_1^{s_1} x_2^{s_2} ... x_n^{s_n} \in \underline{r}$ and coset gU give rise to a word*

$$(r, gU) = (x_1^{s_1}, x_2^{s_2}... x_n^{s_n} gU)...(x_{n-1}^{s_{n-1}}, x_n^{s_n} gU)(x_n^{s_n}, gU) \qquad (6)$$

in the groupoid generators. These words (r, gU) *are the relators for the groupoid.*

Proof. Let $F(\underline{x})$ be the free group on $\underline{x}$. Let R denote the normal subgroup of F normally generated by $\underline{r}$. It yields

$$F/R \cong tt = (\underline{x} \mid \underline{r})$$

Let U be a subgroup of the group tt and let tt/U be the set of left cosets of U in tt.

Let G denote the finitely presented groupoid $Gpd(tt, U)$. By definition G is generated by the set

$$\underline{x}' = \{(x, gU) \mid x \in \underline{x}, gU \in tt/U\}.$$

Let F be the free groupoid generated by $\underline{x}'$ (i.e. $F = Gpd(F, U)$). So each arrow $a \in F$ can be expressed as

$$a = (g_i, S_j),$$

where $S_j \in F/U$ and

s s s

$$g = x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \ldots x_{i_k}^{\epsilon_k} \in F$$

There is a groupoid homomorphism $\varphi : F \to G$ such that the kernel of φ consists of all arrows of the form (g_i, gU) for which the source and target is gU. That means $\varphi(g_i) = 1_{gU}$ and that yields $g_i \in R$. It is readily seen that $\langle r \rangle = R$.

4. Algorithms and implementation

In order to get the presentation of a subgroup H of a finite fp group tt, we need to create an fp groupoid induced by the group action of tt on H. We then evaluate the vertex group on the subgroup under consideration. This is one of the applications of the groupoid techniques. We implement Propositions 3.1 and 3.3 This implementation follows the Algorithm 1.

Algorithm 1: Fp groupoid induced by group action

Result: Fp groupoid

procedure obj(G) = tt/H; gens(G) = [ ];

for *x in GeneratorsOfGroup(tt)* **do**
  for *c in obj(G)* **do**
    add(gens(G), $^x c$);
  end
end

rels(G) = [ ];

for *r in RelatorsOfFpGroup(tt)* **do**
  for *c in obj(G)* **do**
    add(rels(G), $^r c$);
  end
end

return *FpGroupoid(obj(G), gens(G), rels(G))*;

end procedure

Example 4.1. Consider the Mathieu group M_{11} which is generated by two generators, say a and b. Let $L = [a^{-1}ba, (ab)^{-1}b]$ is a set of some members of M_{11}. The presentation for the subgroup $S = M_{11}/L$ can be calculated using our algorithm which is implemented in GAP as a function FpGroupoid, the input Mathieu group M and a subgroup $S \leq M$ and it returns a presentation for the groupoid $G(M, S)$ and finally calculate the presentation for the vertex group, using our GAP function VertexGroup which serves as the presentation for the subgroup S.

$$S = \langle x, y \mid (yx)^2, x^4, y^4, yx^2yx^{-1}y \rangle$$

The calculation is shown in the following GAP session:

```
gap> M:=MathieuGroup(11);;

gap>
H:=Image(IsomorphismFpGroup(M))
;; gap> h:=GeneratorsOfGroup(H);;

gap> L:=[h[1]^-1*h[2]*h[1],h[2]^-
1*h[1]^-1*h[2]]; [ a^-1*b*a, b^-1*a^-
1*b ]

gap> U:=Subgroup(H,L);

Group([ a^-1*b*a, b^-1*a^-1*b ])



gap> G:=FpGroupoid(H,L);;

gap> v:=Source(GeneratorsOfGroupoid(G)[1]);;

gap> S:=VertexGroup(G,v);; S:=SimplifiedFpGroup(S);

<fp group on the generators [
f1, f2 ]> gap>
RelatorsOfFpGroup(S);

[ (f2*f1)^2, f1^4, f2^4, f2*f1^2*f2*f1^-1*f2 ]
```

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Iteration Variational Method for Solving Two-Dimensional Partial Integro-Differential Equations

(1)AMINA KASSIM HUSSAIN, (2)FADHEL SUBHI FADHEL, (3)NURSALASAWATI RUSLI,(3)ZAINOR RIDZUAN YAHYA

Mustansiriyah University, Material Engineering Department, 10052 Baghdad, Iraq

(2)Al-Nahrain University, College of Science, Mathematics and Computer Applications Department,10070 Al-Jadriya, Baghdad, Iraq

(3)Universiti Malaysia Perlis, Institute of Engineering Mathematics,02600 Arau,Perlis,Malysia

Email:[1]amina1975_kas@yahoo.com,amina.kass@uomustansiriyah.edu.iq,[2]fsf@sc.nahrainuniv.edu.iq, dr_fadhel67@yahoo.com

[3]nursalasawati@unimap.edu.my,[3]zainoryahya@unimap.edu.my

ABSTRACT

The two-dimensional integro-differential partial equations is one of the so difficult problems to be solved analytically and/or approximately, and therefore, a method that is efficient for solving such type of problems seems to be necessary. Therefore, in this paper, the iteration methods, which is so called the variational iteration method have been used to provide a solution to such type of problems approximately, in which the obtained results are very accurate in comparison with the exact solution for certain well selected examples which are constructed so that the exact solution exist. Main results of this work is to derive first the variational iteration formula and then analyzing analytically the error term and prove its convergence to zero as the number of iteration increases.

Keywords: *Variational Iteration Method, Partial Integro-Differential Equations, Two-Dimensional Integro-Differential Partial Equation.*

1. INTRODUCTION

In applied mathematics, an interesting attempts that concerning real life phenomena's usually leads to functional equations, such as ordinary and differential partial equations, integro-differential and integral equations and others [1], [2]. Several formulations that are mathematical of such phenomena leads to integro-differential equations [3], [4

]. In some cases, the solution that is analytical could cause difficulty to evaluate; for this reason, approximate and numerical methods appear to be helpful to use which highlight the problem that is under consideration. Mathematicians focus their attention on the development of more efficient and advanced and methods for integro-differential and integral equations, such as semi numerical analytical techniques, Adomian's decomposition method, method of homotopy perturbation. The Homotopy method perturbation and the method of Adomian's decomposition were used for the solution of integral equation by Poushokouhi etal.[5], variational iteration method (VIM) have been used by Xu L. for the solution of Fredholm and Volterra Integral equations of the second type [6] and for solving Volterra integral equations by Abbasbandy [7], while for the two-dimensional integro-differential and integral equations equation which is an extension of the previously proposed methods for solving one-dimensional cases. Also, there is many studies has been done for the solution of a class of two-dimensional problems for example using the VIM for solving mixed nonlinear Volterra-Fredholm integral equation [8], by using transform method that is deferential for the solution of nonlinear and linear two-dimension Volterra integral equations [9], solving two-dimensional Volterra integral equations by using iterated collocation and collocation method [10], providing a solution of a class of two-dimensional nonlinear Volterra integral equations by using Legendre polynomials [11], providing a solution of mixed nonlinear Volterra-Fredholm integral equations with block-pulse functions that are two dimensional by using a method that is direct [12].

Whenever very little attempts have been paid to give a solution to the partial two-dimensional integral equations, for example, d'Halluin in 2004 [13] solved the integro-differential two-dimensional equations by using a semi-Lagrangian approach. The VIM that has been proposed by Ji-Huan recently. In 1998 he studied and used intensively by several engineers and scientists, which is favorably applied to several types of nonlinear and linear problems.

In this paper, the VIM will be used to provide a solution to partial two-dimensional integro-differential equations in which the analysis is based on deriving first the iterated formulas for evaluating the sequence of iterated approximate solutions, and then it will be used to prove the obtained sequence convergence to the precise solution.

The method may be considered as a modified approach to the method of General Lagrange multiplier into a method of iteration in correction with variational approach to derive the so called the correction functional, where the form of considered integro-differential two-dimensional equation is as follows:

$$\frac{\partial u(x,t)}{\partial t} = g(x,t) + \int_{x_0}^{x}\int_{t_0}^{t} k(u(s,y))\,dy\,ds,\ x \in [0,b],\ t \in [0,T] \qquad ...(1)$$

with the condition that is initial:

$$u(x,0) = u_0(x) \qquad ...(2)$$

where k is represents function of kernel, g is the function that is given and u stands for real unknown function to be evaluated.

Several studies were achieved to compare the method of VIM with available techniques, and it is reflected by all that this method gives precise solutions that are faster than other methods, in which the concept of convergence has been emonstrated to be an amount that is substantial for work of research and the studies of the VIM have been directed by many remarkable researchers, [14].

The VIM has been applied successfully to many kinds of problem, for instance, He first proposed the VIM to provide a solution for the nonlinear and linear integral and differential equations. In 1998, He used this method to solve some well known problems for example the classical Blasiu's equation with more accurate results and then extensively used in 1999 by him to study and solve some non-linear well known problems. In 2000, VIM was used by him to solve systems of autonomous differential ordinary equations. In 2006, Soliman applied the VIM to solve equation of kdv-Burger and then to solve equation Lax's seventh-order, Abulwafa and Momani used the VIM to give a solution to coagulation nonlinear problem that is with mass loss. In addition, in 2006, Odibat et al used the VIM to give a solution differential nonlinear equations of order that is fractional and the VIM has been used to give a solution to several types of problems, such as providing a solution to nonlinear PDE's by Bildiki et al., for solving the equation of Fokker-Plank by Dehghan and Tateri, for solving differential equation of quadratic Riccati with constant coefficients by Abbasbandy. In 2007, Wang and He applied VIM to solve integro-differential equations, while Sweilam used VIM to solve boundary value problems of the nonlinear and linear fourth order equations that are integro-differential. In 2009, Wen-Hua Wang used the VIM to solve certain types of fractional integro-differential equations, [15], [16], [17]. Muhammet Kurrulay and Adin Secer in 2011 used the VIM to solve nonlinear integro-differential equations of fractional order, [18] and A.Husaain et al in 2016 applied the VIM for solving one-dimensional partial integro-differential equations,[19].

2. The Main Aspects of the VIM for Solving Two-Dimensional Integro-Differential Partial Equations

As it is said previously, the VIM which was suggested has been illustrated to easily and effectively solve a large class of nonlinear and linear problems, where it may happen that one or two iteration may result in accurate high solutions. Generally, procedure of the solution of the VIM is very operative, convenient and straightforward for most problems given in advanced forms as a functional form, [20,21].

The non-linear general equation below that is given in operator form could be regarded to show the basic idea of the VIM:

$$L(u(x)) + N(u(x)) = g(x),\ x \in [a,b] \qquad ...(3)$$

where L represents a linear operator, N stands for an operator nonlinear and g represents any function that is given and named the non-homogenous term.

Now, rewrite equation (3) as shown below

$$L(u(x)) + N(u(x)) - g(x) = 0 \qquad ...(4)$$

and let u_n be the n-th equation approximate solution (4), and it is then shown as follows:

$$L(u_n(x)) + N(u_n(x)) - g(x) = 0 \qquad ...(5)$$

and therefore the functional correction connected with equation (5), is provided by:

$$u_{n+1}(x) = u_n(x) + \int_a^x \lambda(s)\left\{L(u_n(s) + N(\tilde{u}_n(s)) - g(s)\right\} ds,\ n = 0, 1, ... \qquad ...(6)$$

where λ is recognized as the general Lagrange multiplier, which can be optimally specified by the calculus of variation theory, and $\tilde{u}_n$ is regarded as a variation that is restricted that satisfy $\delta\tilde{u}_n = 0$, [20].

Generally, it is plain now that the essential steps of the method of He's variational iteration require first optimal determination of the multiplier value of Lagrangian λ. After recognizing the multiplier of Lagrang, the approximations that are successive u_{n+1}, for all $n = 0, 1, ...$ of the solution u will be obtained rapidly by the use of any function that is selective u_0, which is favored to be equal to the terms that are non homogenous for the integral equations. Thus, it could be demonstrated that the solution u_n show convergence to the exact solution u as $n \longrightarrow \infty$.

In the next theorem, the equation approximate solutions general form (1) by the use of the correction functional (6) is obtained which is based on the evaluation of the Lagrange multiplier that is general and that is connected with the integro-differential partial equation (1).

<u>Theorem (1):</u>

Consider the nonlinear partial two-dimension integro-differential equation (1) with initial condition (2). Then the sequence of iterative approximate solutions using VIM is provided by:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n}{\partial \xi}(x,\xi) - g(x,\xi) - \int_0^x \int_0^\xi k(\tilde{u}_n(s,y)\,dy\,ds \right] d\xi \qquad ...(7)$$

for all $n = 0,1,\ldots$

Proof:

The correction that is functional (6) connected with equation (1) is provided by:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left[\frac{\partial u_n}{\partial \xi}(x,\xi) - g(x,\xi) - \int_0^x \int_0^\xi k(\tilde{u}_n(s,y))\,dyds \right] d\xi \qquad ...(8)$$

where λ represents the general Lagrange multiplier, that must be evaluated using calculus that is variational, the subscript n indicates the n^{th} approximation and $\tilde{u}_n(t)$ is regarded as the variation that is restricted.

Now, by having the first variation δ with regard to u_n for the two sides of equation (8) and setting $\delta u_n = 0$, provides:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(\xi) \left[\frac{\partial u_n}{\partial \xi}(x,\xi) - g(x,\xi) - \int_0^x \int_0^\xi k(\tilde{u}_n(s,y))\,dy\,ds \right] d\xi \qquad ...(9)$$

and noting that $\delta \tilde{u}_n = 0$, which will consequently reduce equation (9) to:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(\xi) \frac{\partial u_n}{\partial \xi}(x,\xi)\, d\xi \qquad ...(10)$$

Thus, by using the integration method by parts, equation (10) will have the form:

$$\int_0^t \lambda(\xi) \frac{\partial u_n}{\partial \xi}(x,\xi)\, d\xi = \lambda(\xi) u_n(x,\xi) - \int_0^t u_n(x,\xi)\, \lambda'(\xi)\, d\xi \qquad ...(11)$$

and substituting equation (11) back into equation (10) will give:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \lambda(\xi) u_n(x,t) - \delta \int_0^t u_n(x,\xi) \lambda'(\xi)\, d\xi \qquad ...(12)$$

Consequently, the following stationary conditions is gained:

$$\lambda'(\xi) = 0 \qquad ...(13)$$

with initial condition:

$$1 + \lambda(\xi)\big|_{\xi = t} = 0 \qquad ...(14)$$

Now, providing solution to the ordinary differential equation (13) will provide the general Lagrange multiplier value connected with equation (1) to be:

$$\lambda(\xi) = -1 \qquad \text{...(15)}$$

Consequently, substituting $\lambda(\xi) = -1$ into the correction functional (8) will lead to the following approximate solution in the form that is iterated:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n}{\partial \xi}(x,\xi) - g(x,\xi) - \int_0^x \int_0^\xi k(\tilde{u}_n(s,y)\, dy\, ds \right] d\xi\,. \quad \blacksquare$$

3. Convergence of the Sequence of Approximate Iterated Solutions

In this section, the sequence convergence of approximate iterated solution (7) using the VIM for solving partial integro-differential two-dimensional equation will be demonstrated. The central proof idea depends on the evaluation of the error term upper bound between the exact approximate solution of equation (1) which is demonstrated to be zero as $n \longrightarrow \infty$.

<u>Theorem (2):</u>

Let u, $u_n \in C_t^n([a,b]\times[0,T])$ be the approximate and equation exact solutions (1) and (7), respectively. If $E_n(x,t) = u_n(x,t) - u(x,t)$, for all $n = 0, 1, \ldots$ and the kernel k satisfies Lipschitz condition with constant M. Afterwards, the sequence of the approximate solutions $\{u_n\}$, $n = 0, 1, \ldots$ shows convergence to the solution that is exact u.

Proof:

From theorem (1), the approximate solution using the VIM is provided by:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n}{\partial \xi}(x,\xi) - g(x,\xi) - \int_0^x \int_0^\xi k(u_n(s,y)\, dy\, ds \right] d\xi \qquad \text{...(16)}$$

and since u is the exact solution of the equation (1), thus it satisfies VIM formula:

$$u(\mathrm{x},\mathrm{t}) = u(x,t) - \int_0^t \left[\frac{\partial u(x,\xi)}{\partial \xi} - g(x,\xi) - \int_0^x \int_0^\xi k(u_n(s,y)\, dy\, ds \right] d\xi \qquad \text{...(17)}$$

Subtract u from u_{n+1} and recall that $E_n(x,t) = u_n(x,t) - u(x,t)$,indicate:

$$u_{n+1}(x,t) - u(x,t) = u_n(x,t) - u(x,t) - \int_0^t \left[\frac{\partial u_n(\mathrm{x},\xi)}{\partial \xi} - \frac{\partial u(\mathrm{x},\xi)}{\partial \xi} - g(x,\xi) - g(x,\xi) - \int_0^x \int_a^y k(u_n(s,y)) - k(u(s,y))\, ds\, dy \right] d\xi \qquad \text{...(18)}$$

Thus:

$$E_{n+1}(x,t) = E_n(x,t) - \int_0^t \left[\frac{\partial E_n(\mathrm{x},\xi)}{\partial \xi} - \int_0^x \int_a^y k(u_n(s,y)) - k(u(s,y))\, ds\, dy \right] d\xi \qquad \text{...(19)}$$

$$= E_n(x,t) - E_n(x,t) - E_n(x,0) + \int_0^t \int_0^x \int_0^\xi \left[k(u_n(s,y)) - k(u(s,y)) \right] ds\, dy\, d\xi \qquad \text{...(20)}$$

$$= \int_0^t \int_0^x \int_0^\xi \left[k(u_n(s,y)) - k(u(s,y))\right] ds\, dy\, d\xi \text{, where } E_n(x,0) = 0 \qquad ...(21)$$

Taking the norm to the both equation sides (21), give:

$$\left\|E_{n+1}(x,t)\right\| = \left\|\int_0^t \int_0^x \int_0^\xi \left[k(u_n(s,y)) - k(u(s,y))\right] ds\, dy\, d\xi\right\| \qquad ...(22)$$

$$\leq \int_0^t \int_0^x \int_0^\xi \left\|\left[k(u_n(s,y)) - k(u(s,y))\right]\right\| dy\, ds\, d\xi$$

$$\leq M \int_0^t \int_0^x \int_0^\xi \left\|u_n(s,y) - u(s,y)\right\| dy\, ds\, d\xi$$

Therefore:

$$\left\|E_{n+1(x,t)}\right\| \leq M \int_0^t \int_0^x \int_0^\xi \left\|E_n(s,y)\right\| dy\, ds\, d\xi \text{, for all } n = 0, 1, ...$$

Now, if $n = 0$, then:

$$\left\|E_1(x,t)\right\| \leq M \int_0^t \int_0^x \int_0^\xi \left\|E_0(s,y)\right\| dy\, ds\, d\xi$$

$$= M \left\|E_0(s,y)\right\| \int_0^t \int_0^x \int_0^\xi dy\, ds\, d\xi$$

$$= M \left\|E_0(s,y)\right\| x \frac{t^2}{2!}$$

If $n = 1$, then:

$$\left\|E_2(x,t)\right\| \leq M \int_0^t \int_0^x \int_0^\xi \left\|E_1(s,y)\right\| ds\, dy\, d\xi$$

$$\leq M^2 \left\|E_0(s,y)\right\| \frac{x^2 t^4}{4}$$

If $n = 2$, and then:

$$\left\|E_3(x,t)\right\| \leq M \int_0^t \int_0^x \int_0^\xi \| E_2(s,y) \| dyds d\xi$$

$$\leq M^3 \left\|E_0(s,y)\right\| \frac{x^3 t^6}{6}$$

If $n = 3$, then:

$$\left\|E_4(x,t)\right\| \leq M \int_0^t \int_0^x \int_0^\xi \| E_3(s,y) \| dyds d\xi$$

$$\leq M^4 \left\|E_0(s,y)\right\| \frac{x^4 t^8}{12}$$

$$\vdots$$

$$\left\|E_n(x,t)\right\| \leq M^n \left\|E_0(x,t)\right\| \frac{x^n}{n!} \frac{t^{2n}}{(2n)!} \text{, } x \in [0,b],\ t \in [0,T]$$

therefore having the supermom value of x and t over [0,b] and [0,T] respectively to obtain

$$\|E_n(x,t)\| \le M^n \|E_0(x,t)\| \frac{b^n}{n!} \frac{T^{2n}}{(2n)!}$$

and as $n \longrightarrow \infty$ implies to $E_n \longrightarrow 0$, i.e., $u_n \longrightarrow u$, as $n \longrightarrow \infty$. ■

4. Illustrative Examples

In the present section, three examples that are illustrative are considered to examine the validity and illustrate the convergence of the variation iteration formula given by equation (8) for linear and nonlinear two-dimensional partial integro-differential equations.

Example (1):

Consider the linear partial integro-differential two-dimensional equation:

$$\frac{\partial u(x,t)}{\partial t} = x + \frac{tx(t-x)(tx+3)}{6} + \int_0^x \int_0^t (s-y)u(s,y)\,dyds\,,\ (x,t) \in [0,1]\times[0,1] \qquad ...(23)$$

with initial condition:

$u(x,0) = 1,\ 0 \le x \le 1$

For the purpose of comparison, the exact solution of equation (23) is provided by:

$u(x,t) = 1 + xt$

Hence iteration formula of equation (23) that is related and variational is provided by:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n}{\partial \xi}(x,\xi) - x - \frac{\xi x(\xi - x)(\xi x + 3)}{6} - \int_0^x \int_0^\xi (s-y)u_n(s,y)\,dyds \right] d\xi$$

and consider the initial approximation $u_0(x) = u(x,0) = 1$, then:

$$u_1(x,t) = \frac{t^4x^2}{24} - \frac{t^3x^3}{18} + tx + 1$$

$$u_2(x,t) = \frac{t^4x^2}{24} - \frac{t^3x^3}{18} + tx - \frac{t^3x^2(100t^4x - 245t^3x^2 + 168t^2x^3 + 12600t - 16800x)}{302400} + 1$$

$\vdots$

Table (1) presents the results that are numerical for the approximate and exact solutions u, u_1, u_2, u_3 and u_4 for different values of x and t between 0 and 1. While table (2) shows the absolute error between u and u_1, u_2, u_3, u_4, respectively.

Table (1)

Numerical results of the approximate and exact solutions of example (1)

x	*T*	*u(x,t)*	$u_1(x,t)$	$u_2(x,t)$	$u_3(x,t)$	$u_4(x,t)$
0	0	1	1	1	1	1
0	0.25	1	1	1	1	1
0	0.5	1	1	1	1	1
0	0.75	1	1	1	1	1
0	1	1	1	1	1	1
0.25	0	1	1	1	1	1

0.25	0.25	1.0625	1.062497	1.0625	1.0625	1.0625
0.25	0.5	1.125	1.124932	1.125	1.125	1.125
0.25	0.75	1.1875	1.187225	1.1875	1.1875	1.1875
0.25	1	1.25	1.249295	1.25	1.25	1.25
0.5	0	1	1	1	1	1
0.5	0.25	1.125	1.125054	1.125	1.125	1.125
0.5	0.5	1.25	1.249783	1.25	1.25	1.25
0.5	0.75	1.375	1.373535	1.374999	1.375	1.375
0.5	1	1.5	1.49566	1.499993	1.5	1.5
0.75	0	1	1	1	1	1
0.75	0.25	1.1875	1.187958	1.1875	1.1875	1.1875
0.75	0.5	1.375	1.375366	1.374999	1.375	1.375
0.75	0.75	1.5625	1.560028	1.562496	1.5625	1.5625
0.75	1	1.75	1.739746	1.749968	1.75	1.75
1	0	1	1	1	1	1
1	0.25	1.25	1.251736	1.249997	1.25	1.25
1	0.5	1.5	1.503472	1.499992	1.5	1.5
1	0.75	1.75	1.75	1.749985	1.75	1.75
1	1	2	1.986111	1.999924	2	2

Table (2)

The absolute error between the approximate and exact solutions of example (1)

x	*T*	$\|u(x,t) - u_1(x,t)\|$	$\|u(x,t) - u_2(x,t)\|$	$\|u(x,t) - u_3(x,t)\|$	$\|u(x,t) - u_4(x,t)\|$
0	**0**	**0**	**0**	**0**	**0**
0	**0.25**	**0**	**0**	**0**	**0**
0	**0.5**	**0**	**0**	**0**	**0**
0	**0.75**	**0**	**0**	**0**	**0**
0	**1**	**0**	**0**	**0**	**0**
0.25	**0**	**0**	**0**	**0**	**0**
0.25	**0.25**	**0.000003**	**0**	**0**	**0**
0.25	**0.5**	**0.000068**	**0**	**0**	**0**
0.25	**0.75**	**0.000275**	**0**	**0**	**0**
0.25	**1**	**0.000705**	**0**	**0**	**0**
0.5	**0**	**0**	**0**	**0**	**0**
0.5	**0.25**	**0.000054**	**0**	**0**	**0**
0.5	**0.5**	**0.000217**	**0**	**0**	**0**
0.5	**0.75**	**0.001465**	**0**	**0**	**0**
0.5	**1**	**0.00434**	**0**	**0**	**0**
0.75	**0**	**0**	**0**	**0**	**0**
0.75	**0.25**	**0.000458**	**0**	**0**	**0**
0.75	**0.5**	**0.000366**	**0**	**0**	**0**
0.75	**0.75**	**0.002472**	**0**	**0**	**0**
0.75	**1**	**0.010254**	**0**	**0**	**0**
1	**0**	**0**	**0**	**0**	**0**
1	**0.25**	**0.001736**	**0**	**0**	**0**
1	**0.5**	**0.003472**	**0**	**0**	**0**
1	**0.75**	**0**	**0**	**0**	**0**
1	**1**	**0.013889**	**0**	**0**	**0**

Example (2):

Consider the nonlinear partial two-dimensional integro-differential equation:

$$\frac{\partial u(x,t)}{\partial t} = x - \frac{t^2x^2(4tx+9)}{36} + \int_0^x\int_0^t \left[sy + u^2(s,y) \right] dyds\, ,\ (x,t) \in [0,1]\times[0,1] \qquad ...(24)$$

with initial condition:

$u(x,0) = 0,\ 0 \le x \le 1$

For the purpose of comparison, the equation exact solution (24) is provided by:

$u(x,t) = xt$

Thus the related the iteration variational formula of equation (24) is provided by:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n}{\partial \xi}(x,\xi) - x - \frac{\xi x(\xi - x)(\xi x+3)}{6} - \int_0^x\int_0^\xi \left[sy + u^2(s,y) \right] dyds \right] d\xi$$

and consider the approximation that is initial $u_0(x) = u(x,0) = 0$, then:

$$u_1(x,t) = \frac{x(t^3x^2 - 36)}{36} - \frac{t^3x^3}{12}$$

$$u_2(x,t) = \frac{t^4x^3}{36} - \frac{t^7x^5}{3780} + \frac{t^{10}x^7}{816480} - \frac{tx(t^3x^2-36)}{36}$$

$\vdots$

Table (3) shows results that numerical for the approximate and exact solutions u, u_1, u_2, u_3 and u_4 for different values of x and t between 0 and 1. While table (4) presents the error that is absolute between the exact solution u and solutions that are approximate u_1, u_2, u_3, u_4, respectively.

Table (3)

Numerical results of the approximate and exact solutions of example (2)

x	*T*	*u(x,t)*	*u₁(x,t)*	*u₂(x,t)*	*u₃(x,t)*	*u₄(x,t)*
0	0	0	0	0	0	0
0	0.25	0	0	0	0	0
0	0.5	0	0	0	0	0
0	0.75	0	0	0	0	0
0	1	0	0	0	0	0
0.25	0	0	0	0	0	0
0.25	0.25	0.0625	0.062498	0.0625	0.0625	0.0625
0.25	0.5	0.125	0.124986	0.125	0.125	0.125
0.25	0.75	0.1875	0.187454	0.1875	0.1875	0.1875
0.25	1	0.25	0.249891	0.25	0.25	0.25
0.5	0	0	0	0	0	0
0.5	0.25	0.125	0.124973	0.125	0.125	0.125
0.5	0.5	0.25	0.249783	0.25	0.25	0.25
0.5	0.75	0.375	0.374268	0.375	0.375	0.375
0.5	1	0.5	0.498264	0.499998	0.5	0.5
0.75	0	0	0	0	0	0
0.75	0.25	0.1875	0.187363	0.1875	0.1875	0.1875
0.75	0.5	0.375	0.373901	0.374999	0.375	0.375
0.75	0.75	0.5625	0.558792	0.562492	0.5625	0.5625
0.75	1	0.75	0.741211	0.749965	0.75	0.75
1	0	0	0	0	0	0

1	0.25	0.25	0.249566	0.25	0.25	0.25
1	0.5	0.5	0.496528	0.499992	0.5	0.5
1	0.75	0.75	0.738281	0.749937	0.75	0.75
1	1	1	0.972222	0.999737	0.999999	1

Table (4)
The absolute error between the approximate and exact solutions of example (2)

x	T	$\|u(x,t) - u_1(x,t)\|$	$\|u(x,t) - u_2(x,t)\|$	$\|u(x,t) - u_3(x,t)\|$	$\|u(x,t) - u_4(x,t)\|$
0	0	0	0	0	0
0	0.25	0	0	0	0
0	0.5	0	0	0	0
0	0.75	0	0	0	0
0	1	0	0	0	0
0.25	0	0	0	0	0
0.25	0.25	0.000002	0	0	0
0.25	0.5	0.000014	0	0	0
0.25	0.75	0.000046	0	0	0
0.25	1	0.000109	0.000002	0	0
0.5	0	0	0	0	0
0.5	0.25	0.000027	0	0	0
0.5	0.5	0.000217	0.000001	0	0
0.5	0.75	0.000732	0.000008	0	0
0.5	1	0.001736	00.000035	0	0
0.75	0	0	0	0	0
0.75	0.25	0.000137	0	0	0
0.75	0.5	0.001099	0.000001	0	0
0.75	0.75	0.003708	0.00008	0	0
0.75	1	0.008789	0.000035	0	0
1	0	0	0	0	0
1	0.25	0.000434	0	0	0
1	0.5	0.003472	0.000008	0	0
1	0.75	0.011719	0.000063	0	0
1	1	0.027778	0.000263	0.000001	0

5. Conclusions

This paper has two main goals. The first goal is to employ the variational iteration method to investigate nonlinear and linear two-dimensional equations that are Volterra integro-differential and partial as well as studying the convergence of this method. The second goal is to show significant features of this method and its power. The VIM gives convergent that is rapid, successive, and approximate without any restrictive transformation or assumptions that could change physical behaviour of the problem. Generally, the procedure of VIM solution is very straightforward, convenient, and effective. Numerical results and a comparison with the exact solution are provided, which reveal its efficiency.

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The Space of Strongly Prime Gamma Subacts

Mehdi S Abbas

Department of Mathematics, College of Education, Al-Zahraa University for women, Iraq

E-mail: mhdsabass@gmail.com

Abstract .

In this work we consider and study the structure space of gamma acts by considering strongly prime gamma subacts. Also we study compactness and connectedness properties of this space as well as the separation axioms.

Key words : gamma semigroups, gamma acts, (strongly) prime gamma subacts, Noetherian gamma acts, multiplication gamma act and uniserial gamma act.

1. Introduction

The Hausdorff property for the ring C(X) of continues real-valued functions on X has been studied by L. Gillman in [1]. C .W . Kohls in [2] studied the space of prime ideals of an arbitrary ring while S. Chattopadhyay and S. Kar introduced and studied the structure space of gamma semigroups [3].

In this work, we introduce and study the structure space of gamma acts. For this object, let M be an S_Γ-act, we consider the collection SP(M) of all strongly prime gamma subacts. By means of intersection and inclusion we define a closure operator on SP(M) and give a topology $\tau_{SP(M)}$ on SP(M). We call this topological space (SP(M), $\tau_{SP(M)}$) the structure space of the gamma act M. We discuss separation axioms in this space, also we consider the properties of connectedness and compactness.

2. Basic Concept .

Let S and Γ be nonempty sets. Recall that S is Γ-semigroup if $a\alpha b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all a, b, c $\in$ S and $\alpha, \beta \in \Gamma$. S is a Γ-semigroup with zero element if there is an element $0 \in S$ such that $0\alpha a = a\alpha 0 = 0$ for all $a \in S$ and $\alpha \in \Gamma$. A Γ-semigroup S is commutative if $a\alpha b = b\alpha a$ for all a, b $\in$ S, and $\alpha \in \Gamma$ [4].

Let S be a semigroup and A a nonempty set. If we have a mapping $\mu : S \times A \to A$, (s, a) $\mapsto$ sa = μ(sa) such that (st)a = s(ta) for all s, t $\in$ S and a $\in$ A, we call A is a left S-act and write ${}_S^l A_S$. [5]

The notion of gamma acts which is a generalization of acts as well as gamma semigroups has been introduced in [6].

2.1. Definition. If S is a Γ-semigroup . A nonempty set M is called a left gamma acts over S, denoted by ${}_{S_\Gamma}^{l}M$, if there is a mapping S × Γ × M → M , (s, α, m) ↦ sαm (s ∈ S, α Γ and m ∈ M) such that $s_1\alpha_1(s_2\alpha_2 m) = (s_1\alpha_1 s_2)\alpha_2 m$ for all s_1, s_2 ∈ S, α_1, α_2 ∈ Γ and m ∈ M.

2.2. Examples (2.2).

1. Let S = { 5n + 4 | n ∈ Z^+ }, Γ = { 5n + 1 | n ∈ Z^+ }. Then S is a Γ-semigroup where $s_1\alpha s_2 = s_1 + \alpha + s_2$ (usual addition of integers). Now, let M = { 5n | n ∈ Z^+ }. Then M is an S_Γ-act , but M is not Γ-semigroup with usual addition of integers.

2. Let M be the set of all negative rational numbers. It is clear that M is not M-act under usual multiplication of rtional numbers. Let Γ = { $-\frac{1}{p}$ | p is prime } and define the mapping M × Γ × M → M by (x, α, y) ↦ xαy (usual multiplication of rational numbers). It is an easy matter to see that M is M_Γ-act.

A nonempty subset N of S_Γ-act M is called S_Γ-subact, if SΓN ⊆ N where SΓN = { sαn | s ∈ S, α ∈ Γ and n ∈ N }. An S_Γ-subact N of an S_Γ-act M is proper if N ≠ M.

For S_Γ-acts M and N. A mapping f :M⟶N is called S_Γ-homomorphism if f(sαm) = sαf(m), for all s ∈ S, α ∈ Γ and m ∈ M. We denote Hom(M, N) the set of all S_Γ-homomorphisms from M into N.

2.3. Definition . Let N be an S_Γ-subact of an S_Γ-act M. Define (N $:_S$ M) ={ s ∈ S |s Γ M ⊆ N }. In particular, for m ∈ M (N $:_S$ m) ={ s ∈ S | s Γ m ⊆N }.

Recall that a nonempty subset I of a Γ-semigroup S is called ideal if IΓS ⊆ I and SΓI ⊆ I.

We introduce the following

2.4. Definition . Let M be an S_Γ-act. A proper S_Γ-subact P of M called prime if for any ideal I of S and any S_Γ-subact N of M, IΓN ⊆ P implies that N ⊆ P or I ⊆ (P$:_S$M).

In the following, the concept of prime gamma subacts can be reduces to elements

2.5. Proposition . Let P be a proper S_Γ-subact of an S_Γ-act M. Then P is prime if and only if sΓSΓm ⊆ P implies that m ∈ P or s ∈ (P $:_S$M) for all s∈S and m∈M .

Proof. Assume that sΓSΓm ⊆ P where s ∈ S and m ∈ M. Primerss of P implies that m ∈ P or s ∈ (P$:_S$M). Conversely, assume IΓV ⊆ P for an ideal I of S and S_Γ-subact V of M. If V ⊈ P, then there is an element x ∈ V and x ∉ P. Then for any a ∈ I we have aΓSΓx ⊆ IΓV ⊆P , thus a∈ (P$:_S$M).

Recall that a proper ideal T of Γ-semigroup S is prime if for any two ideals I and J of S , IΓJ ⊆T implies that I ⊆ T or J ⊆ T. Then we have the following corollary

2.6. Corollary . A proper ideal T of Γ-semigroup S is prime if and only if s_1ΓSΓs_2 ⊆T implies that s_1 ∈ T or s_2 ∈ T for all s_1, s_2 ∈ S.

2.7. Lemma. Let M be an S_Γ-act. If P is a prime S_Γ-subact of M. then $(P :_s M)$ is a prime ideal of S. **Proof.** Let $s_1, s_2 \in S$ with $s_1\Gamma S\Gamma s_2 \subseteq (P:_s M)$. Then $s_1\Gamma S\Gamma s_2\Gamma M \subseteq P$. Since P is prime, then by Proposition (2.6) we have either $s_2\Gamma M \subseteq P$ or $s_1\Gamma S\Gamma M \subseteq P$ and hence $s_2 \in (P :_s M)$ or $s_1 \in (P:_s M)$.

For the converse we consider the following

2.8. Definition. An S_Γ-act M is called multiplication if for any S_Γ-subact N of M, there is an ideal I of S such that $N = I\Gamma M$.

It is easy matter that an S_Γ-subact N of a multiplication s_Γ-act M is of the form $N = (N :_s M)\Gamma M$.

2.9. Theorem . If M is a multiplication S_Γ-act, then an s_Γ-subact P of M is prime if and only if $(P :_s M)$ is a prime ideal of S.

Proof. Assume that $(P :_s M)$ is a prime ideal of S , and there exist an ideal I of S and s_Γ-subact V of M with $V \nsubseteq P$, $I \nsubseteq (P:_s M)$ and $I\Gamma V \subseteq P$. Since M is multiplication, then $V = J\Gamma M$ for some ideal J of S. Thus $I\Gamma V = J\Gamma J\Gamma M$ so $I\Gamma J \subseteq (P:_s M)$, but $(P :_s M)$ is a prime ideal of S and $I \nsubseteq (P :_s M)$, then $J \subseteq (P :_s M)$. Therefore $V = J\Gamma M \subseteq P$ which is a contradiction . Thus P is prime .

It is easy matter to see that if I and J are two ideals of a Γ-semigroup S and P is a prime ideal of S with $I \cap J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$. This statement is no larger hold if we replace ideals of Γ-semigroup by s_Γ-subact S of s_Γ-act. However we have the following

2.10. Theorem . Let N be a prime s_Γ-subact of a multiplication s_Γ-act M. If N_1, N_2 are s_Γ-subacts of M with $N_1 \cap N_2 \subseteq N$, then either $N_1 \subseteq N$ or $N_2 \subseteq N$.

Proof. Since $(N_1 \cap N_2 :_s M) = (N_1 :_s M) \cap (N_2 :_s M) \subseteq (N :_s M)$ and $(N :_s M)$ is a prime ideal of S, then either $(N_1 :_s M) \subseteq (N :_s M)$ or $(N_2 :_s M) \subseteq (N :_s M)$. Thus either $N_1 = (N_1 :_s M)\Gamma M \subseteq (N :_s M)\Gamma M = N$ or $N_2 = (N_2 :_s M)\Gamma M \subseteq (N :_s M)\Gamma M = N$.

We introduce the following

2.11. Definition . An s_Γ-subact N of s_Γ-act M is called strongly prime (or finitely prime), if S and Γ contain finite subset $\bar{S}$ and $\bar{\Gamma}$ respectively such that $s\bar{\Gamma}\bar{S}\bar{\Gamma}m \subseteq N$ implies that $m \in N$ or $s \in (N:_s M)$ for all $s \in S$ and $m \in M$.

2.12. Proposition . Every strongly prime S_Γ-subact of S_Γ-act M is prime.

Proof. Let N be a strongly S_Γ-subact of s_Γ-act M. For $s \in S$ and $m \in M$, if $s\Gamma S\Gamma m \subseteq N$, then there are finite subsets $\bar{S}$ and $\bar{\Gamma}$ of S and Γ respectively and $s\bar{\Gamma}\bar{S}\bar{\Gamma}m \subseteq s\Gamma S\Gamma m \subseteq N$. This implies that $m \in N$ or $s \in (N:_s M)$.

In the following consider intersection of (strongly) prime gamma subacts.

2.13. Proposition. Let $\{N_\alpha : \alpha \in \Lambda\}$ be a collection of prime S_Γ-subacts of an S_Γ-act M such that $\{N_\alpha : \alpha \in \Lambda\}$ forms a chain . Then $\cap_{\alpha\in\Lambda} N_\alpha$ is a prime S_Γ-subact of M.

Proof: For any ideal I of S and s_Γ-subact V of M, if $I\Gamma V \subseteq \cap_{\alpha\in\Lambda} N_\alpha$ with $I \nsubseteq (\cap N_\alpha :_s M)$ and $V \nsubseteq \cap N_\alpha$, then there are $\alpha, \beta \in \Lambda$ such that $I\Gamma M \nsubseteq N_\alpha$ and $V \nsubseteq N_\alpha$. No loss of generality if we assume $N_\alpha \subseteq N_\beta$. This implies that $V \nsubseteq N_\beta$ a contradiction. Thus $\cap_{\alpha\in\Lambda} N_\alpha$ is a prime S_Γ-subact of M.

A s_Γ-act M is called uniserial, if for any two S_Γ-subact N and K of M , either $N \subseteq K$ or $K \subseteq N$

2.14. Corollary . Let M be a uniserial S_Γ-act. If $\{ N_\alpha \mid \alpha \in \wedge \}$ is a family of (strongly) prime S_Γ-subact of M, then $\bigcap_{\alpha \in \wedge} N_\alpha$ is (strongly) prime in M.

3. Structure space of S_Γ-acts.

Let M be an S_Γ-act. Denote by SP(M) the collection of all strongly prime S_Γ-subacts of M. For any N $\subseteq$ SP(M), we define $\overline{N} = \{ K \in SP(M) \mid \bigcap_{K_\alpha \in N} K_\alpha \subseteq K \}$ it is clear that $\overline{\emptyset} = \emptyset$ and $N \subseteq \overline{N}$ for any subset N of SP(M).

3.1. Theorem . For any two subsets N and L of SP(M), the following hold

(1) $\overline{\overline{N}} = \overline{N}$

(2) $N \subseteq L$ implies that $\overline{N} \subseteq \overline{L}$

(3) if M is a multiplication S_Γ-act, then $\overline{N \cup L} = \overline{N} \cup \overline{L}$.

Proof . (1). It is clear that $\overline{N} \subseteq \overline{\overline{N}}$. For other inclusion, let $k_\beta \in \overline{\overline{N}}$. Then $\bigcap_{K_\alpha \in \overline{N}} K_\alpha \subseteq K_\beta$, and $K_\alpha \in \overline{N}$ implies that $\bigcup_{K_\gamma \in N} K_\gamma \subseteq K_\alpha$ for all $\alpha \in \wedge$. Thus $\bigcap_{K_\gamma \in N} K_\gamma \subseteq \bigcap_{K_\gamma \in \overline{N}} K_\alpha \subseteq K_\beta$ that is $\bigcup_{K_\gamma \in N} K_\gamma \subseteq K_\beta$ and so $K_\beta \in \overline{N}$ hence $\overline{\overline{N}} = \overline{N}$.

(2). Suppose $N \subseteq L$ and $K_\alpha \in \overline{N}$. Then $\bigcap_{K_\beta \in N} K_\beta \subseteq K_\alpha$. Since $N \subseteq L$, then $\bigcap_{K_\beta \in L} K_\beta \subseteq \bigcap_{K_\beta \in N} K_\beta \subseteq K_\alpha$ and this implies that $K_\alpha \in L$ and hence $\overline{N} \subseteq \overline{L}$.

(3). Clearly by (2) $\overline{N} \cup \overline{L} \subseteq \overline{N \cup L}$. Let $K_\alpha \in \overline{N \cup L}$. Then $\bigcap_{K_\beta \in N \cup L} K_\beta \subseteq K_\alpha$. It is easy to see that $\bigcap_{K_\beta \in N \cup L} K_\beta = (\bigcap_{K_\beta \in N} K_\beta) \cap (\bigcap_{K_\beta \in L} K_\beta) \subseteq K_\alpha$. Since K_α is strongly prime for each α, then K_α is prime , Proposition (2.14). By multiplication property of M and Proposition (1.12), we have $\bigcap_{K_\beta \in N} K_\beta \subseteq K_\alpha$ or $\bigcap_{K_\beta \in L} K_\beta \subseteq K_\alpha$, this is $K_\alpha \in \overline{N}$ or $K_\alpha \in \overline{L}$ and hence $\overline{N \cup L} = \overline{N} \cup \overline{L}$.

3.2. Definition . Let M be a multiplication S_Γ-act. The closure operator $N \to \overline{N}$ gives a topology $\tau_{SP(M)}$ on SP(M). This topology is called the strongly prime topology and the topology space ($\tau_{SP(M)}$, SP(M)) is called the structure space of the S_Γ-act M.

For S_Γ-subact N of an S_Γ-act M. We define $\Delta(N) = \{ N' \in SP(M) \mid N \subseteq N' \}$ and $C\Delta(N) = SP(M) \setminus \Delta(N)$. In the following we describe the closed set in SP(M)

3.3. Proposition . Let M be a multiplication S_Γ-act. Then for any closed set $\overline{W}$ in SP(M), there is an S_Γ-subact N of M such that $\overline{W} = \Delta(N)$.

Proof. Let $\overline{W}$ be a closed subset in SP(M) where $W \subseteq SP(M)$. Then $W = \{ N_\alpha \notin SP(M) \mid \alpha \in \wedge \}$. Let $N = \bigcap_{N_\alpha \in W} N_\alpha$. Then N is an S_Γ-subact of M if $N' \in \overline{W}$, then $\bigcap_{N_\alpha \in W} N_\alpha \subseteq N'$. This implies that $N \subseteq N'$ and hence $N' \in \Delta(N)$ so $\overline{W} \subseteq \Delta(N)$. Conversely, let $N' \in \Delta(N)$. Then $N \subseteq N'$, that is $\bigcap_{N_\alpha \in W} N_\alpha \subseteq N'$, this implies that $N' \subseteq \overline{W}$ and hence $\Delta(N) \subseteq \overline{W}$.

3.4. Corollary (3.4). Any open set in SP(M) is of the form $C\Delta(N)$ for some S_Γ-subact N of multiplication S_Γ-act M.

Let M be an S_Γ-subact and $m \in M$. We define $\Delta(m) = \{ N \in SP(M) \mid m \in N \}$ and $C\Delta(m) = SP(M)/\Delta(m)$

3.5. Proposition . If M is a multiplication S_Γ-act. Then $\{ C\Delta(m) \mid m \in M \}$ forms an open base for the topology $\tau_{SP(M)}$ on SP(M).

Proof. Let $U \in \tau_{SP(M)}$. Then by Corollary (3.4), there is an S_Γ-subact N of M such that $U = C\Delta(N)$. Let $K \in U$ Then $N \nsubseteq K$ and there is $x \in N$ with $x \notin K$. Thus $K \in \Delta C(X)$. To see $C\Delta(m) \subseteq U$. Let $K \in C\Delta(m)$. Then $m \notin K$. It follows that $N \nsubseteq K$ and hence $K \in U$ and so $C\Delta(m) \subseteq U$. Thus $\{ C\Delta(m) \mid m \in M \}$ is an open base for $\tau_{SP(M)}$. □

3.6. Theorem . The space $(SP(M), \tau_{SP(M)})$ is T_0-space for any multiplication S_Γ-act M **.**

Proof. Suppose N_1 and N_2 are two distinct elements in SP(M). Without loss of generality , we assume that there is an element $x \in N_1$, and $x \notin N_2$. Then $C\Delta(x)$ is a neighborhood of N_2 not contain N_1..

3.7. Theorem . The following statements are equivalent for a multiplication S_Γ-act M

(1) $(SP(M), \tau_{SP(M)})$ is T_1-space

(2) No element of SP(M) is contained in any other element of SP(M).

Proof. (1) → (2). Suppose $(SP(M), \tau_{SP(M)})$ is a T_1-space and N_1, N_2 be distance elements of SP(M). Then each of N_1, and N_2 has a neighborhood not containing the other . Since N_1 and N_2 are any elements . This implies that no element of SP(M) is containing in any other element of SP(M).

(2) → (1), assume that no element of SP(M) is contained in any other element of SP(M). Let N_1 and N_2 be two different elements of SP(M). Then by hypothesis , there exist x , y ∈ M with $x \in N_1 \setminus N_2$ and $y \in N_2 \setminus N_1$. Thus, we have $N_1 \subseteq C\Delta(y)$ but $N_1 \notin C\Delta(x)$ and $N_2 \in C\Delta(x)$, but $N_1 \notin C\Delta(y)$.Thus each of N_1 and N_2 has a neighborhood no containing the other. Hence $(SP(M), \tau_{SP(M)})$ is a T_1-sapce. □

3.8. Corollary. Let S be a commutative Γ-semigroup and M a multiplication S_Γ-act. If Max(M) is the class of maximal S_Γ-subacts of M , then (Max(M), $\tau_{Max(M)}$) is a T_1-space where $\tau_{Max(M)}$ is the induced topology on Max(M) from $(SP(M), \tau_{SP(M)})$.

3.9. Theorem . If M is a multiplication S_Γ-act. Then the following conditions are equivalent

(1) $(SP(M), \tau_{SP(M)})$ is a Hausdorff

(2) Any two distinct elements N and K of SP(M), there exist x, y ∈ M such that $x \notin N$, $y \notin K$ and does not exist any W ∈ SP(M) such that $x, y \notin W$.

Proof. (1) → (2). Assume that $(SP(M), \tau_{SP(M)})$ is a Hausdorff space . Then for any two distinct N_1 and N_2 of SP(M), there is an open set CΔ(x) and CΔ(y) such that $N_1 \in C\Delta(x)$, $N_2 \in C\Delta(y)$ and $C\Delta(x) \setminus C\Delta(y) = \emptyset$. This implies that $x \notin N1$ and $y \notin N2$. If there is K ∈ SP(M) such that $x \notin K$, $y \notin K$. Then $K \in C\Delta(x) \cap C\Delta(y) = \emptyset$ a contradiction. Thus there does not exist any K ∈ SP(M) with $x \notin K$ and $y \notin K$. (2) → (1). Assume the given condition holds and $N_1, N_2 \in SP(M)$ with $N_1 \neq N_2$. Let a, b ∈ M with $a \notin N_1$ with $b \notin N_2$ and there does not exist any K ∈ SP(M) such that $a \notin K$, $b \notin K$. This exactly implies $N_1 \in C\Delta(x)$, $N2 \in C\Delta(y)$ and $C\Delta(x) \cap C\Delta(y) = \emptyset$ and hence $(SP(M), \tau_{SP(M)})$ is a Hausdorff space.

3.10. Proposition Let M be a multiplication S_Γ-act and $(SP(M), \tau_{SP(M)})$ is a Hausdorff . Then

(1) No proper strongly prime S_Γ-subact of M contains any other proper strongly prime S_Γ-subact

(2) If $(SP(M), \tau_{SP(M)})$ contains more than one element , then there exist x, y ∈ M where $SP(M) = C\Delta(x) \cup C\Delta(y) \cup \Delta(W)$, where W is the S_Γ-subact of M generating by x and y.

Proof. (1).It's clear by Theorem (3.7) and the fact that every Hausdorff space is a T_1-space.

(2). Let N and K be two distinct strongly prime S_Γ-subacts of M. Then exists an open set CΔ(x) and CΔ(y) such that $N \in C\Delta(x)$, $K \in C\Delta(y)$ and $C\Delta(x) \cap C\Delta(y) = \emptyset$. Suppose W is the S_Γ-subact of M generating by x and y, namely W is the smallest S_Γ-subact of M containing x and y and $W = S\Gamma x \cup S\Gamma y$. Let L ∈SP(M). Then we have the following cases. (1) x , y ∈ L, (2). x ∈ L , $y \notin L$, (3). $x \notin L$, y ∈ L and (4). $x \notin L$, $y \notin L$. Case (4) is not possible since $C\Delta(x) \cap C\Delta(y) = \emptyset$, case (2) implies that $L \in C\Delta(y)$, similarity case (3) implies that $L \in C\Delta(x)$ and finally case (1) implies that L ∈

$\Delta(W)$ and thus $PS(M) \subseteq C\Delta(x) \cup C\Delta(y) \cup \Delta(W)$.

□

3.11. Theorem . The following conditions are equivalent for a multiplication S_Γ-act M.

(1) $(SP(M), \tau_{SP(M)})$ is a regular space

(2) For $N \in SP(M)$ and $x \in M \setminus N$, there exist an S_Γ-subact K of M and $y \in M$ such that $N \in C\Delta(y) \subseteq \Delta(K) \subseteq C\Delta(x)$.

Proof. (1) → (2). Let $N \in SP(M)$ and $x \in M \setminus N$. Then $N \in C\Delta(x)$ and $SP(M)\setminus C\Delta(x)$ is closed set not containing N. By (1) there is disjoint open sets U and V such that $N \in U$ and $SP(M) \setminus C\Delta(x) \subseteq V$. This implies that $SP(M) \setminus V \subseteq C\Delta(x)$. Since $SP(M) \setminus V$ is closed, then by Proposition (3.3), there is an S_Γ-subact K of M such that $SP(M) \setminus V = C\Delta(K)$ and hence we get $\Delta(K) \subseteq C\Delta(x)$. Since $U \cap V = \emptyset$, then $V \subseteq SP(M) \setminus U$. Again since $SP(M) \setminus U$ is closed, then there exists an S_Γ-subact W of M such that $SP(M)\setminus U = \Delta(W)$, this is $V \subseteq \Delta(W)$. Since $N \in U$, then $N \notin SP(M) \setminus U = \Delta(W)$. It follows that $W \nsubseteq N$, and hence there is $y \in W \setminus N$ so $N \in C\Delta(y)$. Now we show that $V \subseteq \Delta(y)$. Let $L \in V \subseteq \Delta(W)$. Then $W \subseteq L$. Since $y \in W$, then $y \in L$ and hence $L \in \Delta(y)$, so $V \subseteq \Delta(y)$ this implies that $SP(M)\setminus\Delta(y) \subseteq SP(M)\setminus V = \Delta(K)$ and hence $C\Delta(y) \subseteq \Delta(K)$. This shows that $N \in C\Delta(y) \subseteq \Delta(K)\ C\Delta(x)$.
(2) → (1). Let $I \in SP(M)$ and $\Delta(K)$ be any closed set not containing I . Since $I \notin \Delta(K)$, we have $K \nsubseteq I$. Then there is an element $a \in K\setminus I$. By (2), there is an S_Γ-subact J of M and $b \in M$ such that $I \in C\Delta(b) \subseteq \Delta(J) \subseteq C\Delta(a)$. Since $a \in K$ and $C\Delta(a) \cap \Delta(K) = \emptyset$, it follows that $\Delta(K) \subseteq SP(M) \setminus C\Delta(a) \subseteq SP(M) \setminus \Delta(J)$. Since $\Delta(J)$ is closed , then $SP(M) \setminus \Delta(J)$ is an open set containing the closed $\Delta(K)$. Clearly $C\Delta(b) \cap (SP(M)\setminus\Delta(J)) = \emptyset$, so we find that $C\Delta(b)$ and $SP(M) \setminus \Delta(J)$ are two disjoint open sets containing I and $\Delta(K)$ respectively . This shows that $(SP(M), \tau_{SP(M)})$ is a regular space. □

3.12. Theorem . Let M be a multiplication S_Γ-act. Then the following are equivalent

(1) $(SP(M), \tau_{SP(M)})$ is a compact space

(2) For any set $\{ x_\alpha \in M \mid \alpha \in \wedge \}$ there is a finite subset $\{ x_i \mid i = 1, 2, \ldots, n \}$ such that for any $N \in SP(M)$, there exists x_i such that $x_i \notin N$.

Proof. (1) → (2). Let $\{ x_\alpha \in M \mid \alpha \in \wedge \}$ and N be any element in SP(M). Then $\{ C\Delta(x_\alpha) \mid x_\alpha \in M, \alpha \in \wedge \}$ is an open cover of $(SP(M), \tau_{SP(M)})$. By (1) $SP(M), \tau_{SP(M)})$ has a finite sub cover $\{ C\Delta(x_i) \mid i = 1, 2, \ldots, n \}$ and hence $N \in C\Delta(x_i)$ for some $x_i \in M$. This implies that $x_i \notin N$.
(2) → (1). Assume that $\{ C\Delta(x_\alpha) \mid x_\alpha \in M, \alpha \in \wedge \}$ is an open cover of SP(M) which has no finite sub cover $\{ C\Delta(x_i) \mid i = 1, 2, \ldots, n \}$ of SP(M). This means that for any finite subset $\{ x_1, x_2, \ldots, x_n \}$ of M, $C\Delta(X_1) \cup C\Delta(X_2) \cup \ldots \cup C\Delta(X_n) \neq SP(M)$ and have $\Delta(x_1) \cap \Delta(x_2) \cap \ldots \cap \Delta(x_n) \neq \emptyset$. Then there is $N \in SP(M)$ such that $N \in \Delta(x_1) \cap \Delta(x_2) \cap \ldots \cap \Delta(x_n)$ .Thus, $x_1, x_2, \ldots, x_n \in N$ which is contradicts (2). This shows that $(SP(M), \tau_{SP(M)})$ is a compact space. □

An S_Γ-act M is called finitely generated if there exists a finite subset X of M such that $M = < X > = \bigcup_{u \in X} S\Gamma u$ where $S\Gamma u = \{ s\alpha u \mid s \in S \text{ and } \alpha \in \Gamma \}$.

3.13. Corollary . If M is a finitely generating multiplication S_Γ-act. Then $(SP(M), \tau_{SP(M)})$ is a compact space **.**

Proof. Let $\{ u_i \mid i = i = 1, 2, \ldots, n \}$ be a generated set of M, and N a strongly prime S_Γ-subact of M. Then there exists some u_i such that $u_i \notin N$. Hence by Theorem (2.13), $(SP(M), \tau_{SP(M)})$ is a compact space **.** □

An S_Γ-act M is called Noetherian if any ascending chain $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ of S_Γ-subacts of M, there is a positive integer n such that $N_m = N_n$ for $m \geqslant n$.

3.14. Theorem . If M is a Noetherian S_Γ-act. Then $(SP(M), \tau_{SP(M)})$ is countably compact.

Proof. Let $\{ \Delta(N_i) \mid i = 1, 2, .., \infty \}$ be a countable collection of closed set in SP(M) with finite intersection property where N_i is an S_Γ-subact of M for each i. Consider the following ascending chain $N_1 \subseteq N_1 \cup N_2 \subseteq N_1 \cup N_2 \cup N_3 \subseteq \ldots$ of S_Γ-subacts of M. Then there is a positive integer n such that $N_1 \cup N_2 \cup \ldots \cup N_n = N_1 \cup N_2 \cup \ldots \cup N_{n+1}$. Thus it follows that $N_1 \cup N_2 \cup \ldots \cup N_n \in \bigcap_{i=1}^{\infty} \Delta(N_i)$. Consequently $\bigcap_{i=1}^{\infty} \Delta(N_i) \neq \emptyset$ and hence (SP(M), $\tau_{SP(M)}$) is countably compact.

□

The following follows from Theorem (3.14) and the fact that a second countable space is compact if it is countably compact.

3.15. Corollary . If M is a Noetherian S_Γ-act and (SP(M), $\tau_{SP(M)}$) is second countable, then it is compact.

3.16. Definition . The structure space (SP(M), $\tau_{SP(M)}$) is called irreducible if for any decomposition SP(M) = $A_1 \cup A_2$ where A_1 and A_2 are closed subsets of SP(M) we have SP(M) = A_1 or SP(M) = A_2 .

3.17. Theorem . Let M be a multiplication S_Γ-act. Then the following statements are equivalent for any closed subset A of SP(M).

(1) A is irreducible

(2) $\bigcap_{N_\alpha \in A} N_\alpha$ is a prime S_Γ-subact of M.

Proof. (1) → (2). Let I ba an ideal of S and V an S_Γ-subact of M with $I\Gamma V \subseteq \bigcap_{N_\alpha \in A} N_\alpha$. Then $I\Gamma V \subseteq N_\alpha$ for each α. Since N_α is a prime, then either $V \subseteq N_\alpha$ or $I\Gamma M \subseteq N_\alpha$ which implies that for $N_\alpha \in$ A, either $N_\alpha \in \{ \overline{V} \}$ or $N_\alpha \in \{ \overline{I\Gamma M} \}$. Hence $A = (A \cap \overline{V}) \cup (A \cap \overline{I\Gamma M})$, since A is irreducible and both $A \cap \overline{V}$ and $A \cap \overline{I\Gamma M}$ are closed. Then it follows that either $A = A \cap \overline{V}$ or $A = A \cap \overline{I\Gamma M}$ and hence $A \subseteq \overline{V}$ or $A \subseteq \overline{I\Gamma M}$. This implies that $V \subseteq \bigcap_{N_\alpha \in A} N_\alpha$ or $I\Gamma M \subseteq \bigcap_{N_\alpha \in \wedge} N_\alpha$ and so $\bigcap_{N_\alpha \in \wedge} N_\alpha$ is a prime in M.

(2) → (1). Assume $A = A_1 \cup A_2$ where A_1 and A_2 are closed of A. Then $\bigcap_{N_\alpha \in A} N_\alpha \subseteq \bigcap_{N_\alpha \in A_1} N_\alpha$ and $\bigcap_{N_\alpha \in A} N_\alpha \subseteq \bigcap_{N_\alpha \in A_2} N_\alpha$. Also $\bigcap_{N_\alpha \in A} N_\alpha = \bigcap_{N_\alpha \in A_1 \cup A_2} N_\alpha = (\bigcap_{N_\alpha \in A_1} N_\alpha) \cap (\bigcap_{N_\alpha \in A_1} N_\alpha)$. For each ideal I of S, $I\Gamma(\bigcap_{N_\alpha \in A_1} N_\alpha) \subseteq \bigcap_{N_\alpha \in A_1} N_\alpha$ and $I\Gamma(\bigcap_{N_\alpha \in A_2} N_\alpha) \subseteq \bigcap_{N_\alpha \in A_2} N_\alpha$ so $I\Gamma(\bigcap_{N_\alpha \in A_1} N_\alpha) \subseteq (\bigcap_{N_\alpha \in A_1} N_\alpha) \cap (\bigcap_{N_\alpha \in A_2} N_\alpha) = \bigcap_{N_\alpha \in A} N_\alpha$. Since $\bigcap_{N_\alpha \in A} N_\alpha$ is prime it follows that $\bigcap_{N_\alpha \in A_1} N_\alpha \subseteq \bigcap_{N_\alpha \in \wedge} N_\alpha$ or $I\Gamma M \subseteq \bigcap_{N_\alpha \in A} N_\alpha$ and hence $\bigcap_{N_\alpha \in A_1} N_\alpha = \bigcap_{N_\alpha \in A} N_\alpha$ and $I\Gamma M \subseteq \bigcap_{N_\alpha \in A} N_\alpha$ similarly $\bigcap_{N_\alpha \in A_2} N_\alpha = \bigcap_{N_\alpha \in A} N_\alpha$ and $I\Gamma M \subseteq \bigcap_{N_\alpha \in A} N_\alpha$. It follows that $\bigcap_{N_\alpha \in A_1} N_\alpha = \bigcap_{N_\alpha \in \wedge} N_\alpha$ and $\bigcap_{N_\alpha \in A_2} N_\alpha = \bigcap_{N_\alpha \in \wedge} N_\alpha$. Let $N_\beta \in A$. Then we have $\bigcap_{N_\alpha \in A_1} N_\alpha \subseteq N_\beta$ or $\bigcap_{N_\alpha \in A_2} N_\alpha \subseteq N_\beta$. Since $A_1, A_2 \subseteq A$, so either $N_\alpha \subseteq N_\beta$ for all $N_\alpha \in A_1$ or $N_\alpha \subseteq N_\beta$ for all $N_\alpha \in A_2$. Thus $N_\alpha \in \overline{A_1} = A_1$ or $N_\beta \in \overline{A_2} = A_2$, since A_1 and A_2 are closed i.e $A = A_1$ or $A = A_2$. This proves (1). □

3.18. Corollary . Let M be a uniserial multiplication S_Γ-act. Then any closed subset of SP(M) is irreducible.

Proof. Let A be a closed subset of SP(M). Then by Corollary (2.14) we have $\bigcap_{N_\alpha \in A} N_\alpha$ is a prime S_Γ-subact of M. Hence by Theorem (3.16) we get A is irreducible.

□

Let M be an S_Γ-act and N, K two S_Γ-subacts of M. We define NK= Hom(M, K)N = ∪ $\{\alpha(N) \mid \alpha : M \to K \}$.

An S_Γ-subact N of M is called idempotent if N = NN = ∪(N) where the union runs among all S_Γ-homomorphism $\varphi : M \to N$. this is equivalent to saying that for each $n \in N$, there exist an S_Γ-homomorphism $\varphi : M \to N$ and an element $n' \in N$ such that $n = \varphi(n')$. An element $m \in M$ is called idempotent if it generates an idempotent S_Γ-subact of M, namely $S\Gamma m$ is idempotent S_Γ-subact of M. We denote e(M) for the set of all idempotent elements of M.

3.19. Definition . An S_Γ-subact N of an S_Γ-act M is called id-full if e(M) ⊆ N.

Let W be the collection of all strongly prime id-full S_Γ-subacts of an S_Γ-act M. Then clearly W ⊆ SP(M) and hence (W, τ_W) is a topological space where τ_W is the subspace topology generally (SP(M), $\tau_{SP(M)}$) is neither compact nor connected. But in particular we have the following results.

3.20. Proposition . Let M be a uniserial multiplication S_Γ-act. Then every closed subset of SP(M) is connected.

Proof. Let A be a closed subset of SP(M). By Theorem (3.17). A is irreducible. Hence A is connected.

3.21. Theorem . Let M be a multiplication S_Γ-act . Then (W, τ_W) is a connected space.

Proof. Let N be the strongly prime S_Γ-subact of M generated by e(M). Since every strongly prime id-full K of M conteuns e(M), contains N. Thus N belongs to any closed subset Δ(N') of W. This implies that any two closed subsets of W are not disjoint. Hence (W, τ_W) is a connected space. □

3.22. Theorem . Let M be a multiplication S_Γ-act. Then (W, τ_W) is a compact space.

Proof. Let { $\Delta(N_\alpha \mid \alpha \in \wedge$ } be any collection of closed subsets on W with finite intersection property, and N be the strongly prime S_Γ-subact generated by e(M). Since any strongly prime id-full S_Γ-subact K contains e(M), contains N. Hence N ∈ $\bigcap_{\alpha \in \wedge} \Delta(N_\alpha)$ and so $\bigcap_{\alpha \in \wedge} \Delta(N_\alpha)$ is nonempty. This implies that (W, τ_W) is a compact space.

□

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On $\mathfrak{I}\mathfrak{R}$ - rings and S $\mathfrak{I}$ F-rings

Raida D.Mahmood   Khedher J.khider
raida.1961@uomosul.edu.iq   khedirjumaa@gmail.com

College of Computer Science and Mathematics

University of Mosul

Mosul . Iraq

Abstract

A ring $\mathfrak{R}$ is said to be $\mathfrak{I}\mathfrak{R}$ - rings , if $\sigma \in \sigma\mathfrak{R}\sigma$ for all $\sigma \in \mathfrak{I}(\mathfrak{R})$ and $\mathfrak{R}$ is called right (left) S$\mathfrak{I}$F-ring , if every simple right (left) $\mathfrak{R}$-module is $\mathfrak{I}$-flat . In this paper , we give some characterization of $\mathfrak{I}\mathfrak{R}$ - rings and S $\mathfrak{I}$ F-rings . Further , it is shown that $\mathfrak{R}$ is $\mathfrak{I}\mathfrak{R}$ - ring if and only if , $\mathfrak{R}$ is S $\mathfrak{I}$ F-ring , with $\ell(\sigma) \subseteq r(\sigma)$ for every $\sigma \in \mathfrak{I}(\mathfrak{R})$ if and only if $\mathfrak{R}$ is $\aleph\mathfrak{I}$ with every essential right ideal is $\mathfrak{I}$-flat . Additionally , we have investigated $\mathfrak{I}\mathfrak{R}$ - rings with simple singular right $\mathfrak{R}$ - modules are $\mathfrak{I}$-flat .

Key words : $\mathfrak{I}\mathfrak{R}$ - rings , $\mathfrak{I}$-flat , $\aleph\mathfrak{I}$-rings , reduced rings .

حول الحلقات من النمط - $\mathfrak{I}\mathfrak{R}$ والحلقات من النمط - S $\mathfrak{I}$ F

أ. د. رائدة داؤد محمود   خضر جمعه خدر

كلية علوم الحاسوب والرياضيات

جامعة الموصل

العراق – الموصل

المستخلص

يقال للحلقة $\mathfrak{R}$ بأنها حلقة من النمط - $\mathfrak{I}\mathfrak{R}$ ، اذا كان $\sigma \in \sigma\mathfrak{R}\sigma$ لكل $\sigma \in \mathfrak{I}(\mathfrak{R})$. ويقال عن $\mathfrak{R}$ بأنها حلقة من النمط - S $\mathfrak{I}$ F ، إذا كان كل مقاس ايمن (ايسر) بسيط في $\mathfrak{R}$ هو مقاس مسطح من النمط - $\mathfrak{I}$. في هذا البحث سوف نعطي بعض مميزات الحلقات من النمط - S $\mathfrak{I}$ F والحلقات من النمط - $\mathfrak{I}\mathfrak{R}$. كذلك بيَنَّا ان $\mathfrak{R}$ حلقة من النمط - $\mathfrak{I}\mathfrak{R}$ اذا وفقط اذا كانت $\mathfrak{R}$ حلقة من النمط - S $\mathfrak{I}$ F و $\ell(\sigma) \subseteq r(\sigma)$ لكل $\sigma \in \mathfrak{I}(\mathfrak{R})$ و اذا وفقط اذا كانت $\mathfrak{R}$ حلقة من النمط - $\aleph\mathfrak{I}$ وكل مثالي ايمن اساسي من $\mathfrak{R}$ هو مسطح من النمط - $\mathfrak{I}$. كذلك سوف نختبر الحلقات من النمط - $\mathfrak{I}\mathfrak{R}$ عندما يكون كل مقاس منفرد بسيط ايمن مسطح من النمط - $\mathfrak{I}$

الكلمات المفتاحية : الحلقات من النمط - $\mathfrak{IR}$ ، المقاسات المسطحة من النمط - $\mathfrak{I}$ ، الحلقات من النمط - $\mathfrak{I}\aleph$ ، الحلقات المختزلة .

1 – المقدمة :

في هذا البحث ، نفترض أن $\mathfrak{R}$ حلقة تجميعية ذات عنصر محايد وكل المقاسات هي مقاسات احادية يمنى . لكل $\sigma \in \mathfrak{R}$ ، $r(\sigma)$ ($\ell(\sigma)$) يرمز للتالف الايمن (الايسر) للعنصر σ في الحلقة $\mathfrak{R}$ ، ورمزنا $\aleph(\mathfrak{R})$ و $\mathfrak{I}(\mathfrak{R})$ لمجموعة العناصر المعدومة القوى وجذر جاكوبسون في $\mathfrak{R}$ ، على التوالي .

الحلقات المسطحة درست من قبل عدة باحثين (مثل [3] ، [5] ، [7] ، [8]) . وكتعميم للحلقات المسطحة درس في المصدر [4] الحلقات من النمط - $\mathrm{S}\mathfrak{I}\mathrm{F}$. الحلقة $\mathfrak{R}$ تسمى حلقة من النمط - $\mathrm{S}\mathfrak{I}\mathrm{F}$ يمنى (يسرى) ، إذا كان كل مقاس ايمن (ايسر) بسيط في $\mathfrak{R}$ هو مقاس مسطح من النمط - $\mathfrak{I}$. كذلك اعطينا تعريف الحلقات من النمط - $\mathrm{SS}\mathfrak{I}\mathrm{F}$ وعلاقتها مع الحلقات من النمط - $\mathfrak{IR}$.

يقال لـ $\mathfrak{R}$ جلقة مختزلة اذ كانت $\aleph(\mathfrak{R}) = 0$ [8] . يقال للحلقة $\mathfrak{R}$ حلقة من النمط - $\mathfrak{I}\aleph$ اذا كانت $\aleph(\mathfrak{R}) \subseteq \mathfrak{I}(\mathfrak{R})$ [1] . في المصدر [6] اعطى الباحث الحلقة المختزلة المركزية من النمط – 2 ، اذا كان لكل $\sigma \in \mathfrak{R}$ و $\sigma^2 = 0$ ، فإن σ هو عنصر مركزي واثبت ان كل حلقة مختزلة تكون مختزلة مركزية من النمط – 2 ، لكن العكس يكون صحيح عندما تكون $\mathfrak{R}$ حلقة شبه اولية (إذا كانت $\mathfrak{R}$ لا تحتوي على مثالي غير صفري معدوم القوى) . ويقال لـ $\mathfrak{R}$ حلقة من النمط - $\mathfrak{IR}$ [10] ، اذا كان $\sigma \in \sigma\mathfrak{R}\sigma$ لكل $\sigma \in \mathfrak{I}(\mathfrak{R})$. ويقال لـ $\mathfrak{R}$ حلقة محلية [2] ، اذا كانت تحتوي على مثالي اعظم وحيد . في المصدر [9] يقال لـ $\mathfrak{R}$ حلقة ديو من النمط – $\aleph$ يمنى (يسرى) ، إذا كان لكل $\sigma \in \aleph(\mathfrak{R})$ فإن $\sigma\mathfrak{R}$ ($\mathfrak{R}\sigma$) مثالي في $\mathfrak{R}$.

2 – الحلقات من النمط - $\mathrm{S}\mathfrak{I}\mathrm{F}$:

نستهل هذا البند بالقضية المساعدة الرئيسية في البراهين :

قضية مساعدة 2.1 [10] : اذا كان I مثالي ايمن في $\mathfrak{R}$ ، فإن $\mathfrak{R}/_I$ هو مقاس مسطح من النمط - $\mathfrak{I}$ إذا وفقط إذا كان $I\sigma = I \cap \mathfrak{R}\sigma$ ، لكل $\sigma \in \mathfrak{I}(\mathfrak{R})$ ■

قضية 2.2 : لتكن $\mathfrak{R}$ حلقة من النمط - $\mathrm{S}\mathfrak{I}\mathrm{F}$ يمنى . إذا كانت $\mathcal{M}$ مثالي ايمن اعظم في $\mathfrak{R}$ و $\mathcal{M} \subseteq \mathfrak{I}(\mathfrak{R})$. فإن $\mathcal{M}$ مثالي معدوم القوى .

البرهان : نفترض أن $\mathcal{M}$ مثالي ايمن اعظم في $\mathfrak{R}$ و $\sigma \in \mathcal{M}$ ، لذلك فإن $\sigma \in \mathfrak{I}(\mathfrak{R})$. وبما ان $\mathfrak{R}$ تمثل حلقة من النمط - $\mathrm{S}\mathfrak{I}\mathrm{F}$. لذلك فأنه يوجد $\mathfrak{b} \in \mathcal{M}$ بحيث ان $\sigma = \mathfrak{b}\sigma$ (حسب قضة مساعدة 2.1) . نضرب طرفي المعادلة بـ σ^{n-1} نحصل على $\sigma^n = \mathfrak{b}\sigma^n$ لبعض n عدد صحيح موجب . بما ان $\mathfrak{b} \in \mathcal{M} \subseteq \mathfrak{I}(\mathfrak{R})$ فإن $(1-\mathfrak{b})$ له معكوس . لذلك فإنه يوجد $v \in \mathfrak{R}$ بحيث ان $v(1-\mathfrak{b})\sigma^n = \sigma^n$ ، يؤدي الى $\sigma^n = 0$. وهذا يعني بأن $\mathcal{M}$ مثالي معدوم القوى ■

الان نقدم العلاقة بين الحلقات من النمط - $\mathrm{S}\mathfrak{I}\mathrm{F}$ والحلقات من النمط - $\mathfrak{IR}$:

مبرهنة 2.3 : لتكن $\mathfrak{R}$ حلقة ، وأن $\ell(\sigma) \subseteq r(\sigma)$ لكل $\sigma \in \mathfrak{I}(\mathfrak{R})$. فإن $\mathfrak{R}$ حلقة من النمط - $\mathfrak{IR}$ إذا وفقط إذا كانت $\mathfrak{R}$ حلقة من النمط - $\mathrm{S}\mathfrak{I}\mathrm{F}$ يمنى .

البرهان : نفترض أن $\Re$ حلقة من النمط - $S\Im F$. و نفترض $0 \neq \sigma \in \Im(\Re)$ ، بحيث ان $\sigma\Re + r(\sigma) \neq \Re$ ، لذلك يوجد مثالي ايمن اعظم K في $\Re$ بحيث ان $\sigma\Re + r(\sigma) \subseteq K$. وبما ان $\Re/_K$ مقاس ايمن مسطح من النمط - $\Im$ على الحلقة $\Re$ ، حسب (قضة مساعدة 2.1) يوجد $b \in K$ بحيث ان $\sigma = b\sigma$ ، $(1 - b) \in \ell(\sigma) \subseteq r(\sigma) \subseteq K$ ، يؤدي الى $1 \in K$. وهذا تناقض . لذلك فإن $\sigma\Re + r(\sigma) = \Re$ ومنها نحصل على ان $\Re$ حلقة من النمط - $\Im\Re$. برهان العكس واضح من (مبرهنة 3.4 ، [10]) ■

الان نعطي شرط اخر لكي تكون الحلقة من النمط - $S\Im F$ حلقة من النمط - $\Im\Re$:

مبرهنة 2.4 : لتكن $\Re$ حلقة ديو من النمط – $\aleph$ يمنى ، $\Im(\Re) \subseteq \aleph(\Re)$. فإن $\Re$ حلقة من النمط - $\Im\Re$ إذا وفقط إذا كانت $\Re$ حلقة من النمط - $S\Im F$ يمنى .

البرهان : نفترض أن $0 \neq \sigma \in \Im(\Re) \subseteq \aleph(\Re)$. نفترض بأن $x \in \ell(\sigma)$. بما ان $\Re$ حلقة ديو من النمط – $\aleph$ يمنى ، وان $\sigma\Re$ هو أي مثالي في $\Re$. وهذ يعني ان $\sigma r = t\sigma$ لبعض $r, t \in \Re$. لذلك فإن $x\sigma r = xt\sigma$ ، يؤدي الى $0r = xt\sigma = 0$. وبهذا برهنّا بإن $\ell(\sigma)$ تمثل مثالي ايمن في $\Re$. وحسب (مبرهنة 3.4 ، [4]) نحصل على أن $\Re$ حلقة من النمط - $\Im\Re$. برهان العكس واضح من (مبرهنة 3.4 ، [10]) ■

قضية مساعدة 2.5 [6]: إذا كانت $\Re$ حلقة مركزية مختزلة من النمط – 2 ، فإن $\Re$ حلقة ابيلية ■

قضية 2.6 : لتكن $\Re$ حلقة من النمط - $\aleph\Im$. فإن $\Re$ حلقة من النمط - $\Im\Re$ إذا وفقط إذا كان كل مثالي ايمن اساسي في $\Re$ يمثل مقاس مسط من النمط - $\Im$.

البرهان : نفترض أن كل مثالي ايمن اساسي في $\Re$ يمثل مقاس مسطح من النمط - $\Im$. ونفترض بأن $\sigma \in \Re$ ، بحيث ان $\sigma^2 = 0$. من الواضح بأن $\sigma\Re \oplus I$ يكون مثالي ايمن اساسي في $\Re$ ، بحيث ان I مثالي ايمن في $\Re$. إذاً $\sigma\Re \oplus I$ يمثل مقاس من النمط - $\Im$. لذلك فإن $\sigma\Re$ مقاس مسطح من النمط - $\Im$ و حسب (قضية مساعدة 2.1) ، $\sigma = b\sigma = \sigma c\sigma$ لبعض $c \in \Re$ ، $b \in \sigma\Re$ ، $b = \sigma c$. وهذا يعني أن $\Re$ حلقة من النمط - $\Im\Re$. برهان العكس واضح ■

الان نعطي التعريف التالي :

تعريف 2.7 : يقال للحلقة $\Re$ بأنها حلقة من النمط - $S\,S\,\Im\,F$ يمنى (يسرى) ، اذا كان كل مقاس ايمن (ايسر) منفرد بسيط يمثل مقاس مسطح من النمط - $\Im$.

مبرهنة 2.8 : إذا كانت $\Re$ حلقة مركزية مختزلة من النمط – 2 ، ومحلية . . فإن $\Re$ حلقة شبه اولية ، إذا كانت $\Re$ حلقة من النمط - $S\,S\,\Im\,F$ يمنى

البرهان : نفترض أن $\Re$ ليست حلقة شبه اولية . فإن $(\sigma\Re)^2 = 0$ لبعض $\sigma \in \Re$ ، $r(\sigma) \neq 0$ ، فإنه يوجد مثالي ايمن اعظم $\mathcal{M}$ في $\Re$ يحوي $r(\sigma)$. بما ان $\sigma\Re\sigma\Re = 0$ ، فإن $\Re\sigma\Re \subseteq r(\sigma)$ $\mathcal{M} = \Im(\Re)$ ، $\mathcal{M}$ اما مثالي ايمن اعظم اساسي او مركبة جمع مباشر في $\Re$. نفترض أن $\mathcal{M}$ ليس اساسياً فإن $\mathcal{M}$ مركبة جمع مباشر ، إذاً $\mathcal{M} = r(e)$ لبعض $e^2 = e \in \Re$ لذلك فإن $e\sigma = 0$. وحسب (قضية مساعدة 2.6) فإن $\Re$ حلقة ابيلية . يؤدي الى $\sigma e = 0$. وهذا يعني $e \in r(\sigma) \subseteq \mathcal{M} = r(e)$ ، لذلك فإن $e = 0$. وهذا تناقض . لذلك فإن $\mathcal{M}$ تمثل مثالي اساسي في $\Re$. وبما أن $\Re/_{\mathcal{M}}$ هو مقاس ايمن مسطح من النمط - $\Im$. وحسب (قضة مساعدة 2.2) فإنه يوج $c \in \mathcal{M}$ بحيث ان $\sigma = c\sigma$ ، $(1 - c)\sigma = 0$ ، وبما ان $\Re$ حلقة مركزية مختزلة من النمط – 2 ، فإن σ هو عنصر مركزي ، $\sigma(1 - c) = 0$. لذلك فإن

$(1-\mathfrak{c}) \in \mathscr{r}(\mathscr{a}) \subseteq \mathcal{M}$ ، يؤدي الى $1 \in \mathcal{M}$. وهذا تناقض ، لذلك فإن $\mathscr{a} = 0$ وعليه $\mathfrak{R}$ حلقة شبه اولية
■

مبرهنة 2.9 : لتكن $\mathfrak{R}$ حلقة مركزية مختزلة من النمط – 2 ، وأن $\mathfrak{I}(\mathfrak{R}) \subseteq \aleph(\mathfrak{R})$ فإن العبارات الاتية متكافئة .

1 - $\mathfrak{R}$ حلقة من النمط - $\mathfrak{I}\mathfrak{R}$.

2 - $\mathfrak{R}$ حلقة من النمط - S ℑ F يمنى .

3 - $\mathfrak{R}$ حلقة من النمط - S S ℑ F يمنى .

البرهان : (1) ← (2) ← (3) البرهان واضح . الان سنبرهن (3) ← (1) : نفترض أن $\mathscr{a} \in \mathfrak{I}(\mathfrak{R}) \subseteq \aleph(\mathfrak{R})$. اذا كانت $\mathscr{a} \neq 0$ فأنه يوجد مثالي ايمن اعظم $\mathcal{M}$ يحوي $\mathscr{r}(\mathscr{a})$ في $\mathfrak{R}$. نلاحظ اولا بأن $\mathcal{M}$ تمثل مثالي ايمن اساسي في $\mathfrak{R}$ ، لانه اذا كانت عكس ذلك فأن $\mathcal{M}$ مركبة جمع مباشر في $\mathfrak{R}$ أي ان $\mathcal{M} = \mathscr{r}(e)$ لبعض $0 \neq e = e^2 \in \mathfrak{R}$ ، وبطريقة مشابهة لبرهان (مبرهنة 2.8) نحصل على أن $\mathcal{M}$ مثالي ايمن اساسي في $\mathfrak{R}$. إذاً $\mathfrak{R}/\mathcal{M}$ تمثل مقاس ايمن منفرد بسيط في $\mathfrak{R}$. لذلك فإنه يوجد $\mathfrak{c} \in \mathcal{M}$ بحيث ان $\mathscr{a} = \mathfrak{c}\mathscr{a}$ ، يؤدي الى $(1-\mathfrak{c})\mathscr{a} = 0$. بما ان $\mathfrak{R}$ حلقة مركزية مختزلة من النمط – 2 فإن $\mathscr{a}^2 = 0$ يؤدي الى ان $\mathscr{a}$ تمثل عنصر مركزي ، اذاً $\mathscr{a}(1-\mathfrak{c}) = 0$ ، يؤدي الى $(1-\mathfrak{c}) \in \mathscr{r}(\mathscr{a}) \subseteq \mathcal{M}$. لذلك فإن $1 \in \mathcal{M}$. وهذا تناقض . لذلك فإن $\mathscr{a} = 0$ وإن $\mathfrak{I}(\mathfrak{R})$ مختزل . الان يجب ان نبرهن على ان $\mathfrak{b}\mathfrak{R} + \mathscr{r}(\mathfrak{b}) = \mathfrak{R}$ لكل $\mathfrak{b} \in \mathfrak{I}(\mathfrak{R})$. نفترض عكس ذلك ، لذلك فإنه يوجد مثالي ايمن اعظم K في $\mathfrak{R}$ بحيث ان $\mathfrak{b}\mathfrak{R} + \mathscr{r}(\mathfrak{b}) \subseteq K$ ، لذلك فإن K مثالي ايمن اساسي في $\mathfrak{R}$. بما انه $\mathfrak{R}/K$ مقاس مسطحة من النمط - ℑ ، فإنه يوجد $\mathfrak{c} \in K$ بحيث ان $\mathfrak{b} = \mathfrak{c}\mathfrak{b}$ ، يؤدي الى $(1-\mathfrak{c}) \in \ell(\mathfrak{b}) = \mathscr{r}(\mathfrak{b}) \subseteq K$ ، وهذا يعني $1 \in K$. وهذا تناقض . لذلك فإن $\mathfrak{b}\mathfrak{R} + \mathscr{r}(\mathfrak{b}) = \mathfrak{R}$ ، وبشكل خاص $\mathfrak{b}x + y = 1 \ , \ x \in \mathfrak{R} , y \in \mathscr{r}(\mathfrak{b})$ ، يؤدي الى $\mathfrak{b}x\mathfrak{b} + y\mathfrak{b} = \mathfrak{b}$ ، يؤدي الى $\mathfrak{b}x\mathfrak{b} = \mathfrak{b}$. لذلك فإن $\mathfrak{R}$ حلقة من النمط - $\mathfrak{I}\mathfrak{R}$ ■

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تعميم مسألة البائع المتجول في دراسة مشكلة تصريف مياه الأمطار في مركز محافظة الديوانية لعام 2019-2020

سعيد عبد الكاظم جوني

مديرية تربية الديوانية

saiedjhonnykh@gmail.com

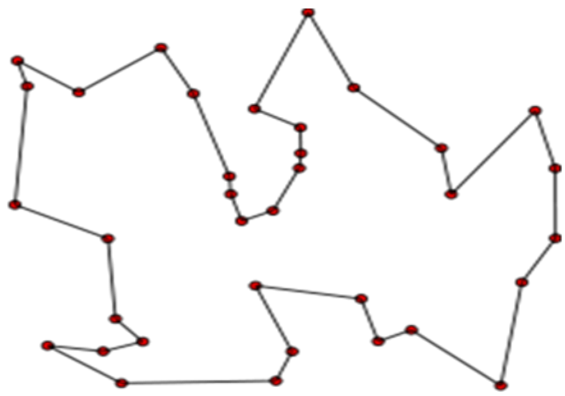
ملخص البحث

تم في هذا البحث توظيف خوارزمية شبيه بتدفق المياه Water Flow-Like Algorithm (In short, WFA) المقترحة لمسألة البائع المتجول Traveling Salesman Problem (In short, TSP) ، خوارزمية جديدة طرحت تعتمد على العمليات الأساسية في التهيئة، تدفق وجريان المياه، دمج التدفق، تبخر المياه وهطول الأمطار لحل مشكلة البائع المتجول (TSP) باعتبارها قائمة على مسألة الرسم البياني. التجارب أجريت على (60) منطقة ضمن مركز محافظة الديوانية وأخذ بيانات المنطقة عددها (12)، حيث تمت مقارنة متوسط وقت حساب (WFA-TSP) مع نظام مستعمرة النحل (ACS). أظهرت النتائج التجريبية التي تم الحصول عليها أن (WFA) المقترحة لحل مسألة البائع المتجول (TSP) حلولاً أفضل ومن ثم الوصول إلى الحل الأمثل بسهولة وأن كفاءة (WFA) المقترحة لحل مسألة البائع المتجول (TSP) زادت الأداء وخفضت الكلفة.

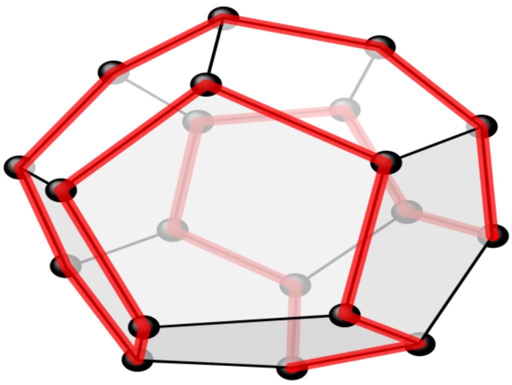
الكلمات المفتاحية : مسألة البائع المتجول، خوارزمية شبيه بتدفق المياه، نظام مستعمرة النمل، خوارزمية الجار الأقرب.

المقدمة :

مسألة البائع المتجول (TSP) [1]، هي أحدى المسائل الكلاسيكية في الرياضيات وعلوم الحاسوب والتي تم تقديمها منذ سنوات عديدة بفرض أن لديك عددا من المدن عليك زيارتها، ولديك مسافة بين كل مدينتين، وتريد المرور بأقصر طريق يمر في كل هذه المدن بحيث لا تمر في المدينة ذاتها مرتين حيث تعود في النهاية الى المدينة التي انطلقت منها بالفعل، و بأقصر طريق و اقل تكلفة و زمن و لا تترك أي مدينة دون زيارة الشكل (1). إن مسألة إيجاد هكذا طريق تسمى بمسألة البائع المتجول (TSP)، يتم التعبير عنه كرسم بياني كامل مع مجموعة من القمم (vertices) وهو مجموعة من الحواف موزونة بالمسافة بين الذروتين (المدن)، للعثور على أقصر طريق من خلال زيارة كل مدينة بالضبط مرة واحدة والعودة للمدينة الأصلية ، ووصفت هذه الجولة بمخطط يعرف بدارة هاملتون (Hamilton Circuit) الشكل (2)، حيث أن الرأس الأول في هذه الدارة هو الرأس الأخير.



الشكل (1)



الشكل (2)

ولطالما أثارت هذه المسألة اهتمام العديد من الباحثين كونها تعتبر من أحدى أهم المسائل في نظرية البيان (Graph Theory). في الآونة الأخيرة، تم اقتراح خوارزمية جديدة تعرف خوارزمية شبيه بتدفق المياه (WFA) [2]، الخوارزمية مستوحاة من محاكاة السلوك الطبيعي لتدفق المياه من المستويات الأعلى الى الأدنى على سطح الأرض، حيث يمكن أن ينقسم التدفق الى عدة تدفقات

فرعية عندما يمر بالتضاريس الوعرة وتندمج هذه التدفقات الفرعية عند وصولها الى نفس الموقع يحكمها الجاذبية مدفوعة بزخم المياه. ستتوقف التدفقات في مواقع ركود (مستويات منخفضة) أذا كان زخمهم لا يستطيع الزخم طرد المياه من الموقع الحالي. يمثل التدفق عامل الحل، ارتفاع التدفق يمثل وظيفة الهدف و مساحة الحل للمشكلة تتمثل في التضاريس الجغرافية. في عام 2010 ، تم تحسين (WFA) [3]، حيث أظهرت النتائج أن يتفوق على الهجين (الخوارزمية الجينية)(Hybrid Genetic Algorithm)، (HGA).

هدف البحث :

يهدف هذا البحث للمساهمة في دراسة توظيف خوارزمية شبيه بتدفق المياه (WFA) المقترحة لمسألة البائع المتجول (TSP) وتطبيق الدراسة على محطات تصريف مياه الأمطار ومساراتها و تحديدا في مركز محافظة الديوانية وأدى تطبيق الخوارزمية الى تقليل عدد التفرعات في مسألة البحث ومن ثم الوصول الى الحل الأمثل بيسر وسهولة. حيث أظهرت النتائج أن الخوارزمية المقترحة كان أداؤها أفضل من طرق القياس المستخدمة في خوارزمية مستعمرة النحل لأيجاد الحل.

أهمية البحث :

ترجع أهمية البحث في كونه يستخدم في العديد من المسائل التطبيقية كونها امتداد لدراسات سابقة قدمها Srour A. وأخرون في عام (2014) [4]، حيث اتخذت العديد من السلوكيات الطبيعية لتدفق المياه.

مواد وطرق البحث :

اعتمدت طرائق البحث على الاطلاع على العديد من المراجع العلمية والبحوث المنشورة والاستفادة من نشرات الأبحاث والمصادر البرمجية المفتوحة من الأنترنت بالإضافة لحصولنا على خرائط محددة ب (GIS) لمحطات تصريف مياه الأمطار لمركز محافظة الديوانية لعام 2020.

النموذج الرياضي لمسألة (WFA-TSP) :

ليكن لدينا المعطيات التالية :

G : حد لتكرار.

W_0 : الكتلة الابتدائية للتدفق الأصلي.

W_i : كتلة التدفق i.

V_0 : السرعة الابتدائية للتدفق الأصلي.

V_i : سرعة التدفق i.

T : الزخم الأساسي.

$\bar{n}$: تدفق الحد الأعلى على عدد التدفقات الفرعية التي يمكن تنقسم من التدفق.

n_i : عدد التدفقات الفرعية المتفرعة من التدفق i.

N : العدد الإجمالي لتدفقات المياه في التكرار الحالي.

X_i : الحل المقابل للتدفق i.

U_{ik}:الحل المقابل للتدفق الفرعي k الذي ينقسم من التدفق i.

W_{ik} : كتلة التدفق الفرعي k الذي ينقسم من التدفق i.

μ_{ik} : سرعة أنياب التدفق الفرعي k الذي ينقسم من التدفق i.

σ_{ik} : انخفاض الارتفاع من التدفق i الى التدفق الفرعي k.

g : تسارع الجاذبية (التعجيل التسارعي).

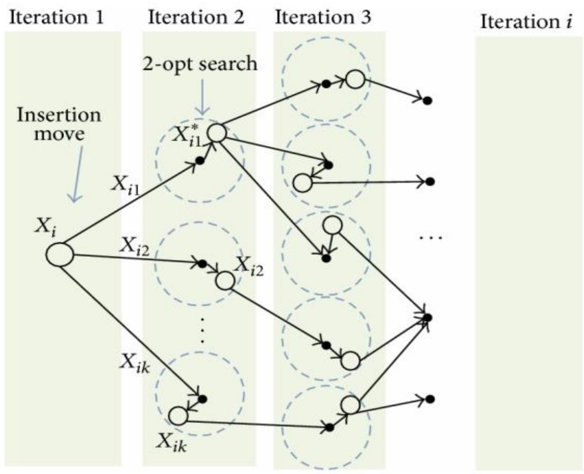
t : عدد التكرارات الموصوفة حتى يتم إزالة التدفق بالكامل عن طريق التبخر.

تعتمد هذه الخوارزمية على العمليات الرئيسية للتهيئة، تقسيم التدفق وانتقاله، دمج التدفق، تبخر المياه و سقوط الأمطار وكما موضح في الشكل (4) حيث يتم تلخيص أفكار التصميم على النحو التالي:

- **تقسيم التدفق وعملية النقل:**

1. من المفترض أن هناك تدفق مياه واحد فقط (وباتجاه واحد) لبدء (WFA) وأن موقعه يتم أنساؤه بشكل عشوائي مدفوع بزحم السوائل والطاقة الكامنة، حيث أن التدفق يبدأ بالانتقال الى المواقع الجديدة لاستكشاف مساحة الحل لحل جديدة وأفضل. يتم ذلك بإعطاء حلول ابتدائية باستخدام مفهوم خوارزمية الجار الأقرب (Nearest Neighbor)، (NN) [5]، وتهيئة (Initialization) معلمات (WFA) ، G ، W_0 و V_0 .

2. في (WFA) ينتج عن أجراء تقسيم التدفق لتدفقات فرعية (Flow splitting and moving)، الشكل (3)، اذا كان لديهم قوة دفع كافية (زخم كافي)، حيث أن التدفق مع الزخم العلوي يولد تدفقات فرعية أكثر من التدفق السفلي .



الشكل (3)

3. العثور على أفضل حل مجاور لجميع التدفقات الفرعية باستخدام اجراء بحث جار (2-opt neighbor search)، أي بعد تنفيذ أو تهيئة تدفق لجميع التدفقات ذات السرع الغير صفرية الى مواقع جديدة باستخدام نقل العملية، حيث أن مواقع التدفقات الفرعية المنقسمة مشتقة من المواقع المجاورة للتدفق الأصلي وألا فأنه يستمر كتيار باتجاه واحد نحو موقع أفضل جار للتدفق الأصلي.

4. من بعد حساب كتلة W_i وسرعة V_i جميع التدفقات الفرعية، فإذا كان N العدد الإجمالي لتدفقات المياه في التكرار الحالي فأن عدد التدفقات الفرعية n_i المتفرعة من التدفق i يتحدد بزخمها وفق العلاقة $T_i = W_i V_i$. التدفق مع زخم صفر يبقى حيث هو ويعتبر حل راكد. يمكن أن ينقسم التدفق الى تدفقات فرعية فقط عندما يتجاوز زخمه الزخم الأساسي T المحدد مسبقاً، أما اذا كان $0 < T_i < T$ فأنه يستمر كتيار باتجاه واحد نحو موقع أفضل جار للتدفق الأصلي.

5. عند أي تكرار عدد التدفقات الفرعية المنقسمة منه يمكن الحصول عليها وفق العلاقة التالية:

$$n_i = min\{max\{1, int(\frac{T_i}{T})\}, \bar{n}\} \;\; ; \; i = 1, 2, \ldots, N \qquad (1)$$

6. عندما ينقسم التدفق i الى تدفقات فرعية فأن كتلته الأصلية يتم توزيعها على التدفقات الفرعية وفق العلاقة التالية:

$$W_{ik} = \left(\frac{n_i+1-k}{\sum_{i=1}^{n_i} r}\right) W_i \;\; ; \; k = 1 , 2 , \dots , n_i \qquad (2)$$

7. يتم حساب سرعة كل تدفق فرعي وفق العلاقة التالية:

$$\mu_{ik} = \begin{cases} \sqrt{V_i^2 + 2g\sigma_{ik}} \;\; ; & V_i^2 + 2g\sigma_{ik} \geq 0 \\ 0 & ; \; otherwise. \end{cases} \qquad (3)$$

أذا كان المقدار $V_i^2 + 2g\sigma_{ik} < 0$ فأن لا يوجد تحسن في الحل على المستوى الأمثل المحلي بدون تقسيم أو انتقال للتدفق أي يبقى راكد.

- **عملية دمج التدفق:**

1. عندما ينتقل تدفقين أو أكثر لنفس الموقع فأنهم سيندمجون في تدفق واحد مع كتلة وزخم أكبر، وبالتالي فأن التدفق يشترك في نفس الموقع مع الأخرين في (WFA) . أذا كان التدفقان i و j يشتركان في نفس الشيء فأنه يتم تحديث الموقع والتدفق i ثم الكتلة والسرعة وفق العلاقات الاتية على التوالي:

$$W_i = W_i + W_j \;\; ; \qquad i , j = 1 , 2 , \dots , N \qquad (4)$$

$$V_i = \frac{W_i V_i + W_j V_j}{W_i + W_j} \;\; ; \qquad i , j = 1 , 2 , \dots , N \qquad (5)$$

2. باستخدام عملية دمج التدفق، يقلل (WFA) من عدد عوامل الحل عندما تؤدي عوامل متعددة الى نفس قيمة الهدف ولتجنب عمليات البحث الزائدة.

- **عملية تبخير المياه :**

1. من الطبيعي أن تتبخر المياه وتعود الى الأرض من خلال هطول الأمطار، حيث أن كل تدفق في (WFA) يخضع لتبخر المياه، حيث يتبخر جزء من المياه في الغلاف الجوي كما أن التدفق سيتم أزالته بالكامل بعد رقم تكرار محدد t . بمعنى يتم تقليل كتل التدفقات بواسطة النسبة $\frac{1}{t}$ التحقق كما موضح في المعادلة (6) في كل مرة يحدث التبخر.

$$W_i = (1 - \frac{1}{t}) \, W_i \; ; \qquad i = 1 , 2 , \dots , N \qquad (6)$$

2. من شروط التبخر، نقوم بأجراء عملية التبخر لكل تدفق.

- **عملية الترسيب :**

1. عندما يتراكم بخار الماء الى حجم معين، فأنه سيعيد نفسه الى الأرض على شكل أمطار.

2. في (WFA) الأصلي يتم تنفيذ نوعين من هطول الأمطار لمحاكاة الدورة الطبيعية للمياه هما هطول الأمطار القسري وهطول الأمطار المنتظم.

3. يتم تنفيذ هطول الأمطار القسري عندما يتم أيقاف جميع التدفقات بدون سرعات، تحت هذه الظروف يتم فرض جميع التدفقات تتبخر في الجو ثم تعود الى الأرض دون تغير عدد التدفقات الحالية. ومع ذلك فأن مواقع هذه التدفقات الراجعة تنحرف بشكل عشوائي عن تلك الأصلية. يتم توزيع كتلة W_0 بشكل متناسب مع التدفقات بناءً على كتلتها الأصلية و بنفس السرعة الابتدائية. ونتيجة لذلك، يمكن تحديد الكتلة المعينة للتدفق i وفق العلاقة التالية :

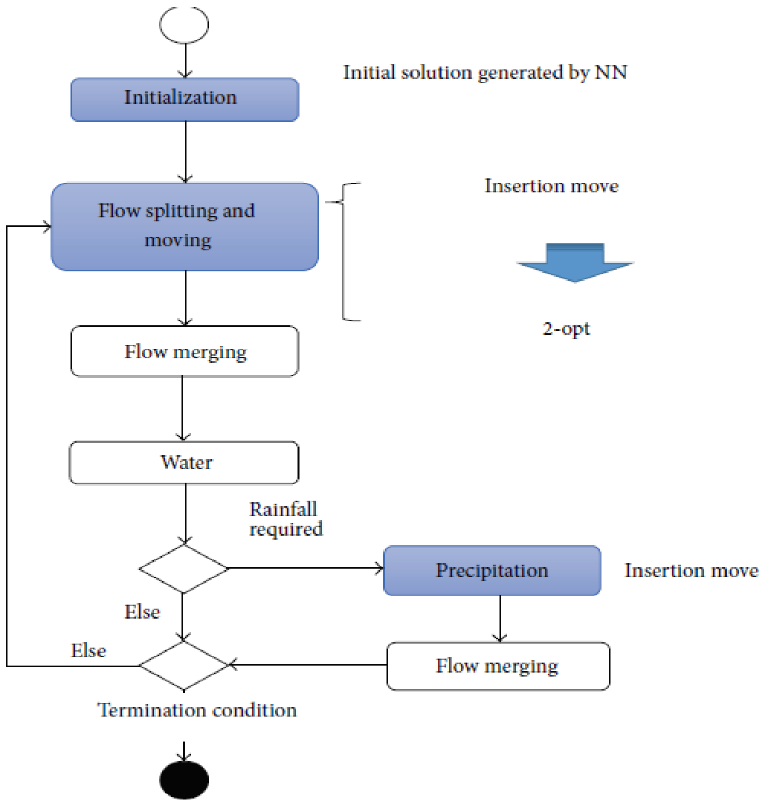
$$W_i' = \left(\frac{W_i}{\sum_{k=1}^{N} W_k}\right) W_0 \; ; \qquad i = 1, 2, \dots, N \qquad (7)$$

4. يتم تنفيذ هطول الأمطار المنتظم بشكل دوري لإعادة المياه المتبخرة من مياه الأرض، حيث أنه في كل تكرار يتم تنفيذ العملية مرة واحدة للحصول على المياه المتبخرة. الكتلة المتراكمة من الماء المتبخر $W_0 - \sum_{k=1}^{N} W_k$ التي تم تعينها لتدفقات الأرض تعطى وفق العلاقات التالية:

$$W_i' = \left(\frac{W_i}{\sum_{k=1}^{N} W_k}\right) W_0 - \sum_{k=1}^{N} W_k \; ; \qquad i = 1, 2, \dots, N \qquad (8)$$

5. بعد اجراء اي نوع من هطول الأمطار، نتحقق مما اذا كانت الحلول الجديدة لها نفس القيمة، أذا كانت الإجابة بنعم، نقوم بأجراء الخطوة (2) من عملية تقسيم التدفق و عملية النقل .

- نكرر الخطوات السابقة حتى تصبح حالة الأنهاء (Termination condition).



الشكل (4) العمليات الرئيسية ل (WFA-TSP)

التجارب و النتائج :

يتم تقييم اداء (WFA-TSP) المقترح بأجراء العديد من التجارب باستخدام المعيار القياسي ل (TSP)، حيث تتوفر مجموعة بيانات من (TSPLIB)، [6]. التجارب أجريت على (11) مجموعة بيانات لمركز مدينة الديوانية المؤلفة من (60) منطقة، حيث أن التجارب تقيس تكلفة الحل ووقت الحساب والتي يتم الحصول عليها من (10) دورات لكل منها مجموعة بيانات، مع (100) تكرار لكل تشغيل مستقل، مطلوب عدد من التكرارات للوصول الى أفضل حل. الحد الأدنى والمتوسط والانحراف المعياري ليتم حساب تكلفة الحل ل (10) دورات مستقلة. يتم حساب المسافة بين أي مدينتين باستخدام المسافة الاقليدية، كما تم تحديد متوسط التكلفة الحسابية. تمت مقارنة النتائج مع نظام مستعمرة النمل (ACS) [7] ، أما إعدادات المعلمات ل (WFA-TSP) المختبرة يتبع نفس اعدادات المعلمات في [8]، وكما موضح في الجدول التالي:

TABLE 2: Parameter settings.

Algorithm	Parameter	Value
WFA-TSP	Base momentum T	20
	Initial mass W_0	8
	Initial velocity V_0	5
	Subflow number limit $\overline{n}$	3
ACS	Number of ants	10
	β	2
	ρ	0.1
	τ_0	$1/nC^{nn}$, where n is the number of cities and C^{nn} is the nearest neighbor value
	ε	0.1

جدول (1)

تظهر التجربة السابقة أن إعدادات المعلمات هذه حصلت على أفضل نتيجة.

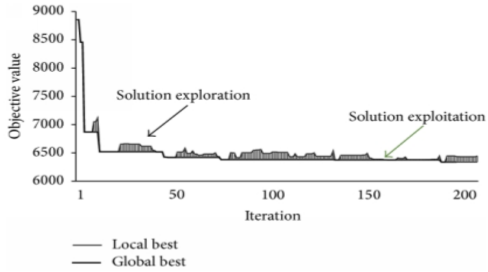
أداء (WFA-TSP) مقارنة مع (ACS) :

هنا يتم تقديم نتائج المقارنة بين (ACS) و(WFA-TSP)، لمجموعات البيانات التي تتضمن مشاكل مع رقم المنطقة من حيث جودة الحل الأفضل، متوسط عدد التكرارات و وقت حساب الخوارزميات. كما يظهر الجدول (2) مقارنة بين (ACS) و (WFA-TSP) من حيث دقة الحل (بالنسبة المئوية) وانحراف الحل لمتوسط القيم بخصوص الحل الأفضل. يمثل (WFA-TSP) كيفية مفهوم السكان الديناميكي في (WFA) يؤثر على سلوك البحث عن الحل، حيث يساعد تغيير عدد التدفقات باستخدام تقسيم التدفق ودمجه تعيين حجم مناسب من السكان على طول عملية التحسين.

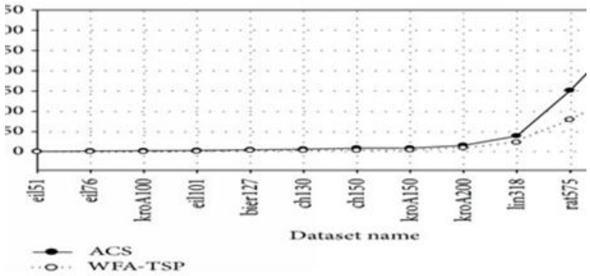
Datasets	ACS							WFA-TSP							P-value		Improvement %	
	Best	Worst	Mean	Std.	Avg. iteration	Avg. time	PD_{avg}	Best	Worst	Mean	Std.	Avg. iteration	Avg. time	PD_{avg}	Time	Solution quality	Accuracy	Time
eil51	427	434	428.80	2.82	5133.5	1.11	0.66	426	427	**426.40**	0.52	939.5	0.22	0.09	-	0.027	0.56	79.8
eil76	538	555	545.70	6.11	5327.1	2.13	1.43	538	538	**538.00**	0.00	1638.5	0.55	0.00	0.006	0.003	1.41	74.0
kroA100	21292	22255	21,585.60	311.51	4546.9	2.84	1.43	21282	21282	**21,282.00**	0.00	1057.7	0.56	0.00	0.001	0.013	1.81	80.2
eil101	638	651	642.30	3.65	5624.2	3.55	2.11	629	638	**630.70**	2.83	2648.8	1.23	0.27	-	-	1.26	65.2
bier127	118616	121876	119,965.10	1,134.08	5634.2	5.13	1.42	118282	118728	**118,451.40**	177.62	5616.9	3.56	0.14	0.092	0.003	2.06	30.6
ch130	6184	6447	6,263.00	89.22	6720.3	6.61	2.50	6110	6201	**6,133.90**	29.29	4892.1	3.45	0.39	0.011	0.004	1.23	47.8
ch150	6556	6756	6,623.50	59.13	7373.7	9.42	1.46	6528	6559	**6,542.20**	14.22	4204.4	3.53	0.22	-	0.001	2.04	62.5
kroA150	26886	27437	27,122.40	170.31	7217.1	9.33	2.26	26524	26719	**26,568.40**	58.45	4901.8	4.41	0.17	-	-	1.91	52.7
kroA200	29691	30556	30,011.90	295.79	7299.7	16.06	2.19	29368	29744	**29,438.20**	114.84	6774.9	9.81	0.24	0.001	-	2.69	38.8
lin318	43119	44089	43,666.10	313.43	8567.7	39.40	3.90	42278	42946	**42,490.10**	208.54	7695.3	23.18	1.10	-	-	2.06	41.1
rat575	7198	7398	7,293.60	62.71	9712.4	151.50	7.69	6971	7042	**7,000.80**	21.73	9090.8	79.43	3.36	-	-	4.01	47.5

جدول (2)

ويمكن ملاحظة أن تعقيد (WFA) زاد مقارنة مع (ACS) وذلك بزيادة عدد المناطق.



شكل (4)



شكل (5)

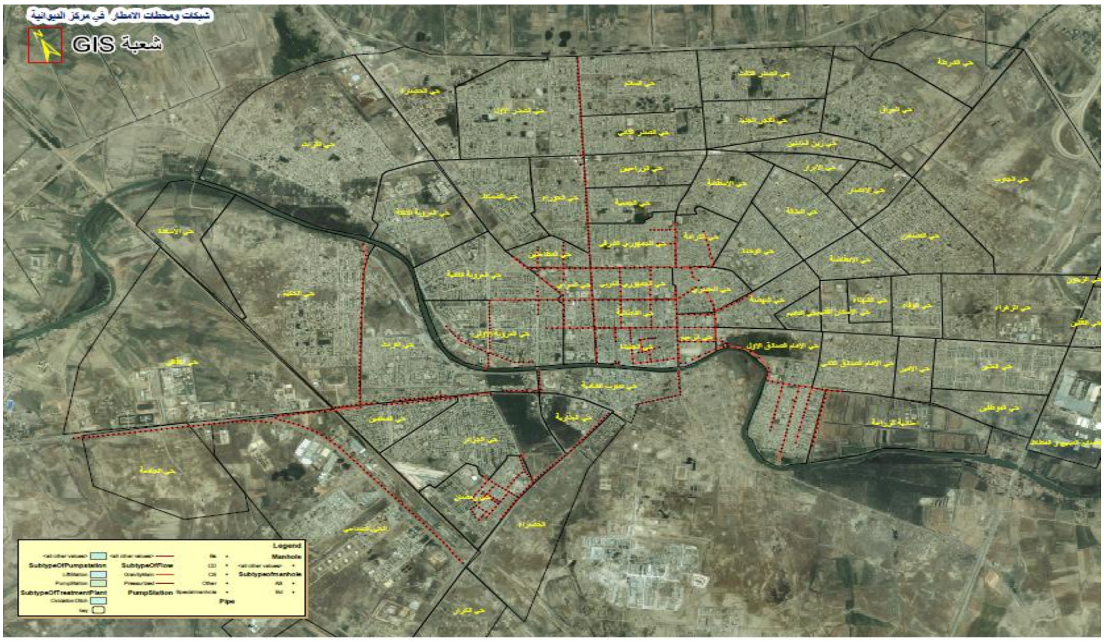
المستخلص :

في هذا البحث قدمنا خوارزمية (WFA-TSP)، والتي تختلف عن خوارزمية (TSP) الأساسية التي مجالها المقابل للمشكلة. وقد أظهرت الدراسة أن (WFA-TSP)، مناسب للحصول على حل جيد. كذلك يتضح لنا أن قوة (WFA-TSP) ، في اظهار وقت حساب سريع حيث يستخدم امكاناته لحل المشكلة التي تتعلق بوقت الحساب.

هنالك العديد من التحسينات المحتملة التي يمكن تقديمها بخصوص خوارزمية (WFA-TSP) ، خاصة تدفق المياه حيث يمكن تحسين اجراء عملية التقسيم والنقل باستخدام استراتيجيات بحث جار أفضل مثل (3-opt) و (4-opt).

الملحقات :

1. خارطة توضح شبكات ومحطات مياه الأمطار لمركز محافظة الديوانية (شعبة GIS) لعام 2019-2020 .



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كيف تفلت من جاذبيةالارض؟

علي حسن عبد الخالق و [2] ايمان سمير بهيه

جامعة بابل/ كلية التربية للعلوم الصرفة

, [2] emanbhaya@itnet.uobabylon.edu.iqali.kufa94@gmail.com

عندما تقفز في الهواء ستسقط مرتطما بالأرض . هذا لا يعني ان قوانين الطبيعة تمنعك من مغادرة الارض، لكن قفزتك لم تكن بالقوة الكافية لكي تجعلك تهرب من جاذبية الارض . لكي تقوم بذلك عليك ان تقفز بسرعة اكبر او تساوي سرعة الإفلات escape velocity للأرض والتي سنحسبها هنا.

يمكن حساب سرعة الإفلات كما يلي:

عندما تقفز في الهواء ستكون طاقتك الحركية kinetic energy هي:

$$E_k = \frac{1}{2}mv^2$$

حيث ان m هي كتلتك و v هي سرعتك.

اما الطاقة الكامنة potential energy التي سوف تواجهها نتيجة لقوة جذب الأرض لك هي

$$E_p = \frac{GMm}{r}$$

حيث m هي كتلتك وM كتلة الارض و r هو نصف قطر الارض.

كي تكون قادرا على الهروب من جاذبية الارض يجب ان تكون طاقتك الحركية E_k اكبر او تساوي الطاقة الكامنة التي سوف تواجهها نتيجة قوة جذب الأرض لك ويمكن التعبير عن ذلك بشكل رياضياتي بالمتراجحه:

$$E_k \geq E_p$$

اي ان

$$\frac{1}{2}mv^2 \geq \frac{GMm}{r}$$

وبحل المتراجحه اعلاه يجب ان تكون سرعتك v اكبر من او تساوي $\sqrt{\frac{GMm}{r}}$ أي ان

$$v \geq \sqrt{\frac{GMm}{r}}$$

ان سرعة الافلات للأرض Earth’s escape velocity هي اصغر سرعة تسمح للشيء بالافلات وسنرمز لها بالرمز

$$V_{Earth} = \sqrt{\frac{GMm}{r}} \qquad (1)$$

الآن لدينا

$$G \approx 6.67 \times 10^{-11} \frac{m^3}{kgs},$$

$$M \approx 5.98 \times 10^{24} kgs,$$

$$r \approx 6.38 \times 10^{6} m.$$

بالتعويض عن القيم اعلاه في المعادلة (1) نحصل على ان

$$V_{Earth} \approx 11182 \frac{m}{s}$$

$$= 40255.2 \frac{km}{h} \; .$$

من اعلاه بإمكاننا أيضا حساب سرعة الافلات لأي جسم كروي عُلمت كتلته ونصف قطره .

لاحظ ان صيغة سرعة الافلات اعلاه لا تعتمد على كتلة الجسم، اي ان حاول الافلات مهما كانت كتلتك. وبالتالي نظريا نحتاج نفس سرعة الافلات للأرض و التي يحتاجها الفيل ايضا. تجدر الاشارة الى اننا في حساباتنا تجاهلنا تأثير مقاومة الهواء التي ستكون مختلفة بيننا وبين الفيل. بالإضافة الى ذلك اذا وصلت الى هذه السرعة العالية داخل الغلاف الجوي للأرض فانك سوف تحترق. ولتجنب ذلك فانه عليك او على

الفيل ان تدخلا في مدار يكون فيه الغلاف الجوي ضعيف او معدوم، بعدها تسارع الى سرعة الافلات التي ستحتاجها للهروب من هذا المدار.

المصادر

تمت الاستعانة بمصادر الانترنيت

How Can You Escape from Earth's Gravity?

Ali Hasan Abdul Khaleq and Eman S. Bhaya. University of Babylon,

College of Education for Pure Sciences. Mathematics Department.

, [2]emanbhaya@itnet.uobabylon.edu.iqali.kufa94@gmail.com

When you jump in the air, you will hit the ground. This does not mean that the laws of nature prevent you from leaving the Earth, but your jump was not strong enough to make you escape the Earth's gravity. To do this you have to jump more quickly or equal the escape velocity of the Earth, which we will calculate here.