KHAZAR UNIVERSITY

IRAQI Al- KHWARIZMI ASSOCIATION

Proceeding Book

4th International Scientific Conference of Iraqi AlKhwarizmi Association

In Cooperation with Khazar University

1-2

Baku - Azerbiagan

KHAZAR UNIVERSITY
Always leading the way!

PROCEEDING

OF

FOURTH INTERNATIONAL SCIENTIFIC CONFERENCE OF IRAQI AL-KHWARIZMI ASSOCIATION IN COOPERATION WITH KHAZAR UNIVERSITY (ICMSPCS 2019)

1-2 August 2019

BAKU, AZERBAIGAN scientific

Supported by

Iraqi Al-Khwarizmi Association

KHAZAR UNIVERSITY
Always leading the way!

CONFERENCE COMMITTEES

1. Prof. Dr. Noori F. Almayahy.
2. Pro. Dr. Mohammad H. Alsharoot.
3. Asst.Prof. Dr . Saleem A. Hussain
4. Asst. Prof. Dr. Hassan R. Yassein.
5. Asst. Prof. Dr. Sataar H. Hamzaa.
6. Asst. Prof. Dr. Boushra Y. Hussein.
7. Asst. Prof. Dr. Ehsaan J. Kadhim.
8. Asst. Prof. Dr. Lmyiaa A. Noor.
9. Asst. Prof. Dr. Mohannad F. Alsaadony.
10. Asst. Prof. Dr. Abdulhussein A. Khdhair.
11. Asst. Prof. Dr. Hameed J. Huzairaan.
12. Asst. Prof. Dr. Mohammad M. Aaraak.
13. Dr. Riyadh D. Ali.
14. Dr. Gorgees S. Mohammad.
15. Dr. Salwa S. Baawi.
16. Dr. Faezah L. Hassan
17. Asst.Lec. Wasan A. Alawsi.
18. Asst.Lec. Zahraa Ch. Elewy.
19. Asst.Lec. Faaz G. Abdulkadhim.
20. Mis. Fadhaa H. Ibrahim.
21. Mr. Hamza Mohsen Alrodhan

INTRODUCTION:

Scientific societies have played a prominent role in advancing science and linking it to society. Scientific societies were built on the basis of volunteerism through the efforts of scientists, researchers, and academics.

At the international level, the Scientific Society is founded on a sincere voluntary desire by groups of learners and intellectuals to contribute to the collection and dissemination of useful information in the field in which we work among members of society without regard to the material output resulting or expected for its members.

Al-Khawarizmi Iraqi Society is one of the scientific societies concerned with the dissemination of the culture of mathematics and its sciences and includes the disciplines of mathematics, computer, statistics, and physics. It was established under the Law of Scientific Associations No. 55 of 1981 and its amendments as per the Ministerial Order No. (3/10793) in 10/10/2012. It works according to an internal system approved by the Legal Department of the Ministry of Higher Education and Scientific Research. The members of Al-Khawarizmi Iraqi Society based in the province of Qadisiyah aims to open other branches to it in the rest of the provinces.

1. Cooperation to raise the level of instruction and scholarship in the subject areas of the Association's specialties.
2. Promoting scientific research and sharing in the research, exchanging its results and linking applied research topics to the needs of Iraqi society.
3. Documenting cooperation between the corresponding departments inside and outside Iraq through the members of the Association.
4. Issue a quarterly Iraqi scientific journal in the specialties of the Society.
5. Unification of scientific terminologies in the specialties of the Association and encourage Arabization and composition in the Arabic language.

6. Attracting the scientific expertise and expertise from outside Iraq to benefit from their potential and to find out what is new in the fields of the Association's specialties.
7. Provide opportunities to complete the graduate studies of the members of the Association in coordination with the members of the Assembly as much as possible.

The Assembly adopted some means to achieve the objectives of the above-mentioned Association, as indicated in the following:-

1. Holding lectures, seminars, and conferences to discuss innovatively and applied research and findings of the member researchers as well as discuss the problems of society.

2. Issuing scientific journals and publications reflecting their activities, including research of their members and their achievements and exchanging with their counterparts locally and internationally.
3. Establishing scientific libraries through exchange or acquisition.
4. Organizing meetings or periodic meetings to study the new scientific terminology.
5. Organizing courses and studies for the activation and scientific modernization of the members.
6. To establish meetings between the Association and some scientific bodies rather than to serve the community.
7. Providing consultancy and technical services.
8. Provide a data room and a specialized library that combines scientific activity in the fields of the Society's specialties.

Prof. Dr. Noori Farhan Al-Mayahi

Table of Contents

16	Numerical Solution of Stochastic Heat Equation	*Nasir I. Naser ,* *Dr. Muhannad Al-Saadony*
17	دراسة تاثير الجرع المختلفة من أشعة كاما على بعض الخصائص البصرية والكهربائية لبولي أكريل أمايد (PAAM)	أنيس عبد المنعم محمد
18	دراسة اولية في استخدام الخوارزميات الغير خطية لقياس قيم تعقيد اشارة الساق عند الاصحاء من خلال معالجة صور التباين الليزرية	عادل ابراهيم خليل وليد احمد النعيمي احمد خضير عباس
19	حماية البيانات في السحابة بإستخدام الاخفاء المعتمد على خوارزمية بحث طائر الوقواق	نادية معن محمد د. نجلاء بديع الدباغ
20	Implicit Hybrid Conjugate Gradient Method For Unconstrained Optimization Problems.	Khalil K. Abbo Edrees M. Nori Yoksal A. Laylani Hisham M.Khudhur
21	goodness of fit approach for testing exponential better than used in convex for life distributions	Gorgees Sh. Mohammad

On fuzzy soft normed space

Noori F. Al-Mayahi and Donia SH. Farhood

Department of Mathematics

College of computer Science and Information Technology

University of AL-Qadisiyah , Diwanyah-Iraq

E-mail:nafm60@yahoo.com

Abstract :

In this paper , we have introduced the definition of fuzzy soft normed space and obtained some new properties of these space by studying the open and closed balls. Moreover , we studied the continuity and the convergences in fuzzy soft normed space .

Keywords: fuzzy soft norme , fuzzy soft set . fuzzy soft continuity

1. Introduction:

In 2002 , Maji et.al gave a new concept called fuzzy soft set , After the rontier work of Maji, many investigator have extended this concept in various branches of mathematics and Kharal and Ahmad in [2] introduced new theories like new properties of fuzzy soft set and then in [3] defined the concept of mapping on fuzzy soft classes and studies of fuzzy soft in topological introduced by Tanay and Kandemir [4].Mahanta and Das [5] continued studier .In all of the above –mentioned works , the researchers used a fuzzy soft vector space or soft vector space ,while in this worke we used a vector space . In this work we introduce Fuzzy soft normed space and discussed the continuity and convergence and bounded

2.Preliminaries

In this work we use the simples X , E , P(x) to denote for an initial universe ,a set of parameters and the collection of all subsets of X , respectively .

Definition (2.1): [1] A fuzzy set A in X is characterized by a function with domain as X and value in I. The collection of all fuzzy sets in X is denoted by I^X

Definition (2.2) [15] : Let X be a universe set and E be a set of parameters , $P(X)$ the power set of X and $A \subseteq E$.A pair (F, A) is called soft set over X with recepect to A and F is a mapping given by $F: A \to P(X)$, $(F, A) = \{F(e) \in P(X): e \in A\}$.

Definition 2.3 [1] : Let A be a subset of E . A pair (F ,A) is called a fuzzy soft set over (X, E) ,if F: $A \to I^X$ is a mapping from A into I^X .The collection of all fuzzy soft sets over (X , E) is denoted by F(X , E)

Definition (2.4)[1]: A Fuzzy soft set (F,A) over (X,E) is said to be absolute fuzzy soft set , if for all e $\in$ A , F(e) is a fuzzy universal set $\tilde{1}$over X and denoted it by $\tilde{E}$

Definition(2.5)[1]: A fuzzy soft set (F,A) over (X , E) is said to be null fuzzy soft set ,if for all e $\in$ A, F(e) is the null fuzzy set $\tilde{0}$ over X .we denoted it by $\tilde{\Phi}$

Definition(2.6)[41] Let X be a non-empty set, $*$ be a continuous t-norm on I= [0, 1]. A function $N : X \times (0, \infty) \to [0, 1]$ is called a fuzzy norm function on X if it satisfies the following axioms: $for\ all\ x, y \in X,\ t, s > 0;$

1) $N(x, t) > 0$.
2) $N(x, t) = 1 \Leftrightarrow x = 0$.
3) $N(\alpha x, t) = N\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \in \mathbb{F}/\{0\}$.
4) $N(x, t) * N(y, s) \leq N(x + y, t + s)$.
5) $N(x, .) : (0, \infty) \to [0, 1]$ is continuous.
6) $\lim_{t\to\infty} N(x, t) = 1$.

$(X, N, *)$ is called a fuzzy normed space.

Definition(2.7) : Let X be a vector space .Then a mapping $\|.\| : X \to R(E)^*$ is said to be a soft norm on X if $\|.\|$ satisfies the following conditions :

1) $\|x\| \geq 0\ for\ all\ x \in X$

2) $\|x\| = 0 \leftrightarrow x = 0$

3) $\|r\,x\| = |r|\|x\|\ for\ all\ x \in X\ and\ for\ every\ soft\ scalar\ r$

4) $\|x + y\| \leq \|x\| + \|y\|\ for\ all\ x, y \in X$

The vector space X with a soft norm $\|.\|$ on X is said to be soft normed space and denoted by (X , $\|.\|$)

3.Main result

Definition(3.1) :Let X be a vector space over the scalar filed K, suppose $*$ is continuous t-norm , and. A fuzzy sub set Γ on X x (0,∞) is called fuzzy soft norm on X if and only if for x_e ,$y_{e'}$ $\in$ X and k $\in$ K the following condition hold

1) E(x_e , t) = 0 $\forall t \leq 0$
2) E(x_e , t) = 1 $\forall t \geq 0$ if and only if $x_e = \theta_0$
3) E($k\, x_e$, t) = E(x_e , $\frac{t}{|k|}$) if $k \neq 0\ \forall\ t > 0$
4) E($x_e \oplus x_{e'}$, t $\oplus$ s) $\geq$E(x_e , t) * E ($y_{e'}$, s) $\forall\, t, s > 0\ and$ x_e ,$y_e \in X$
5) E(x_e , .) is continuous function and $\lim_{t\to\infty} E(x_e, t) = 1$

The triple (X, E , $\|.\|$) will be refered to a fuzzy soft normed space

Definition(3.2): let (X , E, ‖.‖) be a fuzzy soft normed space and $t > 0$ we define an open ball , a closed ball and sphere with center at x_e and radius ∝ as follows

$B(x_{e1}, r, t) = \{y_{e2} \in X: E(x_{e1} - y_{e2}, t) > 1 - r\}$

$\bar{B}(x_{e1}, r, t) = \{y_{e2} \in X: E(x_{e1} - y_{e2}, t) \geq 1 - r\}$

$S(x_{e1}, r, t) = \{y_{e2} \in X: E(x_{e1} - y_{e2}, t) = 1 - r\}$

SFS(B(x_{e1} , r , t)) , SFS(B(x_{e1} , r , t)) and SFS(S(x_{e1} , r , t)) are called fuzzy soft open ball, fussy soft closed ball , fuzzy soft sphere respectively with center x_e and radius r

Definition(3.3):A mapping $\Delta: X \times X \times (0, \infty) \to (0,1)$ is said to be fuzzy soft metric on X if Δ satisfies the following condition

1) $\Delta(x_{e1}, y_{e2}, t) = 0$ for all $t \leq 0$
2) $\Delta(x_{e1}, y_{e2}, t) = 1$ for all $t \geq 0$ if and only if $x_{e1} = y_{e2}$
3) $\Delta(x_{e1}, y_{e2}, t) = \Delta(y_{e2}, x_{e1}, t)$
4) $\Delta(x_{e1}, z_{e3}, s \oplus t) \geq \Delta(x_{e1}, y_{e2}, s) * \Delta(y_{e2}, z_{e3}, t) \ \forall t, s > 0$
5) $\Delta(x_{e1}, y_{e2}, .) : (0, \infty) \to (0,1)$ is continuous .

X with a fuzzy soft metric Δ is called a fuzzy soft metric space and denoted by $(X, \Delta, *)$

Definition(3.4) : Let { x_{ej}^n } be a sequence of vectors in a fuzzy soft normed space (X , E, ‖.‖).Then the sequence convergence to x_{ej}^0 with respect to fuzzy soft norm.

If (x_{ej}^n - x_{ej}^0 ,t)$\geq 1 - \alpha$ for every $n \geq n_0$ and $\alpha \in (0,1]$ where n_0 is positive integer and $t > 0$

Or $\lim_{n \to \infty} E(x_{ej}^n - x_{ej}^0, t) = 1 \ as\ t \to \infty$

Similarly if $\lim_{n \to \infty} \Delta(x_{ej}^n - x_{ej}^0, t) = 1$ as $t \to \infty$,then { x_{ej}^n } is convergent sequence in fuzzy soft metric space $(X, \Delta, *)$

Definition(3.5): A sequence { x_{ej}^n } in a fuzzy soft normed space (X , E, ‖.‖) is said to be a Cauchy sequence with respect to the fuzzy soft norm if

$E(x_{ej}^n - x_{ej}^m, t) \geq 1 - \alpha$ for every $n, m \geq n_0$ and $\alpha \in (0,1]$ where n_0 is positive integer and $t > 0$

Or $\lim_{n, m \to \infty} E(x_{ej}^n - x_{ej}^m, t) = 1$ as $t \to \infty$

Similarly if $\lim_{n \to \infty} \Delta(x_{ej}^n - x_{ej}^0, t) = 1$ as $t \to \infty$ then { x_{ej}^n } is a Cauchy sequence in fuzzy soft metric space $(X, \Delta, *)$.

Definition(3.6): let (X , E, ‖.‖) be a fuzzy soft normed space .Then (X , E , ‖.‖) is said to be complete if every Cauchy sequence in X converge.

Definition(3.7):A Complete fuzzy soft normed space is called a fuzzy soft banach space.

Definition(3.8): let $\{x_{ej}^{n}\}$a sequence in a fuzzy soft metric space $(X, \Delta, *)$. Then the sequence $\{x_{ej}^{n}\}$is said to be a bounded sequence with respect to the fuzzy soft metric Δ if $\|x_{ej}^{n} - x_{ej}^{m}\|_{\alpha} \leq M$

By definition $\|x_{ej}^{n} - x_{ej}^{m}\|_{\alpha}$ = inf {t ; Δ (x_{ej}^{n}, x_{ej}^{m}, t) $\geq \alpha$, $\alpha \in (0,1]$ }

That is $\{x_{ej}^{n}\}$ is said to be bounded if there exist a positive integer N depending on M such that Δ (x_{ej}^{n}, x_{ej}^{m}, t) $\geq \alpha$, $\forall n, m \geq N(M)$.

Theorem(3.9) : Every convergent sequence is Cauchy sequence.

Proof : Let$\{x_{ej}^{n}\}$ be a sequence in a fuzzy soft normed space $(X, E, \|.\|)$. Consider $\{x_{ej}^{n}\}$ converges to x_{ej}^{0} .

Then we have $E(x_{ej}^{n}, -x_{ej}^{0}, t) \geq 1-\alpha$ for every $n \geq n_0$ and $\alpha \in (0,1]$ $where\ n_0 \in N$ and $t > 0$

Therefore

$$E(x_{ej}^{n} - x_{ej}^{m}, t) = E(x_{ej}^{n} - x_{ej}^{m} \oplus x_{ej}^{0} - x_{ej}^{0}, t)$$

$$= E((x_{ej}^{n} - x_{ej}^{0}) \oplus (x_{ej}^{m} - x_{ej}^{0}), t)$$

$$\geq E\left(x_{ej}^{n} - x_{ej}^{0}, \frac{t}{2}\right) * E\left(x_{ej}^{m} - x_{ej}^{0}, \frac{t}{2}\right)$$

$$\geq (1-\alpha) * (1-\alpha)$$

$$= \min\{1-\alpha, 1-\alpha\}$$

$$= 1-\alpha$$

$E(x_{ej}^{n} - x_{ej}^{m}, t) \geq 1-\alpha$ for every n ,m$\geq n_0$ and $\alpha \in (0,1]$

Thus $\{x_{ej}^{n}\}$ is a Cauchy sequence >

Theorem(3.10): limit of a sequence in fuzzy soft normed space if exist is unique.

Proof :

Let$\{x_{ej}^{n}\}$ be a sequence in a fuzzy soft normed space $(X, E, \|.\|)$.

Such that $\lim_{n\to\infty} E(x_{ej}^{n} - x_e, t) = 1$

$\lim_{n\to\infty} E(x_{ej}^{n} - x_{e'}, t) = 1$, are two limits of sequence $\{x_{ej}^{n}\}$.

Then by definition there exist positive integers n_1, n_2 such that

$E(x_{ej}^{n} - x_e, t) \geq 1-\alpha$ for every $n \geq n_1\ and\ \alpha \in (0,1]$

$E(x_{ej}^n - x_{e'}, t) \geq 1-\alpha$ for every $n \geq n_2$ and $\alpha \in (0,1]$

Choose $n \geq n_0$, n_0 = min$\{n_1, n_2\}$

$E(x_e - x_{e'}, t) = E(x_e - x_{ej}^n \oplus x_{ej}^n - x_{e'}, t)$

$= E((x_{ej}^n - x_e) \oplus (x_{ej}^n - x_{e'}), t)$

$\geq E\left(x_{ej}^n - x_e, \frac{t}{2}\right) * E(x_{ej}^n - x_{e'}, \frac{t}{2})$

$\geq (1-\alpha) * (1-\alpha)$

$= \min\{1-\alpha, 1-\alpha\}$

$= 1-\alpha$

$E(x_e - x_{e'}, t) \geq 1-\alpha$

That implies $\lim_{n\to\infty} E(x_e - x_{e'}, t) = 1$

$E(x_e - x_{e'}, t) = 1$

By definition of fuzzy soft normed space

$E(x_e - x_{e'}, t) = 1$ with $t > 0$ if and only if $x_e - x_{e'} = \theta_0$.

Hence $x_e = x_{e'}$

Definition(3.11): Let X and Y be two universe sets . We define two fuzzy soft (F , A) and (G,B) over universe sets , respectively . Let $f: X \to Y$ and $g: A \to B$ be two functions . Then the pair (f, g) is called Fuzzy soft function from

(F,A) to (G,B) and denoted by (f, g): $(F, A) \to (G, B)$ if it is satisfies $f(F(x)) = G(g(x))$ for all $x \in A$.

The image of the fuzzy soft set (F,A) under the fuzzy soft function (f, g), denoted by (f, g)(F,A) $=(f(F), B)$, is a fuzzy soft set over universe set Y and defined by

$$\begin{cases} \vee\, g(x) = y\ f(F(x)) & if\ g^{-1}(y) \neq \emptyset \\ \emptyset & otherwise \end{cases}$$

For all$x \in B$. The pre-image of fuzzy soft set (G,B) under fuzzy soft set (f, g) , denoted by $(f, g)^{-1}$(G,B) $=(f^{-1}(G), A)$ is fuzzy soft set over verse set X and defined by

Definition(3.12): Let (X , E,‖. ‖) and (Y ,E,‖. ‖) be two fuzzy soft normed spaces . A function f : $X \to Y$ is said to be fuzzy soft continuous at $x_0 \in X$ if for every sequence $\{X_{en}^n\}$ in X with $\{X_{en}^n\} \to \{X_{e0}^0\}$ as $n \to \infty$ we have $f(X_{en}^n) \to f(X_{e0}^0)$ as $n \to \infty$. if f is fuzzy soft continuous at each vector of X then f is said to be fuzzy soft continuous function .

Theorem(3.13): If $(X, E, \|.\|)$ be a fuzzy soft normed space then

a) the function $(x_e, y_{e'}) \to x_e \oplus y_{e'}$ is continuous

b) the function $(C, x_e) \to c * x_e$ is continuous where $x_{e1}, y_{e2} \in SSP(X)$ and $C \in K$

Proof :

a) If $x_{en} \to x_e$ and $y_{e_n'} \to y_{e'}$ then as $n \to \infty$

$$E(x_{en} \oplus y_{e_n'}) - (x_e \oplus y_{e'}), t) = E(x_{en} \oplus y_{e_n'} - x_e - y_{e'}, t)$$

$$= E((x_{en} - x_e) \oplus (y_{e_n'} - y_{e'}), t)$$

$$= E(x_{en} - x_e, \tfrac{t}{2}) * E(y_{e_n'} - y_{e'}, t)$$

$$\to 1 \ as\ t \to \infty$$

Thus the function$(x_e, y_{e'}) \to x_e \oplus y_{e'}$ is continuous

Definition(3.14): A fuzzy soft function f $: X \to Y$ is said to be fuzzy soft bounded ,if there exist a fuzzy soft real number M such that $\|f(x_e)\| \leq M \|x_e\|$ for all $x_e \in X$

Theorem(3.15) : The fuzzy soft function f $: X \to Y$ is fuzzy soft continuous if and only if it is fuzzy soft bounded .

Proof: Assume that f $: X \to Y$ be fuzzy soft continuous and f is not fuzzy soft bounded .Thus, there exist at least one sequence $\{X_{en}^n\}$ such that

$\|f(X_{en}^n)\| \geq n \| X_{en}^n \|$ (1)

Where n is a fuzzy soft real number . It s clear that $x_{en}^n \neq \theta_0$.

Let us construct a fuzzy soft sequence as follow :

$$y_{en}^n = \frac{x_{en}^n}{n\|x_{en}^n\|}$$

It is clear that $y_{en}^n \to \theta_0$ as $n \to \infty$. Since f is fuzzy soft continuous , then we have $\|f(y_{en}^n)\| \to 0\ as\ n \to \infty$.

$$\|f(y_{en}^n)\| = \left\| f \frac{x_{en}^n}{n\|x_{en}^n\|} \right\| = \frac{i}{n\|x_{en}^n\|} \|f(x_{en}^n)\| > \frac{n \| X_{en}^n \|}{n \| X_{en}^n \|} = 1$$

Which is a contradiction

Conversely , suppose that f $:X \to Y$ is fuzzy soft bounded and the fuzzy soft sequence $\{X_{en}^{n}\}$ is convergent to the $\{X_{e0}^{0}\}$. In this case

$$\|f(X_{en}^{n}) - f(X_{e0}^{0})\| = \|f(X_{en}^{n} - X_{e0}^{0})\| \le M\| X_{en}^{n} - X_{e0}^{0}\| \to 0$$

Which indicates that f is fuzzy soft continuous .

Definition(3.16): A fuzzy soft function f $:X \to Y$ is said to be fuzzy soft linear function if

1) f is additive , that is $f(x_e + y_e) = f(x_e) + f(y_e)$ for every $x_e, y_e \in X$
2) f is homogeneous ,that is , for every soft scalar r , $f(r\, x_e) = |r|\, f(x_e)$ for every $x_e \in X$,

Theorem(3.17) : Every fuzzy soft normed space is a fuzzy soft metric space.

Proof :

Define the fuzzy soft metric space by $\Delta(x_{e1}, y_{e2}, t) = E(x_{e_1} - y_{e_2}, t)$.......*

for every $x_{e_1}, y_{e_2} \in X$.

Then it is clear to show that the fuzzy soft metric space axioms are satisfied .

1) $\Delta(x_{e1}, y_{e2}, t) = E(x_{e_1} - y_{e_2}, t) = 0$ if $t \le 0$
2) $\Delta(x_{e1}, y_{e2}, t) = E(x_{e_1} - y_{e_2}, t) = 1$ if $t > 0$
3) $\Delta(x_{e1}, y_{e2}, t) = E(x_{e_1} - y_{e_2}, t)$
   $= E(y_{e_2} - x_{e_1}, t)$
   $= \Delta(y_{e_2}, x_{e_1}, t)$
   $\Delta(x_{e1}, y_{e2}, t) = \Delta(y_{e_2}, x_{e_1}, t)$
4) $\Delta(x_{e1}, z_{e_3}, s \oplus t) = E(x_{e_1} - z_{e_3}, t \oplus s)$

   $= E(x_{e_1} - y_{e_2} + y_{e_2} - z_{e_3}, t \oplus s)$

   $\ge E\left(x_{e_1} - y_{e_2}, s\right) * E(x_{e_1} - z_{e_3}, t)$

   $= \Delta(x_{e_1}, y_{e_2}, s) * \Delta(x_{e_1}, z_{e_3}, t)$

   $\Delta(x_{e1}, z_{e_3}, s \oplus t) \ge \Delta(x_{e_1}, y_{e_2}, s) * \Delta(x_{e_1}, z_{e_3}, t)$

5) By the definition * of Δ we get Δ is continuous and

   $\Delta(x_{e_1}, y_{e_2}, .) : (0, \infty) \to [0,1]$

Theorem(3.18): Let $f: X \to Y$ be a fuzzy soft function .Then $\|f\|$ is fuzzy soft norm .

Theorem(3.19): Let $f: X \to Y$ and $g: Y \to Z$ be two fuzzy soft function .Then

a) $\|f \circ g\| \leq \|f\|\|g\|$;

b) If $f: X \to Y$ is fuzzy soft function then $\|f^n\| \leq \|f\|^n$.

Is satisfied .

Proof :

a) $\|f \circ g\| = \sup\{\|f \circ g\ (x_e)\| : \|x_e\| \leq 1\}$

$$= \sup\{\left\|f\left(g(x_e)\right)\right\| : \|x_e\| \leq 1\}$$

$$\leq \sup\{\ \|f\|.\|g(x_e)\| : \|x_e\| \leq 1\}$$

$$\leq \|f\|\|g\|$$

b) If we take $f = g$ then we have $\|f^2\| \leq \|f\|^2$. Then $\|f^n\| \leq \|f\|^n$ is obtained.

References

[1] P.K. Maji, R. Biswas, A.R. Roy, Fuzzy Soft Set ,Journal of Fuzzy Mathematics 9 (3) (2001) 589-602.

[2] B. Ahmad and Athar Kharal, On Fuzzy Soft sets, Advances in Fuzzy Systems, Volume 2009, Article ID 586507.

[3] Athar Kharal and B. Ahmad, Mappings on Fuzzy Soft Classes, Advances in Fuzzy Systems, Volume 2009, Article ID 407890. 3308-3314.

[4] B. Tanay, M. B. Kandemir, Topological structure of fuzzy soft sets, Computers and Mathematics with Applications

61 (2011) 2952-2957.

[5] J. Mahanta and P. K. Das, Results on fuzzy soft topological spaces, arXiv:1203.0634v1 [math.GM] 3 Mar 2012

[6] L.A.Zadeh, Fuzzy sets , Inf. Control, 8 (1965), 338-353

Numerical Solution Based on Backward Differentiation Techniques for Systems of Nonlinear Weakly-Singular Volterra Integral Equations

Merewan A. Saleh[1] , Borhan F. Jumaa[2] and Esa I. Esa [3]

[1, 2]Computer Science Department, College of Computer Science and Information Technology, University of Kirkuk, Kirkuk, Iraq.

[1]marewan_812004abdal@yahoo.com

[2]borhan_nissan@yahoo.com

Abstract and

This paper uses Backward Differentiation Formulas of step (1, 2, 3, 4, 5 and 6) to evaluate a system of three and two second kind of nonlinear Weakly-Singular Volterra integral equations numerically. The system is solved by using MATLAB 2014b software. Finally, a number of examples are proposed to demonstrate the accuracy and effectiveness of this formula.

Keywords: Nonlinear integral equation, Weakly-singular Volterra, Backward differentiation formulas.

الحلول العددية القائمة على تقنيات التمايز الى الوراء لنظام معادلات فولتيرا المفرد ضعيف التكاملية غير الخطية

مريوان عبدل صالح[1] برهان فخرالدين جمعة[2]

[1,2]جامعة كركوك / كلية علوم الحاسوب وتكنولوجيا المعلومات / قسم علوم الحاسوب

[1]marewan_812004abdal@yahoo.com

[2]borhan_nissan@yahoo.com

الملخص

في هذا البحث, استخدمنا صيغ التمايز الى الوراء من الخطوات (6 ,1،2،3،4،5) لحل منظومة مكونة من معادلتين وثلاث معادلات فولتيرا المفرد ضعيف التكاملية غير الخطية من النوع الثاني وقد استخدمنا برنامج matlab14 لحل النظام. وأخيرا، قدمنا العديد من الأمثلة التوضيحية لإظهار فعالية ودقة هذه الصيغ.

الكلمات الدالة: المعادلات التكاملية غير خطية, فولتيرا المفرد ضعيف , صيغ التمايز الى الوراء.

1 Introduction:

The following equation formulates the general nonlinear weakly-singular Volterra integral equations of the second kind:

$$\varphi(x) = g(x) + \int_0^x \frac{\beta}{[f(x)-f(t)]^\alpha} G(\varphi(t))dt \quad , \; 0 < \alpha < 1 \,, \;\; x \in [0, T] \qquad \ldots(1)$$

where β is a constant, and $G(\varphi(t))$ is a nonlinear function of $\varphi(t)$. In the literature, one can find these equations in several mathematical chemistry and physics, such as heat conduction, stereology, radiation of heat from a semi-infinite solid and crystal growth. It is also assumed at the function $g(x)$ is a given real valued function [1, 2].

The books edited by Wazwaz [3] and Linz [4] contain some different methods to solve the system of nonlinear weakly-Singular Volterra integral equations analytically. The author of [5] used the Trapezoidal Predictor-Corrector Method to solve the system of two nonlinear Volterra Integral Equations of the Second Kind. Furthermore, the authors in [6] proposed a numerical solution of System of two nonlinear Volterra integral equations. Moreover, Borhan and Abbas in [7] used non-Polynomial spline method to solve systems of two nonlinear Volterra integral equations. Finally, the authors in [8] used predictor-corrector methods as a numerical solution of these systems.

Accordingly, this work studies the system of two and three nonlinear weakly-singular Volterra integral equations of the second kind. The following form formulates the unknown functions that appear inside and outside the integral sign [8].

$$\varphi_i(x) = g_i(x) + \sum_{j=1}^{m} \int_0^x k_{ij}(x,t) G_{ij}(\varphi_j(t))dt \qquad i = 1,2, \ldots, n \qquad \ldots(2)$$

where $\varphi_i(x)$ are unknown functions, and $G_{ij}(\varphi_j(t))$ is a nonlinear function of $\varphi_j(t)$. The functions $g_i(x), i = 1, 2, 3$ and kernels $k_{ij}(x,t), 1 \le i, j \le 3$ are given real-valued functions on subsets of R^3 *and* R^1, respectively. The kernels k_{ij} comprises of singular kernels where the generalized form is as follows:

$$k_{ij} = \frac{1}{[f(x)-f(t)]^{\alpha_{ij}}} \quad , \qquad 1 \le i, \qquad j \le 3$$

2 Backward Differentiation Formulas [2]:

Let $Q_q(t)$ denote the polynomial of degree $\leq q$ that interpolate Y(t) at the points $t_{n+1}, t_n, \dots, t_{n-q+1}$ for some $q \geq 1$,

$$Q_q(t) = \sum_{j=-1}^{q-1} Y\left(t_{n-j}\right) l_{j,n}(t), \qquad \dots(3)$$

where $\{l_{j,n}(t): j = -1, \dots, q-1\}$ are the Lagrange interpolation basis functions for the nodes $t_{n+1}, t_n, \dots, t_{n-q+1}$. Use

$$Q'_q(t_{n+1}) = Y'(t_{n+1}) = f(t_{n+1}, Y(t_{n+1})) \qquad \dots(4)$$

Combining (4) with (3) and solving for $Y(t_{n+1})$, we obtain

$$Y(t_{n+1}) = \sum_{j=0}^{q-1} \alpha_j Y(t_{n-j}) + h\beta g(t_{n+1}, Y(t_{n+1}))$$

The p-step Backward Differentiation Formulas is given by

$$y_{n+1} = \sum_{j=0}^{q-1} \alpha_j \, y_{n-j} + h\beta g(t_{n+1}, y_{n+1}) \qquad \dots(5)$$

The truncation error for (5) can be obtained from the error formulas for numerical differentiation

$$T_n(Y) = -\frac{\beta}{q+1} h^{q+1} \, Y^{(q+1)}(\rho_n) \text{ for some } t_{n-q+1} \leq \rho_n \leq t_{n+1} \qquad \dots(6)$$

3 Numerical Methods

For simplicity, an interval [a,b] is taken to develop the numerical method for approximation solution of a system of the type (2). Accordingly, a grid of $N+1$ equally spaced points $x_i = a + ih, i = 0,1, \dots, N$ is defined, where $h = \frac{b-a}{N+1}$. Now, the following represents the p-step Backward formula for each i^{th} segment.

$$y_{n+1} = \sum_{j=0}^{q-1} \alpha_j \, y_{n-j} + h\beta g(t_{n+1}, y_{n+1}) \qquad \dots(7)$$

To solve the equation (2) we combine (7) with (2), then we obtain:

$$\varphi_i(x_{n+1}) = g_i(x_{n+1}) + \sum_{j=0}^{q-1} \alpha_j \, \varphi_{n-j} + h\beta \sum_{j=1}^{m} k_{ij}(x_{n+1}, t_{n+1}) G_{ij}(\varphi_j(t_{n+1})) \qquad \dots(8)$$

Accordingly, the system of two and three nonlinear Weakly-Singular Volterra integral equations of the second kind can be approximated via (8). The approximation is defined as NLWSVIEBD algorithm, which is stated as follows:

Algorithm (NLWSVIEBD):

Step 1: Set h $= \frac{b-a}{N}$; $x_i = x_0 + ih,$ i $= 0,1,2,\dots$N, $x_i =$ a, $x_N =$ b.

Step 2: For $q = 1\ to\ 6$ perform the next steps.

Step 3: For $i = 1\ to\ N + 1$ perform the next steps.

Step 4: Set $\varphi_i(x_0) = g_i(x_0)$**.**

Step 5: Evaluate $\varphi_i(x_{n+1})$ using equation (8).

4 Numerical Example:

In this section, we suggest a number of examples to illustrate the methods of Section 3.

4.1 Example (1)

System of two nonlinear Weakly-Singular Volterra integral equations of the second kind is defined as follows:

$$\varphi(x) = x^2 - \frac{2}{5}x^{\frac{5}{2}} - \frac{3}{14}x^{\frac{14}{3}} + \int_0^x \left(\frac{1}{(x^5 - t^5)^{\frac{1}{2}}}\varphi^2(t) + \frac{1}{(x^7 - t^7)^{\frac{1}{3}}}\emptyset^2(t) \right) dt$$

$$\emptyset(x) = x^3 - \frac{4}{15}x^{\frac{15}{4}} - \frac{5}{28}x^{\frac{28}{5}} + \int_0^x \left(\frac{1}{(x^5 - t^5)^{\frac{1}{4}}}\varphi^2(t) + \frac{1}{(x^7 - t^7)^{\frac{1}{5}}}\emptyset^2(t) \right) dt$$

which has the exact solution: $\big(\varphi(x), \emptyset(x)\big) = (x^2, x^3)$

Table (1): The truncation error of the numerical solution via (**NLWSVIEBD**) algorithm.

q	**Error in φ**	**Error in $\emptyset$**

1	0.192913E-02	0.313335E-02
2	0.479822E-03	0.653059E-03
3	0.119803E-03	0.142067E-03
4	0.299413e-04	0.634997E-04
5	0.748472E-05	0.976963E-05
6	0.187687E-05	0.778953E-05

4.2 Example (2):

System of two nonlinear Weakly-Singular Volterra integral equations of the second kind is defined as follows:

$$\varphi(x) = sin^{\frac{1}{2}}x + \frac{3}{2}(cosx - 1)^{\frac{2}{3}} - \frac{3}{2}sin^{\frac{2}{3}}x + \int_0^x \left(\frac{1}{(cosx - cost)^{\frac{1}{3}}}\varphi^2(t) + \frac{1}{(sinx - sint)^{\frac{1}{3}}}\emptyset^2(t) \right) dt$$

$$\emptyset(x) = cos^{\frac{1}{2}}x + 3(cosx - 1)^{\frac{1}{3}} + 3sin^{\frac{1}{3}}x + \int_0^x \left(\frac{1}{(cosx - cost)^{\frac{2}{3}}}\varphi^2(t) - \frac{1}{(sinx - sint)^{\frac{2}{3}}}\emptyset^2(t) \right) dt$$

which has the exact solution: $\big(\varphi(x), \emptyset(x)\big) = (sin^{\frac{1}{2}}x \ , \quad cos^{\frac{1}{2}}x)$

Table (2): The truncation error of the numerical solution via (**NLWSVIEBD**) algorithm.

Q	Error in φ	Error in $\emptyset$
1	0.768544E-01	0.617599E-01
2	0.185549E-01	0.156383E-01
3	0.435448E-02	0.598975E-02
4	0.110156E-02	0.778439E-03
5	0.274531E-03	0.237855E-03
6	0.688776E-04	0.553324E-04

4.3 Example (3):

System of three nonlinear Weakly-Singular Volterra integral equations of the second kind is defined as follows:

$$\varphi(x) = e^{x} - \frac{1}{2}(e^{4x}-1)^{\frac{1}{2}} - \frac{1}{3}(e^{6x}-1)^{\frac{1}{2}} + \int_0^x \left(\frac{1}{(e^{4x}-e^{4t})^{\frac{1}{2}}} \emptyset^2(t) + \frac{1}{(e^{6x}-e^{6t})^{\frac{1}{2}}} \omega^2(t) \right) dt$$

$$\emptyset(x) = e^{2x} - \frac{1}{3}(e^{6x}-1)^{\frac{1}{2}} - (e^{2x}-1)^{\frac{1}{2}} + \int_0^x \left(\frac{1}{(e^{6x}-e^{6t})^{\frac{1}{2}}} \omega^2(t) + \frac{1}{(e^{2x}-e^{2t})^{\frac{1}{2}}} \varphi^2(t) \right) dt$$

$$\varphi(x) = e^{3x} - (e^{2x}-1)^{\frac{1}{2}} - \frac{1}{2}(e^{4x}-1)^{\frac{1}{2}} + \int_0^x \left(\frac{1}{(e^{2x}-e^{2t})^{\frac{1}{2}}} \varphi^2(t) + \frac{1}{(e^{4x}-e^{4t})^{\frac{1}{2}}} \emptyset^2(t) \right) dt$$

which has the exact solution: $(\varphi(x), v(x), w(x)) = (e^{x}, e^{2x}, e^{3x})$

Table (3): The truncation error of the numerical solution via (**NLWSVIEBD**) algorithm.

Q	**Error in φ**	**Error in $\emptyset$**	**Error in ω**
1	**0.582415E-02**	**0.412917E-01**	**0.015870E-00**
2	**0.182492E-02**	**0.209369E-01**	**0.828769E-01**
3	**0.719683E-03**	**0.529830E-02**	**0.232478E-01**
4	**0.357568E-03**	**0.914723E-03**	**0.651729E-02**
5	**0.117323E-03**	**0.467798E-03**	**0.885640E-03**
6	**0.443456E-04**	**0.010317E-03**	**0.336518E-03**

5 Conclusion:

In this paper, we proposed using of Backward Differentiation Formulas method to solve the system of two and three nonlinear Weakly-Singular Volterra integral equations of the second kind numerically. The findings demonstrated the superiority of our proposed methods achieved more better accuracy in solving the systems. Furthermore, q is the key to getting good approximation results, where the increasing of q leads to an increase in the number of points, and the error approaches zero.

References

[1] J.R.Rise, Numerical Method Software and Analysis, macgra Hill, 1985.

[2] K.Atkinson,W. Han,D.Stewart , Numerical Solution of Ordinary Differential Equations, United States of America, 2009.

[3] A.M.Wazwaz, Linear and Nonlinear Integral Equation, Methods and Applications, Springer, 2011.

[4] P.Linz, Analytical and numerical methods for Volterra Equations, studies in Applied Mathematics, SIAM, Philadelphia, 1985.

[5] F.J.Borhan, Solving the System of Two Nonlinear Volterra Integral Equations of the Second Kind Using the Trapezoidal Predictor-Corrector Method, British Journal of Mathematics & Computer Science , vol.20, No.3, P.P. 1-10, 2017.

[6] A. M.Dalal, Numerical Solution of System of Two Nonlinear Volterra Integral Equations, International Journal of Computers and Technology, Vol.12, No.10, P.3967-3975, 2014.

[7] F.J.Borhan, and H.T.Abbas, Non-Polynomial Spline Method for the Solution of the System of two Nonlinear Volterra Integral Equations, Kirkuk University Journal/Scientific Studies(KUJSS) , Vol.11, No.3, P.P.(15-25), 2016.

[8] F.J.Borhan, and H.T.Abbas, Using Predictor-Corrector Methods for Numerical Solution of System of Non-linear Volterra Integral Equations of Second Kind, AL-Rafiden Journal of Computer Sciences and Mathematics, Vol.6, No.3, P.P.(93-109), 2009.

Fuzzy Neutrosophic Soft cosets on HX ring

Dr.Mohammed J.Al-Mesafer , Faleh Abdul Mahdi Jaber
Department of Mathematics , College of Education for Pure Sciences,
University of Thi qar

ABSTRACT: In this paper, we introduce the notion of fuzzy neutrosophic soft HX subring of a HX ring , fuzzy neutrosophic soft coset and pseudo fuzzy neutrosophic soft coset of a HX ring. We explain some related properties and results of fuzzy neutrosophic soft coset and pseudo fuzzy neutrosophic soft coset of fuzzy neutrosophic soft HX subring.

Keywords: fuzzy neutrosophic soft set, fuzzy HX ring , soft set .

1.Introduction

In 1965 Zadah [12] introduced the concept of Fuzzy subset of a set X as a function from universal set into the closed unit interval[0, 1] and studied their properties. Atanassov [1] introduced the developed the theory of intuitionistic fuzzy sets. In 1999 ,[8] Molodtsov initiated the novel concept of soft set theory which is a completely new approach for modeling vagueness and uncertainty. Further P. K. Maji, R. Biswas and A. R. Roy [5] introduced the concept of fuzzy soft sets in 2001. Also, P. K. Maji, R. Biswas and A. R. Roy [6] defined and clarified the concept of intuitionistic fuzzy soft set .In 1988 Professor Li Hong Xing [4] proposed the concept of HX ring and derived some of its properties. In 2014, R. Muthuraj and M. Muthuraman [9] defined the notion of intuitionistic fuzzy HX ring of a HX ring and investigated some of their related properties with the necessity and possibility operators of an intuitionistic fuzzy HX ring. The concept of neutrosophic set (NS) was first introduced by Smarandache [11] which is a generalization of classical sets, fuzzy set, intuitionistic fuzzy set etc. Later, Maji[7] has introduced a combined concept neutrosophic soft set (NSS). .In 2016 the concept of neutrosophic soft ring is introduced by Bera and Mahapatra [2] and some basic properties related to it are established .In 2018 R. Muthuraj and M.S. Muthuraman [10] introduced the concept of intuitionistic fuzzy coset on HX ring and intuitionistic pseudo fuzzy coset of a HX ring and were discussed some related properties of intuitionistic fuzzy coset and of intuitionistic pseudo fuzzy coset of intuitionistic fuzzy HX subring . In this paper, we introduce the notion of fuzzy neutrosophic soft HX subring , fuzzy neutrosophic soft coset , pseudo fuzzy neutrosophic soft coset of a HX ring. We discussed some related properties of fuzzy neutrosophic soft coset and pseudo fuzzy neutrosophic soft coset of fuzzy neutrosophic soft HX subring.

2 Preliminaries

We recall some basic definitions related to, neutrosophic soft ring.

2.1 Definition[9]

A neutrosophic set A=$\{< x, T_A(x), I_A(x), F_A(x) >: x \in R\}$ over a ring $(R,+,.)$ is called a neutrosophic subring of $(R,+,.)$ if the followings hold

i) $\begin{cases} T_A(x+y) \geq \min\{T_A(x), T_A(y)\} \\ I_A(x+y) \leq \max\{I_A(x), I_A(y)\} \\ F_A(x+y) \leq \max\{F_A(x), F_A(y)\} \end{cases}$, ii) $\begin{cases} T_A(-x) \geq T_A(x) \\ I_A(-x) \leq I_A(x) \\ F_A(-x) \leq F_A(x) \end{cases}$

$$iii)\begin{cases} T_A(x.y) \geq \min\{T_A(x), T_A(y)\} \\ I_A(x.y) \leq \max\{I_A(x), I_A(y)\} \\ F_A(x.y) \leq \max\{F_A(x), F_A(y)\} \end{cases}$$

for all $x, y \in R$.

Definition(2.5)[9]

In NSS $H = \{(e, \{< x, T_{fH(e)}(x), I_{fH(e)}(x), F_{fH(e)}(x) >: x \in U\}): e \in E\}$ over a ring $(R, +, .)$ is called a neutrosophic soft ring if $f_{H(e)}$is a neutrosophic subring of $(R, +, .)$ for each e$\in E$.

Proposition(2.5) [9]

An NSS $H = \{(e, \{< x, T_{fH(e)}(x), I_{fH(e)}(x), F_{fH(e)}(x) >: x \in U\}): e \in E\}$ over the ring $(R, +, .)$ is called a neutrosophic soft ring iff following's hold;

$$\text{i)}\begin{cases} T_{fH(e)}(x - y) \geq \min\{T_A(x), T_A(y)\} \\ I_{fH(e)}(x - y) \leq \max\{I_A(x), I_A(y)\} \\ F_{fH(e)}(x - y) \leq \max\{F_A(x), F_A(y)\} \end{cases} \qquad \text{ii)}\begin{cases} T_{fH(e)}(xy) \geq \min\{T_A(x), T_A(y)\} \\ I_{fH(e)}(xy) \leq \max\{I_A(x), I_A(y)\} \\ F_{fH(e)}(xy) \leq \max\{F_A(x), F_A(y)\} \end{cases}$$

for each $x, y \in R$, $e \in E$

3 *Fuzzy Neutrosophic Soft coset of a fuzzy neutrosophic soft HX*

3.1 Definition[4]

Let R be a ring and a non-empty set N$\subset 2^R - \{\Phi\}$ with two binary operation (+) and (.) , then N is said to be HX ring of R if N is a ring respect to the algebraic operation defined by

i) A+B= {a+b: a$\in A$, and $b \in B$} which its null element is denoted by Q , and the negative element of A is denoted by – A .

ii) $A.B = \{a.b\colon\ a \in A\ , b \in B\}$

iii) $A.(B + C) = A.B + A.C \quad and \quad (B + C).A = B.A + C.A$

3.2 Definition

Let R be a ring (F, E) be a fuzzy soft set defined on a R and $N \subset 2^R - \{\Phi\}$ be a HX ring on R . We defin a fuzzy soft HX set $(\lambda_{f(e)})$ such that

$\lambda_{f(e)} = \max\{F_e(x) : for\ all\ x \in A \subset R\ , e \in E\}$

3.3 Definition

Let H be a fuzzy neutrosophic soft set (FNSS) over a ring R such that

$H = \{< x, T_{fH(e)}(x), I_{fH(e)}(x), F_{fH(e)}(x) >: x \in R, e \in E\}$, E be set of parameters and where $T, I, F: R \to [0,1]$ such that $0 \leq T_{fH(e)}(x) + T_{fH(e)}(x) + T_{fH(e)}(x) \leq 3$. Let $N \subset 2^R - \{\Phi\}$ be a HX ring . A fuzzy neutrosophic soft subset λ^H such that

$$\lambda^H = \{< A, \lambda^T_{fH(e)}(A), \lambda^I_{fH(e)}(A), \lambda^F_{fH(e)}(A) >: A \in N, and\ \ \lambda^T_{f0H(e)}(A) + \lambda^I_{fH(e)}(A) + \lambda^F_{fH(e)}(A) \leq 3\}$$

Of N is called on fuzzy neutrosophic soft HX subring induced by H if the following condition are satisfied for A,B $\in N$ $and\ e \in E$

a) $\begin{cases} (i)\ \ \lambda^T_{fH(e)}(A - B) \geq \min\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\} \\ ii)\ \ \lambda^T_{fH(e)}(AB) \geq \min\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\} \end{cases}$

b) $\begin{cases} (i)\ \ \lambda^I_{fH(e)}(A - B) \leq \max\{\lambda^I_{fH(e)}(A), \lambda^I_{fH(e)}(B)\} \\ ii)\ \ \lambda^I_{fH(e)}(AB) \leq \max\{\lambda^I_{fH(e)}(A), \lambda^I_{fH(e)}(B)\} \end{cases}$

c) $\begin{cases} (i)\ \ \lambda^F_{fH(e)}(A - B) \leq \max\{\lambda^F_{fH(e)}(A), \lambda^F_{fH(e)}(B)\} \\ ii)\ \ \lambda^F_{fH(e)}(AB) \leq \max\{\lambda^F_{fH(e)}(A), \lambda^F_{fH(e)}(B)\} \end{cases}$

$$where \qquad \lambda^T_{fH(e)}(A) = \max\{T_{fH(e)}(x): for\ all\ x \in A \subseteq R, e \in E\}$$

$$\lambda^I_{fH(e)}(A) = \min\{I_{fH(e)}(x): for\ all\ x \in A \subseteq R, e \in E\}$$

$$\lambda^F_{fH(e)}(A) = \min\{F_{fH(e)}(x): for\ all\ x \in A \subseteq R, e \in E\}$$

3.4 Definition

Let H be a fuzzy neutrosophic soft set (FNSS) over a ring R such that

$H = \{< x, T_{fH(e)}(x), I_{fH(e)}(x), F_{fH(e)}(x) >: x \in R, e \in E\}$, E be set of parameters and where $T, I, F: R \to [0,1]$ such that $0 \leq T_{fH(e)}(x) + T_{fH(e)}(x) + T_{fH(e)}(x) \leq 3$. Let $N \subset 2^R - \{\Phi\}$ be a HX ring and let

$\lambda^H = \{< A, \lambda^T_{fH(e)}(A), \lambda^I_{fH(e)}(A), \lambda^F_{fH(e)}(A) >: A \in N, and\ \ \lambda^T_{f0H(e)}(A) + \lambda^I_{fH(e)}(A) + \lambda^F_{fH(e)}(A) \leq 3\}$ a fuzzy neutrosophic soft HX subring of a HX ring N . Then the fuzzy neutrosophic soft coset of a fuzzy neutrosophic soft HX subring $\lambda^H$of a HX ring N determined by the element A$\in N$ is denoted by $(A$+$\lambda^H)$ and is defined by

$(A$+$\lambda^H)(X) = \{< X - A, \lambda^T_{fH(e)}(X - A), \lambda^I_{fH(e)}(X - A), \lambda^F_{fH(e)}(X - A) >\}$ for every $X \in N$.

3.5 *Remark*

i) If A=Q then the fuzzy neutrosophic soft coset A+$\lambda^H = \lambda^H$

ii) If λ^H a fuzzy neutrosophic soft HX subring of a HX ring N and $A = Q$ then the neutrosophic soft coset (A+λ^H) is also a fuzzy neutrosophic soft HX subring of a HX ring N.

3.6 $Theorem$

Let R be a ring .Let H be a fuzzy neutrosophic soft set defined on R and λ^{H} be a fuzzy neutrosophic soft HX subring of a HX ring N .Then for any $A, B \in N$,

$(A + \lambda^{H}) = (B + \lambda^{H})$ iff

$$\lambda^{T}_{fH(e)}(B - A) = \lambda^{T}_{fH(e)}(A - B) = \lambda^{T}_{fH(e)}(Q)$$

$$\lambda^{I}_{fH(e)}(B - A) = \lambda^{I}_{fH(e)}(A - B) = \lambda^{I}_{fH(e)}(Q)$$

$$\lambda^{F}_{fH(e)}(B - A) = \lambda^{F}_{fH(e)}(A - B) = \lambda^{F}_{fH(e)}(Q)$$

Proof: Let λ^{H} be a fuzzy neutrosophic soft HX subring of a HX ring N . Let

$(A + \lambda^{H}) = (B + \lambda^{H})$,for any $A, B \in N$, e$\in E$

$(A + \lambda^{T}_{fH(e)})(X) = (B + \lambda^{T}_{fH(e)})(X)$, $(A + \lambda^{I}_{fH(e)})(X) = (B + \lambda^{I}_{fH(e)})(X)$, $(A + \lambda^{F}_{fH(e)})(X) = (B + \lambda^{F}_{fH(e)})(X)$

a)i) $(A + \lambda^{T}_{fH(e)})(B) = (B + \lambda^{T}_{fH(e)})(B)$

$$\lambda^{T}_{fH(e)}(B - A) = \lambda^{T}_{fH(e)}(B - B)$$

$$\lambda^{T}_{fH(e)}(B - A) = \lambda^{T}_{fH(e)}(Q)$$

ii) $(A + \lambda^{T}_{fH(e)})(A) = (B + \lambda^{T}_{fH(e)})(A)$

$$\lambda^{T}_{fH(e)}(A - A) = \lambda^{T}_{fH(e)}(A - B)$$

$$\lambda^{T}_{fH(e)}(Q) = \lambda^{T}_{fH(e)}(A - B)$$

Hence $\lambda^{T}_{fH(e)}(B - A) = \lambda^{T}_{fH(e)}(A - B) = \lambda^{T}_{fH(e)}(Q)$

b)i) $(A + \lambda^{I}_{fH(e)})(B) = (B + \lambda^{I}_{fH(e)})(B)$

$$\lambda^{I}_{fH(e)}(B - A) = \lambda^{I}_{fH(e)}(B - B)$$

$$\lambda^{I}_{fH(e)}(B - A) = \lambda^{I}_{fH(e)}(Q)$$

ii) $(A + \lambda^{I}_{fH(e)})(A) = (B + \lambda^{I}_{fH(e)})(A)$

$$\lambda^{I}_{fH(e)}(A - A) = \lambda^{I}_{fH(e)}(A - B)$$

$$\lambda^{I}_{fH(e)}(Q) = \lambda^{I}_{fH(e)}(A - B)$$

Hence $\lambda^{I}_{fH(e)}(B - A) = \lambda^{I}_{fH(e)}(A - B) = \lambda^{I}_{fH(e)}(Q)$

And in the same way we can prove that $\lambda^F_{fH(e)}(B-A)=\lambda^F_{fH(e)}(A-B)=\lambda^F_{fH(e)}(Q)$

Conversely let

$\lambda^T_{fH(e)}(B-A)=\lambda^T_{fH(e)}(A-B)=\lambda^T_{fH(e)}(Q)$

$\lambda^I_{fH(e)}(B-A)=\lambda^I_{fH(e)}(A-B)=\lambda^I_{fH(e)}(Q)$

$\lambda^F_{fH(e)}(B-A)=\lambda^F_{fH(e)}(A-B)=\lambda^F_{fH(e)}(Q)$

For any $X\in N$ $\quad (A+\lambda^T_{fH(e)})(X)=\lambda^T_{fH(e)}(X-A)$

$=\lambda^T_{fH(e)}(X-B+B-A)$

$\geq \min\{\lambda^T_{fH(e)}(X-B),\lambda^T_{fH(e)}(B-A)\}$

$=\min\{\lambda^T_{fH(e)}(X-B),\lambda^T_{fH(e)}(Q)\}$

$=\lambda^T_{fH(e)}(X-B)$

$=(B+\lambda^T_{fH(e)})(X)$

Hence $\quad (A+\lambda^T_{fH(e)})(X)\geq(B+\lambda^T_{fH(e)})(X)$

Similarly $\quad (A+\lambda^T_{fH(e)})(X)\leq(B+\lambda^T_{fH(e)})(X)$

Hence $\quad \left(A+\lambda^T_{fH(e)}\right)(X)=(B+\lambda^T_{fH(e)})(X)$

And in the same we can prove that $\left(A+\lambda^I_{fH(e)}\right)(X)=(B+\lambda^I_{fH(e)})(X)$

Also $\quad \left(A+\lambda^F_{fH(e)}\right)(X)=(B+\lambda^F_{fH(e)})(X)$

Hence $\quad (A+\lambda^H)=(B+\lambda^H\ )$

3.7 Theorem

Let R be a ring and H be a fuzzy neutrosophic soft set defined on R and λ^H be a fuzzy neutrosophic soft HX subring of a HX ring N .Let for any $A,B\in N$, $(\mathrm{A}+\lambda^H)=(B+\lambda^H\ )$ then

$\lambda^T_{fH(e)}(A)=\lambda^T_{fH(e)}(B)$, $\lambda^I_{fH(e)}(A)=\lambda^I_{fH(e)}(B)$,$\lambda^F_{fH(e)}(A)=\lambda^F_{fH(e)}(B)$

Proof:

.Let λ^H be a fuzzy neutrosophic soft HX subring of a HX ring N .Let for any $A,B\in N$

$(A+\lambda^H)=(B+\lambda^H\ )$ then

$$\lambda^T_{fH(e)}(A)=\lambda^T_{fH(e)}((A-B)+B)$$

$\geq \min\{\lambda^T_{fH(e)}(A-B), \lambda^T_{fH(e)}(B)\}$

$=\min\{\lambda^T_{fH(e)}(Q), \lambda^T_{fH(e)}(B)\}$ (by theorem 3.6)

$=\lambda^T_{fH(e)}(B)$

Hence $\lambda^T_{fH(e)}(A) \geq \lambda^T_{fH(e)}(B)$

Similarly $\lambda^T_{fH(e)}(A) \leq \lambda^T_{fH(e)}(B)$

Hence $\lambda^T_{fH(e)}(A) = \lambda^T_{fH(e)}(B)$

And in the same way we can prove that $\lambda^I_{fH(e)}(A) = \lambda^I_{fH(e)}(B)$, $\lambda^F_{fH(e)}(A) = \lambda^F_{fH(e)}(B)$

4 Homomorphism and anti homomorphism of afuzzy neutrosophic soft coset of a fuzzy neutrosophic soft HX subring of a HX ring

In this section we introduce the concept of an image ,pre-image of a fuzzy neutrosophic soft coset of a HX ring and discuss the properties of homomorphism and anti homomorphism images and pre-images of a fuzzy neutrosophic soft coset of a fuzzy neutrosophic soft HX subring λ^H of a HX ring N determined by the element $A \in N$.

4.1 Theorem: *Let N_1 and N_2 be any two HX rings on the R_1 and R_2 respectively . Let* $g: N_1 \to N_2$ be a homomorphism onto HX rings. And let $(A + \lambda^H)$ be a fuzzy neutrosophic soft coset of a fuzzy neutrosophic soft HX subring λ^H of a HX N_1 determined by the element $A \in N_1$ .Then $g(A + \lambda^H)$ is a fuzzy neutrosophic soft coset of a fuzzy neutrosophic soft HX subring $g(\lambda^H)$ of a HX ring N_2 determined by the element $g(A) \in N_2$ and $g(A + \lambda^H) = (g(A) + g(\lambda^H))$,if λ^H has a supremum property and λ^H is $g-$ invariant .

Proof: Let $g: N_1 \to N_2$ be a homomorphism onto HX rings. Let $(A + \lambda^H)$ be a fuzzy neutrosophic soft coset of a fuzzy neutrosophic soft HX subring λ^H of a HX N_1 determined by the $A \in N_1$. Then

$g(\lambda^H) = \{< g(X) , g(\lambda^T_{fH(e)})g(X), g\left(\lambda^I_{fH(e)}\right)g(X), g\left(\lambda^F_{fH(e)}\right)g(X) >: X \in N_1\}$

Let $A, B \in N_1$, *then* $g(A) , g(B) \in N_2$

Now a) i) $(g(\lambda^T_{fH(e)}))\left(g(A) - g(B)\right) = (g\left(\lambda^T_{fH(e)}\right))(g(A-B))$ (g homomorphism)

$=\lambda^T_{fH(e)}(A-B)$

$\geq \min\{ \lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}$

$=\min\{(g(\lambda^T_{fH(e)}))(g(A)), (g\left(\lambda^T_{fH(e)}\right))(g(B))\}$

Hence $(g(\lambda^T_{fH(e)}))\big(g(A)-g(B)\big) \geq \min\{(g(\lambda^T_{fH(e)}))(g(A)), (g(\lambda^T_{fH(e)}))(g(B))\}$

ii) $(g(\lambda^T_{fH(e)}))\big(g(A)g(B)\big) = (g(\lambda^T_{fH(e)}))(g(AB))$ (g homomorphism)

$$=\lambda^T_{fH(e)}(AB)$$

$$\geq \min\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}$$

Hence $(g(\lambda^T_{fH(e)}))\big(g(A)g(B)\big) \geq \min\{(g(\lambda^T_{fH(e)}))(g(A)), (g(\lambda^T_{fH(e)}))(g(B))\}$

b)) $(g(\lambda^I_{fH(e)}))\big(g(A)-g(B)\big) = (g(\lambda^I_{fH(e)}))(g(A-B))$ (g homomorphism)

$$=\lambda^I_{fH(e)}(A-B)$$

$$\leq \max\{\lambda^I_{fH(e)}(A), \lambda^I_{fH(e)}(B)\}$$

$$=\max\{(g(\lambda^I_{fH(e)}))(g(A)), (g(\lambda^I_{fH(e)}))(g(B))\}$$

Hence $(g(\lambda^I_{fH(e)}))\big(g(A)-g(B)\big) \leq \max\{(g(\lambda^I_{fH(e)}))(g(A)), (g(\lambda^I_{fH(e)}))(g(B))\}$

ii) $(g(\lambda^I_{fH(e)}))\big(g(A)g(B)\big) = (g(\lambda^I_{fH(e)}))(g(AB))$ (g homomorphism)

$$=\lambda^I_{fH(e)}(AB)$$

$$\leq \max\{\lambda^I_{fH(e)}(A), \lambda^I_{fH(e)}(A)\}$$

$$=\max\left\{\left(g(\lambda^I_{fH(e)})\right)(g(A)), (g(\lambda^I_{fH(e)}))(g(B)\right)$$

Hence $(g(\lambda^I_{fH(e)}))\big(g(A)g(B)\big) \leq \max\{(g(\lambda^I_{fH(e)}))(g(A)), (g(\lambda^I_{fH(e)}))(g(B))\}$

And in the same way we can prove that

$$c)\begin{cases}(g(\lambda^I_{fH(e)}))\big(g(A)-g(B)\big) \leq \max\{(g(\lambda^I_{fH(e)}))(g(A)), (g(\lambda^I_{fH(e)}))(g(B))\} \\ (g(\lambda^I_{fH(e)}))\big(g(A)g(B)\big) \leq \max\{(g(\lambda^I_{fH(e)}))(g(A)), (g(\lambda^I_{fH(e)}))(g(B))\}\end{cases}$$

Hence $g(\lambda^H)$ is a fuzzy neutrosophic soft HX subring of N_2

Now, let $A, X \in N_1$ *then* $g(A),\ g(X) \in N_2$

$$\left(g(A)+g(\lambda^T_{fH(e)})\right)(g(X)) = (g(\lambda^T_{fH(e)}))(g(X)-g(A))$$

$$=(g(\lambda^T_{fH(e)}))(g(X-A)) \quad (g \text{ homomorphism})$$

$$=(A+\lambda^T_{fH(e)})(X)$$

$$=g(A+\lambda^T_{fH(e)})g(X)$$

Hence $\left(g(A)+g\left(\lambda^{T}_{fH(e)}\right)\right)(g(X))=g\left(A+\lambda^{T}_{fH(e)}\right)g(X)$ for any g(X)∈ N_2

Hence $g\left(\left(A+\lambda^{T}_{fH(e)}\right)\right)=(g(A)+g(\lambda^{T}_{fH(e)}))$

Also $\left(g(A)+g\left(\lambda^{I}_{fH(e)}\right)\right)(g(X))=(g(\lambda^{I}_{fH(e)}))(g(X)-g(A))$

$=(g(\lambda^{I}_{fH(e)}))(g(X-A))$ (g homomorphism)

$=\left(A+\lambda^{I}_{fH(e)}\right)(X)$

$=g\left(A+\lambda^{I}_{fH(e)}\right)g(X)$

Hence $\left(g(A)+g\left(\lambda^{I}_{fH(e)}\right)\right)(g(X))=g\left(A+\lambda^{I}_{fH(e)}\right)g(X)$

Hence $g\left(\left(A+\lambda^{I}_{fH(e)}\right)\right)=(g(A)+g(\lambda^{I}_{fH(e)}))$

And in the same way we can prove that $g\left(\left(A+\lambda^{F}_{fH(e)}\right)\right)=(g(A)+g(\lambda^{F}_{fH(e)}))$

Hence $g(A+\lambda^{H})=(g(A)+g(\lambda^{H}))$

4.2 Theorem

Let N_1 and N_2 be any two HX rings on the R_1 and R_2 respectively . Let $g: N_1 \to N_2$ be a homomorphism onto HX rings and $(B+\text{Ɣ}^{G})$ be a fuzzy neutrosophic soft coset of a fuzzy neutrosophic soft HX subring Ɣ^{G} of a HX N_2 determined by the element $B\in N_2$ .Then $g^{-1}((B+\text{Ɣ}^{G}))$ is a fuzzy neutrosophic soft coset of a fuzzy neutrosophic soft HX subring $g^{-1}(\text{Ɣ}^{G})$ of a HX ring N_2 determined by the element $g^{-1}(B)\in N_1$ and $g^{-1}((B+\text{Ɣ}^{G}))=(g^{-1}(B)+g^{-1}(\text{Ɣ}^{G}))$.

Proof :Let $g: N_1 \to N_2$ be a homomorphism onto HX rings. Let $(B+\text{Ɣ}^{G})$ be a fuzzy neutrosophic soft coset of a fuzzy neutrosophic soft HX subring λ^{H} of a HX N_2 determined by the element $B\in N_2$. Let $B=g(Y)$, $Y\in N_1$

$g^{-1}(\text{Ɣ}^{G})=\{<X, g^{-1}(\text{Ɣ}^{T}_{fG(e)})(X), g^{-1}(\text{Ɣ}^{T}_{fG(e)})(X), g^{-1}(\text{Ɣ}^{T}_{fG(e)})(X)>: X\in N_1\}$

For any X , $Y \in N_1$, $g(X)$, $g(Y)\in N_2$

Now a) i) $(g^{-1}(\text{Ɣ}^{T}_{fG(e)})(A-B)=(\text{Ɣ}^{T}_{fG(e)})(g(A-B))$

$=(\text{Ɣ}^{T}_{fG(e)})(g(A)-g(B))$ (g homomorphism)

$\geq \min\{\text{Ɣ}^{T}_{fG(e)}(g(A)),\text{Ɣ}^{T}_{fG(e)}(g(B))\}$

$=\min\{(g^{-1}(\text{Ɣ}^{T}_{fG(e)})(A),(g^{-1}((\text{Ɣ}^{T}_{fG(e)}))(B)\}$

Hence $(g^{-1}(\text{Ɣ}^{T}_{fG(e)}))\geq \min\{(g^{-1}(\text{Ɣ}^{T}_{fG(e)})(A),(g^{-1}((\text{Ɣ}^{T}_{fG(e)}))(B)\}$

ii) i) $(g^{-1}(\gamma^{T}_{fG(e)})(AB) = \gamma^{T}_{fG(e)}(g(AB))$

$=\gamma^{T}_{fG(e)}(g(A)g(B))$

$\geq \min\{\gamma^{T}_{fG(e)}(g(A)), \gamma^{T}_{fG(e)}(g(B))\}$

$= \min\{g^{-1}(\gamma^{T}_{fG(e)})(A), g^{-1}(\gamma^{T}_{fG(e)})(B)\}$

Hence $(g^{-1}(\gamma^{T}_{fG(e)})(AB) \geq \min\{g^{-1}(\gamma^{T}_{fG(e)})(A), g^{-1}(\gamma^{T}_{fG(e)})(B)\}$

b) i) $(g^{-1}(\gamma^{I}_{fG(e)})(A-B) = \gamma^{I}_{fG(e)}(g(A-B))$

$= \gamma^{I}_{fG(e)}(g(A) - g(B))$ (g homomorphism)

$\leq \max\{\gamma^{I}_{fG(e)}(A), \gamma^{I}_{fG(e)}(B)\}$

$=\max\{(g^{-1}(\gamma^{I}_{fG(e)}))(A), (g^{-1}(\gamma^{I}_{fG(e)}))(B)\}$

Hence $(g^{-1}(\gamma^{I}_{fG(e)})(A-B) \leq \max\{(g^{-1}(\gamma^{I}_{fG(e)}))(A), (g^{-1}(\gamma^{I}_{fG(e)}))(B)\}$

ii) $(g^{-1}(\gamma^{I}_{fG(e)})(AB) = \gamma^{I}_{fG(e)}(g(AB))$

$=\gamma^{I}_{fG(e)}(g(A)g(B))$ (g homomorphism)

$\leq \max\{\gamma^{I}_{fG(e)}(g(A)), \gamma^{I}_{fG(e)}(g(B))\}$

$= \max\{g^{-1}(\gamma^{I}_{fG(e)})(A), g^{-1}(\gamma^{I}_{fG(e)})(B)$

Hence $(g^{-1}(\gamma^{I}_{fG(e)})(AB) \leq \max\{g^{-1}(\gamma^{I}_{fG(e)})(A), g^{-1}(\gamma^{I}_{fG(e)})(B)\}$

And in the same way we can prove that

c) $\begin{cases} (g^{-1}(\gamma^{F}_{fG(e)})(A-B) \leq \max\{(g^{-1}(\gamma^{F}_{fG(e)}))(A), (g^{-1}(\gamma^{F}_{fG(e)}))(B)\} \\ (g^{-1}(\gamma^{F}_{fG(e)})(AB) \leq \max\{g^{-1}(\gamma^{F}_{fG(e)})(A), g^{-1}(\gamma^{F}_{fG(e)})(B)\} \end{cases}$

Hence $g^{-1}(\gamma^{G})$ is a fuzzy neutrosophic soft HX subring of N_1

Let $X \in N_1$ and $B \in N_2$ .Then

$$\left(g^{-1}(B) + g^{-1}((\gamma^{T}_{fG(e)})\right)(X) = (g^{-1}(g(Y) + g^{-1}((\gamma^{T}_{fG(e)}))(X)$$

$$= (Y + g^{-1}((\gamma^{T}_{fG(e)}))(X)$$

$$= (g^{-1}((\gamma^{T}_{fG(e)}))(X - Y)$$

$$= (\gamma^{T}_{fG(e)}(g(X-Y))$$

$$= (\Upsilon^{T}_{fG(e)}(g(X) - g(Y)) \quad (g \text{ homomorphism})$$

$$= \left(g(Y) + \Upsilon^{T}_{fG(e)}\right)g(X)$$

$$= \left(B + \Upsilon^{T}_{fG(e)}\right)g(X)$$

$$= g^{-1}(\left(B + \Upsilon^{T}_{fG(e)}\right))(X)$$

Hence $\left(g^{-1}(B) + g^{-1}\left((\Upsilon^{T}_{fG(e)}\right)\right)(X) = g^{-1}(\left(B + \Upsilon^{T}_{fG(e)}\right))(X)$

Hence $g^{-1}\left(\left(B + \Upsilon^{T}_{fG(e)}\right)\right) = \left(g^{-1}(B) + g^{-1}\left((\Upsilon^{T}_{fG(e)}\right)\right)$

Also

$$\left(g^{-1}(B) + g^{-1}\left((\Upsilon^{I}_{fG(e)}\right)\right)(X) = (g^{-1}(g(Y) + g^{-1}\left((\Upsilon^{I}_{fG(e)}\right))(X)$$

$$= (Y + g^{-1}\left((\Upsilon^{I}_{fG(e)}\right))(X)$$

$$= (g^{-1}\left((\Upsilon^{I}_{fG(e)}\right))(X - Y)$$

$$= (\Upsilon^{I}_{fG(e)}(g(X - Y))$$

$$= (\Upsilon^{I}_{fG(e)}(g(X) - g(Y))$$

$$= \left(g(Y) + \Upsilon^{I}_{fG(e)}\right)g(X)$$

$$= \left(B + \Upsilon^{I}_{fG(e)}\right)g(X)$$

$$= g^{-1}(\left(B + \Upsilon^{I}_{fG(e)}\right))(X)$$

Hence $\left(g^{-1}(B) + g^{-1}\left((\Upsilon^{I}_{fG(e)}\right)\right)(X) = g^{-1}(\left(B + \Upsilon^{I}_{fG(e)}\right))(X)$

Hence $g^{-1}\left(\left(B + \Upsilon^{I}_{fG(e)}\right)\right) = \left(g^{-1}(B) + g^{-1}\left((\Upsilon^{I}_{fG(e)}\right)\right)$

And in the same way we can prove that $g^{-1}\left(\left(B + \Upsilon^{F}_{fG(e)}\right)\right) = \left(g^{-1}(B) + g^{-1}\left((\Upsilon^{F}_{fG(e)}\right)\right)$

Hence $g^{-1}((\mathrm{B} + \Upsilon^{G})) = (g^{-1}(B) + g^{-1}(\Upsilon^{G}))$

4.3 Theorem

Let N_1 and N_2 be any two HX rings on the R_1 and R_2 respectively . Let $g: N_1 \to N_2$ be anti homomorphism onto HX rings and $(A + \lambda^H)$ be a fuzzy neutrosophic soft coset of a fuzzy neutrosophic soft HX subring λ^H of a HX N_1 determined by the element $A \in N_1$ .Then $g(A + \lambda^H)$ is a fuzzy neutrosophic soft coset of a fuzzy neutrosophic soft HX subring $g(\lambda^H)$ of a HX ring N_2 determined by the element $g(A) \in N_2$ and $g(A+\lambda^H) = (g(A)+g(\lambda^H))$,if λ^H has a supremum property and λ^H is $g-$ invariant .

Proof: Let $g: N_1 \to N_2$ be anti homomorphism onto HX rings. Let $(A+\lambda^H)$ be a fuzzy neutrosophic soft coset of a fuzzy neutrosophic soft HX subring λ^H of a HX N_1 determined by the $A \in N_1$. Then

$$(g\lambda^H) = \{< g(X)\,, g(\lambda^T_{fH(e)})g(X), g(\lambda^I_{fH(e)})g(X), g(\lambda^F_{fH(e)})g(X) >: X \in N_1\}$$

Let $A, B \in N_1$, $then$ $g(A)\,, g(B) \in N_2$

Now a) i) $(g(\lambda^T_{fH(e)}))(g(A)-g(B)) = (g(\lambda^T_{fH(e)}))(g(A-B))$ (g homomorphism)

$$=\lambda^T_{fH(e)}(A-B)$$

$$\geq \min\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}$$

$$=\min\{(g(\lambda^T_{fH(e)}))(g(A)), (g(\lambda^T_{fH(e)}))(g(B))\}$$

Hence $(g\,(\lambda^T_{fH(e)}))(g(A)-g(B)) \geq \min\{(g(\lambda^T_{fH(e)}))(g(A)), (g(\lambda^T_{fH(e)}))(g(B))\}$

ii) $(g\,(\lambda^T_{fH(e)}))(g(A)g(B)) = (g(\lambda^T_{fH(e)}))(g(BA))$ (g anti homomorphism)

$$=\lambda^T_{fH(e)}(BA)$$

$$\geq \min\{\lambda^T_{fH(e)}(B), \lambda^T_{fH(e)}(A)\}$$

$$= \min\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}$$

Hence $(g\,(\lambda^T_{fH(e)}))(g(A)g(B)) \geq \min\{(g(\lambda^T_{fH(e)}))(g(A)), (g(\lambda^T_{fH(e)}))(g(B))\}$

b)) $(g\,(\lambda^I_{fH(e)}))(g(A)-g(B)) = (g(\lambda^I_{fH(e)}))(g(A-B))$

$$= \lambda^I_{fH(e)}(A-B)$$

$$\leq \max\{\lambda^I_{fH(e)}(A), \lambda^I_{fH(e)}(B)\}$$

$$= \max\{(g(\lambda^I_{fH(e)}))(g(A)), (g(\lambda^I_{fH(e)}))(g(B))\}$$

Hence $(g\,(\lambda^I_{fH(e)}))(g(A)-g(B)) \leq \max\{(g(\lambda^I_{fH(e)}))(g(A)), (g(\lambda^I_{fH(e)}))(g(B))\}$

ii) $(g\,(\lambda^I_{fH(e)}))(g(A)g(B)) = (g(\lambda^I_{fH(e)}))(g(BA))$

$$= \lambda^I_{fH(e)}(BA)$$

$$\leq \max\{\lambda^I_{fH(e)}(B), \lambda^I_{fH(e)}(A)\}$$

$$= \max\{\lambda^I_{fH(e)}(A), \lambda^I_{fH(e)}(B)\}$$

$$= \max\left\{\left(g(\lambda^{I}_{fH(e)})\right)(g(A)),(g(\lambda^{I}_{fH(e)}))(g(B)\right)$$

Hence $(g\,(\lambda^{I}_{fH(e)}))(g(A)g(B)) \le \max\{(g(\lambda^{I}_{fH(e)}))(g(A)),(g(\lambda^{I}_{fH(e)})(g(B))\}$

And in the same way we can prove that

c) $\begin{cases}(g(\lambda^{F}_{fH(e)}))(g(A)-g(B)) \le \max\{(g(\lambda^{F}_{fH(e)}))(g(A)),(g(\lambda^{F}_{fH(e)}))(g(B))\} \\ (g(\lambda^{F}_{fH(e)}))(g(A)g(B)) \le \max\{(g(\lambda^{F}_{fH(e)}))(g(A)),(g(\lambda^{F}_{fH(e)})(g(B))\}\end{cases}$

Hence $g\,(\lambda^{H})$ is a fuzzy neutrosophic soft HX subring of N_2.Then $g((A+\lambda^{H}))$ is the fuzzy neutrosophic coset of a fuzzy neutrosophic soft HX ring $g\,(\lambda^{H})$ of a HX ring N_2 determined by the element $g(A) \in N_2$.

To prove that $g(A+\lambda^{H}) = (g(A)+g(\lambda^{H}))$ by similarly to theorem (4.2)

5 Fuzzy neutrosophic soft pseudo coset of a fuzzy neutrosophic HX ring of a HX ring

In this section we introduce the notion of fuzzy neutrosophic soft pseudo coset of a fuzzy neutrosiophic HX ring and discuss its properties.

5.1 Definition Let H be a fuzzy neutrosophic soft set (FNSS) over a ring R .Let $N \subset 2^{R}-\{\Phi\}$ } be a HX ring and λ^{H} a fuzzy neutrosophic soft HX subring of a HX ring N and $A \in N$. Then the fuzzy neutrosophic soft pseudo coset of a fuzzy neutrosophic soft HX subring $\lambda^{H}$of a HX ring N determined by the element $A \in N$ is denoted by $(A+\lambda^{H})^{p}$ and is defined as

$$(A+\lambda^{T}_{fH(e)})^{p}(X) = p(A)\lambda^{T}_{fH(e)}(X)\ ,\ (A+\lambda^{I}_{fH(e)})^{p}(X) = p(A)\lambda^{I}_{fH(e)}(X),$$

$$(A+\lambda^{F}_{fH(e)})^{p}(X) = p(A)\lambda^{F}_{fH(e)}(X)$$

For every $X \in N$, and for some $p \in P$ where $P = \{p(X);\ \ p(X) \in [0,1],\ \forall X \in N\}$

$\mathbf{5.2\,Theorem}$

Let λ^{H} a fuzzy neutrosophic soft HX subring of a HX ring N . Then the fuzzy neutrosophic soft pseudo coset $(A+\lambda^{H})^{p}$ is a fuzzy neutrosophic soft HX subring of a HX ring N determined by the element $A \in N$.

Proof: For every $X, Y, A \in N$ *we have*

a) i) $(A+\lambda^{T}_{fH(e)})^{p}(X-Y) = p(A)\lambda^{T}_{fH(e)}(X-Y)$

$$(A+\lambda^{T}_{fH(e)})^{p}(X-Y) \ge p(A)\min\{\lambda^{T}_{fH(e)}(X),\lambda^{T}_{fH(e)}(Y)\}$$

$$=\min\{\ p(A)\lambda^{T}_{fH(e)}(X), p(A)\lambda^{T}_{fH(e)}(Y)\}$$

$$=\min\{(A+\lambda^{T}_{fH(e)})^{p}(X),(A+\lambda^{T}_{fH(e)})^{p}(Y)\}$$

Hence $(A+\lambda^{T}_{fH(e)})^{p}(X-Y) \geq \min\{(A+\lambda^{T}_{fH(e)})^{p}(X),(A+\lambda^{T}_{fH(e)})^{p}(Y)\}$

ii) $(A+\lambda^{T}_{fH(e)})^{p}(XY) = p(A)\lambda^{T}_{fH(e)}(XY)$

$$(A+\lambda^{T}_{fH(e)})^{p}(XY) \geq p(A)\min\{\lambda^{T}_{fH(e)}(X),\lambda^{T}_{fH(e)}(Y)\}$$

$$=\min\{p(A)\lambda^{T}_{fH(e)}(X),p(A)\lambda^{T}_{fH(e)}(Y)\}$$

$$=\min\{(A+\lambda^{T}_{fH(e)})^{p}(X),(A+\lambda^{T}_{fH(e)})^{p}(Y)\}$$

Hence $(A+\lambda^{T}_{fH(e)})^{p}(XY) \geq \min\{(A+\lambda^{T}_{fH(e)})^{p}(X),(A+\lambda^{T}_{fH(e)})^{p}(Y)\}$

b) i) $(A+\lambda^{I}_{fH(e)})^{p}(X-Y) = p(A)\lambda^{I}_{fH(e)}(X-Y)$

$$(A+\lambda^{I}_{fH(e)})^{p}(X-Y) \leq p(A)\max\{\lambda^{I}_{fH(e)}(X),\lambda^{I}_{fH(e)}(Y)\}$$

$$=\max\{p(A)\lambda^{I}_{fH(e)}(X),p(A)\lambda^{I}_{fH(e)}(Y)\}$$

$$=\max\{(A+\lambda^{I}_{fH(e)})^{p}(X),(A+\lambda^{I}_{fH(e)})^{p}(Y)\}$$

Hence $(A+\lambda^{I}_{fH(e)})^{p}(X-Y) \leq \max\{(A+\lambda^{I}_{fH(e)})^{p}(X),(A+\lambda^{I}_{fH(e)})^{p}(Y)\}$

ii)

$$(A+\lambda^{I}_{fH(e)})^{p}(XY) = p(A)\lambda^{I}_{fH(e)}(X-Y)$$

$$(A+\lambda^{I}_{fH(e)})^{p}(XY) \leq p(A)\max\{\lambda^{I}_{fH(e)}(X),\lambda^{I}_{fH(e)}(Y)\}$$

$$=\max\{p(A)\lambda^{I}_{fH(e)}(X),p(A)\lambda^{I}_{fH(e)}(Y)\}$$

$$=\max\{(A+\lambda^{I}_{fH(e)})^{p}(X),(A+\lambda^{I}_{fH(e)})^{p}(Y)\}$$

Hence $(A+\lambda^{I}_{fH(e)})^{p}(XY) \leq \max\{(A+\lambda^{I}_{fH(e)})^{p}(X),(A+\lambda^{I}_{fH(e)})^{p}(Y)\}$

And in the same way we can prove that

c) $$\begin{cases}(A+\lambda^{F}_{fH(e)})^{p}(X-Y) \leq \max\{(A+\lambda^{F}_{fH(e)})^{p}(X),(A+\lambda^{F}_{fH(e)})^{p}(Y)\}\\ (A+\lambda^{F}_{fH(e)})^{p}(XY) \leq \max\{(A+\lambda^{F}_{fH(e)})^{p}(X),(A+\lambda^{F}_{fH(e)})^{p}(Y)\}\end{cases}$$

Hence the fuzzy neutrosophic soft pseudo coset $(A+\lambda^{H})^{p}$ is a fuzzy neutrosophic soft HX subring of a HX ring N.

6 Homomorphism and anti homomorphism of afuzzy neutrosophic soft pseudo coset of a fuzzy neutrosophic soft HX subring of a HX ring

In this section we introduce the concept of an image ,pre-image of a fuzzy neutrosophic soft pseudo coset of a HX ring and discuss the properties of homomorphism and anti homomorphism images and pre-images of a fuzzy neutrosophic soft pseudo coset of a fuzzy neutrosophic soft HX subring λ^H of a HX ring N determined by the element $A \in N$.

6.1 *Theorem*

Let N_1 and N_2 be any two HX rings on the R_1 and R_2 respectively . Let $g: N_1 \to N_2$ be homomorphism onto HX rings and $(A+\lambda^H)^p$ be a fuzzy neutrosophic soft pseudo coset of a fuzzy neutrosophic soft HX subring λ^H of a HX N_1 determined by the element $A \in N_1$ .Then $g(A+\lambda^H)^p$ is a fuzzy neutrosophic soft pseudo coset of a fuzzy neutrosophic soft HX subring $g(\lambda^H)$ of a HX ring N_2 determined by the element $g(A) \in N_2$ and $g(A+\lambda^H)^p = (g(A)+g(\lambda^H))^p$,if λ^H has a supremum property and λ^H is g –invariant .

Proof : Let $g: N_1 \to N_2$ be a homomorphism onto HX rings. Let $(A+\lambda^H)^p$ be a fuzzy neutrosophic soft pseudo coset of a fuzzy neutrosophic soft HX subring λ^H of a HX N_1 determined by the $A \in N_1$. Then by theorem (4.1) $g(A)$ is a fuzzy neutrosophic HX subring of a HX ring of a HX ring N_2 .Then $g((A+\lambda^H)^p)$ is a fuzzy neutrosophic soft pseudo coset of a fuzzy neutrosophic soft HX subring $g(\lambda^H)$ of a HX ring N_2 determined by the element $g(A) \in N_2$.

Let $A, X \in N_1$ *then* $g(A), g(X) \in N_2$

$$(g(A)+g(\lambda^T_{fH(e)}))^p g(X) = p(g(A))(g(\lambda^T_{fH(e)})(g(X))$$

$$= P(A)\lambda^T_{fH(e)}(X)$$

$$= (A+\lambda^T_{fH(e)})^p(X)$$

$$= g((A+\lambda^T_{fH(e)})^p)g(X)$$

Hence $(g(A)+g(\lambda^T_{fH(e)}))^p g(X) = g((A+\lambda^T_{fH(e)})^p)g(X)$ for any $g(X) \in N_2$

Hence $g((A+\lambda^T_{fH(e)})^p) = (g(A)+g(\lambda^T_{fH(e)}))^p$

Also $(g(A)+g(\lambda^I_{fH(e)}))^p g(X) = p(g(A))(g(\lambda^I_{fH(e)})(g(X))$

$$= P(A)\lambda^I_{fH(e)}(X)$$

$$= (A+\lambda^I_{fH(e)})^p(X)$$

$$= g((A+\lambda^I_{fH(e)})^p)g(X)$$

Hence $(g(A)+g(\lambda^I_{fH(e)}))^p g(X) = g((A+\lambda^I_{fH(e)})^p)g(X)$ for any $g(X) \in N_2$

Hence $g((A+\lambda^I_{fH(e)})^p) = (g(A)+g(\lambda^I_{fH(e)}))^p$

And in the same way we can prove that

$$g((A+\lambda^{F}_{fH(e)})^{p}) = (g(A)+g(\lambda^{F}_{fH(e)}))^{p}$$

$$Hence \quad g((A+\lambda^{H})^{p}) = (g(A)+g(\lambda^{T}_{fH(e)}))^{p}$$

$\underline{6.2 Theorem}$

Let N_1 and N_2 be any two HX rings on the R_1 and R_2 respectively . Let $g: N_1 \to N_2$ be homomorphism onto HX rings and $(B+Ɣ^{G})^{p}$ be a fuzzy neutrosophic soft pseudo coset of a fuzzy neutrosophic soft HX subring $Ɣ^{G}$ of a HX N_2 determined by the element $B \in N_2$ .Then $g^{-1}(B+Ɣ^{G})^{p}$ is a fuzzy neutrosophic soft pseudo coset of a fuzzy neutrosophic soft HX subring g^{-1} $(Ɣ^{G})$ of a HX ring N_1 determined by the element $g^{-1}(B) \in N_1$ and $g^{-1}(B+Ɣ^{G})^{p} = (g^{-1}(B)+g^{-1}(Ɣ^{G}))^{p}$.

Proof: Let $g: N_1 \to N_2$ be a homomorphism onto HX rings. Let $(B+Ɣ^{G})^{p}$ be a fuzzy neutrosophic soft pseudo coset of a fuzzy neutrosophic soft HX subring $Ɣ^{G}$ of a HX ring N_2 determined by the $B \in N_2$. Then by theorem (4.2) $g^{-1}(B)$ is a fuzzy neutrosophic HX subring of a HX N_1 .Then $g((B+Ɣ^{G})^{p})$ is a fuzzy neutrosophic soft coset of a fuzzy neutrosophic soft HX subring g^{-1} $(Ɣ^{G})$ of a HX ring N_1 determined by the element $g^{-1}(B) \in N_1$.

Let $A, X \in N_1$ $\quad then \quad g(A), g(X) \in N_2$

Now $(g^{-1}(B)+g^{-1}(Ɣ^{T}_{fG(e)}))^{p}(X) = P(g^{-1}(B))(g^{-1}(Ɣ^{T}_{fG(e)}))(X)$

$$= P(B)(Ɣ^{T}_{fG(e)}(g(X)))$$

$$= ((B+Ɣ^{T}_{fG(e)}))^{p} g(X)$$

$$= g^{-1}((B+Ɣ^{T}_{fG(e)})^{P})$$

$$(g^{-1}(B)+g^{-1}(Ɣ^{T}_{fG(e)}))^{p}(X) = g^{-1}((B+Ɣ^{T}_{fG(e)})^{P})(X)$$

Hence $g^{-1}((B+Ɣ^{T}_{fG(e)})^{P}) = (g^{-1}(B)+g^{-1}(Ɣ^{T}_{fG(e)}))^{p}$

Now $\quad (g^{-1}(B)+g^{-1}(Ɣ^{I}_{fG(e)}))^{p}(X) = P(g^{-1}(B))(g^{-1}(Ɣ^{I}_{fG(e)}))(X)$

$$= P(B)(Ɣ^{I}_{fG(e)}(g(X)))$$

$$= ((B+Ɣ^{I}_{fG(e)}))^{p} g(X)$$

$$= g^{-1}((B+Ɣ^{I}_{fG(e)})^{P})$$

$$(g^{-1}(B)+g^{-1}(Ɣ^{I}_{fG(e)}))^{p}(X) = g^{-1}((B+Ɣ^{I}_{fG(e)})^{P})(X)$$

Hence $$g^{-1}((B+\Upsilon^{I}_{fG(e)})^{P}) = (g^{-1}(B)+g^{-1}(\Upsilon^{I}_{fG(e)}))^{p}$$

And in the same way we can prove that

$$g^{-1}((B+\Upsilon^{F}_{fG(e)})^{P}) = (g^{-1}(B)+g^{-1}(\Upsilon^{F}_{fG(e)}))^{p}$$

Hence $$g^{-1}(B+\Upsilon^{G})^{p} = (g^{-1}(B)+g^{-1}(\Upsilon^{G}))^{p}$$

$6.3\,Theorem$

Let N_1 and N_2 be any two HX rings on the R_1 and R_2 respectively. Let $g: N_1 \to N_2$ be anti homomorphism onto HX rings.Let $(A+\lambda^{H})^{p}$ be a fuzzy neutrosophic soft pseudo coset of a fuzzy neutrosophic soft HX subring λ^{H} of a HX N_1 determined by the element $A \in N_1$ .Then $g(A+\lambda^{H})^{p}$ is a fuzzy neutrosophic soft pseudo coset of a fuzzy neutrosophic soft HX subring $g(\lambda^{H})$ of a HX ring N_2 determined by the element $g(A) \in N_2$ and $g(A+\lambda^{H})^{p} = (g(A)+g(\lambda^{H}))^{p}$,if λ^{H} has a supremum property and λ^{H} is g –invariant .

Proof: Let $g: N_1 \to N_2$ be anti homomorphism onto HX rings. Let $(A+\lambda^{H})^{p}$ be a fuzzy neutrosophic soft pseudo coset of a fuzzy neutrosophic soft HX subring λ^{H} of a HX N_1 determined by the $A \in N_1$. Then by theorem(4.1) $g(A)$ is a fuzzy neutrosophic HX subring of a HX ring of a HX ring N_2 .Then $g((A+\lambda^{H})^{p})$ is a fuzzy neutrosophic soft pseudo coset of a fuzzy neutrosophic soft HX subring $g(\lambda^{H})$ of a HX ring N_2 determined by the element $g(A) \in N_2$.

Let $A, X \in N_1$ $\quad then \quad g(A), g(X) \in N_2$

$$(g(A)+g(\lambda^{T}_{fH(e)}))^{p} g(X) = p(g(A))(g(\lambda^{T}_{fH(e)})(g(X))$$

$$= P(A)\lambda^{T}_{fH(e)}(X)$$

$$= (A+\lambda^{T}_{fH(e)})^{p}(X)$$

$$= g((A+\lambda^{T}_{fH(e)})^{p})g(X)$$

Hence $$(g(A)+g(\lambda^{T}_{fH(e)}))^{p} g(X) = g((A+\lambda^{T}_{fH(e)})^{p})g(X) \quad \text{for any } g(X) \in N_2$$

Hence $$g((A+\lambda^{T}_{fH(e)})^{p}) = (g(A)+g(\lambda^{T}_{fH(e)}))^{p}$$

$$(g(A)+g(\lambda^{I}_{fH(e)}))^{p} g(X) = p(g(A))(g(\lambda^{I}_{fH(e)})(g(X))$$

$$= P(A)\lambda^{I}_{fH(e)}(X)$$

$$= (A+\lambda^{I}_{fH(e)})^{p}(X)$$

$$= g((A+\lambda^{I}_{fH(e)})^{p})g(X)$$

Hence $(g(A) + g(\lambda^{I}_{fH(e)}))^{p} g(X) = g((A + \lambda^{I}_{fH(e)})^{p}) g(X)$ for any $g(X) \in N_2$

Hence $g((A + \lambda^{I}_{fH(e)})^{p}) = (g(A) + g(\lambda^{I}_{fH(e)}))^{p}$

And in the same way we can prove that $g((A + \lambda^{F}_{fH(e)})^{p}) = (g(A) + g(\lambda^{F}_{fH(e)}))^{p}$

Hence $g((A + \lambda^{H})^{p}) = (g(A) + g(\lambda^{T}_{fH(e)}))^{p}$

REFERENCES

[1] K. Atanassov, Intuitionistic fuzzy sets , Fuzzy Sets and Systems 20 , 87 – 96, 1986.

[2] T. Bera, N.K. Mahapatra ,On neutrosophic soft ring ,DOI 10.1007/s125 97-016-0273-6

[3] Bing-xueYao and Yubin-Zhong, The construction of power ring, Fuzzy information and Engineering (ICFIE), 2007, ASC 40,pp.181-187.

[4] Li Hong Xing ,HX ring , BUSEFAL , 34(1) 3-8 , January 1988.

[5] P. Maji, A. Roy and R. Biswas, Fuzzy soft sets, Journal of Fuzzy Mathematics. 9 (3) 589-602 (2001).

[6] P. Maji, R. Biswas, and A. Roy, "Intuitionistic fuzzy soft sets," The Journal of Fuzzy Mathematics, vol. 9, no. 3, pp. 677-692, 2001.

[7] P. K. Maji, Neutrosophic soft set, Annals of Fuzzy Mathematics and Informatics, 5(1), (2013), 157-168.

[8] D. Molodtsov, "Soft set theory-first results," Computers & Mathematics with Applications, vol. 37, no.4-5, pp. 19–31, 1999.

[9] R. Muthuraj and M. S. Muthuraman, Intuitionistic Fuzzy Hx Ring, (IOSR-JM), Volume 10, Issue 4 Ver. II (Jul-Aug. 2014), PP 03-12.

[10] R. Muthuraj and M. S. Muthuraman, Intuitionistic Fuzzy cosets on HX Ring, International Journal of Mathematical Archive-9(4),125-135, (2018)

[11] Smarandache, F.: Neutrosophic set, a generalisation of the intuitionistic fuzzy sets. Int. J. Pure Appl.Math. 24, 287–297 (2005)

[12] L.Zadeh,Fuzzy sets, Information and Control 8, 338-353, (1965).

The effect of alternative resource and refuge on the dynamical behavior of food chain model

Zahraa Mohammed Hadi and Dahlia Khaled Bahlool

Department of Mathematics, College of science, University of Baghdad, Iraq

E-mail of the corresponding author: dahlia.khaled@gmail.com

Abstract. In this paper a food chain model involving a prey refuge and an alternative food sources is proposed and studied. It is assumed that the number of refuges proportional with the availability of middle predator. The existence and uniqueness of solution are discussed. The bounded of the solution is proved. The local stability of all possible equilibrium points is investigated. Finally the global stability of the locally asymptotically stable points is studied using the Laypunov method.

Keywords: Food chain; refuge; alternative food; stability.

1. Introduction

There has been an increasing interest in the use of mathematical models in ecology as revealed by papers (cf. Holling [1], Tian and Xu [2], Kar[3], Narayan[4]) in the last decade. It has attracted much attention from many scientific directions. Early researches suggested that resource availability plays an important role. Many contributions study the effect of alternative food chain. Researchers found that when resources (food, nesting sites, or refuges) were limited, populations would decline as individuals competed for access to the limiting resources (cf. Senthamara and Vijayalakhmi [5]). Also many contributions to Population ecologists study how births and deaths affect the dynamics of populations and communities. So the food chain is important because each living thing depends on others for survival, no matter how big or how small. There are so many papers introduced readers to food chains and how animals survive by consuming other animals. It teaches humans that every creature depends on another creature for life and sustenance. It is extremely important to understand that even the slightest bit of disturbance in the natural food chain and eating patterns can alter the lifestyles of a lot of creatures (cf. Mohd et al.[6]). In addition researchers interested in refuge(cf. Ghana and Zachilas [7], Sarwardi et al.[8], Huang et al.[9], Kar[10]) which is a term in ecology in which species gain protection by hiding in places difficult to found, Haque et al. in [11] studied refuge on the prey and discussed the effect of coefficient of refuge which plays an important role. Naji and Majeed in[12] proposed and studied refuge in prey as a defensive property against predation and harvesting taken from predator. Sih in [13] studied the influences of prey refuge and harvesting efforts. Das et al. in [14] studied the existence of refuges effects on the coexistence of predators and prey, after that Molla et al.[15] proposed a mathematical model for prey-predator allowing prey refuge depending on both prey and predator species. The results of Ko and Ryu in[16] show that the shape of the functional response plays an important role in determining the dynamics of the system incorporating a prey refuge under homogenous Neumann boundary condition especially, the interesting conclusion is that the prey refuge has a destabilizing effect under some certain conditions.

Kar et al.[17] described a prey-predator model with Holling type II functional response where harvesting of each species is taken in consideration. Also they have considered a gestation delay of predator population and harvesting of both the species. Many researchers have confirmed that the presence of alternative foods can effect bilological control through a variety of mechanism. Agarwal and Kumar in [18] studied the effect of alternative resource for top predator in food chain model with Holling type III functional response. Some theoretical works on alternative food may be found in many papers. MCNair in [19] proposed the effects of refuges on interactions between prey and predator. Alternative food plays a stabilizing role in predator-prey interaction that proposed by Sahoo [20].

Overall the above facts, the researchers motivated to study mathematical model prey refuge with three species food chain with Holling type II functional response for middle predator and top predator and discussed the effect of alternative resource for top predator.

2. Mathematical model

A three species food chain model consisting of prey – middle predator – top predator combining a prey refuge with an alternative food for top predator is formulated mathematically according to the following hypotheses:

1. It is assumed that the model contain three species, where $x(t)$ represents the densities of prey at time t, while $y(t)$ and $z(t)$ represent the densities at time t of the middle predator and top predator respectively. The predation processes have been done according to the Holling type-II functional response where a_1 and b_1 represent the capturing rates for middle predator and top predator respectively, while a and b represent the half saturation constants for middle predator and top

predator respectively. Furthermore, the food is converted to predators with conversion rates given by a_2 and b_2 for middle predator and top predator respectively too.

2. The prey grows logistically in the absence of predator with intrinsic growth rate r and carrying capacity K, while the middle predator grows logistically with intrinsic growth rate s and carrying capacity L in additional to the food gained from their predation on the prey. The middle predator decays exponentially in the absence of their food with natural death rate d_1.
3. The top predator has additional sources of food taken from the environment with alternative food rate A_1 in additional to the food gained from their predation on the middle predator. There is intraspecific competition between the top predator individuals with intraspecific competition rate A_2. Furthermore, the top predator decays exponentially in the absence of their food with natural death rate d_2.
4. Finally, its assumed that the prey specie protect himself through the availability of refuge that proportional with the number of middle predator $x_R = cxy$ with proportionality constant $c \in [0, 1]$ and hence the rest of prey population $x - x_R$ is available for predation by middle predator. Accordingly, we obtain that $y \leq \frac{1}{c}$.

According to the above assumptions, the dynamics of food chain can be described by the following set of nonlinear first order differential equations:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) - \frac{a_1(x-x_R)y}{a+(x-x_R)} \\ \frac{dy}{dt} &= sy\left(1 - \frac{y}{L}\right) + \frac{a_2(x-x_R)y}{a+(x-x_R)} - \frac{b_1 yz}{b+y} - d_1 y \\ \frac{dz}{dt} &= \frac{b_2 yz}{b+y} + A_1 z - A_2 z^2 - d_2 z \end{aligned} \qquad (1)$$

where all the parameters are assumed to be positive constants. Substituting $x_R = cxy$, system (1) will be as follow:

$$\begin{aligned}\frac{dx}{dt} &= x\left[r\left(1-\frac{x}{K}\right)-\frac{a_1(1-cy)y}{a+x(1-cy)}\right] = xF_1\\ \frac{dy}{dt} &= y\left[s\left(1-\frac{y}{L}\right)+\frac{a_2(1-cy)x}{a+x(1-cy)}-\frac{b_1 z}{b+y}-d_1\right] = yF_2 \qquad (2)\\ \frac{dz}{dt} &= z\left[\frac{b_2 y}{b+y}+A_1-A_2 z-d_2\right] = zF_3\end{aligned}$$

with $x(0) \geq 0, y(0) \geq 0$ and $z(0) \geq 0$. Clearly the functions of vector $F = (F_1, F_2, F_3)^T$ are continuous and have continuously differential function on $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$ and hence they are Lipschitzian. So the solution for system (2) exists and is unique.

3. Boundenees of the system

In this section, we had shown in the following theorem all the solutions of system (2) are uniformly bounded:

Theorem (1): All solutions of system (2) in $\mathbb{R}^3_+$ are uniformly bounded.

Proof: Consider the function $w(t)$ which is the sum of solutions in $\mathbb{R}^3_+$ that is given by $w(t) = x(t) + y(t) + z(t)$, then

$$\begin{aligned}\frac{dw}{dt} &= \frac{dx}{dt}+\frac{dy}{dt}+\frac{dz}{dt}\\ &= rx\left(1-\frac{x}{K}\right)-\frac{(a_1-a_2)(x-x_R)y}{a+(x-x_R)}+sy\left(1-\frac{y}{L}\right)\\ &\quad -\frac{(b_1-b_2)yz}{b+y}-d_1 y+A_1 z\left(1-\frac{z}{\left(\frac{A_1}{A_2}\right)}\right)-d_2 z\end{aligned}$$

$$\frac{dw}{dt} \leq \frac{(r+1)^2K}{4r}+\frac{sL}{4}+\frac{{A_1}^2}{4A_2}-\mu(x+y+z) = \rho-\mu w$$

where $\mu = \min\{1, d_1, d_2\}$ and $\rho = \frac{(r+1)^2K}{4r}+\frac{sL}{4}+\frac{{A_1}^2}{4A_2}$, therefore we obtain

$$\frac{dw}{dt}+\mu w \leq \rho$$

Now by solving the above differential inequality we obtain that for $t \to \infty$; $w(t) \leq \frac{\rho}{\mu}$. Thus all solutions are uniformly bounded. ■

Keeping the above theorem in view, system (2) is a dissipative system and hence it has an attractor belongs to $\mathbb{R}_+^3$.

4. Existence and local stability of equilibrium points

There are at most eight equilibrium points for system (2), we described them as follow:

The vanishing equilibrium point $E_0 = (0,0,0)$ and the first axial equilibrium point $E_1 = (\breve{x}, 0,0)$, with $\breve{x} = K$ are exist without any restriction.

The second axial equilibrium point $E_2 = (0, \hat{y}, 0)$ exists provided that

$$s > d_1 \quad (3a)$$

where

$$\hat{y} = \frac{(s-d_1)L}{s} \quad (3b)$$

The third axial equilibrium point $E_3 = (0,0, \bar{z})$ exists if

$$A_1 > d_2 \quad (4a)$$

where

$$\bar{z} = \frac{A_1-d_2}{A_2} \quad (4b)$$

The middle predator free equilibrium point $E_4 = (\bar{\bar{x}}, 0, \bar{\bar{z}})$ exists if

$$A_1 > d_2 \quad (5a)$$

where

$$\bar{\bar{x}} = K; \;\; \bar{\bar{z}} = \frac{A_1-d_2}{A_2} \quad (5b)$$

The prey free equilibrium point $E_5 = (0, \tilde{\bar{y}}, \tilde{\bar{z}})$, where

$$\tilde{\tilde{z}} = \frac{b_2y+(A_1-d_2)(b+y)}{A_2(b+y)} \tag{6a}$$

while $\tilde{\tilde{y}}$ is a positive root of the third degree polynomial that given by

$$B_0y^3 + B_1y^2 + B_2y + B_3 = 0 \tag{6b}$$

where

$$B_0 = -sA_2 < 0$$

$$B_1 = (s-d_1)A_2L - 2sbA_2$$

$$B_2 = 2(s-d_1)A_2Lb - sA_2b^2 - b_1b_2L - Lb_1A_1 + Lb_1d_2$$

$$B_3 = (s-d_1)A_2Lb^2 - b_1A_1bL + Lbb_1d_2$$

So in order to have a positive root and then the prey free equilibrium point $E_5 = (0, \tilde{\tilde{y}}, \tilde{\tilde{z}})$ exists uniquely we should have the following conditions

$$d_2 < \frac{b_2y+A_2(b+y)}{b+y} \tag{6c}$$

And one set of the following conditions

$$B_3 > 0 \text{ and } B_1 < 0 \tag{6d}$$

Or

$$B_3 > 0 \text{ and } B_2 > 0 \tag{6e}$$

The top predator free equilibrium point $E_6 = (\check{\check{x}}, \check{\check{y}}, 0)$, where

$$\check{\check{x}} = \frac{a\left[d_1 - s\left(1-\frac{\check{\check{y}}}{L}\right)\right]}{\left[a_2-d_1+s\left(1-\frac{\check{\check{y}}}{L}\right)\right](1-c\check{\check{y}})} \tag{7}$$

where $\check{\check{y}}$ is appositive root of the fifth degree polynomial

$$\beta_5y^5 + \beta_4y^4 + \beta_3y^3 + \beta_2y^2 + \beta_1y + \beta_0 = 0 \tag{8a}$$

here:

$$\beta_0 = aa_2r(a_2-d_1+s) - \frac{a^2a_2r}{K}(d_1-s)$$

$$\beta_1 = aa_2{}^2cr + aa_2cd_1r - 2a_1a_2d_1 - 2a_1a_2s - a_1d_1{}^2$$
$$+2a_1d_1s - a_1s^2 + aa_2crs + acd_1{}^2r - 2acd_1rs + acrs^2$$
$$-\frac{aa_2rs}{L} - \frac{a^2a_2rs}{KL} + 2a_1a_2d_1$$

$$\beta_2 = \frac{aa_2crs}{L} + \frac{2a_1a_2s}{L} - \frac{2a_1d_1s}{L} + \frac{2a_1s^2}{L} + 2a_1a_2{}^2c$$
$$+4a_1a_2cs + 2a_1cd_1{}^2 - 4a_1cd_1s + 2a_1cs^2 - 4a_1a_2cd_1$$

$$\beta_3 = \frac{-a_1s^2}{L^2} - \frac{4a_1a_2cs}{L} + \frac{4a_1cd_1s}{L} + \frac{2a_1cs^2}{L} - a_1a_2{}^2c^2$$
$$+2a_1a_2c^2d_1 - 2a_1a_2c^2s^2 - a_1c^2d_1{}^2 + 2a_1c^2d_1s - a_1c^2s^2$$

$$\beta_4 = \frac{2a_1cs^2}{L^2} + \frac{2a_1c^2s}{L}(a_2 - d_1 + s)$$

$$\beta_5 = -\frac{c^2a_1s^2}{L^2}$$

Hence there exists a positive root for Eq. (8a) provided that one set of the following sets of condition hold

$$\begin{array}{l} \beta_0 > 0, \beta_1 > 0, \beta_2 > 0, \beta_3 > 0, \beta_4 < 0 \\ \beta_0 > 0, \beta_1 > 0, \beta_2 > 0, \beta_3 > 0, \beta_4 > 0 \\ \beta_0 > 0, \beta_1 > 0, \beta_2 > 0, \beta_3 < 0, \beta_4 < 0 \\ \beta_0 > 0, \beta_1 > 0, \beta_2 < 0, \beta_3 < 0, \beta_4 < 0 \\ \beta_0 > 0, \beta_1 < 0, \beta_2 < 0, \beta_3 < 0, \beta_4 < 0 \end{array} \qquad (8b)$$

Moreover, top predator free equilibrium point $E_6 = (\check{\check{x}}, \check{\check{y}}, 0)$ exists uniquely provided that in addition to condition (8b) the following condition holds

$$s\left(1 - \frac{\check{\check{y}}}{L}\right) < d_1 < a_2 + s\left(1 - \frac{\check{\check{y}}}{L}\right) \qquad (8c)$$

The positive equilibrium point that denoted by $E_7 = (x^*, y^*, z^*)$ exists uniquely in $\mathbb{R}^3_+$, where

$$z^* = \frac{b_2y + (A_1 - d_2)(b + y)}{A_2(b + y)} \qquad (9)$$

While (x^*, y^*) is the positive intersection point of the following two isoclines

$$g_1(x, y) = aKr + Krx - cKrxy - arx - rx^2$$
$$+crx^2y - a_1Ky + a_1cKy^2 = 0$$

$$g_2(x,y) = s - \frac{sy}{L} + \frac{a_2 x(1-cy)}{a+x(1-cy)} - \frac{b_1}{b+y}\left(\frac{b_2 y + (A_1-d_2)(b+y)}{A_2(b+y)}\right) - d_1 = 0$$

Clearly, when $y \to 0$, we obtain that the first isocline $g_1(x,y) = 0$ intersects the x −axis at a positive point:

$$h_1 = -\frac{(a-K)}{2} + \frac{1}{2}\sqrt{(a-K)^2 + 4aK} \qquad (10a)$$

While the second isocline $g_2(x,y) = 0$ intersects the x −axis at the positive point:

$$h_2 = \frac{baA_2(d_1-s)+b_1 a(A_1-d_2)}{bA_2(s+a_2)-b_1(A_1-d_2)-bA_2 d_1} \qquad (10b)$$

Provided that:

$$bA_2 s + b_1 d_2 < b_1 A_1 + bA_2 d_1 < bA_2(s+a_2) + b_1 d_2 \qquad (11a)$$

Therefore in order to have a unique positive equilibrium point we should have in addition to condition (11a) the following set of sufficient conditions holds

$$h_1 < h_2 \qquad (11b)$$

$$\frac{dy}{dx} = -\frac{(\partial g_1/\partial x)}{(\partial g_1/\partial y)} > 0 \qquad (11c)$$

$$\frac{dy}{dx} = -\frac{(\partial g_2/\partial x)}{(\partial g_2/\partial y)} < 0 \qquad (11d)$$

5. Local stability analysis

In this section we discuss the local stability of all possible equilibrium points of system (2) with the use of linearization method by calculating the eigenvalues of the Jacobian matrix of system (2) at those points.

It is easy to verify that the eigenvalues of the Jacobian matrix at the trivial equilibrium point $E_0 = (0,0,0)$, are given by $\lambda_{01} = r > 0, \lambda_{02} = s - d_1$ and $\lambda_{03} = A_1 - d_2$. Therefore, due to existence of positive eigenvalues, E_0 is unstable point.

The eigenvalues of the Jacobian matrix at the first axial equilibrium point $E_1 = (\breve{x}, 0,0)$, are given by $\lambda_{11} = -r < 0, \lambda_{12} = s + \frac{a_2 K}{a+K} - d_1$ and $\lambda_{13} = A_1 - d_2$. Therefore E_1 is locally asymptotically stable provided that

$$s + \frac{a_2 K}{a+K} < d_1 \quad (12a)$$

$$A_1 < d_2 \quad (12b)$$

The eigenvalues of the Jacobian matrix at the second axial equilibrium point $E_2 = (0, \hat{y}, 0)$, are given by $\lambda_{21} = r - \frac{a_1}{a}\hat{y} + \frac{a_1 c \hat{y}^2}{a}, \lambda_{22} = (d_1 - s) < 0$ and $\lambda_{23} = \frac{b_2 \hat{y}}{b+\hat{y}} + A_1 - d_2$. Therefore, the second axial equilibrium point E_2 is locally asymptotically stable provided that the following conditions hold:

$$r + \frac{a_1 c \hat{y}^2}{a} < \frac{a_1}{a}\hat{y} \quad (13a)$$

$$\frac{b_2 \hat{y}}{b+\hat{y}} + A_1 < d_2 \quad (13b)$$

The Jacobian matrix at the third axial equilibrium point $E_3 = (0,0, \bar{z})$ has the following eigenvalues by $\lambda_{31} = r > 0, \lambda_{32} = s - \frac{b_1}{b}\bar{z} - d_1$ and $\lambda_{33} = -A_2 \bar{z} < 0$. Therefore, due to existence of positive and negative eigenvalues, E_3 is unstable saddle point.

Now the eigenvalues of the Jacobian matrix at the middle predator free equilibrium point $E_4 = (\bar{\bar{x}}, 0, \bar{\bar{z}})$, can be written as

$$\lambda_{41} = -r < 0 \quad (14a)$$

$$\lambda_{42} = s + \frac{a_2 \bar{\bar{x}}}{a+\bar{\bar{x}}} - \frac{b_1}{b}\bar{\bar{z}} - d_1 \quad (14b)$$

$$\lambda_{43} = -A_2 \bar{\bar{z}} < 0 \quad (14c)$$

Accordingly, the middle predator free equilibrium point is locally asymptotically stable provided that the following condition holds

$$s + \frac{a_2 \bar{\bar{x}}}{a+\bar{\bar{x}}} < \frac{b_1}{b}\bar{\bar{z}} + d_1 \quad (14d)$$

The characteristic equation of the Jacoblian matrix at the prey free equilibrium point $E_5 = (0, \tilde{\bar{y}}, \tilde{\bar{z}})$, can be written as

$$[r - \lambda][\lambda^2 - T_5 \lambda + D_5] = 0 \quad (15a)$$

where

$$T_5 = \tilde{\tilde{y}} \left[-\frac{s}{L} + \frac{b_1 \tilde{\tilde{z}}}{(b+\tilde{\tilde{y}})^2} \right] - A_2 \tilde{\tilde{z}}$$

$$D_5 = -\tilde{\tilde{y}} \left[-\frac{s}{L} + \frac{b_1 \tilde{\tilde{z}}}{(b+\tilde{\tilde{y}})^2} \right] (A_2 \tilde{\tilde{z}}) + \left[\frac{b b_2 \tilde{\tilde{z}}}{(b+\tilde{\tilde{y}})^2} \right] \left[\frac{b_1 \tilde{\tilde{y}}}{b+\tilde{\tilde{y}}} \right]$$

Therefore Eq. (15a) has three roots (eigenvalues) with negative real parts given by

$$\lambda_{51} = r - \frac{a_1 (1-\tilde{\tilde{y}}) \tilde{\tilde{y}}}{a}, \quad (15b)$$

$$\lambda_{52} = \frac{T_5}{2} - \frac{1}{2} \sqrt{{T_5}^2 - 4D_5} \quad (15c)$$

$$\lambda_{53} = \frac{T_5}{2} + \frac{1}{2} \sqrt{{T_5}^2 - 4D_5} \quad (15d)$$

Provided that the following sufficient conditions hold

$$r + \frac{a_1 c \tilde{\tilde{y}}^2}{a} < \frac{a_1}{a} \tilde{\tilde{y}} \quad (16a)$$

$$\frac{b_1 \tilde{\tilde{z}}}{(b+\tilde{\tilde{y}})^2} < \frac{s}{L} \quad (16b)$$

$$\frac{b_2 \tilde{\tilde{y}}}{b+\tilde{\tilde{y}}} + A_1 < d_2 \quad (16c)$$

Accordingly, the prey free equilibrium point $E_5 = (0, \tilde{\tilde{y}}, \tilde{\tilde{z}})$ is locally asymptotically stable under the conditions (16a)-(16c).

The characteristic equation of the Jacoblian matrix at the top predator free equilibrium point $E_6 = (\breve{\breve{x}}, \breve{\breve{y}}, 0)$, can be written as

$$[\lambda^2 - T_6 \lambda + D_6] \left[\left(\frac{b_2 \breve{\breve{y}}}{b+\breve{\breve{y}}} + A_1 - d_2 \right) - \lambda \right] = 0 \quad (17a)$$

Where

$$T_6 = \breve{\breve{x}} \left[-\frac{r}{K} + \frac{a_1 (1-c\breve{\breve{y}})^2 \breve{\breve{y}}}{[a+\breve{\breve{x}}(1-c\breve{\breve{y}})]^2} \right] + \breve{\breve{y}} \left[-\frac{s}{L} - \frac{a a_2 c \breve{\breve{x}}}{[a+\breve{\breve{x}}(1-c\breve{\breve{y}})]^2} \right]$$

$$D_6 = \breve{\breve{x}}\breve{\breve{y}}\left[-\frac{r}{K}+\frac{a_1(1-c\breve{\breve{y}})^2\breve{\breve{y}}}{[a+\breve{\breve{x}}(1-c\breve{\breve{y}})]^2}\right]\left[-\frac{s}{L}-\frac{aa_2c\breve{\breve{x}}}{[a+\breve{\breve{x}}(1-c\breve{\breve{y}})]^2}\right] + \left[\frac{\breve{\breve{x}}(aa_1(1-2c\breve{\breve{y}})+a\breve{x}(1-c\breve{\breve{y}})^2)}{[a+\breve{\breve{x}}(1-c\breve{\breve{y}})]^2}\right]\left[\frac{aa_2\breve{\breve{y}}(1-c\breve{\breve{y}})}{[a+\breve{\breve{x}}(1-c\breve{\breve{y}})]^2}\right]$$

Therefore Eq. (17a) has three roots (eigenvalues) with negative real parts given by

$$\lambda_{61} = \frac{T_6}{2}-\frac{1}{2}\sqrt{{T_6}^2-4D_6} \quad (17b)$$

$$\lambda_{62} = \frac{T_6}{2}+\frac{1}{2}\sqrt{{T_6}^2-4D_6} \quad (17c)$$

$$\lambda_{63} = \frac{b_2\breve{\breve{y}}}{b+\breve{\breve{y}}}+A_1-d_2 \quad (17d)$$

Provided that the following sufficient conditions hold

$$\frac{a_1(1-c\breve{\breve{y}})^2\breve{\breve{y}}}{[a+\breve{\breve{x}}(1-c\breve{\breve{y}})]^2} < \frac{r}{K} \quad (18a)$$

$$\breve{\breve{y}} < \frac{1}{2c} \quad (18b)$$

$$\frac{b_2\breve{\breve{y}}}{b+\breve{\breve{y}}}+A_1 < d_2 \quad (18c)$$

Accordingly, the top predator free equilibrium point $E_6 = (\breve{\breve{x}},\breve{\breve{y}},0)$ is locally asymptotically stable under the conditions (18a)-(18c).

Finally, the Jacobian matrix evaluated at the positive equilibrium point that denoted by $E_7 = (x^*,y^*,z^*)$ is given by

$$J(E_7) = [a_{ij}]_{3\times 3} \quad (19)$$

where:

$$a_{11} = x^*\left[-\frac{r}{K}+\frac{a_1(1-cy^*)^2y^*}{[a+x^*(1-cy^*)]^2}\right]$$

$$a_{12} = -\frac{a_1x^*[a(1-2cy^*)+x^*(1-cy^*)^2]}{[a+x^*(1-cy^*)]^2}$$

$$a_{13} = 0$$

$$a_{21} = \frac{a a_2 (1 - c y^*) y^*}{[a + x^*(1 - c y^*)]^2} > 0$$

$$a_{22} = y^* \left[-\frac{s}{L} - \frac{a a_2 c x^*}{[a + x^*(1 - c y^*)]^2} + \frac{b_1 z^*}{(b + y^*)^2} \right]$$

$$a_{23} = -\frac{b_1 y^*}{(b + y^*)} < 0$$

$$a_{31} = 0$$

$$a_{32} = \frac{b b_2 z^*}{(b + y^*)^2} > 0$$

$$a_{33} = -A_2 z^* < 0$$

Then the characteristic equation for $J(E_7)$ is given by

$$\lambda^3 + B_1 \lambda^2 + B_2 \lambda + B_3 = 0 \qquad (20)$$

where:

$$B_1 = -(a_{11} + a_{22} + a_{33})$$

$$B_2 = a_{11} a_{22} - a_{12} a_{21} + a_{11} a_{33} + a_{22} a_{33} - a_{23} a_{32}$$

$$B_3 = -(a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33})$$

with:

$$\begin{aligned} \Delta = &-(a_{11} + a_{22})(a_{11} a_{22} - a_{12} a_{21}) - (a_{11} + a_{33}) a_{11} a_{33} \\ &-(a_{22} + a_{33})(a_{22} a_{33} - a_{23} a_{32}) - 2 a_{11} a_{22} a_{33} \end{aligned}$$

Now according to the Routh-Hurwitz criterion all roots of the characteristic equation Eq. (20) have negative real parts and hence the positive equilibrium point that denoted by $E_7 = (x^*, y^*, z^*)$ will be local asymptotically stable provided that $B_1 > 0$; $B_3 > 0$ and $\Delta = B_1 B_2 - B_3 > 0$. Hence the local stability conditions for the positive equilibrium point can be established in the following theorem.

Theorem (2): The positive equilibrium point $E_7 = (x^*, y^*, z^*)$ is locally asymptotically stable in $\mathbb{R}^3_+$ provided that the following sufficient conditions hold

$$\frac{a_1 (1 - c y^*)^2 y^*}{[a + x^*(1 - c y^*)]^2} < \frac{r}{K} \qquad (21a)$$

$$y^* < \frac{1}{2c} \qquad (21b)$$

$$\frac{b_1 z^*}{(b+y^*)^2} < \frac{s}{L} + \frac{a a_2 c x^*}{[a+x^*(1-cy^*)]^2} \quad (21c)$$

Proof: Follow directly from the fact that condition (21a) guarantees that $a_{11} < 0$, condition (21b) guarantees that $a_{12} < 0$ and condition (21c) guarantees that $a_{22} < 0$. Hence straightforward computations show that all the requirements of Routh-Hurwitz criterion are satisfied and hence the positive equilibrium point E_7 is locally asymptotically stable. ■

6. Global stability analysis

In this section the global stability of all locally asymptotically stable is investigated with the help of Lyapunov method for stability. In the following theorem the global stability conditions of the first axial equilibrium point are established.

Theorem (3): Assume that the first axial equilibrium point $E_1 = (\breve{x}, 0,0)$ is locally asymptotically stable. Then it is globally asymptotically stable if the following sufficient condition holds

$$s + \frac{a_2 \breve{x}}{a} < d_1 \quad (22)$$

Proof: Let $V_1 = C_1\left(x - \breve{x} - \breve{x}\ln\frac{x}{\breve{x}}\right) + C_2 y + C_3 z$ be a real valued positive definite function, where the constants $C_i; i = 1,2,3$ are positive to be determine below. Then we have

$$\frac{dV_1}{dt} = -C_1\frac{r}{K}(x-\breve{x})^2 - [C_1 a_1 - C_2 a_2]\frac{x(1-cy)y}{a+x(1-cy)}$$
$$+C_1\breve{x}\frac{a_1(1-cy)y}{a+x(1-cy)} + C_2(s-d_1)y - C_2\frac{s}{L}y^2$$
$$-[C_2 b_1 - C_3 b_2]\frac{yz}{b+y} + C_3(A_1 - d_2)z - C_3 A_2 z^2$$

Then by choosing the positive constants as following $C_1 = \frac{a_2}{a_1}$, $C_2 = 1$ and $C_3 = \frac{b_1}{b_2}$, then by doing some algebraic steps we obtain that

$$\frac{dV_1}{dt} \le -\frac{a_2}{a_1}\frac{r}{K}(x-\breve{x})^2 + \left[\frac{a_2\breve{x}}{a} + s - d_1\right]y + \frac{b_1}{b_2}[A_1 - d_2]z$$

Therefore, according to the given condition a long with local stability condition (12b) we obtain that $\frac{dV_1}{dt} < 0$ is negative definite and hence according to the Laypunov method for stability the first axial equilibrium point $E_1 = (\breve{x}, 0,0)$ is globally asymptotically stable. ■

Theorem (4): Assume that the second axial equilibrium point $E_2 = (0, \hat{y}, 0)$ is locally asymptotically stable. Then it is globally asymptotically stable if the following sufficient condition holds

$$\frac{a_2}{a_1}\left(\frac{rK}{4}\right) < \frac{s}{L}(y-\hat{y})^2 \tag{23}$$

Proof: Let $V_2 = M_1 x + M_2\left(y - \hat{y} - \hat{y}\ln\frac{y}{\hat{y}}\right) + M_3 z$ be a real valued positive definite function, where the constants $M_i; i = 1,2,3$ are positive to be determine below. Then we have

$$\begin{aligned}\frac{dV_2}{dt} &= M_1 rx\left(1-\frac{x}{K}\right) - \frac{x(1-cy)y}{a+x(1-cy)}(M_1a_1 - M_2a_2)\\ &\quad - M_2\frac{s}{L}(y-\hat{y})^2 - M_2\hat{y}\frac{a_2x(1-cy)}{a+x(1-cy)} + M_2\frac{b_1\hat{y}}{b}z\\ &\quad - \frac{zy}{b+y}(M_2b_1 - M_3b_2) - M_3z(d_2 - A_1) - M_3A_2z^2\end{aligned}$$

Then by choosing the positive constants as following $M_1 = \frac{a_2}{a_1}$, $M_2 = 1$ and $M_3 = \frac{b_1}{b_2}$, then by doing some algebraic steps we obtain that

$$\frac{dV_2}{dt} \le \frac{a_2}{a_1}\left(\frac{rK}{4}\right) - \frac{s}{L}(y-\hat{y})^2 - b_1\left[\frac{(d_2-A_1)}{b_2} - \frac{\hat{y}}{b}\right]z$$

Therefore, according to the given condition (23), we obtain that $\frac{dV_2}{dt} < 0$ is negative definite and hence according to the Laypunov method for stability the second axial equilibrium point $E_2 = (0, \hat{y}, 0)$ is globally asymptotically stable.

■

Theorem (5): Assume that the middle predator free equilibrium point $E_4 = (\bar{\bar{x}}, 0, \bar{\bar{z}})$ is locally asymptotically stable. Then it is globally asymptotically stable if the following sufficient condition holds

$$s + \frac{a_2\bar{\bar{x}}}{a} < d_1 + \frac{b_1\bar{\bar{z}}}{b} \tag{24}$$

Proof: Let $V_3 = N_1\left(x - \bar{\bar{x}} - \bar{\bar{x}}\ln\frac{x}{\bar{\bar{x}}}\right) + N_2 y + N_3\left(z - \bar{\bar{z}} - \bar{\bar{z}}\ln\frac{z}{\bar{\bar{z}}}\right)$ be a real valued positive definite function, where the constants $N_i; i = 1,2,3$ are positive to be determine below. Then we have

$$\frac{dV_3}{dt} = -N_1\frac{r}{K}(x - \bar{\bar{x}}) - [N_1 a_1 - N_2 a_2] - \frac{x(1-cy)y}{a+x(1-cy)}$$
$$-N_1\bar{\bar{x}}\frac{a_2(1-cy)y}{a+x(1-cy)} + N_2 s\left(1 - \frac{y}{L}\right)y - N_2 d_1 y$$
$$-[N_2 b_1 - N_3 b_2]\frac{yz}{b+y} - N_3\frac{b_2\bar{\bar{z}}y}{b+y} - N_3 A_2(z - \bar{\bar{z}})^2$$

Then by choosing the positive constants as following $N_1 = \frac{a_2}{a_1}$, $N_2 = 1$ and $N_3 = \frac{b_1}{b_2}$ and then doing some algebraic steps we obtain that

$$\frac{dV_3}{dt} \le -\frac{a_2}{a_1}\frac{r}{K}(x - \bar{\bar{x}})^2 + \left[\frac{a_2\bar{\bar{x}}}{a} + s - d_1 - \frac{b_1\bar{\bar{z}}}{b}\right]y - \frac{b_1}{b_2}A_2(z - \bar{\bar{z}})^2$$

Therefore, according to the given condition (24), we obtain that $\frac{dV_3}{dt} < 0$ is negative definite and hence according to the Laypunov method for stability the middle predator free equilibrium point $E_4 = (\bar{\bar{x}}, 0, \bar{\bar{z}})$ is globally asymptotically stable.

■

Theorem (6): Assume that the prey free equilibrium point $E_5 = (0, \tilde{\tilde{y}}, \tilde{\tilde{z}})$ is locally asymptotically stable. Then it is globally asymptotically stable if the following sufficient condition holds

$$\frac{a_2}{a_1}\left(\frac{rK}{4}\right) < \frac{b_1(b+\tilde{\tilde{y}})}{bb_2}A_2(z - \tilde{\tilde{z}})^2 \qquad (25)$$

Proof: Let $V_4 = \delta_1 x + \delta_2\left(y - \tilde{\tilde{y}} - \tilde{\tilde{y}}\ln\frac{y}{\tilde{\tilde{y}}}\right) + \delta_3\left(z - \tilde{\tilde{z}} - \tilde{\tilde{z}}\ln\frac{z}{\tilde{\tilde{z}}}\right)$ be a real valued positive definite function, where the constants $\delta_i; i = 1,2,3$ are positive to be determine below. Then we have

$$\frac{dV_4}{dt} = \beta_1 rx\left(1 - \frac{x}{K}\right) - \frac{x(1-cy)y}{a+x(1-cy)}(\delta_1 a_1 - \delta_2 a_2)$$
$$-\delta_3 A_2(z - \tilde{\tilde{z}})^2 - \delta_2\left[\frac{s}{L} - \frac{b_1\tilde{\tilde{z}}}{b(b+\tilde{\tilde{y}})}\right](y - \tilde{\tilde{y}})^2$$
$$-\frac{(y-\tilde{\tilde{y}})(z-\tilde{\tilde{z}})}{(b+y)}\left(\delta_2 b_1 - \delta_3\frac{bb_2}{b+\tilde{\tilde{y}}}\right)$$

Then by choosing the positive constants as following $\delta_1 = \frac{a_2}{a_1}$, $\delta_2 = 1$ and $\delta_3 = \frac{b_1(b+\tilde{\tilde{y}})}{bb_2}$, then by doing some algebraic steps we obtain that

$$\frac{dV_4}{dt} = \frac{a_2}{a_1}\left(\frac{rK}{4}\right) - \frac{b_1(b+\tilde{\tilde{y}})}{bb_2}A_2(z-\tilde{\tilde{z}})^2 - \left[\frac{s}{L} - \frac{b_1\tilde{\tilde{z}}}{b(b+\tilde{\tilde{y}})}\right](y-\tilde{\tilde{y}})^2$$

Therefore, according to the given condition (25) and condition (16b), we obtain that $\frac{dV_4}{dt} < 0$ is negative definite and hence according to the Laypunov method for stability the prey free equilibrium point $E_5 = (0, \tilde{\tilde{y}}, \tilde{\tilde{z}})$ is globally asymptotically stable.

■

Theorem (7): Assume that the top predator free equilibrium point $E_6 = (\breve{\breve{x}}, \breve{\breve{y}}, 0)$, is locally asymptotically stable. Then it is globally asymptotically stable if the following sufficient conditions hold

$$q_{12}{}^2 < 4q_{11}q_{22} \quad (26a)$$

$$\frac{r}{K} > \frac{a_1\breve{\breve{y}}(1-cy)(1-c\breve{\breve{y}})}{R\breve{\breve{R}}} \quad (26b)$$

$$d_2 > \frac{b_2}{b_1}\breve{\breve{y}} + A_1 \quad (26c)$$

where $R = a + x(1-cy)$ and $\breve{\breve{R}} = a + \breve{\breve{x}}(1-c\breve{\breve{y}})$, while q_{11}, q_{12} and q_{22} are given in the proof.

Proof: Let $V_5 = \left(x - \breve{\breve{x}} - \breve{\breve{x}}\ln\frac{x}{\breve{\breve{x}}}\right) + \left(y - \breve{\breve{y}} - \breve{\breve{y}}\ln\frac{y}{\breve{\breve{y}}}\right) + \frac{b_1}{b_2}z$ be a real valued positive definite function. Then after doing some algebraic steps we obtain that

$$\frac{dV_5}{dt} \le -q_{11}(x-\breve{\breve{x}})^2 - q_{22}(y-\breve{\breve{y}})^2 + q_{12}(x-\breve{\breve{x}})(y-\breve{\breve{y}}) - \left[\frac{b_1(d_2-A_1)}{b_2} - \breve{\breve{y}}\right]z$$

where:

$$q_{11} = \frac{r}{K} - \frac{a_1\breve{\breve{y}}(1-cy)(1-c\breve{\breve{y}})}{R\breve{\breve{R}}}$$

$$q_{22} = \frac{s}{L} + \frac{aa_2cx}{R\breve{\breve{R}}}$$

$$q_{12} = \frac{aa_2(1-c\breve{\breve{y}}) + a_1c^2\breve{\breve{x}}\breve{\breve{y}}y + a_1c(y+\breve{\breve{y}})(a+\breve{\breve{x}}) - a_1(a+\breve{\breve{x}})}{R\breve{\breve{R}}}$$

Now by using the given conditions we obtains that

$$\frac{dV_5}{dt} \leq -\left[\sqrt{q_{11}}(x-\check{\check{x}})-\sqrt{q_{22}}(y-\check{\check{y}})\right]^2-\left[\frac{b_1(d_2-A_1)}{b_2}-\check{\check{y}}\right]z$$

Therefore, $\frac{dV_5}{dt}<0$ is negative definite and hence according to the Laypunov method for stability the top predator free equilibrium point $E_6=(\check{\check{x}},\check{\check{y}},0)$ is globally asymptotically stable. ■

Theorem (8): Assume that the positive equilibrium point $E_7=(x^*,y^*,z^*)$, is locally asymptotically stable. Then it is globally asymptotically stable if the following sufficient conditions hold

$$\frac{r}{K}>\frac{a_1y^*}{RR^*}(1-cy^*)(1-cy) \qquad (27a)$$

$$\frac{s}{L}+\frac{aa_1cx}{RR^*}>\frac{bz^*}{BB^*} \qquad (27b)$$

$$p_{12}{}^2<2p_{11}p_{22} \qquad (27c)$$

$$p_{23}{}^2<2p_{22}p_{33} \qquad (27d)$$

where $B=b+y$; $B^*=b+y^*$, $R^*=a+x^*(1-cy^*)$, while p_{ij} are given in the proof.

Proof: Let $V_6=\left(x-x^*-x^*\ln\frac{x}{x^*}\right)+\left(y-y^*-y^*\ln\frac{y}{y^*}\right)+\left(z-z^*-z^*\ln\frac{z}{z^*}\right)$ be a real valued positive definite function. Then after doing some algebraic steps we obtain that

$$\frac{dV_6}{dt} \leq -p_{11}(x-x^*)^2+p_{12}(x-x^*)(y-y^*)-p_{22}(y-y^*)^2$$
$$+p_{23}(y-y^*)(z-z^*)-p_{33}(z-z^*)^2$$

where

$$p_{11}=\frac{r}{K}-\frac{a_1y^*}{RR^*}(1-cy^*)(1-cy)$$

$$p_{22}=\frac{s}{L}-\frac{bz^*}{BB^*}+\frac{aa_1cx}{RR^*}$$

$$p_{12}=\frac{a_1}{RR^*}\left[c(y+y^*)(a+x^*)-\left(a+x^*(1+c^2y^*y)\right)+a(1-cy^*)\right]$$

$$p_{23}=\frac{b}{BB^*}(b_2-B^*)$$

$$p_{33}=A_2$$

Using the above conditions, we obtain that

$$\frac{dV_6}{dt} \leq -\left[\sqrt{p_{11}}(x-x^*) - \sqrt{\frac{p_{22}}{2}}(y-y^*)\right]^2 - \left[\sqrt{\frac{p_{22}}{2}}(y-y^*) - \sqrt{p_{33}}(z-z^*)\right]^2$$

Therefore, $\frac{dV_6}{dt} < 0$ is negative definite and hence according to the Laypunov method for stability the positive equilibrium point $E_7 = (x^*, y^*, z^*)$, is globally asymptotically stable. ■

References

[1] C.S. Holling, The functional response of predator to prey density and its role in mimicry and population regulation, Mem. Ent. Sec. Can 45-1-60 (1966).

[2] X.Tian, R.Xu, Global dynamics of a predator-prey system with Holling type II Functional response, nonlinear analysis:Modelling and control, 16 (2), 2011, 242-253.

[3] T.K.Kar, Modelling the analysis of a harvested prey-predator system in corporation a refuge. J. Comp. App. Math., Vol.185 (2006) 19-33.

http://dx.doi.org/10.1016/j.cam.2005.01.035.

[4] K.L.Narayan, A mathematical study of prey-predator, ecological models with partial Covers for the prey and alternative food for the predator, Ph.D thesis, 2004, J.N.T. university.

[5] R. Senthamara,T. Vijayalakhmi, An analytical approach to top predator interference On the dynamics of food chain model, Iop conf. series: Journal of physics: Conf. series 1000 (2018) 012139.

[6] I. B.Mohd, M.Mamat and Z.Salleh, Mathematical model of three species food chain Interaction with mixed functional response, International journal of modern phy sics: conference series Vol.9(2012) 334-340.

[7] A.Gkana, L.Zachilas, Incorporating prey refuge in a prey-predator model with a Holling type I functional response: random dynamics and population outbreaks, J. Biol phys. 2013 Sep.; 39 (4):587-606.

[8] S.Sarwardi, P.K.Mandal, S.Ray, Dynamical prey refuge, J. Biol Phys. 2013 Sep.; 39(4): 701-722.

[9] Y.Huang, F.Chen, L.Zhong, Stability analysis of a prey-predator model with Holling Type III response function incorporating a prey refuge, Applied mathematics and Computation, Vol.(182), Issue 1,1 November 2006 pages 672-683.

[10] T.K.Kar, Stability analysis of a prey-predator model incorporating a prey refuge Communications in nonlinear science and numerical simulation, Vol.10, Issue 6, September 2005, Pges 681-641.

[11] M.Haque, M.S.Rahman, E.Venturino,and B.L.Li, Competition models with niche For squirrel population dynamics, in proceedings of the 11th International conference Of numerical analysis and applied mathematic(ICNAAM 13), Vol.1558 of AIP Conference Proceeding, pp.1818-1821, Rhodes, Greece, September 2013.

[12] R.K. Naji, S. J. Majeed, The dynamical of a prey-predator model with a refuge stage Structure prey population, International journal differential equations,Vol.2016,

Article ID2010464, 10 pages, http://dx.doi.org/101155/2016/2010464.

[13] A. Sih, Prey refuge and predator-prey stability, Theoretical population biology Vol.31. Issue 1, February 19, pages 1-12.

[14] U. Das, T. K. Kar, U.K.Pahari, Global dynamics of an exploited prey-predator with constant prey refuge, ISRN Biomathematics, Vol.2013, article ID 637640,12

pages, http://dx.doi.org/10.1155/2013/637640.

[15] H. Molla, M.S. Rahaman, S. Sarwardi, Dynamics of predator-prey model with Holling type II functional response incorporation a prey refuge depending on both the species, IJNSNS, 2018, http://doi.org/10.1515/ijnsns-2017-0224.

[16] W. Ko, K. Ryu, Qualitive analysis of a predator-prey model with Holling type II functional incorporating a prey refuge , Journal of differential equations, Vol. 231, Issue 2,15, Decembr 2006, Pages 534-550.

[17] T. K. Kar, K. Chakraborty, U. K. Pahari, A prey- predator model with alternative Prey: Mathematical model and analysis, Canaian applied, mathematics quarterly, Vol. 18, Number 2, Summer 2010.

[18] M. Agarwal, A. Kumar, Dynamics of food chain: Role of alternative resource for top predator , International journal of mathematical modeling and computations Vol. 07, No. 02, Summer 2017, 115-128.

[19] J. N. MCNair, The effects of refuges on predator-prey interactions: Areconsideration, Theoretical population biology, Vol. 29,Issue 1,February 1986, pages 38-63.

[20] B. Sahoo, Global stability of predator-pery system with alternative prey, ISRN Biotechnology, Vol. 2013, article ID 898039, 8 pages,

http://dx.doi.org/10.5402/2013/898039.

Lyapunov's Function for Random Dynamical Systems and Pullback Attractors

Mohammed Shannan Imran1 *and* Ihsan Jabbar Kadhim *2*

1,2Department of Mathematics,

College of Computer Science and Information Technology,

University, Al-Qadisiyah

Email:mohammedshannan@yahoo.com1

Email :Ihsan.Kadhim@qa.edu.iq 2

Math. Sub Class :93D30 ,34C35

Abstract. In this paper we study the Lyapunov function for random dynamical systems , where some new properties are proved ,Lyapunov stability theorem ,Rate of pullback convergence.

Key words: random dynamical system ,pullback attractor and Lyapunov function for random dynamical system.

1.Introduction

The concept of random dynamical systems is a comparatively recent development combining ideas and methods from the well developed areas of probability theory and dynamical systems. Due to our inaccurate knowledge of the particular model or due to computational or theoretical limitations (lack of sufficient computational power, in- efficient algorithms or insufficiently developed mathematical and physical theory, for example), the mathematical models never correspond exactly to the phenomenon they are meant to model. Moreover, when considering practical systems we cannot avoid ei- ther external noise or inaccuracy errors in measurements, so every realistic mathematical model should allow for small errors along orbits. To be able to cope with unavoidable uncertainty about the ”correct” parameter values, observed initial states and even the specific mathematical formulation

involved, we let randomness be embedded within the model. Therefore, random dynamical systems arise naturally in the modeling of many
phenomena in physics, biology, economics, climatology, etc.

The concept of random dynamical systems was mainly developed by Arnold [1] and his ”Bremen group”, based on the research of Baxendale [2], Bismut [3], Elworthy [4], Gihman and Skorohod [5], Ikeda and Watanabe [6] and Kunita [7] on two-parameter stochastic flows generated by stochastic differential equations.

Lyapunov’s first method was, however, filled with new life in 1968 when Oseledets [10] proved his celebrated multiplicative ergodic theorem. For (random) dynamical systems under an invariant measure this theorem establishes the existence of Lyapunov exponents as limits and can be used to conclude nonlinear stability from linear stability. A systematic account of the theory of nonlinear random dynamical systems based on Lyapunov’s first method through the multiplicative ergodic theorem is given by Arnold [1]. In contrast to his first method, Lyapunov’s second method turned out to be very successful from the beginning, in particular in numerous applied problems. Early systematic accounts in the West were given by the Springer Grundlehren volumes of Hahn [9] in 1967 and of Bhatia and Szegö [8] (developing Lyapunov’s second method for dynamical systems) in 1970, both of which are still classical references.

D. T. Son (2009)[11] studied the Lyapunov exponents for random dynamical systems. X. Yingchao (2010)[12] used the theory of random dynamical systems and stochastic analysis to research the existence of random attractors and also stochastic bifurcation behavior for stochastic Duffing-van der Pol equation with jumps under some assumptions. I.J.Kadhim and A.H. Khalil(2016)[13] they define the random dynamical system and random sets in uniform space are and proved some necessary properties of these two concepts. Also they sudy the expansivity of uniform random operator.

The structure of this paper is as follows: In Section 2 we recall same basic definition and facts about random dynamical, study the definition of Lyapunov function for random dynamical systems and theorem Lyapunov stability . In Section 3 we will study Lyapunov function for pullback attractor , existence of a pullback absorbing neighborhood system and theorem (Rate of pullback convergence).

2.Lyapunov Functions for Random Dynamical Systems

Definition2.1. A closed random set $M:\Omega \longrightarrow P(x)$ is said to be a semi-weak attractor ,if $\forall x \in M(\omega) \exists$ a Tempered random variable $\delta_x:\Omega \longrightarrow R^+$ and $y \in S(x, \delta_x(\omega))$, there is a sequence $\{t_n\}$ in $\mathbb{R}$, $t_n \longrightarrow +\infty \ni d(\varphi(t_n, \theta_{t_n}\omega)x, M(\omega)) \to 0$ as $t \to +\infty$

i. a semi –attractor , if $x \in M, \exists$ tempered random variable $\delta_x \ni$ $S(x, \delta_x(\omega), d\left(\varphi\left(t, \theta_{t_n}\omega\right)x, M(\omega)\right)) \to 0$ as $t \to +\infty$

ii. a weak attractor , if there is a tempered random variable δ and for each $y \in S\big(M(\omega), \delta(\omega)\big)$, there is a sequence $\{t_n\}$ in $\mathbb{R}$, $t_n \longrightarrow +\infty$ such that $d(\varphi\big(t_n, \theta_{t_n}\omega\big)x, M(\omega) \to 0$

iii. an attractor , if there is a tempered random variable δ such that for each $y \in S\big(M(\omega), \delta(\omega)\big)$, $d(\varphi(t, \theta_t\omega)x, M(\omega)) \to 0$ as $t \to +\infty$

iv. a uniform attractor , if there is a tempered random variable δ such that for each $\varepsilon > 0$, there is T = T $(\varepsilon) > 0$ such that $\{\varphi(t, \theta_t\omega)x : t \in [T, +\infty)\} \subset S(M, \varepsilon)$ for each $x \in S[M, \delta]$

an equi attractor if it is an attractor and if there is $\lambda > 0$ such that for each ε , $0 < \varepsilon < \lambda$ and $T > 0$, there exists tempered random variable δ with $\{\varphi(t, \theta_t\omega)x : t \in [0, T]\} \cap S(M(w), \delta) = \emptyset$ whenever $\varepsilon \leq d(x, M) \leq \lambda$

Definition2.2 (Lyapunov Functions) [1] . Let φ be a random dynamical system in $\mathbb{R}^d$ and A be a random compact set which is invariant under φ. A Function $V: \Omega \times \mathbb{R}^d \to R^+$ is called Lyapunov Functions for A (under φ) if it is has the following properties :

i. $\omega \mapsto V(\omega, x)$is measurable for each $x \in \mathbb{R}^d$, and $x \longmapsto V(\omega, x)$ is continuous for each $\omega \in \Omega$
ii. V is uniformly unbounded , i.e., $lim_{\|x\| \mapsto \infty} V(\omega, x) = \infty$ for all ω.
iii. V is positive-definite, i.e., $V(\omega, x) = 0$ for $x \in A(\omega)$,and $V(\omega, x) > 0$ for $x \notin A(\omega)$.
iv. V is strictly decreasing along orbits of φ i.e, $V\big(\theta_t\omega, \varphi(t, \omega, x)\big)\, V(\omega, x)$ for all $t > 0$ and $x \notin A(\omega)$.

Definition2.3 The derivative of the function $V: \Omega \times \mathbb{R}^d \longmapsto R^+$ along the parametric vector$X(t) = (x_{1\,(t)}, x_{2\,(t)}, \dots, x_{d\,(t)})$ is defined by

$$\dot{V}\big(\omega, X(t)\big) = \frac{d}{dt} V\big(\omega, X(t)\big) = \sum_{i=1}^{d} \frac{\partial v(\omega, x(t))}{\partial x_i} \frac{dx_i}{dt}, \omega \in \Omega. \qquad (1)$$

Theorem2.4 (Lyapunov stability theorem)[1]

Let $x = X(t), x \in S \subseteq \mathbb{R}^d \longmapsto R^+$ has critical point at the origin. If there is function $V: \Omega\, \mathrm{x}\mathbb{R}^d \longmapsto R^+$ such that

i. The partial derivative $\frac{\partial v(\omega, x(t))}{\partial x_i}$, i=1, 2,.,.,d exist and continuous .
ii. V is positive –definite .

iii. $\dot{V}$ is semi-positive –definite .

Then the origin is stable critical point for the system. If (iii) above replaces by a stronger condition (iii*) $\dot{V}$ is negative –definite.

Then the origin is asymptotical stabile critical point for the systems. If the function that satisfies the hypothesis (i), (ii) and (iii) from the above theorem is called weak Lyapunov function and if hypothesis (iii) is replaced by (iii*), then $v(x, y)$ is called strong Lyapunov function**.**

In the following we shall characterize the asymptotically random set in terms the lyapunov function . to this end we first sate and prove the following lemma.

Lemma2.5 let the phase space X be arbitrary and $K \subset X$. Let $V: \Omega \times K \longmapsto R$ be any continuous function defined on K such that $V\big(\theta_t\omega, \varphi(t, \omega, x)\big) < V(\omega, z)$ for all $t > 0$ and $x \notin K(\omega)$. Whenever $\varphi([0, t], \omega, x) \subset K(\omega), t \geq 0$.Then if some $x, \varphi(R^+, \omega, x) \subset K(\omega)$, we have $V(\omega, y) = V(\omega, z)$ for every $y, z \in \Lambda^+(x)$.

Proof. Assume that $V\big(\theta_t\omega, \varphi(t, \omega, x)\big) < V(\omega, x)$. there are indeed sequence$\{t_n\}$ and$\{\tau_n\}$ in $\mathbb{R}$ such that$t_n \to \infty$, $\tau_n \to \infty$ and $\varphi(t_n, \omega, x) \to y$, $\varphi(\tau_n, \omega, x)$ →z . We may assume by taking a subsequence that $t_n < \tau_n$ for each n ,Then clearly $V\left(\theta_{t_n}\omega, \varphi(t_n, \omega, x)\right) \geq V\left(\theta_{\tau_n}\omega, \varphi(\tau_n, \omega, x)\right)$ as

$\varphi(\tau_n, \omega, x) = \varphi((\tau_n - t_n) + t_n, \omega, x) = \varphi(\tau_n - t_n), \theta(t_n)\omega, \varphi(t_n, \omega, x)),$ $\tau_n - t_n > 0$, and

$\varphi([0, \tau_n - t_n], \theta(t_n)\omega, \varphi(t_n, \omega, x)) \subset K(\omega)$, Thus proceeding to the limit we have continuity of V ,$V\big(\theta_t\omega, \varphi(t, \omega, y)\big) \geq V(\omega, z)$. This contradicts the original assumption and the limit is proved .

Theorem2.6. A compact random set $M \subset X$ is asymptotically stable if and only if there exists a function $V: \Omega \times N \longrightarrow \mathbb{R}$,where is a neighborhood $N$of M such that

2.6.1 $V(\omega, .): N \to \mathbb{R}$,is continuous $\forall \omega \epsilon \Omega$ and $V(\omega, x): \Omega \longrightarrow \mathbb{R}$ is measurable $\forall x \epsilon N$.

2.6.2 $V(\omega, x) = 0$ if $\forall x \epsilon M$ and $V(\omega, x) > 0$ if $x \notin M, \forall \omega \epsilon \Omega$

2.6.3 $V\big(\theta_t\omega, \varphi(t, \omega, y)\big) < V(\omega, x)$ For all $t > 0$ and $x \notin M(\omega)$ for $x \notin M, t > 0$ and $\varphi([o, t] , \omega, x) \subset N(\omega) \forall \omega \epsilon \Omega$.

Proof. Assume that a function V as required is given. Choose $\alpha > 0$ such that $S[M,\alpha] \subset N$ and is compact of Let $m = \min\{\Phi(x): x \in H(M,\alpha)\}$,Then by 2.6.2 and continuity of $\Phi, m > 0$. Set $K = \{x \in S[M,\alpha]: \Phi(x) \leq m\}$. Then K is indeed compact and because 2.2.3 k is positively invariant. This establishes that M is stable as K positively invariant neighborhood of. To see that M is an attractor, choose any compact positively invariant neighborhood K of M with $K \subset M$. Then for any $x \epsilon k$, $\emptyset \neq \Lambda^+ \subset K$,and lemma 2.5 shows that Φ is constant on $\Lambda^+(x) \subset M$. Thus M is attractor and consequently asymptotically stable. Now let M be asymptotically stable and $A\,(M)$ its region of attractor. For each $x \in A\,(M)$define

$$\varphi(x) = \sup\{\varphi(t,\theta_t\omega)x, M) : t \geq 0\}$$

Indeed $\varphi(x)$ is defined for each $x \in A\,(M)$ b, because if $\varrho(\pi(t,x),M) = \alpha$, Then there is a $T > 0$ with $\pi(T,+\infty),x) \subset S(t,\alpha)$. Thus

$$\varphi(x) = \sup\{\varphi(t,\theta_t\omega)x, M) : T \geq t \geq 0\}$$

As $\varrho(\varphi(t,\theta_t\omega)x,M)$ is a continuous function of $t, \varphi(x)$ is define. This $\varphi(x)$ has the properties: $\varphi(x) = 0$ for $x \in M, \varphi(x) \geq 0$ for $x \notin M$ and $\varphi\big(\pi(t,x)\big) \leq \varphi(x)$ for $t \geq 0$. This is clear when we remember that M is stable and hence positively invariant and that $A\,(M)$ is invariant. Thus if $\varphi(x)$ is defined for any $x \in M$, it is defined for all xt with$t \in R^+$. We further claim that this $\varphi(x)$ is continuous in M. Indeed stability of M implies continuity of $\varphi(x)$ on M . For $x \notin M$, set $\varrho(\pi(t,x),M) = \alpha(> 0)$ and choose $\varepsilon, \varepsilon > {}^{\alpha}/_{4}$, such that $S[x,\varepsilon]$ is compact subset of $A\,(M)$ is open , since M is uniform attractor , there is a $T > 0$ such that $S[x,\varepsilon]t \subset S[M, {}^{\alpha}/_{4}]$ for all $t \geq T$. Thus for $y \epsilon S[x,\varepsilon]$ we have

$$\varphi(x) - \varphi(y) = \sup\{\varrho(\varphi(t,\theta_t\omega)x,M) : t \geq 0\} - \sup\{\varrho(\varphi(t,\theta_t\omega)y,M) : t \geq 0\}$$

$$= \sup\{\varrho(\varphi(t,\theta_t\omega)x,M) : T \geq t \geq 0\} - \sup\{\varrho(\varphi(t,\theta_t\omega)y,M) : T \geq t \geq 0\}$$

There fore

$$|\varphi(x) - \varphi(y)| \leq \sup\{\varrho(\varphi(t,\theta_t\omega)x,M) - \varrho(\varphi(t,\theta_t\omega)y,M) : T \geq t \geq 0\}$$
$$\leq \varrho(\varphi(t,\theta_t\omega)x, \varphi(t,\theta_t\omega)y) : T \geq t \geq 0\}.$$

The continuity axiom implies that the right hand side of the above inequality tends to zero as $y \to x$, for T is fixed for $y \epsilon S[x,\varepsilon]$ .The function $\varphi(x)$ is therefore continuous in M . However, the above function may not be strictly decreasing along parts of trajectories in $A\,(M)$ which are not in M and so may not satisfy 2.6.3. Such a function can be obtained by setting

$V(\omega,x) = \int_0^\infty V(\theta_{\tau_n}\omega, \varphi(\tau_n,\omega,x)) \exp(-\tau_n)\, d\tau.$

That $V(\omega,x)$ is continuous and satisfies 2.6.2 in $A\ (M)$ is clear. To see that $V(\omega,x)$ satisfies 2.6.3, let $x \notin M(\theta_{-t}\omega)$ and $t > 0$. Then indeed $V\big(\theta_t\omega, \varphi(t,\omega,x)\big) < V(\omega,x)$, holds because $v\big(\theta_t\omega, \varphi(\theta_\tau\omega,x)\big) < v(\omega,x)$ holds.

To rule $V\big(\theta_t\omega, \varphi(\theta_\tau\omega,x)\big) < V(\theta_\tau\omega,x) = V(\omega,x), \forall x \in N$, observe that in this case we must have

$V\big(\theta_{t+\tau}\omega, \varphi(t+\tau\ \ ,\omega,x)\big) \equiv V\big(\theta_\tau\omega, \varphi(\tau\ \ ,\omega,x)\big)$ for all $\tau > 0\ , x \in .$

thus in particular, letting $\tau = 0\ , t\ , 2t\ , \dots$ we get

$\mathrm{V}(\omega\ , x) = V\big(\theta_{nt}\omega, \varphi(nt\ \ ,\omega,x)\big)\ , n\ = 1,2,3, \dots.$ By asymptotic stability of $M(\theta_{-t}\omega)$ implies that for $x\epsilon A(M)$, $\varphi(nt\ \ ,\omega,x) \to 0\ as\ t \to \infty$, $V(\omega,x)$ is continuous. This shows that $V(\omega,x)$ =0. But as $x \notin$ M, we must have $V(\omega,x) > 0$ a contradiction . We have thus proved that $V\big(\theta_t\omega, \varphi(t,\omega,x)\big) < V(\omega,x)$ for$x \notin$ M, and $t > 0$. the theorem is proved.

3.Lyapunov function for pullback attractor

A Lyapunov function characterizing pullback attraction and pullback attractors for a discrete –time process in $\mathbb{R}^d$ will be constructed here.

Consider a non-autonomous difference equation

$x_{n+1}(\omega) = f_n(\theta_n\omega, x_n(\omega))$ (2)

On $\mathbb{R}^d$, where the $f_n\colon \mathbb{R}^d \to \mathbb{R}^d$, are Lipschitz continuous mapping.

This generates a process $\Phi\colon z^2 \times \mathbb{R}^d \to \mathbb{R}^d$ through iteration by $\Phi\big(n, n_0, x_0(\omega)\big) = f_{n-1}{}^\circ \dots {}^\circ f_{n_0}\ (x_0(\omega))$ is Lipschitz continuous for all $n \geq n_0$ the pullback attraction is taken with respect to basin of attraction system, which is define as follows for a process.

Definition 3.1 A basin of attraction system $\mathfrak{D}_{att}$ consist of families $\mathcal{D} = \{D_n\colon n\ \in z\}$ of non empty pounded random set of $\mathbb{R}^d$ with the property that $\mathcal{D}^{(1)} = \left\{D_n^{(1)}\colon n \in z\right\} \in \mathfrak{D}_{att}$if $\mathcal{D}^{(2)} = \left\{D_n^{(2)}\colon n \in z\right\} \in \mathfrak{D}_{att}$and $D_n^{(1)} \subset D_n^2$ for all $n\ \in z.$

Although somewhat complicated, the use of such a basin of attraction system allows both non-uniform and local attraction region, which are typical in non-autonomous system, to be handled.

Definition 3.2 A φ-invariant family of nonempty compact $\mathcal{A} = \{A_n : n \in z\}$ is called a pullback attractor with respect to a basin of attraction system $\mathfrak{D}_{att}$ if it is pullback attracting

$$\lim_{j\to\infty} dist(\varphi\left(n, \theta_{n-j}(\omega), D_{n-j}(\omega)\right), A_n(\omega) = 0 \qquad (3)$$

For all $n \in z$ and all $\mathcal{D} = \{D_n : n \in z\} \in \mathfrak{D}_{att}$.

Obviously $\mathcal{A} \in \mathfrak{D}_{att}$

The construction of the Lyapunov function requires the existence of a pullback absorbing neighborhood family.

Existence of a pullback absorbing neighborhood system

The following lemma shows that three always exists such a pullback absorbing neighborhood system for any given pullback attractor. **Lemma 3.3** if A is a pullback attractor with a basin of attraction system $\mathcal{D}_{att}$ for a process, then there exists a pullback absorbing neighborhood system $B \subset \mathcal{D}_{att}$ of A with random T. φ moreover, B is $\varphi - positive$ invariant

Proof. For each $n_0 \in \mathbb{Z}$ pick $S_{n_0}(\omega) > 0$ such that $B\left[A_{n_0}(\omega); S_{n_0}(\omega)\right] := \{x \in R^d : dist\,(X, A_{n_0}\omega) \leq S_{n_0}\}$ such that $\{B\left[A_{n_0}(\omega);\, S_{n_0}(\omega)\right] : n_0 \in \mathbb{Z}\} \in \mathcal{D}_{att}$, and define $B_{n_0} := \overline{\cup_{j\geq 0}\ \varphi(n_0, \theta_{n_\circ - j}(\omega), B\left[A_{n_0-j}(\omega); S_{n_0-j}(\omega)\right])}$

Obviously $A_{n_0} \subset \text{int}\, B\left[A_{n_0}(\omega); S_{n_0}(\omega)\right] \subset B_{n_0}$.

To show positive invariance the two-parameter semi group property will be used in where follows

$$\varphi\left(n_\circ + 1, \theta_{n_0}\omega, B_{n_0}\right)$$

$$= \overline{\cup_{j\geq 0}\ \varphi(n_0 + 1, \theta_{n_0}\omega, \varphi\left(\theta_{n_0}\omega, \theta_{n_0-j}\omega, B\left[A_{n_0-j}(\omega); S_{n_0-j}(\omega)\right]\right))}$$

$$= \overline{\cup_{j\geq 0}\ \varphi(n_0 + 1, \theta_{n_0-j}\omega, B\left[A_{n_0-j}(\omega); S_{n_0-j}(\omega)\right])}$$

$$= \overline{\cup_{l\geq 1}\ \varphi\left(n_0 + 1, \theta_{n_0+1-j}\omega, B\left[A_{n_0+1-j}(\omega); S_{n_0+1-j}(\omega)\right]\right)}$$

$$\subseteq \overline{U_{l\geq 0}\varphi\left(n_0 + 1, \theta_{n_0+1-j}\omega, B\left[A_{n_0+1-j}(\omega); S_{n_0+1-j}(\omega)\right]\right))} = B_{n_0+1}(\omega)$$

so

$\varphi(n_0+1, \theta_{n_0}\omega, B_{n_0}(\omega)$. This and the two parameter semi group property again gives

$$\varphi(n_0+1, \theta_{n_0}\omega, B_{n_0}(\omega) = \varphi(n_0+1, \theta_{n_0+1}\omega, \varphi(n_0+1, \theta_{n_0}\omega, B_{n_0}(\omega)$$

$$\subseteq \varphi(n_0+2, \theta_{n_0+1}\omega, B_{n_0+1}(\omega) \subseteq B_{n_0+2}(\omega).$$

The general positive invariance assertion then follows by induction. Now referring to the continuity of $\varphi(\theta_{n_0}\omega, \theta_{n_0-j}\omega)$ and the Compactness of $B\left[A_{n_0-j}(\omega); S_{n_0-j}(\omega)\right]$the set $\varphi\left(n_0, \theta_{n_0-j}\omega, B\left[A_{n_0-j}(\omega); S_{n_0-j}(\omega)\right]\right)$ is compact for each $j \geq 0$ and $n_0 \in \mathbb{Z}$. moreover, by pullback convergence , there exists an

$N = N(n_0, S_{n_0}) \in \mathbb{N}$ such that $\varphi(n_0, \theta_{n_0-j}\omega, B\left[A_{n_{0-j}}(\omega), S_{n_0-j}(\omega)\right]) \subseteq$ $B[A_{n_0}(\omega); S_{n_0}(\omega)] \subset \mathrm{B}(\theta_{n_0}\omega)$ for each $j \geq N$. Hence

$$\mathrm{B}(\theta_{n_0}\omega) = \overline{\bigcup_{j\geq 0} \varphi(n_0, \theta_{n_0-j}(\omega), B[A_{n_0-j}(\omega), S_{n_0-j}(\omega))}$$

$$\subseteq B[A_{n_0}(\omega); S_{n_0}(\omega) \cup \overline{\bigcup_{0\leq j<N} \varphi\left(n_0, \theta_{n_0-j}\omega, B\left[A_{n_0-j}(\omega); S_{n_0-j}(\omega)\right]\right)}$$

$= \overline{\bigcup_{0\leq j<N} \varphi(n_0, \theta_{n_0-j}, B[A_{n_0-j}(\omega); S_{n_0-j}(\omega)]}$), which is compact, so $\mathrm{B}(\theta_{n_0}\omega)$ is compact.

To see that $\mathcal{B}$so constructed is pullback absorbing with respect to$\mathfrak{D}_{att}$, let$\mathcal{D} \in \mathfrak{D}_{att}$.

Fix $n_0 \in \mathbb{Z}$. since $\mathcal{A}$ is pullback attracting , there exists an $N(\mathcal{D}, \delta_{n_0}, n_0) \in \mathbb{N}$ such that $\mathrm{dist}(\varphi(n_0, \theta_{n_0-j}\omega, D_{n_0-j}), A_{n_0}) < \delta(\theta_{n_0}\omega)$ for all $j \geq N(\mathcal{D}, S_{n_0}, n_0)$. but $\varphi((n_0, \theta_{n_0-j}\omega, D_{n_0-j}) \subset$ $int\, B[A_{n_0}(\omega), \delta_{n_0}(\omega)]$ and $B[A_{n_0}(\omega), \delta_{n_0}(\omega)] \subset \mathrm{B}(\theta_{n_0}\omega)$, so

$$\varphi(n_0, \theta_{n_0-j}(\omega), D_{n_0-j}) \subset int\, \mathrm{B}(\theta_{n_0}\omega) \text{ for all } j \geq N(D, \delta_{n_0}, n_0)$$

Hence $\mathcal{B}$is pullback absorbing as required .

Necessary and sufficient conditions

The main result is the construction of a Lyapunov function that characterizes this pullback attraction

Theorem. **3.4**. Let f_n be uniformly of Lyapunov continuous on $\mathbb{R}^d$ for each $n \in \mathbb{Z}$ and let φ be the process that they generate. In addition, let $\mathcal{A}$ be a φ –invariant family of nonempty compact random sets that is pullback attracting with respect to φ with a basin of attraction system $\mathfrak{D}_{att}$. Then there exists a Lipschitz continuous function$V: \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$, such that

Property 1 (upper bound): for all $n_0 \in z$ and $x_0 \in \mathbb{R}^d$

$$V(n_0, x_0) \leq dist(x_0, A_{n_0}), \quad (4)$$

Property 2 (lower bound): for each $n_0 \in z$ there exists a function $a(n_0, .): \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $a(n_0, 0) = 0$ and $a(n_0, r) > 0$ for all $r > 0$ which is monotonically increasing in r such that

$$a\left(n_0, dist(x_0, A_{n_0})\right) \leq V(\theta_{n_0}\omega, x_0) \text{ For all } x_0 \in \mathbb{R}^d \quad (5)$$

Property 3 (Lipschitz condition): for $n_0 \in z$ and $x_0 , y_0 \in \mathbb{R}^d$,

$$\left|V\left(\theta_{n_0}\omega, x_0\right) - V(\theta_{n_0}\omega, y_0)\right| \leq \|x_0 - y_0\|, \quad (6)$$

Property 4(pullback convergence): for all $n_0 \in z$ and any $\mathcal{D} \in \mathfrak{D}_{att}$ $limit\ sup_{n\rightarrow\infty}\ sup_{zn_0-n\in D_{n_0-n}} V\left(\theta_{n_0}\omega, \varphi(n_0, \theta_{n_0-n}, zn_0 - n)\right) = 0$ (7)

In addition,

Property 5 (forwards convergence) : there exists $\mathcal{N} \in \mathfrak{D}_{att}$, which is positively invariant under φ and consists of nonempty compact random sets $N(\theta_{n_0}\omega)$ with $A_{n_0}(\omega) \subset intA_{n_0}(\omega)$ for each $n_0 \in \mathbb{Z}$ such that $V\left(\theta_{n_0+1}\omega, \varphi(n_0 + 1, \theta_{n_0}\omega, x_0)\right) \leq e^{-1}V\left(\theta_{n_0}\omega, x_0\right)$ (8)

For all $x_0 \in N(\theta_{n_0}\omega)$ and hence

$$V\left(\theta_{n_0+j}\omega, \varphi(j, \theta_{n_0}\omega, x_0)\right) \leq e^{-j}V\left(\theta_{n_0}\omega, x_0\right) \text{ for all} x_0 \in N(\theta_{n_0}\omega)$$

$$, j \in \mathbb{N} \quad (9)$$

Proof. The aim is to construct a Lyapunov function $V\left(\theta_{n_0}\omega, x_0\right) := \sup_{n\in\mathbb{N}} e^{-T_{n_0}} dist(\left(x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega\right)$ for all $n_0 \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^d$ where $T_{n_0,n} = n + \sum_{j=1}^{n} \alpha_{n_0-j}^{+}$.

With $T_{n_0,0} = 0$. Here $\alpha_n log L_n$ where L_n is the uniform Lipschitz constant f_n on $\mathbb{R}^d$, and$a^+ = (a + |a| / 2$, that is the positive part of a real number a.

Note 4 $T_{n_0,n} \geq n$ and $T_{n_0,n+m} = T_{n_0,n} + T_{n_0-n,m}$ for $n, m \in \mathbb{N}, n_0 \in \mathbb{Z}$.

Proof 1

Since $e^{-T_{n_0},n} \leq 1$ for all $n \in \mathbb{N}$ and $dist\big(x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega\ \big)$

Is monotonically increasing from $0 \leq dist\big(x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega\ \big) \leq 1. dist(x_0, A(\theta_{n_0}\omega)$.

Proof 2

If$x_0 \in A(\theta_{n_0}\omega)$,then $V\big(\theta_{n_0}\omega, x_0\big) = \sup_{n\geq 0} e^{-T_{n_0-n}} dist(\big(x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega\ \big))$ the supremum involves the product of an exponentially quantity bounded below by zero and a bounded increasing function , since the set $\varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega\ )$ are a nested family of compact random sets decreasing to $A(\theta_{n_0}\omega)$ with increasing n. In particular, $dist\big(x_0, A(\theta_{n_0}\omega\big) = \big(x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega\ \big)$ for all $n \in \mathbb{N}$. Hence there exists an $N^* = N^*(n_0, x_0) \in \mathbb{N}$ such that

$\frac{1}{2} dist(x_0, A(\theta_{n_0}\omega) \leq dist(x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega) \leq dist(x_0, A(\theta_{n_0}\omega)$ .For all$n \geq N^*$, but not $n = N^* - 1$. There, from above,

$$V\big(\theta_{n_0}\omega, x_0\big) \geq e^{-T_{n_0-N^*}} dist(\big(x_0, \varphi(n_0, \theta_{n_0-N^*}\omega, B(\theta_{n_0-N^*}\omega)\ \big) \geq \frac{1}{2}\ e^{-T,N^*}(x_0, A\big(\theta_{n_0}\omega\big))$$

Define

$$N^*\big(\theta_{n_0}\omega, r\big) \coloneqq \sup\Big\{N^*\big(\theta_{n_0}\omega, x_0\big): dist\left(x_0, A\big(\theta_{n_0}\omega\big)\right) = r\Big\}$$

Now

$N^*\big(\theta_{n_0}\omega, r\big) <\infty$ For $x_0 \notin A\big(\theta_{n_0}\omega\big)$ with $dist\left(x_0, A\big(\theta_{n_0}\omega\big)\right) = r$ and $N^*\big(\theta_{n_0}\omega, r\big)$ is non decreasing with $r \to 0$. To see this note that by the triangle rule $dist\left(x_0, A\big(\theta_{n_0}\omega\big)\right) \leq dist(\big(x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega)\ \big) + dist\ \big(\ \varphi(n_0, \theta_{n_0-n}\omega, B\big(\theta_{n_0-n}\omega\big)), A\big(\theta_{n_0}\omega\big)\ \big)$.

Also by pullback convergence there exists an $N(n_0, {}^{r}/_{2})$ such that

$dist\ \big(\ \varphi(n_0, \theta_{n_0-n}\omega, B\big(\theta_{n_0-n}\omega\big)), A\big(\theta_{n_0}\omega\big)\ \big) < \frac{1}{2} r$. For all $n \in N(n_0, {}^{r}/_{2})$. Hence for $dist\left(x_0, A\big(\theta_{n_0}\omega\big)\right) = r$ and $n \geq N(n_0, {}^{r}/_{2})$,

That is ${}^{r}/_{2} \leq dist\ \left(x_0, \varphi\left(n_0, \theta_{n_0-n}\omega, B\big(\theta_{n_0-n}\omega\big)\right)\right)$.

Obviously$N^*(\theta_{n_0}\omega, r) \leq N^*(\theta_{n_0}\omega, {}^r/_2)$.

Finally, define $a(n_0, r) := {}^r/_2\, e^{T_{n_0,N^*(n_0,r)}}$. (10)

Note that there is no guarantee here (with further assumption) that $a(n_0, r)$ dose not converge to 0 for fixed $r \neq 0)$ as $n_0 \to \infty$.

Proof 3

$$|V(\theta_{n_0}\omega, x_0) - V(\theta_{n_0}\omega, y_0)|$$

$$= \Big| \sup_{n \epsilon N} e^{-T_{n_0-n}} dist((x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega)\,) -$$

$$\sup_{n \epsilon N} e^{-T_{n_0-n}} dist((y_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega)\,)\Big|$$

$$\leq \sup_{n \epsilon N} e^{-T_{n_0-n}} \left| dist((x_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega)\,) - \ dist((y_0, \varphi(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega)\,) \right|$$

$$\leq \sup_{n \epsilon N} e^{-T_{n_0-n}} \|x_0 - y_0\| \leq \|x_0 - y_0\|$$

Since

$|dist(x_0, C) - dist(y_0, C)| \leq \|x_0 - y_0\|$ for any $x_0, y_0 \in \mathbb{R}^d$ and nonempty compact random subset of $C \in \mathbb{R}^d$.

Proof 4

Assume the opposite. Then there exists an $\varepsilon_0 > 0$, a sequence $n_j \to \infty$ in $\mathbb{N}$ and points $x_j \in \varphi(n_0, \theta_{n_0-n}\omega, D(\theta_{n_0-n}\omega)$ such that $V(\theta_{n_0}\omega, x_0) \geq \varepsilon_0$ for all $\varepsilon_0 \in \mathbb{N}$ since $\mathcal{D} \in \mathfrak{D}_{att}$ and $\mathcal{B}$ is pullback absorbing , there exists an $N = N(\mathcal{D}, n_0) \in \mathbb{N}$ such that $\varphi(n_0, \theta_{n_0-n_j}\omega, D(\theta_{n_0-n_j}\omega) \subset B(\theta_{n_0}\omega)$ for all $n_j \geq$N.

Hence, for all j such that $n_j \geq$ N , it holds $x_j \in B(\theta_{n_0}\omega)$, which is a compact random set , so there exists a convergent subsequence $x_j \to x^* \in B(\theta_{n_0}\omega)$. but also

$$x_j \in \overline{\bigcup_{n_j \geq N} \varphi(n_0, \theta_{n_0-n}\omega, D(\theta_{n_0-n}\omega)})$$

And $\bigcap_{n_j, n \geq n_j} \overline{\bigcup_{n_j \geq N} \varphi(n_0, \theta_{n_0-n}\omega, D(\theta_{n_0-n}\omega)}) \subset A(\theta_{n_0}\omega)$

And the definition of a pullback attractor. Hence $x^* \in A(\theta_{n_0}\omega)$

And $V(\theta_{n_0}\omega, x^*) = 0$. But V is Lipschitz continuous in its second variable by property 3, so $\epsilon_0 \leq V(\theta_{n_0}\omega, x_j) = \|V(\theta_{n_0}\omega, x_j) - V(\theta_{n_0}\omega, x^*)\| \leq \|x_j - x^*\|$, which contradicts the convergence $x_j \to x^*$. Hence, property 4 must hold.

Proof 5

Define

$N_{n_0} := \{ x_0 \in B[B(\theta_{n_0}\omega), 1]: \varphi(n_0 + 1, \theta_{n_0}\omega, x_0) \in B(\theta_{n_0+1}\omega)\}$,

Where $B[B(\theta_{n_0}\omega), 1] = \{x_0 : dist(x_0, B(\theta_{n_0}\omega)) \leq 1\}$ is bounded because $B(\theta_{n_0}\omega)$ is random compact and $\mathbb{R}^d$ is locally compact, so N_{n_0} is bounded. It is also closed, hence compact, since $\varphi(n_0 + 1, \theta_{n_0}\omega, .)\, A(\theta_{n_0}\omega) \subset intB(\theta_{n_0}\omega)$ is continuous and $B(\theta_{n_0}\omega)$ is compact. Now and $B(\theta_{n_0+1}\omega) \subset N_{n_0}$ so $A(\theta_{n_0}\omega) \subset intN_{n_0}$. In addition

$\varphi(n_0 + 1, \theta_{n_0}\omega, Nn_0) \subset B(\theta_{n_0+1}\omega) \subset N_{n_0+1}$, so $\mathcal{N}$ is positive invariant.

It remains to establish the exponential decay inequality (40). This needs the following Lipschitz condition on $\varphi(n_0, \theta_{n_0}\omega, .) \equiv f_{n_0}(.)$:

$\|\varphi(n_0, \theta_{n_0}\omega, x_0) - \varphi(n_0, \theta_{n_0}\omega, y_0)\| \leq e^{\alpha_{n_0}}\|x_0 - y_0\|$.

For all $x_0, y_0 \epsilon\, D(\theta_{n_0}\ \omega)$ .it follows from this that

$dist(\varphi(n_0 + 1, \theta_{n_0}\omega, x_0) - \varphi(n_0 + 1, \theta_{n_0}\omega, C_{n_0}) \leq e^{\alpha_{n_0}} dist(x_0, C_{n_0})$

From the definition of $V, V\left((\theta_{n_0+1}\omega), \varphi(n_0 + 1, \theta_{n_0}\omega, x_0)\right) =$

$$\sup_{n\geq 0} e^{-T_{n_0+1,n}}\, dist(\varphi(n_0 + 1, \theta_{n_0}\omega, x_0), (\varphi\left(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega)\right)$$

Since $\varphi(n_0 + 1, \theta_{n_0}\omega, x_0) \in B(\theta_{n_0+1}\omega)$ when $x_0 \in N(\theta_{n_0}\omega\ )$. Hence re-indexing and then using the two-parameter semi group property and the Lipschitz condition on $\varphi(1, \theta_{n_0}\omega, .)$.

$V\left(\theta_{n_0+1}\omega\ , \varphi(n_0 + 1, \theta_{n_0}\omega, x_0)\right) =$

$$\sup_{j\geq 0} e^{-T_{n_0+1,j+1}} dist(\varphi(n_0+1,\theta_{n_0}\omega,x_0),(\varphi\left(n_0,\theta_{n_0-j-1}\omega,B(\theta_{n_0-j-1}\omega)\right)$$

$$= \sup_{j\geq 0} e^{-T_{n_0+1,j+1}} dist(\varphi(n_0+1,\theta_{n_0}\omega,x_0),(\varphi\left(n_0+1,\theta_{n_0}\omega,\varphi\left(n_0,\theta_{n_0-j}\omega,B(\theta_{n_0-j}\omega)\right)\right)$$

$$\leq \sup_{j\geq 0} e^{-T_{n_0+1,j+1}} e^{\alpha_{n_0}} dist(x_0,\varphi(n_0,\theta_{n_0-j}\omega,B(\theta_{n_0-j}\omega)))$$

Now $T_{n_0+1,j+1} = T_{n_0,j+1} - \alpha_{n_0}{}^{+}$, so , $V(\theta_{n_0+1}\omega,\varphi(n_0+1,\theta_{n_0}\omega,x_0)$

$$\leq \sup_{j\geq 0} e^{-T_{n_0+1,j+1}} e^{\alpha_{n_0}} dist(x_0,\varphi(n_0,\theta_{n_0-j}\omega,B(\theta_{n_0-j}\omega)))$$

$$= e^{-1}\sup_{j\geq 0} e^{-T_{n_0,j}} dist(x_0,\varphi(,\theta_{n_0-j}\omega,B(\theta_{n_0-j}\omega)))$$

$$\leq e^{-1}V(\theta_{n_0}\omega,x_0),$$

Which is the desired inequality. Moreover, since $\varphi(1,\theta_{n_0}\omega,x_0) \in B(\theta_{n_0+1}\omega) \subset N(\theta_{n_0+1}\omega)$, the proof continues inductively to give, $V(\theta_{n_0+j}\omega,\varphi(n_0+1,\theta_{n_0}\omega,x_0) \leq e^{-j}V(\theta_{n_0}\omega,x_0)$, for all $j \in \mathbb{N}$. This completes the proof of theorem 3.4.

Definition. 3.5. A family random sets $\mathcal{D} \in \mathfrak{D}_{att}$ is called past-tempered with respect to Aif

$\lim_{j\to\infty}\frac{1}{j}\log^{+} dist\left((\theta_{n_0}\omega),A(\theta_{n_0-j}\omega)\right) = 0$ .For all $n_0 \in \mathbb{Z}$, or equivalently if $\lim_{j\to\infty} e^{-\gamma j}\ dist\left(D(\theta_{n_0-j}\omega),A(\theta_{n_0-j}\omega)\right) = 0$ for all $n_0 \in \mathbb{Z}, 0 < \gamma$.

Proposition3.6. For past-tempered family random sets, $D \subset \mathcal{N}$ it follows that $\lim_{j\to\infty} dist\varphi\left(n_0,\theta_{n_0-j}\omega,D(\theta_{n_0-j}\omega),A(\theta_{n_0-j}\omega)\right) = 0$

Proof

$V(\theta_{n_0}\omega,\varphi\left(n_0,\theta_{n_0-j}\omega,x(\theta_{n_0-j}\omega)\right) \leq e^{-j} dist\left(D(\theta_{n_0-j}\omega),A(\theta_{n_0-j}\omega)\right) \to 0$ as $j \to \infty$. Hence

$$a(\theta_{n_0}\omega, dist\varphi\left(n_0,\theta_{n_0-j}\omega,x(\theta_{n_0-j}\omega),A(\theta_{n_0-j}\omega)\right)$$

$$\leq e^{-j} dist\left(D(\theta_{n_0-j}\omega),A(\theta_{n_0-j}\omega)\right) \to 0 \text{ as } j \to \infty.$$

Since n_0 is fixed in the lower expression, this implies the pullback convergence $\lim_{j\to\infty} dist\varphi\left(n_0, \theta_{n_0-j}\omega, D(\theta_{n_0-j}\omega), A(\theta_{n_0}\omega)\right) = 0$

Theorem3.7. (Rate of pullback convergence)

If $\mathcal{B}$ is pullback absorbing neighborhood system, then for all $n_0 \in \mathbb{Z}, n \in \mathbb{N}$ and $\mathcal{D} \in \mathfrak{D}_{att}$ there exists an $N(\mathcal{D}, n_0, n) \in \mathbb{N}$ such that $V(\theta_{n_0}\omega, \varphi\left(n_0, \theta_{n_0-m}\omega, Z(\theta_{n_0-m}\omega)\right)$

$\leq e^{-T_{n_0,n}} dist\left(B(\theta_{n_0}\omega), A(\theta_{n_0}\omega)\right)$

Proof

$\varphi\left(n_0, \theta_{n_0-n-m}\omega, D(\theta_{n_0-n-m}\omega)\right)$

$= \varphi\left(n_0, \theta_{n_0-n}\omega, \varphi\left(n_0 - n, \theta_{n_0-n-m}\omega, D(\theta_{n_0-n-m}\omega)\right)\right)$

$\subset \varphi\left(n_0, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega)\right)$

$= \varphi\left(n_0, \theta_{n_0-i}\omega, \varphi\left(n - i, \theta_{n_0-n}\omega, B(\theta_{n_0-n}\omega)\right)\right)$

$\subset \varphi(n_0, \theta_{n_0-i}\omega, B(\theta_{n_0-i}\omega))$ .For every $m \geq N, n \geq i \geq 0$

Where the positive invariance of $\mathcal{B}$ was used the last line. Hence

$\varphi\left(n_0 - n, \theta_{n_0-n-m}\omega, D(\theta_{n_0-n-m}\omega)\right) \subset \varphi\left(n_0 - n, \theta_{n_0-n-m}\omega, D(\theta_{n_0-n-m}\omega)\right)$.

For every $m \geq N(D, n_0, n)$ and $n \geq i \geq 0$ or equivalently

$\varphi\left(n_0 - n, \theta_{n_0-m}\omega, D(\theta_{n_0-m}\omega)\right) \subseteq \varphi\left(n_0 - n, \theta_{n_0-i}\omega, B(\theta_{n_0-i}\omega)\right)$ .For every $m \geq n + N(D, n_0, n)$ and $n \geq i \geq 0$.

This means that for any $Z(\theta_{n_0-m}\omega) \in D(\theta_{n_0-m}\omega)$ the supremum in

$V(\theta_{n_0}\omega, \varphi\left(n_0, \theta_{n_0-m}\omega, Z(\theta_{n_0-m}\omega)\right) = e^{-T_{n_0,i}} dist\left(\varphi(\theta_{n_0}\omega), \theta_{n_0-m}\omega, Z(\theta_{n_0-m}\omega)\right), \varphi\left(n_0 - n, \theta_{n_0-i}\omega, B(\theta_{n_0-i}\omega)\right),$

Need only be consider over $i \geq n$. Hence

$$V\left(\theta_{n_0}\omega, \varphi\left(n_0, \theta_{n_0-m}\omega, Z(\theta_{n_0-m}\omega)\right)\right.$$

$$= \sup_{i\geq 0} e^{-T_{n_0,i}} dist\varphi\left(n_0, \theta_{n_0-m}\omega, Z(\theta_{n_0-m}\omega)\right), \varphi\left(n_0, \theta_{n_0-i}\omega, B(\theta_{n_0-i}\omega)\right)\Big)$$

$$\leq e^{-T_{n_0,n}} \sup_{j\geq 0} e^{-T_{n_0,n,j}} dist\varphi\left(n_0, \left(\theta_{n_0-m}\omega, Z(\theta_{n_0-m}\omega)\right), A(\theta_{n_0}\omega)\right)$$

$$\leq e^{-T_{n_0,n}} dist(B(\theta_{n_0}\omega), A(\theta_{n_0}\omega))$$

Since

$$A(\theta_{n_0}\omega) \subseteq \varphi\left(n_0, \theta_{n_0-i-j}\omega, B(\theta_{n_0-i-j}\omega)\right) \text{and}$$

$$\varphi\left(n_0, \theta_{n_0-m}\omega, Z(\theta_{n_0-m}\omega)\right) \in B(\theta_{n_0}\omega)$$

Thus

$$V(\theta_{n_0}\omega, \varphi\left(n_0, \theta_{n_0-m}\omega, Z(\theta_{n_0-m}\omega)\right) \leq e^{-T_{n_0 n}} dist(B(\theta_{n_0}\omega), A(\theta_{n_0}\omega)) .$$

For every $Z(\theta_{n_0-m}\omega) \in D(\theta_{n_0-m}\omega), m \geq n + N(D, n_0, n)$and $n \geq 0$

Corollary3.8. We can be assumed that the mapping $n \longmapsto n + N(D, n_0, n)$

If monotonic increasing in n (by taking a large $N(D, n_0, n)$ if necessary, and is hence invertible. Let the inverse of $n = M(m) = N(D, n_0, n)$. Then

$$V(\theta_{n_0}\omega, \varphi\left(n_0, \theta_{n_0-m}\omega, Z(\theta_{n_0-m}\omega)\right)$$

$$\leq e^{-T_{n_0,M(m)}} dist(B(\theta_{n_0}\omega), A(\theta_{n_0}\omega))$$

For every $m \geq N(D, n_0, 0) \geq 0$.

Usually $N(D, n_0, 0) > 0$. This expression can be hold for every $m \geq 0$ by replacing $M(m)$ by $M^*(m)$ defined for every $m \geq 0$ and introducing a constant $K_{D,n_0} \geq 1$ to account for the behavior over the finite time set$m \geq N(D, n_0, 0) \geq 0$,for every $m \geq 0$,thisgives

$$V(\theta_{n_0}\omega, \varphi\left(n_0, \theta_{n_0-m}\omega, Z(\theta_{n_0-m}\omega)\right) \leq K_{D,n_0} dist\left(B(\theta_{n_0}\omega), A(\theta_{n_0}\omega)\right).$$

References

[1]. L. Arnold, ‘‘Random Dynamical Systems,’’ Springer-Verlag, Berlin/Heidelberg/New York, 1998.

[2] P. Baxendale, Wiener processes on manifolds of maps. Proc. Roy. Soc. Edinburg Sect. A 87 (1980/1981), no. 1-2, 127–152.

[3] J. M. Bismut, A generalized formula of Ito and some other properties of stochastic flows. Z. Warsch. Verw. Gebiete 55 (1981), no.3, 331–350.

[4] K. D. Elworthy, Stochastic dynamical systems and their flows. Stoch. Anal., Academic Press, New York (1978), 79–95.

[5] I. Gihman and A. Skorohod, Stochastic Differential Equations. Springer-Verlag, Berlin Heidelberg New York, 1972.

[6] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes. North Holland-Kodansha, Tokyo, 1981.

[7] H. Kunita, On the decomposition of solutions of stochastic differential equations. Stochastic Integral (Durham, 1980), 213–255. Lecture Notes in Math., 851, Springer, Berlin-New York, 1981.

[8] N. Bhatia and G. Szegö, ‘‘Stability Theory of Dynamical Systems,’’ Springer-Verlag, Berlin/Heidelberg/New York, 1970.

[9] W. Hahn, ‘‘Stability of Motion,’’ Springer-Verlag, Berlin/Heidelberg/New York, 1967.

[10] V. I. Oseledets, A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems, Trans. Moscow Math. Soc. 19 (1968), 197–231.

[11] D. T. Son," Lyapunov Exponents for Random Dynamical Systems", Phd thesis der Fakult¨at Mathematik und Naturwissenschaften der Technischen Universit¨at Dresden (2009).

[12] X. Yingchao, " The Random Attractors of Stochastic Duffing-Van Der Pol Equations with Jumps", Chinese Journal of Applied Probability and Statistics 26(1) Feb (2010).

[13] I. J. Kadhim and A.H. Khalil," On Expansive Random Operators over a Uniform Random Dynamical Systems", European Journal of Scientific Research, 142 (4) October pp.334-342 (2016)

Studying the Properties of Water Plans in Satellite Images by Adopting Curvelet Transformation

**** Hadia S. AlBadrani(lecturer)***
Comp. Sci. Dept., College of Computer Science and Mathematics - Mosul University
e.mail : hadia_salih@uomosul.edu.iq

*****Najla Matti Safar (lecturer)***
Branch of Basic Sciences - College of Agriculture and Forestry - *Mosul University*
e.mail : najla.matti@uomosul.edu.iq

******Prof. Dr. Khalil I. Alsaif***
Comp. Sci. Dept., College of Computer Science and Mathematics - Mosul University
e.mail : Khalil_alsaif@hotmail.com

ABSTRACT

Features extraction and Texture analysis are viewed as significant activities in image processing field for different PC applications.

In this paper a proposed method for analyzing and extracting the features of water plans images based on Curvelet transform (by decomposing the image into its components then adopting the segmentation algorithms on that components), which offers precisely the edges because it deals with the winding information in all directions. Apply segmentation techniques to get information in region of interest, as it is fragmented or split the image into several sections. Studying the features of texture of water plans in satellite images of northern Mosul (Mosul dam).

Proposed algorithm segmentation need been used to extricate area for premium (water plans) from. images, which contain limitation accuracy for edges, to be studied. The algorithm produces a set of segments, which are stored in the cells array, for extracting features of the textures using a co-occurrence matrix .

The texture features of the image, based on the proposed approach, such as Contrast, Correlation, Energy and Homogeneity, gave an accurate representation of texture class.

دراسة صفات المسطحات المائية في الصور الجوية باعتماد تحويلات الكيرفليت

د. هدية صالح عبدالله* م. نجلاء متي اسحق أ.د.خليل ابراهيم السيف***

***قسم علوم الحاسبات-كلية علوم الحاسبات والرياضيات/جامعة الموصل**

****شعبة العلوم الأساسية - كلية الزراعة والغابات/جامعة الموصل**

الملخص

يعد استخلاص الميزات وتحليل النسجة عمليات مهمة في مجال معالجة الصور لمختلف تطبيقات الكمبيوتر. في هذا البحث اقترحت طريقة لتحليل واستخراج ميزات صور المسطحات المائية بناءً على تحويل كيرفليت (حيث يتم تفكيك الصورة الى معاملاتها ليتم تطبيق خوارزميات التقطيع على تلك المعاملات)، والتي تقدم بدقة الحواف لأنها تتعامل مع معلومات اللف في جميع الاتجاهات. قمنا بتطبيق تقنيات التجزئة للحصول على معلومات في منطقة الاهتمام ، اذ قمنا بتجزئة الصورة إلى عدة اجزاء، ودراسة ملامح قوام المسطحات المائية في صور الأقمار الصناعية لشمال الموصل (سد الموصل).

تم استخدام خوارزمية التجزئة المقترحة لاستخراج منطقة الاهتمام (المسطح المائي) من الصور ، التي تحتوي على دقة لتحديد الحواف ، وذلك لغرض دراستها، اذ تنتج الخوارزمية مجموعة من الاجزاء ، التي يتم تخزينها في مصفوفة الخلايا ، لاستخراج ميزات القوام باستخدام مصفوفة التواجد المشترك.

أعطت ميزات نسيج الصورة المستخلصة (باعتماد الاسلوب القترح) مثل (التباين ، الارتباط ، الطاقة والتجانس) ، تمثيلًا دقيقًا لذلك النسيج.

Introduction

Remote sensing images can be used in the several applications. The main utilization of remotely detected information is to make an order diagram of explicit or important highlights or classifications of land spread sorts in a scene [1].

It is considered, clearly defined is one of the main components of each scene, which plays an important role in the spatial distribution of the flow of surface water and determine the external processes (erosion, accumulation and corrosion etc.). Terrain information is necessary for modeling and understanding many physical processes [2].

Every pixel of image data detected from a separation speaks to a model in a specific spot of the planet. These images have an enormous number of purposes that contain meteorology, mapping and military insight. Satellite pictures can be one of the accompanying: water vapor, infrared and unmistakable light as appeared in Figure (1). image is the most straightforward approach to acquire geographic data. [3]

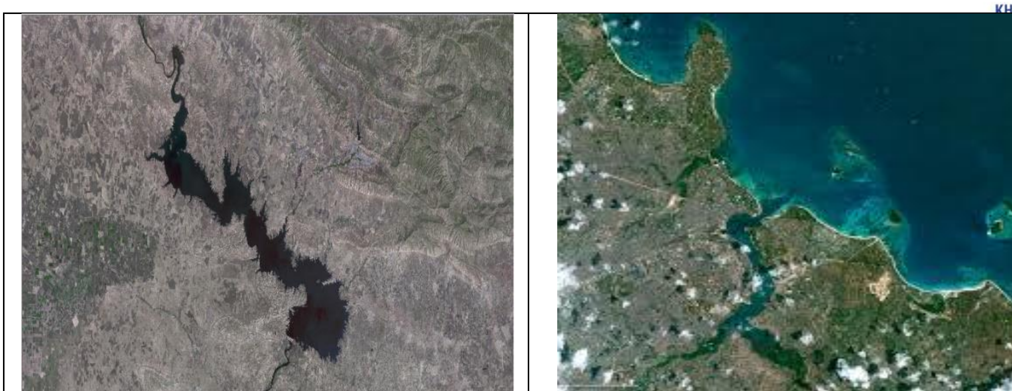


Image1 (Mosul Dam) | Image2 (Tanzanian capital) [4]

Figure1: Satellite Image

Interpretation and understanding of satellite symbolism is a technique for acquiring data about objects and the scene. It is a particular procedure of considering the geographical land reality dependent on the recognition, distinguishing proof and spatial limitation of individual items and territory shapes caught in elevated photos and satellite picture records. Translating the picture means decoding its multifaceted substance from the perspective of the reason which the over viewed learning serves [5].

The information that we are searching for in the pictures are encoded in different shades and surfaces. The translation of computerized images is essentially conceivable in two different ways, more often than not alluded to as visual elucidation and PC understanding. The user is available in both sub-forms yet in every one they have an alternate assignment. Visual elucidation is a less controllable procedure, in which there are numerous variables. [6]

Related Work

In (2019), Ahmed .S and others, was used curvelet transform to disassembled a traffic signs images and get coefficients.[7]

in (2017), Khalil I. Alsaif and Esraa Hussein,was used curvelete Transform to detected kidney stones.[8]

In (2013) A. Djimeli, D. Tchiotsop et.al was concerned with refine edge model dependent on Curvelet coefficients investigation. Results demonstrate that when the decomposition scale increases, their method brings out details on edges.[9]

In (2013) Yan Zhang n, TaoLi et.al presented a way to deal with analysis and division of tire laser stereography image by joining curevelet transform and Canny edge identification to distinguish deserts in tire surface. So this technique would bring about a remade image progressively advantageous for edge detection and the time multifaceted nature is decreased.[10]

In (2010) a researcher was used curvelet transform to estimate the optical flow and founded it much better than most other methods in that area.[11]

Curvelet Transform (CT):

The curvelet transform is a multiscale directional tansform, which permits a practically ideal non versatile scanty portrayal of items with edges. It has produced expanding enthusiasm for the network of connected arithmetic and sign handling over the previous years. [12]

The curvelet transform has two noteworthy ages. Original utilize an unpredictable advances which incorporate the ridgelet transform of radon transform of a image. Second era disregards the utilization of ridgelet transform, the reiteration decreased which prompts expanded speed. [13]

The work all through in two measurements, for example with spatial variable x, with a frequency domain variable, and with r and polar coordinates in the frequency domain. Begin with a couple of windows W(r) and V (t), which we will call the "radial window" and "angular window", separately. These are smooth, nonnegative and genuine esteemed, with W taking positive real arguments and supported on r∈(1/2,2) and V taking real contentions and bolstered on t∈[-1,1]. These windows will consistently comply with the tolerability conditions as in equations (1_a) and (1_b):

$$\sum_{j=\infty}^{\infty} w^2(2^j r) = 1\,, \qquad r \in \left(\frac{3}{4},\frac{3}{2}\right) \ldots (1_a)$$

$$\sum_{j=\infty}^{\infty} V^2(t-1) = 1\,, \qquad t \in \left(-\frac{1}{2},\frac{1}{2}\right) \ldots (1_b)$$

Then, for every $j \geq j_0$, introducing the frequency window U_j defined in the Fourier domain by

$$U_j(r,\theta) = 2^{\frac{3j}{4}} W(2^{-j}r) V\left(\frac{2\lfloor j/2\rfloor\theta}{2\pi}\right) \ldots (2)$$

where $\lfloor j/2 \rfloor$ is the integer part of j/2. Thus the boost of U_j is a polar "wedge" delimit by the support of W and V, the radial and angular windows, applied with scale-dependent window widths in each direction. To acquire real-valued curvelets, work with the symmetrized version namely,

$$U_j(r,\theta) + U_j(r,\theta+\pi)$$

Delimit the waveform $\varphi_j(x)$ by all means of its Fourier transform. is a "mother" curvelet in the sense that every curvelets at scale 2^{-j} are get by rotations and translations of φ_j .

The equispaced sequence of rotation angles $\theta_t = 2\pi . 2^{\lfloor j/2 \rfloor} . l$, with $l = 0,1,\ldots$ such that $0 \leq \theta_t < 2\pi$,and the sequence of translation parameters $k = (k1, k2) \in z^2$. With these notations, defining curvelets (as function of $x = (x1, x2)$) at scale 2^{-j}, orientation θ_t and position $x_k^{(j,l)} = R_{\theta_t}^{-1}\left(k_1 . 2^{-j}, k_2 . 2^{-\frac{j}{2}}\right)$ by

$$\varphi_{j,l,k}(x) = \varphi_j\left(R_{\theta_l}(x - x_k^{(j,l)})\right) \qquad \ldots(3)$$

where R_θ is the rotation by θ radians and R_θ^{-1} its inverse (also its transpose).

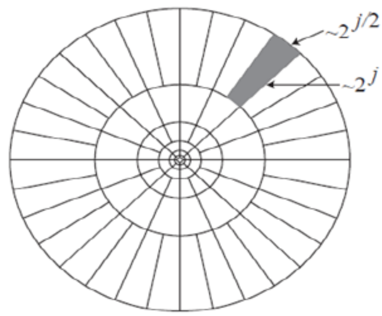


Figure 2: Curvelet tiling of space and frequency [14]

Curvelet transform is another augmentation of wavelet transform which plans to manage fascinating wonders happening along curved edges in 2D images[15]. It is a high-dimensional speculation of the wavelet transform intended to depict images at various scales and various directions (angles). It is seen as a multiscale pyramid with casing components ordered by area, scale, and direction parameters with needle-formed components at fine scales. Curvelets have time-frequency limitation properties of wavelets yet additionally demonstrates a high level of directionality and anisotropy, and its singularities can be all around approximated with not very many coefficients.[16]

Gray Level Co-occurrence Matrix (GLCM)

GLCM is the applied mathematics techniques of analyzing the textures that contain the spatial relationship between the pixels.

GLCM is made from a gray-scale image. The GLCM involve information about how regularly a pixel with gray-level value i happens either on a level horizontally, vertically, or diagonally to adjoining pixels with the value j. Where i and j are the gray level values in an image.
The properties or highlights separated from standardized symmetrical GLCM are appeared in table(1)

Table (1) GLCM properties

properties	formula
'Contrast'	$\sum_{i,j} \lvert i-j \rvert^2 p(i,j)$
'Correlation'	$\sum_{i,j} \frac{(i-\mu i)(j-\mu j)p(i,j)}{\sigma_i \sigma_j}$
'Energy'	$\sum_{i,j} p(i,j)^2$
'Homogeneity'	$\sum_{i,j} \frac{p(i,j)}{1+\lvert i-j \rvert}$

Proposed Algorithm:

The essential thought of the proposed algorithm relies upon the way that the image has many unique attributes. These qualities vary from image to image depending on the color space of the object and the space of the tissue. In this paper the tissue properties obtained from the analysis of water plans images can be used to improve the performance of the classification and knowledge of the water parts in aerial images in future research.

In this research, an algorithm was proposed to extract the properties of the water plans. The algorithm included acquisition aerial images and then performing preliminary treatment by extracting certain parts of the images to be used in the research (These parts were 128×128, 256×256, 512×512), and then apply a linear low pass filter to whole parts in order to remove noise from them, then apply histogram equalization to redistribute the image points evenly within the range 0-256, Then image decomposition using Curvelet transform has been adopted due to its ability to handle the curves contained in the water plans images in order to facilitate the handling of the images, this transform decompose the image to its coefficients. Finally, the GLCM Matlab function was applied to extract the properties from the images (Contrast, Energy, Homogeneity, Correlation coefficient), These properties were stored in a database to be adopted and to get benefiting from it for future research. Figure (3) shows the block diagram of the proposed algorithm.

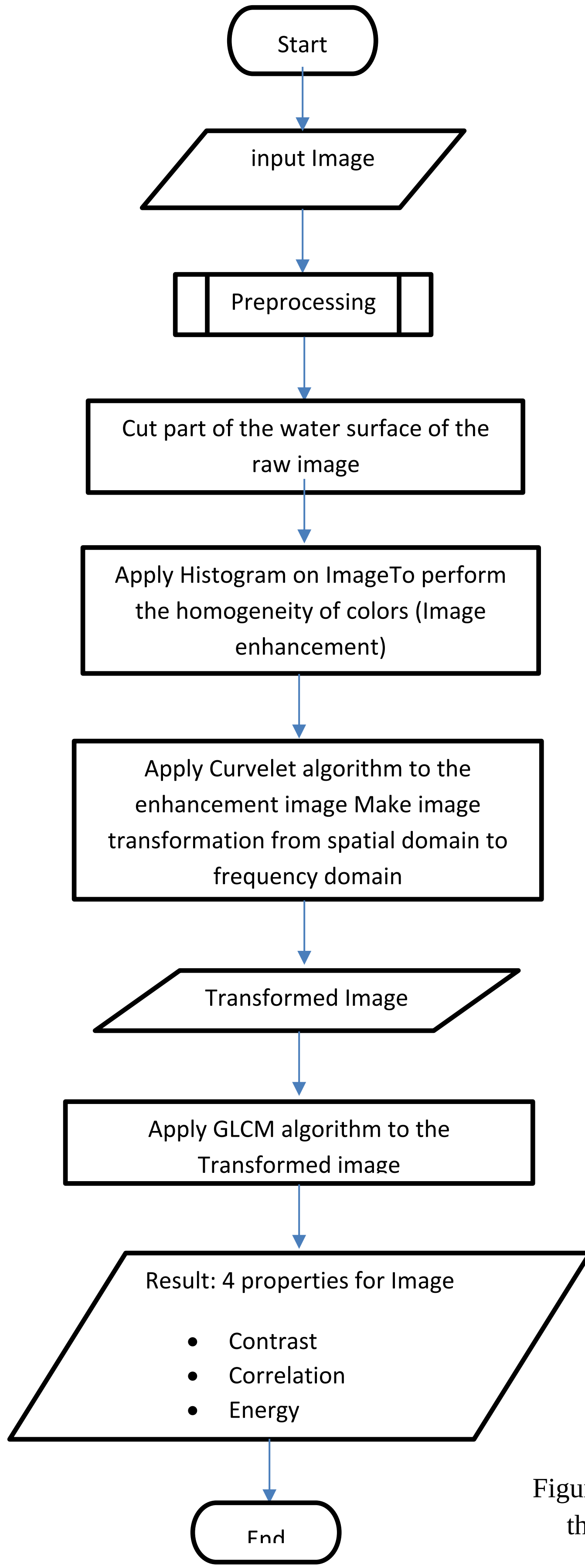


Figure (3) Chart of the proposed algorithm

RESULTS AND DISCUSSION

In the trials, the proposed procedure has been executed to a set of water bodies in satellite images of 128×128, 256×256 and 512×512 sizes. The procedure has been implemented in MATLAB, to get the fast discrete curvelet coefficients.

The accompanying outcomes describe the proposed algorithm for two real of water bodies images:

1. Table(2) indicates estimations of Gray Level Co-Occurrence matrixes (GLCM), the measurable features for contrast, correlation, energy and homogeneity were determined from enhancement water bodies images after applying histogram equalization operations with different sizes of segmented images. Results represent Properties for Image 1 and image 2 : Contrast, Correlation, Energy, homogeneity with sizes (128×128, 256×256, 512×512)
2. Figures (4), and (5) represent Energy properties of image1and image2 respectively, with different sizes of (128×128, 256×256, 512×512). Figures (6) and (7) demonstrate the consequence of applying the proposed calculation on two real of water bodies images contrast properties of image1and image2 respectively, with sizes (128×128, 256×256, 512×512). Figures (8), and (9) represent Correlation properties of image1and image2 respectively, with sizes (128×128, 256×256, 512×512). Figures (10), and (11) represent Homogeneity properties of image1and image2 respectively, with sizes (128×128, 256×256, 512×512). After applying histogram equalization on segmented images, the subsequent image has then issue of more brightness than the original image due to histogram equalizationthe particles give off an impression of being somewhat more brilliant than their unique one. image improvement strategies can be performed on the first image.
3. Energy, contrast and homogeneity, were directly proportional with size of the image, which is clear seen in figures (4,5,6,7,10,11).
4. Contrast and correlation seems to be a good indicator image quality.

Table(2) Results represent Properties for Image 1 and image 2 : Contrast, Correlation, Energy, homogeneity with sizes (128×128, 256×256, 512×512)

Image and size	Contrast	Correlation	Energy	Homogeneity
Image1 128×128	5.1647	0.18	0.0072	0.2364
Image1 256×256	5.69625	0.017	0.0086	0.2444
Image1 512×512	7.23597	0.0441	0.01049	0.2712
Image2 128×128	4.309	0	0.0039	0.21
Image2 256×256	5.213	0.1104	0.0046	0.2351
Image2 512×512	5.698	0.0789	0.0051	0.264

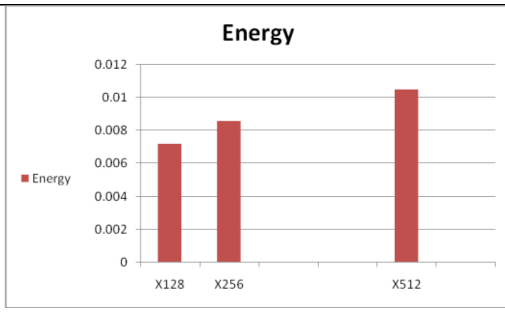


Figure (4) Energy properties of image1 with sizes (128×128, 256×256, 512×512)

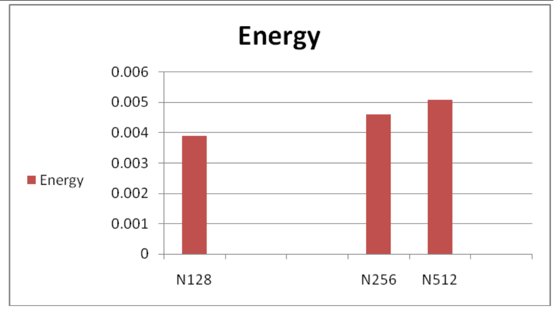


Figure (5) Energy properties of image2 with sizes (128×128, 256×256, 512×512)

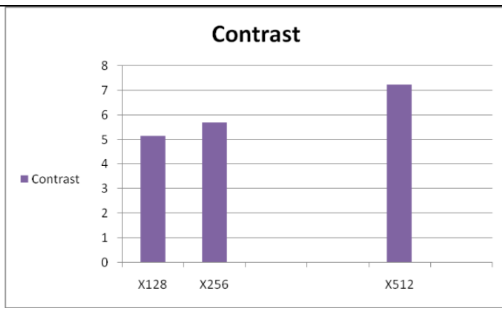


Figure (6) Contrast properties of image1 with sizes (128×128, 256×256, 512×512)

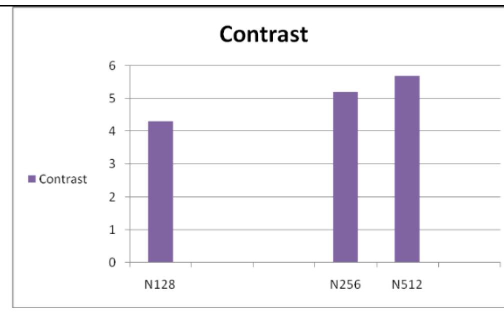


Figure (7) Contrast properties of image2 with sizes (128×128, 256×256, 512×512)

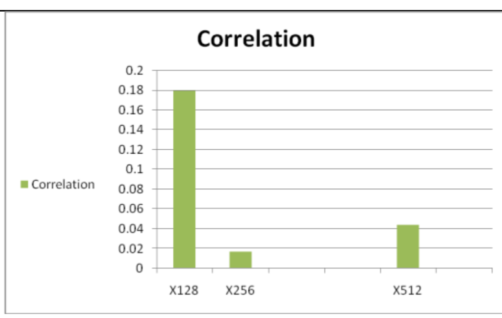


Figure (8) Correlation properties of image1 with sizes (128×128, 256×256, 512×512)

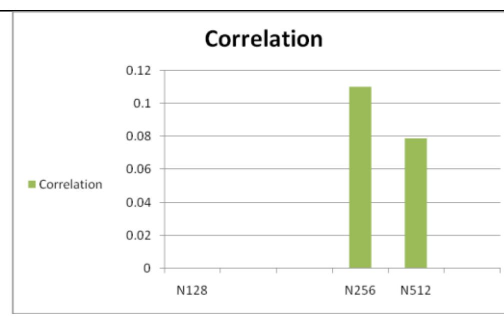


Figure (9) Correlation properties of image2 with sizes (128×128, 256×256, 512×512)

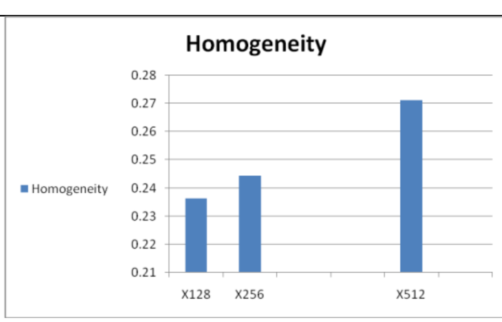


Figure (10) Homogeneity properties of image1 with sizes (128×128, 256×256, 512×512)

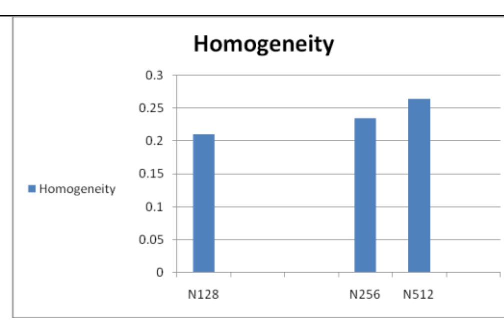


Figure (11) Homogeneity properties of image2 with sizes (128×128, 256×256, 512×512)

Conclusion

Based on the results obtained from the experimental application of proposed algorithm, the following conclusions were obtained:

- The process of dropping the low-frequency of curvelet coefficients is not affected by the level of analysis or the number of angles adopted at decomposition.
- The characteristics of Contrast, Correlation, Energy, and Homogeneity clearly show differences between samples of images.

Future Work

- The properties that obtained in this paper can be used as inputs on a fuzzy system or used in the genetic algorithm that discrimination certain patterns in order to detect some foreign tissue within those sections. Or enter it into a tree resolution to classify images.
- Adopt the algorithm as a diagnostic system so that it can be applied to images of different parts of the body to make satisfactory diagnoses.

References:

1. Perumal, K. and Bhaskaran, R. 2010."Supervised Classification Performance Multispectral Images", *Journal of Computing* ,2, pp:124-129.
2. Blaschke, T., Strobl, J. Defining landscape units through integrated morphometric characteristics. In: Buhmann, E., Ervin, S. (Eds.), Landscape
3. Modeling: Digital Techniques for Landscape Architecture. Wichmann-Verlag,2003, 104–113.
4. https://spacewatch.global/2016/11/uaes-thuraya-satafrik-host-road-show-tanzania/
5. Rehab F. Hassan and Lubna mhammed Bader,2017," Satellite Images Vectorization Based on Clustering and Interpolation Technique"DIYALA Journal for pure sciences, Vol: 13 No:3 , July 2017,p:47-64.
6. H. Svatonova, 2016,"Analysis of visual interpretation of satellite data", The International Archives of the Photogrammetry, Remote Sensing and Spatial Information Sciences, Volume XLI-B2, 2016. XXIII ISPRS Congress, 12–19 July 2016, Prague, Czech Republic.
7. Ahmed saadi Abdullah, Majida Ali Abed and Ahmed Naser Ismael, 2019, Traffic signs recognition using cuckoo search algorithm and Curvelet transform with image processing methods.Journal of AL-Qadisiyah for computer science and mathematics,Vol.11 No.2 Year 2019,ISSN (Online): 2521 – 3504.
8. khalil I. Alsaif and Esraa S. Hussein,2017,"Curvelet Transform for Kidney Stones Detection,International Journal of Computer Science and Information Security (IJCSIS),Vol. 15, No. 1, January 2017.
9. A. Djimeli1, D. Tchiotsop2, R. Tchinda3, "Analysis Of Interest Points Of Curvelet Coefficients Contributions Of Microscopic Images And Improvement Of Edges" ,Signal & Image Processing : An International Journal (Sipij) Vol.4, No.2 , April 2013.
10. Yan zhang n, taoli,qinglingli, "defect detection for tire laser shearography image using curve let transform based edge detector", Elsevier, optics & laser technology, college of electromechanical engineering, qingdao university of science and technology, PP 64–71, ,(2013).

11. Atheer A. Sabri,2010, "Optical Flow Estimation Based on Curvelet Transform and Spatio-temporal Derivatives",Eng. & Tech. Journal, Vol. 28, No.13.

12. Jianwei Ma1, Gerlind Plonka , 2009," A Review of Curvelets and Recent Applications", IEEE Signal Processing Magazine, Citations: 126 - 10 self

13. A. Ein-shoka, H. Kelash, O. Faragallah, H. El-sayed, "Enhancement of IR Images using Homomorphic Filtering in Fast Discrete Curvelet Transform (FDCT)", International Journal of Computer Applications (0975 – 8887) Volume 96, No.8, 2014.

14. D. Vishnuvardhan, Sreenivasan. B, I. Suneetha,2013." Advanced Digital Image Compression Technique Using Curvelet Transform",/ International Journal of Engineering Research and Applications (IJERA) ISSN: 2248-9622 www.ijera.com , Vol. 3, Issue 4, Jul-Aug 2013, pp.795-798

15. E. Candes, L. Demanet, D. Donoho, and L. Ying, 2006, “Fast discrete curvelet transform,” SIAM: Multiscale Modeling and Simulation, vol. 5, no. 3, pp. 861–899.

16. Alzubi Sh., Islam N., Abbod A., 2011, “Multiresolution Analysis Using Wavelet, Ridgelet, And Curvelet Transforms For Medical Image Segmentation”, International Journal Of Biomedical Imaging, Volume 2011 Article Id 136034, 18 Pages.

17. Gonzalez, R. C. and Woods .R. E.,(2002) “Digital Image Processing”, Second Edition .Prentice Hall.

Psoriasis Disease Detection Based On Skin Color

Baidaa M Alsafy[1], Asseel Jabbar Almahdi[2], Wamidh K. Mutlag[3]

[1,2] Computer sciences and mathematics college, University of Thi_Qar, Thi_Qar, Iraq.

[3] Al shatrah Technical Institute, Southern Technical University, Iraq.

[1]baidaaalsafy@utq.edu.iq , [2] asseelalmahdi@gmail.com, [3]wamid.almuhaysen@stu.edu.iq

Abstract

Psoriasis detection and diagnosis is one of significant researches and has interest for medical domain. Image processing plays major role for medical domain to analysis and diagnosis numerous diseases such as in psoriasis diagnosis. Psoriasis is a chronic sore of the human leather in the form of various forms thick red specks, capped stratum of peels with silver color resemble chance (Hence the psoriasis denotation) and accompanies these husks itching, growing the riskiness in ultimate cases particularly cold days through the winter. Psoriasis happens overwhelmingly in adults with both men and women at nearly the same average. In numerous cases, there is a family history of psoriasis, and assured genes have been attached to the disease.

In this paper we built a system to diagnosis of psoriasis skin diseases using image processing techniques where images are loaded from a database on the web specializing in skin diseases and improved this images to make them enjoy the same conditions and then extract qualities based on skin color (12) features that were effective for distinguishing these features are inputs to the neural network first and SVM second which in turn performs the final diagnosis of the classification between psoriasis and other diseases. The results were effective and respond to the algorithm. The percentage for the training stage of ANN was 100% and the testing stage was 95%, the percentage for the training stage of SVM was 100% and the testing stage was 84%.

Keywords: psoriasis; preprocessing; Hair removal; feature extraction; ANN, SVM.

1. Introduction

With progress of medical imaging techniques, the gained information is acquiring very rich to beyond the human's proficiency of visual identification and effective utilization for clinical appreciation. Computer techniques can construct comparatively straightforward abnormality identification and disease recognition of the images, however could not exchange the human capability to examine complicated data. On another side, computer techniques can "view" several specifics in the images, when human might not be capable of view. A computer translator can quickly and continuously understand many images, when human might construct conflicting evaluations in an ineffective method. Therefore, Computer Aided-Detection (CAD) and Computer Aided Diagnosis (CADx) be more beneficial and are now beneath growth via numerous research sets in the world. Computer aided illness detection confirmed to be exact useful and numerous systems which utilize to assist in the detection of numerous illness like psoriasis skin diseases [1] see **Figure (1).**

(a) (b) (c) (d)

Figure (1): Samples with psoriasis

Psoriasis is one of the prototypic papulosquamous skin illness distinguish by erythematous papules or plaques with silvery plates. It is a chronic inflammatory skin illness with growth epidermal proliferation associated to deregulation of the immune system. Psoriasis is said to influence 2% of the world people. Psoriasis has a bimodal age of disease inception. The first top is about 20 and the second top is about 60. People with illness onset about 20 years old have powerful genetic predisposition. Psoriasis is a medical case which happens while skin cells increase in size very rapidly. Damaged signals in the immune system reason current skin cells to compose in days rather than weeks. The body does not get rid of unwanted skin cells, thus the cells pile up on the exterior of the skin and lesions compose [2]. People have psoriasis may attention that occasionally the skin is improving and occasionally it gets worst. Things which can reason the skin to get worst contain: Infections, Stress, variations in weather which dry the skin and certain drugs [3].

2. The Proposed Algorithm

This section provides an overview of the suggested algorithm for the detection of psoriasis skin diseases. The proposed system framework appears in **Figure (2).**

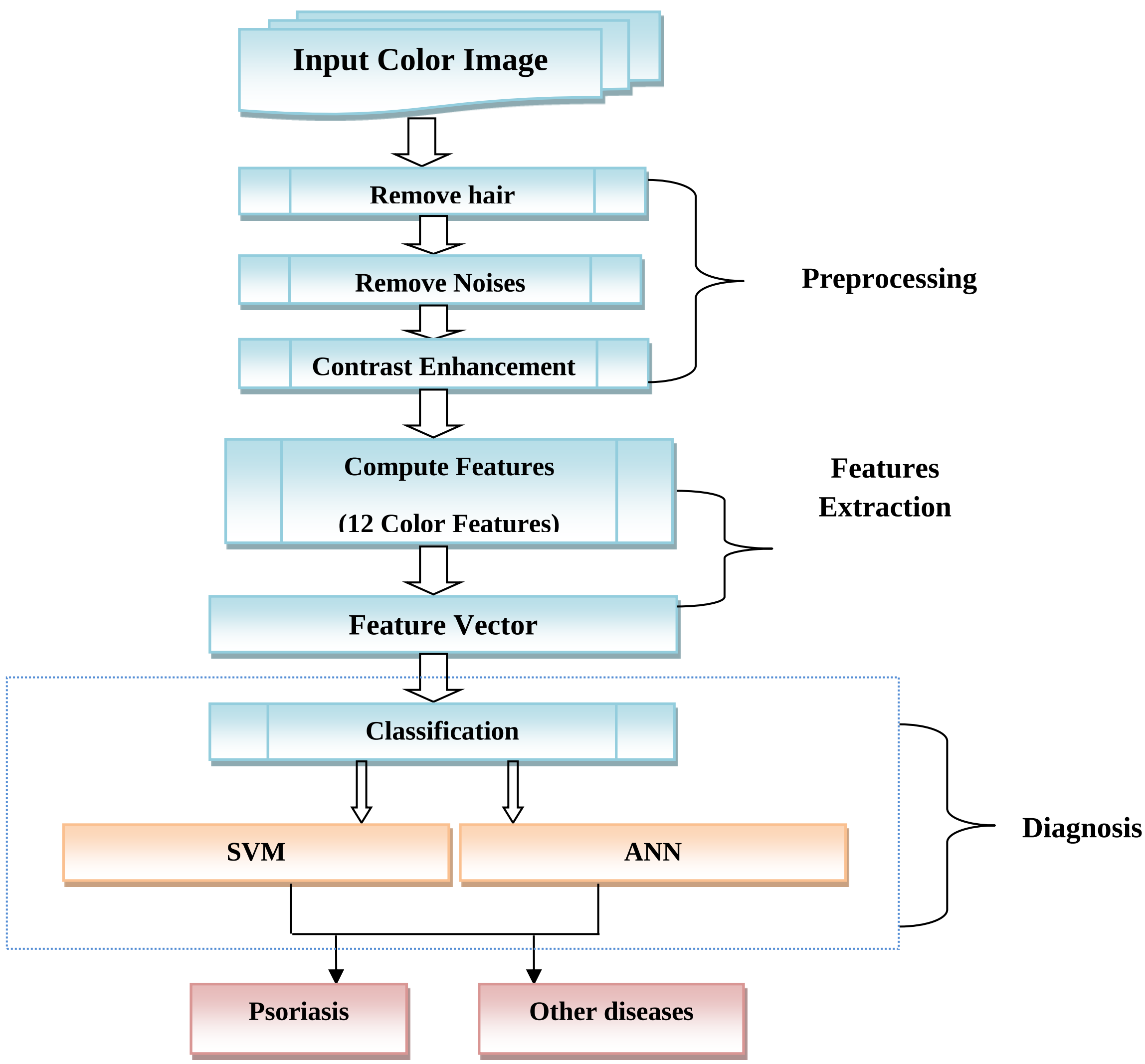


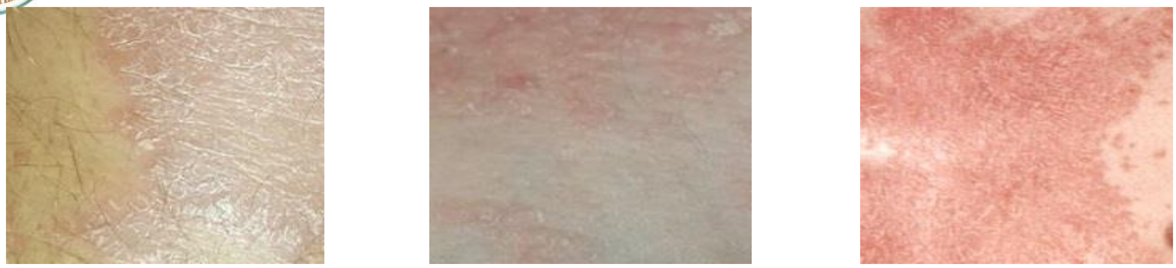
Figure (2): The major steps of the proposed system.

First phase in the Psoriasis diagnosis algorithm is the enter image. Image in digital form is presented as input to the algorithm. Second phase is preprocessing that contain remove the noises, contrast the images and hairs removal from this image) because the noises and hairs in image cause errors in diagnosis. The noises are eliminated via filtering. Filtering technique executed here is the Gaussian Filter; the hairs are eliminated by Dullrazor filtering. Third phase is features extraction. The feature extraction methods applied here are Standard deviation, Variance, Skeweness and mean as color features. The elected features are presented as the input to Artificial Neural Network Categorizer (ANN) and Support vector machine (SVM). The classifiers distinguish the specific datasets into psoriasis and other skin diseases.

2.1 Skin database

The database of skin images is consist of 400 color images of skin images in size 256×512 pixel that resized to 128 x 128 pixel.. The database contain of set of skin kinds of varying human races, several regions of the human body and several illumination cases, with 200 types with psoriasis and 200 types from other several skin diseases, the chosen image in the JPEG format. Types of that data base are shown in **Figure (3).**

(Samples of psoriasis)



(Samples of other diseases)

Figure (3): The skin database samples

2.2 Preprocessing

The objective of pre-processing phase is to improve the images and eliminate undesirable effects; this phase comprises three major methods as the following:

2.2.1 Hair Removal

Because some images in database may that used in this paper contain of hairs (long and short hairs) which cause erroneous diagnosis, Hair removal is achieved by using Dull Razor filtering of these images. So it useful to do the hair elimination before proceeding to other phases. Dull Razor filtering suggested in [4, 5] is medical imaging filter for hair elimination that exchanges hair pixels by neighboring pixels. It gets better diagnoses results, see **Figure (4).**

Algorithm (1) represents the major steps of Dull Razor filtering:

Algorithm (1): Dull Razor Algorithm

Input: Image contains hairs.

Output: Image after hair removal.

Step1: Identify the dark hair places by using morphological closing operation.

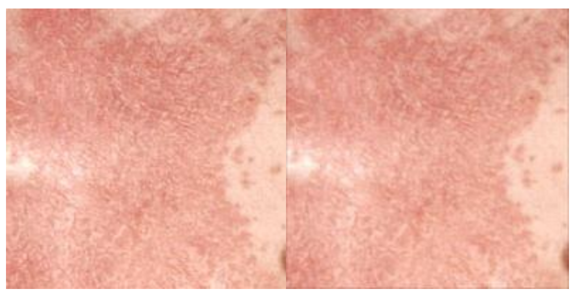
Step2: Smooth the exchanged pixels of hair (final result) using median filter.

(a) Images containing hairs (b) Images after hair removal

Figure (4): Dull Razor Filtering.

2.2.2. Noises Removal

After elimination the hair from images, there may be little noises existent in the image such as scratches and Scars in the skin …etc which compose the noises. Those noises are eliminating through applying Filtering. Filtering methods adopted here is Gaussian filtering suggested in [6] that smoothers the skin image, see **Figure (5).**

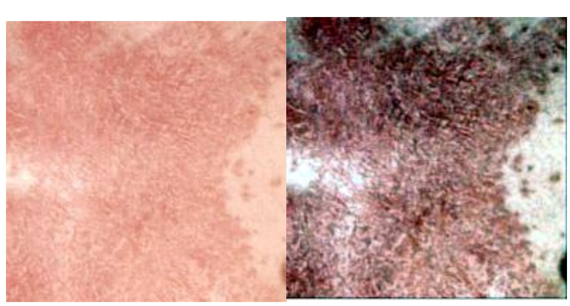


(a) original image (b) after applying Guassian filter

Figure (5): Noises removal.

2.2.3. Contrast Enhancement

Here, image clarity is increased, performance is better and image contrast is high. We utilize image calibration which elimination the undesirable portion of the image which came from several noises. This has been done to improve the edges and the shape of image, see **Figure (6)**.



(a) the original image (b) image after Contrast

Figure (6): enhancement of image.

2.3 Feature Extraction

The third phase in this paper is features extraction. At this stage extracts vital features of images in database which called a features vector that compose an impersonation of the image. The features extraction method suggested here is color features.

2.3.1 Color Features

One early mark of psoriasis is the appearance of color variations in skin color. The color descriptors are mostly statistical parameters computed from varying color bands proposed in [7, 8]. In this paper, color variance of the RGB image has been computed by using HSV bands. The statistical parameters such as variance, mean, standard deviation and Skeweness of the RGB or HSV color model, the corresponding computation can be defined as following:

$$\mu_i = \frac{1}{N}\sum_{j=1}^{N} f_{ij} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\sigma_i = \left(\frac{1}{N}\sum_{j=1}^{N} (f_{ij} - \mu_i)^2\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

$$\gamma_i = \left(\frac{1}{N}\sum_{j=1}^{N} (f_{ij} - \mu_i)^3\right)^{\frac{1}{3}} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

$$\sigma_i^2 = \frac{\sum \quad (f_{ij} - \mu_i)^2}{N} \quad \ldots\ldots\ldots\ldots\ldots (4)$$

Where f_{ij} is the color value of the i-*th* color constituent of the j-*th* image pixel and *N* is the full number of pixels in the image. μ_i, σ_i, γ_i σ_i^2,(*i*=1,2,3) indicate the mean, standard deviation skewness and variance of each channel of an image respectively.

Algorithm (2) illustrates the Algorithm of color features steps

Algorithm (2): Color features Algorithm.

Input: color images of psoriasis and other skin disease after processing.

Output: Twelve features vector.

Step1: Modify the RGB color model into an HSV color model. The column of the output matrix (s) illustrates Hue, Saturation, and Value respectively.

Step2: Calculate mean for every H, S, and V bands by using the equations (1).

Mmean_H, Mean_S, Mean_V

Step3: Calculate standard derivation for every H, S, and V bands by using the equation (2).

STD_H, STD_S, STD_V

Step4: Calculate skewness for every H, S, and V channel as in the equations (3).

Skewness_H, Skewness_S, Skewness_V

Step5: Calculate variance for every H, S, and V channel by using the equation (4).

Var_H,Var_S,Var_V

Step6: The outcome from this algorithm is vector at twelve values.

2.4 Diagnosis

After get features vector and save it, then follows the diagnosis phase which recognition between psoriasis and other skin dieses, in this paper, two techniques were used to classification between psoriasis and other skin dieses. The first technique is neural network and the second method is support vector machine then comparative between the results of these techniques in classification.

2.4.1 Artificial Neural Network

The framework of the neural network: We applied a feed forward back propagation Neural Network (NN) that suggested in [9, 10] with adaptable learning rate. The NN have three layers(input layer with twelve input, four hidden layer and two output layer).The activation function used is the tan sigmoid function, for both the hidden and the output layer. The input to the neural network is the features vector that contains twelve color features; the NN has two output (psoriasis and other skin dieses) as described in **Figure (7)**.

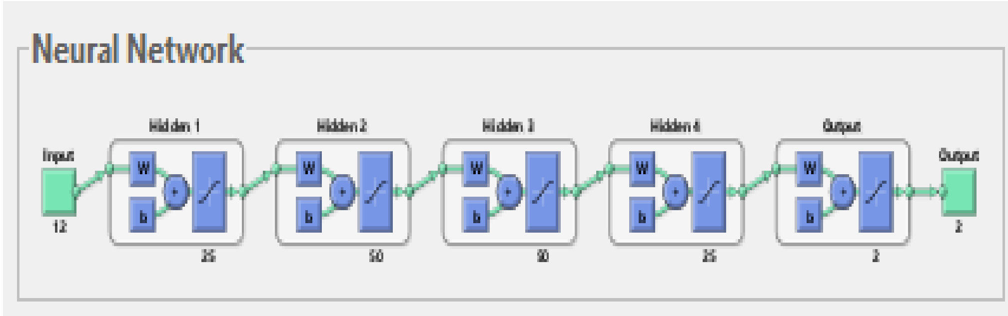


Figure (7): The Neural network structure

Algorithm (3): NN Algorithm

Input: Features vector (twelve features).

Output: Full accuracy of the NN.

Step1: Generate feed forward neural network (NN) that have four hidden layers, each layer have 25,50,50,25 neurons respectively, choosing these number of four layers are after experiment, the input layer of NN is choose based on features vector therefore the input layer neurons are twelve features, the output layer of NN is chosen based on number of class therefore the output layer neurons are two classes (psoriasis and other skin dieses).

Step2: Define the main parameters, these parameters are learning rate that equal to(0.0001), training time which set to infinity , periods (number of iterations) that equal to 1000 , transfer function is sigmoid transfer function , data segmentation function is (divide rand function), training function is back propagation function, activation function for output layer is 'tansig' ,and performance function is (default = 'mse '),weight and bias is producing at random, this parameters are select in our work in order to makes the NN gives effective outcomes.

Step3: Training the NN by using train data and target matrix , the target matrix contains two rows and two columns every row comprise from a vector which have zero values except a 1 in element (i), where i is the class they are to perform.

Step4: Simulates the NN through taking the initiated network and network input matrix , reversion the index to the biggest output as a class predict.

Step5: Calculate the network performance by using (perform).

Step6: Calculate the network accuracy by using confusion matrix.

Step7: Simulates the NN through taking the training network and test data, reversion the index to the biggest output as class predict.

Step8: Calculate the network performance by using (perform).

Step9: Calculate the network accuracy by using confusion matrix**.**

2.4.2 Support Vector Machine

SVM classification technique which suggested in [11,12] three steps are done: in first step applied cross validation (holdout) to divide the database(for psoriasis and other skin daises) to train set and test set, the greatest sequence to test phase and the rest for training phase, in second step gather features for two classes, in third step is train the SVM with train set for the psoriasis and other skin disease and test the SVM with test set for the psoriasis and other skin disease, **the Algorithm (4)** describe SVM Classification.

Algorithm (4): SVM Classification Algorithm

Step1: Train the train data, kind of kernel task which applied is polynomial with order of 2, train data is matrix illustrate features vectors to the twelve features.

Step2: Distinguish test data by using the information generate by step1.

Step3: Use confusion matrix to calculate full accuracy from SVM, in our work the full accuracy is 100% in training phase and the full accuracy is 84% in testing phase.

3. Conclusions and Experiment Results

Psoriasis can lead to death so early detection of the disease is very important. The recognition system by using computer-based methods is more effective than traditional biopsy techniques. The cost and the time used for diagnosis are lower in this suggested system. The system includes artificial intelligence, support vector machine (SVM), and digital image processing to detect psoriasis. The NN and SVM-based workbook has proven to be effective in decision-making and too in pattern identification applications. The accuracy of the suggested method is 95% when using ANN classifier and 84% when using SVM that is higher than the conventional techniques.

Table 1 shows the results of twelve vector features of psoriasis, while Table 2 shows vectors of other skin image features, while Table 3 shows the result of NN and SVM for psoriasis and other skin disease.

Table (1): The features vector to samples of psoriasis images.

Input image	Color features											
	Mean_H	Mean_S	Mean_V	Std_H	Std_S	Std_V						

							Skewness H	Skewness S	Skewness V	Var_H	Var_S	Var_V
Image 1	0.383	0.0525	0.3531	0.1904	0.0435	0.6498	0.609	0.0239	0.0937	0.001	0.0002	0.0001
Image 2	0.3293	0.1132	0.9385	0.2291	0.0415	0.5331	0.5239	0.0178	0.3331	0.0048	0.0001	0
Image 3	0.3594	0.0492	0.2138	0.1871	0.0453	0.3138	0.6413	0.0344	-0.5288	0.0004	0.0002	0.0002
Image 4	0.5442	0.0388	-0.322	0.2494	0.0443	0.5935	0.4508	0.022	0.0083	0.0004	0.0003	0.0001
Image 5	0.5949	0.0496	0.7353	0.2633	0.0291	0.1827	0.5282	0.028	-0.953	0.0003	0.0001	0.0001
Image 6	0.5668	0.053	0.3592	0.2012	0.0396	0.4482	0.6253	0.0348	-0.8012	0.0004	0.0002	0.0001
Image 7	0.4925	0.059	-0.4315	0.2317	0.0337	0.2307	0.5004	0.0203	0.054	0.001	0.0001	0.0001
Image 8	0.4419	0.0855	-0.1482	0.2154	0.0346	0.436	0.5487	0.0237	-0.4317	0.0023	0.0001	0.0001
Image 9	0.5427	0.0668	1.4345	0.2274	0.0313	0.146	0.6326	0.0293	-0.0605	0.0004	0.0001	0.0001
Image 10	0.5389	0.0445	-0.6327	0.2409	0.0379	0.1241	0.4521	0.0273	0.7128	0.0007	0.0002	0.0001

Table (2): The features vector to samples of other skin daises images.

Input image	**Color features**											
	Mean_H	**Mean_S**	**Mean_V**	**Std_H**	**Std_S**	**Std_V**	**Skewness H**	**Skewness S**	**Skewness V**	**Var_H**	**Var_S**	**Var_V**
Image 1	0.3322	0.0959	-0.2673	0.1664	0.0537	-0.5752	0.6498	0.0246	-0.6753	0.0029	0.0003	0
Image 2	0.5511	0.0864	0.9335	0.2526	0.0504	0.5193	0.5307	0.027	-0.3334	0.0024	0.0003	0.0001
Image 3	0.6066	0.0232	0.8122	0.3661	0.0335	-0.6454	0.5922	0.0186	-0.653	0.0001	0.0002	0.0001
Image 4	0.3957	0.086	**-0.2065**	0.2976	0.0728	0.8059	0.6936	0.0569	-0.4128	0.0026	0.0009	0.0004

Image 5	0.636	0.0478	-0.2254	0.1912	0.0282	0.0043	0.5876	0.0224	0.0666	0.001	0.0001	0.0001
Image 6	0.2237	0.0624	0.6522	0.3813	0.0258	-0.0371	0.83	0.0562	-0.3782	0.0019	0.0001	0.0004
Image 7	0.5508	0.0506	0.073	0.1189	0.0578	0.5328	0.7076	0.0237	-0.1233	0.0013	0.0004	0
Image 8	0.4547	0.0632	1.3553	0.2265	0.026	-0.3485	0.5781	0.0308	-0.2827	0.001	0.0001	0.0001
Image 9	0.4864	0.1424	0.5041	0.2826	0.0545	0.67	0.6393	0.049	-1.0103	0.004	0.0003	0.0002
Image 10	0.5394	0.0663	1.9985	0.2102	0.0297	0.4571	0.5862	0.0221	-0.0856	0.0017	0.0001	0.0001

Table (3): The result of NN & SVM Classification for psoriasis and other skin daises.

Data type	**Classifier**	**Feature type**	**Classification Accuracy %**	
			Train Data	**Test Data**
(psoriasis and other skin daises)	**ANN**	**Color Features**	**100**	**95**
	SVM	**Color Features**	**100**	**84**

4. References

[1] Nidhal K. Al Abbadi, Nizar Saadi Dahir, Muhsin A. AL-Dhalimi and Hind Restom,'' **Psoriasis Detection Using Skin Color and Texture Features''**, Journal of Computer Science 6 (6): 626-630, 2010.

[2] Dr.King-man HO**,"Medical Bulletion",** publishing in Hong Kong,May 2010.

[3] Hail Aljazi,**"Type Skin diseases",** publishing ISBN 245-87- 7451-322-2 2015

[4] T.Y.Satheesha, D. Satyanarayana and M. N. Giriprasad,**"A Pixel Interpolation Technique for curved hair removal in skin images to support melanoma detection"**, Journal of Theoretical and Applied Information Technology, 2005.

[5] AlAsadi, A.H.H. and AL-Safy, B.M.R,'' **Early detection and classification of melanoma skin cancer''**,LAP LAMBERT Academic Publishing.(2016).

[6] Estevão S. Gedraite, Murielle Hadad,’’ **Investigation on the Effect of a Gaussian Blur in Image Filtering and Segmentation’’,** Students at University of São Paulo, Av.Luciano Gualberto, travessa 3, nº 380, Brazil**,** January 2011.

[7] Dong ping Tian, **“A Review on Image Feature Extraction and Representation Techniques”**, International Journal of Multimedia and Ubiquitous Engineering Vol. 8, No. 4, July, 2013.

[8] Mathcentre, **"Variance and standard deviation (ungrouped data)"**, published online under www.mathcentre.ac.uk June 9, 2003.

[9] Simon Haykan,**" Neural Networks Comprehensive Foundation"**, 2nd Edition, Pearson Education, Inc., Pearson Prentice Hall, ISBN 81-7808-300-0, 2001.

[10] Devendra K. Chaturvedi**," Soft Computing Techniques and its Applications in Electrical Engineering"**, Springer-Verlag Berlin Heidelberg**,** ISBN 978-3-540-77480-8, 2008.

[11] Hasimah , Momoh J. E. Salami," **Iris Recognition System Using Support Vector Machines**"**,** InTech. ,China, ISBN: 978-953-307-542-6**,** 2011.

[12] Shaker K. Ali, Hazeem B. Taher, Baidaa M. Alsafy,’’ **Atherosclerosis Disease Diagnosis by Using SVM and ANN’’,** Jour of Adv Research in Dynamical & Control Systems, Vol. 10, No. 11, 2018.

Solve Fractional Linear Programming by using Branch and Bound method

Waleed Khalid Jaber

University of Thi-Qar, Computer Science and Mathematics College, Iraq

waleedkhalidj@utq.edu.iq

Abstract

Many methods can be adopted to solve the Fractional Linear Programming Problems (FLPP). Where the solution is optimum of the problem and the values of variables are real numbers not integer numbers. But, when there are conditions in the problem that require the result is to be optimum integer solution, so the resulted variables values are numerical integer. At this point, we must turn to a method that ends to the integer solution of the problem. This is briefly the topic of the research where we will employ an algorithm of the method (Solve (FLP) by using Branch and Bound method) to explore an approach integer solution of Fractional Programming problems.

Keyword: Applied Mathematics, Operation research

Introduction: *[1,3,4,6,7,8,9, 11, 12, 15,18,19,20 and 22]*

If the objection function is two (LF) ratio, the constraints then are linear, and the variables are positive, so this matter is called ***(FLPP).***

One of the methods to find out solution to the problems of (FP) in operations research is the method of approximation of the objection function. However, the method of solving, the method of the stages and the approximation of the objection function, depend initially on the objection function as a primary solution and then compensate this solution in the math model. The finding is the same as the previous one itself. Hence, we stop as the solution is applicable. If, it is not applicable, then we put new item to the objection function and do this case again until the solution is optimum is obtained.

The problem of (FLP) is shown in the following mathematical model:

$$Max\ Z = \frac{\sum_{i=1}^{n} C_i x_i + \alpha}{\sum_{i=1}^{n} D_i x_i + \beta}$$

According to restrictions: … (1.1)

$$Ax_i \leq b$$

$$x_i \geq 0$$

Where :

X : means the "Variables" in the objection function and constraints of the math model.

$C_i . D_i . \alpha . and\ \beta$ Mean the "real numbers" .

A: means the "Matrix variables coefficients for restrictions of problem".

B: means the "Matrix of Absolute boundary for restrictions of problem".

1.2 *Algorithm of approximates the target function* : [20]

1.2.1- To calculate the value of $L\ (x)$ from the following equation:

$$L(x) = \frac{\langle C, D \rangle}{\langle D, D \rangle} \quad \ldots (1.2)$$

Where :

$$\langle C, D \rangle = \sum_{i=1}^{n} c_i d_i$$

$$\langle D, D \rangle = \sum_{i=1}^{n} d_i^2$$

1.2.2- Converting the fractional objection function to a close-line function after offsetting its value in the following equation:

$$Max\,F = \langle c_j - L(x) * d_j \;,\; x_j \rangle \qquad \dots (1.3)$$

$$j = 1, \dots, n$$

Where *F* is a new name for the close-up function.

Then add the original for restrictions of problem:

$$\boldsymbol{Ax_i \leq b} \qquad \boldsymbol{x_i \geq 0}$$

1.2.3. The solved objection function with the restrictions of problem in the simplified way as a linear programming (LP) model to reach the optimum sol. Then, we obtain the values $\boldsymbol{x_j}$, j = 1,2, ..., n

1.2.4. Compensating the $\boldsymbol{x_j}$ values of the sol. is optimum in the simplified way to obtain a value

L (x *) = Max Z

1.2.5- Comparing (Lx) = L (x *) , If L (x) = L (x *) stops, the output is the sol. to the problem, or we have a new objection function(F) depending on L(x*).

In equation (1.3) the simplified method is re-used once more till the sol. is optimum and extracting the value of L (x **) and comparing it with the value of L (x *). If the equal is stopped; a new objection function is extracted based on L (x **). The previous steps are done again using the simplified method until you obtain the sol. is optimum and so on ...

To make this method clearer, we take the following example:

<u>Example (1.1)</u> : [20]

Solve the following math model:

$$Max\ Z = \frac{3x_1 + 2x_2 + 1}{x_1 + x_2 + 1}$$

$$\begin{aligned} S.T.\ \ & 5x_1 + x_2 \le 2 \\ & 2x_1 + 3x_2 \le 3 \qquad (1.1.1) \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\left.\begin{aligned} & c_j = (3,2) \quad ,\alpha = 1 \ ,\beta_j = (1,1) \quad ,\beta = 1 \\ & \langle c, c\rangle = \sum_{i=1}^{n} c_i^2 = 3^2 + 2^2 = 13 \\ & \langle \beta, \beta\rangle = \sum_{i=1}^{n} \beta_i^2 = 1^2 + 1^2 = 2 \\ & \langle c, \beta\rangle = \sum_{i=1}^{n} c_i \times \beta_i = 3\times 1 + 2\times 1 = 5 \\ & L(x) = \frac{\langle c, \beta\rangle}{\langle \beta, \beta\rangle} = \frac{5}{2} = 2.5 \end{aligned}\right\}$$

… (1.1.2

Use the simplex method , we have The sol. is optimum of the first stage are:

x_1 = 0.40 , x_2 = 0

Substituting in (1.1.1), we have

$$\boldsymbol{Max\ Z = \frac{(3 \times 0.4 + 1)}{0.4 + 1} = 1.57 \quad \ldots (1.1.3)}$$

Let   Z = L(x*)

Since *L(x)* = 2.5 , *Then L(x) # L(x*)*

It has been observed that the new objection function count on the new phase as following:

The second stage :

$$Max\ F = \langle c_i - L(x^*) \times \beta_i \ ,\ x_i \rangle$$
$$= (3 - 1.57 \times 1)\, x_1 + (2 - 1.57 \times 1)\, x_2$$
$$= 1.43\, x_1 + 0.43\, x_2$$
$$\therefore Max\ F = 1.43\, x_1 + 0.43\, x_2$$
$$S.t. \quad 5x_1 + x_2 \le 2$$
$$2x_1 + 3x_2 \le 3$$
$$x_1,\ x_2 \ge 0$$

Using the simplex method , we have The sol. is optimum of stage two are

$x_1 = 0.2307692$, $x_2 = 0.8461539$

Substituting in (1.1.1) we get :

$$Max\ Z = \frac{((3 \times 0.231) + 2 \times 0.846 + 1}{0.231 + 0.846 + 1}\ 1.62963 \quad \dots (1.1.4)$$

Let , Z= L(x**)

Since L(x*) = 1.5

Then L(x*) # L(x**)

It also has been observed that the new objection function count on the second stage above:

The Third stage :

$$Max\ F = \langle c_i - L(x^{**}) \times \beta_i \ , \ x_i \rangle$$
$$= (3 - 1.62963 \times 1)\, x_1 + (2 - 1.62963 \times 1)\, x_2$$
$$= 1.37\, x_1 + 0.37\, x_2$$
$$\therefore Max\ F = 1.37\, x_1 + 0.37\, x_2$$
$$S.t. \quad 5x_1 + x_2 \le 2$$
$$2x_1 + 3x_2 \le 3$$
$$x_1, \ x_2 \ge 0$$

Using the simplex method, we have the sol. is optimum of stage two are

$x_1 = 0.2307692$, $x_2 = 0.8461539$

Substituting in (1.1.1) we get :

$$\boldsymbol{Max\ Z = \frac{(3 \times 0.231) + 2 \times 0.846 + 1}{0.231 + 0.846 + 1} = 1.62963} \quad \ldots (\mathbf{1.1.5})$$

Let $Z = L(x^{***})$

Since $L(x^{**}) = 1.62963$

Then $L(x^{***}) = L(x^{**})$

Hence, we stop here and have the best sol. in three stages and four tables are:

$x_1 = 0.2307692$, $x_2 = 0.8461539$

Max Z= 1.62963 ***…(1.1.6)***

Then , this solution is optimum

2.1. Branch and Bound methods:

A Branch and Bound method (B&B method) is also a technicality for solving integer linear programming (ILPP). Therefore, number of reasonable sol. is finite since the resolution variables in (IPP) are separate. The sol.'s may be enumerated If there is small number. Generally speaking, there are many reasonable sol.'s to permit a completion of counting. (B&B method) provides a systematic counting procedure that considers bounds on the objection function for various subdivision of sol.'s and eliminates the subdivision of not optimum sol.'s.
Bradley, Hax, and Magnanti (1977, pp. 387–395); Garfnkel & Nemhauser (1972) , Hillier & Lieberman (1980); Murty (1976); Nemhauser & Wolsey (1988); Ozan (1986).

The gist of (B&B method) is to divide the reasonable region into more subdivisions.

2.2 Branch and Bound Algorithm: [1, 2,3,4,5,8,10,11, 16,14, 16,17 and 21]

The characterization below of this method establishes the most prosperous one applied to date. The idea of it is quite generic and being applied to more other unattached optimization problems (e.g. wandering Salesman, job market scheduling).

Accepting that are attempting to tackle the blended whole integer issue. It is called issue P_0 . The initial step is to take care of the issue of the 'continuous' LP which is gotten by overlooking the integrality limitations. On the off chance that in the ideal solution, at least one of the whole number factors are changed over to be non-number, subsequently, pick one such factor and use it to isolate the given issue P0 into two 'sub-issues' P_1 and P_2 . Assume that the variable chosen is y_j and it takes the non-integral value β_j in the uninterrupted optimum.
Then P_1 and P_2 defined as follows:

$$P_1 \equiv P_0 \text{ with the added constraint } y_i \leq \lfloor \beta_j \rfloor \qquad \ldots(2.2.1)$$
$$P_2 \equiv P_0 \text{ with the added constraint } y_i \geq \lceil \beta_j \rceil \qquad \ldots(2.2.2)$$

Presently any answer for P_0 is either an answer of P_1 *or* P_2 implying that P_0 can be understood by explaining P_1 *or* P_2

The individual limits xj are limitations added to P_0 to make P_1 *and* P_2 so it very well may be taken care of algorithmically as opposed to as express imperatives (see LP with a wide range of factors).

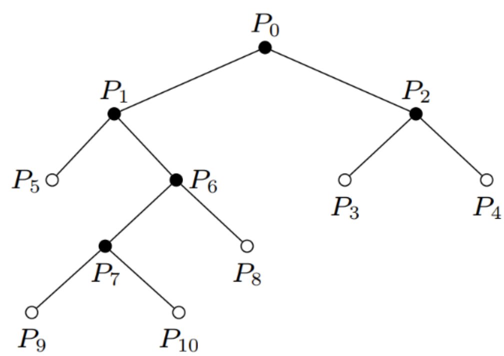
Note that, if xj is a parallel variable (2.2.1) and (2.2.2) diminish to including imperatives $x_j = 0$ ***and*** $x_j = 1$, individually ,

i.e., at that point the variable is being fixed at one of its possible qualities .

Continuing and illuminating LP (P_1) or LP (P_2). Accepting P_1 was picked. The arrangement of LP (P_1) is assessed similarly as that of (LP) P_0. On the off chance that x is as yet not all basic, we separate P_1, just as making P_3 ***and*** P_4 in a comparable soul, etc . . ..

The present procedure can be appeared as the development of a parallel tree of sub-issues whose terminal hubs, called holding up hubs, compares to issues that should be illuminated and assessed .

At one of these stages, we face an all-whole number arrangement (IS). It could or couldn't be an ideal answer for P_0. In the event that there are pending hubs on the web; every one of them must be analyzed as they could achieve a superior May be. Be that as it may, knowing an IS (with ideal worth Z) can significantly put down the age of new hubs. To be specific:



It has been observed that the newly created sub-problems have restriction other than any of their predecessors. Hence, the LP relaxations of these problems have worse (not better) optimum values.

The algorithm must be finished since the integer variables are all bounded because as one proceeds for further steps; the tree the bounds on the variables become close to each other, and finally they become exact if the LP solutions were never integer before.

It is important to note that the newly created sub-problems shouldn't be solved from scratch, but it must be started from the solution is optimum of the predecessor problem.

For the most part, there are numerous approaches to isolate the FP, and thus there are quantities of methods for B&B calculations. We should mull over this B&B calculations method, for issues with just paired factors. A whole number LP is a LP further compelled by the integrality limitations. In a boost issue, the estimation of the complaint capacity will consistently be an upper bound on the ideal whole number programming objective at the straight program ideal. Furthermore, any whole number doable point is generally a lower bound on the ideal straight program target esteem. The possibility of B&B is to utilize these perceptions to methodicallly make allotments of the LP FR, making appraisals of the number programming issue dependent on these subdivisions. The technique is depicted effectively in the event that we think about the model (1.1.1) from past segment. First, giving an optimum value Z^* . this gives the upper bound on

$$Z^* \leq Max\, Z = 1.63 \quad .$$

Since the coefficients in the objection function are integral, Z^* must be integral and this implies that $Z^* \leq 1$.

The solution is optimum of LP for ***ex.(1.1.1)*** has

$$x_1 = 0.2307692 \,.\, x_2 = 0.8461539 \;,\; Max\, Z = 1.62963$$

All these variables must be integer in the optimum solution, and we can partition the FR trying to make either integral. It has been realized that, in any (IP) solution, x_1 must be either an integer ≤ 0 or an integer ≥ 1. Hence, the first subdivision is into the regions where $x_1 \leq 0$ and $x_1 \geq 1$, also $x_2 \leq 0$, $x_2 \geq 1$

as displayed by **Fig. (2.2.1)** here:

A_0 : It speaks to the relationship of LP, which is their ideal arrangement has been incorporated inside the A_0 , and the upper bound on Z^* is appeared to one side of the case. The demonstrated boxes beneath are correspondence to the new subdivisions; the limitations that subdivide A_0 are incorporated by the lines joining the containers. In this manner, the limitations of A_0 are those of A_0 together with the imperative, while the requirements of A_1 are those of A_0 together with the requirement x_1 ***and*** x_2 **.**

The methodology that must be pursued is clear: Simply treat every subdivision as we did the first issue. Consider A_1 first .

Graphically, from (Fig. 2.1.1) we see that since x_1 isn't whole number, we subdivide A_0 further, into the districts A_1 with $x_1 \leq 0$ ***and*** A_2 with and $x_1 \geq 1$

A_2 is an impracticable issue thus this part of the identification tree never again ought to be taken into contemplations.

In regions A_1 it has been noticed that the optimum (LP) solution depends on the second constraint with $x_1 = 0$ *and* $x_2 = 0$, which it gives the objective value

$$Max\ Z = \frac{1}{1} = 1$$

Since x_2 is not integer, we subdivide ***A0*** into the regions ***B1***

with $x_2 \leq 0$ and B_2 with *and* $x_2 \geq 1$

In regions B_1 it has been noticed that the optimum (LP) solution depends on the second constraint

with $x_1 = 0.4$ *and* $x_2 = 0$, which it gives the objective value

$$Max\ Z = \frac{3 * 0.4 + 1}{0.4 + 1} = 1.57 \qquad ...(2.2.3)$$

And In regions ***B2*** what has been noticed is that the optimum (LP) solution depends on the second constraint

with $x_1 = 0$ *and* $x_2 = 1$, which it gives the objective value

$$Max\ Z = \frac{0 + 2 * 1 + 1}{1 + 1} = 1.5 \qquad ...(2.2.4)$$

It has been observed that the region B_2 refers to fact that the region shouldn't be subdivided or, equivalently, that the tree won't be extended from this box. At this point, subdivisions B_2 and L_4 must be taken into considerations. We may select one arbitrarily; however, practically, a number of useful heuristics are applied to make this choice. To simplify this issue, let us choose the subdivision that has been recently generated, here B_1.

Since x_1 is not integer, B_2 must be further subdivided into C_1 with

$x_1 \leq 0$ **and** C_2 with *and* $x_1 \geq 1$.

C_2 is an impracticable problem and so this branch of the enumeration tree no longer should be considered.

In regions C_1 we see that the optimum (LP) solution depends on the second constraint with $x_1 = 0$ *and* $x_2 = 0$, it gives the objective value

$$Max\ Z = \frac{1}{1} = 1$$

is a workable solution to the original problem, $z^* \geq 1.5$ and we have now the bounds

$1 \leq Z^* \leq 1.5$. Without further analysis, we could end with the IS

$x_1 = 0$ *and* $x_2 = 1$, realizing that the target estimation of this point is inside 0,1 % of the genuine ideal. For accommodation, the lower bound $z^* \geq 1.5$ simply decided has been attached to one side of the C_1 confine the specification tree.

In spite of the fact that $x_1 = 0$ and $x_2 = 0$, is the best number point in C_1, the locales $A1$ and C_1 may contain better FSs, and we should proceed with the system by investigating these districts. In C_1, the main doable point is $x_1 = 0$ and $x_2 = 0$, giving a target esteem Max $Z = 1.$ This is superior to anything the past number point and in this way the lower bound on z^* improves, with the goal that $1 \leq Z^* \leq 1.5.$

Further discussion

Basically, there are three that presently can't seem to be considered with respect to the (B&B method) strategy:

Could the (LP) s relate to the subdivisions that has been fathomed effectively?

What is the ideal approach to subdivide a given district, and which unanalyzed subdivision ought to be considered straightaway?

Could the upper bound ($z = 1.5$, in the model) on the ideal worth z^* of the whole number program be improved while taking care of the issue?

The response to the main inquiry is an unfit yes. When moving from a district to one of its subdivisions, we include one limitation that isn't fulfilled by the ideal direct programming arrangement over the parent locale.

Also, this was one inspiration for the double simplex calculation, and it is normal to embrace that calculation here. Alluding to the example issue will delineate the technique. The initial two subdivisions A_1 and A_2 in that model were created by adding the accompanying requirements to the first issue: For subdivision

$1 : x_1 \leq 0$

$2 : x_1 \geq 1$

Outline:

The fundamental thought of **(B&B** method) is to subdivide the FP to create limits $z < z^* < z$ *on* z^*

For an expansion issue, the lower bound z is the most elevated estimation of any doable number point experienced. The upper bound is given by the ideal estimation of the related LP or by the biggest incentive for the protest work at any "hanging" box. In the wake of thinking about a subdivision, we should branch to (move to) another subdivision and break down it. Additionally, assuming either the LP over A_j is impracticable;

the ideal direct programming arrangement over A_j is whole number; or

the estimation of the direct programming arrangement Z_j over A_j fulfilled

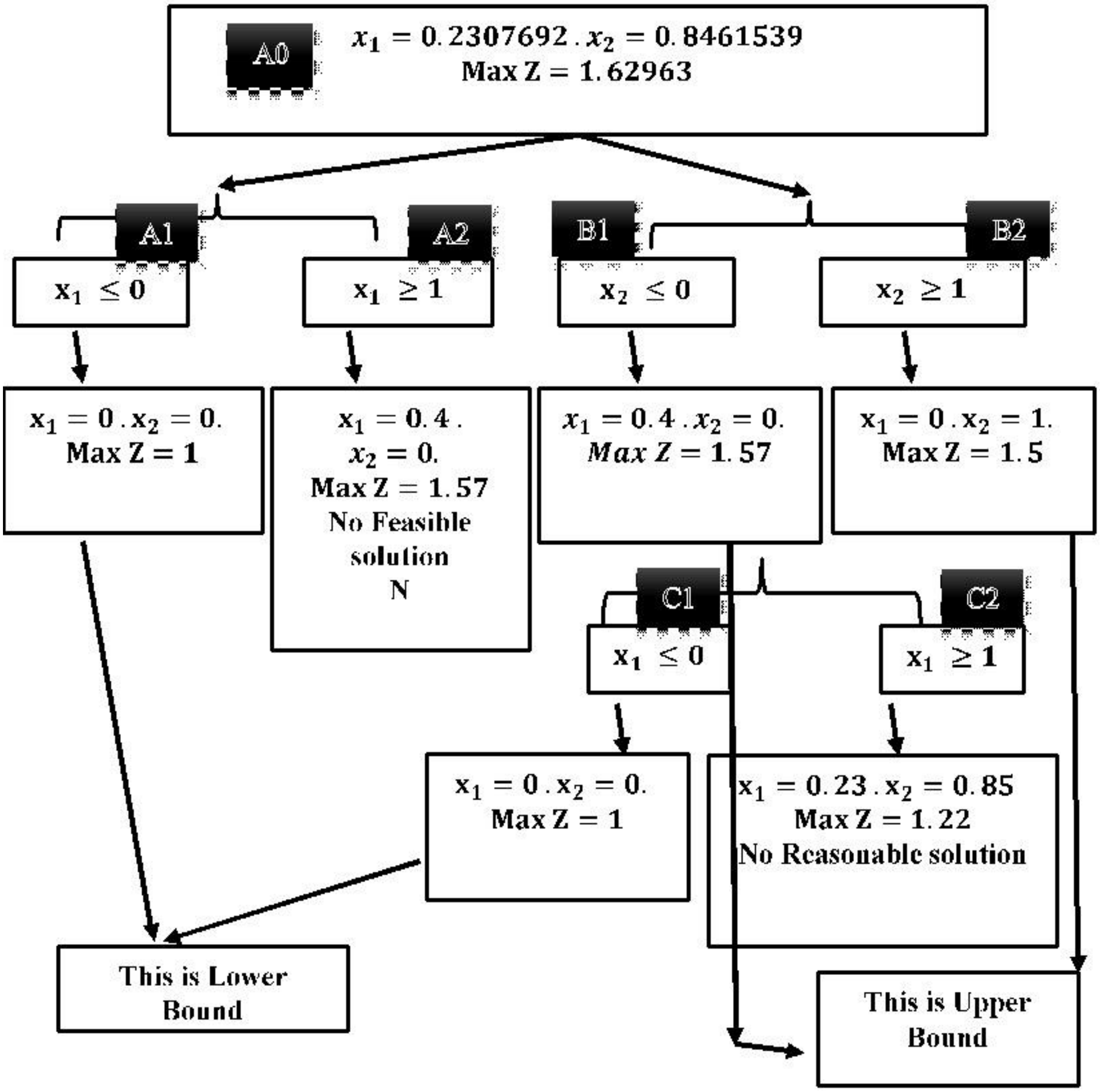
zj ≤ z (on the off chance that expanding), at that point A_j shouldn't be subdivided. In these cases, whole number programming phrasing says that A_j has been comprehended.

Case

(i) is named conception by infeasibility,

(ii) Fathoming by entirety.

(iii) Fathoming by limits.



Then the solution is optimal is

$x_1 = 0$

$x_2 = 1$

Max Z = 1.5

Fig. (2.2.1)

Reference

1. Alper Atamturk And Martin W. P. Savelsbergh , Integer–Programming Software Systems , Department of Industrial Engineering and Operations Research , University of California , USA , 2004.
2. A. Ravi Ravindran , The Operations Research Series , Second edition , Dept. of Industrial & Manufacturing Engineering The Pennsylvania State University, USA , 2009.
3. Asra'a Hadi Hassan, Using Modify Surrogate Cutting Plane (MSCPA) Method To Finding The Integer Solution For (LP) Problems with Unrestricted Variables, Al-Mansour Journal, Iraq, 2008
4. Beale, E. M. L. Branch and Bound Methods for Mathematical Programming Systems. – Annals of Discrete Mathematics, 5, 1979, 201-219.
5. David G. Luenberger , and Yinyu Ye, Linear and Nonlinear Programming , Third Edition , Stanford University, Springer, 2008.
6. Dr. Mouhamed Maziad Dribate , The new Algorithm cutting and unloading to solve (ILP) problems , Tishreen University Journal for Research and Scientific Studies - Basic Sciences Series Vol. (36) No. (3) 2014 , 2014.
7. Esmail, Ahmed Abduljabar, Altimimy, Majeda Abdulamir, Application of operation research on the computer, Amman , Jordan, 2007.
8. Gerard CORNUEJOLS & Reha TUT ¨ UNC ¨ U¨ . Optimization Methods in Finance, Cambridge University Press, Cambridge (USA), 2007
9. G´erard Cornu´ejols , Michael A. Trick , Matthew J. Saltzman , A Tutorial on (IP) , 1995
10. Hamdy A.T.," operations research an introduction" seventh ed. Canada, by Maxwell publishing company, (2004) .
11. Hassan , Abbas Ahmed, Asra'a Hadi Hassan, Modify Surrogate Cutting Plane to solve linear (IP) problems , Geometry and technology Journal, Iraq, Vol. 26, No. 4, 2008
12. Herbert Simon , Optimization Methods in Management Science, Spring 2013.
13. Hong-Wei Jiao , An Effective Branch and Bound Algorithm for Minimax Linear Fractional Programming , Department of Mathematics, Henan Institute of Science and Technology, China, Hindawi Publishing Corporation Journal of Applied Mathematics , Volume 2014
14. Istv´an Maros, Advanced Operations Research, Department of Computing Imperial College, London, 2008.
15. Krasimira Genova, Vassil Guliashki , Linear (IP) Methods and Approaches – A Survey, Cybernetics And Information Technologies , Volume 11, No 1, Sofia • 2011
16. Mehra A. , Chandra S. و Bector C. R. ," Acceptable optimality in linear fractional programming with fuzzy coefficients",Fuzzy Optimization and Decision Making,Vol. 6, no. 1,pp. 5-16, (2007).
17. Nawar N. A. Fatohi , A Hybrid method for combined Cutting Planes and Branch and Bound for (IP) Problems , A Thesis Presented To The Council of the College of Computers and Mathematical sciences, University of Mosul , Iraq , 2005.
18. Saeed Bastani , Optimization: (IP) , Spring 2017.
19. Tommi Sottinen , Operations Research with GNU (LP) Kit , University of Vaasa , 2009.
20. Waleed Khalid Jaber, The methods of solution for the (FLPP), Thesis Submitted to the School of Applied Sciences of The University of Technology in Partial Fulfillment of The Requirements for the Degree of Master Degree of Science In Applied Mathematics, 2004.
21. Yazan Khalid Abed-Allah Migdad , Operations Research , Yarmouk University , Jordan , 2017.
22. Yong-Hong Zhang, and Chun-Feng Wang , A New Branch and Reduce Approach for Solving , Generalized Linear Fractional Programming, 2017

Tabu Search Method to Solve Machine Scheduling Problem under Fuzzy Processing Times

Al-zuwaini Mohammed Kadhim [1], Shaker K. Ali [2], Marwa Mohammed Kassim[3]

[1,2,3]Computer Sceinces and Mathematics college , University of Thi-Qar, Iraq.

[1]mkzz50@yahoo.com,[2] shaker@utq.edu.iq, [3]mm8764270@gmail.com

Abstract

In the problem of scheduling a single machine to minimize multiple objective function (MOF). There are n jobs to be processed, each of which has fuzzy processing time, integer penalty number of early jobs, weighted number of late jobs and an integer due date. The objective is to find the approximate solutions which minimize the sum of penalty number of early jobs and weighted number of tardy jobs with fuzzy processing time. This problem with normal processing time is strongly NP-hard. Tabu search method was used to find on approximate solutions, The problem was solved with up to 12100 jobs in a short time.

Keywords: Scheduling; Single machine; Fuzzy processing time; Tabu Search method; Weighted number of early jobs; Weighted number of tardy jobs.

1. Introduction

In this paper, we will study a single machine with its' scheduling for n independent jobs (fuzzy processing time, due date, weighted of early jobs and weighted of tardy jobs (i.e. the same data calculated and adopted in [8]). To minimize the sum of two functions, our objective is to get a schedule which is given an approximate solution for the problem sum weighted number of early jobs and weighted number of tardy jobs under fuzzy processing time, this problem is strongly NP-hard and denoted by 1/ / $\sum_{j=1}^{n} N_{h_j E_j} + \sum_{j=1}^{n} N_{w_j T_j}$ (which we described in [8]) by using Tabu Search method.

2. Formulation of the problem [8]

Single machine scheduling models seem to be very important for understanding and modeling multiple machines models. A set N={1,2,…,n} of n independent jobs has to be scheduled on a single machine in order to minimize a given criterion. This study concerns the one machine scheduling problem with multiple objectives function which is denoted by (1/ /$\sum_{j=1}^{n} N_{h_jE_j} + N_{w_jT_j}$).

In this problem, preemption is not allowed, no precedence relation among jobs is assumed and all jobs are available at the time zero, (that is r_j=0 $\forall j$). Each job j has fuzzy processing time a_j, b_j, c_j, due date d_j, weighted of early job j h_j and weighted of tardy job j w_j.

The start time of job j is denoted by t_j, where t_1=0 and $t_j = \sum_{i=1}^{j-1} p_i$ and its completion time by C_j, ($C_j = t_j + p_j$), if job j completed befor it's due date ($C_j < d_j$then N_{E_j}=1) and job j is said to be early otherwise ($C_j \geq d_j$, then N_{E_j}=0), and job j is said to be tardy job or just in time, for each job j can be calculate the slack time $Sl_j = |A_j - d_j|$. In this paper, next jobs are scheduled in increasing order of their slack time. Consider the processing time in fuzzy environment (a, b, c), which is integer numbers. We calculate the Average High Ranking (AHR) Method to generate a processing time from a fuzzy processing time.

$$AHR = [3b + (c - a)]/\ 3 \qquad (1)$$

Where a, b, c is fuzzy processing time.

For a given schedule we can calculate weighted number of early jobs $N_{h_jE_j}$, as follow:

$$N_{h_jE_j} = \begin{cases} h_j & if \quad d_j > C_j \\ 0 & if \quad d_j \leq C_j \end{cases} \qquad (2)$$

And weighted number of tardy jobs $N_{w_jT_j}$:

$$N_{w_jT_j} = \begin{cases} w_j & if \quad C_j > d_j \\ 0 & if \quad C_j \leq d_j \end{cases} \qquad (3)$$

The objective is to find the schedule $\pi = (\pi(1), \pi(2), \dots, \pi(n))$ of the jobs that minimize the total cost R which is formulated in mathematic form as:

$$\left.\begin{array}{l} \min R = \min_{\pi\in\delta}\{\sum_{j=1}^{n} N_{h_j E_j} + \sum_{j=1}^{n} N_{w_j T_j}\} \\ \textbf{subject to} \\ \quad C_{\pi(j)} \geq p_{\pi(j)} \qquad\qquad j = 1,2,\dots,n \\ \quad C_{\pi(j)} = C_{\pi(j-1)} + P_{\pi(j)} \qquad j = 2,3,\dots,n \\ \quad N_{E_{\pi(j)}} \in \{0,1\} \qquad\qquad j = 1,2,\dots,n \\ \quad N_{T_{\pi(j)}} \in \{0,1\} \qquad\qquad j = 1,2,\dots,n \\ \quad p_{\pi(j)} > 0,\ d_{\pi(j)} > 0,\ w_{\pi(j)} > 0,\ h_{\pi(j)} > 0 \end{array}\right\} \dots\dots\dots (H)$$

Where δ the set of all feasible solutions, π(j) denoted the position of j in the ordering π.

3. Tabu Search (TS)

By the year of 1986 Fred Glover [1], was willing to overcome local optima arising in the methods of local search, so he proposed a novel methodology by developing over a number of his earlier research, and he named it as Tabu Search. Actually, a numerous foundations of this leading TS proposal, and also a particular foundations of advanced TS elaborations, were a novelty presented by Gendreau in (2003) [2], comprising of a temporary memory to avoid the reversal of contemporary moves and a longer term frequency memory to strengthen attractive components. Hence one finds several references in the literature that addresses algorithm of tabu search, (2002) Beausoleil [3], used the tabu search to solve the problem of weighted tardiness sequence-dependent setups and compare with Re-start method. (2006) AL-Anzi, and A. Allahverdi [4], studied the problem of total completion time; the suggested heuristics is to be comparing with the current heuristics tabu search with simulated annealing this shown to be more efficient. (2013) Haung, and Yu [5], presented a novel algorithm of tabu search in order to solve a scheduling problem that possesses a several robust project that is constrained resourcefully. (2016) Selt [6], proposed two algorithms to solve the NP-hard problem and compare them with tabu search.

Local (neighborhood) searches tackled the problem and found a probable solution by checking its instant neighbors (i.e., the solutions are alike, apart from very little minor specifics) with the anticipation of discovering an enhanced solution. Current applications of Tabu Search span areas related to telecommunication, resource-planning, scheduling, financial enquiry, space development, molecular engineering, pattern recognition, energy sharing, classification, malleable manufacturing, waste management, logistics, biomedical examination, environmental safeguarding, mineral investigation, and plenty of additional applications.

Tabu search improves the above mentioned techniques performance by making use of the memory structures that are designated to the previous solutions or to the groups of instructions provided by Glover (1989) [7].

It's worth mentioning that when a probable solution was formerly visited contained by a specific short duration period or when the solution was violating a rule, then it's labeled as "tabu" (prohibited) in such a way that the algorithm will potentially not take care of that possibility continually.

4. Proposed TS Algorithm

Tabu search is a local search method used for mathematical optimization, it was invented by Fred Glover in 1986 [1].

The algorithm requires certain number of parameters for being set, as follow,

<u>Algorithm:</u> [9]

Assumptions,

S_C	The schedule of the candidate.
$G(S_C)$	The value of schedule of the candidate.
S_0	The best schedule initiated up until now.
$G(S_0)$	The value of best schedule (aspiration criterion).
K	The number of iterations.
S_K	The schedule created through K iterations.
$G(S_K)$	The value of schedule assembled by K iterations.

<u>First Step :</u> <u>Initialization</u>

Set $K = 1$

From starting sequence by any heuristic; call it S_1

Let $S_0 = S_1$,

Then $G(S_0) = G(S_1)$

<u>Second Step :</u>

Select S_C from neighborhood of S_K

IF move from S_K to S_C is not allowed in the Tabu List.

THEN $S_{K+1} = S_K$, GOTO third Step

IF $G(S_C) < G(S_0)$,

THEN $S_0 = S_C$

$G(S_0) = G(S_C)$

Delete the oldest Tabu move in the Tabu List.

GOTO Third Step

Third Step :

Set K = K+1

IF $K \leq N$

THEN GOTO Second Step

ELSE Stop

Example 1: Using the data in Table (3) and algorithm in [8] for getting the starting sequence to solve the problem (1/ / $\sum_{j=1}^{n} N_{h_j E_j}$ + $\sum_{j=1}^{n} N_{w_j T_j}$) by employing the Tabu Search method, then apply the procedure for four iterations, in addition making the Tabu list length equal to two.

Solution:

The example can be set to solve as the following sequence

First iteration

Tabu List = $\emptyset$,

$S_1 = \{j_2, j_1, j_3, j_4\}$

Let, $S_0 = S_1$.

So, $S_0 = \{j_2, j_1, j_3, j_4\}$

Then, $G(S_0) = 14$, $G(S_1) = 14$.

Find neighborhoods of S_1 By using adjacent pairwise interchange, as shown in Figure 1 below:

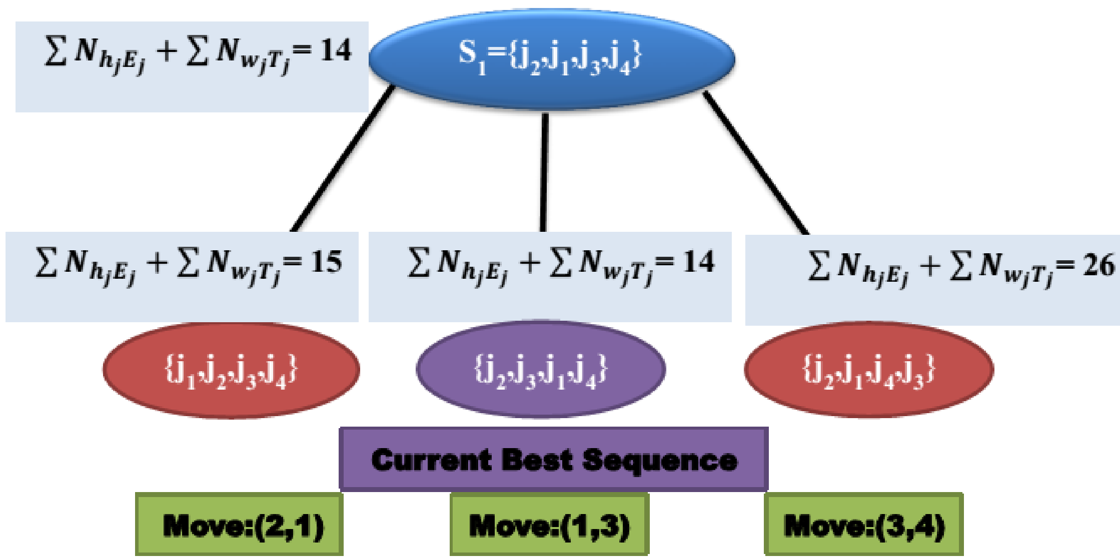


Figure 1 Illustrative approach for finding the sequences of S_1 in example 1 using adjacent pair wise interchange.

Let S_C = Current Best Sequence (CBS)

$S_C = \{j_2, j_3, j_1, j_4\}$.

$G(S_C) = 14$.

Since, $G(S_C) = G(S_0)$,

$S_0 = S_C$, and, $G(S_0) = 14$.

Then, Tabu Move = (1,3) and Tabu List = {(1,3)}

Let, $S_2 = S_C = \{j_2, j_3, j_1, j_4\}$, $G(S_0) = 14$.

<u>Second iteration</u>

Generating all the sequences of S_2 using interchanging adjacent pairwise as shown in the list below.

Sequence S_2	Pair to be interchanged	New Sequence

{j2, j3, j1, j4}	(2,3)	{j3, j2, j1, j4}
{j2, j3, j1, j4}	(3,1)	{j2, j1, j3, j4}
{j2, j3, j1, j4}	(1,4)	{j2, j3, j4, j1}

Then, The CBS is = {j3, j2, j1, j4}

$G(S_C) = 16$

Since, $G(S_C) > G(S_0)$,

$G(S_0) = 14$.

Then, Tabu Move = (2,3) and Tabu List = {(1,3), (2,3)}.

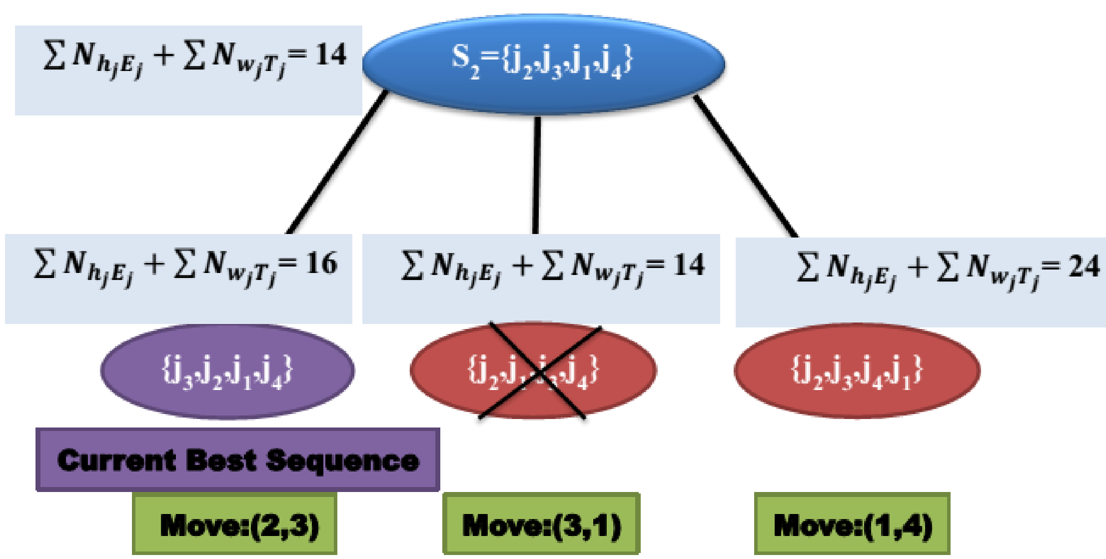


Figure 2 Illustrative approach for finding the sequences of S_2 in example 1 using adjacent pairwise interchange

Let, $S_3 = S_C$ = {j3, j2, j1, j4}, $G(S_0) = 14$.

Third iteration

Generating all the sequences of S_3 using interchanging adjacent pairwise as shown in Figure 3

Only possible move is (2,1)

Then, The CBS is {j3, j1, j2, j4}

$G(S_C) = 16.$

Since, $G(S_C) > G(S_0)$,

$G(S_0) = 14.$

Then, The new Tabu Move = (2,1) and, the new Tabu List = {(2,3), (2,1)}.

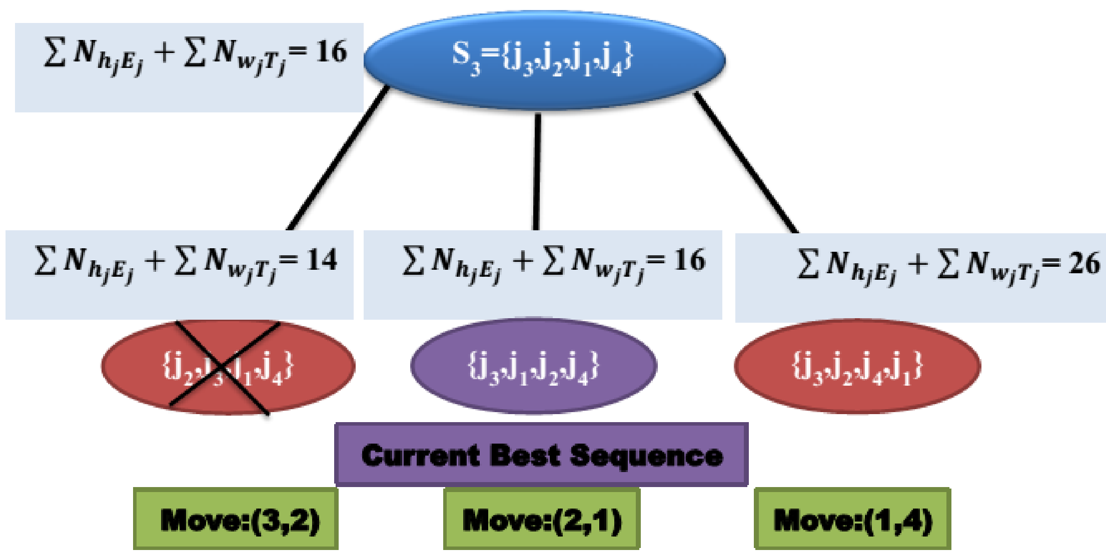


Figure 3 Illustrative approach for finding the sequences of S_3 in example 1 using adjacent pairwise interchange (the current best sequence is shown in green).

Let, $S_4 = S_C = \{j_3, j_1, j_2, j_4\}$, $G(S_0) = 14.$

<u>Fourth iteration</u>

Generating all the sequences of S_4 using interchanging adjacent pairwise as shown in Figure 4

Only possible move is (3,1)

Then, The CBS is $\{j_1, j_3, j_2, j_4\}$

$G(S_C) = 15,$

Since, $G(S_C) > G(S_0)$,

$G(S_0) = 14,$

Then, new Tabu Move = (3,1) and, new Tabu List = {(2,1), (3,1)}.

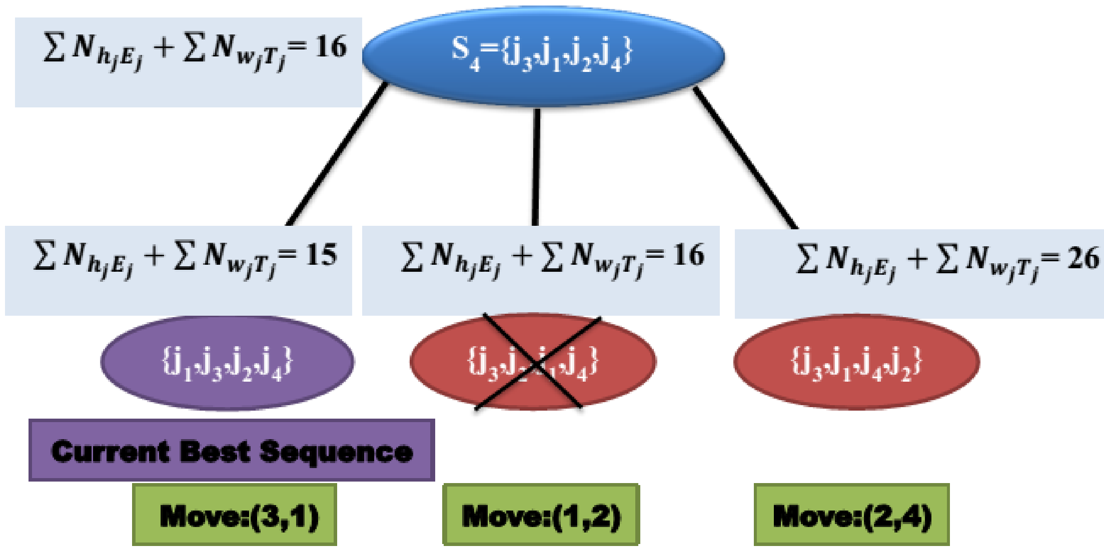


Figure 4 Illustrative approach for finding the sequences of S_4 in example 1 using adjacent pairwise interchange.

Let, $S_5 = S_C = \{j_1, j_3, j_2, j_4\}$. And $G(S_0) = 14$.

At the end of iteration 4, $G(S_0) = 14$,

Best Overall Sequence BOS = $\{j_2, j_1, j_3, j_4\}$.

5. Computational Results of Local Search Algorithms and Comparison

5.1 Test Problems

We use the test problems that are introduced in [8] for the local search algorithms.

5.2 Computational Results

In this section, we provide our results that were obtained for the simulated and Tabu Search algorithms with our proposed function. The test problems are provided in this chapter and chapter three. It's worth mentioning that we used the MATLAB 7.10.0 (R2010a) software for coding the algorithms and the processing personal computer was of 4 GB RAM with a CPU of 2.5 GHz speed. In our computations we proved that the Tabu search algorithm is active for 12100 jobs to a processing time of 30 seconds.

Table Abbreviations:

n	The number of jobs
No. of iterations	The number of iterations
No.	The number of problems
AV. of Cost	The average of cost
AV. of Time	The time average for ten Tabu search algorithm examples in seconds.

The solution of this problem with h_j, w_j by Tabu search algorithm for n ϵ {10,50,100,500,1000,5000,10000,12000,12100} is as revealed by the Table 1 .

Table (1) : The implementation of cost average and average of time in Tabu search algorithm for n ϵ { 10,50,100,500,1000,5000,10000,12000,12100}

n	AV. of Cost	AV. of Time
1	39.1000	0.1540
50	220.7000	0.2121
100	452.3000	0.2484
500	2264.4	0.4850
1000	4455.8	1.0531
5000	22460	10.7370
10000	45015	29.7351
12000	53989	29.4697
12100	54419	29.9722

Now, we find the solution of this problem with h_j, w_j to Tabu search algorithm for n = 500, s.t (n= number of jobs) as revealed by the Table 2.

Table (2): The algorithm implementation of Tabu search for n= 500 jobs

n	No.	Cost	Time	No. Of Iteration
500	1	2295	0.8385	10
	2	2220	0.4426	10
	3	2261	0.4424	10
	4	2189	0.4441	10
	5	2220	0.4658	10
	6	2255	0.4407	10
	7	2348	0.4251	10
	8	2365	0.4649	10
	9	2284	0.4440	10

	10	2207	0.4422	10
AV. of Cost	2264.4			
AV. of Time	0.4850			

5.3 Comparative Effective of Local Search Algorithms

The best solution for examples is provided in Table 5 next page which displays the results of between SA and TS algorithms and the amount of time for establishing the best value for a size of problem " n " where:

Best	The calculated best value.
SA	The value found by simulation annealing.
TS	The value found by Tabu search.
No. of Best	The number of examples that performs the best value.
Av. Time	The time average for ten examples in both algorithms in seconds.

Table (3): Performance of both LS methods with the best solution for n ∈ {10, 50, 100, 500, 1000, 5000,10000,12000}

n	10										No. of Best	Avarage time
Number	1	2	3	4	5	6	7	8	9	10		
Best	38	42	39	20	36	46	37	29	36	28		
SA	55	48	49	41	36	47	51	29	36	54	3	0.2758
TS	38	42	39	20	49	46	37	45	47	28	7	0.1540
n	50											
Best	237	227	205	219	207	195	213	183	194	229		
SA	238	230	205	219	233	237	213	252	194	231	4	2.8994
TS	237	227	225	253	207	195	231	183	220	229	6	0.2121
n	100											
Best	469	430	417	432	429	424	436	414	447	457		
SA	480	430	417	436	498	435	504	419	447	457	4	9.0964
TS	469	460	430	432	429	424	436	414	504	525	6	0.2484
n	500											
Best	2295	2220	2261	2189	2186	2210	2348	2327	2271			

SA	2307	2306	2270	2345	2186	2210	2408	2327	2271	2290	4	29.1419
TS	2295	2220	2261	2189	2220	2255	2348	2365	2284	2207	6	0.485
n	1000											
Best	4514	4376	4471	4370	4529	4426	4437	4482	4419	4460		
SA	4531	4376	4505	4542	4578	4578	4492	4571	4498	4496	1	29.1454
TS	4514	4450	4471	4370	4529	4426	4437	4482	4419	4460	9	1.0531
n	5000											
Best	22260	22282	22452	22402	22101	22130	22555	22680	22592	22119		
SA	22325	22311	22595	22402	22101	22555	22735	22680	22659	22459	3	29.4675
TS	22260	22282	22452	22414	22944	22130	22555	22852	22592	22119	7	10.737
n	10000											
Best	44925	44945	44911	44638	44852	45253	44938	44926	44407	44881		
SA	45153	45401	45094	44986	45228	45253	44938	45313	44407	44946	3	29.7156
TS	44925	44945	44911	44638	44852	45602	45415	44926	45056	44881	7	29.7351
n	12000											
Best	54009	54010	54286	54040	53760	54194	53820	53921	53930	53455		
SA	54009	54650	54453	54882	53837	54194	54723	54111	53930	54125	3	30.4109
TS	54059	54010	54286	54040	53760	54577	53820	53921	53957	53455	7	29.4697

5.4 Summary of Experimental Evaluation of Local Search Methods

Since the Table 3 is very condensed we provide a summary for the table as shown in Table 4 below focusing on the number of times that produces the best value with their summation. This is done to all the above mentioned number of jobs, showing both simulated annealing and Tabu search side by side for comparison purposes.

Table (4): The summary of Table 3 results.

n	SA	TS
10	3	7
50	4	6

100	4	6
500	4	6
1000	1	9
5000	3	7
10000	3	7
12000	3	7
Sum	25/80	55/80

In the Table 4 we give the activity of local search algorithms, (i.e. give the maximum number of jobs "n ") that the local search algorithms can solve the problem (1/ / $\sum_{j=1}^{n} N_{h_jE_j} + \sum_{j=1}^{n} N_{w_jT_j}$) with 30 seconds.

Table (5): Activity of the local search methods.

Algorithm	Active until (maximum no. of jobs)
SA	12000
TS	12100

By comparing results of both LS methods namely, Simulated annealing and Tabu search and accurate revision of result tables provided earlier, we confidently deduce that for the functional problem (1/ / $\sum_{j=1}^{n} N_{h_jE_j} + \sum_{j=1}^{n} N_{w_jT_j}$) the Tabu search has achieved the optimum results for a problem size comprising of 12100 jobs through 30 seconds.

6. Conclusion

In this paper, we have developed approximate solutions for the problem of scheduling n independent jobs on one machine to minimize the sum of weighted number of early jobs and weighted number of tardy jobs under the fuzzy processing time. To resolve it we proposed the Average High Ranking method to obtain a processing time generated from fuzzy processing time, calculate the costs and reach to penalty cost. we report on the results of extensive computations tests of the method: Tabu Search. The main conclusion to be drawn from our computation results is that the SA algorithms can solved the problem (1/ / $\sum_{j=1}^{n} N_{h_j E_j} + \sum_{j=1}^{n} N_{w_j T_j}$) to 12000 jobs in 30 seconds

References:

[1] Glover, F. (1986). Future Paths for Integer Programming and Links to Artificial Intelligence; Computers and Operations Research, Vol. 13, No. 5, Pp. 533-549.

[2] Gendreau, M. (2003). An Introduction to Tabu Search; In: F. Glover, G.A. Kochenberger (eds) Handbook of Metaheuristics, International Series in Operations Research and Management Science, Springer, Boston, Vol. 57, Pp. 37-54.

[3] Beausoleil, R.P. (2002). A Tabu Search Approach for the Weighted Tardiness with Sequence-Dependent Setups in One-Machine Problem; Journal of Mathematics: Theory and Applications, Vol. 9, No. 1, Pp. 35–46.

[4] Al-Anzi, F.S. and Allahverdi, A. (2006). A Hybrid Tabu Search Heuristic for the Two-Stage Assembly Scheduling Problem; International Journal of Operations Research, Vol. 3, No. 2, Pp. 109-119.

[5] Huang, R.H. and Yu, T.H. (2013). A Novel Tabu Search Algorithm for Solving Robust Multiple Resource-Constrained Project Scheduling Problem; International Journal of Computer and Communication Engineering, Vol. 2, No. 2, Pp. 213-215.

[6] Selt, O. (2016). Numerical Solution for Solving Scheduling Problem; Global Journal of Pure and Applied Mathematics, Vol. 12, No. 5, Pp. 4325–4333.

[7] Glover, F. (1989). Tabu Search–Part I; Operations Research Society of America Journal on Computing, Vol. 1, No. 3, Pp. 190-206.

[8] Al- zuwaini M.K, Ali, S.K and Kassim, M.M. (2018). Solving Machine Scheduling Problem under Fuzzy Processing Time using the Simulated Annealing Method; Journal of Progressive Research in Mathematics (JPRM), Vol. 14, No. 1, Pp. 23081-23171

[9] Al-Harkan, I.M. (1997). Algorithms for Sequencing and Scheduling; University of Riyadh, College of Engineering, Industrial Engineering Department, King Saud, Riyadh, Saudi Arabia, Ch. 8, Pp. 8-1, 8-19.

Compact Linear Operator on Modular Spaces

Noori F. Al-Mayahi and Al-ham S . Nief

Department of Mathematics College of Computer Science

and Information Technology

University of AL -Qadisiyah , Diwanyah -Iraq

E-mail: nafm60@yahoo.com

Abstract:

This study tackles Compact linear operator on modular space , we have introduced the definition of Compact linear operator on modular space and also define bounded and continuous linear operator on modular space and proves some new results related with them .

M.S.C: 46A80 .

Keywords: modular space , Compact linear operator on modular space , bounded and continuous linear operator on modular space .

1.Introduction

The theory of modular space was introduction by Nakano [2] in 1950 in the connection with the theory of order spaces and redefined and generalization by Musielak and Orlicz [4] in 1959 , many other mathematicians have studied modular space from several point of view , there are is a large set of known application of modular space in various part of analysis , probability and mathematical statistics .

In this effort , we introduction the notion Compact linear operator on modular space and bounded linear operator on modular space and also define continuous linear operator in modular space .

2. Main Results

Definition (2. 1)[5]:

Let X be a linear space over afield $\mathbb{F}$. A function $M: X \to [0, \infty]$ is called modular if:

1. $M(x) = 0 \iff x = 0$.

2. $M(\alpha x) = M(x)$ for $\alpha \in F$ with $|\alpha| = 1$, for all $\alpha \in F$.

3. $M(\alpha x + \beta y) \leq M(x) + M(y)$ iff $\alpha , \beta \geq 0$,for all $x \in X$.

Example (2.2) [1]:

Let $X = R^2$ with $(x, y) = |x| + |y|$, for any pair $(x - y)$ in X, then X_M is modular space .

Example (2.3) [5]:

As a classical example we mention to the Orlicz modular defined for every measurable real function f by the formula

$$M(f) = \int \emptyset(|f(t)|)\, d\mu(t),$$

where μ denotes the Lebesgue`s measure in $\mathbb{R}$ and $\emptyset: \mathbb{R} \to [0, \infty)$ is continuous . we also assume that $\emptyset(y) = 0$ if and only if $y = 0$ and

$\emptyset(t) \to \infty$ as $t \to \infty$

Definition (2.4) :

Let X and Y be a modular spaces and $T: X \to Y$ a linear operator .The operator T is said to be bounded if there exist a real number r Such that $M\big(T(x)\big) \leq rM(x)$.

Example (2.5) :

1. The identity operator $I: X \to X$ on a modular space $X \neq \{0\}$ is bounded.
2. The zero operator $0: X \to Y$ on a modular space X is bounded .

Definition (2.6) :

Let X and Y be a modular spaces. A operator $T : X \to Y$ is called compact linear operator if for every bounded subset B of X that $\overline{T(B)}$ is compact in Y.

Definition (2.7) :

Let (X, M_1) and (Y, M_2) be a modular spaces over the same field F, The operator $T: (X, M_1) \to (Y, M_2)$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon \in (0,1)$ and all $t > 0$ there exist $\delta \in (0,1)$ such that for all $x \in X$:

$$M_1(x - x_0) < \delta \implies M_2\big(T(x) - T(x_0)\big) < \varepsilon$$

Lemma (2.8) :

Let X and Y be a modular spaces , then Every compact linear operator $T: X \to Y$ is bounded , hence continuous .

Theorem (2.9) :

Let $T: X \to Y$ be a linear operator and X, Y are modular spaces .Then

1. T is continuous if and only if T is bounded .
2. If T is continuous at a single point ,it is bounded .

Definition (2.10) [3]:

Let $T: X \longrightarrow Y$ be a linear operator ,then null space of T is the set of all $x \in X$ such that $Tx = 0$.

Corollary (2.11) [3]:

Let T be a bounded linear operator .Then :

1. $x_n \to x$ implies $T_{x_n} \to T_x$.
2. The null spaces $\mathcal{N}(T)$ is closed .

Definition (2.12) :

Let (X, M) be a modular space. A subset A of X is said to be compact if any sequence $\{x_n\}$ in A has a subsequence converging to an element of A.

Theorem (1.13) :

Let (X, M) be a modular spaces ,then

1 . If $x_n \to x$, $y_n \to y$,then $x_n + y_n \to x + y$.

2 . If $x_n \to x$ then $x_n \to cx, c \in F/\{0\}$.

Proof :

1. Let $x_n \to x$ and $y_n \to y$

$$M((x_n + y_n) - (x + y)) = M((x_n - x) + (y_n - y))$$
$$\leq M(x_n - x) + M(y_n - y)$$

Since $M(x_n - x) \to 0$ and $M(y_n - y) \to 0$

Then $M((x_n + y_n) - (x + y)) \to 0$ as $n \to \infty$

Then $x_n + y_n \to x + y$.

2. Let $x_n \to x$

$$M(cx_n - cx) = M(c(x_n - x)) = M(x_n - x)$$

Since $M(x_n - x) \to 0$ as $\to \infty$, then $M(cx_n - cx) \to 0$ as $n \to \infty$

Then $cx_n \to cx$.

Theorem (2.14) :

Let X, Y be a modular spaces and $T : X \to Y$ is linear operator .Then T is compact linear operator if and only if it maps every bounded sequence $\{x_n\}$ in X onto a sequence $\{T(x_n)\}$ in Y which has a convergent subsequence.

Proof :

If T is compact linear operator and $\{x_n\}$ is bounded ,then the closure of $\{T(x_n)\}$ in Y is compact and from definition (2.6) shows that $\{T(x_n)\}$ contains a convergent subsequence.

Conversely ,suppose that every bounded sequence $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ such that $\{T(x_{n_k})\}$ converges in Y.Considerevery bounded subset $A \subset X$,and let $\{y_n\}$ be any sequence in $T(A)$ .Then $y_n = T(x_n)$ for some $x_n \in A$, and $\{x_n\}$ is bounded

since A is bounded . By a ssumption , $\{T(x_n)\}$ contains a convergent subsequence.

Hence $\overline{T(A)}$ by definition (2.12) because $\{y_n\}$ in $T(A)$ was arbitrary. By definition , this shows that T is compact linear operator.

Theorem (2.15):

Let X and Y be a modular spaces and $T_g : X \to Y$ is compact linear operator where $g = 1,2$ .Then $T_1 + T_2$ is compact linear operator and also cT_g is compact linear operator, where c any scalar $c \in F - \{0\}$, (F is field and $g = 1,2$).

Proof

Let $\{x_n\}$ bounded sequence in modular space X

Since $T_g : X \to Y$ is compact linear operator where $g = 1,2$

Then from theorem (2.14) we have $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ such that $\{T_1(x_{n_k})\}$ and $\{T_2(x_{n_k})\}$ are converges in Y

then from theorem (2.13) we have $\{T_1(x_{n_k}) + T_2(x_{n_k})\}$ is converges in $Y \Longrightarrow$ $\{(T_1 + T_2)(x_{n_k})\}$ is converges in Y

Therefore from theorem (2.14) we have $T_1 + T_2$ is compact linear operator .

Also since $\{T_g(x_{n_k})\}$ is converges in Y where $g = 1,2$.

Then by theorem (2.13) $\{cT_g(x_{n_k})\}$ is converges in Y where where c any scalar $c \in F - \{0\}$, (F is field)

Then from theorem (2.14) we have cT_g is compact linear operator .

Theorem (2.16) :(Riesz Lemma)

Let C be a closed proper subspace of modular space and Let λ a real numbers such that $0 < \lambda < 1$ then there exists a vector $x_\lambda \in X$ such that $M(x_\lambda) > 0$ and $M(x - x_\lambda) \geq \lambda$ for all $x \in C$.

Proof:

Since C be a closed proper subspace of $X \Longrightarrow C \neq X$

There exist $x_0 \in X$ such that $x_0 \notin C$

Let $d = \inf\{M(x - x_0): x \in C\}$

Since $x_0 \notin M \Longrightarrow d > 0$, since $0 < \lambda < 1 \Longrightarrow \frac{d}{\lambda} > d$

By the definition of infimum, there exist $x_1 \in C$ such that $d < M(x - x_1) \leq \frac{d}{\lambda}$

Let $x_\lambda = K(x_0 - x_1)\ where\ K = M(x_0 - x_1)^{-1} > 0$

Then $M(x_\lambda) = M\big(K(x_0 - x_1)\big) = M(x_0 - x_1) > 0$

Let $x \in C \Longrightarrow k^{-1}x + x_1 \in C$

$M(x - x_\lambda) = M\big(x - k(x_0 - x_1)\big)$

$= kM(k^{-1}x + x_1) - x_0) \geq kd$

We have $\frac{1}{k} \leq \frac{d}{\lambda}$ so $kd \geq \lambda$

Hence $M(x - x_\lambda) \geq \lambda$

Theorem (2.17) :

Let X be a modular space and assume that $A = \{x: M(x) = 1\}$ is compact then X is finite dimensional .

Proof:

Suppose that X is not finite dimensional

Choose $x_1 \in A$ and let M_1 be the subspace spanned by x_1

Then M_1 is proper subspace of X

Since M_1 is finite dimensional $\Longrightarrow M_1$ is complete $\Longrightarrow M_1$ is closed

By Riesz Lemma then there exist $x_2 \in A$ such that $M(x_2 - x_1) \geq \frac{1}{2}$

Let M_2 be the closed proper subspace of X generated by $\{x_1, x_2\}$

Then $\exists x_3 \in A$ such that $\mu(x_3 - x) \geq \frac{1}{2}$

It follows that neither the sequence $\{x_n\}$ nor its any subsequence converges , this is contradiction

Then X is finite dimensional.

Lemma (2.18) :

Let $T: X \to X$ be a compact linear operator and $S: X \to X$ be a bounded linear operator on a modular spaces X .Then ST are compact linear operator.

Proof:

To prove that ST is compact linear operator

Let (x_n) be any bounded sequence in X

Then (Tx_n) has convergent subsequence (Tx_{nk}) by theorem (2.14) and (STx_{nk}) converges by theorem (2.15)

Then ST is compact (by theorem(2.15)) .

References

[1] Krar . E . Abdul Sada , Invariant Best Approximation in Modular spaces , M. S.C. Thesis , University of Baghdad , (2018) .

[2] Nakano , H , Modular semi –ordered spaces , Tokyo Mathematical Book Series , Maruzen Co.Ltd ,Tokyo , Japan .

[3] E. Kreyszig, Introductory functional analysis with applications, John Wiley and Sons, Inc. ,(1978).

[4] J . Musielak and W . Orlicz , some remarks on modular spaces , Bull . A cad . polon . Sci .Ser .Sci . Math . Astronom . Phys . 7 (1959) , pp .661-668 .

[5] Chen , R . Wang , X . (2013) , Fixed point of nonlinear contraction in modular spaces , J. of Ineq . And Appl .

Estimators of some inequality dynamical system

Sameer Qasim Hasan
Department of Mathematics
University of Mustansiriyah
Baghdad, Iraq
dr.sameer_kasim@yahoo.com
https://orcid.org/0000-0002-2613-2584

Abstract:

The aim of this paper to presented the estimation of some classes for some inequality dynamical system based on some inequality formulation that will make important role in solvability of these types of dynamical system.

1. ***Introduction:***

The inequalities played important role in the development of all branches of mathematics, the successfully of mathematical development availability of some kinds of inequalities and derivatives, differential equations have become a major tool in analysis of the differential equations that occur in nature and applications fields.

Most of the differential inequalities developed by many literature such as [1], [3],[5] which is provide explicit known bounds on other equations and we have found wide sprouted types of inequalities have been mad a chive to develop various branches of mathematics such as differential equations to application in science and engineering practice also used the inequality to approximating various functions.

The applications of gronwall inequalities with nonlinear kernels of Lipschitz type to the problems of boundedness and convergence to zero at infinity of the solutions of certain volterra integral equations. The all types of stability are also investigated in[1].

Our aim to make a new approach to compute the estimators of the solutions of inequality dynamical system with different methods that not explained before by using the approach of gronwall and generalization of gronwall inequalities in [1,3,5].

Lemma(1.1), [3]:

Let α, a, b and h be nonnegative constants and $u: [\alpha, \alpha + h] \to [0, \infty)$ be continuous if $0 \le u(t) \le \int_\alpha^t [bu(s) + a]ds, \alpha \le t \le \alpha + h.$

, $\alpha \le t \le \alpha + h.$ 0$\le u(t) \le ahe^{bh}$ Then

Lemam(1.2), [3]:

Let α, β are any constants and c be nonnegative constants and $u, f: [\alpha, \beta] \to$
$u(t) \le c + \int_\alpha^t f(s)u(s)ds, \alpha \le t \le \beta.$ $[0, \infty)$ be nonnegative continuous if

, $\alpha \le t \le \beta.$ $u(t) \le ce^{\int_\alpha^t f(s)ds}$ Then

Lemma (1.3), [7]:

If $u(t)$ and $\alpha(t)$ are real valued continuous functions on $[a, b]$, $\alpha(t)$ is non-decreasing, and $\beta(t) \geq 0$ is integrable on $[a, b]$ with

$a \leq t \leq b.\ u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds,$

$a \leq t \leq b. \quad ,u(t) \leq \alpha(t)e^{\int_a^t \beta(s)ds}$ Then

Lemma (1.4), [6]:

Let u satisfies the inequality $u'(t) \leq B(t)u(t)$, and bounded

$.u(t) \leq u(a)e^{\int \quad atB(s)ds}$ Then

Lemma (1.5), [1]:

Let $w(s), u(s) \geq 0$ and $u(t) \leq w(t) + \int_{t_0}^t v(s)u(s)ds.$

$.u(t) \leq w(t) + \int_{t_0}^t v(s)w(s)e^{\int_s^t v(x)dx}ds$ Then

Lemma (1.6), [4]:

If u satisfies the differential inequality

$u'(t) \geq f(u(t), t)$, and y a solution of $y'(t) = f(y(t), t)$ under the boundary $u(t) \geq y(t). \quad u(t_0) = y(t_0),$ for all $t < t_0$

Lemma(1.7),[7]:

Let $v(t)$ $t_0 \leq t \leq t_0 + a$, piecewise continuous , u(t) and $u'(t)$ continuous on some $u(t) \leq u(t_0)e^{\int_{t_0}^{t} v(x)dx}$.interval . If $u'(t) \leq v(t)u(t)$ where K is a constant then

Remark(1.8),[1]

Let $u(t)$ $a \leq t \leq b$, be a real-valued function and suppose that $u'(t) \leq Ku(t)$ $u(t) \leq u(a)e^{K(t-a)}, a \leq t \leq b.$ where K is a constant then

2.Main results:

The following results are presented in details to compute the estimators of the solutions of some types for dynamical systems with suitable spaces and conditions.

Theorem(2.1):

Let $x(.) \in C[a, a+h]$ and $x'(t) \leq Ax(t) + b$, where $b \in R^{n\times 1}$, $A \in R^{n\times n}$ are semipositive matrices and $x(t) \in R^n$, a a constant and

$$0 \leq x(t) \leq x(a) + \int_a^t [Ax(s) + b]ds.$$

$$0 \leq x(t) \leq hbe^{hA}.$$ Then

Proof:

Suppose that $x(t) = \left(z - z(a)\right)^T e^{A(t-a)}$ (1)

such that $\max z =$ maximum z at $t = t_1$, thus

$$0 \leq \left(\max z^T - z(a)\right)e^{A(t_1-a)} \leq \int_a^{t_1} \left[A \ \left(\max z(s) - z(a)\right)^T e^{A(s-a)} + b\right] ds \leq$$

$$\left(\max z - z(a)\right)^T \int_a^{t_1} A \ e^{A(s-a)} ds + \int_a^{t_1} b ds$$

$$= \left[\left(\max z - z(a) \right)^T \left(e^{A(t_1 - a)} - 1 \right) \right] + b(t_1 - a).$$

Then

$$\left(\max z - z(a) \right) e^{A(t_1 - a)} \leq \left(\max z - z(a) \right) e^{A(t_1 - a)} - \left(\max z - z(a) \right) + b(t_1 - a) ,$$

therefore

$\left(\max z - z(a) \right) \leq b(t_1 - a) \leq bh$, hence

From (1), we get

$x(t) = \left(\max z - z(a) \right) e^{Ah} \leq bhe^{Ah}$.Therefore

$$x(t) \leq bhe^{Ah}.$$

Theorem (2.2):

Let $x(t) \in C\left[[a, b], R_0^+\right)$ and $x'(t) \leq A(t)x(t), \quad x(a) \geq 0$

where $A(t)$ is semipositive continuous matrices

$.s \in J = [a, b] \quad x(t) \leq x(a) e^{\int_a^t A(s)ds},$ Then

Proof:

$x(t) \leq z(t)$ Define $z(a) = x(a),$

(2) $t \in J \qquad z'(t) = A(t)x(t) \leq A(t)z(t),$ and

multiply (2) by $e^{-\int_a^t A(s)ds}$, we get

$$z'(t) e^{-\int_a^t A(s)ds} - A(t)z(t) e^{-\int_a^t A(s)ds} = \frac{d}{dt}\left[z(t) e^{\left(-\int_a^t A(s)ds \right)} \right] \leq 0$$

we obtain from (2)

(3) $$\frac{d}{dt}\left[z(t)e^{\left(-\int_a^t A(s)ds\right)}\right] \leq 0$$

Integrating of (3) from a to t , we get

, since $(a) = x(a)$, we get $z(t)e^{-\int_a^t A(s)ds} - z(a) \leq 0$

, which implies that $z(t)e^{-\int_a^t A(s)ds} - x(a) \leq 0$

$z(t) \leq x(a)e^{\int_a^t A(s)ds}$, hence $z(t)e^{-\int_a^t A(s)ds} \leq x(a)$

So , since $x(t) \leq z(t) \leq x(a)e^{\int_a^t A(s)ds}$

Then $x(t) \leq x(a)e^{\int_a^t A(s)ds}$.

Theorem (*2.3*):

Let $x(t) \in C([a,b], R_0^+)$ and $f \ and \ G \ \in C([a,b], R_0^+)$

$x'(t) \leq G(\tau)A(t)x(t), \quad \tau, t \in J = [a,b]$

$x(a) = f(t) \in C([a,b], R_0^+)$

$e^{\int_s^t A(\sigma)G(\sigma)d\sigma} ds.x(t) \leq f(t) + G(\tau)\int_a^t A(s)f(s)$ Then

Proof;

Define a function $z(t)$ by $z(t) = \int_a^t A(s)x(s)ds$. Thus

, $x(t) \leq f(t) + G(\tau)\int_a^t A(s)x(s)ds z(a) = 0$

Therefore

(4) $x(t) \leq f(t) + G(\tau)z(t)$

So, $z'(t) = A(t)x(t) \leq A(t)f(t) + A(t)G(\tau)z(t)$ (5)

Multiply inequality equation (5) by $e^{-\int_a^t A(\sigma)G(\sigma)d\sigma}$, we have that

$\frac{d}{dt}\left[z(t)e^{-\int_a^t A(\sigma)G(\sigma)d\sigma}\right] \leq A(t)f(t)e^{-\int_a^t A(\sigma)G(\sigma)d\sigma}$ (6) set $t = s$ in inequality equation (6), and integrated with respect to s from a to t , we obtain,

$$z(t)e^{-\int_a^t A(\sigma)G(\sigma)d\sigma} \leq \int_a^t A(s)f(s)e^{-\int_a^s A(\sigma)G(\sigma)d\sigma}ds$$

$$= z(t) \leq e^{\int_a^t A(\sigma)G(\sigma)d\sigma} \int_a^t A(s)f(s)e^{-\int_a^s A(\sigma)G(\sigma)d\sigma}ds$$

$$e^{\int_a^s A(\sigma)G(\sigma)d\sigma + e^{\int_s^t A(\sigma)G(\sigma)d\sigma}} \int_a^t A(s)f(s)e^{-\int_a^s A(\sigma)G(\sigma)d\sigma}ds$$

from (4), we get

$$x(t) \leq f(t) + G(\tau)\int_a^t A(s)f(s)e^{\int_s^t A(\sigma)G(\sigma)d\sigma}ds$$

<u>Theorem (2.4):</u>

Let $x(t) \in C([a,b],R)$ and $A(t-s)$ be a continuous semi positive matrix on $\Delta:\ a \leq s \leq t \leq b.$

$$x'(t) \leq A(t-s)x(t),$$

$$x(a) = c$$

$.x(t) \leq x(a) + \int_a^t A(t-s)x(s)ds$ Then

<u>Proof</u>

Fix T such that $a \leq T \leq b$, then $a \leq t \leq T$, thus

$$x(t) \leq c + \int_a^t A(T-s)x(s)ds$$

Now, let

$z(a) = c$, $z(t) = c + \int_a^t A(T-s)x(s)ds$

, for $a \leq t \leq T$ $x(t) \leq z(t)$

$z'(t) = A(T-s)x(t)$

(7) $\leq A(T-s)z(t),$ for $a \leq t \leq T$

set $t = s$ in (1), and integrating with respect to s from a to t , we get

(8) $z(T) \leq c\, e^{\int_a^T A(T-s)ds}$

since T is arbitrary then from (8) and $x(t) \leq z(t)$

$x(T) \leq z(T) \leq c\, e^{\int_a^T A(T-s)ds}$

. $a \leq t \leq T$ $x(t) \leq c\, e^{\int_a^t A(t-s)ds}$ Hence,

3.Conclusion:

The estimators of some types of dynamical systems have been specified in some technical of inequality mathematical for integral and differential inequality that needed in studied the boundedness and stability of the trajectory of dynamical system.

References:

[1] Dragomir. S.S, ' Some gronwall type inequalities and applications',

School of communicating and informatics, Victoria, University of

Technoffrential equations: Gradulogy,2002.

[2] Hu. Jishan and Wei-Ping Li. 'Theory of ordinary differential equations',The Hong Kong University of science and Technology,2004.

[3] Pachpatte. B.G. 'Inequalities for differential and integral equations', mathematics in science and engineering ,volume 197,1997

[4] Petrovitsch, ' Sur une maniere detendre theorem de lamoyence aux equations differentielles du premier order ', ANN. OF Math., 54:3(1901)pp.417-436

[5] Szarski.J, 'Differential inequalities' ,1965

[6] Gronwall . T.H.,' Note on the derivatives with respect to aa parameter of the solutions of system of differential equations', Ann of Math ., 20:2(1991)pp. 292-296.

[7] Yanovsky. I., 'Ordinary differential equations: Graduate Level Problems',2005

المجموعات النتروسوفيكية الهشة المفتوحة من النمط Rα في الفضاءات التبولوجيا النتروسوفيكية الهشة الثنائية

د.رياض خضر الحميدو – سوريا - جامعة الفرات – كلية العلوم- قسم الرياضيات

ملخص : في هذا البحث سنعرف وندرس المجموعات النتروسوفيكية الهشة المفتوحة من النمط Rα، ومتمماتها المجموعات النتروسوفيكية الهشة المغلقة من النمط Rα، وسنوجد الخصائص الأساسية لها، وسندرس علاقتها مع المجموعات النتروسوفيكية الهشة المفتوحة والمغلقة والمجموعات النتروسوفيكية الهشة الفا مفتوحة والفا مغلقة في الفضاءات التبولوجية النتروسوفيكية الهشة الثنائية.

الكلمات المفتاحية : المجموعات النتروسوفيكية الهشة المفتوحة من النمط Rα , المجموعات النتروسوفيكية الهشة المغلقة من النمط Rα , الفضاء التبولوجي النتروسوفيكي الهش الثنائي.

Neutrosophic crisp open sets of the pattern Rα in Neutrosophic crisp Bi-topological spaces

Dr.Riad Khidr Al-Hamido <riad-hamido1983@hotmail.com

Al-Furat University, Faculty of science , Department of mathematics, Dair AlZur, Syria.

Abstract:

In this paper we introduced new types of neutrosophic crisp open and closed sets of the pattern Rα in neutrosophic crisp BI-Topological spaces, and we studied the basic properties of these new types of sets, as the we have created the relationship between them and neutrosophic crisp open and closed sets from one saide and we have created the relationship between them and neutrosophic crisp alpha open and closed sets in these neutrosophic crisp bi-topological spaces from other saide.

Key Word:

neutrosophic crisp BI-Topological space, Neutrosophic crisp open sets of the pattern Rα, Neutrosophic crisp closed sets of the pattern Rα.

مقدمة:

عمم J. C. Kelly مفهوم الفضاء التبولوجي في عام 1963، حيث أدخل مفهوم الفضاء التبولوجي الثنائي في [13].

أيضاً عرف N. A. Jabbar و A. I. Nasir في عام 2010 في [12] المجموعات المفتوحة من النمط N في الفضاء التبولوجي الثنائي.

أيضاً عرف R. KH. AlHamido و Q. Imran عام 2017 في[2] المجموعات المفتوحة من النمط N لكن في الفضاء التبولوجي الثلاثي .

A. A. Salama and F. Smarandache , and Valeri Kroumov عام 2014 عرفوا مفهوم الفضاء التوبولوجي النتروسوفيكي الهش كما عرفوا المجموعة النتروسوفيكية الهشة المفتوحة والمغلقة والعمليات عليها في [21] .

R. Kh. AlHamido عام 2018 عرف مفهوم الفضاء التوبولوجي النتروسوفيكي الهش الثنائي كما عرف المجموعة النتروسوفيكية الهشة الثنائية المفتوحة والمغلقة [40] .

1. تعاريف ومفاهيم أساسية في النتروسوفيك الهش:

تعريف $(\mathbf{1.1})$:[19]

لتكن $X \neq \emptyset$ مجموعة ما, عندئذ: المجموعة النتروسوفيكية الهشة A (التي يرمز لها أختصاراً NCS) هي ثلاثية مرتبة $A =< A_1, A_2, A_3 >$ ، حيث A_1, A_2, A_3 هي مجموعات جزئية من X.

تعريف $(\mathbf{2.1})$:[19]

لتكن $X \neq \emptyset$ مجموعة ما, عندئذ:

1) تعرف المجموعة الخالية النتروسوفيكية الهشة (التي يرمز لها أختصاراً $\emptyset_N$) ، بأحد الأشكال :

- $\emptyset_N =< \emptyset, \emptyset, X >$ او
- $\emptyset_N =< \emptyset, X, X >$ او
- $\emptyset_N =< \emptyset, X, \emptyset >$ او
- $\emptyset_N =< \emptyset, \emptyset, \emptyset >$.

2) تعرف المجموعة الشاملة النتروسوفيكية الهشة X_N ، بأحد الأشكال :

- $X_N =< X, \emptyset, \emptyset >$ او
- $X_N =< X, X, \emptyset >$ او
- $X_N =< X, \emptyset, X >$ او
- $X_N =< X, X, X >$.

تعريف $(\mathbf{3.1})$:[19] لتكن $X \neq \emptyset$ مجموعة ما, تدعى المجموعة النتروسوفيكية الهشة (Neutrosophic crisp set) A التي لها الشكل $A =< A_1, A_2, A_3 >$ ، حيث A_1, A_2, A_3 هي مجموعات جزئية من X:

a) مجموعة نتروسوفيكية هشة من النمط الأول (التي يرمز لها أختصاراً $NCS - Taype1$) ، إذا حققت: $A_1 \cap A_2 = \emptyset$, $A_1 \cap A_3 = \emptyset$, $A_2 \cap A_3 = \emptyset$.

b) مجموعة نتروسوفيكية هشة من النمط الثاني (التي يرمز لها أختصاراً $NCS - Taype2$) ، إذا حققت: $A_1 \cap A_2 = \emptyset$, $A_1 \cap A_3 = \emptyset$, $A_2 \cap A_3 = \emptyset$, $A_1 \cup A_2 \cup A_3 = X$.

c) مجموعة نتروسوفيكية هشة من النمط الثالث (التي يرمز لها أختصاراً $NCS - Taype3$) ، إذا حققت: $A_1 \cap A_2 \cap A_3 = \emptyset$, $A_1 \cup A_2 \cup A_3 = X$.

تعريف $(\mathbf{4.1})$:[19]

لتكن $X \neq \emptyset$ مجموعة ما, ولتكن A , B مجموعتين نتروسوفيكيتين هشتين من الشكل

$A =< A_1, A_2, A_3 >$ و $B =< B_1, B_2, B_3 >$, عندئذ:

الاحتواء $A \subseteq B$ ، يعرف بأحد الشكلين :

- $A \subseteq B \Leftrightarrow A_1 \subseteq B_1,\ A_2 \subseteq B_2, A_3 \supseteq B_3$.
- $A \subseteq B \Leftrightarrow A_1 \subseteq B_1, A_2 \supseteq B_2, A_3 \supseteq B_3$.

تعريف ($\mathbf{5.1}$):[19]

لتكن $X \neq \emptyset$ مجموعة ما, ولتكن A , B مجموعتين نتروسوفيكيتين هشتين من الشكل

$A =< A_1, A_2, A_3 >$ و $B =< B_1, B_2, B_3 >$, عندئذ:

a) التقاطع $A \cap B$ يعرف بأحد الشكلين :

- $A \cap B =< A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3 >$.
- $A \cap B =< A_1 \cap B_1, A_2 \cup B_2, A_3 \cup B_3 >$.

b) الاجتماع $A \cup B$ يعرف بأحد الشكلين :

- $A \cup B =< A_1 \cup B_1, A_2 \cap B_2, A_3 \cap B_3 >$.
- $A \cup B =< A_1 \cup B_1, A_2 \cup B_2, A_3 \cap B_3 >$.

تعريف ($\mathbf{6.1}$):[21]

لتكن $X \neq \emptyset$ مجموعة ما, عندئذ التبولوجيا النتروسوفيكية الهشة على X (التي يرمز لها أختصاراً NCT) هي أسرة مجموعات نتروسوفيكية هشة Γ من X ، تحقق:

1) $\emptyset_N , X_N \in \Gamma$.
2) $A \cap B \in \Gamma$ لأية مجموعتين A،B من Γ .
3) $\bigcup_i A_i \in \Gamma$ لأية مجموعات A_i من Γ .

- ندعو في هذه الحالة (X, Γ) فضاء تبولوجياً نتروسوفيكياً هشاً على X (أو أختصاراً $NCTS$) ، كل عنصر من Γ يدعى مجموعة نتروسوفيكية هشة مفتوحة (أو أختصاراً $NCOS$) ، وتدعى متممتها مجموعة نتروسوفيكية هشة مغلقة (أو أختصاراً $NCCS$).

تعريف ($\mathbf{7.1}$):[21]

لتكن $X \neq \emptyset$ مجموعة ما, ولتكن $A =< A_1, A_2, A_3 >$ مجموعة نتروسوفيكية هشة, عندئذ:

1) متممة المجموعة A يرمز له بالرمز $\grave{A}$ أو A^c وتعرف بأحد الأشكال :

- $A^c =< {A_1}^c, {A_2}^c, {A_3}^c >$.
- $A^c =< A_3, A_2, A_1 >$.

- $.A^c =< A_3, {A_2}^c, A_1 >$

تعريف (8.1): لتكن X مجموعة ما غير خالية, وليكن كلاً من Γ_1, Γ_2 تبولوجيا نتروسوفيكية هشة على X، عندئذ:
ندعو (X, Γ_1, Γ_2) فضاء تبولوجي نتروسوفيكي هش ثنائي على X (أو أختصاراً Bi-NCTS).

- تدعى المجموعة الجزئية Aمن $\Gamma_1 \cup \Gamma_2$ مجموعة ثنائية نتروسوفيكية هشة مفتوحة (أو أختصاراً Bi-NCOS).

تعريف (9.1): ليكن (X, Γ_1, Γ_2) فضاء تبولوجياً نتروسوفيكياً هشاً ثنائياً على X ، عندئذ:
اجتماع كل المجموعات الثنائية النتروسوفيكية الهشة المفتوحة المحتواه في المجموعة النتروسوفيكية الهشة A، تدعى الداخلية الثنائية النتروسوفيكية الهشة للمجموعة النتروسوفيكية الهشة A، ويرمز لها بالرمز($NC^{Bi}Int(A)$)، أي :

$NC^{Bi}Int(A) = \cup$ {B : B⊆A ; مجموعة ثنائية نتروسوفيكية هشة مفتوحة B} .

تعريف (10.1): ليكن (X, Γ_1, Γ_2) فضاء تبولوجياً نتروسوفيكياً هشاً ثنائياً على X ، عندئذ:
تقاطع كل المجموعات الثنائية النتروسوفيكية الهشة المغلقة التي تحوي المجموعة النتروسوفيكية الهشة A، تدعى اللصاقة الثنائية النتروسوفيكية الهشة للمجموعة A، ويرمز لها بالرمز($NC^{Bi}Cl(A)$)، أي :

$NC^{Bi}cl(A) = \cap$ {B : B⊇A ; مجموعة ثنائية نتروسوفيكية هشة مغلقة B} .

2. المجموعات النتروسوفيكية الهشة المفتوحة من النمط Rα :

تعريف (1.2): ليكن (X, τ_1, τ_2) فضاء تبولوجياً نتروسوفيكياً هشاً ثنائياً على X، عندئذ:
المجموعة الجزئية A من X تدعى مجموعة نتروسوفيكية هشة مفتوحة من النمط $R\alpha$ (أو اختصاراً $NCR\alpha$-مفتوحة) إذا كانت مجموعة نتروسوفيكية هشة الفا مفتوحة في الفضاء التبولوجي
$(X, \tau_1 \vee \tau_2)$ حيث:

$\tau_1 \vee \tau_2$: is the supremum Neutrosophic Crisp topology on X contains τ_1, τ_2

- متممة المجموعة النتروسوفيكية الهشة المفتوحة من النمط $R\alpha$، هي مجموعة نتروسوفيكية هشة مغلقة من النمط $R\alpha$ في الفضاء التبولوجي النتروسوفيكي الهش الثنائي(أو اختصاراً $NCR\alpha$-مغلقة).
- يرمز لأسرة كل المجموعات النتروسوفيكية الهشة المفتوحة من النمط $R\alpha$ بـ $NCR\alpha.O(X)$.
- يرمز لأسرة كل المجموعات النتروسوفيكية الهشة المغلقة من النمط $R\alpha$ بـ $NCR\alpha.C(X)$.

مثال (2.2):

$X=\{1,2,3,4\}$, $\Gamma_1 = \{\emptyset_N, X_N, D, C\}$, $\Gamma_2 = \{\emptyset_N, X_N, A, B\}$ $A = <\{1\},\{2,4\},\{3\}> = C$, $B = <\{1\},\{2\},\{3,4\}>$, $D = <\{1\},\{2\},\{3\}>$

إن كلاً من (X, Γ_1) , (X, Γ_2)فضاء تبولوجي نتروسوفيكي هش، وبالتالي فإن (X, Γ_1, Γ_2) هو فضاء تبولوجي نتروسوفيكي هش ثنائي.

بعض المجموعات المفتوحة من النمط $R\alpha$ فيه هي:

$$\emptyset_N, X_N, A, B, D$$

مبرهنة(3.2) :

1) إن المجموعات النتروسوفيكية الهشة المفتوحة من النمط R هي مجموعات نتروسوفيكية هشة مفتوحة من النمط $R\alpha$ أي$R.O(X) \subset NCR\alpha.O(X)$، لكن العكس غير صحيح.
2) إن المجموعات النتروسوفيكية الهشة المغلقة من النمط R هي مجموعات نتروسوفيكية هشة مغلقة من النمط $R\alpha$ أي$R.C(X) \subset NCR\alpha.C(X)$، لكن العكس غير صحيح.

البرهان :

1) لتكن Aمجموعة نتروسوفيكية هشة مفتوحة من النمط R .

نعلم أن $A \subseteq NC^{Bi}cl(A)$ ومنه

$$NC^{Bi}int(A) \subseteq NC^{Bi}int(N - cl(A))$$

لكن بما أن A مجموعة نتروسوفيكية هشة مفتوحة من النمط R فإن

$$NC^{Bi}int(A) = A \text{ ومنه } A \subseteq NC^{Bi}int(NC^{Bi}cl(A)) \ldots\ldots(1)$$

أيضاً لدينا $NC^{Bi}int\left(NC^{Bi}cl(A)\right) = NC^{Bi}int\left(NC^{Bi}cl\left(NC^{Bi}int(A)\right)\right) \ldots\ldots(2)$

ومن ثم ينتج عن $(1),(2)$ أن :

$$A \subseteq NC^{Bi}int(NC^{Bi}cl(NC^{Bi}int(A)))$$

ومنه A هي مجموعة نتروسوفيكية هشة مفتوحة من النمط $R\alpha$.

2) بالطريقة نفسها يتم البرهان.

مثال يثبت أن العكس غير صحيح :

مثال (4.2):

$X=\{1,2,3,4\}$, $\tau_1 = \{\emptyset_N, X_N\}$, $\tau_2 = \{\emptyset_N, X_N, A, B\}$ $A = <\{1\},\{2\},\{3\}>$, $B = <\{1\},\{2,4\},\{3\}>$

واضح أن كلاً من: (X,τ_1) , (X,τ_2) فضاءات تبولوجية نتروسوفيكية هشة لذلك فإن (X,τ_1,τ_2) فضاء نتروسوفيكي هش ثنائي.

إن $M = <\{1\},\{4\},\{3\}>$ مجموعة نتروسوفيكية هشة مفتوحة من النمط $R\alpha$، لكنها ليست مجموعة نتروسوفيكية هشة مفتوحة من النمط R.

إن $\grave{M}$ مجموعة نتروسوفيكية هشة مغلقة من النمط $R\alpha$، لكنها ليست مجموعة نتروسوفيكية هشة مغلقة من النمط R.

نتيجة (5.2):

إن المجموعات النتروسوفيكية الهشة المفتوحة من النمط $R\alpha$ ، تختلف عن المجموعات النتروسوفيكية الهشة المفتوحة من النمط R فيه، أيضاً المجموعات النتروسوفيكية الهشة المغلقة من النمط $R\alpha$ و المجموعات النتروسوفيكية الهشة المغلقة من النمط R فيه هما مفهومان مختلفان.

ملاحظة (6.2) :

1) المجموعة النتروسوفيكية الهشة المفتوحة من النمط $R\alpha$ في الفضاء التبولوجي النتروسوفيكي الهش الثنائي (X,τ_1,τ_2) ليس بالضرورة مجموعة نتروسوفيكية هشة مفتوحة في أي من الفضاءات النتروسوفيكية الهشة (X,τ_i) حيث $i = 1,2$.

2) المجموعة النتروسوفيكية الهشة المغلقة من النمط $R\alpha$ في الفضاء التبولوجي النتروسوفيكي الهش الثنائي (X,τ_1,τ_2) ليس بالضرورة مجموعة نتروسوفيكية هشة مغلقة في أي من الفضاءات النتروسوفيكية الهشة (X,τ_i) حيث $i = 1,2$.

مثال (7.2) :

في المثال (4.2)

إن $M = <\{1\},\{4\},\{3\}>$ مجموعة نتروسوفيكية هشة مفتوحة من النمط $R\alpha$، لكن ليست مجموعة نتروسوفيكية هشة مفتوحة في أي من الفضاءات (X,τ_i) حيث $i = 1,2$.

إن $\grave{M}$ مجموعة نتروسوفيكية هشة مغلقة من النمط $R\alpha$، لكنها ليست مجموعة نتروسوفيكية هشة مغلقة في أي من الفضاءات (X,τ_i) حيث $i = 1,2$.

نتيجة (8.2):

1) المجموعة النتروسوفيكية الهشة المفتوحة من النمط αR في الفضاء التبولوجي النتروسوفيكية الهشة الثنائي (X,τ_1,τ_2) ليس بالضرورة مجموعة نتروسوفيكية هشة مفتوحة فيه.

2) المجموعة النتروسوفيكية الهشة المغلقة من النمط αR في الفضاء التبولوجي النتروسوفيكية الهشة الثنائي (X,τ_1,τ_2) ليس بالضرورة مجموعة نتروسوفيكية هشة مغلقة فيه.

البرهان: ينتج مباشرة عن الملاحظة (6.2).

مثال (9.2):

في المثال (4.2)

إن $M = <\{1\},\{4\},\{3\}>$ مجموعة نتروسوفيكية هشة مفتوحة من النمط $R\alpha$، لكن ليست مجموعة نتروسوفيكية هشة مفتوحة في أي من الفضاءات (X,τ_i) حيث $i = 1,2$ وبالتالي ليست مجموعة نتروسوفيكية هشة مفتوحة فيه.

إن $\grave{M}$ مجموعة نتروسوفيكية هشة مغلقة من النمط $R\alpha$، لكنها ليست مجموعة نتروسوفيكية هشة مغلقة في أي من الفضاءات (X,τ_i) حيث $i=1\ ,2$ وبالتالي ليست مجموعة نتروسوفيكية هشة مغلقة فيه.

ملاحظة (10.2):

1) المجموعة النتروسوفيكية الهشة الفا مفتوحة في أي من الفضاءات (X,τ_i) حيث $i=1\ ,2$ ليست بالضرورة مجموعة نتروسوفيكية هشة مفتوحة من النمط $R\alpha$ في الفضاء التبولوجي النتروسوفيكي الهش الثنائي (X,τ_1,τ_2) .
2) المجموعة النتروسوفيكية الهشة الفا مغلقة في أي من الفضاءات (X,τ_i) حيث $i=1\ ,2$ ليست بالضرورة مجموعة نتروسوفيكية هشة مغلقة من النمط $R\alpha$ في الفضاء التبولوجي النتروسوفيكي الهش الثنائي (X,τ_1,τ_2) .

مبرهنة (11.2): ليكن (X,τ_1,τ_2) فضاء تبولوجياً نتروسوفيكياً هشاً ثنائياً على X، ولتكن A مجموعة جزئية من X عندئذ:

A مجموعة $NCR\alpha-$ مفتوحة $\Leftrightarrow$ توجد مجموعة G نتروسوفيكية هشة مفتوحة من النمط R تحقق $G\subseteq A\subseteq NC^{Bi}int(N-cl(G))$.

البرهان :

$\Leftarrow$: لتكن A مجموعة $NCR\alpha-$ مفتوحة ومنه حسب تعريف المجموعة $NCR\alpha-$ مفتوحة، فإنه يتحقق $A\subseteq NC^{Bi}int(NC^{Bi}cl(N-int(A)))$ ومنه نجد أن:

$N-int(A)\subseteq A\subseteq N-int(N-cl(NC^{Bi}int(A)))$ بوضع $G=N-int(A)$ نجد أن $G\subseteq A\subseteq NC^{Bi}int(NC^{Bi}cl(G))$.

$\Rightarrow$: لنفرض أن G مجموعة نتروسوفيكية هشة مفتوحة من النمط R تحقق

$G\subseteq A\subseteq NC^{Bi}int(NC^{Bi}cl(G))$ ، نعلم أن $NC^{Bi}int(A)$ هي أكبر مجموعة نتروسوفيكية هشة مفتوحة من النمط R محتواه في A وأن G مجموعة $R-$ مفتوحة تحقق $G\subseteq A$ ومن ثم $G\subseteq NC^{Bi}int(A)$ ومنه $NC^{Bi}cl(G)\subseteq NC^{Bi}cl(NC^{Bi}int(A))$ ومنه

$NC^{Bi}int(NC^{Bi}cl(G))\subseteq NC^{Bi}int(NC^{Bi}cl(NC^{Bi}int(A)))$ ، لكن $G\subseteq A\subseteq NC^{Bi}int(NC^{Bi}cl(G))$ ومن ثم $A\subseteq NC^{Bi}int(NC^{Bi}cl(NC^{Bi}int(A)))$ ومنه A مجموعة $R\alpha-$ مفتوحة.

ملاحظة (12.2):

1) المجموعة النتروسوفيكية الهشة المفتوحة من النمط $R\alpha$ في الفضاء التبولوجي النتروسوفيكي الهش الثنائي (X,τ_1,τ_2) ليس بالضرورة مجموعة نتروسوفيكية هشة الفا مفتوحة في أي من الفضائين (X,τ_i) حيث $i=1\ ,2$.
2) المجموعة النتروسوفيكية الهشة المغلقة من النمط $R\alpha$ في الفضاء التبولوجي النتروسوفيكي الهش الثنائي (X,τ_1,τ_2) ليس بالضرورة مجموعة نتروسوفيكية هشة الفا مغلقة في أي من الفضائين (X,τ_i) حيث $i=1\ ,2$.

مثال (13.2):

$X=\{a,b,c,d\},\ \tau_1=\{\emptyset_N, X_N\}, \tau_2=\{\emptyset_N, X_N, A, B\}\ A=<\{a\},\{b,d\},\{c\}>\ , B=<\{a\},\{b\},\{c\}>$

واضح أن كلاً من: (X,τ_1) , (X,τ_2) فضاءات تبولوجية نتروسوفيكية هشة لذلك فإن (X,τ_1,τ_2) فضاء نتروسوفيكي هش ثنائي.

1) إن $L=<\{a\},\{d\},\{c\}>$ مجموعة نتروسوفيكية هشة مفتوحة من النمط $R\alpha$، لكنها ليست مجموعة نتروسوفيكية هشة الفا مفتوحة في أي من الفضائين (X,τ_i) حيث $i=1\ ,2$.
2) إن $\grave{M}$ مجموعة نتروسوفيكية هشة مغلقة من النمط $R\alpha$، لكنها ليست مجموعة نتروسوفيكية هشة الفا مغلقة في أي من الفضائين (X,τ_i) حيث $i=1\ ,2$.

ملاحظة (14.2):

1) كل مجموعة نتروسوفيكية هشة مفتوحة في أي من الفضاءات (X,τ_i) حيث $i=1\ ,2$ هي مجموعة نتروسوفيكية هشة مفتوحة من النمط $R\alpha$ في الفضاء التبولوجي النتروسوفيكي الهش الثنائي (X,τ_1,τ_2).

2) كل مجموعة نتروسوفيكية هشة مغلقة في أي من الفضاءات (X,τ_i) حيث $i=1,2$ هي مجموعة نتروسوفيكية هشة مغلقة من النمط $R\alpha$ في الفضاء التبولوجي النتروسوفيكي الهش الثنائي (X,τ_1,τ_2).

مبرهنة (15.2):

ليكن (X,τ_1,τ_2) فضاء تبولوجياً نتروسوفيكياً هشاً ثنائياً على X، عندئذ:

كل مجموعة نتروسوفيكية هشة مفتوحة في الفضاء التبولوجي النتروسوفيكي الهش الثنائي هي مجموعة مفتوحة نتروسوفيكية هشة من النمط $R\alpha$ فيه.

البرهان: لتكن A مجموعة نتروسوفيكية هشة مفتوحة ولنبرهن أنها مجموعة نتروسوفيكية هشة مفتوحة من النمط $R\alpha$, بما أن A مجموعة نتروسوفيكية هشة مفتوحة فإن $A\in\tau_1\cup\tau_2$ ، لكن

$\tau_1\cup\tau_2\subset\tau_1\vee\tau_2$ ومنه $A\in\tau_1\vee\tau_2$ ومن ثم :

A مجموعة نتروسوفيكية هشة مفتوحة من النمط R ومن ثم هي مجموعة نتروسوفيكية هشة مفتوحة من النمط $R\alpha$.

ملاحظة (16.2):

عكس المبرهنة (15.2)، ليس صحيحاً بشكل عام كما يوضح المثال الآتي.

مثال (17.2):

في مثال (4.2):

إن $M=<\{1\},\{4\},\{3\}>$ مجموعة نتروسوفيكية هشة مفتوحة من النمط $R\alpha$، لكنها ليست مجموعة نتروسوفيكية هشة مفتوحة في الفضاء التبولوجي النتروسوفيكي الهش الثنائي.

إن $\grave{M}$ مجموعة نتروسوفيكية هشة مغلقة من النمط $R\alpha$، لكنها ليست مجموعة نتروسوفيكية هشة مغلقة في الفضاء التبولوجي النتروسوفيكي الهش الثنائي.

مبرهنة(18.2) : ليكن (X,τ_1,τ_2) فضاء تبولوجياً نتروسوفيكياً هشاً ثنائياً على X، عندئذ:

A مجموعة $R\alpha-$ مفتوحة $\Leftrightarrow$ $A\subseteq NC^{Bi}int\left(NC^{Bi}cl\left(NC^{Bi}int\,(A)\right)\right)$.

البرهان : بما أن A مجموعة نتروسوفيكية هشة مفتوحة من النمط $R\alpha$ في الفضاء التبولوجي الثنائي فإن A مجموعة نتروسوفيكية هشة الفا مفتوحة في الفضاء التبولوجي $(X,\tau_1\vee\tau_2)$ ومنه يتم المطلوب.

مبرهنة(19.2): ليكن (X,τ_1,τ_2) فضاء تبولوجياً نتروسوفيكياً هشاً ثنائياً على X، عندئذ:

اجتماع أسرة مجموعات نتروسوفيكية هشة مفتوحة من النمط $R\alpha$ هو مجموعة نتروسوفيكية هشة مفتوحة من النمط αR.

البرهان : لتكن $\{A_\alpha / \alpha\in I\}$ أسره مجموعات نتروسوفيكية هشة مفتوحة من النمط $R\alpha$, فيكون لأجل كل $\alpha\in I$,

$$\Rightarrow A_\alpha\subset NC^{Bi}int\left(NC^{Bi}cl\left(NC^{Bi}int\,A_\alpha\right)\right)$$

$$\Rightarrow \cup A_\alpha\subset\cup\left(NC^{Bi}int\left(NC^{Bi}cl\left(NC^{Bi}int\,A_\alpha\right)\right)\right)$$ (اخذنا الاجتماع للطرفين)

$$\Rightarrow \cup A_\alpha\subset NC^{Bi}int\left(\cup\left(NC^{Bi}cl\left(NC^{Bi}int\,A_\alpha\right)\right)\right)$$ (حسب مبرهنة $(15.1.1)$)

$$\Rightarrow \cup A_\alpha\subset NC^{Bi}int\left(NC^{Bi}cl\left(\cup\left(NC^{Bi}int\,A_\alpha\right)\right)\right)$$

[لان $NC^{Bi}cl\,(A)\cup NC^{Bi}cl\,(B)\subset NC^{Bi}cl\,(A\cup B)$]

$$\Rightarrow \cup A_\alpha\subset NC^{Bi}int\left(NC^{Bi}cl\left(\left(NC^{Bi}int(\cup A_\alpha)\right)\right)\right)$$ (حسب مبرهنة(15.1))

ومنه $\cup A_\alpha$ مجموعة نتروسوفيكية هشة مفتوحة من النمط $R\alpha$.

نتيجة (20.2): ليكن (X,τ_1,τ_2) فضاء تبولوجياً نتروسوفيكياً هشاً ثنائياً على X، عندئذ:

تقاطع أي عدد من مجموعات $NCR\alpha$–مفتوحة هو مجموعة $NCR\alpha$–مفتوحة.

البرهان :

لتكن $\{A_\alpha / \alpha \in I\}$ أسره مجموعات $NCR\alpha$–مفتوحة , فيكون لأجل كل $\alpha \in I$,

$$\Rightarrow A_\alpha \subset NC^{Bi}int\left(NC^{Bi}cl\left(NC^{Bi}int\, A_\alpha\right)\right)$$

$$\Rightarrow \cap\ A_\alpha \subset \cap\left(NC^{Bi}int\left(NC^{Bi}cl\left(NC^{Bi}int\, A_\alpha\right)\right)\right)$$

$$\Rightarrow \cap\ A_\alpha \subset NC^{Bi}int\left(\cap\left(NC^{Bi}cl\left(NC^{Bi}int\, A_\alpha\right)\right)\right)$$

$$\Rightarrow \cap\ A_\alpha \subset NC^{Bi}int\left(NC^{Bi}cl\left(\cap\left(NC^{Bi}int\, A_\alpha\right)\right)\right)$$

$$\Rightarrow \cap\ A_\alpha \subset NC^{Bi}int\left(NC^{Bi}cl\left(\left(NC^{Bi}int(\cap\, A_\alpha)\right)\right)\right)$$

ومنه $\cap\ A_\alpha$ مجموعة $NCR\alpha$–مفتوحة .

ملاحظة (21.2): ليكن (X,τ_1,τ_2) فضاء تبولوجياً نتروسوفيكياً هشاً ثنائياً على X، عندئذ:

1) إن X و$\emptyset$ مجموعتان نتروسوفيكيتان هشتان مفتوحتان من النمط $R\alpha$.

2) إن X و$\emptyset$ مجموعتان نتروسوفيكيتان هشتان مغلقتان من النمط $R\alpha$.

مبرهنة(22.2): ليكن (X,τ_1,τ_2) فضاء تبولوجياً نتروسوفيكياً هشاً ثنائياً على X، عندئذ:

المجموعات النتروسوفيكية الهشة المفتوحة من النمط $R\alpha$ تشكل تبولوجيا نتروسوفيكية هشة على X.

البرهان : ينتج عن مبرهنة (19.2) و مبرهنة (20.2) و مبرهنة (21.2).

References

[1]. S. A. Alblowi, A. A. Salama, and M. Eisa. 2013, "New concepts of neutrosophic sets". *International Journal of Mathematics and Computer Applications Research (IJMCAR)* 3, 95-102.

[2]. R. Kh. AlHamido and Q. H. Imran, 2017," N-Open Sets and S-Open Sets in Tri-topological Spaces", *University of Babylon* J. for Pure & Appl. Sci., Vol.25, No.5.

[3]. W. Al-Omeri. 2016, "Neutrosophic crisp Sets via Neutrosophic crisp Topological Spaces NCTS", Neutrosophic Sets and Systems, Vol.13, pp.96-104

[4]. K. Atanassov. 1986, "intuitionistic fuzzy sets", fuzzy sets and systems 20, 87-96.

[5]. I. Arokiarani, R. Dhavaseelan, S. Jafari and M. Parimala. 2017,"On Some New Notions and Functions in Neutrosophic Topological Spaces", Neutrosophic Sets and Systems, Vol.16, pp.16-19.

[6]. K. Atanassov. 1983, "intuitionistic fuzzy sets". in V.Sgurev, ed., Vii ITKRS Session, Sofia (June 1983 central Sci. and Techn. Library, Bulg. Academy of Sciences.

[7]. K. Atanassov. 1984, "intuitionistic fuzzy sets", Sofia central Sci. and Techn. Library,

Bulg. Academy of Sciences .
[8]. I. M. Hanafy, A. A. Salama, and K. M. Mahfouz. 2013, "Neutrosophic Crisp Events and Its Probability". International Journal of Mathematics and Computer Applications Research (IJMCAR), Vol. 3, Issue 1, 171-178.
[9].I. Hanafy, A.A. Salama and K. Mahfouz,2012, "Correlation of neutrosophic Data",International Refereed Journal of Engineering and Science (IRJES), Vol.(1), Issue 2 , 39-43.
[10].P. Iswarya, K. Bageerathi. 2016,"On Neutrosophic Semi-Open sets in Neutrosophic Topological Spaces", International Journal of Mathematics Trends and Technology (IJMTT), Vol.37, No.3, pp.214-223.
[11]. N. a. Jabbar and Al-Tabatabai, N. M. 2015, "N^{α} −Open Sets And N^{α} −Regularity In Topological Spaces", International Journal of Advanced Scientific and Technical Research., Vol. 3, No.5.
[12]. N. A. Jabbar and A. I. Nasir, 2010, "Some Types of Compactness in Bitopological Spaces", Ibn AL-Haitham J. For Pure & Appl. Sci., Vol.23, (1), 321-327.
[13]. J. C. Kelly, 1963, "Bitopological spaces", Proc. London Math. Soc.,13, 71-89.
[14]. Martin M. Kovar, 2000, "On 3-Topological Version Of q-Regularity", Internat. J. Math. & Math. Sci., Vol.23, No.6, 393-398.
[15]. Mrsevic and I. L. Reilly, 1996, "Covering and Connectedness Properties of a Topological Space and it is Associated Topology of α-subsets", *Indian J. Pure Appl*. Math., 27(10), 995-1004.
[16]. O. Njastad, 1965, "On some classes of nearly open Sets", *Pacific Journal of math*., Vol.15, No.3, 961-970.
[17]. T. Noiri, 1988, "Characterizations of extremally disconnected spaces", Indian J. Pure Appl. Math., Vol.19, No.4, 325-329.
[18]. A. A Salama and F. Smarandache, 2013,"Filters via Neutrosophic Crisp Sets". Neutrosophic Sets and Systems 1, 34-38.
[19]. A. A Salama, F.Smarandache, 2015,"Neutrosophic crisp Set and Theory", Educational Publisher Columbus.
[20]. A. A. Salama and F. Smarandache , 2013, " Intuitionistic Fuzzy Ideals Topological Spaces " . Advances in Fuzzy Mathematics, Volume 7, Number 1, 51-60.
[21]. A. A. Salama and F. Smarandache , and Valeri Kroumov 2014, " Neutrosophic crisp Sets & Neutrosophic crisp Topological Spaces" . Neutrosophic Sets and Systems, Vlo. 2, 25-30.
[22]. A. A. Salama, 2015,"Basic Structure of Some Classes of Neutrosophic Crisp Nearly Open Sets and Possible Application to GIS Topology". *Neutrosophic Sets and Systems* 7, 18-22.
[23]. A. A. Salama and S. A. Alblowi, 2012,Generalized Neutrosophic Set and Generalized Neutrousophic Topological Spaces. Journal computer Sci. Engineering 2, 29-32.
[24]. A. A. Salama and S. A. Alblowi, 2012,"Intuitionistic Fuzzy Ideals Topological Spaces". Advances in Fuzzy Mathematics 7, 51- 60.

[25]. A. A. Salama, 2013,"Neutrosophic Crisp Points and Neutrosophic Crisp Ideals". *Neutrosophic Sets and Systems* 1, 50-54.

[26]. A. A. Salama and S. A. Alblowi, 2012,"Neutrosophic set and neutrosophic topological space". ISORJ, Mathematics 3, 31-35.

[27]. A. A. Salama, 2015, " Basic Structure of Some Classes of Neutrosophic Crisp Nearly Open Sets & Possible Application to GIS Topology " . Neutrosophic Sets and Systems Vol. 7, 18-22.22

[28]. A. A. Salama, and H. Elagamy, 2013,"Neutrosophic Filters". *International Journal of Computer Science Engineering and Information Technology Reseearch* (IJCSEITR) 3, 307-312.

[29]. D. Sarker, 1997,"Fuzzy ideal theory, Fuzzy local function and generated fuzzy topology". *Fuzzy Sets and Systems* 87, 117-123.

[30]. F. Smarandache, 2002, "Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics", University of New Mexico, NM 87301, USA.

[31]. F. Smarandache, 1999,"A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set", *Neutrosophic Probability. American Research Press*, Rehoboth, NM.

[32]. F. Smarandache, 2011,"An introduction to the Neutrosophy probability applid in Quntum Physics". International Conference on introducation Neutrosoph Physics, Neutrosophic Logic, Set, Probability, and Statistics, University of New Mexico, Gallup, NM 87301, USA2-4 December.

[33]. F. Smarandache, 2011,"An introduction to the Neutrosophy probability applid in Quntum Physics". International Conference on introducation Neutrosoph Physics, Neutrosophic Logic, Set, ProbaTrility, and Statistics, University of New Mexico, Gallup, NM 87301, USA2-4 December.

[34]. F. Smarandache, 2002,Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics, University of New Mexico, NM 87301, USA.

[35]. F. Smarandache. 1999, "A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability". American Research Press, Rehoboth, NM.

[36]. M. L. Thivagar, S. Jafari, V. Antonysamy and V.Sutha Devi, 2018,"The Ingenuity of Neutrosophic Topology via N-Topology", Neutrosophic Sets and Systems, Vlo.19, pp. 91-100.

[37]. M. L. Thivagar, V. Ramesh, M. D. Arockia, "On new structure of N –topology", 2016,Cogent Mathematics (Taylor and Francis),3,1204104.

[37]. V. Venkateswara Rao and Y. Srinivasa Rao, 2017,"Neutrosophic Pre-open Sets and Pre-closed Sets in Neutrosophic Topology", International Journal of ChemTech Research, Vol.10, No.10, pp.449-458.

[39]. L.A. Zadeh, 1965, "Fuzzy Sets" , Inform and Control 8, 338-353.

[40]. R. Kh. AlHamido. 2018, "Neutrosophic crisp Bi-Topological Spaces NCTS", Neutrosophic Sets and Systems, Vol.21.

Generalized Right n-derivations in prime near- rings

Enaam F. Adhab

Directorate General of Education in Qadisiyah / Department of supervisory specialization

enaam1972mayahi@gmail.com

Abstract

This research paper presents the conceptions of generalized right n- derivation in near ring as well as generalized right derivation. This further indicates that a prime near rings sustaining various differential identities on generalized right n- derivations are termed as commutative rings.

الخلاصة

في هذا البحث قدمنا تعريف لتعميم المشتقات اليمنى وبينا ان الحلقات المقتربة الاولية مع تعميم المشتقات اليمنى مع بعض الفرضيات عليها تكون حلقات ابدالية .

Keywords: prime near-ring,generalized rightderivations. generalizedright n-derivations.

1. Introduction

"A near-ring is defined to be a set N along with two other binary operations (.) & (+)". Such as (i) is a not necessary abelian group (N, +), (ii) a semi group which is (N, .) and (iii) this group is for all a, b, c ϵ N , thus a.(b+ c) = a.b + b.c ". According to this research paper, N will be 0 symmetric near-ring for instance, "(N satisfy the property 0.x =0 for every x$\in$ N)". The elements' product such as x & y in N will be x.y which is specified by xy". The centre of multiplicative of N will be represented by Z which is $Z = \{x \in N \mid xy = yx for all y \in N\}$.Thus, for each x, y $\in$ N, the symbol [x, y] = xy – yx & (x, y) = x + y – x-y symbolizes the commutator & additive commutator for x & y correspondingly. N is referred as prime near-ring in case of xNy = {0} infers as y = 0 or x = 0. "A nonempty/ full subset U of N is named as semigroup left ideal and resp. semigroup right ideal in case of NU $\subseteq$ U (resp. UN $\subseteq$ U) , but if U represents both semigroup right and left ideal then it will be termed as semigroup ideal." Regarding terminologies of near-rings, Pilz is referred [1].

The additive mapping " d :N $\longrightarrow$N" is known as a derivation in case of "d(xy) = d(x)y + xd(y)", (or equally"d(xy) = xd(y) + d(x)y for every x, y, ϵN" as distinguished in [2].

"The derivation concepts' generalization has been done through various means according to different authors".

"Thus, Ashraf M & Siddeeque M A well-defined the ideas of (σ,τ)-n- derivation, generalized n-derivation in near ring and n-derivation in [4], [5] and [3] correspondingly. Also various properties of such derivations were also examined. "Similarly, Abdul Rehman H.M & Enaam F.A already presented the notions of right n-derivations as well as right derivations in near-ring [6].

In our current study, "the basic concepts of generalized right &generalized right n-derivations in near-rings have been defined. Further, we explored the near-rings' multiplication and addition's commutativity to fulfill the identities while comprising the generalized right n-derivation on prime near rings".

Definition 1.1 Let N be a near-ring, d :N $\longrightarrow$N be right derivation of N , an additive mapping g :N $\longrightarrow$N is said to be generalized right derivation of N associated with d if g(xy) = d(x)y + g(y)x , for all x, yϵ N .

Definition 1.2 If d is right n-derivation of a prime near-ring N . An n-additive(i.e.; additive in each argument) mapping

$g:\underbrace{N \times N \times \ldots \times N}_{n-times} \longrightarrow N$ is said to be generalized right n-derivation of N associated with d if the relations

$g(x_1 x_1', x_2, \ldots, x_n) = d(x_1, x_2, \ldots, x_n)x_1' + g(x_1', x_2, \ldots, x_n)x_1$

$g(x_1, x_2x_2', \ldots, x_n) = d(x_1, x_2, \ldots, x_n)x_2' + g(x_1, x_2', \ldots, x_n)x_2$

$\vdots$ $g(x_1, x_2, \ldots, x_nx_n') = d(x_1, x_2, \ldots, x_n)x_n' + g(x_1, x_2, \ldots, x_n')x_n$

hold for all $x_1, x_1', x_2, x_2', \ldots, x_n, x_n' \in N$.

Example 1.3. Let S be a 2-torsion free zero-symmetric left near-ring .

Let us define :

$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, x, y, 0 \in S \right\}$ is zero symmetric near-ring with regard to matrix addition and matrix multiplication .

Now we define d , f: N → N by

$$d\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$g\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Define $d_1 : \underbrace{N \times N \times \ldots \times N}_{n-times} \longrightarrow N$ such that

$$d_1\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & x_1 x_2 \ldots x_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$g_1 : \underbrace{N \times N \times \ldots \times N}_{n-times} \longrightarrow N$ such that

$$g_1\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & y_1 y_2 \ldots y_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It can be easily seen that g is a nonzero generalized right derivation associated with right derivation d of near-ring N and g_1 is a nonzero generalized right n-derivation associated with right n-derivation d_1 of near-ring N .

2. Preliminary results

We begin with the following lemmas which are essential for developing the proofs of our main results .

Lemma 2.1[7] Let N be a near-ring . If there exists a non-zero element z of Z such that z + z ∈ Z , then (N , +) is abelian .

Lemma 2.2 [8] Let N be a prime near-ring . If z∈ Z \{0} and x is an element of N such that xz ∈ Z or zx ∈ Z, then x∈ Z .

Lemma 2.3 [8] Let N be a prime near-ring and *Z* containsa nonzero semigroup left ideal or nonzero semigroup left ideal , then N is a commutative ring .

Lemma 2.4.[9] Let N be a prime near-ring , d a nonzero n-derivation of N and $x \in N$.

(i) If $d(N, N, . . ., N)x = \{0\}$, then $x = 0$.

(ii) If $xd(N, N, . . ., N) =\{0\}$, then $x = 0$.

Lemma 2.5. Let N be a prime near-ring , g is a nonzero right generalized n-derivation associated with right n–derivation d of N, and $a \in N$. If $g(N, N, . . ., N)a = \{0\}$, then $a = 0$.

Proof . Given that $g(N, N, . . ., N)a = \{0\}$,i.e.;

$$g(x_1, x_2, . . ., x_n)a = 0 \text{ , for all } x_1, x_2, . . ., x_n \in N \quad (1)$$

Putting ax_1 in place of x_1 in relation (1) , we get

$$0 = g(ax_1, x_2, . . ., x_n)a$$

$$= (d(a, x_2, . . ., x_n)\, x_1 + g(x_1, x_2, . . ., x_n)\, a)a$$

$$= d(a, x_2, . . ., x_n)\, x_1a$$

So we get $d(a, x_2, . . ., x_n)\, Na = \{0\}$ for all $x_2, . . ., x_n \in N$. Primeness of N implies that

either $a = 0$ or $d(a, x_2, . . ., x_n) = 0$ for all $x_2, . . ., x_n \in N$.

If $d(a, x_2, . . ., x_n) = 0$ for all $x_2, . . ., x_n \in N$ (2)

Since $g(x(ay), x_2, . . ., x_n) = g((xa)y, x_2, . . ., x_n)$ for all $x, y, x_2, . . ., x_n \in N$, we get

$$d(x, x_2, . . ., x_n)ay + g(ay, x_2, . . ., x_n)x= d(xa, x_2, . . ., x_n)y + g(y, x_2, . . ., x_n)xa \text{ , i.e.;}$$

$$d(x, x_2, . . ., x_n)ay + (d(a, x_2, . . ., x_n)y + g(y, x_2, . . ., x_n)a)\, x =$$

$$(d(x, x_2, . . ., x_n)a + d(a, x_2, . . ., x_n)x)y + g(y, x_2, . . ., x_n)xa$$

Using relations (1) and (2) in previous relation we get $g(y, x_2, . . ., x_n)Na =\{0\}$ for all $y, x_2, . . ., x_n \in N$. Since $g \neq 0$, primeness of N implies that $a = 0$.

So we can considered the following Lemma as a result of Lemma 2.5

Lemma 2.6[6 : Lemma 2.5]. Let N be a prime near-ring , d is a nonzero right n-derivation of N and $a \in N$. If $d(N, N, . . ., N)a = \{0\}$, then $a = 0$.

As a result of Lemma 2.5 we can prove the following Lemma

Lemma 2.7 Let N be a prime near-ring , g is a nonzero generalized rightderivation of N associated with right derivation d of N. and a ϵN . If g(N)a = {0} , then a = 0.

3.Main Results

Theorem 3.1 Let N be a prime near-ring and g is a nonzero generalized rightn-derivation associated with right n-derivation d of N . If $g(N,N,...,N) \subseteq Z$, then N is a commutative ring .

Proof. Since $g(N,N,...,N) \subseteq Z$ and g is a nonzero rightgeneralized n-derivation, then there exist a nonzero elements $x_1, x_2, \ldots, x_n \in N$, such that $g(x_1, x_2, \ldots, x_n) \in Z\backslash\{0\}$. We have

$g(x_1+ x_1,x_2, \ldots,x_n) = g(x_1,x_2, \ldots,x_n) + g(x_1, x_2, \ldots,x_n) \in Z$. By Lemma 2.1 we obtain that (N , +) is an abelain group .

By hypothesis we get

$$g(y_1,y_2, \ldots,y_n)y = yg(y_1,y_2, \ldots,y_n) \text{ for all } y,y_1, y_2, \ldots,y_n \in N. \quad (3)$$

Now replacing y_1 by y_1y_1' where $y_1' \in N$ in (.3) we have"

$$(d(y_1,y_2, \ldots,y_n)y_1' + g(y_1',y_2, \ldots,y_n)y_1)y$$

$$= y(d(y_1,y_2, \ldots,y_n)y_1' + g(y_1',y_2, \ldots,y_n)y_1) .$$

for all $y,y_1,y_1',y_2, \ldots, y_n \in N$. (4)

By definition of d we get for all $y_1,y_1',y_2, \ldots, y_n \in N$

$$d(y_1 y_1',y_2, \ldots,y_n)= d(y_1, y_2, \ldots,y_n)y_1' + d(y_1', y_2, \ldots,y_n)y_1 \quad (5)$$

and

$$d(y_1'y_1,y_2, \ldots,y_n)= d(y_1',y_2, \ldots,y_n)y_1 + d(y_1,y_2, \ldots,y_n)y_1' \quad (6)$$

Since(N, +) is an abelaian group , from (5) and (6) we conclude that

$d(y_1 y_1',y_2, \ldots, y_n)= d(y_1'y_1,y_2, \ldots, y_n)$ for all $y_1,y_1',y_2, \ldots,y_n \in N$.

So we get

$$d([y_1 , y_1'],y_2, \ldots,y_n)= 0 \text{for all } y_1,y_1',y_2, \ldots,y_n \in N . \quad (7)$$

Replacing y_1' by $y_1 y_1'$ in (7) we get

$0 = d([y_1 , y_1y_1'],y_2, . . .,y_n)$

$= d(y_1[y_1 , y_1'],y_2, . . .,y_n)$

$= d(y_1, y_2, . . ., y_n)[y_1 , y_1'] + d([y_1, y_1'], y_2, . . ., y_n)y_1$

$= d(y_1, y_2, . . ., y_n)[y_1, y_1']$ that means

$d(y_1, y_2, . . ., y_n)y_1 y_1'= d(y_1, y_2, . . ., y_n)y_1' y_1$ for all $y_1,y_1',y_2, . . .,y_n\in N$.

Replacing $y_1'$by $y_1'z$, where $z\in N$, in last equation and using it again, we get

$d(y_1,y_2, . . .,y_n)N[y_1 , z]= \{0\}$ for all $y_1,y_1',y_2, . . .,y_n, z\in N$. Primeness of N implies that for each $y_1\in N$ either $d(y_1, y_2, . . ., y_n)= 0$ for all $y_2, . . ., y_n\in N$ or $y_1\in Z$. If $d(y_1,y_2, . . .,y_n)= 0$, then equation (4) takes the form $g(y_1{}',y_2, . . .,y_n)N[y, y_1] = \{0\}$. Since $g \neq 0$, primeness of N implies that $y_1\in Z$. Hence we find that N = Z, and N is a commutative ring .

Corollary 3.2[6,Theorem 3.1] Let N be a prime near-ring and d is a nonzero right n-derivation of N . If $d(N,N,...,N) \subseteq Z$, then N is a commutative ring .

Corollary 3.3 Let N be a prime near-ring and g is a nonzero right generalizedderivation associated right derivation d of N . If $g(N) \subseteq Z$, then N is a commutative ring .

Theorem 3.4 Let N be a prime near-ring and g_1 and g_2 be any two nonzero generalized right n-derivations associated with right derivations d_1 and d_2 respectively". If $[g_1(N,N, . . .,N) , g_2(N,N, . . .,N)] = \{0\}$. then (N,+) is an abelian group .

Proof. Assume that $[g_1(N,N, . . .,N) , g_2(N,N, . . .,N)] = \{0\}$. If both z and z + z commute element wise with $g_2(N, N, . . ., N)$, then for all $x_1,x_2, . . .,x_n\in N$ we have

$zg_2(x_1,x_2, . . .,x_n) = g_2(x_1,x_2, . . .,x_n)z$ (8)

and

$(z + z)g_2(x_1,x_2, . . .,x_n) = g_2(x_1,x_2, . . .,x_n)(z + z)$ (9)

Substituting $x_1 + x_1'$ instead of x_1 in (9) we get

$(z + z)g_2(x_1 + x_1',x_2, . . .,x_n) = g_2(x_1 + x_1',x_2, . . .,x_n)(z + z)$ for all $x_1,x_1',x_2,\ldots,x_n\in N$.

From (8) and (9) the previous equation can be reduced to

$zg_2(x_1 + x_1'- x_1- x_1',x_2, . . .,x_n) = 0$ for all $x_1,x_1',x_2,\ldots,x_n\in N$., i.e.;

$zg_2((x_1, x_1'), x_2, \ldots, x_n) = 0$ for all $x_1, x_1', x_2, \ldots, x_n \in N$.

Putting $z = g_1(y_1, y_2, \ldots, y_n)$ we get

$g_1(y_1, y_2, \ldots, y_n)g_2((x_1, x_1'), x_2, \ldots, x_n) = 0$ for all $x_1, x_1', x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$.

By Lemma 2.5 we conclude that

$g_2((x_1, x_1'), x_2, \ldots, x_n) = 0$ for all $x_1, x_1', x_2, \ldots, x_n \in N$ (10)

Since we know that for each $w \in N$

$w(x_1, x_1') = w(x_1 + x_1' - x_1 - x_1')$

$= wx_1 + wx_1' - wx_1 - wx_1'$

$= (wx_1, wx_1')$

Which is again an additive commutator , putting $w(x_1, x_1')$ instead of (x_1, x_1') in (10) we get

$g_2(w(x_1, x_1'), x_2, \ldots, x_n) = 0$ for all $w, x_1, x_1', x_2, \ldots, x_n \in N$. i.e.;

$d_2(w, x_2, \ldots, x_n)(x_1, x_1') + g_2((x_1, x_1'), x_2, \ldots, x_n)w = 0$, using (10) in previous equation yields $d_2(w, x_2, \ldots, x_n)(x_1, x_1') = 0$. Using Lemma 2.6 we conclude that $(x_1, x_1') = 0$. Hence $(N, +)$ is an abelain group .

Corollary 3.5 [6,Theorem 3.3] Let N be a prime near-ring and d_1 and d_2 be any two nonzero right n-derivations . If $[d_1(N,N, \ldots,N), d_2(N,N, \ldots,N)] = \{0\}$ then $(N,+)$ is an abelian group".

Corollary 3.6 Let N be a prime near-ring and g_1 and g_2 be any two nonzero generalized right derivations associated with right derivations d_1, d_2 respectively. If $[g_1(N),g_2(N)] = \{0\}$ then $(N,+)$ is an abelian group .

Theorem 3.7 Let N be a prime near-ring and g_1 and g_2 be any two nonzero right generalized n-derivations associated with right n-derivation d_1 and d_2 respectively . If $g_1(x_1,x_2, \ldots,x_n)g_2(y_1,y_2, \ldots,y_n) + g_2(x_1,x_2, \ldots,x_n)g_1(y_1,y_2, \ldots,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n \in N$. Then $(N,+)$ is an abelian group .

Proof. By our hypothesis we have ,

$g_1(x_1,x_2, \ldots,x_n)g_2(y_1,y_2, \ldots,y_n) + g_2(x_1,x_2, \ldots,x_n)g_1(y_1,y_2, \ldots,y_n) = 0$

for all $x_1,x_2,\ldots,x_n, y_1,y_2,\ldots,y_n \in N$. (11)

Substituting $y_1 + y_1'$ instead of y_1 in (11) we get

$g_1(x_1,x_2, \ldots,x_n)g_2(y_1 + y_1',y_2, \ldots,y_n) +$

$g_2(x_1,x_2, \ldots,x_n)g_1(y_1 + y_1',y_2, \ldots,y_n) = 0$

for all $x_1,x_2,\ldots,x_n,y_1,y_1',y_2,\ldots,y_n \in N$.

Therefore

$g_1(x_1,x_2,\ldots,x_n)g_2(y_1,y_2,\ldots,y_n) + g_1(x_1,x_2,\ldots,x_n)g_2(y_1',y_2,\ldots,y_n) +$

$g_2(x_1,x_2,\ldots,x_n)g_1(y_1,y_2,\ldots,y_n) + g_2(x_1,x_2,\ldots,x_n)g_1(y_1',y_2,\ldots,y_n) = 0$

for all $x_1,x_2,\ldots,x_n, y_1,y_1',y_2,\ldots,y_n \in N$.

using (11) again in preceding equation we get

$g_1(x_1,x_2,\ldots,x_n)g_2(y_1,y_2,\ldots,y_n) + g_1(x_1,x_2,\ldots,x_n)g_2(y_1',y_2,\ldots,y_n) +$

$g_1(x_1,x_2,\ldots,x_n)g_2(-y_1,y_2,\ldots,y_n) + g_1(x_1,x_2,\ldots,x_n)g_2(-y_1',y_2,\ldots,y_n) = 0$

for all $x_1,x_2,\ldots,x_n,y_1,y_1',y_2,\ldots,y_n \in N$.

Which means that

$g_1(x_1, x_2, \ldots, x_n)\, g_2((y_1, y_1'),y_2,\ldots,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1,y_1',y_2,\ldots,y_n \in N$. By Lemma 2.5 we obtain

$g_2((y_1,y_1'),y_2,\ldots,y_n) = 0$ for all $y_1,y_1',y_2,\ldots,y_n \in N$.

Now putting $w(y_1,y_1')$ instead of (y_1,y_1'), where $w \in N$ in previous equation we get $g_2(w(y_1,y_1'),y_2,\ldots,y_n) = 0$ for all $y_1,y_1',y_2,\ldots,y_n \in N$. So we have $d_2(w, y_2, \ldots, y_n)(y_1, y_1') = 0$, using Lemma 2.6; we conclude that (N, +) is abelain .

Corollary 3.8[6, Theorem 3.5] Let N be a prime near-ring and d_1 and d_2 be any two nonzero right n-derivations . If $d_1(x_1,x_2,\ldots,x_n)d_2(y_1,y_2,\ldots,y_n) + d_2(x_1,x_2,\ldots,x_n)d_1(y_1,y_2,\ldots,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n \in N$. Then (N,+) is an abelian group .

Corollary 3.9 Let N be a prime near-ring and g_1 and gd_2 be any two nonzero generalized right derivations associated with right derivation d_1 and d_2 respectively . If $g_1(x)g_2(y) + g_2(x)g_1(y) = 0$ for all $x, y \in N$. Then (N,+) is an abelian group .

Theorem 3.10 Let N be a prime near-ring , let g_1 be a nonzero generalized right n-derivation associated with right n-derivattion d_1 and g_2 be a nonzero generalized n-derivation associated with n-derivation d_2 .

(i) If $g_1(x_1,x_2,\ldots,x_n)g_2(y_1, y_2,\ldots,y_n) + g_2(x_1,x_2,\ldots,x_n)g_1(y_1,y_2,\ldots,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n \in N$. Then (N,+) is an abelian group .

(ii) If $g_2(x_1,x_2,\ldots,x_n)g_1(y_1,y_2,\ldots,y_n) + g_1(x_1,x_2,\ldots,x_n)g_2(y_1,y_2,\ldots,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n \in N$. Then (N,+) is an abelian group .

Proof. (i) By our hypothesis we have ,

$g_1(x_1,x_2,...,x_n)g_2(y_1,y_2,...,y_n) + g_2(x_1,x_2,...,x_n)g_1(y_1,y_2,...,y_n) = 0$

for all $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n \in N$. (12)

Substituting $y_1 + y_1'$, where $y_1' \in N$, for y_1 in (12) we get

$g_1(x_1,x_2,...,x_n)g_2(y_1 + y_1',y_2,...,y_n) +$

$g_2(x_1,x_2,...,x_n)g_1(y_1 + y_1',y_2,...,y_n) = 0$

for all $x_1,x_2,\ldots,x_n,y_1,y_1',y_2,\ldots,y_n \in N$.

So we have

$g_1(x_1,x_2,...,x_n)g_2(y_1,y_2,...,y_n) + g_1(x_1,x_2,...,x_n)g_2(y_1',y_2,...,y_n) + g_2(x_1,x_2,...,x_n)g_1(y_1,y_2,...,y_n) + g_2(x_1,x_2,...,x_n)g_1(y_1',y_2,...,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1,y_1',y_2,\ldots,y_n \in N$.

Using (12) in previous equation implies

$g_1(x_1,x_2,...,x_n)g_2(y_1,y_2,...,y_n) + g_1(x_1,x_2,...,x_n)g_2(y_1',y_2,...,y_n) + g_1(x_1,x_2,...,x_n)g_2(-y_1,y_2,...,y_n) + g_1(x_1,x_2,...,x_n)g_2(-y_1',y_2,...,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1, y_1',y_2,\ldots,y_n \in N$.

Which means that

$g_1(x_1, x_2, ..., x_n)g_2((y_1,y_1'),y_2,...,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1,y_1',y_2,\ldots,y_n \in N$.

Now using Lemma 2.5 we conclude that

$g_2((y_1 , y_1'),y_2,...,y_n) = 0$ for all $y_1,y_1',y_2,\ldots,y_n \in N$. (13)

Now putting $w(y_1, y_1')$ instead of (y_1, y_1'), where $w \in N$ in (13) we get $g_2(w(y_1, y_1'),y_2,...,y_n) = 0$ for all $w,y_1,y_1',y_2,\ldots,y_n \in N$, so we have $d_2(w, y_2, ..., y_n)(y_1, y_1') = 0$, using Lemma 2.4(i); we conclude that $(y_1, y_1') = 0$ for all $y_1,y_1' \in N$. Thus (N, +) is an abelain group .

(ii) If N satisfies

$g_2(x_1,x_2,...,x_n) g_1(y_1,y_2,...,y_n) +$

$g_1(x_1,x_2,...,x_n) g_2(y_1,y_2,...,y_n) = 0$

for all $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n \in N$, then again using the same arguments as in (i) we get the required result .

Corollary 3.11[6, Theorem 3.7] Let N be a prime near-ring, let d_1 be a nonzero right n-derivation and d_2 be a nonzero n-derivation .

(i) If $d_1(x_1,x_2,...,x_n)d_2(y_1,y_2,...,y_n) + d_2(x_1,x_2,...,x_n)d_1(y_1,y_2,...,y_n) = 0$ for all $x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n \in N$. Then (N,+) is an abelian group .

(ii) If $d_2(x_1,x_2,...,x_n)d_1(y_1,y_2,...,y_n) + d_1(x_1,x_2,...,x_n)d_2(y_1,y_2,...,y_n) = 0$ for all $x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n \in N$. Then (N,+) is an abelian group .

Corollary 3.12 Let N be a prime near-ring, let g_1 be a nonzero right generalized derivation associated with right derivation d_1 of N, and g_2 be a nonzero generalized derivation associated with derivation d_2 of N .

(i) If $g_1(x)d_2(y) + g_2(x)d_1(y) = 0$ for all $x,y \in N$. Then (N,+) is an abelian group .

(ii) If $g_2(x)d_1(y) + g_1(x)d_2(y) = 0$ for all $x,y \in N$. Then (N,+) is an abelian group .

Theorem 3.13 Let N be a prime near-ring, let g be a nonzero generalized right n-derivation associated with nonzero right n-derivation d of N ,If $g([x,y],x_2,...,x_n) = 0$ for all $x,y,x_2,\dots,x_n \in N$. Then N is a commutative ring .

Proof. By hypothesis, we have

$g([x,y],x_2,...,x_n) = 0$ for all $x,y,x_2,\dots,x_n \in N$. (14)

Replace y by xy in (14) to get

$g([x,xy],x_2,...,x_n) = 0$ for all $x,y,x_2,\dots,x_n \in N$.

Which implies that $g(x[x,y],x_2,...,x_n) = 0$ for all $x,y,x_2,\dots,x_n \in N$.

Therefore

$d(x,x_2,...,x_n)[x,y] + g([x,y],x_2,...,x_n)x = 0$ for all $x,y,x_2,\dots,x_n \in N$.

Using (14) in previous equation we get

$d(x,x_2,...,x_n)[x,y] = 0$ for all $x,y,x_2,\dots,x_n \in N$, or equivalently

$d(x,x_2,...,x_n)xy = d(x,x_2,...,x_n)yx$

for all $x,y,x_2,\dots,x_n \in N$. (15)

Replacing y by yz in (15) and using it again, we get

$d(x,x_2,...,x_n)y[x,z] = 0$ for all $x,y,z,x_2,\dots,x_n \in N$.

Hence we get

$d(x,x_2,...,x_n)N[x,z] = \{0\}$ for all $x,z,x_2,\dots,x_n \in N$ (16)

This yields that

for each fixed $x \in N$

either $d(x,x_2, . . .,x_n) = 0$ for all $x_2,\ldots,x_n \in N$ or $x \in Z$. (17)

If $d(x, x_2, . . ., x_n) = 0$ for all $x_2,\ldots,x_n \in N$ and for each fixed $x \in N$. We get $d = 0$, leading to a contradiction as d is a nonzero right n-derivation of N . Therefore there exist $x_1,x_2,...,x_n \in N$, all being nonzero such that $d(x_1,x_2,\cdots,x_n) \neq 0$ such that $x_1 \in Z$.

Since $x_1 \in Z$, we conclude that $[x_1y,z] = x_1[y, z]$, where $y,z \in N$, by hypothesis we get

$g([x_1y, z] , x_2, . . ., x_n) = 0$

This implies that

$0 = g(x_1[y, z],x_2, . . .,x_n)$

$= d(x_1,x_2, . . .,x_n)[y, z] + g([y, z],x_2, . . .,x_n)x_1$

$= d(x_1,x_2, . . .,x_n)[y, z]$ for all $y,z,x_2,\ldots,x_n \in N$.

Which implies that

$d(x_1, x_2, . . ., x_n)yz = d(x_1, x_2, . . ., x_n)zy$ for all $y,z,x_2,\ldots,x_n \in N$.

Replacing z by zt, where $t \in N$, in previous equation, and using it again, we get

$d(x_1, x_2, . . ., x_n)z[y, t] = 0$ for all $y,t,z,x_2,\ldots,x_n \in N$, i.e.;

$d(x_1,x_2, . . .,x_n)N[y, t] = \{0\}$ for all $y,t,x_2,\ldots,x_n \in N$, since $d(x_1,x_2,\cdots,x_n) \neq 0$, primeness of N implies that N = Z .

By Lemma 2.3, we conclude that N is a commutative ring .

Corollary 3.14 [6,Theorem 3.11] Let N be a prime near-ring, let d be a nonzero right n-derivation d ,If $d([x , y],x_2, . . .,x_n) = 0$ for all $x,y,x_2,\ldots,x_n \in N$, then N is a commutative ring .

Corollary 3.15 Let N be a prime near-ring . Let g be a nonzero generalized right derivations associated with nonzero right derivation d of N , If $g[x , y] = 0$ for all $x, y \in N$. Then N is a commutative ring .

Theorem 3.16 Let N be a 2-torsion free prime near-ring . Then N admits no nonzero generalized right n-derivation g associated with nonzero right derivation d such that $g(xoy, x_2, . . ., x_n) = 0$ for all $x, y, x_2,\ldots,x_n \in N$.

Proof. Assume that

$g(xoy,x_2,...,x_n) = 0$ for all $x,y,x_2,\ldots,x_n \in N$. (18)

Replace y by xy in (18) to get

$g(xoxy,x_2,...,x_n) = 0$ for all $x,y,x_2,\ldots,x_n \in N$.

Which implies that $g(x(xoy),x_2,...,x_n) = 0$ for all $x,y,x_2,\ldots,x_n \in N$.

$d(x,x_2,...,x_n)(xoy) + g(xoy,x_2,...,x_n)x = 0$ for all $x,y,x_2,\ldots,x_n \in N$.

Using (18) in previous equation we get

$d(x, x_2, ..., x_n)(xoy) = 0$ for all $x, y, x_2,\ldots,x_n \in N$, or equivalently ,

$d(x,x_2,...,x_n)yx = - d(x,x_2,...,x_n)xy$

for all $x,y,x_2,\ldots,x_n \in N$. (19)

Replacing y by yz, where $z \in N$, in (19) we get

$d(x,x_2,...,x_n)yzx = - d(x,x_2,...,x_n)xyz$

$= d(x,x_2,...,x_n)xy(-z)$

$= d(x, x_2, ..., x_n)y(-x)(-z)$ for all $x,y,z,x_2,\ldots,x_n \in N$.

In last equation using the fact

$- d(x,x_2,...,x_n)yzx = d(x,x_2,...,x_n)yz(-x)$ for all $x,y,z,x_2,\ldots,x_n \in N$, implies

$d(x, x_2, ..., x_n)yz(-x) - d(x, x_2, ..., x_n)y(-x)z = 0$

for all $x,y,z,x_2,\ldots,x_n \in N$, which implies that

$d(x,x_2,...,x_n)y[-x, z] = 0$ for all $x,y,z,x_2,\ldots,x_n \in N$.

Replacing x by $-x$ in previous equation we get

$d(-x,x_2,...,x_n)y[x, z] = 0$ for all $x,y,z,x_2,\ldots,x_n \in N$.

Hence we get

$d(-x,x_2,...,x_n)N[x, z] = \{0\}$ for all $x,z,x_2,\ldots,x_n \in N$. (20)

This yields that

For each fixed $x \in N$ either $d(-x, x_2, ..., x_n) = 0$ for all $x_2,\ldots,x_n \in N$ or $x \in Z$.

Since $d(x, x_2, \ldots, x_n) = -d(-x, x_2, \ldots, x_n) = 0$, so we get :

for each fixed $x \in N$ either $d(x, x_2, \ldots, x_n) = 0$ for all $x_2,\ldots,x_n \in N$ or $x \in Z$.

If $d(x, x_2, \ldots, x_n) = 0$ for all $x_2,\ldots,x_n \in N$ and for each fixed $x \in N$, we get $d = 0$, leading to a contradiction as d is a nonzero right n-derivation of N . Therefore there exist $x_1,x_2,\ldots,x_n \in N$, all being nonzero such that $d(x_1,x_2,\cdots,x_n) \neq 0$ and $x_1 \in Z$. Since $x_1 \in Z$, we conclude that $(x_1 y o z) = x_1(y o z)$, where $y,z \in N$.

By hypothesis we get

$g(x_1 yoz,x_2, \ldots,x_n) = 0$ for all $x_1,y,z,x_2,\ldots,x_n \in N$.

Therefore

$0 = g(x_1(yoz),x_2, \ldots,x_n)$

$= d(x_1,x_2, \ldots,x_n)(yoz) + g(yoz,x_2, \ldots,x_n)x_1$

$= d(x_1,x_2, \ldots,x_n)(yoz)$ for all $y, z, x_2,\ldots,x_n \in N$.

Which implies that

$d(x_1,x_2, \ldots,x_n)yz = -d(x_1,x_2, \ldots,x_n)zy$ for all $y,z,x_2,\ldots,x_n \in N$. Replacing z by zt, where $t \in N$, in previous equation and using it again, we get $d(x_1, x_2, \ldots, x_n)z[y, t] = 0$ for all $x_1,x_2,\ldots,x_n,y,z,t \in N$. i.e.; $d(x_1, x_2, \ldots, x_n)N[y, t] = \{0\}$ for all $x_1,x_2,\ldots,x_n,y,z,t \in N$. Since $d(x_1,x_2,\cdots,x_n) \neq 0$, primeness of N implies that N = Z. So we conclude that N is a commutative ring in view of Lemma 2.3 . In this case, return to hypothesis we find that $2d(xy,x_2,\ldots,x_n) = 0$ for all $x,y,x_2,\ldots,x_n \in N$.

Since N is 2-torsion free we get $d(xy,x_2,\ldots,x_n) = 0$ for all $x,y,x_2,\ldots,x_n \in N$, hence we get

$d(x,x_2,\ldots,x_n)y + d(y,x_2,\ldots,x_n)x = 0$ for all $x,y,x_2,\ldots,x_n \in N$, replacing x by zx, where $z \in N$, in previous equation yields

$d(zx,x_2,\ldots,x_n)y + d(y,x_2,\ldots,x_n)zx = 0$ for all $x,y,z,x_2,\ldots,x_n \in N$.

Which implies that

$d(y,x_2,\ldots,x_n)zx = 0$ for all $x,y,z,x_2,\ldots,x_n \in N$.

Which means that

$d(y,x_2,\ldots,x_n)Nx = \{0\}$ for all $x,y,x_2,\ldots,x_n \in N$. Since N is prime and $d \neq 0$, we conclude that $x = 0$ for all $x \in N$, a contradiction .

Theorem 3.17 Let N be a prime near-ring , let g be generalized n-derivation associated with right n-derivation d of N . If $[g(x,x_2,...,x_n), y] \in Z$ for all $x,y,x_2,\ldots,x_n \in N$. Then N is a commutative ring .

Proof. Assume that

$[g(x, x_2, . . ., x_n) , y] \in Z$ for all $x,y,x_2,\ldots,x_n \in N$. (21)

Hence $[[g(x, x_2, . . ., x_n), y], t] = 0$ for all $x,y,t, x_2,\ldots,x_n \in N$. (22)

Replacing y by $g(x,x_2, . . .,x_n)y$ in (22) we get

$[g(x,x_2, . . .,x_n)[g(x,x_2, . . .,x_n), y], t] = 0$

for all $x,y,t,x_2,\ldots,x_n \in N$. (23)

In view of (21), equation (23) assures that

$[g(x,x_2, . . .,x_n), y] N[g(x,x_2, . . .,x_n), t] = \{0\}$

for all $x,y,t,x_2,\ldots,x_n \in N$. (24)

Primeness of N implies that

$[g(x, x_2, . . ., x_n) , y] = 0$ for all $x, y, x_2,\ldots,x_n \in N$.

Hence $g(N,N,...,N) \subseteq Z$ and application of Theorem 3.1 assures that N is a commutative ring .

Corollary 3.18 [6, Theorem 3.15] Let N be a prime near-ring , let d be right n-derivation . If $[d(x,x_2, . . .,x_n), y] \in Z$ for all $x,y,x_2,\ldots,x_n \in N$. Then N is a commutative ring .

Corollary 3.19 Let N be a prime near-ring, let g be a right generalized derivation associated with right derivation d of N . If $[g(x), y] \in Z$ for all $x, y \in N$. Then Nis a commutative rings .

Theorem 3.20 Let N be a prime near-ring , let g benonzero generalized right n-derivation associated with right n-derivation d of N . If $g(x,x_2, . . .,x_n)oy \in Z$ for all $x,y,x_2,\ldots,x_n \in N$. Then N is a commutative ring .

Proof. Assume that

$g(x,x_2, . . .,x_n)oy \in Z$ for all $x,y,x_2,\ldots,x_n \in N$. (25)

(a) If Z = 0, then equation (25) reduces to

$yg(x,x_2, . . .,x_n) = - (g(x,x_2, . . .,x_n)y) = g(x,x_2, . . .,x_n)(-y)$

for all $x,y,x_2,\ldots,x_n \in N$. (26)

Substituting zy for y in (26) and using it again , we obtain

$zyg(x,x_2,...,x_n) = -(g(x, x_2, ..., x_n)zy)$

$= g(x, x_2, ..., x_n)z(-y)$

$= zg(-x, x_2, ..., x_n))(-y)$ for all $x,y,z,x_2,…,x_n \in N$.

Using the fact $-zyg(x, x_2, ..., x_n) = zyg(-x, x_2, ..., x_n)$ in previous equation, implies that

$zyg(-x,x_2,...,x_n) = zg(-x,x_2,...,x_n))y$ for all $x,y,z,x_2,…,x_n \in N$.

Which implies that

$z(yg(-x,x_2,...,x_n) - g(-x,x_2,...,x_n)y) = 0$

for all $x,y,z,x_2,…,x_n \in N$. (27)

Taking -x instead of x in (27), we get

$zN(yg(x, x_2, ..., x_n) - g(x, x_2, ..., x_n)y) = \{0\}$

for all $x, y, z,x_2,…,x_n \in N$. (28)

Primeness of N implies that

$g(N,N,...,N) \subseteq Z$ and application of Theorem 3.1 assures that N is a commutative ring .

(b) Suppose that $Z \neq 0$, then there exists $0 \neq z \in Z$, by hypothesis we have $g(x,x_2,...,x_n)oz \in Z$ for all $x,x_2,…,x_n \in N$.

Thus we get $g(x,x_2,...,x_n)z + zg(x,x_2,...,x_n) \in Z$ for all $x,x_2,…,x_n \in N$. Since $z \in Z$, we get

$z(g(x, x_2, ..., x_n) + g(x, x_2, ..., x_n)) \in Z$ for all $x,x_2,…,x_n \in N$.

By Lemma 2.2 we conclude that

$g(x,x_2,...,x_n) + g(x,x_2,...,x_n) \in Z$ for all $x,x_2,…,x_n \in N$ (29)

By (25), we get

$g(x+x,x_2,...,x_n)y + yg(x+x,x_2,...,x_n) \in Z$ for all $x,x_2,…,x_n \in N$. (30)

Using equation (29) in (30) we conclude that

$y(g(x+x,x_2,...,x_n) + g(x+x,x_2,...,x_n)) \in Z$ for all $x,y,z,x_2,…,x_n \in N$.(31)

Therefore , for all $x,y,t,x_2,\ldots,x_n \in N$, we get

$ty(g(x+x,x_2,...,x_n)+g(x+x,x_2,...,x_n)) = y(g(x+x,x_2,...,x_n)+g(x+x,x_2,...,x_n))t = (g(x+x,x_2,...,x_n)+g(x+x,x_2,...,x_n))yt$.

Which implies that

$(g(x+x,x_2,...,x_n)+g(x+x,x_2,...,x_n))N[t,y]=\{0\}$ for all $x, y, t, x_2,\ldots,x_n \in N$. (32)

Primeness of N implies that

either $g(x+x, x_2, ..., x_n) + g(x+x, x_2, ..., x_n) = 0$ and thus $g = 0$, a contradiction , or $N = Z$, hence $g(N,N,\ldots,N) \subseteq Z$ and application of Theorem 3.1 assures that N is a commutative ring

Corollary 3.21 [6, Theorem 3.17] Let N be a prime near-ring , let d be nonzero right n-derivation . If $d(x,x_2,...,x_n)oy \in Z$ for all $x,y,x_2,\ldots,x_n \in N$. Then N is a commutative ring .

Corollary 3.22 Let N be a prime near-ring , let gbe generalized right derivation associated right derivation d of N . If $g(x)oy \in Z$ for all $x, y \in N$. Then Nis a commutative ring .

The following example demonstrates that N to be prime is essential in the hypothesis of our results

Example 3.23 Let S be a 2-torsion free zero-symmetric left near-ring . Let us define :

$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, x, y, 0 \in S \right\}$ is zero symmetric left near-ring with regard to matrix addition and matrix multiplication .

Define $d_1, g_1, d_2, g_2 : \underbrace{N \times N \times \ldots \times N}_{n-times} \longrightarrow N$ such that

$$d_1\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & x_1 x_2 \ldots x_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$g_1\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & y_1 y_2 \ldots y_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$d_2\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & x_1 x_2 \ldots x_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$g_2\left(\begin{pmatrix}0 & x_1 & y_1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\begin{pmatrix}0 & x_2 & y_2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ldots,\begin{pmatrix}0 & x_n & y_n\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right)=\begin{pmatrix}0 & 0 & y_1y_2\cdots y_n\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

It can be easily seen that g_1 ,g_2 are nonzero right generalized n-derivations associated with right n-derivation d_1, d_2 respectively of near-ring N which is not prime . We also have

(i) Let $A\epsilon N$, $A=\begin{pmatrix}0 & x & y\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ such that x, y ≠ 0, we can see that $g_1(N,N,...,N)A = 0$. However $A \neq 0$ and

(ii) $g_1(N,N,...,N) \subseteq Z$.

(iii) $[g_1(N, N, \ldots, N) , g_2(N, N, \ldots, N)] = \{0\}$,

(iv) $g_1(A_1,A_2,\ldots,A_n)g_2(B_1,B_2,\ldots,B_n) + g_2(A_1,A_2,\ldots,A_n)g_1(B_1,B_2,\ldots,B_n) = 0$ for all $A_1,A_2,\ldots,A_n, B_1,B_2,\ldots,B_n \epsilon N$.

(v) $A_1g_1(B_1,B_2,\ldots,B_n) = g_1(A_1,A_2,\ldots,A_n)B_1$ for all $A_1,A_2,\ldots,A_n,B_1,B_2,\ldots,B_n \epsilon N$.

(vi) $g_1([A , B],A_2,\ldots,A_n) = 0$ for all $A,B,A_2,\ldots,A_n \epsilon N$.

(vii) $g_1(AoB,A_2,\ldots,A_n) = 0$ for all $A,B,A_2,\ldots,A_n \epsilon N$.

However (N , +) is not abelain group .

References

[1] Pilz G. Near-Rings. 2nded. North Holland .American Elsevier. Amsterdam 1983.

[2] Wang XK. Derivations in prime near-rings. Proc. Amer. Math. Soc. 1994; 121 (2) : 361–366.

[3] Asraf A, Siddeeque MA. On permuting n-derivations in near-rings. Commun. Kor. Math. Soc. 2013; 28 (4) : 697–707.

[4]Ashraf A, Siddeeque MA. On (σ,τ)- n-derivations in near-rings. Asian-European Journal of Mathematics. 2013: 6 (4) 1-14.

[5] Ashraf M, Siddeeque MA. On generalized n-derivations in near-rings.Palestine Journal of Mathematics. 2014; 3(1): 468–480.

[6] Majeed AH, Adhab EF. Right n-derivations in prime Near–Rings. Journal Al-Qadisiyah / Pure Sciences.2016;2 (3): 31-41

[7] Bell HE, Mason G. On derivations in near-rings, Near-rings and Near-fields, ed. G. Betsch. (North-Holland/American Elsevier, Amsterdam).1987; 137 : 31–35.

[8] Bell HE. On Derivations in Near-Rings II. Near-rings, Near-fields and k-loops. Kluwer Academic Publishers. Dordrecht. 1997; 426: 191–197.

[9] Ashraf A,Sideeque MA, Parveen N.On semigroup ideals and n-derivations in near-rings. Science Direct Journal of Taibah University for Science.2015 (9): 126–132.

APPLYING SUBORDINATION ON SUBCLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH RUSCHEWEYA OPERATOR

BY

MOHAMMAD FALIH HANI AL'MURSHIDY

Email:mohfalih17@yahoo.com

ABSTRACT. In the present paper, we introduce and investigate an interesting subclass $\mathcal{B}_{\Sigma}^{p,q}(\lambda)$ of the function class Σ of bi-univalent functions defined in the open unit disk U, which are associated with the Ruscheweya operator, satisfying subordinate conditions, furthermore we find estimate on the Taylor – Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this subclass.

KEY WORKS AND PHRASES. Subordination, analytic functions, univalent functions, Bi-univalent functions, Taylor – Maclaurin series, coefficients estimates Ruscheweya operator.

AMS Subject classifications : 30C50

1- INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \ and \ |z| < 1\}$. Further, by S we shall denote the class of all functions in $\mathcal{A}$ which are univalent in U.

It is well know that every function $f \in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, \quad (z \in U),$$

and

$$f^{-1}(f(w)) = w, \quad \left(|w| < r_0(f);\ r_0(f) \geq \tfrac{1}{4}\right),$$

where :

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots \ . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi - univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of bi – univalent functions in U given by (1) (see [8]).

For a brief history and interesting examples of function in the class Σ (see [3,8]) and the references there in.

The study of the coefficient problems involving bi - univalent functions was revived recently by Brannan and Taha [1], Srivastava et al. [8], Xu et al. [9], Jothibasu [4], Bulut [2] and Magesh et al. [5]. Various subclasses of the bi - univalent function class Σ were introduced and non - sharp estimates on the first two Taylor –Maclaurin coefficients $|a_2|$ and $|a_3|$. But the coefficient problem for each of the following Tayler Maclaurin coefficients $|a_n|(n \in N\backslash\{1,2\}; N = \{1,2,3, \dots\})$ is still an open problem.

The object of the present paper is to introduce new subclass of the function class Σ associated with Ruscheweyh Derivative operator, where Ruscheweyh [7] observed that

$$D^n f(z) = \frac{z(z^{n-1} f(z))^n}{n!} \ , \tag{3}$$

for $n \in N_0 = \{0,1,2 \dots\}$. This symbol $D^n f(z), n \in N_0$ is called the n[th] order Ruscheweya operator of $f(z)$.

We not that $D^0 f(z) = f(z), D^1 f(z) = zf'(z)$ and

$$D^n f(z) = z + \sum_{k=2}^{\infty} \sigma(n,k) a_k z^k, \qquad (4)$$

where

$$\sigma(n,k) = \binom{n+k-1}{n},$$

and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these subclass of the function class Σ employing the techniques used by Bulut [2] and Magesh et. al. [5].

An analytic function f is subordinate to an analytic function g written as $f(z) \prec g(z) \quad (z \in U)$, if there is an analytic function W defined on U with $W(0) = 0$ and $|W(z)| < 1, \quad z \in U$ such that $f(z) = g(W(z))$. In particular, if g is univalent in U then we have the following equivalence :

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

DEFENTION 1. Let $f \in \mathcal{A}$ and the functions $p, q: U \to \mathbb{C}$ be so constrained that min $\{R(p(z), R(q(z)\} > 0, (z \in U)$ and $p(0) = q(0) = 1$. We say that $f \in \mathcal{B}_{\Sigma}^{p,q}(\lambda)$ if the following conditions are satisfied :

$$f \in \Sigma, \frac{z(D^n f(z))'}{(1-\lambda)D^n f(z) + \lambda z(D^n f(z))'} \prec p(z), \qquad (0 \le \lambda < 1; z \in U), \qquad (5)$$

and

$$g \in \Sigma, \frac{w(D^n g(w))'}{(1-\lambda)D^n g(w) + \lambda w(D^n g(w))'} \prec q(w), \qquad (0 \le \lambda < 1; w \in U), \qquad (6)$$

where the function g is the extension of f^{-1} to U is defined by (2).And

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots$$

$$q(w) = 1 + q_1 w + q_2 w^2 + \ldots \quad ,$$

Remark 1. There are many choices of the functions p and q which would provid interesting know and new subclasses of the analytic function class $\mathcal{A}$. For example, if we set

$$p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \text{ and } q(w) = \left(\frac{1-w}{1+w}\right)^{\alpha} \quad , \quad (0 < \alpha \le 1, \ z, w \in U) \, ,$$

in the class $\mathcal{B}_{\Sigma}^{p,q}(\lambda)$ then we have $\mathcal{B}^{p,q} S_{\Sigma}^{*}(\alpha, \lambda)$. Also $f \in \mathcal{B}^{p,q} S_{\Sigma}^{*}(\alpha, \lambda)$ if the following conditions are satisfied :

$$f \in \Sigma, \left| arg\left(\frac{z(D^n f(z))'}{(1-\lambda)D^n f(z) + \lambda z(D^n f(z))'}\right)\right| < \frac{\alpha\pi}{2} \ \ (0 < \alpha \le 1; 0 \le \lambda < 1; z \in U), \quad (7)$$

and

$$\left| arg\left(\frac{w(D^n g(w))'}{(1-\lambda)D^n g(w) + \lambda w(D^n g(w))'}\right)\right| < \frac{\alpha\pi}{2} \ \ (0 < \alpha \le 1; 0 \le \lambda < 1; w \in U), \qquad (8)$$

where g is the extension of f^{-1} to U.

Similarly, if we set

$$p(z) = \frac{1 + (1-2\beta)z}{1-z} \text{ and } q(w) = \frac{1-(1-2\beta)w}{1+w} \quad (0 \le \beta < 1; \ z, w \in U),$$

in the class $\mathcal{B}_{\Sigma}^{p,q}(\lambda)$ then we get $\mathcal{B}^{p,q} S_{\Sigma}^{*}(\beta, \lambda)$. Further, we say that $f \in \mathcal{B}^{p,q} S_{\Sigma}^{*}(\beta, \lambda)$ if the following conditions are satisfied :

$$f \in \Sigma, R\left(\frac{z(D^n f(z))'}{(1-\lambda)D^n f(z) + \lambda z(D^n f(z))'}\right) > \beta \ \ (0 \le \beta < 1; 0 \le \lambda < 1; z \in U), \qquad (9)$$

and

$$R\left(\frac{w(D^n g(w))'}{(1-\lambda)D^n g(w)+\lambda w(D^n g(w))'}\right)>\beta \ \ (0\le\beta<1; 0\le\lambda<1; w\in U), \tag{10}$$

where g is the extension of f^{-1} to U.

We note that :

i) For $\lambda=0, \mathcal{B}_{\Sigma}^{p,q}(0)=B_{\Sigma}^{p,q}$ was introduce and studied by Bulut [2].

ii) For $n=0$, the class $\mathcal{B}_{\Sigma}^{p,q}(\lambda)=\mathcal{G}_{\Sigma}^{p,q}(\lambda)$ was introduced and studied by Magesh et al. [5].

iii) The class $\mathcal{B}^{p,q}S_{\Sigma}^{*}(\alpha,\lambda)=S_{\Sigma}^{*}(\alpha,\lambda)$ and $\mathcal{B}^{p,q}S_{\Sigma}^{*}(\beta,\lambda)=S_{\Sigma}^{*}(\beta,\lambda)$ were introduced and studied by Murugusundaramoothy et al. [6].

iv) The class $\mathcal{B}^{p,q}S_{\Sigma}^{*}(\alpha,0)=S_{\Sigma}^{*}(\alpha)$ and $\mathcal{B}^{p,q}S_{\Sigma}^{*}(\beta,0)=S_{\Sigma}^{*}(\beta)$ are strongly bi - starlike functions of order α and bi - starlike functions of order β respectively, were introduced and studied by Braman and Taha [1].

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{B}_{\Sigma}^{p,q}(\lambda)$.

2- A SET OF GENERAL COEFFICIENT ESTIMATES

THEOREM 1. Let $f(z)$ be of the form (1). If $f\in\mathcal{B}_{\Sigma}^{p,q}(\lambda)$, then

$$|a_2|\le min\left\{\frac{\sqrt{|p'(0)|^2+|q'(0)|^2}}{\sqrt{2}(n+1)(1-\lambda)},\sqrt{\frac{|p''(0)|+|q''(0)|}{4(n+1)(1-\lambda)\big(1-\lambda(n+1)\big)}}\right\}, \tag{11}$$

and

$$|a_3|\le min\left\{\frac{|p'(0)|^2+|q'(0)|^2}{2(n+1)^2(1-\lambda)^2}+\frac{|p''(0)|+|q''(0)|}{4(n+1)(n+2)(1-\lambda)},\frac{|p''(0)|+|q''(0)|}{4(n+1)(n+2)(1-\lambda)}+\frac{|p''(0)|+|q''(0)|}{4(n+1)(1-\lambda)\big(1-\lambda(n+1)\big)}\right\}. \tag{12}$$

PROOF. Since $f(z)\in\mathcal{B}_{\Sigma}^{p,q}(\lambda)$. From (5) and (6), we have

$$\frac{z(D^n f(z))'}{(1-\lambda)D^n f(z)+\lambda z(D^n f(z))'}=p(z), \quad (z\in U), \tag{13}$$

and

$$\frac{w(D^n g(w))'}{(1-\lambda)D^n g(w)+\lambda w(D^n g(w))'}=q(w), \quad (w\in U), \tag{14}$$

where

$$p(z)=1+p_1z+p_2z^2+\ldots$$
$$q(w)=1+q_1w+q_2w^2+\ldots \ ,$$

satisfy the conditions of Definition 1.

Now, upon equaliting the coefficients in (13) and (14), we get

$$(n+1)(1-\lambda)a_2=p_1 \tag{15}$$
$$(n+1)(n+2)(1-\lambda)a_3-(n+1)^2(1-\lambda^2){a_2}^2=p_2 \tag{16}$$
$$-(n+1)(1-\lambda)a_2=q_1 \tag{17}$$
$$(n+1)(n+2)(1-\lambda)(2{a_2}^2-a_3)-(n+1)^2(1-\lambda^2){a_2}^2=q_2. \tag{18}$$

From (15) and (17), we get

$$p_1=-q_1 \quad , \tag{19}$$

and

$${p_1}^2+{q_1}^2=2(n+1)^2(1-\lambda)^2{a_2}^2. \tag{20}$$

Adding (16) to (18), we obtain

$$2(n+1)(1-\lambda)\big(1-\lambda(n+1)\big)a_2{}^2 = p_2 + q_2. \qquad (21)$$

Therefore, we find from the equations (20) and (21) that

$$a_2{}^2 = \frac{p_1{}^2 + q_1{}^2}{2(n+1)^2(1-\lambda)^2}, \qquad (22)$$

and

$$a_2{}^2 = \frac{p_2 + q_2}{2(n+1)(1-\lambda)\big(1-\lambda(n+1)\big)}, \qquad (23)$$

respectively. Since $p(z) \in p(U)$ and $q(w) \in q(U)$, we have

$$|a_2|^2 \le \frac{|p'(0)|^2 + |q'(0)|^2}{2(n+1)^2(1-\lambda)^2} \;, \qquad (24)$$

and

$$|a_2|^2 \le \frac{|p''(0)| + |q''(0)|}{4(n+1)(1-\lambda)\big(1-\lambda(n+1)\big)} \;, \qquad (25)$$

respectively. So we get the desired estimate on the coefficient on $|a_2|$ as asserted in (11).

Next, in order to find the bound on $|a_3|$, by subtracing (18) from (16), we get

$$2(n+1)(n+2)(1-\lambda)a_3 - 2(n+1)(n+2)(1-\lambda)a_2{}^2 = p_2 - q_2\,. \qquad (26)$$

Upon substituting the value of $a_2{}^2$ from (22) and (23) into (26), we have

$$a_3 \;\; = \frac{p_1{}^2 + q_1{}^2}{2(n+1)^2(1-\lambda)^2} + \frac{p_2 - q_2}{2(n+1)(n+2)(1-\lambda)}, \qquad (27)$$

and

$$a_3 = \frac{p_2 - q_2}{2(n+1)(n+2)(1-\lambda)} + \frac{p_2 + q_2}{2(n+1)(1-\lambda)\big(1-\lambda(n+1)\big)} \;, \qquad (28)$$

respectively. Since $p(z) \in p(U)$ and $q(w) \in q(U)$. We readily get

$$|a_3| \le \frac{|p'(0)|^2 + |q'(0)|^2}{2(n+1)^2(1-\lambda)^2} + \frac{|p''(0)| + |q''(0)|}{4(n+1)(n+2)(1-\lambda)} \;\;, \qquad (29)$$

and

$$|a_3| \le \frac{|p''(0)| + |q''(0)|}{4(n+1)(n+2)(1-\lambda)} + \frac{|p''(0)| + |q''(0)|}{4(n+1)(1-\lambda)\big(1-\lambda(n+1)\big)}, \qquad (30)$$

respectively. So we get the desired estimate on the coefficient on $|a_3|$ as asserted in (12).

This complete the proof of Theorem 1.

We choose

$$p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \text{ and } q(w) = \left(\frac{1-w}{1+w}\right)^{\alpha} \;, \quad (0 < \alpha \le 1; z, w \in U),$$

in Theorem 1, we have the following corollary

<u>COROLLARY 1.</u> Let $f(z)$ be of the form (1) and in the class $\mathcal{B}^{p,q}S_{\Sigma}^{*}(\alpha,\lambda)$.

Then

$$|a_2| \le min\left\{\frac{2\alpha}{(n+1)(1-\lambda)}, \frac{2\alpha}{\sqrt{2(n+1)(1-\lambda)(1-\lambda(n+1)}}\right\},$$

and

$$|a_3| \le min\left\{\frac{4\alpha^2}{(n+1)^2(1-\lambda)^2}, \frac{2\alpha^2}{(n+1)(1-\lambda)(1-\lambda(n+1)}\right\}.$$

If we choose

$$p(z) = \frac{1+(1-2\beta)z}{1-z} \text{ and } q(w) = \frac{1-(1-2\beta)w}{1+w}, \quad (0 \le \beta < 1, z, w \in U),$$

in Theorem 1, we readily have the following Corollary

COROLLARY 2. Let $f(z)$ be of the form (1) and in the class $\mathcal{B}^{p,q}S_{\Sigma}^{*}(\beta,\lambda)$. Then

$$|a_2| \le min\left\{\frac{2(1-\beta)}{(n+1)(1-\lambda)}, \frac{2\sqrt{1-\beta}}{\sqrt{2(n+1)(1-\lambda)\big(1-\lambda(n+1)\big)}}\right\},$$

and

$$|a_3| \le min\left\{\frac{4(1-\beta)^2}{(n+1)^2(1-\lambda)^2}, \frac{2(1-\beta)}{(n+1)(1-\lambda)(1-\lambda(n+1))}\right\}.$$

Remark 2. The estimates on the Coefficients $|a_2|$ and $|a_3|$ of Corollaries 1 and 2 are improvement of the estimates obtained in [5]. Taking $\lambda = 0$ in Corollaries 1 and 2, the estimates of the Coefficients $|a_2|$ and $|a_3|$ are improvement of the estimates in [2].Similarly, various other interesting corollaries and consequences of our main result can be derived by choosing different p and q.

REFERENCES

[1] D. A. Brannan and T.S. Taha, On some classes of bi - univalent functions, studia Unviersitatis Babes - Bolyai, Mathmatical, 31, (1986), 70 – 77.

[2] S. Bulut, Coefficient estimates for a class of analytic and bi - univalent functions, Novi Sad Journal of Mathmatics, 43, (2013), 59 – 65.

[3] B.A. Farsin and M. K. Aouf., New subclasses of analytic and bi - univalent functions, Appl. Math. Lr., 24, (2011), 1569 – 1573.

[4] J.Jothibasu, Certain subclasses of bi - univalent functions defined by Sãlãgean operator, EJMAA, Vol. 3(1) Jan. 2015, pp, 150 - 157.

[5] N. Magesh and J. Yamini, Coefficient estimates for a certain genral subclass of analytic and bi -univalent functions, Applied Mathmatics, (2014), 5, 1047 – 1052.

[6] G. Murugusundaramoorthy, N. Magesh and V. parameela, Coefficient Bounds for certain of subclasses of bi - univalent founctions, Abstract and Applied analytics, (2013), article ID : 573017, 3 pages.

[7] S. Rucheweyh, New criteria for univalent functions, proc. Amer. Math. Soc., (1975) pp. 109 – 115.

[8] H.M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi - univalent functions, Appl. Math. Lr. 23, (2010), 1188 – 1192.

[9] Q.H., Xu, Y. C. Gui. and H.M. Srivadava, Coefficient estimates for a certain subclass of analytic and bi - univalent functions, Appl. Math. Lett. 25 (2012), 990 – 994.

Method of creating a digital watermark for official documents

[1]Samer Saud Dakhel , [1]Zainab saad Rubaidi , [1]Rasha Muhseen Hadi
[1]College of Agriculture /Al-Muthanna University

Abstract

In this paper we have proposed a method for creating the watermark for official documents. For example, which are placed when authentication the documents, as well as use for secure communication. We have described the procedure for the exchange of information between two parties. The watermark is configured in a way that does not allow an unauthorized person to detect the existence of the watermark and extract the embedded information, and the watermark showed its ability to work of the proposed method.

Keywords: *authenticity of electronic documents, steganography, stegosystem, stegocontainer, digital watermark* .

Introduction

The question of determining the authenticity of documents, is now rising increasingly. This is due to the increase in the volume of document circulation between organizations, as well as the development of document exchange technologies. In this regard, there are many different methods of protecting documents from forgery.

For example, the following methods can be used to protect paper documents:

- technological - the use of watermarks, a protective thread / fiber, a certain composition of colorants, etc .;
- graphic - protective mesh, irregular raster, corrosion, combination of fonts, etc .;
- chemical - the appearance of a color reaction when the document interacts with another substance, for example, water;

- physical - use of elements with holograms, substances with different magnetic properties, etc .;
- combined - all possible combinations of methods described above.

To protect electronic documents, it is possible to use:

- cryptographic method - use of electronic digital signature.
- steganographic methods [1,2] - embedding text in an electronic document (image) or embedding a digital watermark, as well as introducing special noise.

Typically, the use of methods to protect against falsification of physical documents is a costly procedure, because this requires the availability of specialized equipment and a certain skill in the implementation of these methods of protection.

To determine the authenticity of an electronic document signed by an electronic digital signature, it is necessary to resort to the services of a third party (services of certification centers), which in turn can be expensive and time-consuming, in addition, the user can not trust a third party.

Currently, most electronic documents are distributed in such formats as pdf, bmp, jpg and tif, i.e. in the form of an image, but using any compression algorithms. In view of the foregoing, it is possible to propose a universal way of determining the authenticity of both physical and electronic documents by introducing watermarks (for electronic documents) in them. Moreover, watermarks being introduced can be used both for proving the authenticity of a document and for embedding certain information in them, for example, for a banknote, you can build a series and number.

Currently, many methods of embedding information into an image are offered. For example, such spatial methods as:

- LSB (Last Significant Bit) method.

- random interval method.
- method of pseudo-random permutation (choice).
- The method of block hiding.

The above methods are the closest to the proposed method.

The essence of the LSB method is to replace the least significant bits in the image bytes (container) responsible for color coding by the bits of the hidden message. The main advantages of this method are:

1) the fact that the human eye in most cases is not able to notice changes in the least significant bits;

2) the simplicity of the method itself;

3) the ability to hide relatively large amounts of information in relatively small images.

The main drawback of the LSB method is its high sensitivity to container distortion.

To reduce this sensitivity, noise immunity coding is often used. In addition, the LSB method has low steganographic resistance to attacks of passive and active violators.

Unlike the LSB method, in which each bit of a hidden message is written to consecutive low-order bits, the random interval method allows random distribution of bits of this message across the container, resulting in a random distance between the two built-in bits of the hidden message. But there is a disadvantage of this method - the bits of the hidden message in the container are placed in the same sequence as in the most concealed message. Therefore, in order to avoid this drawback, one resorts to the method of pseudo-random permutation (choice), the essence of which is that by using the pseudo-random number generator a sequence of indices $j_1, j_2, ..., j_k$ is formed and the k bit of the message in pixel with the index j_k .

The essence of the method of block hiding is as follows: the original image is divided into (l_m) non-intersecting blocks $\Delta i \quad (1 \le i \le l_m)$ of arbitrary configuration, for each of which the parity bit $b(\Delta i) = \sum_{j \in \Delta_i}^{\bmod 2} LSB(C_j)$. In each block, one secret bit (M_i) is hidden.

If the parity bit is $b(\Delta i) \neq M_i$, then one of the least significant bits of the block Δi is inverted, resulting in $b(\Delta i) = M_i$. The selection of the block can occur pseudo-randomly using a stegano key.

In fact, this method has the same resistance to distortion as the methods described above, but in comparison with them it has a number of advantages:

Firstly, it is possible to modify the value of such a pixel in the block, the change of which will lead to a minimum change in the statistics of the container; secondly, the impact of the consequences of embedding secret data in the container can be reduced by increasing the block size.

It should be noted that methods of random interval, pseudo-random permutation (choice) and block concealment are a kind of complication of the LSB method.

The most ideologically close to the method proposed by the authors are, for example, the Patch Work method [3] and its modification PatchTrack [4].

The essence of the PatchWork method is as follows: firstly, in a pseudo-random manner, two pixels of the image are selected in accordance with the key. Then

the brightness value of one of them increases by a certain amount (from 1 to 5), the other - decreases by the same amount. This operation is repeated many times (about 10,000 times), then the sum of the values of all the differences is:

$$S_n = \sum_{i=1}^{n}((a_i + c) - (b_i - c)) = 2cn + \sum_{i=1}^{n}(a_i - b_i)$$

Where a_i, b_i are the brightness values of the two selected pixels in step i, c is the increment value by which the brightness changes at each step of the algorithm.

The mathematical expectation of the sum of the differences in the brightness values of the pixels in the unfilled container $\sum_{i=1}^{n}(a_i - b_i)$ is close to zero for a sufficiently large value of n. Thus, in the presence of the Digital watermark, the value of S_n is much greater than zero [3].

The main advantage of the PatchWork method is sufficient resistance to the operations of compression, truncation and changing the contrast of the image. The drawbacks of the method include its instability to affine transformations (rotation, shift, scaling), as well as its small bandwidth (20,000 pixels are required to transmit 1 bit of a hidden message). The main distinguishing feature of PatchTrack from PatchWork is the principle of forming a pseudo-random sequence, according to which pixels of the image are selected. A pseudo-random sequence is formed in such a way that the obtained values can be recovered after a series of geometric attacks [4].

Also, the proposed method is ideologically comparable with the technology of marking (yellow dots) with color laser printers on each page they print. Typically, the printer implements a code that contains information about its serial number, as well as the date and time the document was printed. In 2005, specialists from the Electronic Frontier Foundation deciphered the code that was printed by the Xerox DocuColor family of printers.

The principle of embedding a watermark

As a general rule, the implementation of the Digital Watermark is carried out to the area of the original image without using cumbersome calculations and due to certain transformations of image brightness and color components of the image (r, g, b). It should be noted that one of the important steps in constructing a stegosystem is to determine the optimal size of the Digital Watermark, which can be embedded in the source document (stegokonteiner) without compromising the system's reliability. In cases where it is not possible to determine the optimal size of the embedded Digital Watermark and there is a suspicion that the stegosystem will be less reliable, it is possible to resort to introducing "side noise", for example, additional points in the document.

1) Consider a rectangular region in the Cartesian coordinate system (X , Y), divided into MN squares (Fig. 1). Further, this rectangular region will be called a document (stegokonteyner).

2) We arbitrarily select (K) the squares, then we will call these squares "shaded". Selected squares should be the only ones in the column.

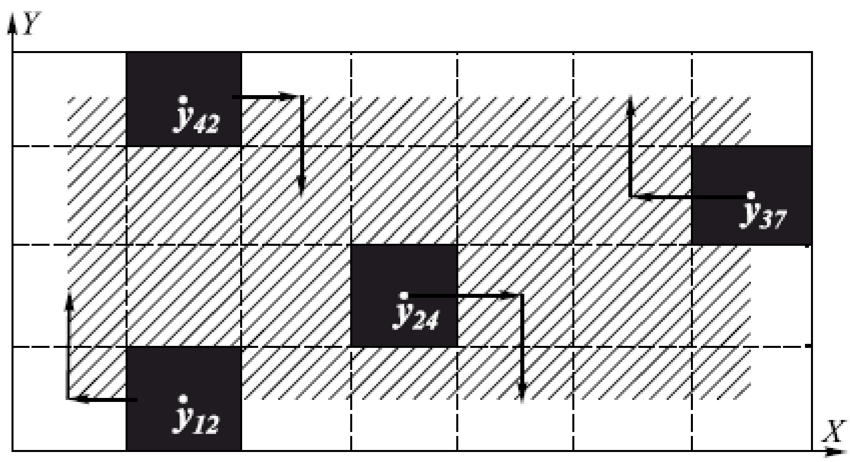


Fig.1. The rectangular region divided into MN squares

3) We calculate the sum of the distances R from the centers of these squares to any of the coordinate axes, for example, to the X axis :

$\sum_{i,j\in\Xi} y_{ij} = R$, Where Ξ is the set that indexes the selected squares, the pair (ij) are the natural indices of the squares in the total numbering, and y_{ij} are the coordinates of the centers of the squares along the Y axis. For example, for "shaded" (marked in black) squares in Fig. 1, sum of distances :

$$R = \sum_{i,j\in\Xi} y_{ij} = y_{12} + y_{42} + y_{24} + y_{36}$$

4) We will implement the introduction of "some noise" by creating additional displaced "shaded" squares.

When squares are displaced, it is necessary to take into account the condition that the original and obtained biased squares should be unique in the column.

With the help of pseudorandom value sensors, we form two sets of integers.

The set $I=(r_k, k=1,..,mes(\Xi))$ is formed in such a way that $\sum_k r_k = 0$ and r_k are selected without any restrictions on the value, but with the following conditions:

- If If ($x_{ij} + r_k$) > N, then $x_{ij}' = (x_{ij} + r_k) - N \cdot \left[\frac{x_{ij} + r_k}{N}\right]$
- If ($x_{ij} + r_k$) < 0, then $x_{ij}' = (x_{ij} + r_k) - N \cdot \left[\frac{|x_{ij} + r_k|}{N}\right] + N$
- If 0 <= ($x_{ij} + r_k$) <= N, then $x_{ij}' = (x_{ij} + r_k)$

Where x_{ij} : the coordinates of the centers of the "Shaded" squares along the X axis, x' : coordinates of the centers of the "shaded" centers displaced along the X axis.

The r_k are formed in way that the coordinates of the centers of the displaced "shaded" squares x'_{ij} along the X axis obtained according to the above conditions must correspond to the following condition:

$x_{11} \le x_{ij}' \le x_{1N1}$, x_{1N} – coordinates of the first and last centers of squares of the first row, obtained by dividing the rectangular area into MH squares.

The set $J=(s_k, k=1,..,mes(\Xi))$ is formed as follows $\sum_k s_k = 0$. Then the center of the "shaded" squares along the Y axis is converted to S_k : $y_{ij}' = y_{ij} + s_k$ where y'_{ij} are the coordinates of the centers of the displaced "shaded" squares along the Y axis.

The S_k in the set ($J = |(s_k, k=1,..,mes(\Xi))$) are formed in way that the obtained coordinates of the centers of the displaced "shaded" squares y'_{ij} along the Y axis must correspond to the following condition: $\min y_{ij} \le y_{ij}' \le \min(M - y_{ij})$

Thus, the centers of the displaced "shaded" squares remain within a certain limited area (the shaded area in Figure 1).

As shown, for the newly obtained set of squares, the following condition will also be fulfilled: $\sum_{i,j \in \Xi} y_{ij}' = R$

Suppose that the document is marked with an independent number W, and we will interpret this number as the clock of the operation of a certain generator specifying the set of indices Ξ and R. The correspondence between the number W and the sets of "shaded" squares can serve as a criterion for the authenticity of the document, in other words, act as a procedure for verifying the watermark. It should be noted that when performing the authentication procedure, the inspector knows in advance the coordinates of the "shaded" squares.

The obtained sets can be subjected to random tests, iteratively select I, J so that the tests are successful. But what happens if the square obtained during the movement coincides with some originally "shaded" square?

In this case, as a criterion for displaced squares, we should choose a proximity to the value $R / mes\ (\Xi)$.

Embedding information in the digital watermark

An optimal container can be any document that has external identification features. For example, Issuing a certain scientific journal T38N1 (that is, volume 38 (T38) number 1 (N1))and the page of the electronic journal in the format *.bmp containing some texts. The binary representation of this number is 1010100 110011111000 1001110 110001, a dimension is 32 bits. As an example, we embed the Digital Watermark containing this unique number into the specified page.

First, in accordance with the method described above, the Digital watermark is introduced into the original container. Then, from pixels with coordinates x_{ij} or x'_{ij} arbitrarily selected 32 pixels, but the choice is made with the following conditions:

1- if the "1" must be written in the Digital watermark, then the coordinates or $y_{ij} > \frac{L}{2}$, re $y_{ij}' > \frac{L}{2}$ ely, must be arbitrarily chosen coordinates x_{ij} or x'_{ij}, - middle of the $\frac{L}{2}$ stegocontainer (document).

2- if you want to write "0" in the Digital watermark, then the coordinates or $y_{ij} < \frac{L}{2}$, re $y_{ij}' < \frac{L}{2}$ ely, must be arbitrarily chosen coordinates x_{ij} or x'_{ij} .

For the subsequent restoration of information, the user must store not only the coordinates of the pixels carrying this information, but also their sequence.

In Fig. 2 is a stagokonteiner with a Digital Watermark embedded in it, which in turn carries information about the identification number T38N1. For more illustration, Fig. 2 includes pixels with embedded information.

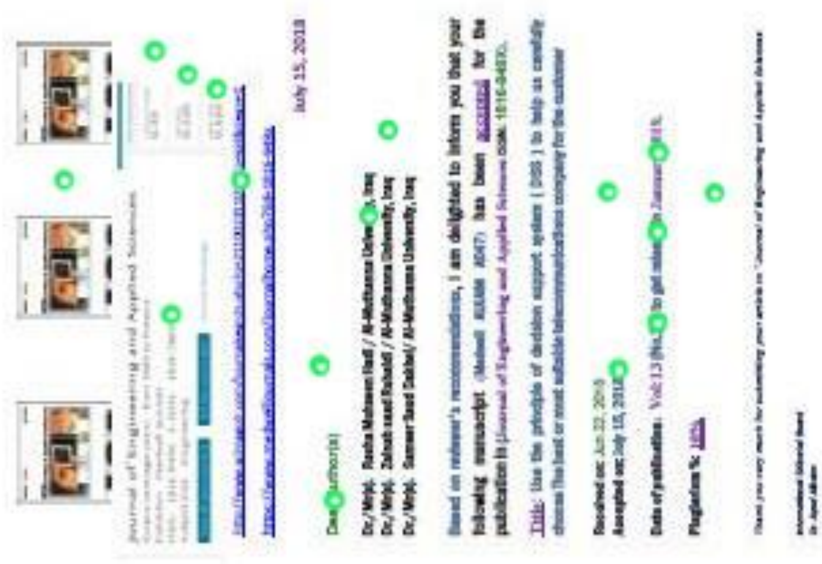


Fig. 2. Digital watermark containing information

It should be noted that the pixels used to transmit information are no different from the pixels of digital watermark, because the choice of "shaded" pixels required for information transfer is carried out among the entire set of "shaded" pixels of Digital watermark.

The main criterion is the location of the selected "shaded" pixel relative to the middle of the image. For the selected pixels, only the X-axis coordinates are remembered.

Also, to insert additional information into the container it is possible to use the third coordinate Z if you place the stack container, for example, as follows in (Fig. 3).

As parameters for the Z coordinate, for example, the brightness of the pixels of the Digital Watermark with the information embedded in them can be calculated using the formula : $I = 0{,}299 \cdot R + 0{,}587 \cdot G + 0{,}114 \cdot B$, where : I – the brightness value of the pixel. R, G, B - red, green and blue channels of this pixel respectively.

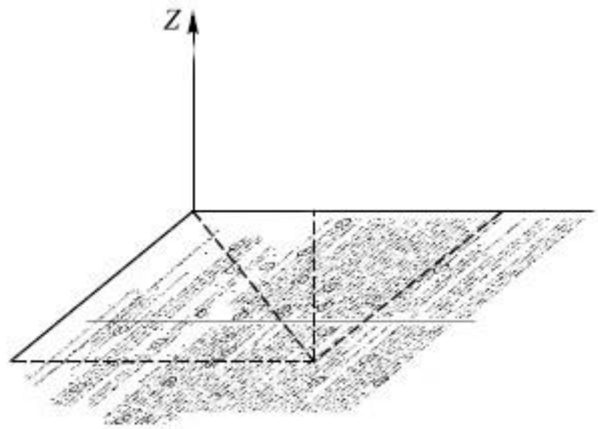


Fig. 3. Embedding additional information with the help of an additional Z coordinate

Description of the embedding system of Digital watermark in the proposed method, Evaluation of the detection capability of Digital watermark

Often in practice, there is no task of introducing an absolutely stable Digital watermark. Sometimes it is enough to develop such a system of embedding a digital watermark, in which an attacker can not completely destroy or replace digital labels. The main purpose of embedding the Digital Watermark in the method proposed is to protect the document from forgery, as well as the transfer of information embedded in it, for example, for the subsequent identification of this document.

Typically, the following types of embedding systems are distinguished: keyless systems, public key systems, systems with a private key, and mixed systems.

The method proposed provides for a system of embedding a digital watermark with a private key. According to the Kerkhoffs principle [5], with respect to the embedding systems of the Digital Watermark, their durability is based on some secret information, without knowledge of which it is impossible to extract information embedded in it from the container. when assessing the reliability of the embedding system of Digital watermark, it should be assumed that the attacker has complete information about the structure and algorithms of this system. the Digital watermark in the method proposed, arbitrarily selected "shaded".

Practical application of the proposed method

The method proposed can be used to identify pre-registered products, including the integration in the Digital watermark of the individual numbers of each manufactured product and the subsequent labeling of each product in the manufacturing process. More precisely, the process can be described as follows: in the manufacture and packaging of products, an individual number is assigned to each product. Then, according to the method described above, a Digital Watermark is formed with an individual number embedded in it . For example, a digital watermark can be embedded in the product barcode. When selling, the seller reads the bar code itself and the Digital Watermark embedded in it and the data on the sold goods are transferred to a special database.

The buyer discovers the individual number assigned to this product after opening the package. Then, for example, using the mobile device camera, it reads out the Digital Watermark and the information embedded in it, then reconciles the numbers on the package and the number obtained after the recognition of the Digital Watermark. If they coincide, then it is possible with confidence to declare the authenticity of the product.

Conclusion

In this article, a universal way of forming a watermark for official documents was proposed, in addition, an embedded watermark is capable of carrying in itself any information, for example, a certain document number or text.

A preliminary analysis of the complexity of the recognition of the digital watermark and the embedded information contained in it is made.

To demonstrate the proposed method, a program was developed that implements the actions described above.

References

1. **Cox, I.J.** Digital watermarking and steganography / I.J. Cox, M. Miller, J. Bloom, J. Fridrich. – San Francisco: Morgan KaufmannPublishing, 2008. 624 p
2. **Fridrich, J.** Methods for Tamper Detection in Digital Images/ J. Fridrich // Proceedings of ACM Workshop on Multimedia and Security. –1999. Vol. 1 , P. 19-23.
3. **Bender, W.** Techniques for Data Hiding / W. Bender, D. Gruhl, N. Morimoto, A. Lu // IBM Systems Journal. 1996. Vol. 35. P. 313-336.
4. **Bender, W.** Applications for Data Hiding / W. Bender, W. Butera, D. Gruhl, R. Hwang, F.J. Paiz, S. Pogreb // IBM Systems Journal. – 2000. Vol. 39, No.3&4., P. 547-568.
5. **Kerckhoffs, A.** La cryptographie militaire / A. Kerckhoffs// Journal des Sciences Militaries., 1883. Vol. IX. – P. 5-38, 161-191.

طريقة إنشاء علامة مائية رقمية للوثائق الرسمية

سمير سعود داخل[1] زينب سعد ربيدي[1] رشا محسن هادي[1]

[1]كلية الزراعة / جامعة المثنى

في هذه الورقة ، اقترحنا طريقة لإنشاء العلامة المائية للوثائق الرسمية. على سبيل المثال ، يتم وضعها عند مصادقة المستندات ، وكذلك استخدامها للاتصال الآمن. لقد وصفنا الإجراء الخاص بتبادل المعلومات بين الطرفين. يتم تكوين العلامة المائية بطريقة لا تسمح لشخص غير مصرح له باكتشاف وجود العلامة المائية واستخراج المعلومات المضمنة ، وأظهرت العلامة المائية قدرتها على العمل بالطريقة المقترحة.

Numerical Solution of Stochastic Heat Equation

***Nasir I. Naser*[a] *, Dr. Muhannad Al-Saadony*[b]**

[a]*Msc student in Statistics, University of Al-Qadisiyah, Departmenet of Statistics, Iraq. Email: naibna82@gmail.com*

[b]*Assist. Prof. in the Statistics Department, University of Al-Qadisiyah, Departmenet of Statistics, Iraq. Email: Muhammand.alsaadony@gmail.com*

A R T I C L E I N F O

Article history:
Received: dd /mm/year
Revised form: dd /mm/year
Accepted: dd /mm/year
Available online: dd /mm/year

Keywords:

Stochastic Partial Differential Equations (SPDEs), Heat equation, Numerical solution of stochastic, partial differential equation

A B S T R A C T

This paper will use the numerical methods to solve the stochastic heat equation driven by (Brownian motion, Brownian bridge, and Reflected Brownian motion). The numerical schema depends on the impersonation of the equation solution and includes a stochastic portion of the noise and a partial differential equation. We will apply our methods using daily temperatures in Iraq for the year 2018.

DOI :

1 . Introduction

In this paper, we want to take a quick look at the numerical solution of stochastic partial differential equations. Working on the numerical solution of stochastic partial differential equations faces many difficulties. On the one hand, we should think of the known problem of numerical solution of inevitable differential equations. On the other hand, we face the challenge of solving the random term.

The SPDEs used as a model in differenced applications. Motivated the mathematics field in particularly through the need to study characterize stochastic the phenomena in normal sciences such as biology, alchemy, physics, and used in statistics and finance science.[2]

Consider the SPDE with the white noise [10]

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \sigma B(x,t)\,,(x,t) \in [0,T] \times [0,1] \quad \dots (1)$$

Where $T > 0$, and $B(x,t)$ is white noise. The initial condition is getted through continuous function $u_0: [0,1] \to R$ and we consider Dirichlet boundary conditions.

$$u(0,x) = u_0(x)\,, x \in [0,1]$$

$$u(t,0) = u(t,1) = 0, t \in [0,T]$$

The real-value stochastic domain solution to equation (1) denoted by $\{u(t,x),(t,x) \in [0,T] \times [0,t]\}$. $\sigma > 0$ is constant. We suppose that $\{B(t,x),(t,x) \in [0,T] \times [0,1]\}$ is a Brownian motion.

Numerical experiments indicating that there is a convergence between the approximate difference in the solution of (1). The aim of the present sheet is to investigate the best possible outcome.

2. Stochastic Integral With Brownian Motion

Definition 1 Brownian motion[1] [2] is a stochastic process $\{B_t: t \geq 0\}$, satisfying

I. $B_0 = 0$,

II. B_t has independent increments.

III. $B_t$is continuous in t.

IV. For any $s = t_0 \leq t_1 \leq \cdots \leq t_{n-1} \leq t_n = t$ the increments $\Delta B_t = B_t - B_s$, $0 \leq s \leq t$ are normal distribution with $\mu = 0$ and $\sigma = t - s$, $B_t - B_s \sim N(0, t-s)$

V.

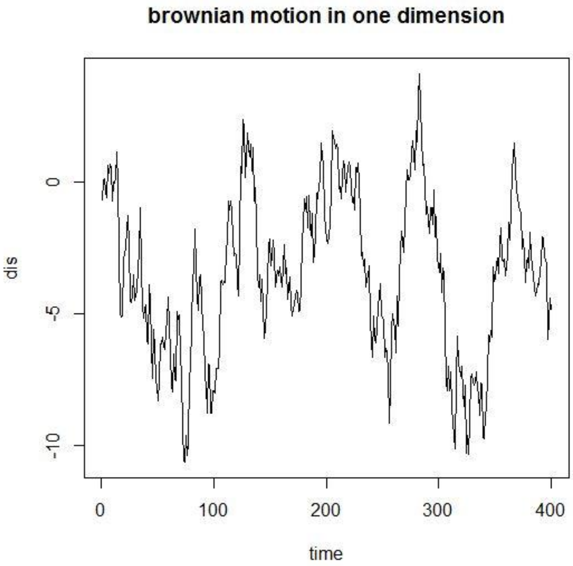


Fig.1 Brownian motion

Definition 2 Brownian Bridge.[3][6]

The process $X_t = B_t - tB_1$ is called the Brownian bridge where $X_t \in (0,1)$ [see 3,p 59]

i.e.,

$$X_t = B_t - tB_t - tB_1 + tB_t$$

$$= (1-t)(B_t - B_0) - t(B_1 - B_t)$$

Because of the increments $B_t - B_0$ and $B_1 - B_t$ are independent and normally distribution with $B_t - B_0 \sim N(0,t), Then\ B_1 - B_t \sim N(0,1-t)$

And the expected and variance are follows [1]

$$E[X_t] = (1-t)E(B_t - B_0) - tE(B_1 - B_t) = 0$$

$$Var[X_t] = (1-t)^2 var[(Bt - B0)] + t^2 var[(B_1 - B_t)]$$

$$= (1-t)^2(t) + t^2(1-t)$$

$$= t(1-t)$$

This can too be stated by saying that the Brownian bridge attached at 0 and 1 is a Gaussian process with drift 0 and variation $t(1-t)$, i.e., $X_t \sim N\big(0, t(1-t)\big)$.

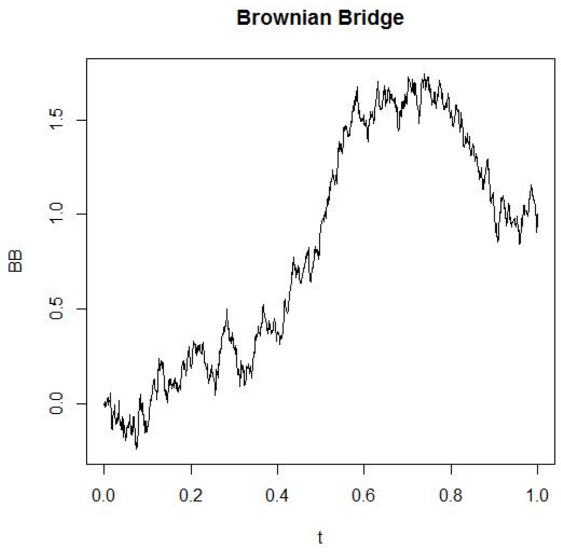


Fig.2 Brownian bridge

Definition 3(The Itô process) [2] [7][10]

A stochastic process X_t, $0 \leq t \leq T$ is called an Itô process with respect to $\{B_t, P, F_t\}$, where F_t is adapted to B_t, relative to $f(s), g(s)$ if

$$X_t = X_0 + \int_0^t f(s)ds + \int_0^t g(s)dB_s \quad, 0 \leq t \leq T \ldots (2)$$

Definition 4.[2][7][8]

denote by $L^2(\Omega \times [s,t])$ the space of all adapted stochastic process f_t such that

$$E\left[\int_s^t f_t^2 dt\right] < \infty \ for\ s < t \quad \ldots (3)$$

And let $\Delta t = \{s = t_0 < t_1 < \cdots < t_{n-1} < t_n = t\}$ and $\Delta B_i = B_{ti} - B_{ti-1}$. we define the Itô integral of B . when $B \in H^2(\Omega \times [s,t])$ with respect to Brownian motion

$$I(f) = \int_s^t f_t dB_t = \sum_{i=1}^{n} f_{t_{i-1}} \Delta_i B \quad \ldots (4)$$

Whenever the limit in probability exists.

3. Stochastic Partial Differential Equations Solutions Types. [1][4][5]

The SPDE of following

$$dX_t = [AX_t + F(X_t)]dt + \sigma dB_{(x,t)} \quad \ldots (5)$$

Have several notions from, solution as we will see below

Definition (5): *G(A)*- valued expected process X_t , $t \in [0,T]$ is named a strong analytical solution of the eq.(3)

$$X_t = \int_0^t [AX_s + F(X_s)]ds + \int_0^t \sigma dB_{(x,s)} \quad \ldots (6)$$

Where $A = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i}$, F is Laplace location and $B_{(x,s)}$ Brownian motion.

In special, the integration has to be well-define, [3]

Definition (6): F-valued expected process X_t , $t \in [0,T]$ is named a weak analytical solution of the eq.(3) if

$$\langle X_t , \delta\rangle = \int_0^t \left[\langle X_s, A^{'}\delta\rangle + \langle F(X_s), \delta\rangle\right]ds + \int_0^t \langle \delta, \sigma\, dB_s\rangle \quad \dots (7)$$

For each$\delta \in G(A^{'})$, the integral has to be well-define [4]

Definition (7): F-valued expected process X_t , $t \in [0,T]$ is named a mild analytical solution of the eq.(3) if

$$X_t = \int_0^t \left[e^{A(t-s)}F(X_s)\right]ds + \int_0^t e^{A(t-s)}\sigma dB_s \quad \dots (8)$$

the integral has to be well-defined, [13]

4. Numerical Solution of Stochastic Heat Equation. [2][5]

The stochastic heat equation given by a space-time white noise

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \sigma\, B_{(x,t)} \quad , 0 < x < b \, , t > 0 \dots (8)$$

Subject to the initial and boundary condition.

$$u(0,t) = u(b,t) = g(t) \quad , t > 0$$

$$u(0,0) = f(x) \qquad 0 < x < b$$

Where $B_{(x,t)}$ is two-dimensional white noise, by using Taylor Theorem we can find sa follows :

$$u_{i,j+1} = u_{i,j} + r\left[\, u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] + k\sigma B_{i,j}$$

$$When\ i = 1,2,3, \dots , n-1\ and\ j = 1,2,3, \dots , m-1\ and\ r = \frac{k}{h^2}$$

5. Application of the Temperatures. [2]

In this part, we used temperatures on "SPDEs", some Brownian motion types with additional noise.
We first discussed the "SPDEs" for a reference on the quality of the work already submitted. We look at the local white noise with stochastic heat equation in the field [0,1] during the period [0, T] when T = 1.

Let the "SPDE"

$$dX_t = [A\Delta X_t + F(X_t)]dt + \sigma dB_{(x,t)} \quad \dots (9)$$

for $X_0(x) = 0\ and\ X_t(0) = X_t(1) = 0\ for\ x \in (0,1), t \in [0,T)$.

$F(X_t) = 0, \sigma = \sqrt{q}$, $q = jh$, where the $B_{(x,t)}$ here is the white noise.

Then the SPDE

$$dX_t = [A\Delta X_t]dt + \sigma dB_{(x,t)} \quad \dots (10)$$

has singular mild solution $X: [0,T] \times \Omega \to H_B$

example (Example to illustrate the applied side of a paper).

we attempt to discover the numeral solution for SPDE with additional noise, by taking the temperature for 2018.

$$\frac{\partial u}{\partial t}(x,t) = (\partial^2 u/\partial x^2) + \sigma B(x,t) \,, (x,t) \in [0,T] \times [0,1] \quad \dots (11.)$$

With

$$u(0,x) = u_0(x)\,, x \in [0,1]$$

$$u(t,0) = u(t,1) = 0, t \in [0,T]$$

$B(x,t)$ is white noise and $\sigma = \sqrt{q}$.

By employing the equation (2). We find

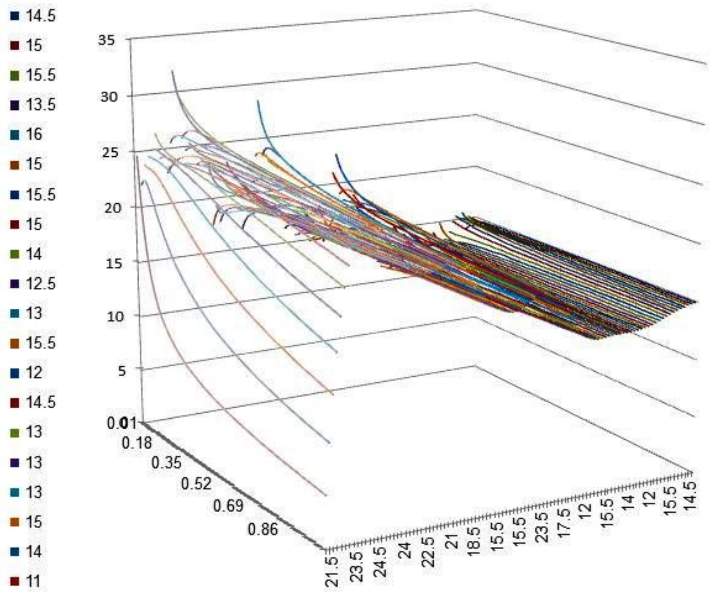


Fig.3 solution of SPDE with Brownian motion

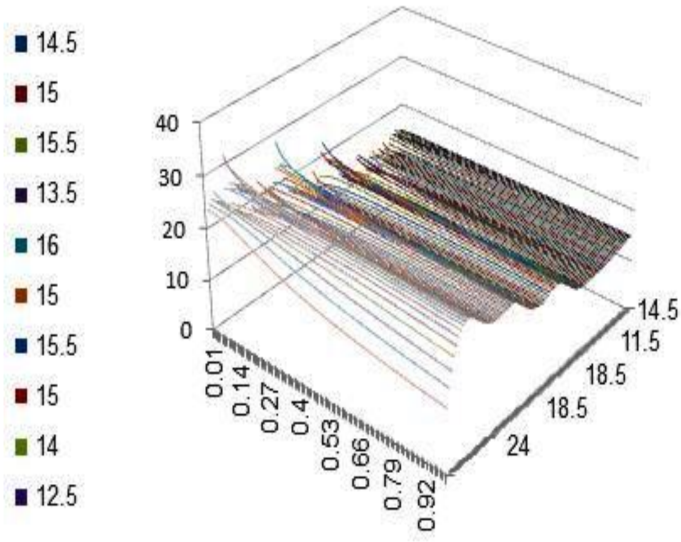


Fig.3 solution of SPDE with Brownian bridge

Table(1)mean square errors

Number	**City**	**MSE**	
		Brownian motion	**Brownian Bridg**
1	**Qadisiyah**	4.024871066	35.15392897
2	**Basrah**	4.372799879	30.62404944
3	**Erbil**	4.574865516	31.24863633

6. Conclusions

Applying a technique for solve the stochasti partial differential equations. And find numerical solutions using heat equation with the added Brownian motion, it was noted that this technique has benefits in general applications, and we find that the random factors have an effect on the temperature of the area under study, through the results shown in the application example.

The main problem we encountered during the application of numerical solutions to random differential equations is the process of noise generation, such as (Brownian motion or Brownian bridge). Typically, noise values should be distributed with an average of zero and a variation of dt, for example, N (0, dt). To achieving this value should be minimal and minimal change to get the most benefits over the specified period, and these problems affect the shape and distribution of noise and show their effect on the final solutions.

References

[1] L.R.Boris and A.L.Sergey, (2017),"Stochastic Partial Differential Equations ", Springer International Publishing AG.

[2] H. A. Eman and Alrajhi. M. Y. (2016) "Numarical Solution for Stochastic Partial Differenatial Equations Via Acelerated Gentic Algorithm" International Journal of Engineering Sciences &Research Technology 2277-9655.

[3] Calin O. (2015)."An Informal Introduction to Stochastic Calculus with Applications", World Scientific.

[4] L.Wei &R.Mihael,(2015)," Stochastic Partial Differential Equations :An Introduction", Springer International Publishing Suitzerland.

[5] C.Pao-Liu,(2015)," Stochastic Partial Differential Equations ", Taylor & Francis Group.

[6] S.Benedykt, (2012)," The new stochastic integral and anticipating stochastic differential equations",A Dissertion Doctor of Philosophy in the Department of Mathematics, Louisiana state University.

[7] D. Nualart and L. Q. sardanyons ,(2009) "Gaussian density estimates for solutions to quasi-linear stochastic partial differential equation " Stochastic Process and their Applications 119(2009) 3914-3038.

[8] M.lacus Stefano. (2008)."Simulation and Inference for Stochastic Differential Equation with R Examples", Springer Series in Statistics.

[9] S.David,(2005),"Stochastic Process & Models", Oxford University press.

[10] A. M. Davie and J. G. Gaines (2000) "CONVERGENCE OF NUMERICAL SCHEMES FOR THE SOLUTION OF PARABOLIC STOCHASTI PARTIAL DIFFERENTIAL EQUATION", mathematics of computation.

دراسة تاثير الجرع المختلفة من أشعة كاما على بعض الخصائص البصرية والكهربائية لبولي أكريل أمايد (PAAM)

أنيس عبد المنعم محمد ..... جامعة ذي قار/ كلية العلوم/ قسم الفيزياء

anees_mn2000@yahoo.com

فؤاد نمر عجيل ..... جامعة سومر/ كلية العلوم/ قسم الفيزياء

علاء محسن خضير..... جامعة سومر/ كلية العلوم/ قسم الفيزياء

عباس شويع علوان ..... جامعة ذي قار/ كلية العلوم/ قسم الفيزياء

الخلاصة:

يتضمن هذا البحث دراسة الخصائص البصرية والكهربائية لمادة "بولي أكريل أمايد" (PAAM) كمحلول, وذلك قبل وبعد تشعيع المادة بثلاث جرع إشعاعية مختلفة من أشعة كاما. قسم مسحوق المادة إلى أربع عينات متساوية, حيث استخدمت العينة الأولى لدراسة الخصائص قبل التشعيع, بينما شععت العينات الثلاث الأخرى بـ rad (2000, 4000, 6000) من جرع أشعة كاما على التوالي. بعد ذلك تم اذابة عينات المادة الأربع في الماء المقطر وتحضير تراكيز من (1-6%) لكل عينة, وبذلك أصبح مجموع التراكيز (24) تركيزا.

أظهرت النتائج أن قيم بعض هذه الخصائص البصرية والكهربائية تزداد خطيا بزيادة تركيز المحلول قبل وبعد التشعيع وتزداد كذلك بزيادة الجرعة الاشعاعية بعد التشعيع, وذلك بسبب التشابك الحاصل في سلاسل البوليمر. أما البعض الآخر من هذه الخصائص فقد أظهر سلوكا معاكسا تماما لهذه الخصائص.

الكلمات المفتاحية: أشعة كاما, PAAM, الخصائص البصرية, الخصائص الكهربائية.

Abstract:

This research includes a study of the optical and electrical properties for material "Poly Acryl Amide" (PAAM) as a solution before and after irradiating it by three different doses of gamma rays. The material powder is divided into four equal samples. The first sample was used for the study of the properties before irradiation, while the other three samples are irradiated with (2000, 4000, 6000) rad of gamma rays doses respectively. Afterward, the four samples of the material were dissolved in the distilled water and prepared concentrations about (1-6%) for each sample. Therefore, the total number of the concentrations became (24) concentration.

The results show that the values of all these optical and electrical properties increase linearly with the increasing of the solution concentration before and after irradiation and also increase with the increasing of radiation dose after irradiation, which related it to the crosslinking yields in the polymer chains, except the critical angle which shows exactly opposite behavior of these properties.

Keywords: gamma rays, PAAM, optical properties, electrical properties.

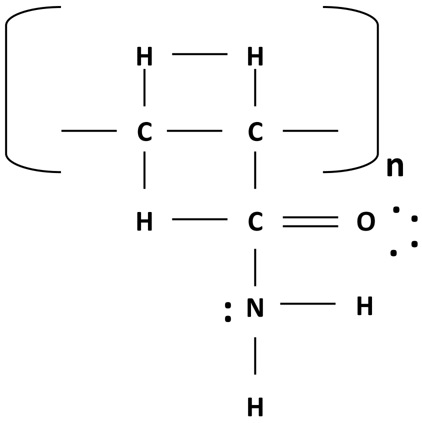
1- المقدمة: Introduction

استخدمت البوليمرات المذابة في الماء بشكل واسع في مجالات متباينة في الصناعة والزراعة مثال ذلك الزغب (الشبيه بالصوف) لتنقية المياه، وفي استخلاص الزيت، وفي تخفيض المقاومة الهايدروداينميكية (القوة المائية) في الأنبوب، وفي استخلاص ايونات الفلز... الخ، أضافة الى استخدامها في علوم الحياة والطب كبدائل عن البلازما، في ترشيح وتنقية الخمائر وفي اطالة مفعول المستحضرات الطبية, فالمركبات البوليمرية ذات الوزن الجزيئي العالي المذابة في الماء يمكن أن تفيد كأساس في خلق المحفزات البوليمرية لنماذج بسيطة من الخمائر. وتستعمل الألكتروليتات المتعددة في بناء الأنسجة الكيميائية لنقل الطاقة الكيميائية الى الميكانيكية وبالعكس، والبولي الكتروليتات تعمل

كمحفزات في بعض تفاعلات المواد ذات الوزن الجزيئي الواطئ ، ففي المحفزات البوليمرية الصناعية وتحت شروط معينة، تمنحها خواصا شبيهة بتلك المحفزات الحياتية والخمائر. [1].

ان التأثير الرئيسي للاشعاع على البوليمرات هو انحلالها Degradation او تشابكها (Cross linking). ويمثل هذان التأثيران التغيرات الرئيسة في خواص البوليمر. هذا بالاضافة الى أن التفكك قد يؤدي الى تحرر نواتج غازية وتكون عادة ذات جزيئات صغيرة مثل غاز الهيدروجين والميثان وثاني أوكسيد الكربون [2].

لقد حظيت بولي أكريل أمايد (PAAM) ومشتقاتها باهتمام متزايد خلال السنوات الماضية والى وقتنا هذا بسبب تطبيقاتها الواسعة في المجالات الصناعية والحياتية وبالأخص في مجالات الاستخدامات الطبية. الصيغة الكيميائية العامة له:



شكل رقم (1): الصيغة الكيميائية لمادة بولي أكريل أمايد (PAAM) [3]

ففي المجالات الصناعية تستخدم هذه البوليمرات ومشتقاتها في معالجة المياه (Water treatment)، التعدين (Mining)، استخراج النفط (Oil recovery), وفي صناعة البوليمرات المتصلبة حرارياً (Thermosetting), فضلاً عن ذلك تستخدم بوصفها موادّ مانعة للاكسدة [3] ومواد محورة (Modifiers) لخواص البوليمرات المختلفة. ومؤخراً تم استخدامها في بلمرة تطعيم السطوح [4] ويستعمل (PAAM) عاملاً مثخناً للمواد ومواد مزغبة [5].

كما تستخدم طبياً في مجال صناعة الأنسجة الحية الطرية (Soft tissues)، وفي صناعة القرنية الصناعية للعين (Artificial corneas)، أو صناعة العدسات اللاصقة, وفي صناعة الأنسجة الخاصة بتغطية الحروق [6].

ولقد وجدت علاقة بين التركيب الكيميائي لمشتقات الأكريل أمايد المعوضة والبوليمرات الناتجة عنها وبين التجانس البايولوجي, اذ أن هذه المشتقات فعالة للاستخدامات الطبية ولاسيما تلك التي تتبلمر بالجذور الحرة لما تمتلكه من خصائص ملائمة لهذا المجال, ومع ذلك فان تطبيقاتها تكون مفيدة نتيجة لخصائصها الميكانيكية الضعيفة من قوة الشد الضعيفة ومقاومتها الضعيفة للضغط وقلة استطالتها [7].

والأكريل أمايد من البوليمرات المذابة في الماء وهو لا يشبه المونومير وغير سام لاحتوائه على نسبة من النتروجين نظرياً (19.7%) وتبلغ نسبة احتوائه على مجاميع الهيدروكسيد (3.6%) وهذه الأخيرة تمنح البوليمر تقنية بخواص متعددة الأقطاب [5]

2- الجانب النظري: Theoretical part

أولا: الخصائص البصرية: Optical Properties

الامتصاصية (A) Absorbance تحسب عن طريق معادلة بير- لامبرت. فعند مرور أشعة خلال محلول معين فأن كمية الضوء الممتص أو النافذ تكون حالة دالة أسية لتركيز المذاب (شرط أن يكون المذيب شفافاً للاشعة في تلك النقطة). كذلك يكون دالة لطول المسافة التي تقطعها الأشعة خلال النموذج, وتوجد صيغ مختلفة لقانون بير- لامبرت [8]:

$$I = I_o \exp(-\alpha_{op}.C_{ml}) \quad \text{...(1)}$$

Or:

$$-\log(I_o/I) = \alpha_{op}.C_{ml} \quad \text{...(2)}$$

حيث أن:

I_o = تمثل شدة الاشعة الساقطة, I = يمثل شدة الاشعة النافذة.

C_m = يمثل التركيز المولاري , l= يمثل (طول المسار البصري) بالسنتمتر.

$$A = \alpha_{op}.C_m l \quad \text{……} (3)$$

حيث أن α_{op} هو معامل الامتصاص البصري للمادة المراد دراسة الضوء الممتص لها.

ان النسبة I/Io تعرف بالنفاذية Transmittance (T) كما يطلق على لوغاريتم النسبة المقلوبة I/Io بالامتصاصية (Absorbance A) وعليه فان:

يعتمد معامل الامتصاص البصري على نوع المادة الماصة للضوء والطول الموجي ونوع المذيب ودرجة الحرارة [8].

أما النفاذية Transmittance (T) فهي النسبة بين شدة الضوء النافذ في المحلول الى شدة الضوء الساقط ويعبر عنه رياضياً بالعلاقة [9]:

$$T = I / I_o \quad \text{……} (4)$$

$$\text{Log}(1/T) = A = -\alpha l C \quad \text{……} (5)$$

ثانيا: الخصائص الكهربائية: Electrical Properties

تعتمد التوصيلية الكهربائية Electrical Conductivity على عاملين رئيسيين هما حاملات الشحنة والتي تكون الكترونات أو فجوات أو ايونات متولدة ذاتياً أو من الشوائب وعلى حركيتها (Mobility) ويعبر عن التوصيلية الكهربائية بالعلاقة [10]:

$$\sigma = \Sigma \; n_i q_i \, \mu_i \quad \text{……} (6)$$

حيث:

n = كثافة حاملات الشحنة, q = الشحنة الكهربائية وتساوي $1.602 * 10^{-19}$ col , μ_i = حركية الشحنة.

ان التيار الكهربائي الناتج I بواسطة فرق جهد معين E فانه يتناسب معه طرديا وفقا للعلاقة الآتية [11]:

$$I = \sigma E \quad \text{……} (7)$$

أما ثابت التناسب σ فيدعى التوصيلية الكهربائية للمحلول. ان العادلة (7) هي في الحقيقة اعادة صياغة لقانون أوم (E = RI) [11]. ان التوصيلية σ تقاس بوحدة $S.m^{-1}$ أو بوحدة $ohm^{-1}.m^{-1}$ وتعتمد على تركيز المحلول ودرجة حرارته [12]. أما المقاومية الكهربائية Electrical Resistivity (ῤ) للمحلول فهي معكوس التوصيلية الكهربائية وتعطى بالصيغة التالية[13 ' 14]:

$$\acute{\rho} = \frac{1}{\sigma} \quad \text{……} (8)$$

التوصيلية المولارية Molar Conductivity (Λ) تعرف بانها النسبة بين التوصيلة الكهربائية وتركيز المحلول المولاري أي ان [15]:

$$\Lambda = \frac{\sigma}{C_m} \quad \text{……} (9)$$

حيث أن Cm هو التركيز المولاري ويقاس بوحدات ($\mu s.\ L.\ Cm^{-1}.\ mol^{-1}$).

ثابت العزل الكهربائي: هو عبارة عن نسبة سعة مكثف كهربائي ذو عازل بين قطبيه الى السعة في حالة وجود هواء بدلاً من العازل بين القطبين أو الفراغ. وتعطى سعة المتسعة بدلالة أبعادها في الفراغ بالعلاقة [16]:

$$C_o = \epsilon_o \frac{A_o}{d} \quad \text{……} (10)$$

حيث ان:

C_o = سعة المتسعة في الفراغ بين اللوحين, ϵ_o سماحية الفراغ ومقدارها ($8.85 * 10^{-12}\ col^2/N.m^2$),

A_o = مساحة أي من لوحي المتسعة المتوازيين, d = المسافة العمودية بين لوحي المتسعة.

أما عند وضع مادة عازلة بين لوحي المتسعة فان سعتها الكهربائية تصبح [17, 18]:

$$C_D = \epsilon \frac{A_o}{d} \quad \ldots\ldots (11)$$

حيث أن: C_D = السعة الكهربائية للمتسعة عند وضع المادة العازلة بين اللوحين. ϵ = سماحية المادة العازلة.

ويعطى ثابت العزل الكهربائي (D) dielectric constant بالعلاقة التالية [17, 18]:

$$D = \frac{C_D}{C_o} = \frac{\epsilon}{\epsilon_o} \quad \ldots\ldots (12)$$

3- الجانب العملي: Experimental Part

أولا: تشعيع العينات:

لقد تم تشعيع العينات باستخدام المصدر المشع Co^{60} , حيث وضعت عينات المادة في مساحة قدرها $(32x32)\ cm^2$ وبمعدل اشعاع قدره (36 rad/min) وقد تم تشعيع المادة بثلاث جرع اشعاعية مختلفة, بحيث شعت العينة الأولى لمدة (55. 33 min) للحصول على عينات مشععة بمقدار (2000 rad), و شععت العينة الثانية لمدة (111.7 min) للحصول على عينات مشععة بمقدار (4000 rad), أما العينة الثالثة فقد شععت لمدة (166.40 min) للحصول على عينات مشععة بمقدار (6000 rad).

ثانيا: تحضير المحاليل:

تم تحضير المحاليل باضافة أوزان معينة من البوليمير من g (6-1) الى (100 ml) من الماء المقطر, فقد استخدم الميزان الالكتروني الحساس نوع (Sartorius) الذي يتحسس الى أربع مراتب عشرية لغرض قياس كتل المادة المراد تحضير المحاليل منها. وقد استخدمت العلاقة الآتية لاستخراج التراكيز الوزنية:

$$\text{التركيز الوزني\%} = \frac{\text{كتلة المذاب (g)}}{\text{كتلة المذاب (g)} + \text{كتلة المذيب (g)}} \times 100 \quad \ldots\ldots (13)$$

أو بحساب عدد غرامات المادة المذابة في حجم محدد من المذيب [19].

أما التركيز المولاري فقد تم حسابه باستخدام العلاقة الآتية [20]:

$$\text{التركيز المولاري}(C_m) = \frac{\text{كتلة المادة (g)}}{\text{الوزن الجزيئي للمادة (g/mol)} * \text{حجم المادة (lit)}} \quad \ldots\ldots (14).$$

جدول رقم (1): يبين قيم التراكيز الوزنية وما يقابلها بالتركيز المولاري.

التركيز الوزني % بدون وحدات	التركيز المولاري C_m x 10^{-5} mole/L
1	0.33
2	0.66
3	1.00
4	1.33
5	1.66
6	2.00

ثـالـثـا: الأجـهـزة المستعملة في القياس:

جهاز التشعيع:

تم تشعيع العينات باستخدام جهاز التشعيع نـــوع (CIRUS) بمعدل طـاقة 1.25) Mev) وبـاسـتـخدام المصدر المشع Co^{60}.

مقياس الامتصـاصـية والنفاذية:

تم قياس الامتصاصية والنفاذية باستخدام المطيـــــــاف (uv-160A/visible recording spectrophotometer) والذي تتراوح الأطوال الموجية فيه (190-1000) nm أي في المنطقة فوق البنفسجية (UV) والمنطقة المرئية (V.L.) وهو مبرمج أوتوماتيكياً للقيام بعملية المســح لكافة الأطوال الموجية واعطاء الطول الموجي الذي يحدث عنده أقصــى امتصـــاص وأقل نفوذية. يحتوي الجهاز على خليتين الأولى لوضع المذيب النقي لغرض تصفير الجهاز أي بجعل قراءة الامتصاصية تساوي صفراً لكل طول موجي, ثم يوضع محلول البوليمر في الخلية الثانية وتقاس الامتصاصية، وبعد الحصول على طيف الامتصاص يتم تحديد الطول الموجي الذي تحدث عنده أعلى امتصاصية (λ max) ثم تثبت هذه القيمة على الجهاز وتؤخذ قياسات الامتصاصية لتراكيز وزنية مختلفة من المحاليل المشععة وغير المشععة.

مقياس التوصيلية الكهربائية:

هو جهاز رقمي نوع (LF-91) صـــنع المانيا يحتوي على مجس لقياس التوصـــيلية الكهربائية بشـــكل مباشـــر. يغمر ذلك المجس في المحلول البوليمري ومن ثم يتم قراءة الجهاز مباشرة, وتكرر هذه العملية لجميع التراكيز الوزنية للنماذج المشععة وغير المشععة.

مقياس سعة المتسعة الكهربائية:

هو جهاز رقمي من نوع (Digital 9205 A) صـيني الصـنع، يحتوي على قطبين يتم توصـيلهما الى متسـعة معدة مسـبقاً وبأبعاد x 2.23 1.55 x 3.75) cm)، وقد تم قياس سـعة المتسـعة لجميع التراكيز المشـعة وغير المشـعة وذلك بوضـع المحلول البوليمري في داخل المتســعة ويتم توصــيل قطبي الجهاز الى طرفي المتســعة التي تحتوي على لوحين من الالمنيوم بابعاد 4.45 x 0.9) cm)، وتؤخذ القراءات مباشرة من الجهاز.

وقد تمت جميع القياسات بدرجة حرارة 25 ^{o}C.

4- النتائج والمناقشة: Results and Discussion

بالنسبة للخصائص البصرية فقد تم قياس كل من الامتصاصية والنفاذية عمليا باستخدام المطياف الآنف الذكر, وبعد ذلك تم حساب معامل الامتصــاص البصــري رياضــيا. الشـكل (2) يوضـح زيادة في قيم امتصــاصــية المحلول بزيادة تركيز المادة المذابة قبل وبعد التشـعيع بعد تثبيت $λ_{max}$ والتي تسـاوي (195 nm) عند أعلى تركيز وهو 6 . ان السـبب في ذلك يعود الى العلاقة الطردية للامتصــاصــية مع التركيز المولاري وحسب العلاقة (3). أما بعد التشعيع بجرع مختلفة من أشعة كاما فيلاحظ أيضا الزيادة في قيم الامتصاصية بسبب حدوث التشابك في سلاسل البوليمر والذي يؤدي الى زيادة الكثافة والوزن الجزيئي للبوليمر (PAAm) والذي يقلل من عملية تشتت الضوء [21].

ويمثل الشكل (3) رسماً بيانياً لتغير النفاذية مع تركيز البوليمر المذاب, ويلاحظ نقصان في قيم النفاذية مع التركيز الوزني وكذلك نقصان القيم بعد التشعيع بجرع مختلفة. ولنفس السبب الذي ادى الى زيادة الامتصاصية لان الزيادة في الامتصاصية يقابلها نقصان في النفاذية حسب المعادلة (5).

ويمثل الجدول (2) قيم معامل الامتصاص البصري بعد التشعيع بثلاث جرع والتي تم حسابها من خلال رسم العلاقة البيانية بين الامتصاصية والتركيز الوزني ثم اخذ الميل للمستقيمات الناتجة بحيث تظهر قيمة واحدة لكل جرعة اشعاعية, وعليه فان المعادلة رقم (5) تصبح كالآتي [22] :

$$\alpha_{op} = slope / \ell \qquad (15)$$

ويلاحظ من الجدول (2) نقصان قيم معامل الامتصاص البصري بعد التشعيع والذي يعود سببه الى التفاعل الداخلي الذي يسبب اندماجاً بين جزيئات البوليمر والماء قد يكون مسؤولاً عن نقصان معامل الامتصاص بزيادة تركيز المحاليل [23].

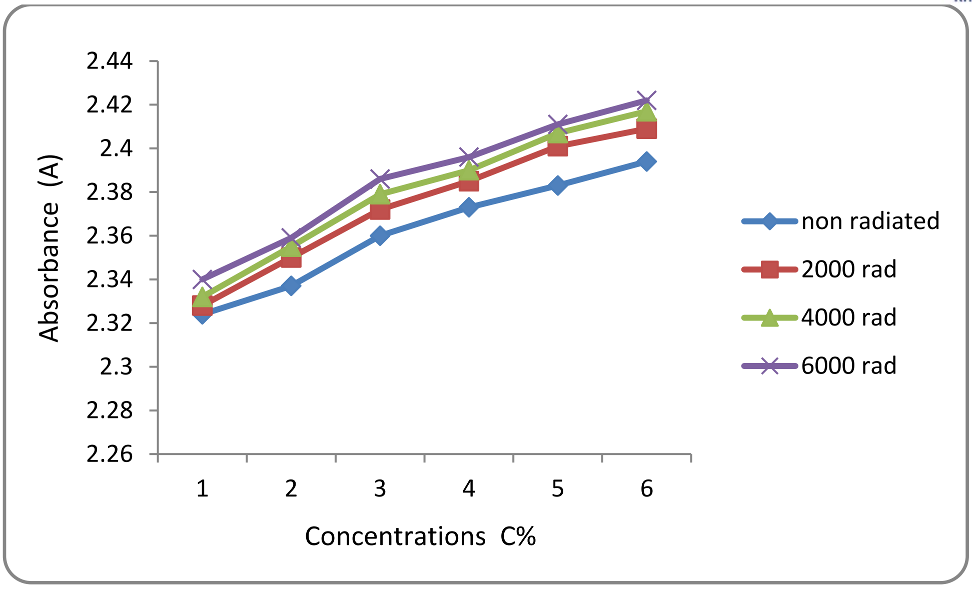
بالنسبة للخصائص الكهربائية فقد تم قياس التوصيلية الكهربائية وسعة المتسعة الكهربائية عمليا, وبعد ذلك تم حساب المتغيرات الكهربائية الأخرى رياضيا. حيث يبين الشكل (4) زيادة في قيم التوصيلية الكهربائية بزيادة تركيز المحلول وجرعة التشعيع. فبالنسبة لزيادة التركيز سوف تزيد من التوصيلية الكهربائية حيث أن محاليل البوليمر (PAAM) هي محاليل قطبية, اذ يمكن اعتبار الجزيء قضيباً مشحوناً كهربائياً لعزم ثنائي القطب الكهربائي. أما زيادة التشعيع فقد أدت الى زيادة التوصيلية الكهربائية بفعل تهيج الجزيئات وتأينها وانبعاث الالكترونات بسرع واطئة نسبياً [24].

أما الشكل رقم (5) فهو يوضح العلاقة بين التغير في قيم سعة المتسعة الكهربائية مع زيادة التركيز الوزني, ويلاحظ الزيادة الخطية في سعة المتسعة الكهربائية من حيث زيادة التركيز والتشعيع, حيث تؤدي زيادة التركيز الى زيادة في عدد الجزيئات المستقطبة, أما زيادة التشعيع فتؤدي الى تهيج الالكترونات في البوليمر وتوليد الكترونات حرة بفعل التشعيع [25].

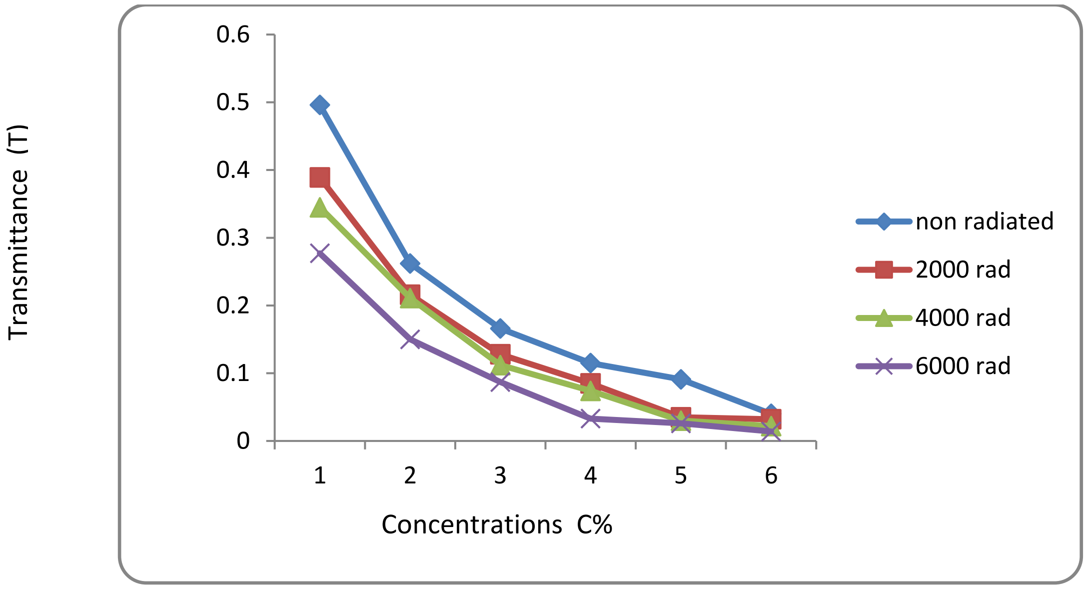
الشكل (6) يوضح نقصانا واضحا في قيم المقاومية الكهربائية مع زيادة كل من تركيز المحلول البوليمري وزيادة الجرعة الاشعاعية, وذلك لأن المقاومية الكهربائية هي معكوس التوصيلية الكهربائية وفقا للمعادلة رقم (8) لذلك فانهما يتصرفان باسلوب معاكس لأحدهما الآخر.

أما الشكل البياني رقم (7) فيوضح العلاقة بين التوصيلية المولارية لمختلف الجرع الاشعاعية و التركيز الوزني. ان التغير هذا يكون طرديا مع زيادة التشعيع و عكسيا مع زيادة التركيز, حيث تتبع التوصيلية المولارية سلوك التوصيلية الكهربائية كونها المتغير الاساسي في المعادلة (9).

أخيرا فان الشكل رقم (8) يبين الزيادة الخطية لثابت العزل الكهربائي قبل التشعيع وبعد التشعيع بزيادة التركيز. ويعود سبب الزيادة قبل التشعيع عند زيادة التركيز الى زيادة عدد الجزيئات المستقطبة, أما بعد التشعيع فيعود السبب في زيادة ثابت العزل الكهربائي الى تهيج الكترونات البوليمر وتوليد الكترونات حرة بعد تشعيعه [26].



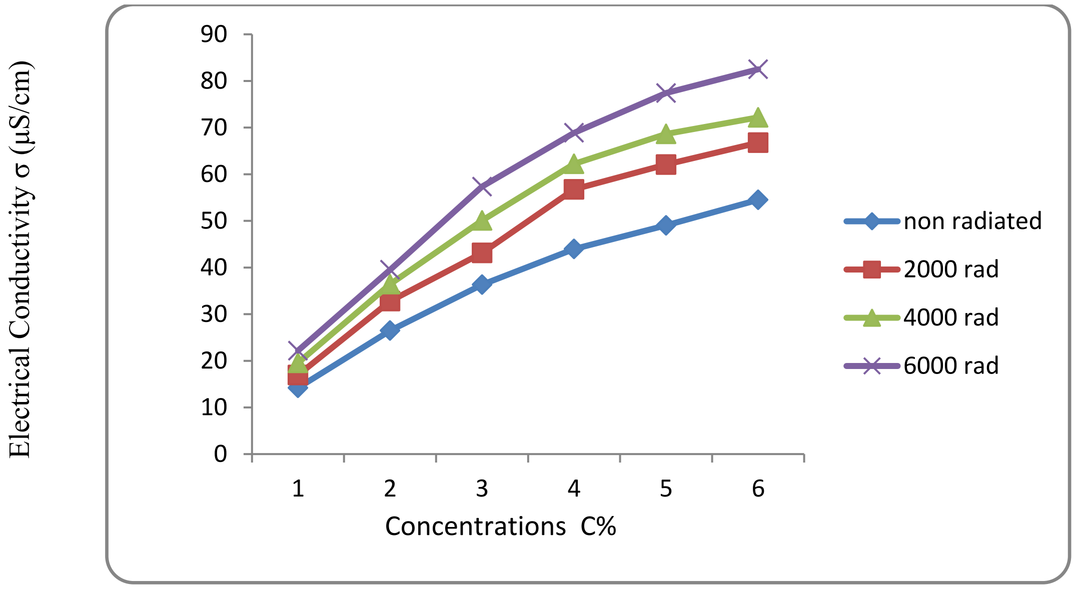
شكل رقم (2): تغير الامتصاصية مع التركيز.



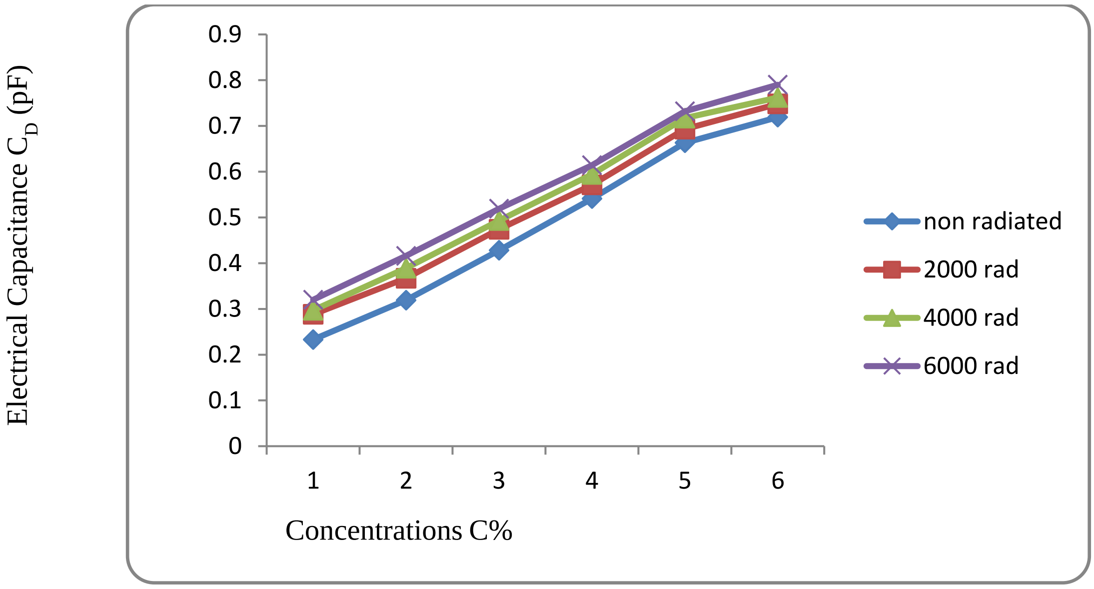
شكل رقم (3): تغير النفاذية مع التركيز.

جدول رقم (2): قيم معامل الامتصاص البصري قبل وبعد التشعيع.

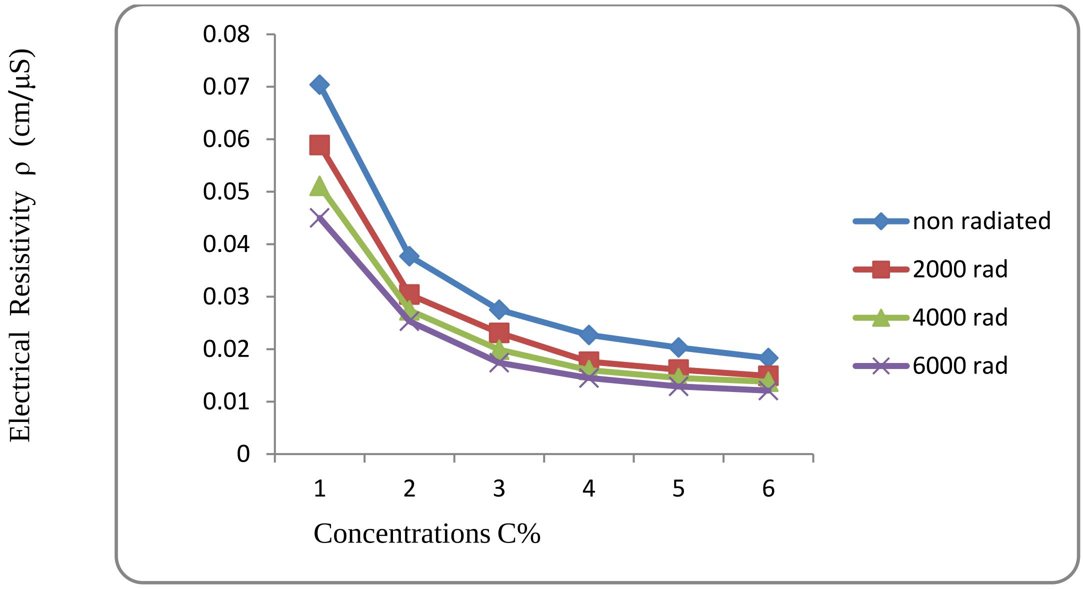
Absorption Coefficient α_{op} x10^5 (L/mole.cm)			
Non radiated	Irr. With 2000 rad.	Irr. With 4000 rad.	Irr. With 6000 rad.
0.173	0.154	0.146	0.140



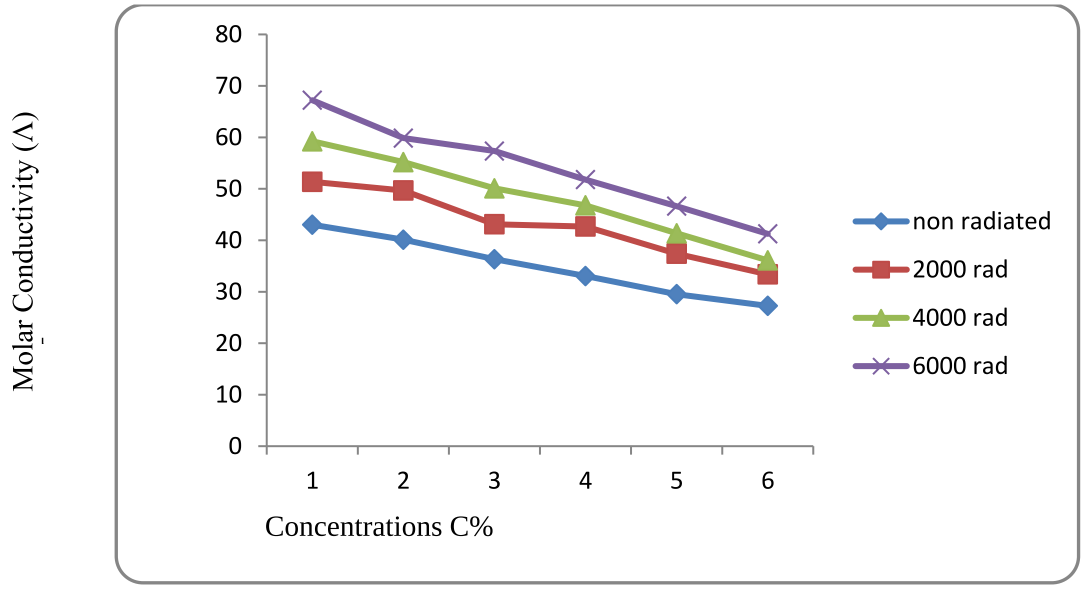
شكل رقم (4): تغير التوصيلية الكهربائية مع التركيز.



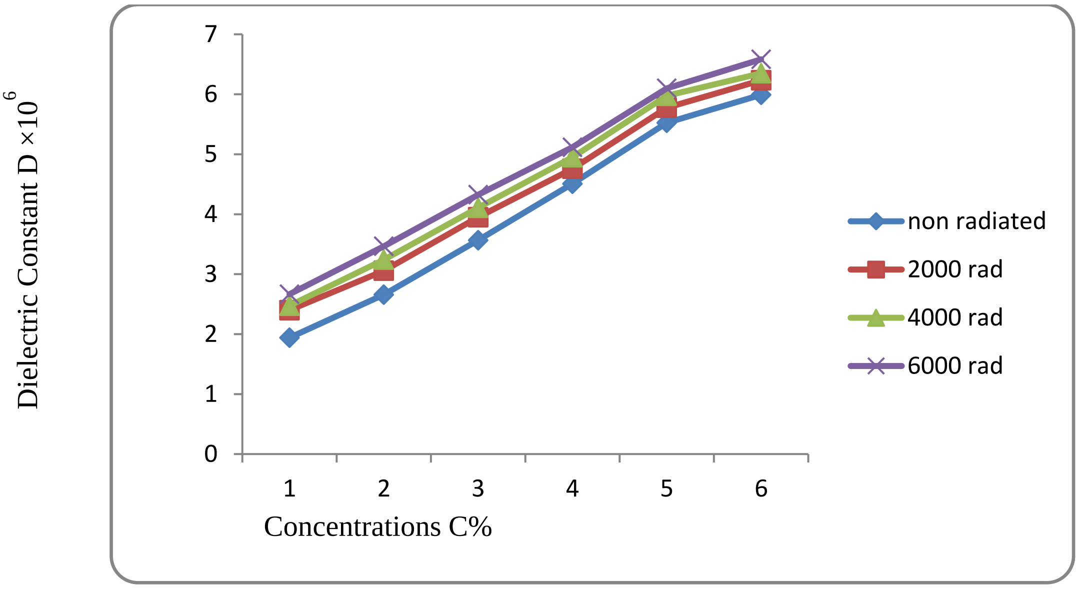
شكل رقم (5): تغير سعة المتسعة الكهربائية مع التركيز.



شكل رقم (6): تغير المقاومية الكهربائية مع التركيز.



شكل رقم (7): تغير التوصيلية المولارية مع التركيز.



شكل رقم (8): تغير ثابت العزل الكهربائي مع التركيز.

5- الاستنتاجات: Conclusions

بالنسبة للخصائص البصرية, فان التشعيع أدى الى تقليل عملية تشتت الضوء من خلال زيادة الامتصاصية ونقصان النفاذية بسبب التشابك الحاصل, وبالتالي فان هذا سيزيد من امتصاصية البوليمر للضوء وتقليل تشتته. أما بالنسبة للخصائص الكهربائية فان التشعيع أدى الى تهيج الكترونات البوليمر وتوليد الكترونات حرة, وذلك من خلال زيادة التوصيلية الكهربائية للبوليمر بعد التشعيع.

من خلال الرسومات البيانية, يظهر أن تأثير زيادة جرعة الاشعاع يكون متقاربا نوعا ما في أغلب الخصائص الفيزيائية المدروسة في هذا البحث. وبشكل عام فان التشعيع بأشعة كاما قد حسن من الخصائص البصرية والكهربائية للبوليمر المستخدم.

المصادر: References

[1] كوركيس عبد ال آدم، حسين كاشف الغطاء: "تكنولوجيا وكيمياء البوليمرات"، جامعة البصرة, (1983).

[2] اودنين وسانكستر: "مبادىء في الكيمياء الاشعاعية", ترجمة د. خالد يونس اليامور جامعة الموصل, كلية العلوم, (1985).

[3] F. Billmeyer: "text book of polymer", 3rd. ed., Wiley N.Y., USA, (1985).

[4] V. Herman: "Polyolefins structure and properties", Elsevier publishing company, Amsterdam (1966).

[5] Fred W. Bilmeyer: "Textbook of polymer science", 2nd ed., Joihn Wiley & Sons. Inc., New York (1971).

[6] M. F. Refojo; Surv. Ophthalmol; 16, 529, (1972).

[7] K. Ulbrich; M. Iiavsky; K. Dusek; J. Kopecek: Eur. Polym. J.; 13, 579, (1977).

[8] A.R. Daniels: "Physics Chemistry", Fifth Edition, John wiely and Sons Ins, U.S.A, (1980).

[9] Reilly P.M., and Van Der B.M., "J. of Applied polymer Sci" 24 (101), (1977).

[10] S. Wany: "Solid State Electronics", Mc Graw. Hill, New York (1966).

[11] E. Kirk Roberts: "Principles of Physical Chemistry", Stanford University, Allyn and Bacon, Inc. (1984).

[12] V. Kubasov and S. Zaretsky: "Introduction to Electrochemistry", Mir Publishers (1987).

[13] V. Kubasov and S. Zaretsky: "Introduction to Electrochemistry", Mir Publishers (1987).

[14] M. Razeghi: "Fundamentals of Solid State Engineering", 3rd. Ed., Business Media, LLC, (2009).

[15] أدمون ميخائيل حنا: "الكيمياء الكهربائية"، الطبعة الاولى، دار الحكمة للطباعة والنشر، بغداد، (1992).

[16] سعد عبد الباري توفيق: "دراسة الخواص البصرية والكهربائية لمادة استينات الكادميوم بطريقة التبخير الثنائي"، أطروحة دكتوراه, الجامعة المستنصرية, كلية التربية، (1996).

[17] G. M. Barrow: "Physical Chemistry", 3rd. Ed., Mc Graw-Hill, Inc., (1973).

[18] D. Hallidy and R. Resnick: "Fundamentals of Physics", John Wiley & Sons, Inc., New York (1981).

[19] Al-Berty, R.A.: "Physical Chemistry". John Wiley & Sons, Inc, Ed. 7. (1987).

[20] سعاد مصلح الدين عبد المجيد: "دراسة تأثير أشعة كاما على الخصائص الفيزيائية لبعض البوليمرات الذائبة في الماء (بولي فاينيل بايروليدون PVPD) باستخدام تقنية الموجات فوق السمعية"، رسالة ماجستير، كلية العلوم، الجامعة المستنصرية، (1993).

[21] G.R. Fowels: "Intordutcion to Modern Optics", 2nd. Edition, Holt Rinehert and Winston, Inc. U.S.A. (1975).

[22] P.M. Reilly, and B.M. Van Der, J. of Applied polymer Sci" 24(10) (1977).

[23] S.K. Hassun, and S.O., Iraqi J.Sci., V.30, No.3. (1989).

[24] د. علي عبد الحسين: "الكيمياء الاشعاعية"، جامعة البصرة (1983).

[25] R.O. Bolt; J.G. Carrol: "Radiation Effect of Organic Materials", Academic Press, New York, (1963).

[26] K. J. Laidler: "Physical Chemistry with Biological Applications", Uni. of Ottawa (1982).

a primary study on using nonlinear analysis methods to measure signal complextiy values of leg by processing laser speckle contrast images

5.1.1 **Adil I. Khalil*** **Waleed A. Al-nuaami** **Ahmed K. Abbas**

University of Diyala, College of Education for Pure Science, Computer Science Deparment – Diyala-Iraq

Nebras J. Ibrahim
University of Diyala, Electronic computer center – Diyala-Iraq

*Please address correspondence to Adil Ibrahim Khalil, adil.khalil.uod@gmail.com
adil.khalil@yahoo.com
Tel: +964 (0) 7705559307

Summary

Purpose: The physiological signals are considered as a sensitive measure for describing human state. The evaluation of such signals can be accomplished by monitoring peripheral blood flow in the skin. Laser speckle contrast imaging (LSCI) is a powerful optical imaging tool that provides two-dimensional information on microvascular blood flow. By applying entropy-based complexity measure to LSCI time series, we present herein a primary study measure signal complexity values obtained from leg into two age healthy groups.

Methods: Leg skin microvascular blood flow was studied with LSCI in 8 healthy subjects. The subjects were subdivided into two age groups; younger (20–30 years old, n=4) and older (50– 68 years old, n=4). To compute complexity values of microvascular blood flow, we applied entropy-based complexity algorithm to LSCI time series obtained from laser speckle contrast images of leg.

Results: The application of entropy-based complexity algorithm to LSCI time series presented higher entropy values obtained from young group than the ones obtained from aged group. However, there was no significant difference between these two age groups (p=0.649).

Conclusion: The impact of aging on microcirculation could be estimated by applying entropy-based complexity algorithms to LSCI time series of leg. However, there was no significant difference on complexity values between aged and younger groups. Further studies with more subjects are needed to confirm the results presented in this paper.

Key words:

1) Laser speckle contrast imaging 2) Image processing 3) Microvascular blood flow 4) Entropy based compexity measures

دراسة اولية في استخدام الخوارزميات الغير خطية لقياس قيم تعقيد اشارة الساق عند الاصحاء من خلال معالجة صور التباين الليزرية

عادل ابراهيم خليل* **وليد احمد النعيمي** **احمد خضير عباس**[1]
جامعة ديالى – كلية التربية للعلوم الصرفة – قسم الحاسوب / العراق – ديالى

نبراس جليل ابراهيم
جامعة ديالى – مركز الحاسبة الالكترونية / العراق – ديالى

لمزيد من المعلومات يرجى مراسلة عادل ابراهيم خليل adil.khalil.uod@gmail.com
adil.khalil@yahoo.com

الخلاصة

الغرض من البحث: تعتبر الإشارات الفسيولوجية اداة دقيقة لوصف حالة الانسان الصحية. يمكن الحصول على هذه الإشارات من خلال مراقبة تدفق الدم المحيطي في الجلد. تعتبر تقنية التصوير الليزرية (Laser speckle contrast imaging (LSCI أداة تصوير بصرية رائدة توفر معلومات ثنائية الأبعاد حول تدفق الدم في الأوعية الدموية الدقيقة. نقدم هنا دراسة أولية لقياس تعقيد الإشارة التي تم الحصول عليها من الصور الليزرية لتدفق الدم في الاوعية الدموية الدقيقة للساق. لهذا الغرض تم تطبيق احدى الخوارزميات القائمة على مبدأ الانتروبي على سلاسلLSCI الزمنية التي تم الحصول عليها من تدفق الدم المحيطي في الساق لفئتين عمريتين مختلفة سالمة من الامراض.

الطريقة: تم الحصول على صور تدفق الدم في الأوعية الدموية الدقيقة للساق باستخدام تقنية LSCI على 8 اشخاص سليمين من اية امراض. تم تقسيم الاشخاص إلى فئتين عمريتين. الشباب (20-30 سنة ، العدد = 4) والمسنين (50-68 سنة ، عدد = 4). لحساب قيم تعقيد اشارة تدفق الدم في الاوعية الدموية الدقيقة ، طبقنا خوارزمية تعقيد مستندة إلى الانتروبي تسمى (Multiscale entropy (MSE على سلسلة LSCI زمنية تم الحصول عليها من تدفق الدم المحيطي في الساق.

النتائج: بين تطبيق خوارزمية MSE على سلسلة LSCI الزمنية تفوق قيم الانتروبي التي تم الحصول عليها من مجموعة الشباب على قيم الانتروبي التي تم الحصول عليها من مجموعة المسنيين. مع ذلك ، لم يكن هناك فرق احصائي مؤثر بين هاتين الفئتين العمريتين(P = 0.649).

الاستنتاجات: بين هذا البحث امكانية دراسة تأثير الشيخوخة على عمل الأوعية الدموية الدقيقة عن طريق تطبيق خوارزميات التعقيد المستندة إلى الانتروبي على سلسلة LSCI الزمنية. مع ذلك ، لم يكن هناك فرق احصائي واضح على قيم الانتروبي بين الفئات العمرية. هناك حاجة إلى عمل دراسة معمقة اكثر تحتوي على عدد اكبر من الاشخاص لاجراء بحث موسع على النتائج المعروضة في هذه الورقة.

الكلمات المفتاحية :

1- جهاز التصوير الليزري Laser speckle contrast imaging 2- معالجة الصور الليزرية
3- تدفق الدم في الاوعية الدموية الدقيقة 4- تعقيد الاشارة 5- الانتروبي

1- مقدمة

دخول التقنيات البصرية في مجال التصوير الطبي وفر إمكانيات مثيرة للحصول على معلومات حول الأنسجة . على مدى السنين القليلة الماضية ، تم تطوير العديد من هذه التقنيات لمراقبة تدفق الدم في الاوعية الدموية الدقيقة (Allen 2014) (Bi 2015). تستند العديد من هذه التقنيات على نمط التداخل العشوائي - المعروف باسم نمط الرقعة speckle pattern الذي يتم إنشاؤه من الضوء المنتشر للأشعة المرتدة. Laser speckle contrast imaging (LSCI) هي تقنية تصوير معاصرة توفر صورة ثنائية الابعاد ذات دقة عالية لتدفق الدم في الأوعية الدقيقة بتكلفة منخفضة (Richards 2013). تستخدم LSCI على نطاق واسع في البحوث الطبية بسبب الاداء العالي وعدم الحاجة الى جراحة. تستغل هذه التقنية التقلبات التي تظهر في نمط الرقطة speckle pattern لتوفير معلومات حول تدفق الدم في الأنسجة السطحية (Bi 2015). يتم تسجيل هذه التقلبات في speckle pattern باستخدام كاميرا . يتم حساب تعبير speckle contrast (*K*) بالاعتماد على وقت التعرض للكاميرا (Briers 2007).

لتقييم عمل الأوعية الدموية الدقيقة في الجلد ، فإن المهمة الصعبة هي كيفية الحصول على المعلومات الفسيولوجية ذات الاهتمام من الصور الطبية. بالتالي تم اقتراح العديد من طرق معالجة الإشارات والصور من أجل السماح لفهم أفضل للخصائص الفسيولوجية الأساسية. من بين هذه الطرق ، أصبح استخدام نظرية المعلومات (Information theory) وخصوصا الانتروبي (Entropy) ذو أهمية كبيرة في المجال الطبي. من بين هذه الادوات هي Sample Entropy (SampEn). مع ذلك فان SampEn يوفر تحليل وحيد النطاق بينما يظهر نظام القلب والاوعية الدموية نطاقات زمنية متعددة لزيادة قدرته على التكيف في بيئة متطورة. بالتالي فإن تعقيد نظام القلب والأوعية الدموية يعمل على نطاقات زمنية متعددة. لهذا السبب لا توفر تحليلات Sample Entropy بمقياس واحد معلومات متعددة المستوى عن سلوك النظام الفسيولوجي المعقد. لذلك تم تقديم Multiscale Entropy (MSE) متعددة النطاقات كأداة مفيدة لمعالجة الإشارات الفسيولوجية في نطاقات زمنية متعددة ، بالاعتماد على نفس المبادئ الاحصائية ل SampEn (Costa 2002). تُستخدم تحليلات MSE على نطاق واسع في البيانات المسجلة من الدورة الدموية الكبرى و لتشخيص أنواع مختلفة من الأمراض ولكن أيضًا يمكن توظيفه لدراسة وظيفة وبنيه الاوعية الدموية الدقيقة.

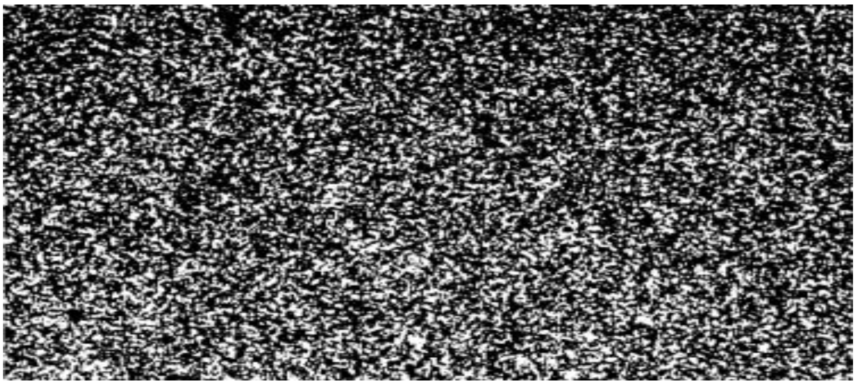
هذا البحث يقدم دراسة اولية لقياس تعقيد الاشارة الفسيولوجية لتدفق الدم في الاوعية الدموية الدقيقة للساق عند الاصحاء. لذلك تم التقاط مجموعة من صور التباين الليزرية (LSCI) لتدفق الدم في الاوعية الدموية الدقيقة للساق. لقياس تعقيد اشارة تدفق الدم في الساق تم تطبيق خوارزمية MSE على بيانات LSCI التي تم الحصول عليها من نظام الأوعية الدموية الدقيقة في جلد الساق. اجريت الدراسة على فئتين عمريتين خالية من الامراض (فئة الشباب , فئة المسنين).

2- الخلفية النظرية

2.1. ما هو speckle ؟

خلال تطور تقنية الليزر التي ظهرت في أوائل 1960، لاحظ الباحثون ظاهرة جديدة سميت فيما بعد speckle. يتم إنتاج speckle من خلال تماسك ضوء الليزر. عندما يقوم مصدر الليزر بأسقاط الضوء فوق جزء خشن ، فإن فوتونات ضوء الليزر سوف ترتد من قبل كل من الجزيئات المتحركة والثابتة. يشكّل الضوء المنعكس نمط تداخل من البكسلات الساطعة والداكنة - تسمى speckle - على الكاميرا (Briers 2007). تؤدي الحركات داخل الجسم المضيء إلى تقلبات زمنية في نمط التداخل وهذا النمط هو نمط ديناميكي. وبدلاً من ذلك ، إذا لم تحدث أي حركات داخل الجسم المضيء وكان ضوء الليزر مستقرا فإن نمط التداخل لا يتذبذب بمرور الوقت. وبالتالي يكون ثابت عند وصف النمط. يوضح الشكل (1) speckle pattern نموذجية.

Laser speckle هي ظاهرة عشوائية ويمكن وصفها إحصائياً فقط. لذلك يوفر التحليل الإحصائي في المجال الزمني أو في المجال المكاني معلومات أساسية عن حركة جسيمات الانتشار (Stern 1975).



الشكل 1: speckle pattern نموذجي

2.2. ما هو speckle contrast ؟

عندما يتم تسجيل speckle pattern باستخدام كاميرا CCD ، وبسبب حركات الجسيمات داخل الوسط (مثل حركة خلايا الدم الحمراء في الأنسجة) ، فان نمط speckle pattern يتغير مع تغير عامل الزمن. بالتالي يتم الحصول على صور ديناميكية على الكاميرا. يؤدي وقت التعرض للكاميرا exposure time, T) عادة من 1 إلى 10 مللي ثانية إلى عدم وضوح أنماط speckle patterns . يتم حساب تباين الرقطة (speckle contrast, K) لتحديد مقدار درجة الضبابية (Briers 1996). حيث يمثل K نسبة الانحراف المعياري σ إلى متوسط الكثافة $\langle I \rangle$:

$$K = \frac{\sigma}{\langle I \rangle} = \frac{\sqrt{\langle I^2 \rangle - \langle I \rangle^2}}{\langle I \rangle} \quad (1)$$

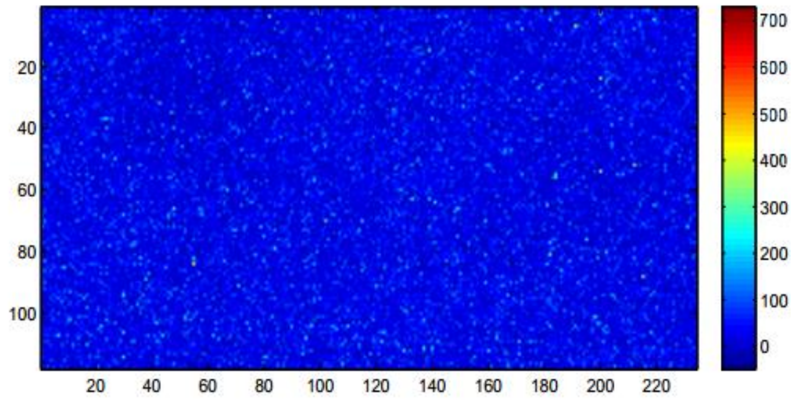
في المعادلة السابقة , يتم حساب الانحراف المعياري ومتوسط كثافة البكسل في منطقة N حول البكسل P(X,Y).

2.3. حساب perfusion من خلال speckle contrast

إذا افترضنا أن الجسيمات المتحركة تتبع توزيع (Lorentzian), فأن حساب نضح الدم perfusion يتم من خلال المعادلة التالية :

$$Perfusion \sim \frac{1}{K} - 1 \quad (2)$$

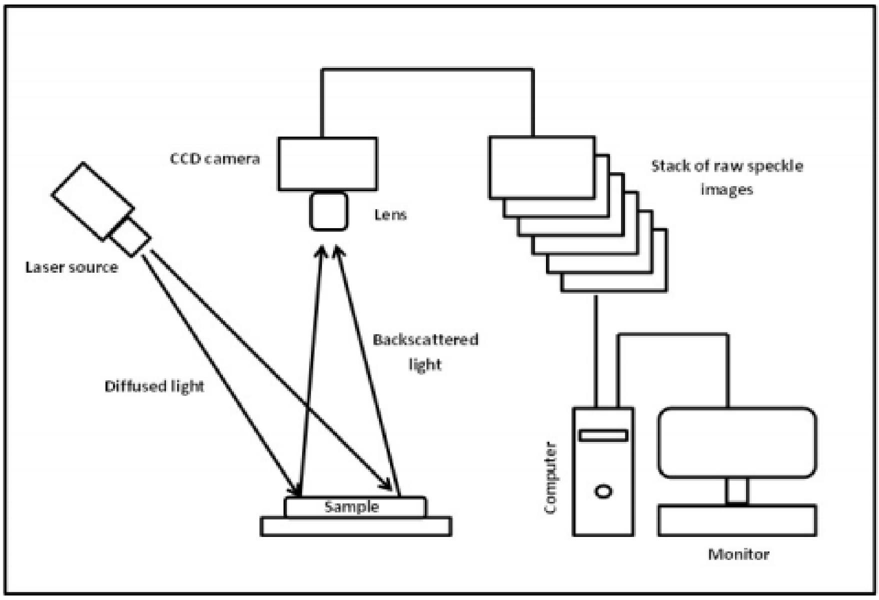
الشكل ادناه يوضح صورة عملية لنضح الدم (**perfusion**)



الشكل 2: صورة نضح الدم للساق (بحجم 118 × 234 بكسل) عند الاسترخاء حيث تم الحصول عليها باستخدام LSCI.

2.4. تقنية Laser speckle contrast imaging

أصبحت تقنية LSCI معتمدة على نطاق واسع في مجال الطب الحيوي لبساطة الأجهزة المطلوبة لرصد تدفق الدم. تتكون أجهزة LSCI من مصدر ليزر لإلقاء الضوء على المنطقة ذات الاهتمام مثل الأنسجة ، وكاميرا لاستشعار الضوء المبعثر ، وعدسة لجمع الضوء على مستشعر الكاميرا .(يعرض الشكل 3 رسما تخطيطيًا لتقنية LSCI) . يتم اسقاط شعاع الليزر والتحكم به لإلقاء الضوء على المنطقة المدروسة ، والتي قد تختلف من بضعة ملليمترات إلى عدة سنتيمترات. وفقًا لشروط النظام يكون ضوء الليزر عادة في المنطقة الحمراء ليقترب من منطقة الأشعة تحت الحمراء لتقليل التأثيرات الناتجة عن امتصاص الهيموغلوبين. يمكن استخدام كاميرا CCD (كاميرا قياسية غير مكلفة تستخدم للحصول على صور ممتازة لتدفق الدم) في LSCI . بالإضافة إلى ذلك ، أثبتت الأبحاث الحديثة أن LSCI يمكنها توفير خرائط عالية الجودة لنضح الدم حتى مع أجهزة تصوير بسيطة مثل الكاميرات الملونة وكاميرات الويب وكاميرات الهواتف (Briers 1996).



الشكل 3: رسم تخطيطي لنظام LSCI.

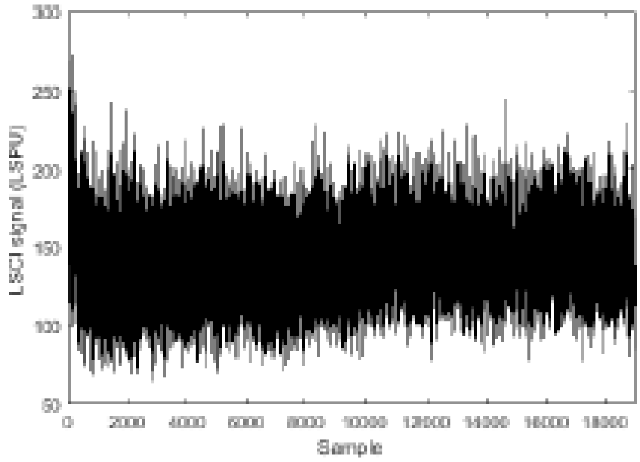
3- منهجية البحث

3.1. المجموعة التي تم اختبارها

تم مشاركة ثمانية اشخاص في هذه الدراسة في حالتهم الصحية الطبيعية دون تاريخ مرضي معروف. تم تقسيم الاشخاص إلى فئتين عمريتين : مجموعة شابة و مجموعة مسنين . شملت المجموعة الأصغر اربعة اشخاص تتراوح أعمارهم بين 20 و 30 سنة. تضمنت المجموعة الأكبر سنًا اربعة اشخاص تتراوح أعمارهم بين 50 و 68 عامًا. قبل المشاركة أعطى جميع المشاركين موافقتهم الخطية ، وتم إجراء الدراسة وفقًا لإعلان هلسنكي.

3.2. بروتوكول الاختبار

لتطبيق MSE على سلسلة LSCI زمنية ، تم الحصول على جميع صور تدفق الدم من الساق باستخدام تقنية LSCI. يبلغ طول موجة التصوير بالليزر 785 نانومتر ووقت التعرض (Exposure time, *T*) 6 مللي ثانية. في هذا الجهاز يتم الحصول على الصور الاولية (speckle pattern) في المنطقة المضيئة باستخدام كاميرا CCD بحجم 1038 × 1388 بكسل ، ويتم حساب التباين (speckle contrast, *K*) بعد ذلك مكانيًا. يتم حساب تدفق الدم (perfusion) من قيم التباين (المحسوب من عكس التباين K). تقنية التصوير بالليزر(LSCI) بحكم تعريفها فهي حساسة للغاية للحركات. لذلك طُلب من الأشخاص أن يكونوا مستلقين دون اجراء اي حركة أثناء الحصول على البيانات. قبل معالجة بيانات LSCI باستخدام المقاييس المستندة ، لم يتم إجراء أي معالجة مسبقة لإزالة التواجد المحتمل للقيم المتطرفة (حرصنا على التحقق من أن القيم المتطرفة إذا كانت موجودة كانت قليلة للغاية وذات سعة منخفضة ،(انظر الشكل 4).



الشكل 4: الدورة الزمنية النسبية لتدفق الدم محسوبة من بيانات LSCI خلال 20 دقيقة من الاسترخاء.

في هذا البحث ، تم معالجة 19000 صورة (حوالي 20 دقيقة) لكل شخص.

3.3. طريقة معالجة الصور الليزرية

لتحليل مدى تعقيد سلاسل LSCI الزمنية تم استخدام خطوات معالجة الصور التالية :

1. في أول صورة نضح perfusion image لكل شخص تم اختيار بكسل واحد بشكل عشوائي ، وتم تتبع نضحه مع الزمن لجميع الصور المتعاقبة.

2. للحصول على إشارة معقولة وتقليل التباين المكاني لتدفق الدم ، تم حساب متوسط قيمة النضح داخل منطقة مربعة (ROI) حول كل من البكسلات المختارة في الخطوة 1 مع الزمن. لهذا الغرض تم اختيار منطقة مربعة بالحجم (31 x 31) بكسل.

3. يتم حساب قيم MSE لكل سلسلة زمنية وعرضها كدالة لعامل القياس τ (1—267).

3.4. التحليل الاحصائي

تم إجراء تحليلات إحصائية باستخدام تحليل t-test لمقارنة النتائج بين مجموعة الشباب والمسنين. بالنسبة للمجموعتين (مجموعة الشباب ومجموعة المسنين) تم حساب متوسط قيم MSE على المقياس الذي تم دراسته من 1 الى 267 . تم اجراء تحليلًا إحصائيًا على هذا الفهرس لمقارنة النتائج بين مجموعة الشباب ومجموعة الأكبر سناً. بالنسبة لجميع التحليلات الإحصائية ، اعتبرت قيمة $p < 0.05$ قيمة مؤثرة.

3.5. نظرية المعلومات (Entropy)

المفهوم الرئيسي لنظرية المعلومات هو الإنتروبي وهو مقياس الشك المرتبط بمتغير عشوائي. وفي هذا السياق، فإن المصطلح يشير عادة إلى شانون انتروبي ، الذي يحدد كمية القيمة المتوقعة للمعلومات الواردة في رسالة ما.

إنتروبي المتغير العشوائي X هو:

$$H(X) = -\sum_{i=1}^{n} p_i \log p_i \qquad (3)$$

حيث p_i هو احتمالية حدوث المخرج x_i.

الانتروبي لديه الخصائص التالية:

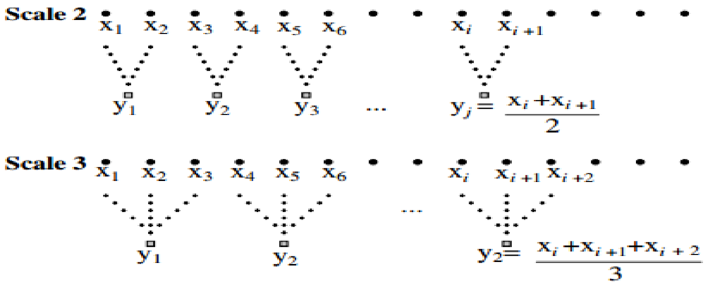
1. $H(x) \geq 0$
2. $H(x) = 0$ إذا كان وفقط إذا كان المتغير "العشوائي X " لديه نتيجة واحدة فقط (Certain event)
3. اعلى قيمة للانتروبي يمكن الحصول عليها عندما يمتلك المتغير العشوائي احتمالية متساوية لكافة النتائج.

3.6. الانتروبي متعدد النطاق MSE

يهدف نهج MSE إلى دراسة تعقيد النظام الديناميكي عبر نطاقات زمنية متعددة (انظر الشكل 5). بالنسبة لمتجه البيانات أحادي البعد ، { x1، ..... ، xi ، .... xN} ، فإن مجموعات من النقاط المتتالية يتم تجميعها معا لتشكيل مجموعات اصغر وعلى مستويات مختلفة بمقدار (τ). لهذا الغرض ، تنقسم السلاسل الزمنية الأصلية إلى مجموعات غير متداخلة الطول. يتم حساب متوسط نقاط البيانات داخل كل مجموعة. يتم إنجاز الخطوات المذكورة أعلاه لإنشاء المجموعات الزمنية الغير متداخلة باستخدام المعادلة التالية

$$y_j^{(\tau)} = \frac{1}{\tau} \sum_{i=(j-1)\tau+1}^{j\tau} x_i, \qquad 1 \leq j \leq N/\tau. \qquad (4)$$

أخيرًا ، يتم تقييم كل سلسلة زمنية من عن طريق حساب مقياس إنتروبي (SampEn). يتم عرض النتيجة مقابل عامل القياس τ.



الشكل 5: مخطط يوضح طريقة توليد السلاسل الزمنية لحساب MSE (Costa 2002) .

خوارزمية SampEn هي مفهوم الاحتمال الشرطي : اذا كان هناك مجموعتين فرعيتين مضمنتين قريبتين من بعضهما البعض بالنسبة للنقاط المتعاقبة m ، ضمن حد تسامح معطى r ، فان هاتين المجموعتين ستبقى أيضًا قريبة من بعضها البعض إذا تم تضمين نقطة جديدة أخرى في كل مجموعة فرعية.

بالنسبة لسلسلة لبيانات N ، فان SampEn يتم حسابه بالشكل التالي

$$SampEn(m, r) = \lim_{N \to +\infty} \left\{ -\ln \left[\frac{B^{m+1}(r)}{B^m(r)} \right] \right\}. \qquad (5)$$

حساب SampEn لمجموعة محددة من البيانات يتم عن طريق المعادلة التالية (Moorman 2000)

$$SampEn(m, r, N) = -\ln \left[\frac{B^{m+1}(r)}{B^m(r)} \right]. \qquad (6)$$

وبالتالي يتم حساب MSE لكل سلسلة زمنية مصنوعة تم إنشاؤها كما في المعادلة التالية .

$$MSE(x,\tau,m,r) = -\ln\left(\frac{n_\tau^{m+1}}{n_\tau^m}\right), \quad (7)$$

حيث يمثل n_τ^m العدد الإجمالي لأزواج المتجهات المتطابقة (السلاسل الزمنية) عند عامل مقياس τ.

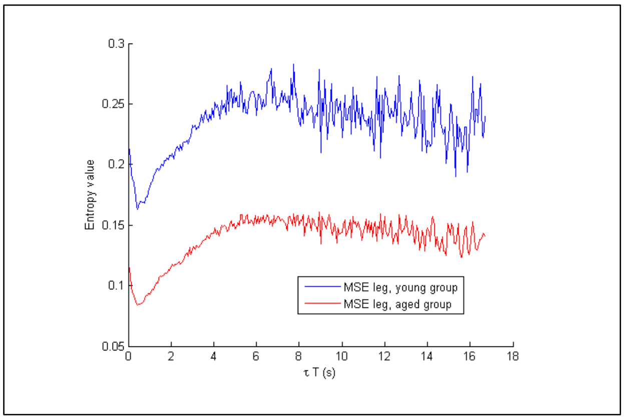
في خوارزمية MSE ، يتم رسم القيم المقدرة لـ SampEn مقابل عوامل القياس τ. تُستخدم قيم الانتروبي هذه لتقييم درجة تعقيد السلاسل الزمنية الطبيعية. يشير السلوك المتزايد أو الثابت لقيم الإنتروبي مقابل زيادة عامل القياس τ إلى أن السلسلة الزمنية الأصلية معقدة للغاية ، وتحتوي على معلومات عبر جداول زمنية متعددة. في المقابل يظهر انخفاض قيم التعقيد مقابل زيادة في قيم عامل القياس τ ان السلسلة الزمنية الأصلية تحمل معلومات فقط عند معامل القياسات الصغيرة τ (Costa 2002).

4. النتائج والمناقشة

يعرض الشكل (5) النتائج العملية المتوسطة لـقيم الانتروبي التي تم الحصول عليها عند تطبيق MSE على سلاسل LSCI الزمنية للساق للمجموعات العمرية التي تم اختبارها (الشباب والمسنين). من هذا الشكل يمكن ملاحظة انحدار مستمر لقيم الاننتروبي لتستقر عند الفاصل الزمني ($\tau T = 1$) ثم تعاود الارتفاع مجددا لتستقر على وتيرة ثابتة على طول مقايس النطاق الزمنية التي تم اختبارها. يعود الانحدار في قيم الانتروبي نتيجة عمل القلب المنتظم الذي تم ملاحظته في ابحاث سابقة عند هذا الفاصل الزمني. اضافة الى ذلك يمكن ملاحظة فقدان في قيم تعقيد الانتروبي مع تقدم العمر. يمكننا أن نلاحظ أن قيم الانتروبي التي تم الحصول عليها من المجموعة الأصغر سنا (الأزرق) أعلى من القيم التي تم الحصول عليها من المجموعة العمرية الاكبر سنا (الحمراء). نظام الكائن الحي هو نظام معقد للغاية. يأتي هذا التعقيد من مجموعة واسعة من ردود الفعل التكيفية مع المتغيرات الفسيولوجية المختلفة في البيئة الخارجية. لذلك فإن التعقيد الفسيولوجي للنظام الحي يعكس قدرته على التكيف مع الظروف المتغيرة باستمرار ، والتي ستكون ضرورية لدمج العمليات متعددة النطاقات. بدلا من ذلك ، في ظل حالة خط الأساس ، يعكس الانخفاض المستمر في التعقيد ضعف الاستجابات الفسيولوجية للكائن الحي للتغيرات في البيئة الخارجية . من خلال تطبيق MSE على بيانات الأوعية الدموية للساق ، فقد لوحظ فقدان التعقيد في إشارة الاوعية الدموية الدقيقة بسبب الشيخوخة .

فقدان التعقيد الذي لوحظ مع مجموعة المسنين قد يكون بسبب حدوث خلل في عمل وهيكلية الأوعية الدموية الدقيقة مع تقدم العمر. لقد لوحظ تدهور في أنشطة الأوعية الدموية الدقيقة مع تقدم السن في العديد من الدراسات السابقة (Tikhonova 2010). مع تقدم السن قد يحدث انخفاض في كمية الأكسجين التي تصل إلى الأنسجة ، وعدم الاتزان في عمليات التمثيل الغذائي المهم في عملية البناء والهدم (Harris) 2010. علاوة على ذلك ،مع تقدم العمر قد يظهر انخفاض في أعداد الشعيرات الدموية العاملة ، وعيوب في وظائفها الأساسية بسبب ظواهر مثل فقدان الانتظام , تدمير وترهل الأوعية الدموية (Tikhonova 2010). وقد تم الإشارة ايضا إن لشيخوخة ترتبط بتثبيط وظيفة بطانة الأوعية الدموية والعمليات الكيميائية الخلوية ، وتدهور الجهاز العصبي(Kenney 2003) .

أن تطبيق MSE على البيانات LSCI يمكن أن يفرق بين المجموعات الأصغر والأكبر سنا باستخدام بيانات LSCI للساق : تقلبات المجموعة الأصغر سنا تظهر تعقيدات أعلى من تلك التي تم الحصول عليها من المجموعة الاكبر سنا. يمكن تفسير فقدان التعقيد في إشارة تدفق الدم في الأوعية الدموية الدقيقة للمسنيين كنتيجة للتغيرات التي تحدث داخل القلب والأوعية الدموية مع التقدم في العمر. مع ذلك ، لم تكن هناك فروقات ذات دلالة إحصائية بين المجموعات الأصغر سنا والأكبر سنا على قيم مؤشر الانتروبي التي تم الحصول عليها من MSE على بيانات LSCI للساق (p=0.649 ، انظر الشكل5).



الشكل 5 : متوسط قيم الانتروبي التي تم الحصول عليها من الساق لمجموعتي الأصحاء من الاشخاص : المجموعة الأصغر سنا (الأزرق) والمجموعة الأكبر سنا (الأحمر) ل 4 اشخاص في كل مجموعة. لكل مجموعة تم تطبيق MSE على بيانات LSCI. نطاق القياس المستخدم من ($\tau = 1$ to $\tau = 267$) .

5. الاستنتاجات والتوصيات

بينت هذه الدراسة أن تقلبات الاشارة الفسيولوجية للاوعية الدموية الدقيقة لدى الأشخاص الأصحاء الشباب معقدة للغاية لكن هذا التعقيد يتناقص مع تقدم العمر. لقد تم التوصل سابقًا الى وجود علاقة وثيقة بين الدورة الدموية الكبرى والصغرى واعتبارهما نظامين مترابطين (Khalil 2014). لذلك فان هذه النتائج تعطي دلالة على امكانية التنبوء بامراض قد تصيب القلب من خلال دراسة عمل الاوعية الدموية الدقيقة. اضافة الى ذلك بين هذا البحث امكانية استخدام الخوارزميات الغير خطية لدراسة عمل الاوعية الدموية الدقيقة من خلال معالجة صور التباين الليزرية لتدفق الدم في الساق.

من اهم المحددات في هذا البحث هو حجم العينه الصغير نسبيا. لذلك نوصي بضرورة تطبيق نفس الدراسة على عينة اكبر بعد التاكد من التوزيع الطبيعي لهذه العينه. اضافة الى ذلك نوصي بتطبيق خوازميات غير خطية جديدة على الصور الليزرية ومقارنتها مع النتائج المعروضة في هذا البحث. تطبيق خوارزميات غير خطية جديدة على اشخاص يعانون من امراض تصيب الاوعية الدموية الدقيقة مثل السكري قد يعطي تصور دقيق عن طرق العلاج والوقاية المبكرة.

.6 المصادر

6

7 Allen, J., and Kevin H. 2014. "Microvascular imaging: techniques and opportunities for clinical physiological measurements." *Physiological measurement* 35 (7): R91.

Bi, R. et al. 2015. "Optical methods for blood perfusion measurement theoretical comparison among four different modalities." *JOSA* 32 (5): 860-866.

Briers, J. D. and Webster, S. 1996. "Laser speckle contrast analysis (LASCA): a non scanning, full-field technique for monitoring capillary blood flow." *Journal of biomedical optics* 1 (2): 174-180.

Briers, J. D. 2007. "Laser speckle contrast imaging for measuring blood flow." *Optica Applicata* 37.

Costa, M., Goldberger, A.L., Peng C.-K. 2002. "Multiscale entropy to distinguish between physiologic and synthetic RR time series." *Computers in cardiology* 137-140.

Harris, Norman R., and Rolando E. Rumbaut. 2010. "Age-related responses of the microcirculation to ischemia-reperfusion and inflammation." *Pathophysiology* 8 (1): 1--10.

Kenney, W. Larry, and Thayne A. Munce. 2003. "Invited review: aging and human temperature regulation." *Journal of applied physiology* 95 (6): 2598-2603.

Khalil, A., Humeau-Heurtier, A., Mahé, G., and Abraham, P. 2014. "Laser speckle contrast imaging: age-related changes in microvascular blood flow and correlation with pulse-wave velocity in healthy subjects." *Journal of biomedical optics* 20 (5): 051010.

Moorman, J. S. Richman and J. R. 2000. "Physiological time-series analysis using approximate entropy and sample entropy." *Am. J. Physiol* 278: H2039–H2049.

Richards, L. M., Kazmi, S. M. S., Davis, J. L., Olin, K. E., and Dunn, A. K. 2013. "Low-cost laser speckle contrast imaging of blood flow using a webcam." *Biomedical optics express* 2269-2283.

Stern, M. D. 1975. ""In vivo evaluation of microcirculation by coherent light scattering."." *Nature* 254.5495.

Tikhonova, I. V., Arina V. T., and Nikolay K. C. 2010. "Time–amplitude analysis of skin blood flow oscillations during the post-occlusive reactive hyperemia in human." *Microvascular research* . 80 (1): 58-64.

8

حماية البيانات في السحابة بإستخدام الاخفاء المعتمد على خوارزمية بحث طائر الوقواق

نادية معن محمد د. نجلاء بديع الدباغ
قسم هندسة البرمجيات قسم علوم الحاسوب
كلية علوم الحاسوب والرياضيات/ جامعة الموصل

الملخص

في هذا البحث تم حماية البيانات السرية لمؤسسة ما في السحابة بإستخدام طريقة مقترحة لإخفاء المعلومات تعتمد إحدى خوارزميات مابعد الحدس (metaheuristic) خوارزمية بحث طائر الوقواق ((Search Cuckoo (CS) وهي مستوحاة من سلوك تربية طائر الوقواق. تعتمد الطريقة المقترحة على خوارزمية بحث طائر الوقواق للبحث عن المواقع المثلى للاخفاء في الصورة الغطاء لغرض اخفاء المعلومات السرية فيها، حيث يتم اولا استخدام المسافة الاقليدية (Euclidian Distance) لتحديد أفضل نقطة (pixel) واستخدام رحلة ليفي Lévy العشوائية لتحقيق التحرك العشوائي من نقطة إلى آخرى. ثانيا يتم التضمين في مكونات RGB بإستخدام تقنية الاستبدال الهاشية الأقل أهمية Hash Least Significant Bit technique) (HLSB2−3−3)). وأخيرًا، يتم رفع الصورة الى السحابة بعد ان تم تضمين البيانات السرية فيها. ان الاخفاء باستخدام هذه الطريقة اثبت كفاءته من خلال قيم نسبة الإشارة إلى الضوضاء (Peak Signal Noise Ratio(PSNR) التي تم الحصول عليها مقارنة مع الطرائق التي تعتمد الخوارزميات الاخرى.

Protecting Data in Cloud using Steganography based on Cuckoo Search Algorithm

Abstract

In this research, a corporation's confidential data is protected in the cloud using a proposed method of hiding information based on one of the metaheuristic algorithms its Cuckoo Search (CS) algorithm, which is inspired by Cuckoo's breeding behavior. The proposed method is based on a Cuckoo Search algorithm to search for the optimal hiding locations in the cover image for the purpose of embedding the secret information in it. In the proposed method, first, the Euclidian Distance is used to determine the best pixel and use the Lévy random flight to achieve random movement from pixel to pixel. Second, the data are embedded in the RGB components using the Hash Least Significant Bit technique (HLSB 3-3-2). Finally, the image (after hiding) is uploaded to the cloud after the secret data has been embedded in it. Steganography using this method proved its efficiency through Peak Signal Noise Ratio (PSNR) values which is obtained in comparison with the methods based on other algorithms.

Keywords:
Cloud Computing, Data Storage, Security, Steganography, Cuckoo Search.

1. المقدمة

تعتبر الحوسبة السحابية نموذجاً مشهوراً جداً ولايزال مهمًا للغاية فهي تتيح للمؤسسات والأفراد الاستفادة من الموارد المتاحة على السحابة لإنقاذهم من تكلفة شراء هذه الموارد والمحافظة عليها وصيانتها. هناك عدة أنواع من نماذج نشر الحوسبة السحابية مثل السحابة الخاصة، السحابة العامة، سحابة المجتمع والسحابة الهجينة. تتكون البنية التحتية للحوسبة السحابية من بعض نماذج الخدمة الأساسية مثل البرمجيات كنموذج خدمة (SaaS)، النظام الأساسي كنموذج خدمة (PaaS) والبنية التحتية كخدمة (IaaS). يتم استخدام الموارد المستخدمة في الحوسبة السحابية من قِبل جهة خارجية، وهذا يصبح مصدر قلق بالغ للمستخدم نظرًا لأن المستخدم قد لا يعرف من الذي يصل إلى بياناته في السحابة وما إذا كانت التغييرات قد أجريت على البيانات من الطرف الثالث الذي لديه سيطرة كاملة على البيانات بمجرد إرسالها إلى السحابة[1].

مشكلة كبيرة في الحوسبة السحابية هي الأمن، عندما تكون البيانات موجودة في أي خادم، فهناك فرصة أكبر لتعرضها للهجوم. يجب أن تكون البيانات آمنة خلال جميع مراحل المعالجة بما في ذلك: التخزين والمعالجة والتحميل. هناك نوعان من هجمات التهديد في الحوسبة السحابية. التهديد الأول هو التهديد الداخلي الذي يمثل تهديدًا كبيرًا من قبل المسؤول (موفر الخدمة السحابية) والذي قد يقوم ببعض عمليات الوصول غير القانونية إلى بيانات المستخدم أو حتى إجراء تغييرات على البيانات دون موافقة المستخدم لأن المستخدم لا يتحكم في بياناته في

السحابة. التهديد الثاني هو التهديد الخارجي، هذا النوع من التهديد هو في الأساس من المتسللين الذين قد يحصلون على وصول غير قانوني إلى المورد في السحابة دون أن يدفعوا مقابل ذلك أو يكون لديهم وصول غير قانوني إلى بيانات المستخدم في السحابة وحتى تلف البيانات أو إجراء تغييرات عليها. ومع ذلك، نظرًا لأن الوصول إلى الموارد السحابية يزداد يوميًا، تصبح الحاجة إلى حماية هذه الموارد من المستخدمين غير المصرح لهم أمراً ضرورياً لتجنب تداخل وتعديل الموارد من قِبل المستخدمين الضارين[2].

يمكن للعملاء تخزين كمية كبيرة من البيانات في مراكز تخزين البيانات السحابية، لكن العديد من المستخدمين لايقومون بالخزن في الحوسبة السحابية بسبب الضعف في حماية البيانات، ولهذا أصبح أمن المعلومات جزءًا من حياتنا اليومية. تم استخدام طرق مختلفة مثل التشفير وإخفاء المعلومات والترميز وما إلى ذلك لهذا الغرض. ومع ذلك، في السنوات الأخيرة، اجتذب إخفاء المعلومات مزيدًا من الاهتمام، حيث تتمثل الطريقة الجيدة لتأمين البيانات السرية في الحوسبة السحابية في إخفاء البيانات في صورة أو صوت أو فيديو بواسطة تقنيات التغطية (Steganography)[3].

في هذا البحث، تم اقتراح طريقة لإخفاء البيانات لزيادة أمان البيانات في الإرسال ولزيادة أمان البيانات المخزونة في السحابة. وفيها استخدمت خوارزمية بحث طائر الوقواق (التي تعد واحدة من الخوارزميات الذكية) لايجاد أفضل المواقع (النقاط) في الصورة الغطاء لإخفاء الرسالة السرية فيها. يتم الإخفاء في وحدات البكسل الناقلة ضمن مكونات RGB بإستخدام تقنية الاستبدال الهاشية الاقل اهمية 3-3-2 HLSB. في هذه الطريقة، يتم تضمين البتات الثلاث الأولى من الرسالة في الأجزاء الثلاثة المختارة من المكون الأحمر، البتات الثلاث الثانية في الأجزاء الثلاثة المختارة من المكون الأخضر والبتان الأخيرتان في اول 2 بت من اللون الأزرق. وهذا يعني تمكننا من إخفاء البايت في كل بكسل من الصورة الملونة الحقيقية (24 بت).

تم تنظيم باقي البحث على النحو التالي: توضح الفقرة 2 الأعمال السابقة. تشرح الفقرة 3 خوارزمية بحث طائر الوقواق المستخدمة في الطريقة المقترحة. تصف الفقرة 4 الطريقة المقترحة. اما الفقرة 5، فتقدم نتائج وتقييم لأداء الطريقة المقترحة وتختتم الفقرة 6 استنتاجات البحث.

2. الاعمال السابقة

أجريت العديد من الأعمال البحثية لتأمين البيانات في السحابة. في هذه الفقرة، يتم تقديم تقنيات إخفاء المعلومات سواءاً التي تم تنفيذها لتأمين البيانات في السحابة او التي نفذت بإعتماد الخوارزميات الذكائية. في [1]، اقترح المؤلفون تقنية إخفاء معلومات لتحسين الأمان في البيانات أثناء الاستراحة. يضمن النموذج المقترح تقريبًا تكامل البيانات عند إنشائها في مركز البيانات لأي مزود خدمة سحابية. ولكن هذه التقنية يمكن أن تتعامل مع عدد محدود من التهديدات الأمنية في بيئة صغيرة. وفي [2]، اقترح المؤلفون طريقة لإخفاء البيانات الحساسة داخل صورة ضمن السحابة العامة. في هذه الطريقة، يتم إخفاء البيانات الحساسة داخل صورة تشاركها المؤسسات وستكون هذه الصورة واضحة فقط للعملاء. استخدام الإخفاء ادى إلى تقليل الوصول غير المصرح به للمهاجمين على البيانات الحساسة.

اقترح الباحثون في [3] تقنية لتطوير الأمن في الإطار السحابي باستخدام إخفاء المعلومات. في التقنية المقترحة، قاموا باستخدام إخفاء المعلومات إلى جانب مراقبة مالك المعلومات للعملاء الداخليين أو الخارجيين. حيث أنشؤوا صورة الغطاء ديناميكيًا لتعزيز الأمان، وتغطي هذه الصورة الرسالة المراد إخفاءها. هذه التقنية أدت إلى تطوير الأمن. قام الباحثون في [4]، بتحسين تقنية 3-3-2 LSB باستخدام الخوارزمية الجينية (GA) للحصول على إدراك مثالي للبيانات المخفية. وقد أظهرت النتائج التي تم الحصول عليها أن PSNR تقع بين 20 و40 ديسيبل. وفي [5]، قام الباحث بتطوير طريقة البت الأقل أهمية بإستخدام امثلية اسراب الطيور لإخفاء المعلومات داخل الصور. وأظهرت النتائج التي تم الحصول عليها أن قيم PSNR عند إستخدام امثلية اسراب الطيور تقع بين 57.34 و62.87 ديسيبل.

في [6]، قام الباحث بتحسين تقنية 3-3-2 LSB باستخدام خوارزمية حشرة اليراعة وخوارزمية سرب القطط لأخفاء المعلومات داخل الصور. أظهرت النتائج التي تم الحصول عليها أن قيم PSNR عند إستخدام خوارزمية حشرة اليراعة تقع بين 77.44 و78.08 ديسيبل وقيم PSNR عند إستخدام خوارزمية سرب القطط تقع بين 77.21 و79.57 ديسيبل. أما في [7]، فقد تم تحسين تقنية 3-3-2 LSB بإستخدام الدالة الهاشية لإيجاد مواقع الإخفاء في LSB bits، وقد أظهرت النتائج ان قيم PSNR التي تم الحصول عليها تقع بين 42.66 و45.67

ديسيبل. واخيرا في [8]، إستخدمت خوارزمية بحث طائر الوقواق لحل مشكلة الإخفاء في الفيديو، وقد أظهرت النتائج التي تم الحصول عليها أن قيم (PSNR) تقع بين 47.74 و 54.23 ديسيبل.

3. خوارزمية بحث طائر الوقواق (CS)

اقترحت هذه الخوارزمية لأول مرة بواسطة يانغ وديب. تعتمد هذه الخوارزمية على السلوك الطفيلي الملتوي الموجود في بعض أنواع الوقواق، وفي الوقت نفسه تجمع بين سلوك رحلة ليفي Lévy المكتشفة. إن خوارزمية بحث طائر الوقواق CS هي خوارزمية قائمة على السكان، بطريقة مشابهة للخوارزمية الجينية (GA) وخوارزمية الاسراب (PSO). إن التوزيع العشوائي في خوارزمية بحث طائر الوقواق أكثر كفاءة حيث أن طول الخطوة يتم التحكم به حسب القيمة المعطاة، وأي خطوة كبيرة ممكنة. عدد المعلمات المطلوب ضبطها في خوارزمية بحث طائر الوقواق أقل من GA وPSO، وبالتالي من المحتمل أن يكون أكثر عمومية للتكيف مع فئة أوسع من مشكلات التحسين. بالإضافة إلى ذلك، يمكن أن يمثل كل عش مجموعة من الحلول، بالتالي يمكن توسيع نطاق بحث طائر الوقواق CS إلى خوارزمية التعداد السكاني. يتم وصف القواعد المثالية الثلاثة لخوارزمية CS على النحو التالي [9].

- يضع كل طائر وقواق بيضة واحدة في كل مرة، ويقذف بيضه في عش تم اختياره عشوائياً.
- تنتقل أفضل الأعشاش ذات الجودة العالية للبيض إلى الأجيال القادمة.
- تثبيت عدد الأعشاش المضيفة المتاحة، ويتم اكتشاف البيض الموضوع بواسطة طائر من الطيور المضيف مع احتمال قيمة $P_a \in [0,1]$.

في هذه الحالة، يمكن للطيور المضيفة إلقاء البيضة أو التخلي عن العش، وبناء عش جديد تمامًا.استنادًا إلى هذه القواعد الثلاثة، يمكن تلخيص الخطوات الأساسية لـ CS كخوارزمية كما هو موضح في الشكل (1) [10].

Begin
Objective function $f(x)$, $x = (x1, x2, ..., xd)^T$;
Generate initial population of n host
nests x_i $(i = 1, 2, ..., n)$.
While ($t < MaxGeneration$) or (*stop criterion*)
Get a cuckoo randomly by Lévy flights;
Evaluate its quality/fitness F_i;
Choose a nest among n (*say*, j) randomly.
If $(F_i > F_j)$,
Replace j by the new solution;
End
A fraction (P_a) of worse nests are abandoned
and new ones are built;
Keep the best solutions (or nests with quality solutions);
Rank the solutions and find the current best.
End while
Postprocess results and visualization.
End

الشكل (1): خوارزمية بحث طائر الوقواق

خوارزميًا، يتم تعريف كل عش مضيف (n) على أنه عامل يمكن أن يحتوي على بيضة بسيطة (x). تعتمد تقنية بحث طائر الوقواق على رحلة ليفي Lévy كمسار عشوائي، والذي يستخدم لإنتاج جيل (خليط) جديد من الجيل الحالي وفقًا للمعادلة (1).

$$X_i^{(t+1)} = X_i^{(t)} + \alpha \oplus \text{Lévy}(\lambda) \quad \text{.............................} \quad (1)$$

حيث ان $X_i^{(t+1)}$ هو طائر الوقواق ذو التسلسل (i)، على سبيل المثال t + 1

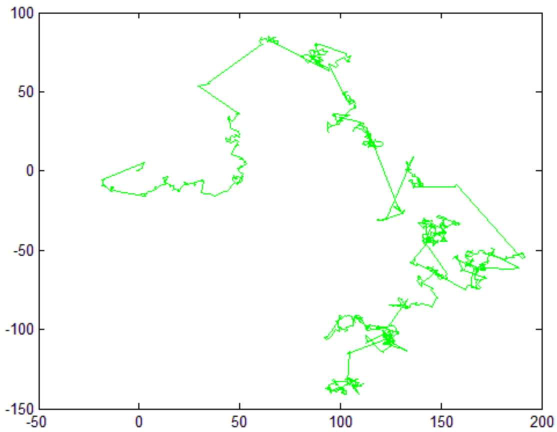
α هو حجم الخطوة،

λ هو معامل توزيع ليفي.

توفر رحلة ليفي lévy بشكل أساسي نزهة عشوائية، بينما يتم رسم طول الخطوة العشوائية من توزيع ليفي في المعادلة (2).

$$Lévy \sim u = t^{-\lambda}, \qquad (1 < \lambda <= 3), \qquad \ldots\ldots\ldots\ldots(2)$$

إن السير العشوائي عبر رحلة Lévy أكثر كفاءة حيث أن طول خطها أطول بكثير على المدى الطويل، لاحظ الشكل (2) [11].



الشكل (2): مثال على رحلة Lévy بدءًا من [0 ، 0]

4. الطريقة المقترحة

تستخدم الطريقة المقترحة إخفاء المعلومات في الصور لانها الأكثر شعبية بسبب توافرها على الإنترنت. في هذا البحث، تم التركيز على زيادة الأمان من خلال نقل البيانات وتخزينها في مركز بيانات التخزين السحابي، حيث تهدف الطريقة إلى تأمين البيانات في الاستراحة وأثناء النقل من خلال إخفاء البيانات السرية داخل الصورة.

قمنا بإقتراح طريقة فعالة استخدمت مواقع عشوائية (بكسل) في الصورة يتم إيجادها بإستخدام خوارزمية بحث طائر الوقواق لإخفاء البيانات السرية. حيث تحتوي كل بكسل من الصور الملونة على ثلاث بايتات RGB: الأحمر والأخضر والأزرق، كل لون لديه 8 بت. ملخص عمل هذا البحث، استخدام LSB مع خوارزمية بحث طائر الوقواق والدالة الهاشية لإخفاء المعلومات السرية في الصورة الغلاف ثم إرسال هذه الصور إلى مساحة التخزين في الحوسبة السحابية.

1.4 ايجاد المواقع المثلى بإستخدام خوارزمية بحث طائر الوقواق

الهدف الرئيسي لخوارزمية بحث طائر الوقواق CS في هذه الورقة هو البحث عن المواقع المناسبة في الصورة الغطاء لتضمين البيانات السرية، ويتم استخدام هذه المواقع كمفتاح سري. تستخدام هذه الخوارزمية كالاتي:

1. تهيئة معلمات CS: عدد الأعشاش ($n = 5$)، حجم الخطوة ($\alpha = 1$)، عدد الأجيال = 100 واحتمال اكتشاف بيض أجنبي ($P_a = 0.25$).
2. إنشاء أعشاش أولية: يتم ضبط CS على عدد السكان ($n = 5$). سيتم تقييم الأعشاش الأولية باستخدام دالة الهدف، وهي المساهم الرئيسي في CS.
3. دالة اللياقة: تستخدم المسافة الاقليدية بين البيكسل LSB2−3−3 والبايت السري لتحديد جودة الحل.
4. جيل الوقواق: يتم إنتاج الوقواق بشكل عشوائي عن طريق رحلة Lévy. يتم تقييم الوقواق باستخدام دالة اللياقة.

5. الاستبدال: يتم اختيار العش بشكل عشوائي، إذا كانت جودة الحل الجديد في العش المختار أفضل من الحل القديم ، فسيتم استبداله بالحل الجديد. لاحظ أنه يتم استبدال جميع الأعشاش باستثناء أفضل واحد على أساس نوعية بيض الوقواق الجديد.

6. اكتشاف المضيف لبيض الكوكو: يتم التخلي عن الأعشاش الأسوأ على أساس الاحتمال (P_a = 0.25) ويتم بناء اعشاش جديدة.

7. الإنهاء: في هذه الطريقة، يتم تعيين معيار الإنهاء على 100 جيل. بمجرد الوصول إلى عدد التكرارات التدريبية في CS يتم إيقاف عملية التحسين. العش ذو أقل قيمة لياقة في التكرار النهائي هو الموقع الامثل. يتم تكرار الخطوات من 3–7 لحين الانتهاء من إيجاد مواقع مثلى بعدد مساوي لطول الرسالة السرية.

2.4 تقنية الإستبدال الهاشية الاقل اهمية HLSB

يتم الحصول على مواقع LSB bits لتضمين البايت السري الحالي في مكونات (RGB 3−3−2 LSB) للبيكسل المحددة بإستخدام الدالة الهاشية [7]. والدالة الهاشية المستخدمة:

K= p % n……………………………………….(3)

حيث أن K هي موقع LSB bit ضمن النقطة (pixel)،

P هي الموقع لكل بايت سري،

n هي عدد bits لـ LSB (هنا تمثل 4)

لو اعتبرنا ان نقطة RGB (في الصورة الغطاء):

B								G								R							
LSB				MSB				LSB				MSB				LSB				MSB			
1	0	0	1	0	0	1	0	1	0	0	0	1	0	1	0	1	0	1	0	0	0	1	1

وافترضنا ان قيمة البايت السري هي 92 وقيمته الثنائية 01011100، والتي سيتم تضمينها في نقاط RGB باستخدامHLSB 2−3−3 وبالتتابع. ولو كانت قيم الدالة الهاشية للمعادلة (3):

K لـ R = 1،2،3،

K لـ G = 1،2،4

K لـ B (ثابتة) =1،2

اذن، ستكون قيمة نقطة RGB (في الصورة بعد الإخفاء) بعد تضمين البيانات السرية في مواقع قيمة RGB المحددة (حسب قيم K) كالاتي:

B								G								R							
LSB				MSB				LSB				MSB				LSB				MSB			
1	0	0	1	0	0	**0**	**1**	1	0	0	1	**0**	0	**1**	**1**	1	0	1	0	0	**1**	**0**	**0**

3.4 خوارزمية عملية التضمين

المدخلات: ملف الصورة الغطاء، ملف الرسالة السرية.

الإخراج: ملف الصورة بعد الإخفاء (Stego).

في هذه الطريقة، يتم إخفاء بايت واحد من الرسالة السرية في بكسل واحد من الصورة بشكل عشوائي اعتماداً على خوارزمية بحث طائر الوقواق وتقنية HLSB وحسب الخطوات الاتية:

الخطوة 1: فتح ملف البيانات السرية.

الخطوة 2: قراءة البيانات السرية.

الخطوة 3: تقسيم البيانات السرية الى مجموعة من البايت (Bytes).

الخطوة 4: حساب طول الرسالة السرية بالبايت.

الخطوة 5: تقسيم البايت السري إلى أقسام 2-3-3 بت.

الخطوة 6: فتح ملف الصورة الغطاء.

الخطوة 7: قراءة الصورة الغطاء.

الخطوة 8: تقسيم كل بايت إلى مقاطع 2-3-3 بت (الأحمر(R)، الأخضر(G) والأزرق(B) على التوالي).

الخطوة 9: تنفيذ خوارزمية بحث طائر الوقواق للحصول على موقع التضمين الامثل في الصورة الغطاء.

الخطوة 10: إستخدام الدالة الهاشية للحصول على مواقع LSB bits لتضمين البايت السري الحالي في مكونات (RGB 3-3-2 LSB) للنقطة (البيكسل) المحددة.

الخطوة 11: في حين لم يتم الوصول الى نهاية ملف الرسالة السرية يتم تنفيذ الخطوات التالية.

الخطوة 12: قراءة بايت من ملف الرسائل السرية بالتسلسل، وتحويله إلى شكل ثنائي.

الخطوة 13: إخفاء الحرف الأول (البايت) من الرسالة في نقطة (بكسل) من الصورة الغطاء تم اختياره بواسطة خوارزمية بحث طائر الوقواق وبإستخدام تقنية HLSB.

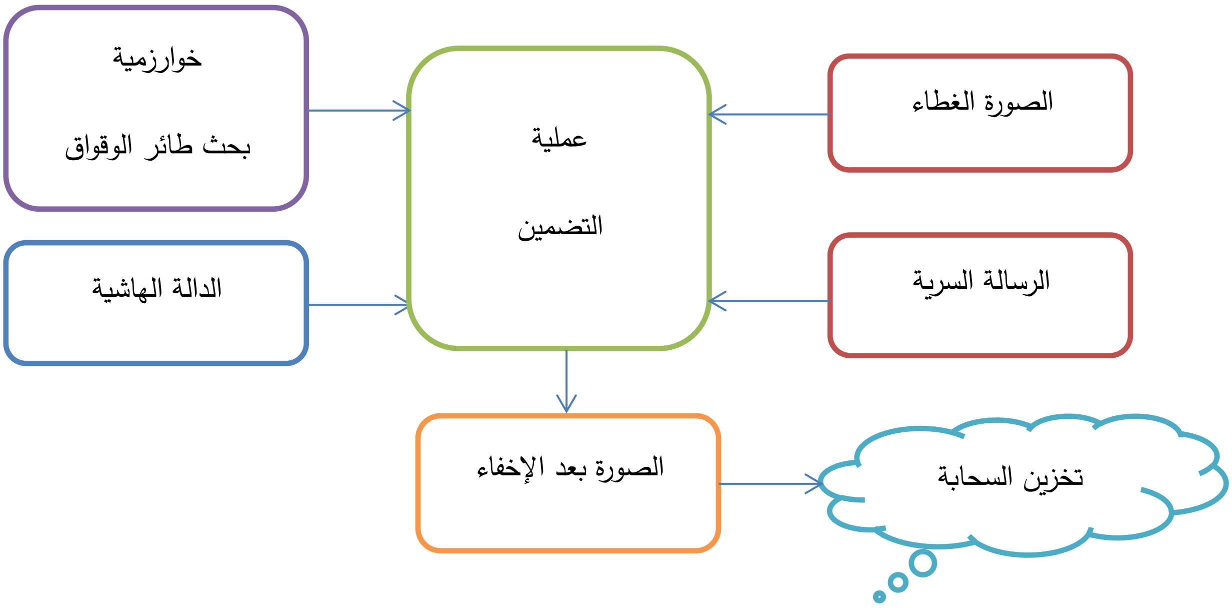
الخطوة 14: الرجوع الى الخطوة 9 وتكرار الخطوات (9 – 14) لحين الوصول لنهاية ملف الرسالة السرية.

الخطوة 15: الحصول على الصورة بعد الإخفاء (Stego).

الخطوة 16: خزن الصورة بعد الإخفاء في ملف.

الخطوة 17: إنشاء مجلد في حساب Goggle Drive.

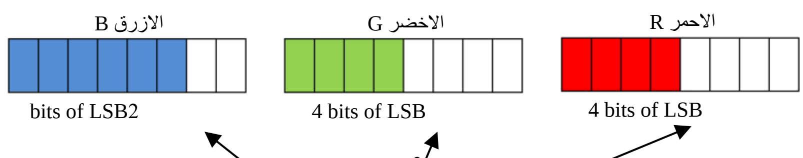
الخطوة 18: تحميل ملف الصورة بعد الإخفاء إلى المجلد المنشئ في الخطوة السابقة. لاحظ الشكل (3).



الشكل(3): مخطط عملية التضمين

بإستخدام الطريقة المقترحة، يمكننا تضمين البايت حمر والأخضر والأزرق) في الترتيب 3 و 3 و 2 على التوالي. في البداية، يتم إخفاء ثلاث بتات من البايت السري داخل ثلاث بتات (محددة عن طريق الدالة الهاشية) LSB من من البكسل الأحمر، ثم ثلاث بتات في البتات الثلاثة المحددة (عن طريق الدالة الهاشية) من LSB من البكسل الأخضر. وأخيراً، يتم إخفاء البتتين المتبقيتين من البايت السري في جزئين من LSB من البكسل الأزرق. ، لاحظ الشكل (4).

نقاط RGB للصورة الغطاء



4.4 خوارزمية عملية الاسترجاع

المدخلات: ملف الصورة بعد الإخفاء (Stego).

الإخراج: الرسالة السرية.

سيتم وصف عملية الاسترجاع بالخطوات الاتية:

الخطوة 1: فتح المجلد الذي تم تكوينه مسبقا في Goggle Drive.

الخطوة 2: تنزيل ملف الصورة بعد الإخفاء.

الخطوة 3: فتح ملف الصورة بعد الإخفاء.

الخطوة 4: قراءة الصورة بعد الإخفاء.

الخطوة 5: قراءة طول الرسالة السرية.

الخطوة 6: إسترجاع مواقع الاخفاء (المضمنة بعد الرسالة السرية).

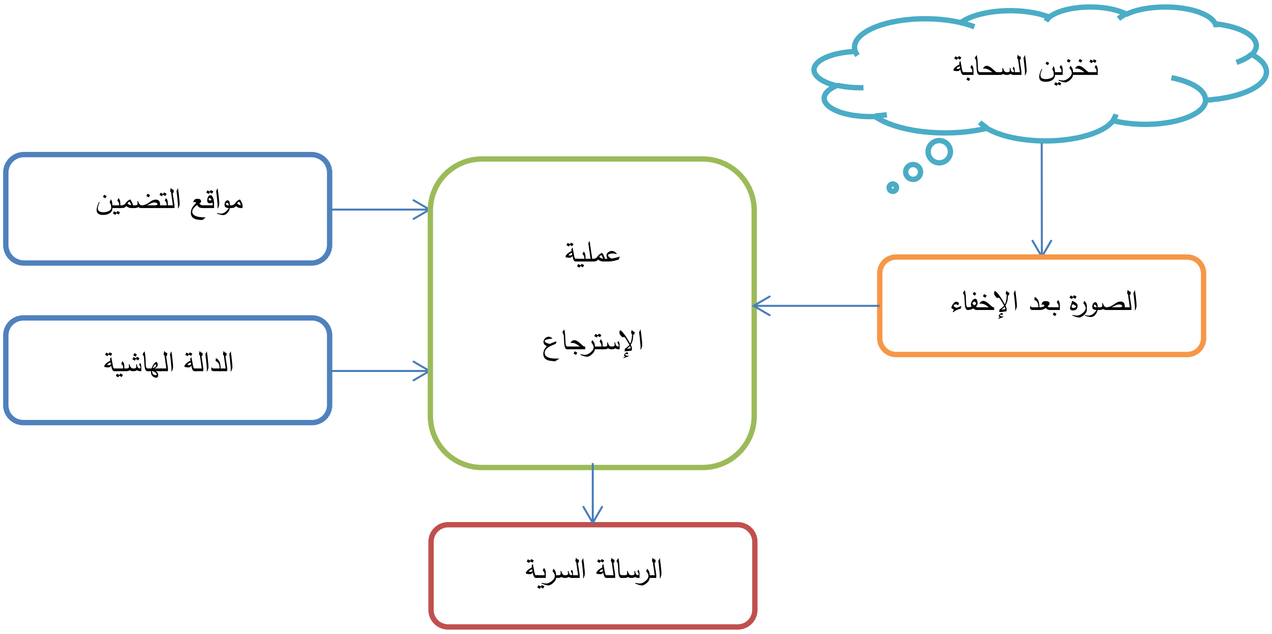
الخطوة 7: تنفيذ الدالة الهاشية للحصول على مواقع حشر LSBbits.

الخطوة 8: إسترجاع البايت السري باستخدام تقنية الإستبدال الهاشية HLSB 2−3−3.

الخطوة 9: وفقًا لحجم البيانات السرية، سيتم استرجاع البيانات السرية من الصورة بعد الإخفاء ووفقًا للمواضع العشوائية التي تم الحصول عليها من خوارزمية بحث طائر الوقواق وتقنية الإستبدال الهاشية.

الخطوة 10: الحصول على الرسالة السرية.

الخطوة 11: عرض الرسالة السرية.



الشكل(5): مخطط عملية الإسترجاع

5.4 المسار الوظيفي للطريقة المقترحة

- إنشاء مجلد في حساب Goggle Drive الخاص بنا.
- اختيار ملف الصورة الغطاء.
- اختيار ملف النص السري المراد إخفاءه.
- تنفيذ خوارزمية بحث طائر الوقواق للحصول على المواقع (Bytes) المثلى للاخفاء.
- بعد الحصول على المواقع (Bytes) المثلى للاخفاء، يتم تنفيذ تقنية 3−3−2 HLSB لتضمين LSB bits في الصورة الغطاء.
- الحصول على الصورة بعد الإخفاء.
- تحميل الصورة بعد الإخفاء إلى المجلد الذي تم إنشاؤه في حساب Goggle Drive.
- تنزيل الصورة بعد الإخفاء إلى نظام الملفات المحلي لدينا.
- إسترجاع مواقع الإخفاء المضمنة.
- تنفيذ تقنية HLSB للحصول على مواقع تضمين LSB bits.
- إسترجاع النص السري بإستخدام تقنية 3−3−2 HLSB.
- محتوى الملف الذي تم إسترجاعه سيكون هو نفسه الملف الأصلي.

5. النتائج والتقييم

لتقييم أداء الطريقة المقترحة لتحديد المواقع المثلى لتضمين البيانات السرية، تم استخدام الصور القياسية مثل Lena ،Peppers و Baboon بحجم 512 × 512 كصور غطاء موضحة في الشكل (6) (أ− ج).

(أ) (ب) (ج)

الشكل (6): الصور الغطاء (أ) لينا ، (ب) الفلفل ، (ج) البابون

الجدول (1): قياس الأداء للطريقة المقترحة

أسم الصورة	حجم الصورة	MSE	PSNR	NC	BER

0	1	79.39	0.0008	512*512	Lena.png
0	1	74.57	0.0007	512*512	Peppers.png
0	1	81.42	0.0009	512*512	Baboon.png

يوضح الجدول (1) قيم المقاييس (BER ,PSNR, MSE) التي تم الحصول عليها بعد تنفيذ الطريقة المقترحة. وفيه نلاحظ ان قيم المقاييس للطريقة المقترحة هي كالاتي: قيم MSE قريبة من الصفر، قيم PSNR تتراوح بين 74.57 و81.42 ديسيبل (dB)، قيم NC مساوية للواحد وقيم BER مساوية للصفر وهي تعتبر نتائج جيدة فيما يتعلق بجودة العمل وكفاءته.

الجدول (2): مقارنة الأداء (قيم PSNR) للطريقة المقترحة مع الطرائق الأخرى

الصورة الغطاء / الطريقة	PSNR Lena	PSNR Peppers	PSNR Baboon
الطريقة المقترحة	79.39	74.57	81.42
[8] **LSB & CS**	51.62	47.74	54.23
[4] **LSB 2,3,3**	38.03	32.67	39.21
[7] **HLSB 2,3,3**	44.34	45.67	42.66
[4] **LSB & GA**	39.37	34.37	41.61
[5] **LSB & PSO**	57.34	53.56	62.87
[6] **LSB & FA**	77.44	72.32	78.08
[6] **LSB & CA**	77.21	71.82	79.57

اما الجدول(2) فيوضح مقارنة بين قيم PSNR للصور المستخدمة (Baboon ,Peppers,Lena) في الطريقة المقترحة وبين قيم PSNR للطرائق الأخرى، ونلاحظ ان أفضل قيم لـ PSNR هي القيم التي تم الحصول عليها بإستخدام الطريقة المقترحة، تليها طريقة LSB بإستخدام خوارزمية اسراب القطط، ثم طريقة LSB بإستخدام خوارزمية حشرة اليراعة، ومن ثم طريقة LSB بإستخدام خوارزمية أسراب الطيور إنتهاءاً بتقنية 3-3-2 LSB مما يثبت كفاءة الطريقة المقترحة.

6. الاستنتاجات

لتحسين الأمن في مركز البيانات والإرسال في السحابة السحابية، تم إقتراح طريقة تستخدم خوارزمية بحث طائر الوقواق للحصول على مفتاح الإخفاء لإخفاء البيانات السرية، وتم تخزين الصور بعد الإخفاء في مركز البيانات السحابية. عندما يتم الاحتفاظ بهذه الصور في مراكز البيانات السحابية، لايمكن عرض المحتوى الأصلي للرسالة بواسطة الأشخاص غير المصرح بهم والبيانات السرية المخفية فيها تتمتع بأمان كبير من أي هجمات يقوم بها المعارضون.

تم اقتراح واختبار وتقييم الطريقة المقترحة لتحديد المواقع المثلى لإخفاء البيانات السرية بناءً على خوارزمية بحث طائر الوقواق وتقنية 3-3-2 HLSB. تستخدم هذه الخوارزمية التي تم تقديمها مؤخرًا لتحسين أداء الإخفاء والعثور على أفضل عش، وقد قدمت تحسينات ملحوظة على الأجيال المتعاقبة. إن نتيجة تقنية التحسين هي المواقع المثلى للصور لتضمين البيانات السرية والتي تستخدم كمفتاح سري مرتبط.

أظهرت النتائج التجريبية، أن الطريقة المقترحة عالية الكفاءة للتضمين نظرًا لوجود اختلاف طفيف جدا في الصورة قبل وبعد الإخفاء مما يدل على جودة عالية. من الواضح جدًا أن جودة الصورة قد تحسنت بشكل كبير حتى عند تضمين المعلومات بداخلها، يتم تحقيق هذه الجودة عن طريق اختيار مواقع التضمين المثلى بإستخدام خوارزمية بحث طائر الوقواق مع الحفاظ على المتانة عند مستويات عالية.. قيم PSNR وMSE للطريقة المقترحة جيدة، مما يدل على جودة بصرية. قيم PSNR للطريقة المقترحة أعلى من قيمها في تنفيذ الأخفاء بإستخدام الخوارزميات الأخرى، وكلما زادت قيمة PSNR كانت النتيجة أفضل. قيم BER جميعها مساوية للصفر مما يعني ان الرسالة السرية تم استرجاعها بشكل صحيح وبدون اي خطأ.

7. المصادر

1. Hassan Hadi Saleh, ”Increasing Security for Cloud Computing By Steganography in Image Edges”, Al-Mustansiriyah Journal of Science (ISSN: 1814-635X) Vol. 27, No 4, 2016.
2. Mohis, M., & Devipriya, V. S., “An improved approach for Enhancing Public Cloud Data Security through Steganographic Technique”. IEEE In Inventive Computation Technologies (ICICT), 2016.
3. Sarkar, M. K., & Chatterjee, T., “Enhancing Data Storage Security in Cloud Computing Through Steganography”. International Journal of Network Security, Vol. 5, No. 1, 13–19, 2014.
4. Kousik Dasgupta, Jyotsna Kumar Mondal, Paramartha Dutta, 2013, “Optimized Video Steganography using Genetic Algorithm (GA)”, International Conference on Computational Intelligence: Modeling, Techniques and Applications (CIMTA), Procedia Technology 10 (2013) 131 – 137, 2013.
5. البك، يحيى سعيد، "تطوير طريقة البت الأقل أهمية بإستخدام امثلية اسراب الطيور لإخفاء المعلومات داخل الصور"، رسالة ماجستير مقدمة الى كلية علوم الحاسوب والرياضيات، قسم علوم الحاسوب، جامعة الموصل، 2014.
6. جاسم، جاسم ابراهيم، "تطبيق الكتابة المغطاة بإعتماد تقنيات الذكاء الاصطناعي"، رسالة ماجستير مقدمة الى كلية علوم الحاسوب والرياضيات، قسم علوم الحاسوب، جامعة الموصل، 2017.
7. Kousik Dasgupta, J.K. Mandal and Paramartha Dutta,” HASH BASED LEAST SIGNIFICANT BIT TECHNIQUE FOR VIDEO STEGANOGRAPHY(HLSB)”, International Journal of Security, Privacy and Trust Management (IJSPTM), Vol. 1, No 2, 2012.
8. Sameh A. Abbas, Taha I. B. El Arif, Fayed F. M. Ghaleb and Sohier M. Khamis, “Optimized Video Steganography Using Cuckoo Search Algorithm”, IEEE Seventh International Conference on Intelligent Computing and Information Systems (ICICIS'15), 2015.
9. M. Mareli and B. Twala, “An adaptive Cuckoo search algorithm for optimisation”, Applied Computing and Informatics 14, 107–115, 2018.
10. Pooja Prashar, Nayan Jain, Shivanku Mahna, “Image Optimization using Cuckoo Search and Levy Flight Algorithms”, International Journal of Computer Applications (0975 – 8887) Volume 178 – No.4, 2017.
11. Venkata Vijaya Geeta. Pentapalli1 and Ravi Kiran Varma,” Cuckoo Search Optimization and its Applications: A Review”, International Journal of Advanced Research in Computer and Communication Engineering ISO 3297:2007 Certified Vol. 5, Issue 11, 2016.

Implicit Hybrid Conjugate Gradient Method For Unconstrained Optimization Problems.

Khalil K. Abbo[1] Edrees M. Nori[2] Yoksal A. Laylani[3] Hisham M.Khudhur[4]

Abstract

In the relevant paper to the conjugate gradient methods, we suggest a new implicit hybrid conjugate gradient method, which is based on Fletcher-Reveere and Hestenes-Stiefel methods. The descent property and global convergence with Wolfe line search are proved. The numerical results show that the proposed algorithm is efficient.

1- Introduction

We deal with the following unconstrained optimization problems;

$$\min f(x) \quad x \in R^n \tag{1}$$

Where $f : R^n \to R$ is continuously differentiable function, bounded from below. Starting from an initial guess, a nonlinear conjugate gradient (CG)algorithm generates a sequence of points $\{x_k\}$, according to the following recurrence formula[**R. Fletcher**,1987] ,[**J. Nocedal, S. J. Wright**,1999]

$$x_{k+1} = x_k + \alpha_k d_k \tag{2}$$

Where $d_k \in R^n$ search direction and $\alpha_k \in R$ is a step length, usually obtained by the Wolfe [**P. Wolfe**, 1969] and [**P. Wolfe**,1969], the standard Wofle condition (SWC)

$$f(x_k + \alpha_k d_k) - f(x_k) \le \rho \alpha_k g_k^T d_k \tag{3}$$

$$g_{k+1}^T d_k \ge \sigma g_k^T d_k \tag{4}$$

With $0 < \rho < \sigma < 1$, or by strong Wolfe condition(STWC)

$$f(x_k + \alpha_k d_k) - f(x_k) \le \rho \alpha_k g_k^T d_k \tag{5}$$

$$g_{k+1}^T d_k \ge \sigma g_k^T d_k \tag{6}$$

and the direction computed as

$$d_{k+1} = -g_{k+1} + \beta_k s_k, \; d_0 = -g_0. \tag{7}$$

Here β_k is scalar known as the conjugate gradient parameter, $g_k = \nabla f(x_k)$ and $s_k = x_{k+1} - x_k$. Different conjugate gradient algorithms corresponding to different choices for the parameter β_k see [**Andrei**, 208], [**Y.H. Dai, Y.Yuan**, 1996], [**Y. H. Dai,, L. Z. Liao**,2001], [**Y. H. Dai, Y. Yuan**,1999], [**W. W. Hager, H. Zhang**,2006], [**W.W. Hager and H. Zhang**,2006] and [**Y. Liu and C. Storey**,1991]. Therefore, a crucial element in any conjugate gradient algorithm is the formula definition of β_k . The well known formulas for β_k are as follows;

$$\beta_k^{FR} = \frac{g_{k+1}^T g_{k+1}}{g_k^T g_k}, \; \beta_k^{PR} = \frac{y_k^T g_{k+1}}{g_k^T g_k}, \quad \beta_k^{HS} = \frac{y_k^T g_{k+1}}{y_k^T d_k}$$

FR denotes Fletcher and Reeves [**R. Fletcher and C. Reeves**, 1964], HS denotes Hestenes and Steifel [**M. R. Hestenes, E. L. Stiefel**, 1952], and PR denotes Polak and Ribiere [**E. Polak, G. Ribiére**, 1969]. Note that these formulas for β_k are equivalent if the objective function is a strictly convex quadratic function and α_k is the exact one-dimensional minimizer[**W. W. Hager, H. Zhang**, 2003].

The structure of the paper is as follows. Section 2 introduce our suggested method(IHCG say), and proves that it generates descent directions. In section 3 its convergence analysis is shown. In section 4 some numerical experiments are presented.

2- Implicit Hybrid Conjugate Gradient Method

The numerical performance of the Fletcher-Reeves (β_k^{FR}) method is somewhat erratic, it is sometimes as efficient as the Polack-Ribier and Hestenes-Stiefel methods, but it is much slower [**J. C. Gilbert, J. Nocedal**, 1992]. In this section we try to overcome to the disadvantages of the Fletcher-Reeves method by introducing implicit hybrid CG algorithm as follows:

Consider the following search direction defined by

$$d_{k+1} = -\theta_k \, g_{k+1} + (1-\theta_k)\beta_k^{FR} s_k \qquad (8)$$

To find the value of θ_k , we use the pure conjugacy condition, with simple algebra we obtain

$$\theta_k = \frac{\beta_k^{FR}}{\beta_k^{HS} + \beta_k^{FR}} \qquad (9)$$

Note that if $\beta_k^{HS} < 0$, then $\theta_k > 1$ which implies that $1-\theta_k < 0$, to avoid this situation we take β_k^{HS+} that is

$$\beta_k^{HS+} = \max\{ \beta_k^{HS}, 0\} \qquad (10)$$

Therefor $0 < \theta_k \leq 1$, and equation (9) becomes

$$\theta_k = \frac{\beta_k^{FR}}{\beta_k^{HS+} + \beta_k^{FR}} \qquad (11)$$

We call the method defined by equations (8) and (11) (IHCG) method.

On some studies of the conjugate gradient methods, the sufficient descent condition

$$d_{k+1}^T g_{k+1} \leq -c \left\| g_{k+1} \right\|^2 \qquad (12)$$

Plays an important role [Jinkui L. and Shacheng W, 2011]. Unfortunately, this condition is hard

3- Descent Property and Global Convergence analysis

Next we will show that our CG method (8) satisfies the descent property and global converges.

Assumption(1)

Assume f is bound below in the level set $S = \{x \in R^n : f(x) \le f(x_\circ)\}$; In some neighborhood N of S, f is continuously differentiable and its gradient is Lipshitz continuos, there exist $L > 0$ such that:

$$\|g(x) - g(y)\| \le L\|x - y\| \quad \forall x, y \in N \ . \tag{13}$$

or equivalently

$$y_k^T s_k \ge \mu\|s_k\|^2 . \quad \text{and} \quad \mu\|s_k\|^2 \le y_k^T s_k \le L\|s_k\|^2 . \tag{14}$$

From (13) we get

$$y_k^T y_k \le L y_k^T s_k . \tag{15}$$

On the other hand, under Assumption(1) , It is clear that there exist positive constant B such

$$\|x\| \le B \ , \forall x \in \Omega \tag{16}$$

$$\underline{\gamma} \le \|g(x)\| \le \overline{\gamma} \ , \forall x \in \Omega \tag{17}$$

Suppose that Assumption (1) holds and if the line search satisfies the Wolfe condition.

It follows from [Polak E., and Ribiere, G., 1969]that

$$s_k^T y_k = s_k^T (g_{k+1} - g_k) \ge (\sigma - 1) s_k^T g_k \tag{18}$$

Lemma(1)

Suppose that Assumption(1) and equation (16) hold .consider any conjugate gradient method in from (2)and(7) ,where d_k is a descent direction and α_k is obtained by the strong Wolfe (SW) .If

$$\sum_{k>1} \frac{1}{\|d_{k+1}\|^2} = \infty \tag{19}$$

Then we have

$$\lim_{k \to \infty} \left(\inf \| g_k \| \right) = 0 \ . \tag{20}$$

More details can be found in [Tomizuka, H and Yabe, H, 2004], [Dai, Y and Lio, L, 2001]

Theorem(1)

Let $\{x_k\}$ and $\{d_k\}$ be generated by the equation (2), (3) and α_k satisfies Wolfe line search conditions (SWLSC) (5) and (6) , then $d_k^T g_k < 0$ hold for all $k \geq 1$.

Proof . The conclusion can be proved by induction . When $k = 1$, we have $d_1^T g_1 \geq -\|g_1\|^2 < 0$

Suppose that $d_k^T g_k < 0$ hold for all k. From (8) we have

$$d_{k+1} = -\theta_k \ g_{k+1} + (1-\theta_k)\beta_k^{FR} s_k$$

$$\theta_k = \frac{\beta_k^{FR}}{\beta_k^{HS} + \beta_k^{FR}}$$

$$\beta_k^{HS+} = \max\{ \beta_k^{HS}, 0\}$$

$$\theta_k = \frac{\beta_k^{FR}}{\beta_k^{HS+} + \beta_k^{FR}}$$

$$g_{k+1}^T d_{k+1} = -\theta_k \ g_{k+1}^T g_{k+1} + (1-\theta_k)\beta_k^{FR} g_{k+1}^T s_k$$

$$g_{k+1}^T d_{k+1} = -\left(\theta_k \ - (1-\theta_k) \frac{1}{g_k^T g_k} g_{k+1}^T s_k \right) g_{k+1}^T g_{k+1}$$

Let $c = \left(\theta_k \ - (1-\theta_k) \frac{1}{g_k^T g_k} g_{k+1}^T s_k \right)$

$$g_{k+1}^T d_{k+1} \leq -\left(\theta_k \ - (1-\theta_k) \frac{1}{g_k^T g_k} g_{k+1}^T s_k \right) g_{k+1}^T g_{k+1}$$

Theorem(2)

Suppose that assumption(1) holds, and consider the new algorithm (New), where α_k is computed by the Wolfe line search conditions (5) and (6) then:

Proof:-

$$\|d_{k+1}\|^2 = \|-\theta_k \ g_{k+1} + (1-\theta_k)\beta_k^{FR} s_k\|^2$$

$$\|d_{k+1}\|^2 \le |\theta_k| \|g_{k+1}\|^2 + |(1-\theta_k)| |\beta_k^{FR}| \|s_k\|^2$$

$$\|d_{k+1}\|^2 \le |\theta_k| \bar{\gamma}^2 + |(1-\theta_k)| \frac{\|s_k\|^2}{\|g_k\|^2} \bar{\gamma}^2$$

Let $u = |\theta_k| + |(1-\theta_k)| \frac{\|s_k\|^2}{\|g_k\|^2}$

$$\|d_{k+1}\|^2 \le \bar{\gamma}^2 u$$

$$\sum_{k\ge1} \frac{1}{\|d_{k+1}\|^2} \ge \frac{1}{\bar{\gamma}^2 u} \sum_{k\ge1} 1 = \infty$$

$$\lim_{k\to\infty} \left(\inf \|g_k\|\right) = 0$$

4- Numerical results and comparisons

In this section, we compare the performance of new formal developed a New method of conjugate gradient method to conjugate gradient method (KHY). we have selected (75) large scale unconstrained optimization problem, for each test problems taken from [Andrei N., 2008]. For each test function we have considered numerical experiments with the number of variables $n = 100 ,..., 1000$. These new version is compared with well-known conjugate gradient algorithms (HS and FR). All these algorithms are implemented with standard Wolfe line search conditions (5) and (6) with. In all these cases, the stopping criteria is the $\|g_k\| = 10^{-6}$. All codes are written in doble precision FORTRAN Language with F77 default compiler settings. The test functions usually start point standard initially summary numerical results recorded in the figures (1),(2). The performance profile by [Dolan. E. D and Mor´e. J. J, 2002] is used to display the performance of the developed a New method of conjugate gradient algorithm with (HS and FR) algorithms. Define $p = 750$ as the whole set of n_p test problems and $S = 3$ the set of the interested solvers. Let $l_{p,s}$ be the number of objective function evaluations required by solver s for problem p. Define the performance ratio as

$$r_{p,s} = \frac{l_{p,s}}{l_p^*} \qquad (21)$$

Where $l_p^* = \min\{l_{p,s} : s \in S\}$. It is obvious that $r_{p,s} \geq 1$ for all p,s . If a solver fails to solve a problem, the ratio $r_{p,s}$ is assigned to be a large number M . The performance profile for each solver s is defined as the following cumulative distribution function for performance ratio $r_{p,s}$,

$$\rho_s(\tau) = \frac{size\{p \in P : r_{p,s} \leq \tau\}}{n_p} \qquad (22)$$

Obviously, $p_s(1)$ represents the percentage of problems for which solver s is the best. See [Dolan. E. D and Mor´e. J. J, 2002] for more details about the performance profile. The performance profile can also be used to analyze the number of iterations, the number of gradient evaluations and the cpu time. Besides, to get a clear observation, we give the horizontal coordinate a log-scale in the following figures.

Fig.1: Performance profiles with respect to the number of iterations

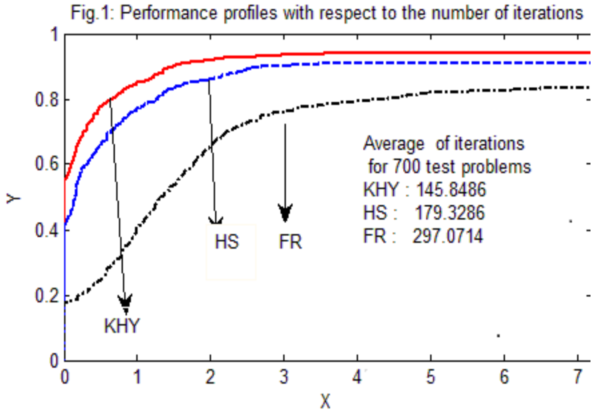
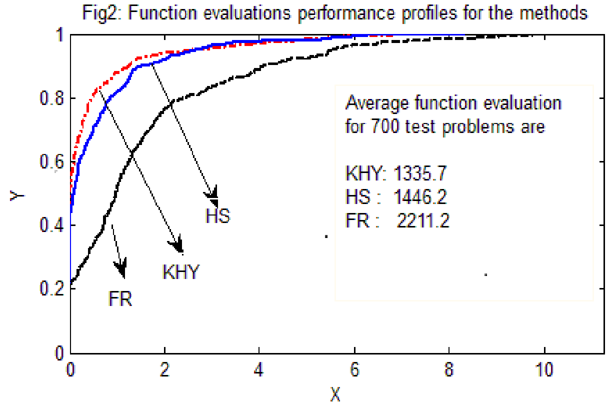


Fig2: Function evaluations performance profiles for the methods



References

[1] Andrei N. (2008), "An Unconstrained Optimization test function collection". Adv. Model. Optimization. 10. pp.147-161.

[2] Dai ,Y. and Liao, L. (2001) "New conjugacy conditions and related non-linear CG methods" Appl. Math optim.,(43). Spring-verlag, New York.

[3] Dolan. E. D and Mor´e. J. J, "Benchmarking optimization software with performance profiles", Math. Programming, 91 (2002), pp.

[4] E. Polak, G. Ribiére, Note sur la convergence de méthodes de directions conjugués, Revue Française d'Informatique et de Recherche Opérationnelle, 16 (1969), 35-43.

[5] J. C. Gilbert, J. Nocedal, Global convergence properties of conjugate gradient methods for optimization, SIAM Journal of Optimization, 2 (1992), 21-42.

[6] J. Nocedal, S. J. Wright, Numerical Optimization, Springer, 1999.

[7] Jinkui L. and Shacheng W. (2011) Modified non-linear conjugate gradient method with sufficient descent Condition for unconstrained optimization. J. of Inequalities and Applications 57.

[8] M. R. Hestenes, E. L. Stiefel, Methods of conjugate gradients for solving linear systems, J. Research Nat. Bur. Standards, 49 (1952), 409-436.

[9] N. Andrei, 40 Conjugate Gradient Algorithms for unconstrained optimization, A survey on their definition, ICI Technical Report, 13/08, 2008.

[10] P. Wolfe, Convergence conditions for ascent methods, SIAM Review, 11 (1969), 226-235.

[11] P. Wolfe, Convergence conditions for ascent methods. II: Some corrections, SIAM Review, 11 (1969), 226-235.

[12] Polak E., and Ribiere, G., (1969), "Not sur la convergence de directions conjugate". Rev. Franaise Informant Researche operationelle, 3e Anne'e., Vol. 16, pp. 35-43.

[13] R. Fletcher and C. Reeves, Function minimization by conjugate gradients, Comput. J., 7 (1964), 149-154.

[14] R. Fletcher, Practical methods of optimization vol. 1: Unconstrained Optimization, John Wiley and Sons, New York, 1987.

[15] Tomizuka. H and Yabe. H. (2004). "A Hybrid Conjugate Gradient method for unconstrained Optimization".

[16] W. W. Hager, H. Zhang, A new conjugate gradient method with guaranteed descent and an efficient line search, SIAM J. Optim., 16, 1 (2003), 170-192.

[17] W. W. Hager, H. Zhang, CG-DESCENT, a conjugate gradient method with guaranteed descent, ACM Transactions on Mathematical Software, 32, 1 (2006), 113-137.

[18] W.W. Hager and H. Zhang, A survey of nonlinear conjugate gradient methods, Pacific journal of Optimization, 2 (2006), 35-58.

[19] Y. H. Dai, Y. Yuan, A nonlinear conjugate gradient method with a strong global convergence property, SIAM J. Optim., 10 (1999), 177-182.

[20] Y. H. Dai,, L. Z. Liao, New conjugacy conditions and related nonlinear conjugate gradient methods, Appl. Math. Optim. 43 (2001), 87-101.

[21] Y. Liu and C. Storey, Efficient generalized conjugate gradient algorithms, Part 1: Theory, JOTA, 69 (1991), 129-137.

[22] Y.H. Dai, Y.Yuan, Convergence properties of the Fletcher-Reeves method, IMA J. Numer. Anal., 16 (1996), 155-164.

GOODNESS OF FIT APPROACH FOR TESTING EXPONENTIAL BETTER THAN USED IN CONVEX FOR LIFE DISTRIBUTIONS

By

Gorgees Shaheed Mohammad

AL –Qadisiya University ,College of Education ,Department of mathematics,

Abstract

Test statistic is derived for testing exponentially against exponential better (worse) than used in convex EBUC (EWUC). Selected critical values are tabulated for sample size *n* =5(1)25(5)50. The Power calculations of the test are simulated for some commonly used distributions in reliability. The pitman asymptotic relative efficiency based on other classes are studied. An example of 40 patients of blood cancer disease demonstrates the practical application of the proposed test in medical sciences.

Key Words and Phrases: Asymptotic normality, asymptotic efficiency critical values, life distribution classes, convex ordering, EBUC, EWUC, power of the test, product limit estimator, goodness of fit test.

1 .Introduction

Concept of aging describes how a population of units or systems improves or deteriorates with age**.** Many classes of life distributions are categorized and defined in the literature according to their aging properties. The notion of stochastic aging is important in any reliability analysis. Many statistics have been developed for testing exponentially against various aging alternatives. In developing test for various classes of life distributions, almost without exception the exponential distribution is used as a boundary member of an aging class $\hat{S}$ a usual format for testing is

H_0: F is exponential; Versus (vs),

H_1: $F \in \hat{S}$ but F is not exponential.

Testing exponentiality against the classes of life distributions has received a good deal of attention. For testing against new better than used (NBU), we refer to Hollander and Proschan (1972), and Ahmed (1994) among others. For new better than used in expectation (NBUE), we refer to Koul (1978), and Ahmed et al. (1999) among others. For harmonic new better than used in expectation (HNBUE), we refer to Klefsjo (1982), and Hendi et al. (1998), among others.

Definition 1.1: Let X be a non-negative random variable representing the life of an item with distribution function F(x) (X>0) and let F(x) =1-F(x).

The residual life X_t at age (t > 0) is the random variable with distribution function $F_t(x)$, and survival function $F_t(x)$.

It is well known that F belongs to increasing (decreasing) failure rate IFR (DFR) class if X_t is decreasing (increasing) in $t \geq 0$ on stochastic sense, F belongs to NBU (NWU) class if X_t is smaller (larger) than X for any $t \geq 0$ in convex ordering (see Cao and Wang (1991).

Definition 1.2: Let X and Y be two random variables with distribution functions F and G, respectively. We say that X is less variable than Y (or X is smaller in convex functions h (.). we write X $\leq$co Y. Clearly X $\leq$co Y

Iff $_x\int F(u)\,du \leq {}_x\int G(u)\,du$ for all $x \geq 0$, [see Stoyan] (1983)]

Using the above two definitions one may redefine old classes or introduces new ones. Of the newly defined classes, we say that a random variable X is " Exponential better than used in convex ordering" (EBUC) if $x_t \leq Y$ for all t > 0. Thus X is EBUC iff

$_x\int F_t(u)\,du \leq {}_x\int G(u)\,du$, $x, t \geq 0$,

Where X_t is the life length of a unit of age t compared with another unit of life length y which is exponential with the same mean of X. when x =0 in (1.2), We get

$_0\int F_t(u)\,du \leq u$, $x, t > 0$,

Which is NBUE condition.

The implications among EBUC, NBUB and HNBUB classes of life distributions are:

EBUC ⟶ NBUE ⟶ HNBUE

Similarly, we get the implications among dual classes EWUC, NWUE and HNWUE, respectively, by reversing the inequalities and direction of monotonicity.

Definition 1.3: A life distribution (cdf), F and its survival function (sf),

F = 1 - F with finite mean (u)du are said to be exponential better (worse) than used in convex if The exponential distribution (with mean μ) is the only distribution where equality is attained in (1.2) to (1.4).

Hence we test H_0: F is exponential (μ) against H_1: F is EBUC and not exponential. In order to test H_0 against H_1, we use the following measure of departure from H_0

However, in contrast to goodness of fit problems, where the test statistic is based on a measure of departure from H_0 that depends on both H_0 and H_1, most test in life testing setting, including those referenced above, did not use the null distribution in devising the test statistics. Which resulted in test statistics that are often difficult to work with and require programming to evaluate.

Recently Ahmed et al.(2001) used a new methodology for testing H_0 against the alternatives that a life distribution is increasing failure rate (IFR), NBU or new better that used in convex ordering (NBUC) and (NBUE) or (NBUEH) and for testing against the decreasing mean residual life time distribution (DMRL),we refer to EL-Bassiouny and Alwasel (2003). Here we consider the problem of testing exponentiality versus the exponential better (worse) than used in convex EBUC (EWUC).

In section 2, we present a test statistic based on the goodness of fit approach. In addition, we use Monte Carlo method to compute the critical point for sample size n=5(1)25(5)50, the power of the test is also simulated for some commonly used distributions in reliability and the efficiencies of the test statics is calculated based on common alternatives. An application in medical science is given in section 3.

2. Testing Against the EBUC Class

In this section a test statistic based on the goodness of fit approach for testing exponentiality or H_0 against H_1 or EBUC (EWUC) class (not exponentiality) since F is EBUC (EWUC). We use the parameter as a measure of departure from H_0 from (1.5).

In a similar fashion, if we denote by F_0, the exponential distribution we can take in place (2.1) the following measure of departure.

The measure given in (2.2) can be used for testing the hypothesis that H_0: F is exponential against H_1: F is EBUC and not exponential. Since this measure is scale invariant then without loss of generality we take u = 1 and thus $F_0(x) = 1- e^{-x}$. in order to derive an expression for we need the following lemma.

Based on a random sample $X_{1,}, X_n$ from a distribution F. we wish to test H_0 against H_1. Thus, we may be testing an its estimate.

Proof: It is straightforward, by noting that is just an average, thus using the central limit theorem the result follows. For the variance

To perform the above test: Reject H_0, if the calculated $\sqrt{(\frac{216n}{67})\hat{S}}$ value exceeds the upper a-percentile of the standard normal variate.

To compare this procedure to others in the literature we use the concept of pitman asymptotic efficacy defined.

Carrying out the efficacy calculation for the above three alternatives, namely Weilbull, linear failure rate, and Makeham we get 0.466, 0.378, 0.116, respectively.

It is also easy to see that our proposed test is consistent and unbiased. For samples size n=5(1)25(5)50 and using 1000 replications, the percentiles of the statistic , and its power for samples of sizes n=10,20 and 30 at significant level 95% upper percentile are given in the following Tables 1 and 2, respectively.

Table 1: Critical values of $\hat{S}$

N	1%	5%	10%	90%	95%	99%
5	-0.715	-0,430	-0,263	0,161	0.175	0.191
6	-0.681	-0,364	-0,221	0,154	0.165	0.187
7	-0.709	-0,356	-0,227	0,152	0.164	0.188
8	-0.539	-0,310	-0,207	0,144	0.156	0.177
9	-0.558	-0,304	-0,238	0,140	0.154	0.175
10	-0.565	-0,278	-0,212	0,138	0.154	0.169
11	-0.503	-0,280	-0,185	0,134	0.146	0.168
12	-0.426	-0,243	-0,167	0,133	0.145	0.165
13	-0.369	-0,209	-0,153	0,127	0.144	0.160
14	-0.380	-0,228	-0,156	0,125	0.137	0.163
15	-0.387	-0,223	-0,150	0,125	0.139	0.159
16	-0.309	-0,197	-0,132	0,122	0.134	0.151
17	-0.383	-0,235	-0,154	0,117	0.130	0.156
18	-0.314	-0,214	-0,158	0,117	0.129	0.152
19	-0.363	-0,209	-0,154	0,112	0.124	0.147
20	-0.366	-0,211	-0,135	0,111	0.126	0.150
21	-0.298	-0,195	-0,139	0,109	0.125	0.152
22	-0.342	-0,195	-0,139	0,105	0.122	0.145
23	-0.287	-0,173	-0,129	0,101	0.117	0.140
24	-0.347	-0,199	-0,134	0,102	0.116	0.136

25	-0.283	-0,183	-0,117	0,100	0.112	0.138
30	-0.280	-0,166	-0,111	0,095	0.111	0.131
35	-0.248	-0,146	-0,108	0,086	0.103	0.124
40	-0.202	-0,129	-0,091	0.087	0.102	0.125
45	-0.191	-0,131	-0,094	0.078	0.092	0.115
50	-0.182	-0,116	-0,084	0.076	0.091	0.116

Table 2: Power estimates of $\hat{S}$

Distribtion	Parameter	Sample Size n		
		10	20	30
F_1 Linear Failure rale	2	0.878	0.999	1.000
	3	0.970	1.000	1.000
	4	0.977	1.000	1.000
F_2 Makeham	2	0.738	0.963	0.996
	3	0.969	1.000	1.000
	4	0.988	1.000	1.000
F_3 Weibull	2	0.643	0.947	0.996
	3	0.992	1.000	1.000
	4	1.000	1.000	1.000

Thus We have Shown that, based on Monte Carlo methods, the test statistic $\hat{S}$ has not only simplicity advantages over earlier ones but also better asymptotic relative efficiency.

3- Application

In this section we calculate the $\hat{S}$ test statistic for the data represents 40 patients suffering from blood cancer from one ministry of Health Hospitals in Saudi Arabia see Abouammoh et al (1994)> The ordered life times (in days) are

115, 181, 225, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1578, 1599, 1605, 1696, 1735, 1799, 1815, 1852.

It was found that the test statistics for the set of data by using equation (2.5) is $\hat{S}$ = - 1135, it is clear from the computed value of the test statistics that we reject H_1 which states that the set of data don't have EBUC property under significant level α=0.05.

References:

1. Abouammoh, A. M. (1994). "Partial orderings and testing for renewal failure rate classes of life distributions". Dirasat, 21B, 175 – 187.
2. Ahmad, I. A. (1994). " A class of statistics useful in testing increasing failure rate average and new better than used life distributions, J. Statist Plann. Inder, 41, 141 – 149.
3. Ahmad, I. A. Hendi, M. I, and Al – Nachawati, H. (1999). "Testing new better than used classes of life distributions derived from convex ordering using Kernel methods". J. Nonparametr, Statist, 11, 393 – 411.
4. Ahmad, I.A, Alwasel, I.A. and Mugdadi, A.R. (2001)" A goodness of fit approach to major life testing problems". Int, J, Reliab. Appl. 2, 81 – 97.
5. Cao, I. H. and Wang, Y. D. (1991). "The NBUC and NWUC classes of life distributions". J. Appl. Probab, 29, 473 – 479.
6. El-Bassiony, A. H. and Alwasel, I. A (2003). " A goodness of fit approach to testing mean residual times", Aool. Math and Compu 134, 301 – 307.
7. Hendi, M, I, Al – Nachawati, H, Montassar, M. and Alwasel, I. A. (1998) " An exact test for HNBUE class of life distributions". J. Statist. Comput. Simul, 60, 261 – 275.
8. Hollander, M. and Proschan, F (1972). Testing whether new better than used". Ann. . Math Statist, 43, 1136 – 1146.
9. Klefsjo, B. (1982). "The HNBUE and HNBUE and HNWUE class of life distributions". Naval Res. Logist. Quanrterly, 21, 331 – 344.
10. Koul, H.L (1978). "Testing for new is better than used in expectation" Comm. Statist. Theor. Meth, 7, 685 – 701.
11. Stoyan, D (1983). Comparisons Methods for Queues and Their Stochastic Models, Wiley, New York.