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Table Content

Approximation of Fractal Interpolation Functions via Artificial Neural Network

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Abstract_ For centuries, numerical interpolations have had many physical applications. The task of numerical interpolations has been to evaluate experimental points. Several methods have been proposed, the most famous of which are Newton's differences formulas, Spline's functions, and Biezer's formula, followed by Barnsley's method for fractal interpolation functions via iterated functions system. It is suitable for representing natural phenomena such as smoke, wave movement, mountain peaks and clouds, and it is also a true expression of market curves. In this research, the functions of the linear fractal curve were found by using the algorithm of artificial neural network. Several examples were studied to demonstrate the efficiency of the new method.

1 Introduction

The interpolation of laboratory mesh points to produce a curve passing through them is considered one of the good applied mathematical tools [1]. Michael Barnsley [2-3] was the first to introduce fractal interpolation functions using a system of iterated functions system in a complete space. Several formulas were calculated for the interpolation of a set of points using an arbitrary fractal variable whose value is between zero and one, called the fractional scale factor. The fractional scale factor gives the method great flexibility in modeling natural phenomena by including a few points, perhaps as few as three only. And then several studies have been done. For instance, Prusinkiewicz and Hammal designed the curves of mountains and rivers [4]. Craig studied IFS fractal interpolation for 2D and 3D visualization [5]. Bindlish and Barros studied the aggregation of Digital Terrain Data [6]. Zhao studied each of the construction and applications of fractal interpolation surfaces [7]. Zair and Tosan studied computer aided design with IFS technique [8]. Alfonseca and Ortega studied turtle L-System [9]. Berkner studied the fractional wavelet [10]. Xie and Sun studied the fractal interpolation surfaces [11]. Cochran etal approximated the rough curve [12]. Feng and Xie introduced the Stability of fractal interpolation functions [13]. Mela and Louie studied the power spectra of data [14]. Ortega etal studied the design Fractal Curves from a given its dimension [15]. Vinoy etal studied Hilbert curve fractal antenna [16]. Jovanvic studied the chaotic via fractal conjecture [17]. Kim etal introduced the concept of downscaling of Soil Remotely Sensed to test the properties of soil using FIFs [18]. Navascas and Victoria introduced the convergent of cubic spline with FIFs [19]. Yong studied the swelling clay minerals [20]. Guedri etal introduced ECG compression Algorithm using Fractal Interpolation [21]. Polychronis etal studied Financial time series modeling using FIFs [22]. In this research , approximation of fractal interpolation functions is introduced by using artificial neural network. Arulprakash etal Fractal calculus is implemented on fractal interpolation functions and Weierstrass functions, which may be non-differentiable and non-integrable in the sense of ordinary calculus [23].

2 Basic Concepts of Artificial Neural Network

The artificial neural network is based on the inner-workings of the human brain and its decision- making process [23-24] .The allure of neural networks is that they are meant to mimic the human brain, which can perform computations much faster than the most powerful supercomputers in existence today. Although the silicon logic gates can be made to run at a speed of one nanosecond as compared to the human neuron's millisecond range, the efficiency and overall speed comes from the fact that there are so many neurons with so many interconnections which more than make up for the relative slow timing [34] .The power of the neural network comes from its many interconnections along with its ability to generalize based on what it has already learned. Even though neural networks cannot solve all complex problems , they can be utilized to solve portions of a whole problem that has been broken down into
smaller pieces . First of all , a neuron is defined as the fundamental Information processing unit for the, neural, network.

Each neuron has Inputs and outputs. The interconnections, between neuron layers in the neural network are characterized by their respective weights . For instance, say the input layer below has the neurons labeled x_1, x_2 x_3, x_4, and the top neuron in the hidden layer is called y_1. Also , the weights from the input layer neurons connecting to y_1 would be w_{11}, w_{12}, w_{13}, and w_{14}. Then the effect of neuron x_1 on y_1 would be the product of $x_1 * w_{11}$. Note that when classifying the addresses of the weight synapses, the rule of thumb is to list the destination address first and then the source address. The total effect of all the input layer neurons on y_1 is

$$\sum_{i=1}^{4} x_i w_{1i} \tag{1}$$

The rest of the products for the hidden layer neurons can be found through similar calculations.

2.1 Properties of Artificial Neural Network

1. Nonlinearity : Because a neuron is essentially a nonlinear mechanism , a network of interconnected neurons is nonlinear and leads for a model to solve nonlinear types of problems.

2. Input-Output Mapping : Through supervised learning, there is a set of input-output pairs that we want the neural network to learn. A pair is randomly selected and its input is presented to the neural network .When the neural network creates an output , if the output does not match the input , the neural network interconnection weights are adjusted to help the actual output converge to the desired output . The process is continued for all the wanted input- output pairs until the neural network reaches a point at which the actual output matches the wanted output within a certain amount of error . At this point we say the neural network has completed the learning process and successfully approximated the wanted mapping of input-output pairs.

3. Adaptivity: Neural networks are able to adapt their interconnection weights to alterations in their environments . If new requirements are provided , the neural network can inherently change its interconnection weights in response.

4. Fault Tolerance: An implemented neural network is fault-tolerant in that since the data or knowledge is spread over a wide range of interconnection weights, the degradation of the performance will be confined and not as harmful as compared to if there were a problem with coding or the absence of a line of code, the failure would be devastating.

5. VLSI Implementability : Because of the parallel nature and its ability to perform fast computations, neural networks can be used for very large scale integration (VLSI) . VLSI allows capturing behavior in a hierarchic al fashion for uses in many real-time data fields such as control, pattern recognition, and signal processing.

6. Uniformity of Analysis and Design :The universality of the neural network structure allows its use in many different fields for different problems. Because all neural networks have neurons, structures , and learning algorithms that do not change as we go from one field to another, they are useful and are easily reused for a variety of independent applications.

7. Neurobiological Analogy : Advances in the understanding of how the human brain works by neurobiologists have led to the development of faster and more powerful neural network models. Engineers rely upon neurobiologists for new models while neurobiologists rely upon engineers for verification that a new, more powerful model works .The cooperation between scientist and engineer leads to continuously improving technology and understanding in the field of neural networks.

3 Fractal Interpolation Functions

The fractal interpolation technique is an iterative affine mapping procedure to construct a synthetic deterministic small-scale field (in general fractal provided certain conditions are met) given a few large-scale interpolating points (mesh points). For an excellent treatise on this subject, the reader is referred to the book by Barnsley [2-3]. One has a set of data, and wants to interpolate to find a function that fits it. One can arrive at an IFS that interpolates the data,

$$\{(x_i, y_i) \in R^2 : i = 0,1,2,3,\cdots,N; N \in Z^+; x_0 < x_1 < \cdots < x_N\} \tag{2}$$

Definition(1) The set $G = \{(x, f(x)) : x \in [x_0, x_N]\}$ is a graph of function $f : [x_0, x_N] \to [a,b]$ if there is only y-value which corresponds to each x-value , where $y \in [a,b]$.

There is an IFS : $\{R^2; w_n, n = 1,2,3,\cdots,N\}$ which has the form:

$$w_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ c_n & d_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_n \\ f_n \end{pmatrix} \tag{3}$$

with condition

$$w_n\begin{pmatrix}x_0\\y_0\end{pmatrix}=\begin{pmatrix}x_{n-1}\\y_{n-1}\end{pmatrix} \quad and \quad w_n\begin{pmatrix}x_N\\y_N\end{pmatrix}=\begin{pmatrix}x_n\\y_n\end{pmatrix} \tag{4}$$

So ,

$$a_n x_0 + e_n = x_{n-1}$$

$$a_n x_N + e_n = x_n \tag{5}$$

$$c_n x_0 + d_n y_0 + f_n = y_{n-1} \tag{6}$$

$$c_n x_N + d_n y_N + f_n = y_n \tag{7}$$

therefore there are five unknown variables , four equations , and so one free (independent) will pick and choose d_n.Looking at , that acts vertically - stretches or compresses vertical line segments ,acts like a shear. One can study $0 \le d_n < 1$ and show that one has contraction maps .

Condition (4) yields that:

$$a_n = \frac{x_n - x_{n-1}}{x_N - x_0} \tag{8}$$

$$e_n = \frac{x_N x_{n-1} - x_0 x_n}{x_N - x_0} \tag{9}$$

$$c_n = \frac{y_n - y_{n-1}}{x_N - x_0} - d_n \frac{y_N - y_0}{x_N - x_0} \tag{10}$$

$$f_n = \frac{x_N y_{n-1} - x_0 y_n}{x_N - x_0} - d_n \frac{x_N y_0 - x_0 y_N}{x_N - x_0} \tag{11}$$

Theorem(1) Let N be a positive integer greater than one. Let $\{R^2; w_n, n = 1,2,\cdots,N\}$ denote the IFS defined in equations (2.3.1)-(2.3.3) , associated with the data set $\{(x_n, y_n): n = 0,1,2,\cdots,N\}$. *Let* $0 \le d_n < 1$ for $n = 1,2,\cdots,N$ .Then there exists a metric d on R^2; equivalent to the Euclidean metric, such that the IFS is hyperbolic with respect to d. Therefore, there is a unique nonempty compact set $G \subset R^2$; such that $G = \bigcup_{n=1}^{N} w_n(G)$.

4 Artificial Neural Network Technique for Approximation of Fractal Interpolation Functions

Following a proposed technique for approximation natural curve use the artificial neural network .The neural method benefits from finding the parameters of fractal interpolation functions.

1) Taking two affine transformations :

$$g_i\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e_i \\ f_i \end{bmatrix} \quad \text{for } i = 1,2. \tag{12}$$

2) Reading three points from original curve denoted by p_1, p_2, p_3.

For flexibility one can put p_1 as the original point .

3) Letting $g_1(p_1) = p_1, g_2(p_1) = p_2$ (13)

then

$$\begin{bmatrix} e_1 \\ f_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \tag{14}$$

4) Taking $g_1(p_3) = p_2, g_2(p_3) = p_3$. (15)

Then

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \tag{16}$$

Therefore

$$\begin{aligned} x_3 a_1 + y_3 b_1 &= x_2 \\ x_3 c_1 + y_3 d_1 &= y_2 \\ x_3 a_2 + y_3 b_2 + x_2 &= x_3 \\ x_3 c_2 + y_3 d_2 + y_2 &= y_3 \end{aligned} \tag{17}$$

5) Transforming the system (17) into the artificial neural network diagram in figure1 for n = 2, and m = 4.

6) Letting $b_1 = 0, b_2 = 0$, then n =2,m =4 and

$$a_1 = \frac{x_2}{x_3}, a_2 = \frac{x_3 - x_2}{x_3} \tag{18}$$

and then we have four unknown variables and two equations .

7) Taking $(x_i^*, y_i^*)_{i=1}^N$ to be the original data and $(x_i, y_i)_{i=1}^N$ to be approximation data , so the total error δ is :

$$\delta = \left(\sum_{i=1}^{N} \left(y_i^* - y_i\right)^2\right)^{\frac{1}{2}} \qquad (19)$$

8) Computing the parameters of each transformation g_1 and g_2 if $\delta < \alpha$ for small real number α.

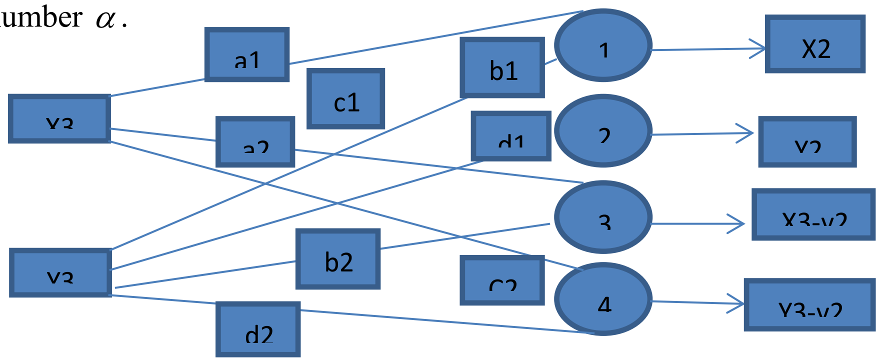

Fig. 1 ANN for modeling FIF

5 Implementation

Following many examples for testing the efficiency of our proposed method which programmed using ML.

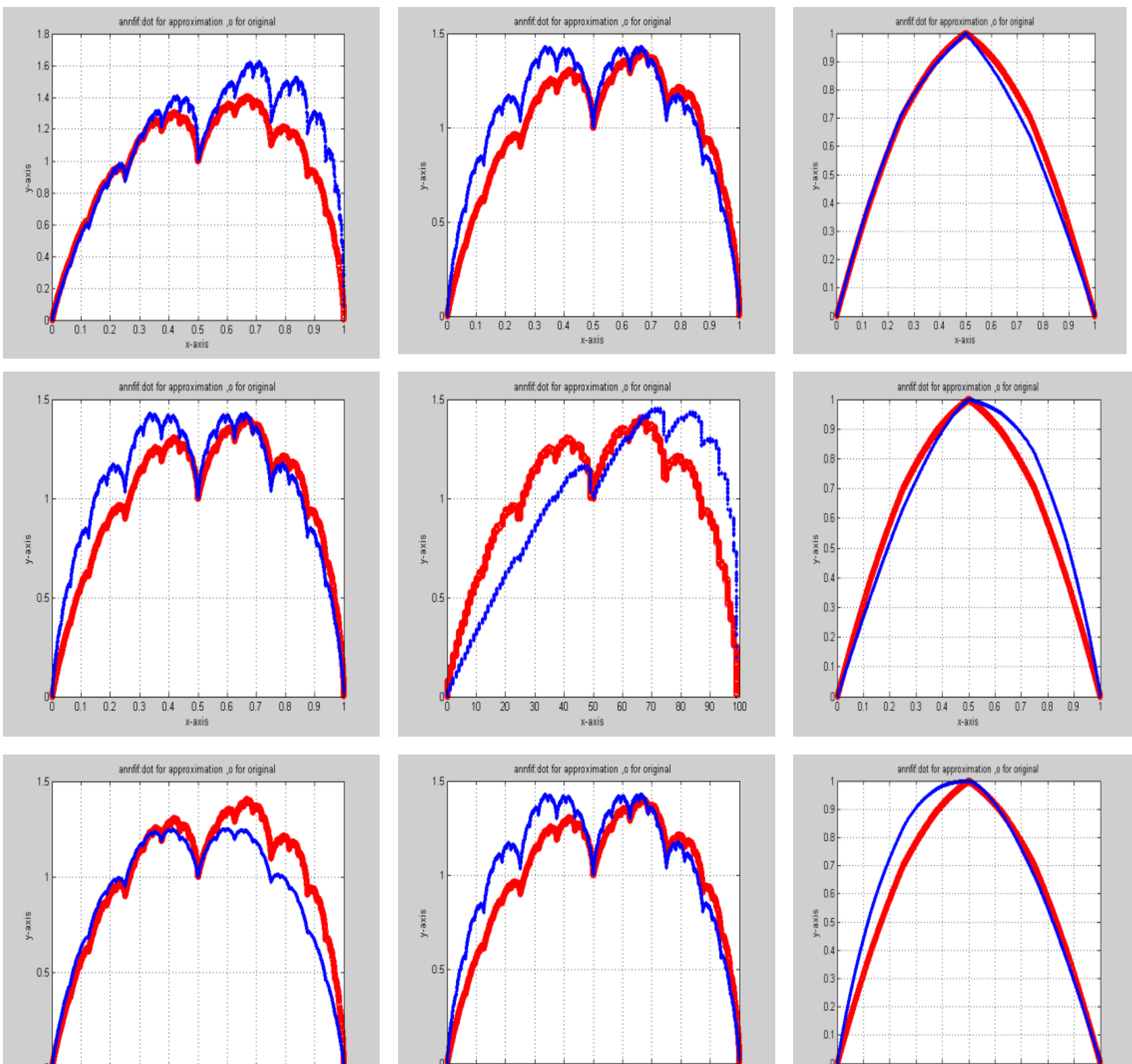

Fig. 2 approximations of FIF curves via ANN

6 Conclusions

To test the efficiency of the proposed method in this research, which is based on the ANN algorithm, 12 curves in red color were adopted. These curves vary in their characteristics, some of them are smooth, some are sharp, and some are not differentiable. By fixing three interpolation mesh points that were derived from the curve and using ANN equations of type basic delta rule, the approximated curve in blue was derived. All curves. All curves converged strongly.

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A Hybrid Approach for Opinion Leader Detection Integrating Semantic, Behavioral, and Topological Features in Online Social Networks

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Abstract:

The rapid expansion of social media has created highly dynamic environments in which a small set of opinion leaders can disproportionately shape public opinion and consumer behaviour. Identifying these individuals within large, noisy interaction networks remains a major challenge. This paper proposes a hybrid frame- work that integrates deep content-based indicators with structural graph analysis. Opinion leadership is mod- elled through four dimensions:expertise, novelty, influence, and activity extracted from users' textual and behavioural patterns and fused into a composite score that guides the HITS algorithm for refined authority propagation. Experiments on the RepLab banking and automotive domains show that the proposed method substantially outperforms structural, content-only, and conventional social-metric baselines, achieving relative gains of 68% and 24% over standalone HITS. These results demonstrate the framework's superior precision, robustness across topics, and resilience to social media noise.

1 Introduction

The rapid expansion of social media platforms has transformed digital communication, making social in- formation seeking a collaborative, data-driven pro- cess. Large-scale analyses of interaction patterns, sentiment trends, and engagement dynamics (Fadhli et al., 2023a) now reveal communication hierarchies and influential actors, while predictive models es- timate emerging trends and user impact (Landolsi et al., 2024). Influencers and opinion leaders play an increasingly central role in shaping consumer be- haviour and digital communication (Karoui et al., 2022), and the diversification of platforms such as Facebook, LinkedIn, and X has further intensified the need for data-driven strategies. Recent studies em- phasise the importance of integrated, context-aware models for opinion leader detection (Jin et al., 2025; Kunz-Lomelin et al., 2025), recognising that leader- ship depends not only on prominence but also on ex- pertise, activity, and innovation within specific do- mains

(Riquelme and Gonza´lez-Cantergiani, 2016). These developments motivate the design of reliable computational frameworks for opinion leader detec-

tion across sectors such as marketing, healthcare, and political communication. The remainder of this pa- per is structured as follows: Section 2 reviews related work, Section 3 introduces the hybrid framework and feature extraction, Section 4 describes the OLSD al- gorithm, Section 5 presents experimental results on the RepLab dataset, and Section 6 concludes the pa- per.

2 Related work

Opinion leader detection has been examined from several perspectives. Early work relied on graph- theoretic centrality measures such as degree, be- tweenness, closeness, and eigenvector centrality (Freeman, 1978; Bonacich, 2007), which capture structural prominence but often conflate popularity with genuine credibility or expertise. To address these limitations, later studies incorporated behavioural, temporal, and semantic cues. Research analysing user activity, interaction patterns, and diffusion cascades (Cha et al., 2010; Weng et al., 2010; Bakshy et al., 2011), as well as sentiment and content-based sig- nals (Liu, 2012), offers a more nuanced view of in- fluence. However, these methods still struggle to dis- tinguish highly visible users from trusted, domain- relevant opinion leaders and remain sensitive to the

dynamic nature of social media environments.

2.1 Trust-based approaches

Trust-based approaches provide an alternative per- spective by modelling interpersonal trust as a key de- terminant of influence. Extensions of the Advogato metric (Al-Oufi et al., 2012) and fuzzy trust–based methods (Jain et al., 2020) improve leader identifica- tion but rely on binary or static trust structures. Propa- gation models (Aghdam et al., 2016) capture trust dif- fusion yet overlook temporal and semantic variabil- ity, while multi-parameter frameworks

(Dutta and Ku- maravel, 2016) assume stable trust dynamics. More recent work integrates community and semantic infor- mation (Yang et al., 2022), yielding richer trust repre- sentations but at the cost of increased computational complexity and limited scalability.

2.2 Opinion analysis-based approaches

A second line of research focuses on opinion and content analysis. User characteristics such as ex- pertise, novelty, influence, and activity have been modelled through textual and behavioural signals (Li et al., 2013), though such approaches often lack generalisability across domains. Joint text-mining and network-based methods (Bodendorf and Kaiser, 2010) better capture opinion dynamics but remain limited in highly dynamic social platforms. More adaptive models like POLD (Huang et al., 2014) in- corporate thematic structure and temporal cues to im- prove stability, albeit with increased structural com- plexity. Related work on author profiling and cred- ibility assessment (Ouni et al., 2023; Fadhli et al., 2023b; Fadhli et al., 2023c) highlights the value of combining semantic and behavioural indicators, even though these studies do not explicitly target topic- specific opinion leader detection.

2.3 Propagation-based approaches

Propagation-based approaches model how informa- tion or behavioural signals diffuse through networks. Time-aware diffusion models (Tsai et al., 2014) cap- ture long-term adoption effects but rely on com- plex simulations and strong diffusion assumptions. Diffusion-speed methods (Cho et al., 2012) offer mar- keting value but depend on limited centrality fea- tures. More specialised techniques such as OLFinder (Aleahmad et al., 2016) and signal-diffusion mod- els (Sun and Bin, 2018) improve ranking accuracy through topical or multi-relational analysis, albeit with higher computational cost. Topic-specific propa-

2.4 Structure-based and learning-based approaches

Several studies refine structural approaches by broad- ening the notion of influence. Domain-sensitive mod- els (Miao et al., 2013) and longitudinal methods such as LUCI (Shafiq et al., 2013) capture specialised content and behavioural patterns, yet remain limited by context dependence and lack of thematic adapt- ability. Algorithms including Influence Rank (Cao et al., 2007), proximity-based techniques (Yang et al., 2018), and firefly-inspired metaheuristics

(Jain and Katarya, 2019) improve upon basic centrality but suf- fer from complexity and limited scalability, while community-based models (Chen et al., 2017) still rely on simple descriptors. Machine learning and deep learning approaches, such as DPIN (Karoui et al., 2022), demonstrate the value of integrating struc- tural and semantic cues but often address general influence rather than domain-specific opinion lead- ership. Overall, these works underscore the poten- tial of multi-modal features yet rarely operationalise the full multidimensional construct of opinion lead- ership—encompassing influence, expertise, activity, and innovation.

2.5 Motivation and contribution

Opinion leaders play a decisive role in online social networks, shaping public discourse and consumer be- haviour while driving engagement on platforms such as X and Facebook. Identifying these actors is cru- cial for applications ranging from viral marketing to misinformation control, yet remains challenging: centrality- and activity-based metrics often conflate popularity with authority, and content-only methods struggle to separate genuine expertise from high- volume posting. Consequently, recent work supports hybrid models that combine structural, behavioural, and linguistic cues (Hlaoua, 2025).In this study, we define opinion leaders not merely by visibility but by the combined presence of expertise, novelty, influ- ence, and sustained activity within a given domain. We propose a hybrid framework that aggregates these four dimensions into a composite score and refines candidate rankings using network-level authority pat- terns, enabling more accurate and context-aware iden- tification of opinion leaders.

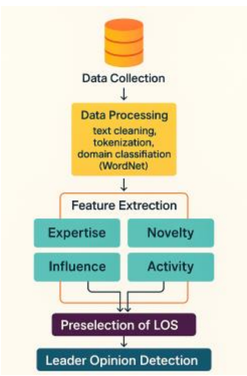

Figure 1: Pipeline of opinion leader detection model

The proposed model offers a structured hybrid ap- proach to identifying domain-specific opinion lead- ers by integrating content-based signals with network- level engagement features. It combines preprocessing, multidimensional feature extraction, candidate preselection, and graph-based ranking to differentiate authoritative leaders from general influencers. The pipeline consists of four steps: (1) semantic pre- processing of user content, (2) extraction of influ- ence, expertise, activity, and novelty indicators, (3) aggregation into a composite score for candidate fil- tering, and (4) refinement via the HITS algorithm, which distinguishes true opinion leaders through au- thority–connectivity analysis.

Figure 1 illustrates the pipeline of our model

3.1 Data Preprocessing

The detection pipeline begins with a rigorous prepro- cessing stage that converts raw user-generated content into clean and semantically coherent text by apply- ing normalization, tokenization, and noise removal. A key element of this stage is the integration of Word- Net (Miller, 1995), whose hierarchical semantic struc- ture enables effective computation of similarity mea- sures. Leveraging WordNet supports more accurate domain classification and introduces meaningful se- mantic cues, thereby strengthening the reliability of subsequent feature extraction.

3.2 Feature Extraction

The model then derives four key indicators: influ- ence, expertise, activity, and novelty, reflecting com- plementary facets of user behaviour. We posit that true opinion leaders combine high expertise, contin- uous activity, and meaningful novelty, making these dimensions fundamental to robust leader characteri- sation.

3.2.1 Influence

Influence is quantified through X’s social interactions using the Retweet Impact method (Riquelme and Gonza´lez-Cantergiani, 2016), which measures the ef- fect of user i’s tweets based on retweet activity. The influence score is given by:

$$Influence(i) = RT2 \times \log(RT3) \qquad (1)$$

Where RT2 reflects the volume of user i’s content retweeted by others, RT3 measures the number of dis- tinct retweeters, and logarithmic scaling mitigates the impact of disproportionately high retweet activity.

3.2.2 Expertise

User expertise is computed as the sum of domain- membership probabilities of user i's tweets, normal- ized by the total volume of domain-related tweets.*n*

$$Expert(i) = \sum^{n}_{t=1} ([\text{Probabilityo f atweet} \in \text{domain}] \quad (2)$$
$$\text{Numbero f tweets} \in \text{domain}$$

pipeline consists of four steps: (1) semantic pre- processing of user content, (2) extraction of influ- ence, expertise, activity, and novelty indicators, (3) aggregation into a composite score for candidate fil- tering, and (4) refinement via the HITS algorithm, which distinguishes true opinion leaders through au- thority–connectivity analysis.
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3.1 Data Preprocessing
The detection pipeline begins with a rigorous prepro-

This score prioritizes domain-specialized contrib [Probability(tweett/tweet)])….(4)

This score prioritizes domain-specialized contrib- utors, rewarding users who consistently produce con- tent aligned with the target topic.

3.2.3 Activity

This metric evaluates user i's activity in domain x by assigning WordNet-based domain probabilities to each tweet and aggregating their distances to the tar- get domain. It is computed as follows:

Activity(i) = ∑nt=1 ([Probability(tweett/tweet)])….(3)

Unlike expertise, which focuses on quality and knowledge, activity quantifies the volume of relevant participation, indicating the user's consistency and visibility in domain discussions.

3.2.4 Novelty

We define innovation as the probability that a tweet contributes genuinely new content—distinct from temporal trends. The innovation score for user i is given by:

$$Novelty(i) = 1 - \sum^{n}_{t=1} ([Probability(tweett/tweet)])….(4)$$

A high score indicates that a user consistently pro- duces unique and less redundant content, an important attribute of thought leaders who introduce novel per- spectives rather than echo prevailing narratives.

3.3 Preselection of opinion leaders

Once the four characteristic indicators have been computed, the model proceeds to a preselection stage designed to isolate users who exhibit promising lead- ership potential. Rather than relying on a single be- havioral attribute, the proposed framework adopts a balanced, multi-criteria strategy by aggregating the normalized scores of influence, expertise, activity, and novelty. This aggregation yields a global Opin- ion Leadership Prior defined as:

$$\text{ScoreUser}(i) = \text{sinf}(i) \times \text{sexp}(i) \times \text{sact}(i) \times \text{snov}(i)$$

The multiplicative formulation acts as an implicit regularizer, requiring users to demonstrate balanced strength across all four dimensions rather than ex- celling in only one. This mitigates common biases of popularity-driven rankings by penalising users who are active but lack expertise, influential but unorig- inal, or narrowly engaged. The resulting composite score produces a semantically grounded initial rank- ing and a refined preselection pool, reducing noise and ensuring that subsequent structural analysis fo- cuses solely on candidates with genuine leadership potential.

3.4 Leader Opinion Detection

While preselection identifies users with strong be- havioural and content-based profiles, structural anal- ysis is required to distinguish genuine opinion leaders from merely popular users. To achieve this, we apply the HITS algorithm (Kleinberg, 1999) to a directed user–retweet graph, where authority scores capture informational credibility and hub scores reflect the ability to disseminate authoritative content. Based on mutual reinforcement—strong hubs pointing to strong authorities and vice versa—HITS iteratively updates these weights until convergence. Normalised author- ity and hub scores yield a stable structural ranking that separates domain-relevant leaders from interme- diaries and filters out popularity-driven influence. Ap- plied to the preselected candidates, HITS refines the final set of opinion leaders, producing a robust and interpretable influence hierarchy.

4 Proposed OLSD algorithm

To operationalise the framework, we introduce the Opinion Leader Scoring and Detection (OLSD) al- gorithm, which integrates social characteristics with graph-based authority analysis. OLSD first computes

four social scores: influence, expertise, activity, and novelty and aggregates them into a global leadership score used to rank users. The top-ranked candidates are then processed by the HITS algorithm, which iter- atively refines authority and hub values on the inter- action graph. Users with the highest authority scores are ultimately identified as opinion leaders.

5 Experimental study and results analysis

The objective of our experimental study is to evaluate the effectiveness of the proposed framework in identi- fying opinion leaders under realistic social media con- ditions. This section outlines the evaluation metrics, describes the experimental protocol, and discusses the results obtained on the CLEF RepLab 2014 dataset.

5.1 Evaluation scenario

Our evaluation proceeds in two stages. First, we ex- tract and compute four social scores—influence, in- novation, activity, and expertise—by applying Equa- tions 1, 3, 4, and 2, each capturing complementary behavioural and content-based aspects of user leader- ship. These characteristics are then aggregated into a composite score used to preselect candidate opinion leaders. Next, follow relationships are used to con- struct the social network graph, and the HITS algo- rithm is applied to refine authority estimates. The top K candidates (K=10) are selected as the final opin- ion leaders. We then analyse the individual impact of HITS and the contribution of social factors to overall detection performance.

5.2 Results analysis and discussion

In our work, we evaluated the Recall rate and Preci- sion rate by aggregating social characteristics. The HITS algorithm produced superior results.

Table 1: Improvement rate of our opinion leader detection on the Banking and Vehicle datasets

Dataset	Rate of Improvement (%)
Banking Dataset	67.51
Vehicle Dataset	24.8

Precision and Recall results show that our social- characteristic model surpasses HITS in both domains, detecting 3/10 and 4/10 leaders versus 6/10 and 8/10 for HITS. The hybrid approach achieves perfect iden- tification (10/10) and yields relative gains of 68%

(banking) and 24% (automotive), underscoring its ro- bustness and cross-domain reliability.

Comparing our approach to three previous works (2), we observed that 39% of tweets had been deleted during RepLab data enrichment, impacting results. This missing information likely reduced performance.

6 Conclusion

This paper introduced a hybrid framework for opinion leader detection that integrates behavioural signals with structural graph analysis to capture the multidi- mensional nature of influence. By modelling leader- ship through four core dimensions: expertise, novelty, influence, and activity, and guiding a refined HITS- based ranking, the framework outperforms both stan- dalone HITS and score-only baselines on two RepLab domains. These findings demonstrate the effective- ness of combining semantic, behavioural, and topo- logical cues to identify influential users in noisy social settings. Although promising, performance remains sensitive to platform-specific interaction patterns and data quality. Future work will explore broader bench- mark evaluations, incorporate transformer-based lan- guage models for deeper content analysis, and extend the framework to opinion-level and dynamic influ- ence modelling, while addressing challenges related to noise, scalability, and privacy.

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Table 2: Results of the different approaches with respect to the different metrics considered.

Approach	Dataset	F1-score	Recall	Precision	Accuracy	G-mean
Ram´ırez et al. (2014)	Bank/Vehicule dataset	0.694,0.599	0.710,0.601	**0.788**,0.598	–	–
Our approach	Bank/Vehicule dataset	**0.780**,**0.784**	**0.789**,**0.793**	0.771,**0.776**	0.811,0.783	0.682,0.693
Ram´ırez et al. (2014)	Miscellaneous	0.764	0.751	0.780	–	–
Cosu et al. (2016)	Miscellaneous	0.792	–	–	–	–
Azaz et al. (2019)	Miscellaneous	0.754	0.770	0.739	–	–
Our approach	Miscellaneous	**0.835**	**0.799**	**0.875**	0.789	0.706

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Enhancing the Fréchet Distribution: Properties, Computational Methods, and Applications in Cancer Survival and Material Strength

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Abstract:

This research aims to develop the Enhancing Fréchet distribution (EF) by combining a hybrid Weibull-G (HWG) family with classical Fréchet distribution, resulting in EF with high flexibility in representing data with heavy tails and high positive skewness. The mathematical properties of EF distribution were studied, including the survival and hazard functions, moments, the extended CDF, extended pdf, and quantile function. the results showed that the enhanced distribution has a high ability to represent extreme and asymmetric data. As verified by the graphs of the different functions, which showed a variety of shapes depending on parameter values. Moments, variance, skewness, and kurtosis were also calculating, confirming the presence of heavy tails and strong positive skewness in EF distribution. In addition, the distribution parameters were estimated using three methods, followed by a simulation that showed that the MLE method was the most accurate and least biased compared to the other methods, especially with increasing sample size. On the applied side, the performance of the new distribution was tested on two real data sets where it achieved the lowest information criterion values and the highest p-values, confirming its high fit to the data.

Keywords: Hybrid Weibull-G, Monte Carlo, Estimation, Cancer Survival, and Material Strength.

1. Introduction

Statistical distributions have long formed the backbone of probability theory and mathematical statistics, providing mathematical frameworks for describing the behavior of random variables. The journey began with simple distributions such as the Bernoulli and binomial distributions, which were sufficient to describe specific phenomena. As data complexity increased in the 20th century, the need arose for more flexible distributions capable of representing diverse properties such as over-variance, asymmetry, and heavy tails.

To overcome these limitations, statisticians began developing extended distribution families by adding new parameters. One of the most important of these methods, introduced in the last two decades, is the T-X method, introduced by researchers Alzaatreh in 2013 [1] as a general framework for generating new distributions. This method relies on transforming a random variable, T, using an appropriate generating function, combining it with a variable, X, with a known distribution, and then creating a new distribution that retains the properties of the original distribution while adding flexibility. Examples of statistical families that have used this method include can see sources ([2], [3], [4], [5], [6], [7], [8], [9], [10], and [11]). Among these families emerged the Hybrid Weibull-G (H.W.G) family with CDF and pdf function defined as follows [12], [13]:

$$F(x) = 1 - e^{-\omega\left[-G(x).\log(1-G(x))\right]^{\delta}} \tag{1}$$

$$f(x) = \frac{\omega\delta\, g(x)\left[\frac{G(x)}{1-G(x)} - \log\left(1-G(x)\right)\right]}{\left[-G(x).\log\left(1-G(x)\right)\right]^{1-\delta}} e^{-\omega\left[-G(x).\log(1-G(x))\right]^{\delta}} \tag{2}$$

where $G(x)$ and $g(x)$ are CDF and pdf for any baseline distribution, respectively, with $x \in \mathbb{R}^{+}$ and $\omega, \delta > 0$ are shape parameters for H.W.G.

This study aims to provide a qualitative addition to the field of statistical modeling by enhancing the traditional Brushy distribution and expanding its scope of practical applications. The study integrates the Brushy distribution with the hybrid Weibull-G (H.W.G) family. The work will focus on exploring the in-depth mathematical properties of the new distribution, including deriving a number of properties of the enhancing distribution. On the applied side, the study will include a comprehensive evaluation of the performance of three different estimation methods (weighted least squares, least squares, and maximum likelihood method) through advanced Monte Carlo simulation. To demonstrate the feasibility of the proposed distribution, it will be applied to two real-world data sets: cancer patient survival data and engineering materials durability data. The performance of the new distribution will be compared with a set of competitive distributions using accuracy criteria.

2. Enhancing Fréchet Distribution

The Fréchet distribution belongs to the family of extreme value distributions of the second kind. The CDF and pdf of the distribution is defined as for a positive random variable $X \in (0, \infty)$, the functions are defined respectively, by the formula:

$$G(x) = e^{-\alpha x^{-\theta}}, x \in (0, \infty) \tag{3}$$

$$g(x) = \alpha\theta x^{-(\alpha+1)} e^{-\alpha x^{-\theta}} \tag{4}$$

where α, θ respectively are shape parameter and scale parameter.

As previously defined, the CDF of the Enhancing Fréchet (EF) Distribution is found by combining Equation 1 with Equation 3 as follows:

$$F_{EF}(x) = 1 - e^{-\omega\left[-e^{-\alpha x^{-\theta}}.\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]^{\delta}} \tag{5}$$

By deriving the above equation, the distribution function pdf is obtained as follows:

$$f_{EF}(x) = \frac{\omega\delta\,\alpha\theta e^{-\alpha x^{-\theta}}\left[\frac{e^{-\alpha x^{-\theta}}}{1-e^{-\alpha x^{-\theta}}} - \log\left(1-e^{-\alpha x^{-\theta}}\right)\right]}{x^{\alpha+1}\left[-e^{-\alpha x^{-\theta}}.\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]^{1-\delta}} e^{-\omega\left[-e^{-\alpha x^{-\theta}}.\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]^{\delta}} \tag{6}$$

The first step in demonstrating the elasticity of a distribution is to plot the CDF and PDF functions with different parameter values, which are shown in Figures 1 and 2.

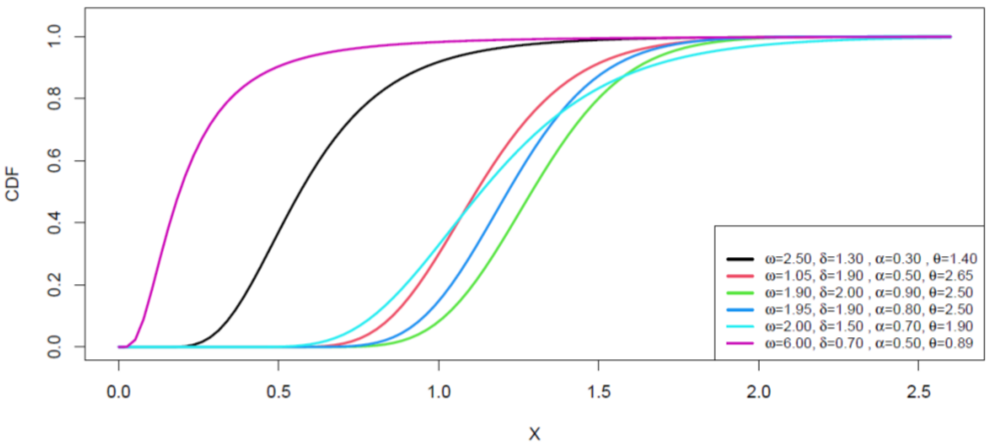

Figure 1. plot the CDF for EF Distribution

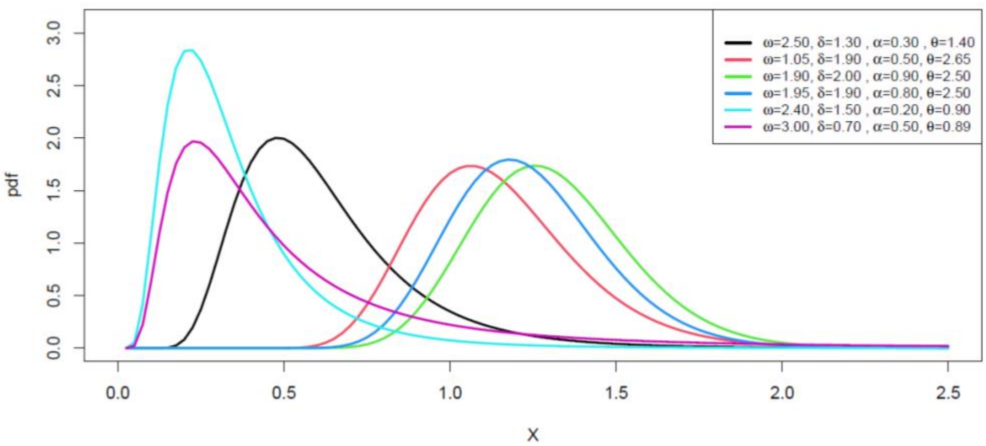

Figure 2. plot the pdf for EF Distribution

Figure 1 shows the behavior of the cumulative distribution function (CDF). We notice that all curves start from zero and gradually converge toward 1 as the x value increases. This reflects the fundamental nature of cumulative distribution functions, as the rate of convergence toward 1 varies with different parameters. Some curves converge faster, while others converge more slowly, demonstrating the flexibility of the distribution to represent different data patterns.

Figure 2 shows the behavior of the probability density function (PDF) for the same set of parameters. The curves exhibit a variety of shapes, ranging from heavy-tailed curves with sharp peaks to more elongated curves with less sharp peaks. This variety of shapes highlights the ability of the boosted distribution to model different types of data, especially those with extreme values or asymmetric distributions.

Now we find the survival function by form [14], [15]:

$$S_{EF}(x) = 1 - F_{EF}(x)$$

$$S_{EF}(x) = e^{-\omega\left[-e^{-\alpha x^{-\theta}}.\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]^{\delta}} \tag{7}$$

And the hazard function can be finding as [16], [17]:

$$h_{EF}(x) = \frac{f_{EF}(x)}{S_{EF}(x)}$$

$$h_{EF}(x) = \frac{\omega\delta\ \alpha\theta e^{-\alpha x^{-\theta}}\left[\dfrac{e^{-\alpha x^{-\theta}}}{1-e^{-\alpha x^{-\theta}}} - \log\left(1-e^{-\alpha x^{-\theta}}\right)\right]}{x^{\alpha+1}\left[-e^{-\alpha x^{-\theta}}.\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]^{1-\delta}} \tag{8}$$

Plotting the survival function and hazard of an EF distribution is not just a visual representation; it is a powerful analytical tool for understanding when events occur (from the survival function) and how their risks change over time (from the hazard function). This makes it essential for decision-making in sensitive areas that rely on the time-course analysis of data. Figures 3 and 4 represent the plots of these functions.

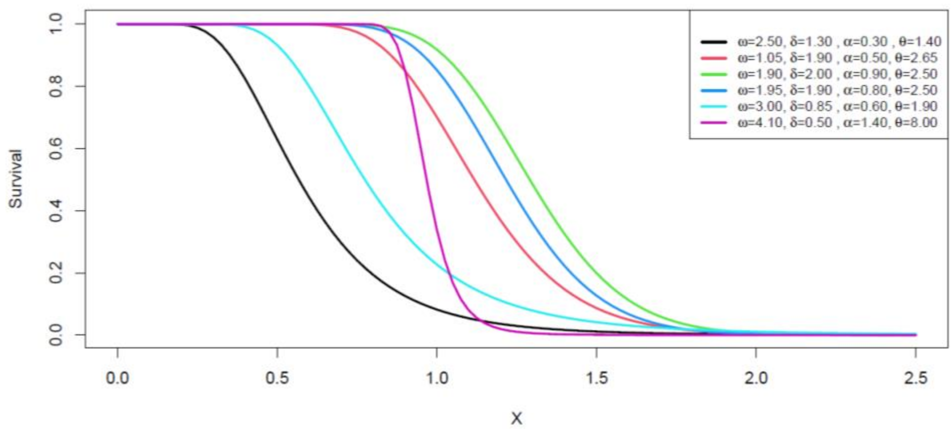

Figure 3. plot the survival for EF Distribution

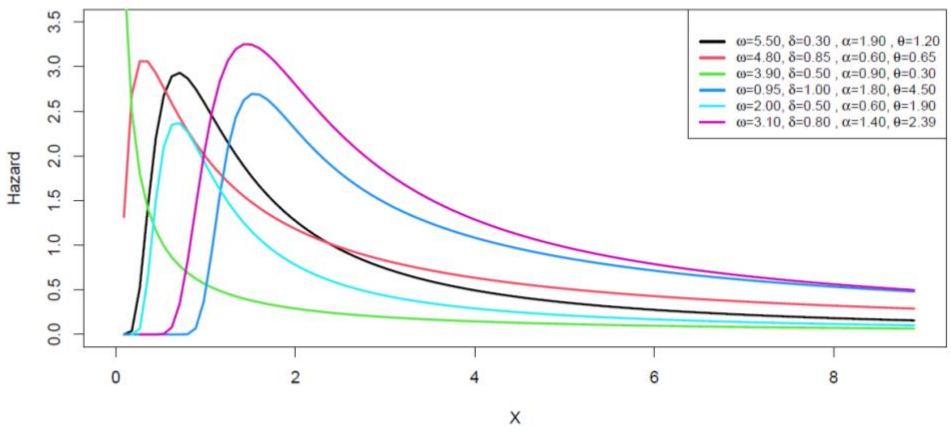

Figure 4. plot the hazard for EF Distribution

Figure 3 illustrates the probability of values remaining for more than a certain time (x). We note that all curves start from a value of 1 and gradually decrease toward zero, with different rates of decrease depending on the parameter values.

Figure 4 reveals the instantaneous rate of occurrence (such as failure or death) subject to survival until that time. The curves show diverse hazard patterns: curves that start with high risk and then decrease, curves whose risk gradually increases over time, and curves that show a peak at medium hazard before stabilizing. This diversity confirms the ability of the boosted distribution to simulate complex risk behavior in real-world applications.

Some special cases of the distribution can be obtained by substituting values for its parameters in CDF, and pdf distribution, as follows:

Table 1. some special sub models of MODLBIII

Model	**Fixed Parameters**	$G(x)$	**Applications**
Classical Fréchet	$\omega = \delta = 1$	$1 - e^{-\alpha x^{-\theta}}$	Modeling extreme values
Hybrid Weibull Fréchet	$\delta = 1$	$1 - e^{-\omega\left[-e^{-\alpha x^{-\theta}}.\log\left(1-e^{-\alpha x^{-\theta}}\right.\right.}$	Survival and reliability analysis
Exponential Asymptotic Limit	$\theta \to \infty^{+}$	$1 - e^{-\omega y e^{-y}}$	Modeling failure rates
Heavy Tail	$\omega = \delta = \alpha = 1$	$1 - e^{\left[e^{-x^{-\theta}}.\log\left(1-e^{-x^{-\theta}}\right)\right]}$	Financial and physical data
Separate	$\theta \to \infty^{+}$	$1 - e^{-\omega x^{-\delta}}$	Correct meters and data

representation			

3. Some Properties for Enhancing Fréchet Distribution

3.1 CDF and pdf expansion

Considering the importance of CDF and pdf of EF distribution functions in finding the mathematical properties of EF distribution, as well as the difficulty of these functions, these functions are expanded to deal with them using the binomial expansions, exponential expansion, and logarithmic expansion as [18], [19] and accordingly the CDF function is expansion for EF distribution in form:

$$F_{EF}(x) = 1 - He^{-\alpha(j+2i\delta)x^{-\theta}} \quad (9)$$

where $H = \sum_{i=j=0}^{\infty} \frac{(-1)^{i+\delta+j}}{i!} d_{i\delta,j}\, \omega^i$, $d_{i\delta,j} = j^{-1} \sum_{s=1}^{j} \frac{[s(i\delta+1)-j]}{s+1}$ for $j \geq 0$ and $d_{i\delta,0} = 1$

by same way to expand pdf to get a formula:

$$f_{EF}(x) = Ne^{-\alpha(k+2i\delta+2\delta+z)x^{-\theta}}\, x^{-(\alpha+1)} - Me^{-\alpha(k+2i\delta+2\delta+1)x^{-\theta}} x^{-(\alpha+1)} \quad (10)$$

where $N = \sum_{i=k=z=0}^{\infty} \frac{(-1)^{i+2\delta+k+z-1}}{i!} d_{i\delta+\delta-1,k}\, \omega^{i+1}\delta\, \alpha\theta$, and

$M = \sum_{i=j=0}^{\infty} \frac{(-1)^{i+2\delta+j-1}}{i!} d_{i\delta+\delta,j}\, \omega^{i+1}\delta\, \alpha\theta$, such as $d_{i\delta+\delta-1,k} = k^{-1} \sum_{s=1}^{k} \frac{[s(i\delta+\delta-1+1)-k]}{s+1}$ for $k \geq 0$ and $d_{i\delta+\delta-1,0} = 1$, and $d_{i\delta+\delta,j} = j^{-1} \sum_{l=1}^{j} \frac{[l(i\delta+\delta+1)-j]}{l+1}$ for $j \geq 0$ and $d_{i\delta+\delta,0} = 1$

The $F_{EF}^t(x)$ has a form:

$$F_{EF}^t(x) = \left(1 - e^{-\omega\left[-e^{-\alpha x^{-\theta}} .\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]^{\delta}}\right)^t \quad (11)$$

Also can be expanded to form:

$$F_{EF}^t(x) = \Theta e^{-\alpha(u+2l\delta)x^{-\theta}} \quad (12)$$

where $\Theta = \sum_{q=l=0}^{\infty} \frac{(-1)^{q+l+\delta+u}}{i!} \binom{t}{q} d_{l\delta,u} \omega^l q^l$, and $d_{i\delta,u} = u^{-1} \sum_{s=1}^{u} \frac{[s(l\delta+1)-u]}{s+1}$ for $u \geq 0$ and $d_{l\delta,0} = 1$

3.2 Moments for EF distribution

Moments are numerical measurements used to describe the shape and properties of probability distribution of random variable. These moments express various properties of distribution. the moment is defined as follows: let X be a random variable, the n^{th} moment of any distribution is defined by formula [20], [21]:

$$\mathcal{M}_n = E(x^n) = \int_{-\infty}^{\infty} x^n f(x)dx \quad (13)$$

To obtain the moment function of EF distribution, substitute equation 10 into equation 13 as follows:

$$\mathcal{M}_n = E(x^n) = N\int_0^\infty e^{-\alpha(k+2i\delta+2\delta+z)x^{-\theta}}\, x^{n-(\alpha+1)}dx$$
$$- M\int_0^\infty e^{-\alpha(k+2i\delta+2\delta+1)x^{-\theta}}\, x^{n-(\alpha+1)}dx$$

$$\mathcal{M}_n = \frac{1}{\theta}\Gamma\left(\frac{\alpha-n}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-n}{\theta}} - M\big(\alpha(k+2im+2m+1)\big)^{\frac{\alpha-n}{\theta}}\right\} \quad (14)$$

From equation 14 can be get:

$$\mathcal{M}_1 = \frac{1}{\theta}\Gamma\left(\frac{\alpha-1}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-1}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-1}{\theta}}\right\} \quad (15)$$

$$\mathcal{M}_2 = \frac{1}{\theta}\Gamma\left(\frac{\alpha-2}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-2}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-2}{\theta}}\right\} \quad (16)$$

$$\mathcal{M}_3 = \frac{1}{\theta}\Gamma\left(\frac{\alpha-3}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-3}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-3}{\theta}}\right\} \quad (17)$$

$$\mathcal{M}_4 = \frac{1}{\theta}\Gamma\left(\frac{\alpha-4}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-4}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-4}{\theta}}\right\} \quad (18)$$

So the variance, skewness and kurtosis functions of EF distribution can be obtained as follows [22], [23]:

$$\sigma^2 = \frac{1}{\theta}\Gamma\left(\frac{\alpha-2}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-2}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-2}{\theta}}\right\} - \left(\frac{1}{\theta}\Gamma\left(\frac{\alpha-1}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-1}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-1}{\theta}}\right\}\right)^2 \quad (19)$$

$$S = \frac{\frac{1}{\theta}\Gamma\left(\frac{\alpha-3}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-3}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-3}{\theta}}\right\}}{\left(\frac{1}{\theta}\Gamma\left(\frac{\alpha-2}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-2}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-2}{\theta}}\right\}\right)^{\frac{3}{2}}} \quad (20)$$

$$K = \frac{\frac{1}{\theta}\Gamma\left(\frac{\alpha-4}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-4}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-4}{\theta}}\right\}}{\left(\frac{1}{\theta}\Gamma\left(\frac{\alpha-2}{\theta}\right)\left\{N\big(\alpha(k+2i\delta+2\delta+z)\big)^{\frac{\alpha-2}{\theta}} - M\big(\alpha(k+2i\delta+2\delta+1)\big)^{\frac{\alpha-2}{\theta}}\right\}\right)^{2}} - 3 \quad (21)$$

In order to illustrate nature of EF distribution in the study, table 2 was prepared to show the first four moments with variance, skewness and kurtosis for different values of the parameters.

Table2: the Moments for EF distribution with different values of the parameters

ω	δ	α	θ	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	σ^2	S	K
1.8	1.6	1.4	0.1	0.021554	0.018802	0.016632	0.014887	0.018337	6.451481	42.11197
			0.2	0.021816	0.019215	0.01713	0.015429	0.018739	6.431403	41.79026
		1.6	0.3	0.011337	0.010179	0.009221	0.008418	0.01005	8.979007	81.24768
			0.4	0.011429	0.010331	0.009412	0.008634	0.0102	8.963453	80.89674
	1.9	1.8	0.5	0.002102	0.001949	0.001815	0.001698	0.001945	21.10036	447.065
			0.6	0.002112	0.001967	0.001839	0.001726	0.001962	21.08359	446.1635
		1.9	0.7	0.001431	0.001341	0.001262	0.00119	0.001339	25.68288	661.6519

0.8	0.003014	0.002819	0.002646	0.002491	0.00281	17.67929	313.5805

Form table 2 all moments are positive but very small (less than 1), indicating that the data are concentrated near zero with long tails (typical of extreme value increase distributions such as Fréchet). The effect of θ is greatest, as increasing (holding other parameters constant), the moments decrease, indicating an increasing concentration of data near zero. The variance is relatively small, but increases as the parameters change, reflecting the flexibility of EF distribution to represent heterogeneous data. All skewness values are positive and large, indicating strong positive skewness (longer right tails), larger θ implies greater skewness, confining increasing asymmetry. The kurtosis values are very high, indicating heavy tails (the presence of more extreme values than in normal distribution).

Therefore, EF distribution can be said to have long, heavy tails and strong positive skewness (the data are concentrated on the left, with rare large values).

3.3 Moments Generating function for EF distribution

The MGF of EF distribution can be found using the moments function in equation 14 and expanding the exponential function to get [24]:

$$M_x(y)_{\mathrm{EF}} = \sum_{s=0}^{\infty} \frac{y^s}{s!}\left[\frac{1}{\theta}\Gamma\left(\frac{\alpha-n}{\theta}\right)\left\{NA_1^{\frac{\alpha-n}{\theta}} - MA_2^{\frac{\alpha-n}{\theta}}\right\}\right] \tag{22}$$

3.4 Characteristic function for EF distribution

The Characteristic function of EF distribution can be found using the moments function in equation 14 and expanding the exponential function to get [25]:

$$C_x(y)_{EF} = Q_x(t) = \sum_{v=0}^{\infty} \frac{(it)^v}{v!}\left[\left[\frac{1}{\theta}\Gamma\left(\frac{\alpha-n}{\theta}\right)\left\{NA_1^{\frac{\alpha-n}{\theta}} - MA_2^{\frac{\alpha-n}{\theta}}\right\}\right]\right] \tag{23}$$

where $A_1 = \alpha(k + 2i\delta + 2\delta + z)$, and $A_2 = \alpha(k + 2im + 2m + 1)$

3.5 Incomplete Moments for EF distribution

Let X be a random variable, the incomplete moment of any distribution is defined by formula [20], [26]:

$$I_n = \int_{-\infty}^{y} x^n f(x)dx \tag{24}$$

To obtain the incomplete moment function of EF distribution, substitute equation 10 into equation 24 as follows:

$$I_N = \frac{N}{\theta} A_1^{\frac{n-\alpha}{\theta}} \Gamma\left(\frac{\alpha-n}{\theta}, \frac{A_1}{y^\theta}\right) - \frac{M}{\theta} A_2^{\frac{n-\alpha}{\theta}} \Gamma\left(\frac{\alpha-n}{\theta}, \frac{A_2}{y^\theta}\right) \qquad (25)$$

where $y \to \infty$ then $I_N \to \mathcal{M}_n$ (i.e. equation 25 converge to equation 13 or incomplete moment converge to moments)

3.6 Probability Weighted Moments

We may calculate the probabilistic weighted moments of the EF distribution by applying the following equation [27]:

$$\tau_{m,t} = E\left(x^m F_{EF}^t(x)\right) = \int_{-\infty}^{\infty} x^m f_{EF}(x) F_{EF}^t(x) dx$$

By substitute equation 10 and 12 in above equation to get a formula:

$$\tau_{m,t} = \int_0^\infty \left(e^{-\alpha(k+2i\delta+2\delta+zu+2l\delta)x^{-\theta}} x^{m-(\alpha+1)} - Me^{-\alpha(k+2i\delta+2\delta+1+u+2l\delta)x^{-\theta}} x^{m-(\alpha+1)}\right) dx$$

Then, integrate the preceding equation to obtain the final formula:

$$\tau_{m,t} = \frac{\Theta}{\theta} \Gamma\left(\frac{\alpha-m}{\theta}\right)\left(NC_1^{\frac{m-\alpha}{\theta}} - MC_2^{\frac{m-\alpha}{\theta}}\right) \qquad (26)$$

where $C_1 = \alpha(k + 2i\delta + 2\delta + zu + 2l\delta)$, and $C_2 = \alpha(k + 2i\delta + 2\delta + 1 + u + 2l\delta)$

3.7 Quantile function of EF distribution

The quantile function, also known as the inverse CDF , is a fundamental concept in statistics and probability theory. This function is used to determine the value that random variable will not exceed with a given probability. Its function $Q(p)$ for a random variable X is defined as follows [28], [29]:

$$Q(p) = F^{-1}(x)$$

Where $F(x)$ for each $p \in (0,1)$. Then the Quantile function of EF distribution by form:

$$Q(x) = \left[\frac{1}{\alpha} \log\left\{\frac{\pi - W_{-1}(-\beta e^\pi)}{\pi}\right\}\right]^{-\frac{1}{\theta}}, \pi = \left(-\frac{\log(1-u)}{\omega}\right)^{\frac{1}{\delta}} \qquad (27)$$

where $W_{-1}(-\beta e^\pi)$ is the secondary real branch of the Lambert W function, defined for $-\beta e^\pi \in [-e^{-1}, 0)$.

The measurements of skewness and kurtosis based on Quantile function were defined as follows:

$$S = \frac{Q\left(\frac{6}{8}\right) - 2Q\left(\frac{4}{8}\right) + Q\left(\frac{2}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \tag{28}$$

$$K = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \tag{29}$$

Table 3 was prepared to show the Quantile function for different values of the parameters.

Table 3: explanation of the quantile function for specific EF distribution with different parameter values

	$(\omega, \delta, \alpha, \theta)$				
u	(1.8,1.9,1.4,1.7)	(1.6,1.4,1.4,1.5)	(1.5,1.6,1.5,1.6)	(1.7,1.5,1.8,1.7)	(1.7,1.9,1.7,1)
0.1	1.317960	1.194348	1.332529	1.386355	1.970648
0.2	1.491886	1.407464	1.545657	1.592880	2.439520
0.3	1.631171	1.590061	1.723840	1.764469	2.845417
0.4	1.759989	1.769166	1.894843	1.928387	3.244388
0.5	1.889081	1.959118	2.072376	2.097946	3.666552
0.6	2.027135	2.174631	2.269273	2.285399	4.142399
0.7	2.185507	2.43857	2.504351	2.508552	4.718758
0.8	2.386139	2.800418	2.816750	2.804250	5.494824
0.9	2.694769	3.423055	3.330425	3.289162	6.787325

From above table, it is noted that the large u, the large value of quantile, as expected (the quantile is an increasing function), it demonstrates the flexibility of EF distribution in representing different ranges of data and the ability to model extreme value (long tails), especially at small values of θ.

The Median of the EF distribution can be found by substituting $u = 0.5$ into Equation 27, to get:

$$Q(x) = \left[\frac{1}{\alpha}\log\left\{\frac{\pi - W_{-1}(-\beta e^{\pi})}{\pi}\right\}\right]^{-\frac{1}{\theta}}, \pi = \left(\frac{0.6931}{\omega}\right)^{\frac{1}{\delta}} \tag{30}$$

4. Estimation parameters

4.1 Maximum Likelihood Estimation - MLE

The technique known as MLE involves maximizing the likelihood function, which expresses the likelihood of witnessing the provided data under the suggested model, in order to find the values of unknown parameters of statistical model. in mathematically, the MLE is expressed as [28], [29] if we have probability distribution $f(x/\vartheta)$ with ϑ being parameters and a sample of data $X = (x_1, x_2, \ldots, x_n)$:

$$L(\vartheta, x) = \prod_{i=1}^{n} f_{EF}(x)$$

$$L(\vartheta, x) = \prod_{i=1}^{n} \frac{\omega\delta\,\alpha\theta e^{-\alpha x_i^{-\theta}}\left[\frac{e^{-\alpha x_i^{-\theta}}}{1-e^{-\alpha x_i^{-\theta}}} - \log\left(1-e^{-\alpha x_i^{-\theta}}\right)\right]}{x^{\alpha+1}\left[-e^{-\alpha x_i^{-\theta}}.\log\left(1-e^{-\alpha x_i^{-\theta}}\right)\right]^{1-\delta}} e^{-\omega\left[-e^{-\alpha x_i^{-\theta}}.\log\left(1-e^{-\alpha x_i^{-\theta}}\right)\right]^{\delta}}$$

Compute the log- likelihood in form:

$$\begin{aligned} L = {} & n\log\omega + n\log\delta + n\log\theta + n\log\alpha - \alpha\sum_{i=1}^{n} x_i^{-\theta} - (\alpha+1)\sum_{i=1}^{n} x_i \\ & + \sum_{i=1}^{n} log\left\{\frac{e^{-\alpha x_i^{-\theta}}}{1-e^{-\alpha x_i^{-\theta}}} - log\left(1-e^{-\alpha x_i^{-\theta}}\right)\right\} \\ & - \omega\sum_{i=1}^{n}\left[-e^{-\alpha x_i^{-\theta}}.\log\left(1-e^{-\alpha x_i^{-\theta}}\right)\right]^{\delta} \\ & + \sum_{i=1}^{n} log\left\{\left[-e^{-\alpha x_i^{-\theta}}.\log\left(1-e^{-\alpha x_i^{-\theta}}\right)\right]^{1-\delta}\right\} \end{aligned} \quad (31)$$

4.2 Ordinary Least Squares - LSE

The equation for this method is given by the form [30], [12]:

$$\varphi(\vartheta) = \sum_{i=1}^{m}\left[F(x_i) - \frac{i}{n+1}\right]^2$$

$$\varphi(\vartheta) = \sum_{i=1}^{n}\left[1 - e^{-\omega\left[-e^{-\alpha x^{-\theta}}.\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]^{\delta}} - \frac{i}{n+1}\right]^2 \quad (32)$$

4.3 *Weighted Least Squares - WLS*

The equation for this method is given by the form [30], [12]:

$$W(\vartheta) = \sum_{i=1}^{n} W_i\left[F(x_i) - \frac{i}{n+1}\right]^2$$

$$W(\vartheta) = \sum_{i=1}^{n} W_i\left[1 - e^{-\omega\left[-e^{-\alpha x^{-\theta}}.\log\left(1-e^{-\alpha x^{-\theta}}\right)\right]^{\delta}} - \frac{i}{n+1}\right]^2 \quad (33)$$

Equations 31, 32, and 33 are derived for the four parameters and then set equal to zero to obtain an estimate of parameters for EF model.

5. Simulation

Monte Carlo techniques are used for estimating complicated distributions using random sampling approaches. The accuracy with which three estimators (MLE, LSE, and WLSE) product parameters EF distribution over sample size $N = 70, 140, 210, 280, \ldots$ up to 100 was evaluated for this work using these techniques. In addition to bias calculations [33], the results were assessed using mean square error (MSE) [33], and root it (RMSE) [25]. According to the simulation findings, samples $N = 70, 140, 210$, and 280 yielded the best accurate estimates of parameters ω, δ, α, and θ for compared parameters. The sample sizes are shown in table 4.

Table 4 : Monte Carlo simulations conducted for EF distribution

ω =1.6, δ =1.3, α =1.9, θ =1.1					
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**
70	**Mean**	$\hat{\omega}$	1.4752145	1.8614104	1.65949808
		$\hat{\delta}$	1.22835499	1.1135785	1.0718747
		$\hat{\alpha}$	2.3646745	2.0241325	2.1300302
		$\hat{\theta}$	1.11962212	1.08169871	1.07847996
	MSE	$\hat{\omega}$	0.4345895	0.6392675	0.51416972
		$\hat{\delta}$	2.04734115	0.6896355	0.4155742
		$\hat{\alpha}$	0.8640154	0.3044784	0.3516827
		$\hat{\theta}$	0.12778147	0.04918305	0.04569176
	RMSE	$\hat{\omega}$	0.6592340	0.7995421	0.71705629
		$\hat{\delta}$	1.43085330	0.8304430	0.6446505
		$\hat{\alpha}$	0.9295243	0.5517956	0.5930284
		$\hat{\theta}$	0.35746534	0.22177252	0.21375631
	Bias	$\hat{\omega}$	0.1247855	0.2614104	0.05949808
		$\hat{\delta}$	0.07164501	0.1864215	0.2281253
		$\hat{\alpha}$	0.4646745	0.1241325	0.2300302
		$\hat{\theta}$	0.01962212	0.01830129	0.02152004

140	**Mean**	$\hat{\omega}$	1.53952724	1.7250935	1.61235302
		$\hat{\delta}$	1.4086616	1.1287	1.1367819
		$\hat{\alpha}$	2.2042555	2.0975041	2.2004221
		$\hat{\theta}$	1.09014082	1.06284354	1.02318118
	MSE	$\hat{\omega}$	0.25843474	0.3274072	0.18176731
		$\hat{\delta}$	1.6269457	0.4979029	0.4210810
		$\hat{\alpha}$	0.6301536	0.2780136	0.3535798
		$\hat{\theta}$	0.06320890	0.06203874	0.03729340
	RMSE	$\hat{\omega}$	0.50836477	0.5721951	0.42634178
		$\hat{\delta}$	1.2755178	0.7056224	0.6489075
		$\hat{\alpha}$	0.7938221	0.5272699	0.5946258
		$\hat{\theta}$	0.25141380	0.24907578	0.19311498
	Bias	$\hat{\omega}$	0.06047276	0.1250935	0.01235302
		$\hat{\delta}$	0.1086616	0.1713000	0.1632181
		$\hat{\alpha}$	0.3042555	0.1975041	0.3004221
		$\hat{\theta}$	0.00985918	0.03715646	0.07681882
210	**Mean**	$\hat{\omega}$	1.603106833	1.56756871	1.61401493
		$\hat{\delta}$	1.5193209	1.0933770	1.1856953
		$\hat{\alpha}$	2.1132552	2.0918893	2.1095845
		$\hat{\theta}$	1.090619255	1.08959176	1.06341091
	MSE	$\hat{\omega}$	0.233121709	0.12152551	0.12865889
		$\hat{\delta}$	1.6409932	0.4489576	0.4801535
		$\hat{\alpha}$	0.5356266	0.1887299	0.2365791
		$\hat{\theta}$	0.072765749	0.04564394	0.04383683
	RMSE	$\hat{\omega}$	0.482826789	0.34860509	0.35869053
		$\hat{\delta}$	1.2810126	0.6700430	0.6929311
		$\hat{\alpha}$	0.7318651	0.4344305	0.4863940
		$\hat{\theta}$	0.269751273	0.21364444	0.20937246
	Bias	$\hat{\omega}$	0.003106833	0.03243129	0.01401493
		$\hat{\delta}$	0.2193209	0.2066230	0.1143047

		$\hat{\alpha}$	0.2132552	0.1918893	0.2095845
		$\hat{\theta}$	0.009380745	0.01040824	0.03658909
280	**Mean**	$\hat{\omega}$	1.61721954	1.64095786	1.57583611
		$\hat{\delta}$	1.4972093	1.20968389	1.21383508
		$\hat{\alpha}$	2.0856541	2.1042104	2.0366132
		$\hat{\theta}$	1.106398438	1.04535746	1.08562392
	MSE	$\hat{\omega}$	0.31906997	0.18466328	0.08248827
		$\hat{\delta}$	2.0465212	0.46596600	0.32846800
		$\hat{\alpha}$	0.4463212	0.2124728	0.1252380
		$\hat{\theta}$	0.056805371	0.03206397	0.02236085
	RMSE	$\hat{\omega}$	0.56486279	0.42972466	0.28720771
		$\hat{\delta}$	1.4305667	0.68261702	0.57312128
		$\hat{\alpha}$	0.6680727	0.4609478	0.3538898
		$\hat{\theta}$	0.238338775	0.17906414	0.14953546
	Bias	$\hat{\omega}$	0.01721954	0.04095786	0.02416389
		$\hat{\delta}$	0.1972093	0.09031611	0.08616492
		$\hat{\alpha}$	0.1856541	0.2042104	0.1366132
		$\hat{\theta}$	0.006398438	0.05464254	0.01437608

Monte Carlo simulation results for estimating EF distribution parameters show that the estimates converge toward the true values of parameters as the sample size increases, demonstrating the consistency of the estimators. Bias decreases with increasing sample size, especially for MLE, confirming the unbiasedness of this method compared to LSE and WLSE. MLE exhibits superior performance in most case, especially for parameters δ and α, where MSE and RMSE values are smaller compared to LSE and WLSE. Thus, MLE has best accuracy and consistency, followed by WLSE and LSE.

6. Application

The demonstrate the effectiveness of EF distribution in practical applications, it was applied to two data sets. The first set represented medical data, representing the remission times (in months) of a random sample of 128 bladder cancer patients. The second set represented engineering data, including the failure times of compressive carbon fibers for 69 components with a length of 50 mm. The descriptive statistics for the two sets of data are shown as follows:

Data	Var	N	Mean	SD	Median	Trimmed	Mad	Min	Max	Range	SK	KU	Se
Data set	1	127	9.45	10.53	6.54	7.52	5.62	0.05	79.05	79	3.23	15.06	0.93
Data set	1	65	2.24	0.42	2.27	2.24	0.47	1.34	3.17	1.83	0.02	-0.53	0.05

The results of the proposed distribution were compared with six other distributions. table 4 presents the CDF functions of these comparative distributions, as follows:

Table 5 : CDF function for comparative distributions

Distribution	CDF
Beta Fréchet distribution (BeF)	$\beta\left(e^{-\alpha x^{-\theta}}, \omega, \delta\right)$
Kumaraswamy Fréchet distribution (New) (KuF)	$1-\left(1-\left(e^{-\alpha x^{-\theta}}\right)^{\omega}\right)^{\delta}$
Exponential Generalized Exponential Fréchet distribution (New) (EGF)	$\left(1-\left(e^{-\alpha x^{-\theta}}\right)^{k}\right)^{c}$
Log Gamma Fréchet	$\frac{1-e^{1-\left(1-\omega e^{-\alpha x^{-\theta}}\right)^{\delta}}}{1-e^{1-(1-\omega)^{\delta}}}$
[0,1]Truncated Exponentiated Exponential Fréchet (New) ([0,1]TEEF)	$\frac{\left(1-e^{-\alpha\omega x^{-\theta}}\right)^{\delta}}{(1-e^{-k})^{c}}$
Fréchet distribution (F)	$e^{-\alpha x^{-\theta}}$

This comparison uses eight measures. The two statistical measures used are the Kolmogorov-Smirnov (KS) statistic and the Anderson-Darling (A) statistic. The Cramer-von-Mises (W) statistic and the HQIC, BIC, AIC, and CAIC information criteria were also considered [35]. Additionally, the application uses the probability value (p-value) derived from the Kolomogrov-Smirnov test [36]. These measures are often used to assess the goodness of fit and are shown in table 5 and 6, respectively

Table 6. results of the criteria for the distributions

Data	Dist.	-L	AIC	CAIC	BIC	HQIC
Data set 1	**EF**	411.7591	831.5182	831.8461	842.895	836.1405
	BeF	415.4692	838.9383	839.2662	850.3151	843.5605
	KuF	414.0446	836.0892	836.4171	847.466	840.7114
	EGF	435.8316	879.7564	880.0842	891.1331	884.3786
	N[0,1] NHF	413.961	835.9219	836.2498	847.2987	840.5442
	N[0,1] TEEF	415.7867	839.5735	839.9014	850.9502	844.1957
	F	460.964	925.928	926.0248	931.6164	928.2391
Data set 2	**EF**	34.51252	77.02519	77.69186	85.72274	80.45693
	BeF	38.0213	84.0682	84.73487	92.76575	87.49994
	KuF	36.99326	81.99025	82.65692	90.6878	85.42199
	EGF	37.90646	83.81551	84.48218	92.51306	87.24725
	N[0,1] NHF	40.77596	89.55381	90.22048	98.25136	92.98555
	N[0,1] TEEF	35.51778	79.03631	79.70298	87.73386	82.46806
	F	43.86001	91.72003	91.91358	96.0688	93.4359

Table 6 show EF distribution achieves the lowest values for all metrics in both datasets, indicating a clear advantage in data fitting. EF outperforms all comparable distributions in modeling medical and engineering data, making it the optimal model according to information metrics.

Table 7. results of the criteria for the distributions

Data	Dist.	W	A	K-S	p-value
Data set 1	EF	0.1287589	0.9580693	0.0769401	0.4397636
	BeF	0.1902141	1.505737	0.08963973	0.259238
	KuF	0.1571864	1.27655	0.07946541	0.3989351
	EGF	0.7006894	4.782829	0.1407645	0.01303947
	N[0,1] NHF	0.1736346	1.304028	0.09233958	0.2289807
	N[0,1] TEEF	0.194154	1.545229	0.08825249	0.2759158
	F	1.404492	8.756085	0.1898896	0.0002105896
Data set 2	EF	0.03360679	0.2259957	0.06545795	0.9433906
	BeF	0.1380418	0.8230522	0.09955464	0.5398611
	KuF	0.1104121	0.6573354	0.09323381	0.6243478
	EGF	0.1306284	0.7766481	0.09202611	0.640751
	N[0,1] NHF	0.173459	1.071343	0.1308811	0.2154578
	N[0,1] TEEF	0.02880857	0.2971873	0.04868391	0.9978748
	F	0.275234	1.651979	0.124985	0.2618771

1

Table 7 presents the results of statistical goodness-of-fit tests. EF achieves the lowest K-S value in both datasets, 0.0769 (dataset 1) and 0.0654 (dataset 2), while p-value for EF is high (>0.43 and >0.94), failing to reject the distribution hypothesis (i.e. good fit). EF passes all statistical goodness-of-fit tests with the best performance, supporting its use in modeling survival and material durability data.

As a visual evaluation to illustrate the results of tables 6 and 7, the fitted densities for both types of data are plotted, as well as empirical CDF for the data, which is shown in the figures below:

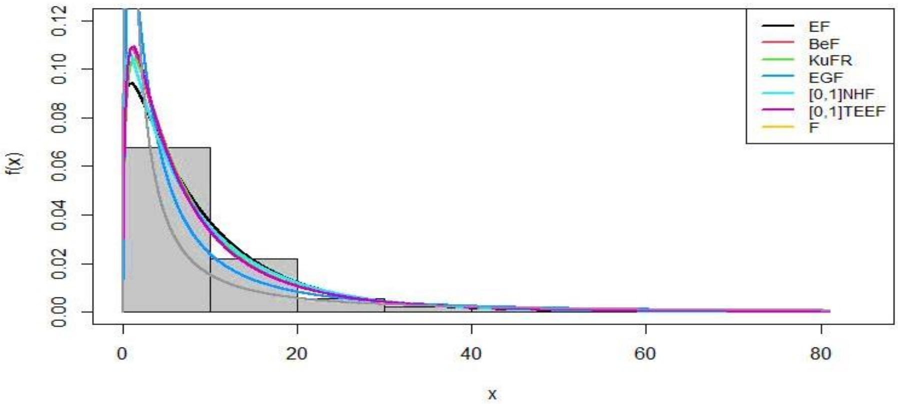

Figure.5 Fitted densities for Data 1

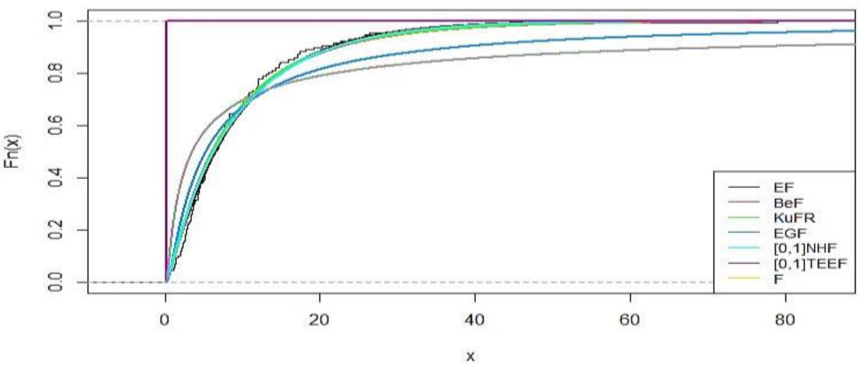

Figure.6 Empirical CDF for Data 1

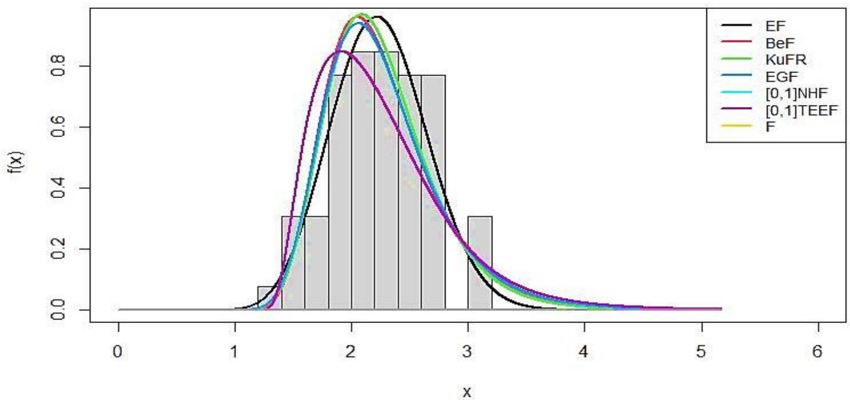

Figure.7 Fitted densities for Data 2

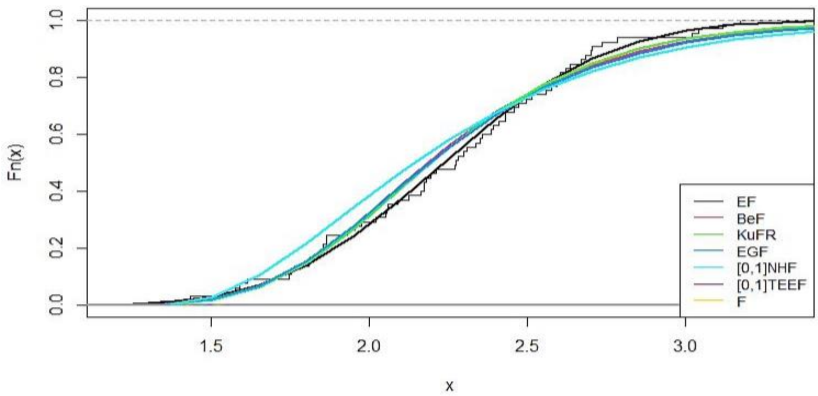

Figure.8 Empirical CDF for Data 2

Figures 5 shows that EF curve accurately captures the long tail of the data, reflecting its ability to model extreme value. This is consistent with the heavy-tailed distribution characteristic of EF, as show in the previous tables. The superiority of EF in this context supports its use in analyzing survival data, where data are often asymmetric and contain extreme values.

Figure 6 shows a high convergence between the two curves, especially in the low and median regions, confirming the goodness of EF's fit to data. This is consistent with the results of K-S test, which showed a low value 0.0769 and a high p-value 0.439, supporting hypothesis that the data follow an EF distribution.

Figure 7 shows EF's flexibility in fitting low-variance data concentrated around the median values, as demonstrated by the convergence between the experimental and theoretical curves. This is consistent with the low standard deviation 0.42 in the engineering data and confirms EF's ability to model engineering data reliability. Figure 8 highlights the accuracy of EF in fitting empirical CDF, especially in the central range. EF's superior performance here mirrors the results from the previous tables. These results confirm that EF is capable of modeling not only extreme data but also more regular data.

Conclusions

This research makes a qualitative contribution to the field of statistical modeling by developing the EF distribution, which combines the flexibility of HWG with properties of classical Fréchet distribution. this was achieved by deriving advanced mathematical properties such as CDF and pdf, survival and hazard functions, as well as moments. The results demonstrate that EF has high flexibility in representing data with heavy tails and large positive skewness, making it suitable for diverse application. Through Monte Carlo simulations, MLE method was found to be most accurate in estimating EF parameters compared to LSE and WLSE, exhibiting lowest bias and lowest MSE, especially with increasing sample size. In practical applications, EF outperformed six competing distributions according to AIC,CAIC, HQIC and BIC and goodness-of-fit tests, confirming its ability to provide accurate modeling of real-data. Figures 5-8 reinforce these results, demonstrating a close match between the theoretical distribution and experimental data in both medical and engineering settings. For example, in the bladder cancer data, EF was able to capture asymmetric behavior and outliers, while in the carbon fiber data, it provided a perfect fit to the data centered around the mean.

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"Generation of Orthonormal Matrices and Pythagorean Tuples Using the Cayley Transform"

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Abstract

This paper explores the relationship between Pythagorean tuples and orthonormal matrices with rational entries. We present a method for generating Pythagorean vectors by constructing orthonormal matrices using the Cayley transform. The necessary conditions for generating such matrices are examined, particularly those derived from skew-symmetric matrices of orders two and three. We demonstrate the possibility of systematically generating Pythagorean triples and quadruples from any rational matrix of order two or three

Key Words:

Orthonormal matrices – Pythagorean vectors – Cayley transform – Pythagorean tuples – Skew-symmetric matrice

1. Introduction

Orthogonal unimodular matrices constitute a fundamental component of the mathematical structure in linear algebra, due to their significant geometric and computational properties. Similarly, Pythagorean triples, rooted in the Pythagorean Theorem, represent a cornerstone in number theory because of their wide-ranging applications. This highlights the importance of linking these two concepts—especially when exploring methods for constructing Pythagorean vectors through matrices**.**

Pythagorean triples have been known for over 4000 years. In 1922, George Arthur Plimpton purchased a Babylonian clay tablet dating back to approximately 1800 BCE from Edgar James Banks [1]. This tablet, now known as Plimpton 322, was written in the sexagesimal numeral system and contains two of the three numbers that are now recognized as a Pythagorean triple**.**

Numerous articles and studies have addressed Pythagorean triples and methods for generating them. These range from Euclid's formula for producing such triples to more advanced methods involving ternary trees constructed using Price matrices [2] and Berggren matrices [3**.[**

Frisch and Vasershtein demonstrated that Pythagorean triples can be represented using a single triple of integer-valued polynomials [4]. They further expanded on this by proving that when or , higher-order Pythagorean n-tuples can be represented using a single polynomial of degree with integer coefficients [5**.[**

At the University of Homs, there has been considerable interest in Pythagorean triples, with numerous studies published on various methods of generating them [6,7,8,9**.[**

In this study, we address the application of the Cayley transform in constructing orthogonal unimodular matrices, and consequently in generating Pythagorean triples. We place special emphasis on the rules and conditions necessary to ensure the validity of this approach**.**

2. Research Objective

This study aims to:

- Present a systematic method for generating Pythagorean vectors based on the Cayley transform.

- Explore the properties of orthogonal unimodular matrices with rational entries.
- Prove that every rational matrix of order two or three can generate Pythagorean triples or quadruples.

3. Discussion and Results

Definition 1: Pythagorean n-tuples are sets of n positive integers x_i that satisfy:

$$\sum_{i=1}^{n-1}(x_i)^2 = (x_n)^2$$

The tuple is primitive if the integers are coprime and arranged in ascending order.

Definition 2: A vector $x = [x_1, \dots, x_n]$ is said to be Pythagorean if, for some $k \in \{1, \dots, n\}$, the following two conditions are satisfied: $x_k \in \square$ and $\|x\|_2 \in \square$.

Definition 3: [10] Two vectors are said to be orthogonal if their inner product is equal to zero.

Definition 4: [10] A numerical matrix Q is said to be orthogonal if its rows and columns are orthogonal vectors, and if:

$$Q^T Q = Q Q^T = I$$

Where Q^T is the transpose of matrix Q, and I is the identity matrix. In this case, Q is called an orthogonal matrix.

One of the most important properties of orthogonal matrices is that their transpose is equal to their inverse. That is, if Q is an orthogonal matrix, then:

$$Q^T = Q^{-1}$$

Moreover, the determinant of an orthogonal matrix is equal to ± 1. However, the converse is not necessarily true; that is, if $\det(A) = \pm 1$, this does not necessarily mean that A is an orthogonal matrix.

Note 1: Let $Q = [q_{ij}]$ be an orthogonal and nonsingular matrix of order n, whose entries are rational numbers.

Assume that d is a common denominator of all entries of Q, Then, the following vectors:

$$q_1 = \begin{pmatrix} q_{11}d \\ q_{21}d \\ \cdots \\ q_{n1}d \end{pmatrix}, \; q_2 = \begin{pmatrix} q_{12}d \\ q_{22}d \\ \cdots \\ q_{n2}d \end{pmatrix}, \; \ldots, \; q_n = \begin{pmatrix} q_{1n}d \\ q_{2n}d \\ \cdots \\ q_{nn}d \end{pmatrix}$$

are Pythagorean vectors.

Example 1: Let the following matrix be given:

$$Q = \begin{pmatrix} \frac{10}{15} & -\frac{10}{15} & \frac{5}{15} \\ -\frac{2}{15} & \frac{5}{15} & \frac{14}{15} \\ \frac{11}{15} & \frac{10}{15} & -\frac{2}{15} \end{pmatrix}$$

This matrix is orthogonal and nonsingular, its entries are rational, and the common denominator is 15. Then, the following vectors:

$$q_1 = \begin{pmatrix} 10 \\ -2 \\ 11 \end{pmatrix}, \; q_2 = \begin{pmatrix} -10 \\ 5 \\ 10 \end{pmatrix}, \; q_3 = \begin{pmatrix} 5 \\ 14 \\ -2 \end{pmatrix}$$

are Pythagorean vectors■.

The importance of Pythagorean vectors appears in the fact that, for any Pythagorean triple (a,b,c), the corresponding matrix:

$$Q = \begin{pmatrix} \frac{a}{c} & -\frac{b}{c} \\ \frac{b}{c} & \frac{a}{c} \end{pmatrix}$$

is orthonormal.

Generating Rational Orthogonal Matrices

The process of constructing orthogonal matrices Q with rational entries is not straightforward, as it requires the existence of orthogonal and nonsingular vectors with rational components.

Cayley's Formula

Cayley [10] introduced a formula for generating an orthogonal matrix. Lebesgue and Osborne [11] showed that any orthogonal matrix—except those involving reflections in the basis—can be obtained using Cayley's formula, provided that a suitable input matrix S is chosen.

Cayley's Theorem 1: Let $S \in \mathbb{R}^{n\times n}$ be a skew-symmetric matrix that does not have 1 as an eigenvalue. Then, the following matrix:

$$Q = (S - I)^{-1}(S + I) \qquad (1)$$

is orthonormal.

Theorem 2: Let $Q \in \mathbb{R}^{n\times n}$be an orthogonal and nonsingular matrix. Then the matrix S , defined by:

$$S = (Q - I)^{-1}.(Q + I) \qquad (2)$$

is a skew-symmetric matrix.

Theorem 3: For the following pair of equations:Then

$$Q = (S - I)^{-1}.(S + I)$$
$$S_1 = (Q - I)^{-1}(Q + I)$$

Then $S_1 = S$.

The question now is:

What are the conditions on S that allow us to construct an orthogonal and nonsingular matrix?

First: The case of a skew-symmetric matrix of order two.

A skew-symmetric matrix of order 2 has the following form:

$$S=\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$$

The eigenvalues of this matrix are the roots of the equation:

$$|S-\lambda I|=0 \quad \Leftrightarrow \quad \begin{vmatrix} -\lambda & a \\ -a & -\lambda \end{vmatrix}=0 \quad \Leftrightarrow \quad \lambda^2+a^2=0$$

Clearly, $\lambda=1$ is **not** an eigenvalue of the matrix S. Therefore, according to **Cayley's theorem**, an orthonormal matrix Q can be constructed using any **skew-symmetric matrix of order 2**, derived from the formula:

$$Q=(S-I)^{-1}(S+I)$$

By substitution, we find:

$$Q=\begin{pmatrix} -1 & a \\ -a & -1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix}=\frac{1}{1+a^2}\begin{pmatrix} -1 & -a \\ a & -1 \end{pmatrix}\begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix}$$
$$=\frac{1}{1+a^2}\begin{pmatrix} a^2-1 & -2a \\ 2a & a^2-1 \end{pmatrix}$$

Result 1: Let $a\in\mathbb{R}$, then the following matrix is an orthonormal matrix:

$$Q=\begin{pmatrix} \frac{a^2-1}{1+a^2} & \frac{-2a}{1+a^2} \\ \frac{2a}{1+a^2} & \frac{a^2-1}{1+a^2} \end{pmatrix}$$

In the special case where a is an integer, the following triples are Pythagorean

$$\left(a^2-1,\ 2a,\ 1+a^2\right),\left(-2a,\ a^2-1,1+a^2\right)$$

Second: The Case of a Third-Order Skew-Symmetric Matrix

A third-order skew-symmetric matrix has the following form:

$$S=\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

The eigenvalues of this matrix are the roots of the equation:

$$|S-\lambda I|=0 \quad \Leftrightarrow \quad \begin{vmatrix} -\lambda & a & b \\ -a & -\lambda & c \\ -b & -c & -\lambda \end{vmatrix}=0$$

$$\Leftrightarrow \quad \lambda^3+\left(a^2+b^2+c^2\right)\lambda=0$$

which can be written in the form:

$$\lambda\left[\lambda^2+\left(a^2+b^2+c^2\right)\right]=0$$

Clearly, for any real values of a,b,c from the set of real numbers, the matrix S cannot have non-zero real eigenvalues. Consequently, no skew-symmetric matrix can have an eigenvalue equal to positive one. This implies that an orthonormal matrix can indeed be constructed according to Cayley's formula:

$$Q=(S-I)^{-1}(S+I)$$

By substitution, we obtain:

$$Q=\begin{pmatrix} -1 & a & b \\ -a & -1 & c \\ -b & -c & -1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{pmatrix}$$

$$=\frac{-1}{1+a^2+b^2+c^2}\begin{pmatrix} c^2+1 & a-bc & b+ac \\ -a-bc & b^2+1 & c-ab \\ b-ac & -c-ab & a^2+1 \end{pmatrix}\begin{pmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{pmatrix}$$

$$=\frac{-1}{1+a^2+b^2+c^2}\begin{pmatrix} 1-a^2-b^2+c^2 & -2(-a+bc) & 2(b+ac) \\ -2(a+bc) & 1-a^2+b^2-c^2 & 2(c-ab) \\ -2(b-ac) & -2(c+ab) & 1+a^2-b^2-c^2 \end{pmatrix}$$

Result 2: Let a,b,c be real numbers, then the following matrix is an orthonormal matrix:

$$Q=\begin{pmatrix} \frac{-1+a^2+b^2-c^2}{1+a^2+b^2+c^2} & \frac{2(-a+bc)}{1+a^2+b^2+c^2} & \frac{-2(b+ac)}{1+a^2+b^2+c^2} \\ \frac{2(a+bc)}{1+a^2+b^2+c^2} & \frac{-1+a^2-b^2+c^2}{1+a^2+b^2+c^2} & \frac{-2(c-ab)}{1+a^2+b^2+c^2} \\ \frac{2(b-ac)}{1+a^2+b^2+c^2} & \frac{2(c+ab)}{1+a^2+b^2+c^2} & \frac{-1-a^2+b^2+c^2}{1+a^2+b^2+c^2} \end{pmatrix}$$

In the special case where a,b,c are integers, the following quadruples are Pythagorean:

$$\left(-1+a^2+b^2-c^2,\ 2(a+bc),\ 2(b-ac),\ 1+a^2+b^2+c^2\right)$$
$$\left(2(-a+bc),\ -1+a^2-b^2+c^2,\ 2(c+ab),\ 1+a^2+b^2+c^2\right)$$
$$\left(-2(b+ac),\ -2(c-ab),\ -1-a^2+b^2+c^2,\ 1+a^2+b^2+c^2\right)$$

Clearly, the Pythagorean fourth term of these quadruples equals:

$$1+a^2+b^2+c^2$$

Example 2: For the numbers $a=1, b=2, c=3$, we obtain the following Pythagorean quadruples:

$$(-5,14,-2,15),\ (10,5,10,15),\ (-10,-2,11,15)$$

Result 3: For any skew-symmetric matrix S of order 2 or 3, Cayley's formula can be applied to construct an orthonormal matrix.

Theorem 4: For any 2nd order matrix A with rational elements, at least two Pythagorean triples can be generated.

Proof: Let A be a 2nd order matrix with rational elements. Then the matrix:

$$A-A^T$$

is skew-symmetric. Based on this, an orthonormal matrix of order 2 can be constructed:

$$Q=\left(A-A^T-I\right)^{-1}\left(A-A^T+I\right)$$

A Pythagorean triple can be derived from each of its columns.

Example 3: For the matrix: $A=\begin{pmatrix}1 & -2\\ 3 & 5\end{pmatrix}$ we can construct an orthonormal matrix Q via:

$$Q=\left(A-A^T-I\right)^{-1}\left(A-A^T+I\right)$$

By substitution, we find:

$$Q=\left(\begin{pmatrix}1 & -2\\ 3 & 5\end{pmatrix}-\begin{pmatrix}1 & 3\\ -2 & 5\end{pmatrix}-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)^{-1}\left(\begin{pmatrix}1 & -2\\ 3 & 5\end{pmatrix}-\begin{pmatrix}1 & 3\\ -2 & 5\end{pmatrix}+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$

$$=\begin{pmatrix}-1 & -5\\ 5 & -1\end{pmatrix}^{-1}\begin{pmatrix}1 & -5\\ 5 & 1\end{pmatrix}=\frac{1}{26}\begin{pmatrix}-1 & 5\\ -5 & -1\end{pmatrix}\begin{pmatrix}1 & -5\\ 5 & 1\end{pmatrix}=\begin{pmatrix}\frac{24}{26} & \frac{10}{26}\\ \frac{-10}{26} & \frac{24}{26}\end{pmatrix}$$

Theorem 5: For any 3rd-order matrix A with rational elements, at least three Pythagorean quadruples can be generated.

Proof: Let A be a 3rd-order matrix with rational elements. Then, the matrix:

$$A-A^T$$

is skew-symmetric. Consequently, based on this, an orthonormal matrix of order 3 can be constructed:

$$Q=\left(A-A^T-I\right)^{-1}\left(A-A^T+I\right)$$

A Pythagorean quadruple can be derived from each of its columns.

Example 4: For matrix $A=\begin{pmatrix}5 & 1 & 3\\ -2 & 1 & 4\\ 3 & -5 & 6\end{pmatrix}$, an orthonormal matrix Q can be constructed, where:

$$S=A-A^T=\begin{pmatrix}5 & 1 & 3\\ -2 & 1 & 4\\ 3 & -5 & 6\end{pmatrix}-\begin{pmatrix}5 & -2 & 3\\ 1 & 1 & -5\\ 3 & 4 & 6\end{pmatrix}=\begin{pmatrix}0 & 3 & 0\\ -3 & 0 & 9\\ 0 & -9 & 0\end{pmatrix}$$

And:

$$S-I=\begin{pmatrix}-1 & 3 & 0\\ -3 & -1 & 9\\ 0 & -9 & -1\end{pmatrix},\quad S+I=\begin{pmatrix}1 & 3 & 0\\ -3 & 1 & 9\\ 0 & -9 & 1\end{pmatrix}$$

$$Q=\begin{pmatrix}-1 & 3 & 0\\ -3 & -1 & 9\\ 0 & -9 & -1\end{pmatrix}^{-1}.\begin{pmatrix}1 & 3 & 0\\ -3 & 1 & 9\\ 0 & -9 & 1\end{pmatrix}$$
$$=\frac{1}{91}\begin{pmatrix}-82 & -3 & -27\\ 3 & -1 & -9\\ -27 & 9 & -10\end{pmatrix}.\begin{pmatrix}1 & 3 & 0\\ -3 & 1 & 9\\ 0 & -9 & 1\end{pmatrix}$$

Therefore:

$$Q=\begin{pmatrix}-\frac{73}{91} & -\frac{6}{91} & -\frac{54}{91}\\ \frac{6}{91} & \frac{89}{91} & -\frac{18}{91}\\ -\frac{54}{91} & \frac{18}{91} & \frac{71}{91}\end{pmatrix}$$

Clearly, the following quadruples are Pythagorean:

$$(-73,6,-54,91),(-6,89,18,91),(-54,-18,71,91)$$

Note 2:"The construction of the orthonormal matrix Q from matrix A is independent of the main diagonal elements of A, as these elements play no role in determining the elements of Q.

Example 5: The two matrices $A_1=\begin{pmatrix}-1 & 2 & 5\\ 3 & 4 & 6\\ 8 & 9 & 5\end{pmatrix}, A_2=\begin{pmatrix}9 & 2 & 5\\ 3 & 1 & 6\\ 8 & 9 & 3\end{pmatrix}$ will generate the same orthonormal matrix Q.

Note 3: Let $A=\left[a_{ij}\right], B=\left[b_{ij}\right]$ be two matrices of the same order (either both 2nd-order or both 3rd-order), such that:

$$a_{ij}-a_{ji}=b_{ij}-b_{ji}\ ;\ i\neq j$$

Then, A, B will generate the same orthonormal matrix Q.

Example 6: For the two matrices:

$$A=\begin{pmatrix}-1 & 2 & 5\\ 3 & 4 & 1\\ 8 & 4 & 5\end{pmatrix},\ B=\begin{pmatrix}9 & 3 & 6\\ 4 & 1 & 3\\ 9 & 6 & 3\end{pmatrix}$$

we observe that:

$$A-A^{T}=\begin{pmatrix}0 & -1 & -3\\ 1 & 0 & -3\\ 3 & 3 & 0\end{pmatrix}=B-B^{T}$$

Therefore, matrices A, B *will generate the same orthonormal matrix* Q.

$$Q=\begin{pmatrix}0 & 1 & 0\\ \frac{4}{5} & 0 & \frac{3}{5}\\ -\frac{3}{5} & 0 & \frac{4}{5}\end{pmatrix}$$

Theorem 6: Every matrix A of order 2 or 3 can be decomposed into the product of two matrices: $Q.M$, where:

- Q is an orthonormal matrix,
- M is a matrix of the same order.

Proof: For any matrix A of order 2 or 3, an orthonormal matrix Q can be constructed, where:

$$Q=\left(A-A^{T}-I\right)^{-1}.\left(A-A^{T}+I\right)$$

Since $\det(Q)=\pm 1$, the inverse Q^{-1} exists. Therefore, by setting:

$$M=Q^{-1}.A$$

This completes the proof.

4. Results

The following conclusions were drawn from this study:

1. For 2×2 matrices: Every skew-symmetric matrix can be transformed via the Cayley transform into a proper orthogonal matrix. Pythagorean triples can be generated from such matrices.
2. For 3×3 matrices: All skew-symmetric matrices lack eigenvalues equal to 1, making them consistently suitable for the application of the Cayley transform. Pythagorean quadruples can be generated from the resulting matrices.
3. Pythagorean sets can be generated from any second- or third-order matrix.
4. The generation process is independent of the main diagonal elements of the skew-symmetric matrix, indicating that these elements do not affect the final proper orthogonal matrix.
5. Every 2×2 or 3×3 matrix can be decomposed into the product of two matrices, one of which is a proper orthogonal matrix, confirming the existence of an inherent Pythagorean structure within every small rational matrix.

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The role of induction in constructing the rigorous definition of the continuity of a function in Tunisian secondary school

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Abstract

This study analyses the role of induction in teaching continuity of functions in Tunisia. It reveals that, although prescribed by curricula and widely used by teachers, the inductive approach often remains incomplete: it serves to build graphical and numerical intuition, but does not enable students to move on to the formal epsilon-delta definition. Praxeological analysis of the school textbook within the framework of anthropological theory of didactics (TAD), and a survey of 67 teachers show that the generalization phase, which is crucial for formalization, is not sufficiently addressed. We conclude that there is a need to better structure the transition from intuition to rigor, in particular through the explicit institutionalization of knowledge and enhanced didactic training for teachers.

Keywords: Induction, continuity, epsilon-delta definition, Anthropological Theory of Didactics (TAD), teaching practices.

ملخص

تحلل هذه الدراسة دور الاستقراء في تدريس اتصال الدوال في تونس .وتكشف أنه على الرغم من أن المنهج الاستقرائي منصوص عليه في المناهج الدراسية ومستخدم على نطاق واسع من قبل المعلمين ، إلا أنه غالبًا ما يظل غير مكتمل :فهو يعمل على بناء الحدس البياني والعددي ، ولكنه لا يمكّن الطلاب من الانتقال إلى التعريف الدقيق إبسيلون - دلتا لاتصال دالة .يُظهر التحليل البراكسيولوجي للكتاب المدرسي في إطار النظرية الأنثروبولوجية للتعلمية (TAD)، و استبيان شمل 67 معلمًا أن مرحلة التعميم التي تعتبر حاسمة في بناء التعريف لم يتم تناولها بشكل كافٍ .نخلص إلى أن هناك حاجة إلى هيكلة أفضل للانتقال من الحدس إلى الصرامة ولا سيما من خلال إضفاء الطابع المؤسسي الصريح على المعرفة وتعزيز التدريب التربوي للمعلمين.

الكلمات المفتاحية :الاستقراء ، الاتصال ، تعريف إبسيلون – دلتا ، النظرية الأنثروبولوجية للتعليم ، الممارسات التدريسية.

1- Introduction

The introduction of the concept of continuity of a real function in Tunisian secondary education faces a fundamental tension between intuition (graphs, movement) and the rigor of the formal definition in epsilon-delta (Selmi, 2004).

The official curricula (2008) advocate an approach in which induction with other types of reasoning contributes to the development of intuition, imagination and abstraction. In this context, inductive reasoning is used to bring out the properties of a concept through conjecture based on observation and analysis of specific cases, thus paving the way for synthesis and formalization. It is therefore a process of discovery: "Students use [...] calculators or software to conduct tests or experiments on simple or specific cases." This corresponds to the observation phase of specific cases, the first step in induction (Jeannotte, 2015).

However, a key question arises: how is induction implemented by teachers in their actual teaching practice? Do they simply illustrate the concept intuitively, or do they use induction as a process of gradually and rationally constructing a rigorous definition? In other words, does induction play the role assigned to it by curricula, or does it remain a simple pedagogical exercise with no real conceptual significance, quickly abandoned in favour of an intuitive approach?

This article aims to contribute to a broader reflection on the conditions for teaching continuity in a way that is both accessible and faithful to the rigour required by the discipline, while being aware of the difficulties associated with the formalism of its definition and the levels of conceptualization of this notion, whose transition to more formal approaches can only take place at university level (Artigue, 1999).

By cross-referencing the analysis of curricula, textbooks and teachers' stated practices, we aim to identify the place and function accorded to induction in official Tunisian texts in particular, in the introductory activity of the concept of continuity in school textbooks, based on the framework of anthropological theory of didactics (TAD). We then aim to evaluate, through a questionnaire survey, how teachers implement this activity and how they perceive its usefulness. Finally, in light of the results, avenues for optimizing the use of induction as a lever for building rigorous mathematical knowledge will be proposed.

2- Theoretical framework and methodology

2-1- Theoretical Framework

To analyse in detail the place and function of induction in continuity teaching, we draw on the theoretical framework of anthropological theory of didactics (TAD), developed by Chevallard (1985). In this framework, knowledge is present in the institution in the form of mathematical organizations (MOs) called praxeologies (Chevallard, 1998; Bosch & Gascon, 2004), where a praxeology is a quadruplet $(T, \tau, \theta, \Theta)$ in which T denotes a type of task; τ denotes a technique that enables the task to be solved; θ denotes a technology that is a discourse justifying the technique or enabling its production; and Θ denotes a theory that supports the technological discourse or enables its production. Applied to our subject, this model allows us to analyse how induction, defined here as a process of generalization from specific cases (Polya, 1954), plays the role of a technology θ between graphical or numerical tasks (T,τ) and the theory Θ, which is the formal epsilon-delta definition of continuity. The central question then becomes: is induction a transitive technology, enabling the construction of theory, or does it remains an intuitive discourse disconnected from formalization?

2-2- Methodology

To answer this question, we combined a qualitative analysis of teaching materials with a quantitative survey of teachers' reported practices.

a) Documentary and praxeological analysis

We analyzed the official Tunisian curricula (2008) and the 3rd year mathematics textbook (2007) in order to identify the prescribed place of induction. The curricula present induction as a ‘preferred approach to discovery,’ recommending that concepts be developed ‘through observation and experimentation on specific cases.’ Based on this observation, we conducted a praxeological analysis of the introductory activity on continuity, identifying the tasks, techniques and technologies used, and assessing the extent to which they prepare students for the theoretical element in question (the formal definition).

b) Questionnaire survey

A questionnaire was administered to 67 secondary school mathematics teachers in Tunisia as part of doctoral research (Selmi, 2024). For this article, we have selected two key questions: Question 10: on pupils' difficulties with certain concepts (approximation, generalization, inequalities, etc.).

Question 11: on the perceived usefulness of the activities in the textbook for constructing the formal definition.

The data were processed using descriptive analysis (frequencies, percentages) and qualitative analysis (analysis of explanations). This dual approach makes it possible to compare the prescribed approach (programmes, textbooks) with the practices reported by teachers, thereby identifying potential discrepancies between the didactic intention and its implementation.

2- Praxeological analysis of the introductory activity

2-1- Presentation and objective of the Activity

The introductory activity in the textbook (p. 23) presents a function defined by pieces and officially aims to study the continuity of the function . The student is asked to complete three main tasks: graph of the function, representing the set of real numbers y such that with and then determining a condition on the real numbers x that ensures the inequality with

2-2- Praxeological analysis

The following table summarizes the praxeological organization and highlights the role of induction using the quadruplet (T, τ, θ, Θ) as a tool for analyzing introductory activity:

Praxeological component Activity content Role of induction

Tasks (T) T1: Graph the function f defined by pieces

T2: Graphically represent the set of real numbers y such that ;

T3 : Graphically determine a sufficient condition on x such that

The tasks are specific

and particular :

then

Techniques (τ) For T1: Plot each of the restrictions of the function f on the same coordinate system.

For T2: Draw a horizontal line whose ordinates of the points are in the interval on the graph.

For T3 : Graphically identify all x-coordinates such that the points lie within the plotted band.

Graphic and visual techniques, rooted in the specific case.

Technology (θ) “For f (x) to be close to f (1), x must be within a certain range”

Inductive reasoning: generalization based on a specific case Induction is the intuitive technology that emerges from observation.

Theory (Θ) :

The knowledge targeted Definition in

For all , il existe tel que :

Absent from the Activity.

The link between inductive technology (θ) and theory (Θ) is not established.

Comment: The analysis reveals that the activity successfully builds an intuitive technology (θ): the idea that proximity can be guaranteed by constraining x within an appropriate interval. This discourse clearly emerges from an inductive process rooted in the observation of specific cases .

However, there is a clear break: the activity stops at these two specific observations. The crucial phase of generalization, i.e. the transition from “for , there exists an interval...” to “for all , there exists ” and formalization in the language of quantifiers is not covered. The leap from intuitive technology (θ) to theory (Θ) is not constructed; it is only announced by the definition that immediately follows the activity, without any explicit link.

2-3- Activity on discontinuity

The following activity (Activity 2, p. 23), which applies the formal definition (which has just been imposed) to the sign function to conclude that there is a discontinuity at 0, confirms this incompleteness. Although it uses the formal definition, it does not exploit the inductive potential to explain the negation of the definition (the existence of a problem). The treatment remains syntactic, without returning to the reasoning that could lead from the graphical observation of a 'break' to its formal characterization.

2-4- Summary

The textbook proposes an approach that presupposes a clear link between graphic intuition, induction based on a few observed cases, and rigorous definition. Our praxeological analysis shows that this link is not constructed pedagogically. Induction is used to produce intuitive meaning, but it is abandoned before it can accomplish its task of generalization. The definition thus appears less as the outcome of an investigation than as a new object to be accepted, creating a cognitive break where the prescription promised continuity.

3- Results

Data collected from 67 Tunisian teachers via a questionnaire (Selmi, 2024) sheds light on their perception of students' difficulties and the perceived usefulness of the textbook activities in constructing the formal definition of continuity.

3-1- Pupils' previous difficulties (Question 10) :

Teachers identify several concepts that hinder learning the definition .

The average responses (on a recognition scale) reveal that the most frequently cited difficulties concern:

Generalization (average = 0.38) and approximation (average = 0.41): these abilities, which are central to the inductive process (moving from the specific to the general, the idea of "as close as one wants"), are perceived as poorly mastered.

The use of graphs (average = 0.28): this difficulty directly calls into question the relevance of a graphical introduction to the introductory activity.

As an example of answer, we have chosen the answer accompanied by an explanation of teacher E48:

English Translation

Formalization: the absence of mathematical logic creates difficulties for students in understanding the formal definition.

Figure 1 - Excerpt from questionnaire Q-E48

The teacher (E48) summarizes this obstacle by pointing to the lack of explicit teaching of logic and quantifiers, which is necessary to move from intuition to formalization.

These results suggest that the cognitive prerequisites for successful inductive reasoning are fragile among students, according to the practices reported by teachers.

3-2- Perceived usefulness of the activities in the manual (Question 11) :

The central question regarding how the activities in the manual can help to construct the formal definition yields the following results:

Justified answers on the contribution of activities to the construction of the formal definition

		Frequency	Percentage	Valid Percentage	Cumulative Percentage
Valid	No	13	19.4	19.7	19.7
	Yes	53	79.1	80.3	100.0
	Total	66	98.5	100.0	
Missing	System	1	1.5		
Total		67	100.0		

Interpretation: A large majority (80.3%) state that the activities in the textbook can help to construct the definition. This high percentage reflects confidence in the textbook and acceptance of the didactic approach it proposes (from graphic intuition to definition).

However, analysis of the justifications provided by the 13 sceptical teachers is revealing. Their doubts do not concern the activity itself, but rather the students' ability to complete the intellectual journey. For example, teacher E20 explains their “No” response as follows: “the pupil is incapable of either assimilating the situation or generalizing it”.

English Translation

No, the pupil is incapable of either assimilating the situation or generalizing it.

Figure 2 - Excerpt from questionnaire Q-E20

This point of view highlights a perceived gap between the design of the activity (which assumes inductive generalization) and the pupils' actual ability to carry it out.

3-3- Summary of Results

The questionnaire therefore reveals a paradoxical situation:

Majority support for the prescribed approach (80.3% consider the activities useful) and an awareness of the obstacles: some teachers perceive that students lack the skills (generalization, abstraction) necessary for induction to go beyond the intuitive stage. However, there is doubt about the outcome: the scepticism expressed by a significant minority (nearly 20%) focuses precisely on the generalization stage, which is the breaking point identified in our praxeological analysis.

Thus, while the majority of teachers say they use and value the inductive approach, their responses suggest a potentially cautious or truncated practice, where induction could remain at the illustration stage without leading to formalization, in anticipation of pupils' difficulties.

4- Discussion

The cross-referencing of our analyses reveals a coherent didactic context, but one marked by a fundamental tension surrounding the status of induction.

4-1- An unfinished teaching scenario

Our results indicate a coherent teaching scenario, but one that reveals a tension. Induction is prescribed, translated into exploratory activity, and declared useful by the majority of teachers. This scenario seems perfect: starting from the observation of specific cases, using induction to formulate a conjecture about the behavior of the function, and thus arriving at the formal definition. However, praxeological analysis shows that in the textbook, this induction constructs an intuitive technology (θ) that does not lead to the intended theory (Θ). The link between the specific case and the formal definition is not established. This disconnect is noticed by some teachers, who anticipate students' difficulties in generalizing, pointing out that induction risks, in practice, being reduced to a routine task serving intuition without rigorous conceptual follow-through.

4-2- Generalization is not supported

Our praxeological analysis identifies the precise breaking point of this organization. The manual activity does indeed construct an intuitive technology (θ) illustrated by the idea of image control by the antecedent based on an example ( ). But it stops at this stage. The phase of generalization (transition from 'for' to 'for everything') and formalization in quantified language is not scripted. The definition in as a theoretical element (Θ) is then stated without the logical-mathematical link with the prior intuition being constructed. This break in the activity is perceived and anticipated by a significant proportion of teachers (19.7%). Their responses to the questionnaire point to the difficulty students have in 'generalizing' and 'assimilating the situation', revealing that they themselves perceive this cognitive leap as problematic, even insurmountable under current conditions.

4-3- The incomplete induction

This configuration explains the paradox observed: majority support for the textbook approach, but doubts about its effectiveness in practice. The risk is that this inductive approach will be reduced to a routine practice: a motivating and intuitive step, quickly overtaken by a lecture on the definition.

This reduction is all the more likely as teachers identify prior cognitive obstacles in pupils (mastery of abstraction, generalization) which precisely hinder the crucial phase of induction.

4-4- Implication: from prescription to implementation

Our discussion leads to a key proposition: the gap between the prescription of the inductive approach and its effective implementation, in the case of continuity, is not due to a rejection of the method, but to a failure to understand its function. The problem is not that induction is absent, but that it is incomplete. The proposed organization does not take on the most important mediating role of induction: producing, from an observed regularity, a conjecture which, once validated, will be the formal definition sought.

This analysis therefore invites us to shift our perspective: it is not enough to prescribe or use induction; we must also explicitly organize and plan its outcome in formalization. It is on this necessity that it will be imperative to structure the transition from the specific to the general by adding a task of generalization. For teachers, our study highlights the crucial importance of the institutionalization phase. This involves orchestrating the reformulation of observations and systematically linking graphic, numerical, linguistic and symbolic registers (Duval, 1993). The immediate use of simple counter-examples then becomes a powerful means of giving meaning to the definition.

These findings therefore argue in favor of strengthening teachers' didactic training by developing their skills in analyzing textbooks to identify these breaking points and designing activities to ensure the continuity of the cognitive journey from intuition to formalization.

4-5- Limitations and prospects

This study, based on an analysis of curricula, textbooks and teachers' statements, calls for further research. Detailed observations of teaching practices would help to understand how this transition is actually negotiated in the classroom. Educational engineering work could be carried out to design and test the effectiveness of resources incorporating the principles mentioned above. Further research could explore how this teaching prepares pupils for the rigorous demands of analysis at university.

Conclusion

This research has characterized the current role of induction in Tunisian continuity teaching: omnipresent in discourse and intentions, but often incomplete in its didactic implementation, leaving a gap between intuition and formalization. By identifying this specific breaking point, the absence of scripting for generalization, our work is not limited to a diagnosis, but opens up concrete avenues for rethinking the link between inductive exploration and rigorous knowledge construction. Beyond continuity, it raises the broader question of learning mathematical rigor in secondary school, inviting us to make induction not just a simple step, but the heart of a true engineering of the concept.

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Bidirectional Relevance Propagation for Passage Retrieval

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Abstract. Passage Retrieval (PR) is at the intersection of Information Retrieval (IR) and Question Answering (QA). It relies on NLP to interpret natural-language queries and IR techniques to extract relevant passages from large collections. Traditional PR systems mainly use lexical relevance, which often fails when passages share keywords without semantic relation or, conversely, when semantically relevant passages lack lexical overlap. To overcome these limitations, we introduce a novel PR approach that jointly exploits lexical, semantic, and contextual signals. Our framework integrates advanced concept detection and relevance propagation to better capture the semantic relationships between query terms and passage content, thereby improving the accuracy of relevance estimation beyond keyword-based methods.

Keywords: Passage Retrieval (PR) • Relevance Propagation • Concept-Based Similarity • MS MARCO Dataset • Ranking Efficiency

1 Introduction

Passage retrieval (PR) is a key task in information retrieval (IR), aiming to extract relevant text segments in response to natural-language queries. With the rapid growth of online information, users increasingly struggle to locate precise answers within massive data collections.

Traditional information retrieval systems (IRSs) can identify documents potentially related to a query, but often return large sets of results that still require manual inspection. Many retrieved passages remain irrelevant, as keyword overlap does not guarantee that the passage answers the user's need. This makes the task both complex and time-consuming.

PR has therefore emerged as a focused research direction that seeks to deliver short, precise answers rather than full documents. Despite significant progress (such as integrating semantic analysis, modeling term dependencies, and comparing structural similarities between queries and passages) identifying truly relevant passages within large datasets remains challenging.

In this work, we introduce a new PR approach centered on concept detection to enhance relevance estimation. Our method employs relevance propagation to effectively assess passages within the hierarchical structure of documents. By using semantic and contextual relationships, the proposed model better captures user intent and delivers more accurate results, especially for complex queries where keyword-based methods often fail.

This paper is structured as follows. Section 2 provides an overview of major related works on passage retrieval. Section 3 outlines our motivation and presents the proposed model. Section 4 describes the first phase of our approach, Data Indexing, while Section 5 details the second phase, Passage Retrieval. Section 6 reports the experimental results and compares our method with benchmark algorithms. Finally, Section 7 concludes the paper and summarizes our main contributions to passage retrieval in IRSs.

2 Related Works

Many methods aim to improve Passage Retrieval (PR) by extracting and ranking relevant passages from document collections, particularly in semi-structured documents, where statistical techniques are applied to identify candidate passages and enhance retrieval accuracy.

2.1 Passage Retrieval Approaches

Passage Retrieval (PR) methods can be categorized into contextual, lexical, and semantic dependencies.

Contextual Dependency: Retrieval is enhanced by analyzing proximity within documents. Fuzzy logic evaluates the closeness of query terms ([7],[2]), and treebased models capture passage relationships, assuming that a relevant passage influences nearby passages, with influence decreasing over distance.

Lexical Dependency: Retrieval is enhanced by analyzing proximity within documents. Fuzzy logic evaluates the closeness of query terms ([7],[2]), and treebased models capture passage relationships, assuming that a relevant passage influences nearby passages, with influence decreasing over distance.

Semantic Dependency: Methods capture deeper semantic relations. Entity recognition with UMLS ([16]), FrameNet-based similarity ([12]), and NLP/semantic techniques ([1],[4]) have been applied. Learning-to-rank frameworks for generative retrieval (LTRGR) [8] further optimize models for passage ranking, achieving state-of-the-art results.

2.2 A Graph-based IR Approaches

Graph-based methods have significantly improved information retrieval by enhancing context understanding, identifying important passages, and refining ranking. [6], [5] represented text as graphs, where nodes are words and edges capture relationships, facilitating identification of key terms. [3] integrated document structure with passage context, combining structured retrieval with passage level analysis to improve relevance estimation. Other studies ([10], [11]) modeled documents as graphs to propagate contextual information across neighboring elements, while relevance propagation methods ([13], [9]) suggested that a node's relevance can be derived from its children's scores.

Building on these ideas, our work introduces a bidirectional relevance propagation mechanism between sections and passages, enhancing query–document matching and refining relevance assessment by considering both hierarchical structures and detailed passage-level context, leading to improved retrieval performance.

3 Motivation of the Proposed Approach

To enhance passage retrieval, we focus on accurately filtering and ranking relevant passages for a query. By leveraging contextual information from document titles and key concepts in parent sections, we propose a relevance propagation model that evaluates and ranks passages based on their contextual relevance, targeting the most pertinent passages with an advanced propagation method.

3.1 Our Model

Our approach aims to extract the most relevant passages to meet user needs by prioritizing pertinent content. Passages are first retrieved and reorganized based on their main concepts, resulting in an annotated collection. The model consists of two phases: Data Indexing, performed once to index the collection, and Passage Retrieval and Ranking, as shown in Figure 1, which identifies the most relevant passages in response to natural-language queries.

3.2 Definition

A semi-structured document combines logical structure and textual content, often organized hierarchically. We focus on two main elements: sections, which are logical units with titles that may contain sub-sections or passages, and passages, which are text segments associated with a single parent section, defined by length or punctuation.

4 Data Indexing

Before retrieval, passages are modeled and indexed by their specific concepts, creating a structured dataset for efficient browsing and matching. This one-time phase involves two steps: "preprocessing" and "concept extraction", the latter being essential for capturing passage themes and aligning them with queries.

4.1 Preprocessing

Preprocessing converts passages and queries into word vectors, representing each passage pi as a set of terms pi = {ti1,....,tim}, and involves two main steps:

– Lexical analysis: Tokenization and stop-word removal organize terms and handle linguistic nuances, including multi-word expressions (e.g., using WordNet).

– Grammatical analysis: Assigns grammatical roles to terms, ensuring accurate interpretation, as word meaning can depend on its syntactic position (e.g., verb vs. noun forms).

4.2 Passage Concept Extraction

This phase links each passage to relevant concepts, assigning a list Cpi = {c1 : pr1,c2 : pr2,...,cj : prj} with relevance scores. Concepts are extracted using WordNet, which organizes terms into synsets, capturing the semantic context of nouns, verbs, adjectives, and adverbs. This approach improves matching accuracy and reduces irrelevant results compared to keyword-based methods, particularly in specialized domains like legal and scientific texts.

– Term Concept Score: Each term is assigned a probability of belonging to its concepts using WordNet, highlighting frequent concepts and reducing less relevant ones as shown in Eq. 1.

(1)

where Ci is the concept of the word W, C is the set of concepts for the word W, and freqCoOcc(Ci,Cj) indicates the frequency of co-occurrence of concept Ci with Cj in the set of passage terms.

– Passage Concept Score: ach passage receives a score for its association with concepts, computed from TCS values of terms, as shown in Eq. 2: .

(2)

where W is the set of words in the passage p, and distance(Wj,Wk) indicates the distance between the two words Wj and Wk.

5 Passage Retrieval

This phase retrieves relevant passages matching a user's natural-language query from an indexed collection. The process begins by assigning an initial query similarity score NQS(q,n) (as shown in Eq. 3) to each node (section or passage). The query–title similarity simtitle (Eq. 6) is propagated to the document root, then distributed to sections and passages via the PPC formula (Eq. 8). Finally, inverse propagation from child to parent nodes (Eq. 9) computes the final passage similarity scores, ensuring accurate relevance ranking through the document hierarchy.

5.1 Query concept Extraction

Query concepts are extracted similarly to passages, despite the short length of queries. Concepts are ranked by their association scores using the same formula as in the Passage Concept Extraction (PCE) phase, ensuring accurate matching with relevant passages.

5.2 Passage Filtering

With the concept sets Cq and Cpi for the query q and each passage pi, we can extract the passages most relevant to the query. This step filters and returns passages that best match the user's query.

A key challenge in information retrieval is minimizing noise and focusing on relevant content. We use relevance propagation from document titles and parent sections to enhance filtering accuracy. This method dynamically ranks passages based on contextual relevance, improving retrieval accuracy and efficiency.

– Node Query Similarity (NQS):

All initial similarities between a node n (section, passage, title or document title) and the query q are calculated using this formula 3.

NQS(q,n) = (Simkeyword(q,n) + simconcept(q,n))/2(3)

where simkeyword(q,n) and simconcept(q,n) represents, respectively, the KeywordBased Similarity and the Concept-Based Similarity between the node and the query.

The Keyword-Based Similarity, calculated using formula 4, is based on Jaccard similarity and evaluates the similarity between two sets of keywords:

(4)

where Vkq is the query keyword vector, and Vkn is the node keyword vector.

On the other hand, Concept-Based Similarity assesses the relevance of shared concepts between the node and the query based on cosine similarity, as shown in formula 5:

(5)

where pri is the percentage associated with the concept ci in the vector Vcn (vector of node concepts), and pri′ is the percentage associated with the concept ci in the vector Vcq (vector of query concepts).

– Propagation from Title to Document:

Relevance propagation from the title to the document is crucial for assessing a document's overall relevance to a given query. We use an advanced representation of document titles Tdoc and sections Ts, expressed as vectors of keywords Vk = {k1,k2,...,kn} and a vector of concepts with associated relevance scores Vc = {c1 : pr1,c2 : pr2,...,cj : prj}. This representation evaluates the similarity between titles and query. The similarity calculation combines the similarity between the document title and the query Simtitle(q,Tdoc) and the similarity between each section title and the query Simtitle(q,Ts) as shown in the formula 6:

(6)

where α is a parameter between 0 and 1 that emphasizes the importance of the document title in determining its overall relevance and nbtitle is the number of titles considered in the document.

The similarity Simtitle(q,T) between a title and a query is calculated based on vectors of keywords and concepts. This similarity considers both the proximity of the keywords and the relevance of the concepts, as shown in formula 7:

$$simtitle(q,T) = (simkeyword(q,T) + simconcept(q,T))/2 \quad (7)$$

where simkeyword(q,T) and simconcept(q,T) represent the Keyword-Based Similarity and Concept-Based Similarity between the title and the query. Relevance propagation assesses keyword and concept similarities between the query and document titles. A near-zero propagation score indicates irrelevance, while a non-zero score indicates relevance, propagating to the document's sections.

– Propagation of Relevance Based on Parent Section Concepts to Child Section (PPC):

After identifying relevant documents, passages are filtered using Concept-Based Similarity. Conceptually irrelevant sections or passages are discarded, while relevant sections propagate their relevance to sub-sections and passages. This ensures that only pertinent passages are retained, providing precise information. For example, in document D (Figure 2), irrelevant sections and passages (e.g., section S11 and passages P1 and Pn2) are removed after relevance propagation.

Figure 2 Propagation example

The relevance of a sub-section (section or passage) is calculated by aggregating the Concept-Base-Similarity score simconcept(q,sf), between the query and the sub-section, and the score of the parent node. The relevance propagation is calculated as follows in formula 8:

$$Propagsection(q,sf) = \alpha \times simconcept(q,sf) + (1-\alpha) \times Propagsection(q,sp) \quad (8)$$

where Propagsp is the relevance score of the parent section sp, simconcept(q,sf) is the Concept-Based Similarity between the query q and the sub-section sf, and Propagsf is the relevance score of the child section after propagation. This formula propagates the relevance score from the parent to the child nodes based on Concept-Based Similarity and the parent's relevance.

5.3 Passage Ranking: Propagation of Relevance Based on Child Section to Parent Section (PCP)

We rank relevant passages by propagating relevance from sub-sections to parent sections, improving similarity calculations and document analysis. This aggregation of sub-section scores refines the relevance estimation of parent sections. The relevance propagation formula is shown in formula 9:

(9)

where Propagsection(q,sp) represents the relevance score propagated from the sub-section sf to the parent section sp, based on the aggregation of the relevance scores of the sub-sections Sf and the distance distance(sp,sf) between the parent section sp and the sub-section sf.

6 Experiments and Results

In this section, we focus exclusively on English-language retrieval tasks to evaluate our proposed relevance propagation approach.

6.1 Experimental design

Dataset

We evaluate our approach on the MS MARCO dataset, a large-scale collection with 8.8 million passages and 3.2 million documents from Bing search results for 1 million real-world queries. Using this dataset, we assess the effectiveness of similarity propagation in passage re-ranking with metrics such as MRR@10, and further analyze its impact using MRR@100 and R@1000.

Baselines

We compared the PCP method with several other passage retrieval techniques, including our proposed NQS and PPC methods. Additionally, we evaluated it against the term-based BM25 method [15], DSI [14] and LTRGR [8]. The baseline results for these methods were obtained from their respective papers.

6.2 Results and discussion

This section presents the performance metrics of different steps of our proposed approach evaluated on the MS MARCO dataset.

Table 1 presents the retrieval performance of various models on the MSMARCO dataset. The PCP model significantly outperforms the BM25 benchmark on all parameters, indicating superior recall and ranking capabilities. This demonstrates the effectiveness of our advanced propagation techniques in accurately identifying and ranking relevant passages. The LTRGR model achieves the highest MRR@10, showcasing superior early precision. This result is expected, as the learning-to-rank approach is designed to prioritize the top results. PCP also shows a significant improvement over both BM25 and DSI. These results highlight the effectiveness of the PCP model in improving both the recall and ranking of relevant documents.

Table 1. Retrieval performance on the MSMARCO dataset

	R@1000	MRR@10	MRR@100
BM25	0.803	0.184	0.166
DSI	-	0.173	-
LTRGR	-	0.255	-
PCP	0.857	0.225	0.189

7 Conclusion

This paper presented a novel Passage Retrieval (PR) approach for semi-structured documents, combining lexical, semantic, and contextual information with dualdirectional similarity propagation. By recalculating relevance scores based on context and query, our method improves passage filtering and ranking, outperforming traditional techniques like BM25.

For future work, we plan to further analyze the bidirectional propagation mechanism, extend comparisons to advanced models such as BERT, and explore real-world applications in search engines, legal document retrieval, and large-scale content management, aiming to enhance scalability, accuracy, and robustness.

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The Evolution of the Definition of Absolute Value in the Tunisian Educational System: A Curricular Perspective

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Abstract – In this text we are interested in the history of the definition of the notion of absolute value in Tunisian curricula. We conducted semiotic analyzes of the different definitions introduced into the programs in order to identify their didactic potentialities. It turns out that the evolution of the definition of the AVN adapts to the trends of the different reforms. These adaptations are marked by three distinct contexts articulated between the numeric-arithmetic context, the algebraic context and the graphic-algebraic context.

Keywords: absolute value, semiotic registers, curricular evolution, historical evolution

I. INTRODUCTION

A curriculum is shaped by political decisions, socio-economic needs, and socio-cultural expectations. It defines the competencies, goals, content, pedagogical methods, teaching materials, assessment approaches, and training structures. According to Assude (2002), curriculum analysis can be approached from a global perspective, connected to social, historical, cultural, and epistemological contexts, or from a local perspective focused on practices and actors embedded within specific institutional and cultural frameworks.

We adopt the view of Assude and Margolinas (2005), who consider the textbook to be a key instrument for curriculum analysis. The textbook reflects the values and goals of a culture as well as the societal choices concerning what should be taught and how. Indeed, analyzing school textbooks enables the identification of the cultural and institutional dimensions of the mathematics being taught, which are themselves the product of a social, cultural, political, and epistemological history (ibid.).

By tracing the evolution of the definition of absolute value through its external didactic transposition in official curricula and Tunisian textbooks, we pursue two main objectives. First, we analyze the semiotic potential of each institutionalized definition, its capacity to support understanding and learning through various semiotic representation registers. Second,

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we identify tensions[3] in its teaching, understood as imbalances between pedagogical goals, instructional methods, and institutional requirements (Assude, 2002). Finally, we examine how this concept interacts with other mathematical notions throughout the curricular reforms of the Tunisian educational system.

The didactic analysis tools employed in this study are partly based on Duval's work (1993–2006) on semiotic registers. His research highlights the importance of analyzing mathematical objects through semiotic representation registers, with particular attention to both inter-register and intra-register articulation. We also draw on the *tool/object* dialectic (Douady, 1984–1992), as we assume that the object dimension of absolute value notion (AVN) deserves further clarification within the knowledge to be taught. Additionally, we refer to the work of Bosch and Chevallard (1999), who emphasized the importance of distinguishing between *ostensive* and *non-ostensive* elements of a mathematical concept. This leads us to raise a legitimate question about the role of each ostensive element mobilized in the definitions of AVN in establishing an appropriate relationship with the associated *non-ostensive* components.

We begin with a brief overview of the major curricular movements that have significantly influenced the content of Tunisian educational programs and mathematics textbooks, as well as the key orientations of various mathematics reforms. In the second part, we provide a short historical account of the emergence of the concept of absolute value within scholarly mathematical knowledge, along with the theoretical frameworks we relied upon to conduct our textbook[4] analyses. In the third part, we examine the semiotic aspects of the various institutionalized definitions of absolute value found in both older and more recent mathematics textbooks, in light of curricular changes over time.

II. Reforms of the Tunisian Educational System

The reforms of the Tunisian educational system have a profound impact on teaching practices. Each curricular reform has been progressively adjusted over time, thereby influencing subsequent reforms.

Following independence, the 1958 reform aimed to combat illiteracy, train a national workforce, and modernize education in alignment with socio-economic needs. In

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[3] Tensions between knowledge domains (arithmetic/algebra, algebra/analysis, etc.), between forms of reasoning (intuition/deduction, concrete/abstract, etc.), and between practices (problem solving/knowledge reorganization, tool/object, expansion/reduction of task types and techniques) (Assude, 2002).

[4] The Tunisian educational context is characterized by the use of a "single" official textbook, which serves as the primary reference for both teachers and students.

mathematics, this "*classical*" period emphasized algebra and analysis as the foundational domains of scientific inquiry.

In 1978, the reform sought to align education with modern life and employment needs. It changed the language of instruction for mathematics at the primary level[5], while at the secondary level, it emphasized hypothetico-deductive reasoning. The mathematical *counter-reform* marginalized sets and algebraic structures in favor of Euclidean geometry, thereby promoting the development of proofs grounded in mental imagery and perceptual experiences.

Framed within a context of socio-economic change, the 1993 reform restructured the education system by introducing compulsory basic[6] education up to the age of 16 and by generalizing the Arabization of mathematics at the lower secondary level. This transition raised challenges related to "bilingualism" and "bilaterality[7]" (Abdeljaoued, 2004). Mathematics instruction began to follow an objectives-based pedagogy, emphasizing problem solving and logic while reducing the dominant presence of linear algebra and set theory.

The innovation of the Tunisian educational system was embodied in the 2002 reform, which institutionalized a new pedagogical orientation: the competency-based approach. New school approaches inspired by socioconstructivist theory emerged and influenced curricula as well as didactic tools still in use today. This reform required an epistemological reflection on the content of mathematics curricula, particularly in both lower and upper secondary cycles. The new mathematics vision places significant emphasis on algebraic modeling in the construction of knowledge. In this perspective, great importance is given to the flexibility between semiotic representation registers, reflecting implicit dialectics between the numerical and algebraic registers on the one hand, and reinforced dialectics between the algebraic and graphical registers on the other (Smida, 2003).

III. A Brief Historical Overview and Semiotic Tools

1. Genesis of the Notion of Absolute Value

[5] Mathematics instruction at the primary level, comprising six years of schooling (for students aged 6 to 13), was conducted in French until that time. The Arabization of scientific content began in 1980.

[6] The basic education system comprises the primary level (for students aged 6 to 12) and the lower secondary level, also known as the preparatory cycle (for students aged 13 to 15), which spans three years (7th, 8th, and 9th grades of basic education). The secondary cycle (for students aged 15 to 20) consists of four academic years.

[7] Abdeljaoued defines bilaterality as the process of reading, within the same sentence, certain sequences from right to left and others from left to right.

Like any mathematical concept, the notion of absolute value (AVN) emerged and evolved in response to societal needs and advances in problem-solving. Its historical development can be divided into three key periods (Gagatsis & Panaoura, 2013). First, the discovery of negative numbers in the 17th century implicitly initiated the genesis of absolute value. In 1673, Wallis attempted a geometric interpretation of positive and negative numbers, defining absolute value as a “signless number” or “distance from zero” (Gagatsis & Panaoura, p. 2). In the 18th century, absolute value acquired an explanatory role in solving algebraic inequalities and approximate equations, particularly in the works of Gauss and Lagrange (Gagatsis & Thomaidis, p. 426). With the evolution of the number concept, Cauchy consolidated AVN as an autonomous mathematical object subject to analytical transformations (Gagatsis & Panaoura, p. 3). Finally, the double-bar notation was introduced by Weierstrass in 1840 and later adopted by Cantor, Kronecker, and Hilbert.

2. The Status of Definitions in Mathematics

In mathematics, a statement may assert a property of a given object or the existence of a relationship between two objects (Ouvrier-Buffet, 2003). These statements can either refer to known objects or serve as definitions introducing new ones. Definitions play a key role in the conceptualization of mathematical notions, as they enable students to acquire concepts and use them in problem-solving or in proving theorems when necessary. As Ouvrier-Buffet (2003) points out, *the communicative function of a definition brings to light various linguistic requirements that students expect from definitions: a definition must be simple, clear, concise, elegant, and familiar (Ouvrier-Buffet, p. 115).* From a logical standpoint, a definition is an open sentence associated with a property of an object or a relationship between objects, which is satisfied by some elements of a given domain and not by others. Therefore, a definition is neither true nor false. What is true or false is whether a particular object satisfies the definition (Durand-Guerrier, Hausberger, Spitalas, 2015).

3. **Semiotic Aspects in Mathematical Activity**

Semiotics provides tools to analyze the denotation of mathematical objects and the evolution of their representations, thereby influencing knowledge construction (Bloch, p. 71). It investigates the production, codification, and communication of signs. Peirce (1978) proposed a semiotic framework suitable for mathematics, based on a triadic model:

- The sign (a symbol representing an idea),
- The object (the concept being referred to),

- The interpretation (the individual understands of the sign).

Duval (1993, 2006) emphasizes that mathematical objects, being abstract and inaccessible from a perceptual or instrumental standpoint, can only be understood through semiotic representations (such as symbols and diagrams).
He distinguishes between several semiotic registers (algebraic, numerical, rhetorical[8]), each organizing knowledge in a specific way. The coordination of these registers is essential, as a single representation cannot fully capture the meaning of a concept (Duval, p. 74).

This is why the coordination of semiotic representation registers plays a crucial and genuinely constructive role in mathematical activity, as a given representation is only activated and developed to the extent that it can be transformed into another (Duval, p. 74).

In the Anthropological Theory of Didactics, Chevallard (1994) distinguishes between "ostensive[9]" objects (manipulable or representable) and "non-ostensive[10]" objects (abstract concepts), suggesting that studying the former facilitates access to the latter.

Finally, Douady (1986–1992) distinguishes between two statuses of a mathematical concept: the "*object status*" (defined independently of any context) and the "*tool status*" (mobilized to solve a problem). Learning is enriched by the tool–object dialectic, which fosters a shift in perspective toward mathematical knowledge.

IV. SEMIOTIC STUDY OF THE EVOLUTION OF THE DEFINITION OF ABSOLUTE VALUE IN TUNISIAN MATHEMATICS TEXTBOOKS

The concept of absolute value has undergone significant evolution in Tunisian mathematics textbooks, reflecting broader changes brought about by national educational reforms. These reforms aimed to enhance mathematics education by diversifying didactic approaches and gradually introducing more formal concepts, while placing particular emphasis on student understanding through the articulation of various semiotic representation registers.

1. *From the 1958 reform to the current one, the definition of absolute value has been in continuous evolution:*

In a secondary year two textbook (students aged 13 to 14) from the early 1960s (1960–1966), after first introducing the notion of an equivalence class[11] of an element a in a set E, followed

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[8] By employing argumentative techniques, analogies, and pedagogical strategies, the rhetorical register in mathematics helps make complex ideas more accessible and engaging for students.

[9] Such as a name, a notation, a graph, a gestural schema, and so on.

[10] Notions, concepts, ideas, and so on.

[11] Let $\boldsymbol{R}$ be an equivalence relation on a set $\boldsymbol{E}$ and let $\boldsymbol{a}$ be an element of $\boldsymbol{E}$. The equivalence class of a modulo R is the set of all elements in E that are equivalent to $\boldsymbol{a}$ modulo $\boldsymbol{R}$: $\boldsymbol{cl(a)} = \{\boldsymbol{x} \in \boldsymbol{E} \mid \boldsymbol{xRa}\}$

by the notion of a signed number[12], absolute value is introduced using a set-theoretic register with ostensives related to the equivalence class modulo R, through the following definition: “The absolute value of the integer a, whose canonical form among the pairs constituting the equivalence class that defines it is (m,0) or (0,m), is defined as the natural number m.” This definition highlights, within the arithmetic framework, an interplay between the numerical and set-theoretic registers. It is clearly non-ostensive, as it is based on an abstract formalization, the equivalence class. Absolute value is presented here as a mathematical object through a rigorous construction, without immediate reference to its utility in problem solving.

In 1969, the grade 9 mathematics textbook (students aged 14 to 15) introduced absolute value as “the number obtained after removing its sign.” The examples provided include both negative and positive numbers from various registers (fractional, decimal). This definition is ostensive and is more oriented toward presenting a simple, procedural tool rather than an abstract or formal mathematical property. While this approach aims to simplify initial understanding, it lacks geometric or symbolic support.

In contrast, the grade 10 textbook (students aged 15 to 16) presents absolute value within a chapter devoted to the set of integers: “The absolute value of a signed integer is the corresponding natural number.” It is clear that the numerical register, reinforced by the rhetorical register of natural language and supported by ostensives tied to the arithmetic domain, predominates over the set-theoretic perspective. This dominant presence of these registers significantly enhances the recognition of the object status of the concept.

The 1979 textbook for grade 8 students introduced absolute value with the following definition: “Given an integer x, the absolute value of $\boldsymbol{x}$, denoted $|\boldsymbol{x}|$, is defined as $\boldsymbol{x}$ if $\boldsymbol{x}$ is positive, and the opposite of $\boldsymbol{x}$ if $\boldsymbol{x}$ is negative.” This definition introduced, for the first time, logical and symbolic structures, marking a shift toward a more formal approach. Absolute value is presented here with a dual character: it is conceptually defined (as an object) while also being readily usable in computations (as a tool). The rhetorical register, supported by algebraic notation, involves new ostensives closely linked to algebraic and logical registers, with the variable x appearing for the first time.

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[12] Integers of the form $cl(m,0)$ are called **positive integers**. We set $cl(m,0) = +m$. Integers of the form $cl(0,m)$ are called **negative integers**. We set $cl(0,m) = -m$.The integers $+m$ and $-m$ are said to be opposites. $cl(3,0) = +3$ *and* $cl(0,3) = -3$ are opposites.

"For every element $\boldsymbol{x} \in \mathbb{Z}$: $\begin{cases} \text{if } \boldsymbol{x} \in \mathbb{Z}^{+}, \text{then } |\boldsymbol{x}| = \boldsymbol{x}. \\ \text{si } \boldsymbol{x} \in \mathbb{Z}^{-}, \text{then} |\boldsymbol{x}| = \text{opp}(\boldsymbol{x}) \end{cases}$ »

In the 1992 mathematics textbook for the third year of lower secondary school, the AVN is introduced within the section devoted to the study of real numbers. For the first time, an entire chapter is dedicated exclusively to this concept, in which the authors present the following definition: *"Given a nonzero real number x, the absolute value of* x*, denoted* $| x |$*, is defined as the greater of the two real numbers x and* $-x$*. Moreover,* $| 0 |= 0$*."*

This definition, equivalent to *"for all real numbers* x*,* $| x |= max(x, -x)$*,"* marks the introduction of mathematical analysis within the set of real numbers. Its non-ostensive nature highlights an analytical property of absolute value. While its use as a computational tool is implicit, it can nonetheless be directly applied to calculations. New ostensives are introduced to define the absolute value of any real number x: grammatical rhetoric employing the superlative ("the greater of...") and the connective "and" borrowed from logical discourse. A few lines later on the same page, a property is presented in the form of a theorem, followed by an exercise for direct application: « Given a real number $\boldsymbol{x}$ $\begin{cases} |x| = x \ if \ x \in \mathbb{R}^{+} \\ |x| = -x \ if \ \in \mathbb{R}^{-} \end{cases}$ ».

The emergence of algebraic notation ensures an intra-register articulation concerning the object status of absolute value, through a formal characterization. Conversely, in the chapter of the 1992 textbook entirely devoted to this notion, its tool status is expanded through the construction of intervals, bounds, and the triangle inequality.

Inspired by the 2002 reform, the new version of the official lower secondary mathematics curriculum, published in 2012, emphasizes the introduction of the absolute value of a number x as the distance OM between point M, with abscissa x, and the origin O on the number line. It also highlights that $|x - y|$ represents the distance MN, where M and N are the points with respective abscissas x and y on this line. This reform establishes a clear articulation between the algebraic and graphical registers, thereby reinforcing students' contextual understanding. In fact, in the current 9th grade textbook, absolute value is defined as: 'Let M be a point on the graduated line (OI) with abscissa x. The absolute value of x is the distance OM, and we write $|x| = OM$.' This definition is purely ostensive, as it introduces a geometric interpretation (distance) and relies on potential visualization. Absolute value is treated as a tool, offering an intuitive interpretation for solving geometric or numerical problems. It is supported by the following properties: for any real numbers a and x, where a is positive:

$|x| = x$ if x is positive
$|x| = -x$ if x is negative
$|x| = a$ implies $x = a$ *or* $x = -a$"

In the given definition and properties, the algebraic register ($|x| = OM$) establishes an abstract link between a mathematical object, the absolute value, and a geometric concept, distance. This connection facilitates the transition from the quantitative register, which measures the distance OM, to a symbolic register represented by the notation $|x|$. We believe that such formulations are simultaneously meaningful and potentially ambiguous for a 9th-grade student who is just beginning to grasp algebraic expressions. Moreover, within the Tunisian curriculum, learners encounter the concept of distance in geometry as early as the second cycle of primary school (3rd and 4th grade, for students aged 8 to 10).

2. *Observations and Analysis*

It emerges from the preceding analysis that the AVN has been a consistent component of the Tunisian mathematics curriculum since 1958, particularly within the lower secondary cycle (either in the 2nd or 3rd year of the former system, or the 8th or 9th year of the current system), typically in chapters devoted to either integers or real numbers. Its definition has undergone significant transformations over time, reflecting both shifts in curricular orientation and the specific features of each period’s mathematics reforms. These definitions do not follow an axiomatic structure; rather, they are predominantly based on the evolution and/or articulation of various semiotic registers: set-theoretic/arithmetic, arithmetic/algebraic, or algebraic/graphical. We have identified a progression in the definition of absolute value along three major axes, each corresponding to a distinct contextual framework: a numerico-arithmetic context, an algebraic context, and a graphico-algebraic context.

In the textbooks from the classical period and the era of modern mathematics, definitions introducing the AVN are situated within an arithmetic framework, initially through the set-theoretic register involving equivalence classes, and subsequently via the rhetorical register, each time supported by the numerical register. This numerico-arithmetic context allows the absolute value to emerge as an object of knowledge closely linked to the concept of number. At this stage, the AVN is primarily considered in its object status, acting on the sign of a number and legitimizing its expression without that sign. Accordingly, obtaining the absolute value of a number depends on how the number is written, with either a “+” or “–” sign, which are not operational symbols but indicators of the nature of the integer. The definitions proposed in textbooks from the 1960s and early 1970s are expressed in a simple written register, adapted to students' level. However, this approach poses several difficulties. It risks generating inappropriate conceptions of the notion (Gharbi & Kouki, 2023), such as confusion between the 'sign' and the 'number' itself. Furthermore, this method does not always take into account the kinds of situations students encounter, particularly when manipulating algebraic expressions involving absolute value.

The counter-reform of mathematics led to a massive algebraization of mathematical discourse at the secondary level, often at the expense of arithmetic approaches. In this algebraic context, the definition of absolute value and the associated properties presented in textbooks have primarily adopted formal algebraic structures. The absolute value of a number is initially defined with limited symbolism, using ostensives related to the numerical register, such as the term *"the opposite of..."* or the expression *"the greater of..."*. These formulations are supported by the logical register, notably through conditional structures like *"if... then..."*, as seen in the 1979 Grade 10 textbook and the 1992 textbook, where the AVN is featured in a dedicated chapter within the section on real numbers and operations in ℝ. The properties accompanying these definitions increasingly rely on purely algebraic notation. It appears that the didactic intention behind organizing this chapter at the beginning of the textbook is to highlight the utility of the absolute value in constructing intervals, bounds, and inequalities, with algebra serving as both a formal tool and language for problem-solving. We believe that the nearly joint treatment of the object and tool dimensions of the AVN is promising. However, the articulation between the numerical and algebraic registers seems to lack efficiency for fostering a proper conceptualization of the notion, particularly with regard to its manipulation in the tasks provided[13]. Indeed, the use of algebraic notation and formal logic, as in the 1992 textbooks, requires a prior understanding of algebraic concepts, which may pose a challenge for lower secondary school students.

During the two most recent curriculum reforms (1993 and 2002), the AVN was introduced with a definition that, for the first time, marked the emergence of a graphical register specific to the number line, supported by algebraic structures and accompanied by algebraic, geometric, and logical ostensives. The absolute value of a number was thus presented as its distance from the origin on the number line. In this context, the concept is considered a modeling tool for the notion of distance. These reforms introduced a geometric dimension to

Tasks that are mostly purely technical, mobilizing syntactic viewpoints (a point to be explored in another text).[13]

the AVN by linking it to the idea of distance on a graduated axis. This approach, further developed in the 2012 curriculum, promotes a smoother articulation between the algebraic and graphical registers, thereby facilitating students' understanding of distance and reinforcing the visual and concrete nature of this mathematical concept.

In this graphico-algebraic context, algebra functions both as a tool and as an object, later employed in transfer situations (Kouki & Hassayoun, 2015). The number line thus serves as a transitional register between its mathematical and physical uses. In the latter case, it does not require formal definition, as it merely illustrates the linearity of experimental points (Malafosse, Lerouge & Dusseau, p. 7). Moreover, Grenier Sénéchaud, Vandebrouk, and Ciiu (2024) emphasize the importance of linking absolute value to the concept of distance, which is essential for understanding other topological notions such as limits and continuity.

The various curricular reforms have also transformed the design of Tunisian mathematics textbooks. For example, the 1992 textbook, which introduced the absolute value notion in a dedicated chapter, marked a shift toward a more formal and theoretical approach. This change required adjustments in teaching practices, as educators needed to incorporate rigorous proofs and apply theoretical properties through practical exercises.

V. CONCLUSION AND PERSPECTIVES

In this article, we have traced the historical evolution of the definition of the absolute value notion within the Tunisian educational system through an analysis of both older and recent mathematics textbooks. This study allowed us to explore the causes behind the transformations made to each definition, as well as their articulation with other mathematical concepts. These changes reflect both the broader movements that influenced the orientations of the Tunisian educational system and the evolution of mathematics teaching methods, taking into account the knowledge objects to be taught and the pedagogical approaches employed.

The semiotic analysis conducted shows that each definition of the absolute value notion is closely linked to a specific mathematical framework and to a dominant semiotic register, often complemented by a secondary register to enhance its explicitation. Furthermore, the definitions proposed in the textbooks are influenced by the curricular choices specific to each reform. We have identified three main contexts structuring the evolution of the absolute value notion: the numérico-arithmetic context, the algebraic context, and the graphico-algebraic context. These contexts reflect progressive changes in pedagogical approaches and institutional priorities.

This work paves the way for further research on the evolution of praxeologies associated with the absolute value notion, particularly by examining the two constitutive elements of the praxeological block: the types of tasks and the techniques employed in textbooks. Despite recommendations from numerous studies advocating the graphical approach for solving situations involving absolute value (W. Ellis & Bryson, 2011; Jupri, Usdiyana & Gozali, 2022; Curtis, 2016; Konyalioğlu, Aksu & Şenel, 2012), its implementation in the Tunisian context has not always led to an optimal conceptualization of the notion (Gharbi & Kouki, 2023). We propose to explore, within a didactic engineering framework, the integration of semiotic aspects and the articulation between semantic and syntactic dimensions developed in the work of Kouki (2008–2018). This could provide a valuable opportunity to improve both procedural and conceptual knowledge of students regarding absolute value, thereby enhancing their overall understanding of this fundamental notion.

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On Compact Set Generated by T-open Sets

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Abstract: In this work, a novel definition of compactness in topological space, termed T-compact set, was introduced by relying on a new type of open set called T-open. Additionally, the properties of T-compact spaces were discussed.

1. **Introduction:** Jingcheng Tong(Jacksonville) [1] initiate the study of t-set in 1989. The concept of T-open set was defined J. M. Saad based on a foundational t-set collection, with subsequent show that the family of T-open sets forms a topological structure, this topology is distinguished by two properties: all closed sets and all t-sets in a space are T-open sets [2].There are types of compact space known in ancient and modern times, such as: S-compact space which introduced by Travis Thompson in 1976 [3], and θ-compact space which defined by Kohli and Das in 2006 [4].

In this paper, we discuss a new types called T-compact set and generalize the characterizations of this notion in light of new open sets. The interior points of A and closure set of A by A°, ($\overline{A}$) with respectively

Definition 1.1 [1]: A is called t-set of $A^{\circ} = \overline{A}^{\circ}$. The family of t-set denoted by ts.

Definition 1.2 [2]: Let (X, τ) be a topological space, $A \subseteq X$. Then the T-closure set of A is defined by $x \in \overline{A}^{T}$ if $\forall U \in ts$, such that $U \cap A \neq \emptyset$.

Definition 1.3 [2]: The set $A \subseteq X$ is called T-closed if $A = \overline{A}^{T}$. The complement of T- closed is called T-open. The family of T-open (T-closed) set is denoted by $TO(X)$. ($TC(X)$].

Proposition 1.4 [2]:A family of T-open set is topology.

Definition 1.5 [2]: Let A subset a space X, a point $x \in A$ is called T-interior point of A if there exists an $U \in ts$ such that $x \in U \subseteq A$, it is denoted $A^{\circ T}$.

Proposition 1.6 [2]: Let (X, τ) be a topological space and $A, B \subseteq X$, then A is T-open set if and only if $A = A^{\circ T}$.

Definition 1.7 [2]: A function $f: X \to Y$ is called T-continuous if the inverse image for any open set in X is T-open in Y.

Note: In fact, every open set in metrizable space is T-open.

Definition 1.8[5]:A family A of subsets of a space X has the finite intersection property provided that every finite subcollection of A has non-empty intersection.

2. T-Compact Spaces

Definition 2.1: Let A be a subset of the topological space X. The T-open cover for A is a collection λ of T-open sets whose union contains A. A subcover derived from the T-open cover λ is a subcollection λ' of λ whose union contains A.

Example 2.2: First of all, it is clear that every open set in any metrizable space is T-open set, therefore if $A = [1,4]$ and consider the T-open cover $\lambda = \{(z-1, z+1) | z \in \mathbb{Z}\}$.

Consider the subcover $\lambda = \{(-1,1), (0,2), (1,3), (2,4), (3,5), (4,6)\}$ is a subcover of A, and happens to be the smallest subcover of λ that covers A.

Definition 2.3: A topological space X is T-compact provided that every T-open cover of X has a finite subcover.

This says that however we write X an a union of T-open sets, there is always a finite subcollection $\{O_i\}_{i=1}^{n}$ of these sets whose union is X. A subspace A of X is T-compact if A is a T-compact space in its subspace topology.

Since relatively T-open sets in the subspace topology are the intersections of T-open sets in X with the subspace A, the definition of T-compactness for subspaces can be restated as follows.

Alternate definition:

A subspace A of X is T-compact if and only if every T-open cover of A by T-open sets in X has a finite subcover.

Example2.4: Every discrete space consisting of a finite number of points is T-compact.

1. The finite complement topology with real line $\mathbb{R}$ is T-compact.
2. An infinite set X with the discrete topology is not T-compact.
3. Any open interval (r_1, r_2) is not T-compact. $\lambda = \left\{\left(r_1 + \frac{1}{s}, r_2\right) | s = 2, \dots, \infty\right\}$ is T-open cover of (r_1, r_2). However, no finite subcollection of these sets will cover (r_1, r_2).
4. $\mathbb{R}^n$ is not T-compact for any positive integer n, since $\lambda = \{B(0, s) | s = 1, \dots., \infty\}$ is T-open cover with no finite subcover.

Proposition 2.5: A space X is T-compact if and only if every family of T-closed sets in X with the finite intersection property has non-empty intersection.

This says that if F is a family of T-closed sets with the finite intersection property, then we must have that $\bigcap_\alpha C_\alpha \neq \emptyset$.

Proof: Assume that X is T-compact and let $F = \{C_\alpha | \alpha \in I\}$ be a family of T-closed sets with the finite intersection property. We want to show that the intersection is empty. Let $\lambda = \{O_\alpha = X \setminus C_\alpha | \alpha \in I\}$ is a collection of T-open sets in X. Then,

$$\bigcup_{\alpha \in I} O_\alpha = \bigcup_{\alpha \in I} X \setminus C_\alpha = X \setminus \bigcap_{\alpha \in I} C_\alpha = X \setminus \emptyset = X$$

Thus, λ is T-open cover for X. Since X is T-compact, it must have a finite subcover; i.e.,

$$X = \bigcup_{i=1}^{n} O_{\alpha_i} = \bigcup_{i=1}^{n} (X \setminus C_{\alpha_i}) = X \setminus \bigcap_{i=1}^{n} C_{\alpha i}$$

This means that $\bigcap_{i=1}^{n} C_{\alpha i}$ must be empty, contradicting the fact that F has the finite intersection property. Thus, if F has the finite intersection property, then the intersection of all numbers of F must be non-empty.

Definition 2.6: A function $f: X \to Y$ is called T-irresolute if the inverse image for any T-open set in Y is T-open in X.

Definition 2.7: A function $f: X \to Y$ is called T-open if the image for any T-open set in X is T-open in Y.

Proposition 2.8: Let X be a T-compact space and $f: X \to Y$ a T-irresolute function from X onto Y. Then Y is T-compac space.

Proof: We will outline this proof. Start with T-open cover for Y.Use the T-irresolution of f to pull it back to an T-open cover of X. Use T-compactness to extract a finite sub cover for X, and then use the fact that f is onto to reconstruct a finite subcover for Y.

Corollary 2.9: Let X be a T-compact space and $f: X \to Y$ a T-irresolute function. The image $f(X)$ of X in Y is a T-compact subspace of Y.

Corollary 2.10: Let X be a compact space and $f: X \to Y$ a T-continuous function from X onto metrizable space Y. Then Y is T-compact.

The T-compactness is not hereditary, because $(0,1)$ is not a T-compact subset of the T-compact space $[0,1]$. It is closed hereditary.

Proposition 2.11: Each T-closed subset of a T-compact space is T-compact.

Proof: Let A be T-closed subset of the T-compact space X and let λ be an T-open cover of A by T-open sets in X. Since A is T-closed, then $X \setminus A$ is T-open and $\lambda^* = \lambda \cup \{X \setminus A\}$ is T-open cover of X. Since X is T-compact, it has a finite subcover, containing only finitely many members $O_1, \dots, O_n$ of λ and may contain $X \setminus A$. Since

$$X = (X \setminus A) \cup \bigcup_{i=1}^{n} O_i$$

It follows that

$$A \subset \bigcup_{i=1}^{n} O_i$$

and A has a finite subcover.

Is the opposite implication true? Is every T-compact subset of a space is T-closed? Not necessarily. The following though is true.

Definition 2.12: A topological space (X, τ) is said to be TT_2- space provided that every pair of distinct points of X, there exists two disjoint T-open sets each one contains one point but not the other.

Proposition 2.13: Each T-compact subset of TT_2-space X is T-closed.

Proof: To evidence that A is T-closed., we will evidence that its complement is T-open. Let $x \in X \setminus A$. Then for each $y \in A$ there are disjoint sets U_y and V_y with $x \in V_y$ and $y \in U_y$. The collection of T-open sets $\{U_y | y \in A\}$ forms an T-open cover of A. Since A is T-compact, this T-open cover has a finite subcover, $\{U_{yi} = | i = 1, \dots, n\}$. Let

$$U = \bigcup_{i=1}^{n} U_{yi} \qquad V = \bigcap_{i=1}^{n} V_{yi}$$

Now, any U_{yi} and V_{yi} are disjoint, we have U and V are disjoint. Also, $A \subset U$ and $x \in V$. Thus, for each point $x \in X \setminus A$ we have found T-open set, V containing x which is disjoint from A. Thus, $X \setminus A$ is T-open, and A is T-closed .

Corollary 2.14: A subset A of any T-compact TT_2- space is T-compact if and only if it is T-closed.

Corollary 2.15: Let X be T-compact space and Y be TT_2-space. Then a function $f: X \to Y$ is T- open (T-closed) function.

Proposition 2.16: If A and B are disjoint T-compact subsets of TT_2-space X, then there exist disjoint T-open sets U and V in X such that $A \subset U$ and $B \subset V$.

Proof: Assume A and B are T-compact sets in X such that $A \cap B = \emptyset$, that is $A \subseteq X \setminus B$ and $B \subseteq X \setminus A$. Since X is TT_2-space, then A and are T-closed sets due to Proposition 2.13, so $X \setminus B$ and $X \setminus A$ are T-open sets.

Corollary 2.17: If A and B are disjoint closed subsets of a T-compact TT_2-space X, then there exist disjoint T-open sets U and V in X such that $A \subset U$ and $B \subset V$.

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The Spectral Origin of Pythagorean Triples: A Duality in Orthogonal Circulant Matrices

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Abstract

This paper presents a novel theoretical framework grounded in linear algebra to interpret and generate Pythagorean triples, offering a rigorous alternative to classical approaches based on number theory. We prove that the orthogonality condition of a real 4×4 circulant matrix is mathematically equivalent to a spectral constraint, which in turn enforces a unifying Diophantine equation on the matrix coefficients: $c_0 c_2 + c_1 c_3 = 0$. Through this equation, we uncover a previously unknown structural duality: two distinct families of matrices emerge from symmetric solutions to the master equation, each capable of generating Pythagorean triples. Furthermore, we demonstrate that both families are fully equivalent in generative capacity and structurally analogous to Euclid's formula, ensuring the completeness of the framework.

Finally, this work offers a new interpretation of Pythagorean triples as the necessary numerical manifestation of a spectral balance in a higher-dimensional space, establishing a direct and intrinsic bridge between matrix spectra and number-theoretic structure.

Keywords
Pythagorean Triples, Linear Algebra, Circulant Matrices, Orthogonal Matrices, Spectral Theory, Diophantine Equations.

Chapter 1: Introduction

1.1 Historical and Foundational Context

Pythagorean triples, which are sets of positive integers (x, y, z) that satisfy the Diophantine equation:

$$x^2 + y^2 = z^2$$

represent one of the most ancient problems in the history of mathematics. The roots of its systematic study trace back to the Babylonian civilization (circa 1800 B.C.), as evidenced by historical analysis of the clay tablet Plimpton 322 (Robson, 2002). Subsequently, Euclid, in his *Elements*, established the theoretical framework for the problem, providing a complete parametric description of all primitive solutions through the famous formula:

$$(m^2 - n^2, 2mn, m^2 + n^2)$$

This formula not only solved the problem but also became the cornerstone from which most subsequent research has departed (Nathanson, 2000).

1.2 Review of Existing Approaches

Despite the completeness of Euclid's solution, the search for alternative methods and deeper algebraic structures governing these triples has never ceased. This continuous pursuit has led to the emergence of diverse categories of approaches (Taghavi, 2024), which can be organized as follows:

- **Direct Number-Theoretic and Algebraic Approaches:** These rely on classical tools from number theory, starting directly from the equation $x^2 + y^2 = z^2$ for analysis within the ring of integers $\mathbb{Z}$ or the ring of Gaussian integers $\mathbb{Z}[i]$ (Nathanson, 2000; Conway & Guy, 1996; Trivedi et al., 2015).
- **Refined Parametric Approaches:** These aim to improve generative efficiency and avoid redundancy by re-parameterizing the coefficients in ways that ensure uniqueness (Overmars et al., 2019).
- **Matrix-Based Approaches:** With the evolution of linear algebra, powerful matrix-based methods have emerged. These can be iterative and transformational (such as Barning's or Berggren's matrices, which use 3×3 matrices to generate the tree of solutions) (Alperin, 2005; Cha et al., 2018) or geometric-relational, studying the properties of triple-preserving matrices (Crasmareanu, 2019).
- **Advanced Structural Approaches and Generalizations:** Other research directions have sought to place the problem in a broader context by generalizing the concept to higher dimensions (Al-Ahmad et al., 2023; Olagunju, 2023) or reinterpreting it with advanced theoretical tools such as projective geometry (Tikoo & Wang, 2009), null vectors (Gerstein, 2005), quaternions (Chamizo & Jiménez-Urroz, 2022), or class group theory (Jaklitsch et al., 2024).

1.3 Research Gap and Motivation

A closer examination of this diverse literature, despite its richness, reveals a common philosophical starting point: all these approaches, without exception, begin with the Pythagorean equation $x^2 + y^2 = z^2$ (or an equivalent structure) as a primary axiom. They then proceed to analyze, solve, or generalize it. They treat the Pythagorean structure as a self-contained mathematical "given," leaving a fundamental question unanswered: Is this numerical structure primary, or is it a derived structure originating from a deeper mathematical principle?

To the best of our knowledge, it has not been explored whether Pythagorean triples are merely a numerical manifestation of a fundamental spectral geometric property in a higher-dimensional space.

1.4 Main Contribution of the Research

This research bridges that knowledge gap by presenting a new perspective that does not start from number theory but rather from the fundamental principles of linear algebra. We do not start with the Pythagorean equation; we arrive at it. This work analyzes families of constrained 4×4 circulant matrices and establishes the following pivotal results:

- **Derivation of a Unified Governing Equation:** We prove that the full orthogonality condition for such matrices, which is equivalent to a "spectral balance" condition, necessarily imposes a single, unified Diophantine equation on their coefficients:

$$c_0 c_2 + c_1 c_3 = 0$$

- **Uncovering a Dual Structure:** We show that this master equation possesses structured solutions arising from simple symmetry constraints, thereby revealing the existence of two distinct families of matrices (M and N), each capable of generating Pythagorean triples.
- **Proof of Completeness and Equivalence:** We demonstrate that both families are fully equivalent in their generative capacity and are structurally analogous to Euclid's classic formula, thereby ensuring the completeness of the new methodology.

Thus, this research does not merely present a new generation method; it offers a structural and original interpretation that links two fundamental mathematical fields via a unified algebraic-spectral structure.

1.5 Organization of the Paper

This paper is organized as follows:

- **Chapter 2:** Introduces the mathematical framework for circulant matrices and the Spectral Theorem.

- **Chapter 3:** Presents the main theorem and provides a rigorous proof for the derivation of the governing equation from the spectral orthogonality condition.
- **Chapter 4:** Unveils the dual structure and proves the completeness and equivalence of the two discovered matrix families with Euclid's formula.
- **Chapter 5:** Provides numerical examples and a clear algorithm for applying the methodology.
- **Chapter 6:** Positions our work within the current literature through a critical comparison, highlighting its originality and conceptual contribution.
- **Chapter 7:** Concludes the paper by summarizing the main findings and proposing avenues for future research.

Chapter 2: Mathematical Framework

In this chapter, we establish the necessary mathematical framework for our work. We begin by defining circulant matrices and their fundamental algebraic properties. We then proceed to an explicit presentation of their spectral theory, with a focus on the fourth-order case, which lies at the core of this research. Finally, we connect the geometric concept of orthogonality to its equivalent spectral condition, thereby paving the way for the proof of the main theorem in the subsequent chapter.

2.1 Definition and Properties of Circulant Matrices

Definition 2.1.1 (Circulant Matrix): (Golub & Van Loan, 2013)

Let $\mathbb{F}$ be a field, and let $\mathbf{c} = (c_0, c_1, \dots, c_{n-1}) \in \mathbb{F}^n$ be a vector. The $n \times n$ circulant matrix $C = \text{circ}(\mathbf{c})$ is a matrix where each row vector is a right cyclic shift of the row vector preceding it. It is completely determined by its first row $\mathbf{c}$ as follows:

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix}$$

In this work, we will focus exclusively on the case $n = 4$ where the coefficients are integers, i.e., $\mathbf{c} \in \mathbb{Z}^4$.

Property 2.1.2 (Commutative Algebra Structure): (Golub & Van Loan, 2013)

The set of $n \times n$ circulant matrices over a field $\mathbb{F}$ forms a commutative algebra of dimension n. This means that the sum and product of any two circulant matrices is also a circulant matrix, and matrix multiplication is commutative: for any two circulant matrices

c_1, c_2, we have $c_1 c_2 = c_2 c_1$. This property distinguishes them as a special class of matrices, which in general do not commute.

2.2 The Spectral Theorem for Circulant Matrices: (Golub & Van Loan, 2013)

The most powerful property of circulant matrices is their diagonalizability by the Discrete Fourier Transform (DFT) matrix, which allows for an explicit characterization of their spectrum (the set of their eigenvalues).

Definition 2.2.1 (Discrete Fourier Transform Matrix):

The $n \times n$ Discrete Fourier Transform (DFT) matrix, denoted F_n, is a unitary matrix whose entries F_{jk} are defined by:

$$F_{jk} = \frac{1}{\sqrt{n}} \omega^{jk}, j, k \in \{0,1, \dots, n-1\}$$

where $\omega = e^{2\pi i/n}$ is the n-th primitive root of unity.

Theorem 2.2.2 (Diagonalization of Circulant Matrices):

Any circulant matrix $C = \text{circ}(c_0, \dots, c_{n-1})$ can be diagonalized by the Fourier matrix as follows:

$$C = F_n^* D F_n$$

where F_n^* is the conjugate transpose (Hermitian adjoint) of F_n, and D is a diagonal matrix whose diagonal entries are the eigenvalues of C.

Corollary 2.2.3 (Eigenvalues and Eigenvectors): As a direct consequence of Theorem 2.2.2: (Golub & Van Loan, 2013)

(i) The eigenvalues of $\boldsymbol{C}$ are the components of a vector $\boldsymbol{\lambda}$ and are given by the formula:

$$\lambda_j = \sum_{k=0}^{n-1} c_k \omega^{jk}, j \in \{0,1, \dots, n-1\}$$

which is simply the Discrete Fourier Transform of the first row vector.

(ii) The eigenvectors of $\boldsymbol{C}$ are the columns of the conjugate transpose of the Fourier matrix, $\boldsymbol{F_n^*}$. The eigenvector $\mathbf{v}_j$ corresponding to the eigenvalue $\boldsymbol{\lambda_j}$ is:

$$\mathbf{v}_j = \frac{1}{\sqrt{n}} (1, \omega^j, \omega^{2j}, \dots, \omega^{(n-1)j})^T$$

These vectors form an orthonormal basis for the space $\mathbb{C}^n$.

Specialization to the case $n = 4$:

In the context of our research, $n = 4$, and the roots of unity are

$$\omega^0 = 1, \omega^1 = i, \omega^2 = -1, \omega^3 = -i$$

Consequently, the four eigenvalues of the matrix $C = \text{circ}(c_0, c_1, c_2, c_3)$ are:

$$\lambda_0 = c_0 + c_1 + c_2 + c_3$$
$$\lambda_1 = c_0 + c_1 i - c_2 - c_3 i$$
$$\lambda_2 = c_0 - c_1 + c_2 - c_3$$
$$\lambda_3 = c_0 - c_1 i - c_2 + c_3 i$$

2.3 Orthogonality and the Equivalent Spectral Condition

We now define the concept that connects geometry to algebra in this work.

Definition 2.3.1 (Orthogonal Matrix): A real square matrix A is said to be an orthogonal matrix if its columns (and rows) form an orthonormal basis. This is equivalent to the algebraic condition:

$$A^T A = AA^T = I$$

where A^T is the transpose of A and I is the identity matrix. (Golub & Van Loan, 2013)

In this research, we will consider a slightly more general case where the norm of the rows is not required to be unity, but rather the rows are mutually orthogonal and have the same norm. This is equivalent to the condition $A^T A = \mathrm{k}^2 I$ for some positive constant k

Theorem 2.3.2 (The Spectral Condition for Generalized Orthogonality):

Let A be an $n \times n$ matrix over a field $\mathbb{F}$ (where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$). Assume A is a normal matrix, i.e., it satisfies the condition $A^* A = AA^*$, where A^* represents the conjugate transpose of A (which reduces to the transpose A^T if A is real). It is worth noting that the class of Circulant Matrices is a special case of Normal Matrices.

In this case, the matrix A is a constant multiple of a unitary matrix (or an orthogonal matrix in the real case), i.e., it satisfies the relation $A^* A = cI$ for a positive real constant c, if and only if all its eigenvalues λ_j (where $j = 1, \dots, n$) have the same absolute value. Equivalently, there exists a positive real value k such that:

$$A^* A = \mathrm{k}^2 I \iff |\lambda_j| = k \; \forall j = 1, \dots, n$$

where $\kappa = \sqrt{c}$. In the real case, the first condition is written as $A^T A = \mathrm{k}^2 I$. (Golub & Van Loan, 2013)

This theorem constitutes the cornerstone of our work, as it provides a mechanism to translate the geometric condition related to the orthogonality of rows (in a generalized sense, where all rows have the same norm κ) into an equivalent and simpler algebraic condition imposed on the matrix's spectrum (i.e., its eigenvalues). This transformation from a geometric to a spectral property greatly facilitates the analysis in the next section. We will explore how a condition related to the balance between only two specific eigenvalues can be sufficient to generate the mathematical structure that is at the heart of this research.

Chapter 3: From Spectral Balance to the Pythagorean Structure

In this chapter, we arrive at the core of this research contribution. We will demonstrate a direct and deterministic causal relationship between a spectral property of circulant matrices and the algebraic structure of Pythagorean triples.

We begin by proving that the orthogonality condition, when translated into a condition on the matrix spectrum, imposes a specific algebraic constraint on the matrix coefficients. We then show how this algebraic constraint is precisely the mechanism that naturally generates Pythagorean triples upon the imposition of simple symmetry conditions.

3.1 The Orthogonality Condition and the Governing Equation

Here, we prove that the full orthogonality condition for a fourth-order circulant matrix imposes a specific Diophantine equation on its coefficients.

Theorem 3.1.1 (The Fundamental Algebraic-Spectral Equivalence Theorem):
Let $C = \text{circ}(c_0, c_1, c_2, c_3)$ be a circulant matrix with integer coefficients. The matrix C is orthogonal (i.e., $C^T C = kI$ for a constant $k > 0$) if and only if its coefficients satisfy the following Diophantine equation, which we shall call the **Governing Equation**:

$$c_0 c_2 + c_1 c_3 = 0 \quad (3.1)$$

Proof:
According to Theorem 2.3.2, the orthogonality condition $C^T C = kI$ for the circulant matrix C is equivalent to the spectral condition:

$$|\lambda_j|^2 = k \text{ for all } j \in \{0,1,2,3\}$$

It is sufficient to equate the squared moduli of any two independent eigenvalues. Let's choose $\lambda_0, \lambda_1, \lambda_2$. From section 2.2, we have:

$$\lambda_0 = c_0 + c_1 + c_2 + c_3$$
$$\lambda_1 = (c_0 - c_2) + i(c_1 - c_3)$$
$$\lambda_2 = c_0 - c_1 + c_2 - c_3$$

The orthogonality condition implies

$$|\lambda_0|^2 = |\lambda_1|^2 \ \ and \ \ |\lambda_2|^2 = |\lambda_1|^2$$

Since λ_0 and λ_2 are real,this means:

$$\begin{cases} (c_0 + c_1 + c_2 + c_3)^2 = (c_0 - c_2)^2 + (c_1 - c_3)^2 & (1) \\ (c_0 - c_1 + c_2 - c_3)^2 = (c_0 - c_2)^2 + (c_1 - c_3)^2 & (2) \end{cases}$$

Summing equations (1) and (2):

$$(c_0 + c_1 + c_2 + c_3)^2 + (c_0 - c_1 + c_2 - c_3)^2 = 2[(c_0 - c_2)^2 + (c_1 - c_3)^2]$$

This can be rewritten by regrouping terms:

$$((c_0 + c_2) + (c_1 + c_3))^2 + ((c_0 + c_2) - (c_1 + c_3))^2 = 2[(c_0 - c_2)^2 + (c_1 - c_3)^2]$$

Applying the identity $(X + Y)^2 + (X - Y)^2 = 2(X^2 + Y^2)$ to the left-hand side, with $X = c_0 + c_2$ and $Y = c_1 + c_3$, we find:

$$2[(c_0 + c_2)^2 + (c_1 + c_3)^2] = 2[(c_0 - c_2)^2 + (c_1 - c_3)^2]$$

Dividing both sides by 2 and expanding the squares:

$$(c_0^2 + 2c_0c_2 + c_2^2) + (c_1^2 + 2c_1c_3 + c_3^2) = (c_0^2 - 2c_0c_2 + c_2^2) + (c_1^2 - 2c_1c_3 + c_3^2)$$

After canceling all identical squared terms from both sides, we are left with the relation:

$$2c_0c_2 + 2c_1c_3 = -2c_0c_2 - 2c_1c_3$$

This implies $4c_0c_2 + 4c_1c_3 = 0$, which necessarily leads to the Governing Equation (3.1).

3.2 Generation of the Pythagorean Identity

We now demonstrate that the Governing Equation (3.1) does not by itself generate a single Pythagorean structure. Rather, it is a prerequisite condition that allows such structures to emerge when additional symmetry constraints are imposed.

Theorem 3.2.1 (The Constrained Pythagorean Structures):
The Governing Equation $c_0c_2 + c_1c_3 = 0$ leads to explicit Pythagorean structures under specific linear constraints. Namely:

(i) Under the linear constraint $c_2 = -c_0$, the Governing Equation (3.1) reduces to $c_1c_3 = c_0^2$. This latter condition is mathematically equivalent to the Pythagorean identity:

$$(c_1 - c_3)^2 + (2c_0)^2 = (c_1 + c_3)^2 \qquad (*)$$

(ii) Under the linear constraint $c_3 = -c_1$, the Governing Equation (3.1) reduces to $c_0c_2 = c_1^2$. This latter condition is mathematically equivalent to the Pythagorean identity:

$$(c_0 - c_2)^2 + (2c_1)^2 = (c_0 + c_2)^2 \ (**)$$

Proof: We need to prove the equivalence for (i) and (ii).

Proof of (i): We must prove the equivalence

$c_1c_3 = c_0^2 \iff (c_1 - c_3)^2 + (2c_0)^2 = (c_1 + c_3)^2$.

($\Longrightarrow$) Necessity: Assume $c_1c_3 = c_0^2$ holds. We need to prove that

$(c_1 - c_3)^2 + (2c_0)^2 = (c_1 + c_3)^2$.

Start with the LHS of the target equation (*):

$$\text{LHS} = (c_1 - c_3)^2 + (2c_0)^2 = (c_1^2 - 2c_1c_3 + c_3^2) + 4c_0^2$$

Substituting from the hypothesis

$(c_0^2 = c_1c_3 \implies 4c_0^2 = 4c_1c_3)$:

$$\text{LHS} = c_1^2 - 2c_1c_3 + c_3^2 + 4c_1c_3 = c_1^2 + 2c_1c_3 + c_3^2 = (c_1 + c_3)^2 = \text{RHS}$$

(⟸) Sufficiency: Assume the identity $(c_1 - c_3)^2 + (2c_0)^2 = (c_1 + c_3)^2$ holds. We need to prove that $c_1c_3 = c_0^2$.
Start from the hypothesis:

$$(c_1^2 - 2c_1c_3 + c_3^2) + 4c_0^2 = c_1^2 + 2c_1c_3 + c_3^2$$

Canceling terms from both sides gives:

$$-2c_1c_3 + 4c_0^2 = 2c_1c_3 \implies 4c_0^2 = 4c_1c_3 \implies c_0^2 = c_1c_3$$

Thus, the reverse direction is proven.

Proof of (ii): We must prove the equivalence

$c_0c_2 = c_1^2 \iff (c_0 - c_2)^2 + (2c_1)^2 = (c_0 + c_2)^2$.

(⟹) Necessity: Assume $c_0c_2 = c_1^2$ holds.
Start with the LHS of (**):

$$\text{LHS} = (c_0 - c_2)^2 + (2c_1)^2 = (c_0^2 - 2c_0c_2 + c_2^2) + 4c_1^2$$

Substituting $c_1^2 = c_0c_2$:

$$\text{LHS} = c_0^2 - 2c_0c_2 + c_2^2 + 4c_0c_2 = c_0^2 + 2c_0c_2 + c_2^2 = (c_0 + c_2)^2 = \text{RHS}$$

(⟸) Sufficiency: Assume $(c_0 - c_2)^2 + (2c_1)^2 = (c_0 + c_2)^2$ holds.
Expanding the identity:

$$(c_0^2 - 2c_0c_2 + c_2^2) + 4c_1^2 = c_0^2 + 2c_0c_2 + c_2^2$$

$$-2c_0c_2 + 4c_1^2 = 2c_0c_2 \implies 4c_1^2 = 4c_0c_2 \implies c_1^2 = c_0c_2$$

Thus, the reverse direction is proven. ■

Section Summary:

We have demonstrated that the orthogonality condition for a fourth-order circulant matrix is equivalent to the unified Diophantine equation $c_0c_2 + c_1c_3 = 0$. We have also shown that this equation functions as a fundamental condition which, upon the imposition of simple, additional symmetry constraints, reduces to a form mathematically equivalent to the Pythagorean identity. This reveals the precise mechanism linking the geometric spectrum of matrices to the algebraic structure of number theory. Thus, we have established that the coefficients of any fourth-order orthogonal circulant matrix must be subject to this algebraic constraint. This equation is not an axiom from which we begin, but rather an inevitable consequence of a fundamental geometric condition.

Chapter 4: A Dual Structure and Generative Equivalence

Having established that the orthogonality condition necessarily imposes the Governing Equation $c_0c_2 + c_1c_3 = 0$, we proceed in this section to explore its structured integer solutions. We will uncover the existence of a **duality**, whereby two distinct families of constrained matrices emerge as symmetric solutions to this equation. We will prove that both families are capable of generating Pythagorean triples, and then demonstrate that their generative capacities are fully equivalent to each other and to Euclid's classical formula, thereby ensuring the completeness of the new methodology.

4.1. The First Family (M): The Central Symmetry Constraint $c_2 = -c_0$

The simplest non-trivial solution to the Governing Equation arises from imposing a linear symmetry constraint.

Theorem 4.1.1 (The First Constrained Solution and its Pythagorean Equivalence): Upon imposing the linear constraint $c_2 = -c_0$ on the Governing Equation (3.1), it reduces to the condition $c_1c_3 = c_0^2$. This condition is mathematically equivalent to the Pythagorean identity:

$$(c_1 - c_3)^2 + (2c_0)^2 = (c_1 + c_3)^2$$

Proof: The equivalence was proven in Theorem 3.2.1 (i). ■

Corollary 4.1.2 (Generation Rule for Family M):

As a direct consequence, any circulant matrix from the constrained family

$$M = \text{circ}(a, b, -a, d)$$

which necessarily satisfies the condition $a^2 = bd$ (where $c_0 = a, c_1 = b, c_3 = d$), generates the Pythagorean triple (X, Y, Z) defined by:

$$(X, Y, Z) = (|b - d|, |2a|, |b + d|)$$

where the sides are positive by definition.

4.2. The Second Family (N): The Central Symmetry Constraint $c_3 = -c_1$

In a similar manner, a second, structured solution to the Governing Equation can be explored.

Theorem 4.2.1 (The Second Constrained Solution and its Pythagorean Equivalence): Upon imposing the linear constraint $c_3 = -c_1$ on the Governing Equation (3.1), it reduces to the condition $c_0c_2 = c_1^2$. This condition is mathematically equivalent to the Pythagorean identity:

$$(c_0 - c_2)^2 + (2c_1)^2 = (c_0 + c_2)^2$$

Proof: The equivalence was proven in Theorem 3.2.1 (ii). ■

Corollary 4.2.2 (Generation Rule for Family N):
As a direct consequence, any circulant matrix from the constrained family

$$N = \text{circ}(p, q, r, -q)$$

which necessarily satisfies the condition $q^2 = pr$ (where $c_0 = p, c_1 = q, c_2 = r$), generates the Pythagorean triple (X, Y, Z) defined by:

$$(X, Y, Z) = (|p - r|, |2q|, |p + r|)$$

4.3. Completeness and Generative Equivalence

Having identified the two generation rules, we now prove that both families are not merely special cases, but are complete and equivalent frameworks.

Theorem 4.3.1 (Equivalence with Euclid's Formula):
Both constrained families, M and N, are capable of generating the complete set of primitive Pythagorean triples, and they are structurally analogous to Euclid's formula.

Proof: To prove this, we use the standard parametric representation from number theory. Any primitive Pythagorean triple (X, Y, Z) can be generated by two integer parameters m, k (where $m > k > 0$, $\gcd(m, k) = 1$, and one is even while the other is odd) according to Euclid's formula:

$$X = |m^2 - k^2|, Y = 2mk, Z = m^2 + k^2$$

(i) **Equivalence of Family M:** To find the triple (X, Y, Z) using family M, we need to find coefficients (a, b, d) that satisfy $a^2 = bd$ and generate the triple according to rule 4.1.2. The parametric solution to $a^2 = bd$ is:

$$a = mk, b = m^2, d = k^2$$

Substituting into the generation rule:

$$(|m^2 - k^2|, |2mk|, m^2 + k^2)$$

This is a perfect match with Euclid's formula. Here, it is required that b and d be perfect squares and a be a composite number.

(ii) **Equivalence of Family N:** To find the same triple (X, Y, Z) using family N, we need coefficients (p, q, r) that satisfy $q^2 = pr$ and generate the triple according to rule 4.2.2. The parametric solution to $q^2 = pr$ is:

$$p = m^2, q = mk, r = k^2$$

Substituting into the generation rule:

$$(|m^2 - k^2|, |2mk|, m^2 + k^2)$$

This is also a perfect match with Euclid's formula. Here, it is required that p and r be perfect squares and q be a composite number.

Note: To ensure the completeness of our theoretical framework and its ability to generate all Pythagorean triples (both primitive and non-primitive), we introduce a positive integer scaling factor $s \in \mathbb{Z}^+$. Any non-primitive triple is simply a scalar multiple s of a primitive triple. Accordingly, the general parametric solution for the coefficients of family M becomes:

$$a = s \cdot m \cdot k, b = s \cdot m^2, d = s \cdot k^2$$

It is important to note that this general solution preserves the fundamental equation for family M:

$$a^2 = (s \cdot m \cdot k)^2 = s^2 m^2 k^2$$

$$b \cdot d = (s \cdot m^2)(s \cdot k^2) = s^2 m^2 k^2$$

Thus, $a^2 = bd$ remains satisfied. Similarly, the general parametric solution for the coefficients of family N is:

$$p = s \cdot m^2, q = s \cdot m \cdot k, r = s \cdot k^2$$

This makes our theoretical framework capable of generating any desired Pythagorean triple by selecting the parameters m, k (to define the primitive triple) and the scaling factor s (to define the desired multiple). ■

Corollary 4.3.2 (The Structural Equivalence between Families M and N):
Since both families M and N are generatively equivalent to Euclid's formula, they are necessarily equivalent to each other.

Family M ⟺ Euclid's Formula ⟺ Family N

This proves that the two families are not distinct structures but are dual manifestations of the same underlying mathematical phenomenon. The complete solution space can be accessed via two different algebraic paths, both of which emerge from the same Governing Equation.

Section Summary:

In this chapter, we have uncovered a dual structure arising from symmetric solutions to the Governing Equation. We have proven that both matrix families do not just generate some Pythagorean triples, but rather they generate the complete set of primitive solutions through a direct structural equivalence with Euclid's formula. With this, the deductive path is complete: starting from the geometric orthogonality condition, through the algebraic Governing Equation, to the full numerical realization of the Pythagorean structure. We have shown that Pythagorean triples are not a mere numerical coincidence, but the necessary consequence of geometric symmetry in the world of numbers.

Chapter 5: Numerical Examples

Example 5.1: Generation of the classic (3,4,5) triple and the manifestation of the dual structure.

(i) Using Family M:

- **Inputs:** To generate the primitive triple (3,4,5), we use the coprime parameters $m = 2, k = 1$ (where $\gcd(m, k) = 1$ and they have opposite parity). Since it is a primitive triple, we choose the scaling factor $s = 1$.
- **Calculate Coefficients** (according to the general parametric solution):

$$a = s \cdot m \cdot k = 1 \cdot 2 \cdot 1 = 2b = s \cdot m^2 = 1 \cdot 2^2 = 4d = s \cdot k^2 = 1 \cdot 1^2 = 1$$

(The condition $a^2 = 4 = bd$ is satisfied).
* **Construct the Matrix:**

$$M = \text{circ}(2,4,-2,1) = \begin{pmatrix} 2 & 4 & -2 & 1 \\ 1 & 2 & 4 & -2 \\ -2 & 1 & 2 & 4 \\ 4 & -2 & 1 & 2 \end{pmatrix}$$

- **Generate the Triple:**

$$\text{X} = |b - d| = |4 - 1| = 3Y = |2a| = |2 \cdot 2| = 4Z = |b + d| = |4 + 1| = 5$$

* The result is the primitive triple (3,4,5).

(ii) Using Family N (Manifestation of Duality):

- **Inputs:** We use the same parameters $m = 2, k = 1, s = 1$.
- **Calculate Coefficients:**$p = sm^2 = 4, q = smk = 2, r = sk^2 = 1$
(The condition $q^2 = 4 = pr$ is satisfied).
* **Construct the Matrix:**

$$N = \text{circ}(4,2,1,-2) = \begin{pmatrix} 4 & 2 & 1 & -2 \\ -2 & 4 & 2 & 1 \\ 1 & -2 & 4 & 2 \\ 2 & 1 & -2 & 4 \end{pmatrix}$$

- **Generate the Triple:**

$$X = |p - r| = |4 - 1| = 3, Y = |2q| = |2 \cdot 2| = 4, Z = |p + r| = 4 + 1 = 5$$

* The result is the same triple (3,4,5), demonstrating that the two families are different paths to the same result.

Example 5.2: Generation of the (5,12,13) triple.

- **Inputs:** We choose $m = 3, k = 2$ and $s = 1$. The conditions ($\gcd(3,2) = 1$ and opposite parity) are met.
- **Calculate Coefficients (Family M):** $a = 6, b = 9, d = 4$.
(The condition $a^2 = 36 = bd$ is satisfied).
- **Construct the Matrix:** $M = \text{circ}(6,9,-6,4)$.
- **Generate the Triple:**$X = |9 - 4| = 5, Y = |2 \cdot 6| = 12, Z = 9 + 4 = 13$
* The result is the primitive triple (5,12,13).

Example 5.3: Cases with non-ideal parameters.

(Case 1: Parameters are not coprime):

- **Inputs:** Choose $m = 6, k = 2$. Here $\gcd(6,2) = 2 > 1$.
- **Calculate Coefficients (Family M):** $a = 12, b = 36, d = 4$.
- **Generate the Triple:**

$$X = |36 - 4| = 32, Y = |2 \cdot 12| = 24, Z = 36 + 4 = 40$$

* **Result:** The triple is (32,24,40). This triple is not primitive and can be reduced by dividing by the greatest common divisor, 8:

$$(32,24,40) = 8 \times (4,3,5)$$

- **Analysis:** The primitive triple (4,3,5) is generated by the reduced parameters

$m' = m/g = 3$ and $k' = k/g = 1$, where $g = \gcd(m, k)$

The direct result shows that violating the gcd condition produces a non-primitive triple.

(Case 2: Parameters with the same parity):

- **Inputs:** Choose $m = 3, k = 1$. Both are odd.
- **Calculate Coefficients (Family M):** $a = 3, b = 9, d = 1$.
- **Generate the Triple:**

$$X = |9 - 1| = 8, Y = |2 \cdot 3| = 6, Z = 9 + 1 = 10$$

* **Result:** The triple is (8,6,10), which is the primitive triple (4,3,5) multiplied by 2.
* **Analysis:** When m and k are both odd, the sides $X = |m^2 - k^2|$ and $Z = m^2 + k^2$ are both even, and the side $Y = 2mk$ is necessarily even. Therefore, all sides of the triple are even, ensuring it is non-primitive.

Section Summary:

These numerical examples have demonstrated how to effectively apply the algorithm derived from our main theorem. We have shown the methodology's capability to generate any Pythagorean triple (primitive or non-primitive), tangibly illustrated the concept of the dual structure, and confirmed the predictable and robust behavior of the system even when ideal parameter conditions are violated.

Chapter 6: Discussion and Future Prospects

After establishing the theoretical framework of the new methodology and proving its completeness, it is now essential to conduct a critical analysis to position this work within the context of current mathematical literature, highlighting its originality and conceptual contribution.

6.1 The Explanatory Perspective: From Spectral Geometry to Number Theory

The new methodology is distinguished by its explanatory power. Unlike most existing approaches that start from axioms in number theory, this work originates from a primary principle in linear algebra and geometry.

- **Traditional Approaches:** Begin with the question: "What are the integer solutions to the equation $x^2 + y^2 = z^2$?" They then employ tools from number theory to arrive at a parametric formula.
- **Our Methodology:** Starts from a different question: "What are the algebraic constraints imposed by the orthogonality condition on the structure of constrained circulant matrices in $\mathbb{R}^4$?" We have proven that this geometric condition, equivalent to a "spectral balance" condition, inevitably generates the Governing Equation $c_0c_2 + c_1c_3 = 0$. This master equation, in turn, under simple symmetry constraints, produces the complete numerical structure of Pythagorean triples.

This direct causal relationship offers a radical new interpretation: the Pythagorean triple is no longer just a solution to an equation, but has become the **necessary numerical manifestation** of the spectral orthogonality condition in the matrix families discovered in this work. In other words, the new methodology changes the nature of the question from "How do we find the solutions?" to a deeper one: "**Why does this mathematical structure exist in the first place?**" The answer provided by this work is: because the orthogonality condition, when applied to the structure of circulant matrices, imposes a strict balance on their spectrum, and this balance cannot be achieved unless the numerical coefficients adhere to these Pythagorean constraints.

6.2 Originality and Comparison with Advanced Approaches

To clarify the position of this work precisely, we now compare it with key trends in the literature:

1. **Comparison with Refined Parametric Approaches:** The new methodology differs fundamentally from methods that seek to improve parametric descriptions, such as the work of (Overmars et al., 2019). While both approaches are capable of generating all solutions, our primary goal is not to optimize the algorithm, but to reveal its structural origin. We do not re-characterize Euclid's formula; we derive it as a necessary solution to a problem in linear algebra.
2. **Comparison with Transformational Matrix Approaches:** The new method is direct and non-iterative, unlike the classical methods established by Berggren (1934) and Barning (1963). Our methodology focuses on the individual embodiment of each triple as a unique orthogonal matrix, independent of its relationship to other solutions, and explains its existence through its spectral properties.
3. **Comparison with Advanced Structural Approaches:** The originality of our methodology becomes evident when compared with deep algebraic research. While recent works such as (Jaklitsch et al., 2024) use advanced tools like "class groups,"

we have shown that the new framework, relying exclusively on elementary principles of linear algebra, is capable of deriving a complete characterization of the solutions. More importantly, in comparison to research that uses quaternions to link orthogonality with numerical structures (Chamizo & Jiménez-Urroz, 2022), our work presents a self-contained matrix model. It not only reveals this relationship but also establishes the existence of a duality not previously described, without the need to import external algebraic structures.

6.3 Theoretical Explanation for Empirically Discovered Structures

One of the most significant strengths of the new methodology is its ability to provide a theoretical framework that explains certain empirically observed phenomena. In the search for orthogonal matrices, specific matrix patterns have been discovered computationally (Olagunju, 2023). It is interesting to note that some of these structures can be considered special cases of the families (M and N) discovered in this work. While these patterns were found as isolated cases, our research provides the master equation ($c_0c_2 + c_1c_3 = 0$) and the symmetry constraints that govern these families in their entirety. This indicates that our methodology not only offers a new generation method but also provides the theoretical framework that explains the emergence of such ordered structures, moving from the mere observation of a "pattern" to the understanding of the "principle" that generates it.

6.4 Conclusion

In conclusion, the primary contribution of this methodology lies in its conceptual depth. By demonstrating that the numerical structure of Pythagorean triples is an inevitable consequence of the spectral orthogonality condition in specific matrix structures, this work provides a direct structural bridge between a fundamental principle in linear algebra and a classical result in number theory. The discovery of the unified Governing Equation and its resulting dual structure enriches the current approaches with a new and original explanatory perspective.

Chapter 7: Conclusion and Future Prospects

7.1 Summary of Main Results

This research has presented a deductive journey that began with elementary principles in linear algebra to arrive at the heart of one of the oldest problems in number theory. We have introduced a new theoretical framework for generating and interpreting Pythagorean triples, and our main contributions can be summarized in the following rigorous points:

1. **Establishment of the Governing Equation:** We proved that the orthogonality condition for a fourth-order circulant matrix, which is equivalent to the spectral balance condition, imposes a single, unified algebraic constraint on its coefficients—the Diophantine Governing Equation: $c_0c_2 + c_1c_3 = 0$.

2. **Discovery of a Dual Structure:** We demonstrated that this master equation possesses structured and symmetric solutions, revealing the existence of two distinct families of constrained matrices (M and N). These two families arise from imposing two simple and symmetric constraints: $c_2 = -c_0$ and $c_3 = -c_1$.
3. **Proof of Equivalence and Completeness:** We proved that both families, despite their different algebraic structures, are generatively equivalent in their entirety. Each is capable of generating the complete set of primitive Pythagorean triples, and both are structurally analogous to Euclid's classical formula.

In its essence, this work offers a new interpretation of Pythagorean triples, not as solutions to an equation, but as the **necessary numerical manifestation** of the spectral balance property imposed by the orthogonality condition on the structure of circulant matrices in a four-dimensional space.

7.2 Future Prospects and Open Questions: The theoretical framework developed here is not the end of the road but rather a starting point that opens the door to several promising research directions:

1. **Generalization to Higher Dimensions:** The most direct and important question is: can this methodology be generalized to generate Pythagorean **n-tuples**? This would require exploring constrained circulant matrices of order $2^k \times 2^k$ (such as $k = 3$ for order 8) and searching for the appropriate analogous constraints imposed by the spectral orthogonality conditions. This path could lead to the derivation of new families of Diophantine equations.
2. **Exploring the Solution Space of the Governing Equation:** We have focused on two symmetric and simple solutions to the equation $c_0c_2 + c_1c_3 = 0$. A study of the complete parametric solution space for this equation may reveal new families of solutions, perhaps non-symmetric, and their corresponding numerical structures with different properties, which would enrich our understanding of this structure.
3. **Generalization to Other Number Fields:** Our work is confined to the ring of integers $\mathbb{Z}$. Replacing it with other algebraic number fields, such as the ring of Gaussian integers $\mathbb{Z}[i]$, and imposing the Hermitian orthogonality condition, could lead to the discovery of complex analogues of Pythagorean triples and their associated algebraic structures.
4. **Potential Applications in Signal Processing and Cryptography:** Orthogonal circulant matrices with integer coefficients have practical importance in fields such as the design of **Digital Filters** and **Lattice-based Cryptography**. The ability of our framework to systematically construct these matrices and link them to structured numerical properties (Pythagorean triples) could open the door to designing new families of orthogonal transforms with desirable properties, such as low computational complexity or specific security structures.

This work concludes by opening a new door for thinking about the deep relationship between linear algebra and number theory. We hope that this research will form a solid foundation for deeper investigations in this direction and inspire more explorations into the other bridges that unite different branches of mathematics.

$$\boxed{C^T C = kI}$$

$$\Downarrow$$

$$\boxed{\left(|\lambda_j|^2 = k\right)}$$

$$\Downarrow$$

$$\boxed{c_0 c_2 + c_1 c_3 = 0}$$

$$\Downarrow$$

$$\boxed{c_2 = -c_0 \quad \text{or} \quad c_3 = -c_1}$$

$$\Downarrow$$

$$\boxed{c_0^2 = c_1 c_3 \quad or \quad c_1^2 = c_0 c_2}$$

$$\Downarrow$$

$$\boxed{X^2 + Y^2 = Z^2}$$

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The Reality of Mathematics Teachers' Utilization of Educational Technology Innovations

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Abstract

The research aimed to identify the extent to which mathematics teachers employ emerging educational technologies and the obstacles they face in doing so. The study sample consisted of 102 male and female mathematics teachers in middle and high schools in Homs Governorate. To achieve the research objectives, the researchers developed a questionnaire comprising three main dimensions: the first relates to the availability of emerging technologies in schools, the second to the degree of their use in teaching, and the third to obstacles hindering their employment.

Key Findings:

- The results showed that the availability of educational technology innovations in middle and high schools under the supervision of the Directorate of Education in Homs Governorate was generally low.
- The results also indicated, based on the calculated means, that the extent to which mathematics teachers use emerging educational technologies in teaching was low.
- Teachers identified a moderate level of obstacles preventing them from utilizing these technologies, with an overall mean of 2.28. Chief among these challenges was the absence of a guidebook detailing the available educational technology tools in the school and how to use them.

Main Recommendations:

- Prepare bulletins that list various educational websites specialized in different topics and facilitate their exchange between teachers and learners, including a description of the educational ideas they offer.
- Foster collaboration between educators and technology specialists in the design and development of innovative educational materials.
- The Ministry of Education should provide various educational technology innovations across all schools.
- Organize training courses supervised by experts and specialists in educational technology.
- Distribute instructional booklets on the availability and proper use of these technological innovations.

Keywords:

Emerging Educational Technologies, Educational Technology, Mathematics Teachers

Introduction:

The world today is witnessing a major technological renaissance, with advancements touching many aspects of life. Much of this development is attributed to mathematics and its various applications. Indeed, most of the modern advancements such as rockets, satellites, space exploration, and electronic computers fundamentally rely on mathematics for their progress.

Mathematics serves as a primary source of human intellectual illumination, contributing significantly to scientific and cognitive advancement. It plays an effective role in solving practical problems in line with the scientific progress that humanity aspires to in various aspects of daily life.

Given that mathematics is a core subject in the school curriculum, it is necessary to improve how it is taught and learned—starting from planning, through implementation and evaluation, to development. This process of improvement (enhancing the quality of teaching and learning mathematics) should address all four components of the mathematics curriculum: objectives, content, instructional methods and activities, and assessment.

Improving school mathematics requires revising its content by updating its educational goals in line with societal needs, as well as enhancing teaching methods and techniques to support the achievement of those goals. Accordingly, assessment methods, which are inherently linked to the objectives, must also be developed to reinforce strengths and address weaknesses in both student performance and educational outcomes.

Consequently, greater attention has been given to preparing mathematics teachers more effectively at academic, professional, cultural, and training levels. This involves developing programs that provide teachers with educational knowledge and professional competencies to fulfill their roles effectively and keep pace with scientific and technological developments. Hyndman and colleagues emphasized the necessity of enhancing teachers' technological competencies within teacher preparation programs (Hyndman et al., 2007, p.25). Similarly, Roob (2001) found that experience in educational technology significantly reduces anxiety about using technology and improves teachers' attitudes toward it.

In this context, the Syrian Arab Republic has sought to modernize its educational system by introducing computers into education and adopting modern, technology-based teaching methods. The importance of using educational technology increases with the abstraction level of the subject, such as in spatial (geometric) mathematics.

Ministry of Education officials have emphasized the importance of spreading computer literacy by making computer studies a core subject in basic education. Despite these efforts, Syria has not achieved the desired success and remains far behind. The number of electronically delivered courses in official institutions is very limited, and there is a general lack of reliance on digital content in school curricula. Modern technological devices are also scarce in schools, and national efforts to integrate information and communication technology in education remain fragmented and insufficient.

The effective use of modern educational technologies in teaching largely depends on teachers' knowledge and skills in utilizing these innovations.

Research Problem:

Many studies have emphasized the importance of incorporating modern technologies into the teaching and learning of mathematics. For example, Al-Tamimi (2007) found that integrating contemporary technologies significantly improves both teacher and student attitudes toward mathematics. Similarly, Al-Qurashi (2007) concluded that students' performance in mathematics could be enhanced, and their attitudes improved, if teachers employ modern technological tools, online learning, and methods that focus on self-directed learning and student motivation.

Through their professional experience in education, the researchers observed a noticeable deficiency in the use of modern educational technologies in teaching in general—and mathematics in particular. Traditional teaching methods and conventional tools still dominate the learning process. According to the researchers, several factors may contribute to this, including a lack of qualified personnel, insufficient knowledge about operating and maintaining equipment, inadequate training, fear of misuse and potential accountability, resistance to change, or teachers' lack of conviction in the value of integrating educational innovations.

In light of the Syrian Ministry of Education's interest in integrating technology into education, the need to investigate this issue became evident.

Therefore, the research problem can be defined as:

The weak integration of modern educational technology in the teaching of mathematics in the Syrian Arab Republic.

Research Significance :

The findings of this research may contribute to the effective integration of educational and information technology in the teaching of modern curricula in schools across the Syrian Arab Republic.

The current study serves as a foundational step among the body of research concerned with bridging the fields of educational and information technology and curriculum teaching and development—ultimately benefiting the latter field.

Research Objectives:

The current research aims to:

- Identify the degree of availability of emerging educational technology in middle and high schools in Homs Governorate.
- Determine the extent to which mathematics teachers use educational technology innovations in teaching.
- Identify the obstacles to using modern educational technology methods in teaching mathematics.

Research Questions:

The research seeks to answer the following questions:

1. What are the emerging educational technologies that should be used in teaching mathematics?

2. What is the degree of availability of educational technology innovations in middle and high schools in Homs Governorate?

3. To what extent do mathematics teachers use educational technology innovations in their teaching?

4. What are the obstacles to using modern educational technology methods in teaching mathematics?

Search Terms:

Emerging Educational Technology Innovations: Abdel-Majeed (2000) defines the term "emerging educational technology innovations" as all that is new and modern in the field of utilizing technological tools in the educational process—such as modern devices, machines, and teaching methods—with the aim of enhancing both the teacher's and learner's ability to engage with the educational process. (p. 309)

The researchers operationally define it as: "Everything that is new and innovative in the use and application of technological tools in mathematics instruction. It is a comprehensive educational system designed to facilitate learning, enhance the teacher's and learner's ability to engage effectively with the educational process, and solve its challenges. It integrates various types of electronically presented educational stimuli—written, auditory, visual, and animated—that can be employed to achieve specific educational objectives."

Utilization of Emerging Educational Technology Innovations: Abdullatif (2005) defines it as "the ability to use—specifically, the ability to use the Internet in all educational processes and activities undertaken by students related to knowledge, information, theories, and facts they encounter."

Al-Kindi (2005) defines it as "the use of modern technological capabilities to support general education, and the use of technology as an educational aid in the teaching of various subjects, whether theoretical or practical, through modern technology, practice, training, or simulation, in a manner that serves the objectives of general education." (p. 6)

The researchers operationally define it as: "The planning, designing, and execution by mathematics teachers of the use of emerging educational technology skills as needed and at the appropriate moment within the instructional context, in an integrated and interactive manner with other learning resources, following a systematically designed plan to ensure effective usage for the purpose of improving teaching and learning."

Barriers to Utilizing Emerging Educational Technology Innovations: The term "barriers" is linguistically derived from the Arabic root meaning "to prevent or hinder." According to Lisan al-Arab, it refers to diverting or restraining someone from doing something (Ibn Manzur, 1405 AH, p. 3173).

The researchers operationally define it as: "The factors or circumstances that hinder mathematics teachers from effectively using innovations in educational technology, thereby limiting the potential benefits of these technologies in building an effective general education that achieves desired outcomes."

Skill in Using Emerging Educational Technology Innovations in Teaching: Zaytoon (2001) defines a skill as "the ability to perform a task that usually consists of several smaller, simpler performance components." (p. 12)

The researchers operationally define the skill of using emerging educational technology in teaching as: "The ability to use and employ emerging educational technologies to effectively serve the educational process."

Research Boundaries:

Human Boundaries:The study was conducted on a sample of 102 male and female mathematics teachers working in official secondary schools in Homs Governorate during the 2022–2023 academic year.

Spatial Boundaries: A sample drawn from official secondary schools and second-cycle basic education schools in Homs Governorate.

Temporal Boundaries: The research tool was applied to the study sample during the first semester of the 2022–2023 academic year.

Previous Studies:

Al-Zahrani Study (2010): This study aimed to identify the extent to which technological innovations were available in science laboratories, the degree to which female teachers used these innovations, and the main challenges associated with their use.

Research Methodology: The researcher employed a descriptive method. The study sample consisted of 22 supervisors and 120 female teachers. A questionnaire was used as the research instrument.

Findings: The study concluded that the availability and use of technological innovations in secondary school science laboratories were low, from the perspectives of both supervisors and teachers.

Al-Qahtani Study (2013): This study aimed to evaluate the reality of integrating technological innovations in teaching mathematics based on the developed curricula, from the viewpoints of teachers and educational supervisors in the Tabuk educational region.

Research Methodology: The researcher employed the descriptive method. The study sample consisted of 62 mathematics teachers and 13 educational supervisors from schools in Tabuk City.

Findings: There was a deficiency in mathematics teachers' performance in using technological innovations to support the educational environment during math classes, which hindered learners from interacting positively.

There was a lack of integration of technological innovations in lesson planning, instructional implementation, and learner performance assessment, as well as in utilizing these technologies for the self-directed professional development of mathematics teachers.

There were statistically significant differences at the 0.01 level between the mean responses of the groups, attributed to academic qualification and teaching experience variables.

There were no statistically significant differences related to the job nature variable (teacher vs. supervisor) in participants' responses.

Moila Study (2006): This study aimed to explore the extent of Information and Communication Technology (ICT) use in mathematics instruction and to develop strategies for the use of ICT in teaching and learning mathematics in rural schools in North Africa.

Research Methodology: The researcher used the descriptive method, and the data collection tool was the interview technique, involving three mathematics teachers at Fosela Secondary School.

Findings: The level of educational technology use at the school was influenced by a variety of factors, including teachers' awareness of educational technology tools, the availability of those tools, and the support provided by school administration in preparing and using them.

Khambaria, Luan, and Ayub Study (2010): This study aimed to identify the benefits and challenges teachers face when integrating technology into mathematics teaching.

Research Methodology: The researchers used a descriptive approach, and the data collection tool was interviews conducted with 172 mathematics teachers from 28 secondary schools in Malaysia to gather their perspectives on using laptop computers.

Findings: Teachers were able to use their laptops along with ICT accessories. The results indicated that the portability of the laptops benefited teachers by allowing them to work wherever they preferred and providing them access to a wide range of information via the Internet.

Commentary on Previous Studies

Previous studies affirm the importance of equipping teachers with the concepts and skills necessary to implement emerging educational technologies in the teaching-learning process.

These studies revealed a general weakness in the actual application of such innovations within educational settings.

The current researchers benefited from these prior studies in designing the research instrument and identifying the appropriate statistical methods for analyzing results.

This study shares a common feature with the previous studies—except for Al-Zahrani's study—in terms of sample composition. However, it differs in the specificity of the implementation context, as it was conducted in the Syrian Arab Republic, specifically in Homs Governorate.

Theoretical Framework

Guidelines for Implementing Emerging Educational Technologies

The successful integration of emerging educational technologies—and the achievement of their intended educational goals—depends on adhering to a set of principles and guidelines, including:

1. Studying the innovation in depth, understanding its features, benefits, and the problems it may help resolve.

2. Evaluating the educational feasibility of the innovation compared to traditional methods.

3. Carefully planning for the gradual integration of the innovation based on the educational environment.

4. Highlighting the positive aspects of the innovation while managing and minimizing its drawbacks.

5. Piloting the innovation on small samples to allow for adjustments and to ensure its effectiveness. (Al-Najjar, 2009, p. 715)

Stages of Integrating Emerging Educational Technologies

It is difficult to implement all educational technologies at once. A gradual approach is necessary. Al-Ghazou (2004) suggested that the integration process goes through five stages:

1. Entry Stage: Use is very limited, confined to tools like overhead projectors, animated films, and slides.

2. Adoption Stage: Teachers begin to use basic technologies, incorporating some devices (e.g., video, computer) into their lesson plans.

3. Adaptation Stage: Some technologies are used in instruction, though the overall teaching approach remains largely traditional.

4. Appropriation Stage: Teachers develop a positive attitude toward the innovation and are able to use technology effectively to enhance learning.

5. Invention Stage: Teachers fully integrate the innovation into teaching, become proficient in all related skills, and are able to design educational programs creatively—based on their own vision—drawing on modern psychological theories like constructivism. (Quoted in Al-Najjar, 2009, p. 716)

The Role of Emerging Educational Technologies in Solving Educational System Challenges

The educational process faces a range of challenges that require mitigation or resolution. Emerging educational technologies play a significant role in addressing these challenges, including:

1. Individual Differences: These technologies deliver content tailored to the learner's abilities and prior knowledge, allowing for personalized learning paths.

2. Learning Time and Place Flexibility: They offer learners the opportunity to learn whenever and wherever they choose, using tools such as the Internet and video conferencing.

3. Learning Pace: Learners can proceed at their own pace, repeating tasks as needed until they achieve mastery.

4. Motivation: To sustain learning, motivation must be maintained throughout the process, which educational technologies can help foster.

5. Continuous Performance Feedback and Improvement: Educational technologies provide ongoing feedback to enhance the performance of both teachers and learners.

6. Transfer of Learning: Many educational technologies facilitate the generalization of learned knowledge to real-life contexts.

7. Mastery Standards: Their use requires clear, specific learning objectives and performance criteria, with mastery being essential for progression.

8. Cost-Effectiveness: Despite high implementation costs, the skills, knowledge, and outcomes these technologies deliver justify their use in the educational process. (Al-Zahrani, 2010, pp. 31–34)

Research Procedures

- Research Methodology: The researchers adopted the descriptive-analytical method, collecting data from male and female mathematics teachers in secondary schools affiliated with the Homs Directorate of Education. Data was gathered using a questionnaire specifically designed for this purpose, as it is a tool well-suited for obtaining large amounts of information in a short time. It also offers ease of analysis using computer software, is easy to manage and organize, and is cost-effective.
- Research Population and Sample : The research population included all mathematics teachers (male and female) in the second cycle of basic education and public secondary schools under the Homs Directorate of Education who were actively working during the academic year 2022/2023. The total number was 929 teachers, consisting of 648 permanent staff and 281 on contractual basis. The sample was selected randomly. The main sample comprised 102 teachers, separate from the pilot sample used to test the validity and reliability of the research instrument.

Research Instrument

In line with the nature and objectives of the research problem, the researchers developed a questionnaire as the primary research instrument. The design of the questionnaire was based on a review of available studies, prior research, and relevant educational literature. Drawing on this body of knowledge, the researchers constructed an instrument consisting of 64 items, distributed across three main axes:

First Axis: Availability of Emerging Educational Technologies in Schools

This axis includes 24 items distributed across three domains:

1. Emerging Educational Devices
2. Emerging Educational Materials
3. Emerging Instructional Methods

Second Axis: Teachers' Use of Emerging Educational Technologies in Schools

This axis also includes 24 items, covering the same three domains:

1. Emerging Educational Devices
2. Emerging Educational Materials
3. Emerging Instructional Methods

Third Axis: Obstacles Faced by Teachers in Employing Emerging Educational Technologies in Schools

This axis includes 16 items addressing the challenges and barriers teachers encounter in effectively integrating these technologies into the teaching and learning process.

Steps for Designing and Constructing the Research Instrument:

First: Determining the Purpose of the Research Instrument A questionnaire was developed with the aim of identifying the following:

- The degree of availability of emerging educational technologies in schools.
- The extent to which teachers use emerging educational technologies in schools.
- The level of difficulties and obstacles teachers face in integrating these technologies into their teaching practices.

Second: Identifying the Sources for Constructing the Research Instrument

The development of the research instrument was based on:

1. Educational journals, periodicals, and previous studies relevant to the current research problem.
2. Interviews with a group of specialists in the field to benefit from their expertise and practical experience.

To measure the degree of availability, use, and obstacles, the researchers employed a three-point Likert scale, where:

- A score of (1) indicates a low level of availability, use, or obstacle.
- A score of (2) indicates a moderate level.
- A score of (3) indicates a high level.

The scoring of responses on the research instrument was calculated as follows:

Range = Highest value of response categories – Lowest value = 3 – 1 = 2

Interval width = 2 ÷ 3 = 0.66

Based on the above calculations, the following interpretive scale was adopted:

- Mean values from 1 to less than 1.66 indicate a low level.
- Mean values from 1.66 to less than 2.32 indicate a moderate level.
- Mean values from 2.32 to 3 indicate a high level.

Third: Instrument Validity: The researchers ensured content validity by presenting the instrument to a panel of expert reviewers (judges), and making the necessary modifications based on their feedback and suggestions.

Fourth: Instrument Reliability: Reliability refers to the consistency and precision of the measurement. To verify the reliability of the instrument, the questionnaire was distributed to a pilot sample of 20 teachers (not included in the main study sample). The Cronbach's Alpha coefficient was then calculated for the total questionnaire as well as for each of its axes, as shown in the following table:

Table (1) shows the Cronbach's alpha reliability coefficient for the research tool.

Dimension	Cronbach's Alpha
Overall Questionnaire Score	0.836
First Dimension	0.766
Second Dimension	0.851
Third Dimension	0.824

The reliability coefficients presented in the table are all high, suggesting an adequate level of internal consistency suitable for research purposes.

Research Findings

Question 1: What is the extent to which educational technology innovations are available in Cycle Two basic education schools?

To answer this question, the mean and standard deviation were calculated for the availability level of each innovation item within the three domains (innovative educational devices, innovative instructional materials, and innovative instructional methods), as well as for each domain as a whole and for the overall dimension.

Table (2): Means and standard deviations of the availability level of educational technology innovations in each of the three domains and in the overall dimension from the perspective of the sample.

Field No.	Domain	Mean	Standard Deviation	Evaluation
1	Innovative Educational Devices	1.82	0.297	Moderate
2	Innovative Instructional Materials	1.63	0.285	Low
3	Innovative Instructional Methods	1.52	0.306	Low
Overall Dimension (First Axis)		1.64	0.185	Low

Based on the previous table, it is evident that the overall availability of educational technology innovations in Cycle Two basic education schools and secondary schools is generally low—particularly in the domains of instructional materials and methods. However, the availability in the domain of educational devices is moderate. This outcome is reasonable, as the Ministry of Education strives to provide devices to schools as much as possible. Nevertheless, the crises that Syria has experienced have limited the availability of such devices across all schools, which explains the moderate rating in this domain.

In contrast, the domain of instructional materials and innovations received a low rating. This may be attributed to the fact that producing educational materials requires technological and pedagogical collaboration, while the materials available on the market are often commercial products that overlook the educational aspect. Similarly, instructional methods also received a low rating, likely because training in these methods requires quality content, high-quality instructional materials, and integration between the materials and available devices.

Table (3): Means and standard deviations of the availability of educational technology innovations from the perspective of the sample.

No.	Emerging Educational Technology	Mean Availability Level	Std. Deviation	Evaluation
Emerging Educational Devices				
	Educational Computer	1.98	0.487	Moderate
2	Interactive Video	1.59	0.586	Low
3	Data Show Projector	2.23	0.716	Moderate
4	Educational Television	1.58	0.516	Low

5	Interactive Whiteboard	2.09	0.565	Moderate
6	Document Camera	1.47	0.54	Low
7	Mobile Learning	1.82	0.776	Moderate
Emerging Instructional Materials				
8	Multimedia	1.75	0.767	Moderate
9	Internet	2.27	0.72	Moderate
10	Virtual Laboratories	1.37	0.486	Low
11	E-Books	1.56	0.739	Low
12	Hypermedia	1.53	0.713	Low
13	E-mail	1.6	0.693	Low
14	Electronic Exams	1.38	0.546	Low
Emerging Instructional Methods				
15	Multi-Path Teaching	1.48	0.641	Low
16	E-Learning	1.5	0.641	Low
17	Educational Software	1.46	0.655	Low
18	E-Learning Bag	1.49	0.656	Low
19	Distance Learning (Video Conf.)	1.28	0.495	Low
20	Virtual Learning	1.21	0.43	Very Low
21	Individualized Learning	1.59	0.495	Low
22	Blended Learning	1.81	0.609	Moderate
23	WebQuests (Online Knowledge Trips)	1.46	0.655	Low
24	Flipped Classroom	2.01	0.838	Moderate

Based on the provided table, it is evident that the availability of educational technology innovations in second-cycle basic education schools and secondary schools ranged between low and medium, with availability means varying between (1.21 and 2.27). The least available educational devices in schools were the document camera, educational television, and interactive video. The researchers note that, despite the presence of cameras, televisions, and videos in most schools, they received a low availability score due to the sample's lack of awareness of their availability for educational purposes.

All educational material innovations were available at a low level, except for the internet and multimedia, which achieved a medium level. This may be attributed to the previously mentioned requirement for collaboration among various entities to produce these innovations.

As for teaching method innovations, they were available at a low level, with the exception of blended learning and flipped learning, which achieved a medium level. This is due to the nature of blended and flipped learning, which combine traditional and innovative approaches. These two methods primarily rely on the internet and multimedia, both of which also received a medium availability score.

The second question: What is the extent of use of educational technology innovations by second-cycle basic education teachers in teaching?

To address this question, the arithmetic mean and standard deviation were used to measure the degree of utilization of innovations for each item within the three domains (educational devices innovations, educational materials innovations, and teaching methods innovations).

Table (4): The arithmetic means and standard deviations for the degree of use of educational technology innovations in each of the three domains and the overall axis, from the perspective of the sample.

Domain No.	Domain	Mean Score	Standard Deviation	Evaluation
1	Emerging Educational Devices	1.8	0.309	Moderate
2	Emerging Instructional Materials	1.58	0.28	Low
3	Emerging Instructional Methods	1.41	0.284	Low
Overall Average for Axis Two		1.57	0.191	Low

Based on the previous table, it is evident that the overall level of use of emerging educational technologies by mathematics teachers is generally low, particularly in the domains of instructional materials and methods. However, in the domain of educational devices, the usage level was moderate. This may be attributed to the moderate availability of such devices in schools, in contrast to the low availability of instructional materials.

Regarding instructional methods, the low usage may be due to the inadequate preparation of teachers and their lack of the necessary competencies to apply these methods. Furthermore, a large number of mathematics teachers do not hold an educational qualification (such as a diploma in educational qualification), which would enable them to keep pace with modern strategies and methodologies in teaching their subject.

Table (5): Means and Standard Deviations for the Degree of Use of Emerging Educational Technologies from the Perspective of the Sample.

No.	Integration of Emerging Technologies in Education	Mean Usage	Std. Deviation	Evaluation
Emerging Educational Devices				
1	Use of Educational Computer	2.03	0.476	
2	Use of Interactive Video	1.57	0.622	Low
3	Use of Data Show Projector	2.25	0.724	
4	Use of Educational Television	1.53	0.502	Low
5	Use of Interactive Whiteboard	1.76	0.76	Moderate
6	Use of Document Camera	1.43	0.498	Low
7	Use of Mobile Learning	2.08	0.592	Moderate
Emerging Instructional Materials				
8	Use of Multimedia	1.69	0.796	
9	Use of Internet	2.18	0.801	
10	Use of Virtual Labs	1.34	0.497	Low
11	Use of E-Books	1.51	0.7	Low
12	Use of Hypermedia	1.5	0.671	Low
13	Use of E-mail	1.53	0.685	Low
14	Use of E-Exams	1.32	0.491	Low
Emerging Instructional Methods				
15	Use of Multi-Path Teaching	1.43	0.622	Low
16	Use of E-Learning	1.44	0.623	Low

17	Use of Educational Software	1.43	0.668	Low
18	Use of E-Learning Bag	1.45	0.639	Low
19	Use of Distance Learning (Video Conf.)	1.23	0.42	Low
20	Use of Virtual Learning	1.18	0.383	Low
21	Use of Individualized Learning	1.53	0.502	Low
22	Use of Blended Learning	1.75	0.62	Low
23	Use of WebQuests (Online Knowledge Trips)	1.3	0.523	Low
24	Use of Flipped Classroom	1.38	0.564	Low

Based on the above table, it is evident that the level of use of educational technology innovations by mathematics teachers ranged between low and moderate. The mean availability scores ranged from 1.23 to 2.25. The most commonly used technological devices in schools were the educational computer, data projector, and mobile learning (mobile phones). The researchers attribute this to the fact that computers and mobile learning tools, such as smartphones and laptops, are part of everyday life skills. As for the higher usage of the data projector compared to other devices, it is due to its suitability for current classroom environments, especially given the overcrowding of students.

The use of innovative instructional materials was generally low, with the exception of the Internet and multimedia tools. This limited use is likely due to the overall low availability of such materials in schools.

On the other hand, the use of innovative teaching methods was also low. The researchers explain this by pointing to inadequate teacher preparation and the lack of necessary competencies to effectively utilize these methods. Additionally, training programs designed to equip mathematics teachers with skills in educational technology are very limited. For example, only one training course was held in 2018 to train mathematics teachers on the use of educational technologies and mathematical devices. This course lasted just three days and included only a limited number of teachers in Homs Governorate.

Question 3: What are the obstacles to using educational technology innovations in teaching?

To answer this question, the mean (arithmetic average) and standard deviation were used to assess the degree of each obstacle, as shown in the following table:

Table (6): Means and standard deviations of the degree of difficulties and obstacles that hinder the use of educational technology innovations in teaching from the perspective of the sample.

No.	Difficulties and Obstacles	Mean Difficulty	Standard Deviation	Difficulty Level
1	Lack of availability of computers.	2.11	0.673	Medium
2	Lack of a learning resource center in schools with internet access for teachers and students.	2.1	0.622	Medium
3	Inadequate classroom spaces for using technological innovations, in terms of space, wiring, or furniture.	2.43	0.653	High
4	Lack of appropriate libraries and electronic books.	2.29	0.683	Medium
5	Failure to adopt modern methods for converting curricula into electronic formats.	2.18	0.681	Medium
6	Inadequate teacher preparation during pre-service education to handle such innovations.	2.48	0.593	High
7	Lack of suitable training courses to provide teachers with e-learning skills.	1.94	0.504	Medium
8	Lack of electronic training programs for effective use and integration of technology in teaching.	2.07	0.585	Medium
9	Need for training teachers on the educational uses of ICT and how to train their students in these.	1.91	0.705	Medium
10	Lack of integration between curricula and the World Wide Web.	2.01	0.621	Medium
11	Scarcity of Arabic websites that support teachers and focus on education.	2.49	0.593	High
12	Scarcity of printed materials related to educational websites on the internet.	2.34	0.814	High
13	Lack of a guidebook detailing available educational technology tools in schools and how to use them.	2.72	0.495	High
14	Lack of information on how to use software programs in teaching.	2.49	0.641	High
15	Perceived lack of importance of employing technology in education.	2.4	0.567	High
16	Teachers’ adherence to traditional educational mindsets.	2.56	0.623	High
	Overall Dimension	2.28	0.164	Medium

According to the previous table, the level of difficulties and obstacles that hinder the use of educational technology innovations by mathematics teachers ranged from moderate to high, with mean scores ranging between 1.91 and 2.72. The top-ranked obstacle was the lack of a guidebook that outlines the available technological devices and tools in the school and how to use them. This aligns with the explanation that, despite the presence of some innovative educational devices such as educational television, educational video, and document camera, they received low usage scores, and many other innovations exist but are not used due to teachers' lack of familiarity with how to operate them. The second-ranked obstacle was teachers' adherence to traditional teaching methods. This can be addressed by organizing training courses and encouraging teachers to take initiative through moral incentives, such as certificates of appreciation, and material incentives, such as rewards.

Research Recommendations:

1. Increasing attention to internet access by providing it free of charge to both teachers and learners, and organizing training courses for both groups to raise awareness of the importance of the internet, how to use it, and how to benefit from it.

2. The Ministry of Education must ensure the provision of various educational technology innovations in all schools.

3. Preparing bulletins that include various educational websites specialized in a range of subjects, and encouraging the exchange of these bulletins between teachers and learners, describing the educational ideas offered by the sites.

4. Emphasizing the need for collaboration between educators and technology specialists in designing innovative educational materials.

5. Enhancing mathematics teachers' knowledge in secondary schools regarding the use of educational technologies through:

- Organizing training courses for all teachers across disciplines to familiarize them with the latest developments in the use of educational technologies.

- Involving qualified experts and specialists in training on educational technology innovations, as some supervisors currently assigned to such training lack proper expertise, which may reduce the effectiveness of the sessions to mere lectures.

- Developing a guide for teachers on the selection and use of educational technologies, including criteria for choosing appropriate tools based on content, and guidelines to be followed before, during, and after use.

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Solitary wave solutions of the nonlinear faom equating in mathematical physics

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Abstract

In this work, we apply two technics namely; the projective Riccati equations method and the $\left(\frac{G'}{G}, \frac{1}{G}\right) -$ expansion method to construct new solitons and other different exact wave solutions of the nonlinear faom equation in fluid mechanics. The acquired exact solutions contain, bell, anti-bell kink, anti-kink and periodic wave solutions. A comparison between our newly obtained results and the well-established ones is anticipated. Finally, several plots of the exact solutions derived were generated using Maple.

Keywords: The Nonlinear foam seepage equation. Generalized projective Riccati equations method; exact solutions; Soliton solutions; Periodic solutions, the $\left(\frac{G'}{G}, \frac{1}{G}\right) -$ expansion method. **Introduction**

The inspection of exact solitary wave solutions to nonlinear partial differential equations PDEs plays an main function in the study of nonlinear physical phenomena. Nonlinear waves appears in various scientific fields, especially in physics in the way that fluid mechanics, body tissue physics, optical fibers and atomic science. In current age, many powerful analytic method have existed settled to determine solitary and intermittent wave resolutions of nonlinear PDEs, to a degree the $\left(\frac{G'}{G}\right) -$ expansion method [1-6], the lengthened auxiliary equation method [7,8], the new plan method [9-11], the generalized projective Riccati equations method [12-17], the $\left(\frac{G'}{G}, \frac{1}{G}\right) -$ expansion method [23] thus. Conte and Musette [12] bestowed an indirect method to find solitary wave solutions of few nonlinear PDEs that maybe signified as polynomials in two basic functions that placate a projective Riccati equation [24, 25]. This method has used to many nonlinear PDEs and the solitary wave solutions of these equations maybe in the direction of [13-17]. Recently, Yan [16] has happened likely a generalization of Conte and Musette's method.

The aim of this article is to search these solutions using the generalized projective Riccati equations method [13-17] to find the solitons, and soliton-like solutions of the following two higher-order nonlinear PDEs. The (NLS) equation with one of fourth part-order dispersion and two-fold capacity control nonlinearity[5]

$$-u_t + \frac{1}{2} u u_{xx} + 2u^2 u_x + (u_x)^2 = 0, \qquad (1.1)$$

Where $u = u(x, t)$ is a main nonlinear evolution equation in the direction of the study of the seepage of liquid foams. These systems are of ultimate direct and effective algebraic orders for verdict the exact solutions, the alone wave solutions and the concerning manipulation of numbers function answers of nonlinear PDEs in mathematical physics.

In the paper, our aim into review to find many new resolutions and different exact wave solutions Eq.(1.1). Therefore, this paper is organized in this manner: In Sec. 2, the description of the statement projective Riccati equatings plan. In Sec. 3, writing the $\left(\frac{G'}{G},\frac{1}{G}\right)$ −expansion method. In Sec. 4, we use the $\left(\frac{G'}{G},\frac{1}{G}\right)$ −expansion method for judgment the exact solution of Eqs. (1.1). In Sec. 5, we draw figures for few answers of Eq.(1.1). In Sect. 6, ends are acquired.

2. Description of the generalized projective Riccati equations method

Consider a nonlinear PDE in the form:

$$P(u,u_x,u_t,u_{xx},u_{tt},\dots)=0, \tag{2.1}$$

place $u=u(x,t)$ is an mysterious function, P is a polynomial in $u(x,t)$ and allure incomplete descendants at which point the capital order descendants and nonlinear topmost conditions are complicated. Let us immediately present the main steps of the generalized projective Riccati equations method [13-17]:

Step 1. We use the following wave transformation:

$$u(x,t)=u(\xi), \qquad \xi=kx+\omega t, \tag{2.2}$$

to weaken Eq.(2.1) to the following nonlinear ODE:

$$H(u,u',u'',\dots)=0, \tag{2.3}$$

Where ω is speed of the diffusion, H is a polynomial of u(ξ) and allure total products $u'(\xi),u''(\xi),\dots$ and $'=\frac{d}{d\xi}$.

Step 2. We assume that the solution of Eq.(2.3) has the form:

$$u(\xi)=\alpha_0+\sum_{i=1}^{N}\sigma^{i-1}(\xi)[\alpha_i\sigma(\xi)+\beta_i\tau(\xi)], \tag{2.4}$$

Where α_0,α_i and $\beta_i(i=1,2,\dots,n)$ are constants expected driven. The functions $\sigma(\xi)$ and $\tau(\xi)$satisfy the following ODEs:

$$\sigma'(\xi)=\varepsilon\sigma(\xi)\tau(\xi), \tag{2.5}$$

$$\tau'(\xi)=R+\varepsilon\tau^2(\xi)-\mu\sigma(\xi), \qquad \varepsilon=\pm1, \tag{2.6}$$

Where

$$\tau^2(\xi)=\varepsilon\left(R-2\varepsilon\mu\sigma(\xi)+\frac{\mu^2+r}{R\sigma^2(\xi)}\right), \tag{2.7}$$

attending $r=\pm1$and R,μ are nonzero constants.

If $R=\mu=0$, before Eq.(2.4) has the established solution:

$$u(\xi) = \sum_{i=0}^{N} A_i \tau^i(\xi), \tag{2.8}$$

where $\tau(\xi)$satisfies the nonlinear ODE:

$$\tau'(\xi) = \tau^2(\xi), \tag{2.9}$$

Step 3. The positive integer number N in (2.4) must be decide by utilizing the comparable balance between the highest-order descendants and the highest nonlinear conditions in Eq.(2.4).

Step 4. Substitute (2.4) in addition to Eqs.(2.5)-(2.7) into Eq.(2.3). Collecting all agreements of the same capacity of $\sigma^i(\xi)\tau^i(\xi)$ $(j = 0,1, \dots; i = 0,1)$.. Setting each coefficient to nothing, yields a set of algebraic equations that maybe solved using Maple to find the principles of $\alpha_0, \alpha_i, \beta_i, \omega, \mu, k$ and R.

Step 5. It is famous [13-17] that Eqs.(2.5),(2.6) grants the following solutions:

Case 1. When $\varepsilon = -1, r = -1, R > 0$,

$$\sigma_1(\xi) = \frac{R\,\mathrm{sech}(\sqrt{R}\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\xi) + 1}, \tau_1(\xi) = \frac{\sqrt{R}\tanh(\sqrt{R}\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\xi) + 1}, \tag{2.10}$$

Case 2. When $\varepsilon = -1, r = 1, R > 0$,

$$\sigma_2(\xi) = \frac{R\,\mathrm{csch}(\sqrt{R}\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\xi) + 1}, \tau_2(\xi) = \frac{\sqrt{R}\coth(\sqrt{R}\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\xi) + 1}, \tag{2.11}$$

Case 3. When $\varepsilon = 1, r = -1, R > 0$,

$$\sigma_3(\xi) = \frac{R\sec(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi) + 1}, \tau_3(\xi) = \frac{\sqrt{R}\tan(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi) + 1}, \tag{2.12}$$

$$\sigma_4(\xi) = \frac{R\csc(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi) + 1}, \tau_4(\xi) = \frac{\sqrt{R}\cot(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi) + 1}, \tag{2.13}$$

Case 4. When $\varepsilon = 1, r = 1$,

$$\sigma_5(\xi) = \frac{C}{\xi}, \tau_5(\xi) = \frac{1}{\varepsilon\xi}, \tag{2.14}$$

Case 5. When $R = \mu = 0$,

where $C$is a nonzero constant.

Step 6. Substituting the values of $\sigma_0, \sigma_i, \beta_i, \omega, \mu, k$ and R in addition to the answers (2.10)-(2.14) into (2.4) we get the exact solutions of Eq.(2.1).

3. Description of the $\left(\frac{G'}{G},\frac{1}{G}\right)$ −expansion method.

Before, we detail the main steps concerning this method, we need the following remarks (see[17-22]):

Remark 1. If we consider the second, order linear ODE:

$$G''(\xi)+\lambda(\xi)=\mu, \tag{3.1}$$

and set $\phi=\frac{G'}{G}, \psi=\frac{1}{G}$, before we receive

$$\phi'(\xi)=-\phi^2(\xi)+\mu\psi(\xi)-\lambda, \qquad \psi'(\xi)=-\phi(\xi)\psi(\xi), \tag{3.2}$$

where λ and μ are constants while $'=\frac{d}{d\xi}$.

Remark 2. If $\lambda<0$, then the common solution of Eq.(2.1) has the form:

$$\phi'(\xi)=-\phi^2(\xi)+\mu\psi(\xi)-\lambda, \qquad \psi'(\xi)=-\phi(\xi)\psi(\xi),$$

$$G(\xi)=A_1\sinh(\xi\sqrt{-\lambda})+A_2\cosh(\xi\sqrt{-\lambda})+\frac{\mu}{\lambda}, \tag{3.3}$$

where A_1 and A_2 are arbitrary constants. Consequently, we have

$$\psi^2=\frac{-\lambda}{\lambda^2\sigma_1+\mu^2}(\phi^2-2\mu\psi+\lambda), \tag{3.4}$$

where $\sigma_1=A_1^2-A_2^2$

Remark 3. If $\lambda>0$, therefore the general solution of Eq.(2.1) has the form:

$$G(\xi)=A_1\sin(\xi\sqrt{\lambda})+A_2\cos(\xi\sqrt{\lambda})+\frac{\mu}{\lambda}, \tag{3.5}$$

and therefore

$$\psi^2=\frac{-\lambda}{\lambda^2\sigma_2+\mu^2}(\phi^2-2\mu\psi+\lambda), \tag{3.6}$$

where $\sigma_2=A_1^2+A_2^2$.

Remark 4. If $\lambda=0$, before the general solution of Eq.(2.1) has the form:

$$G(\xi)=\frac{\mu}{2}\xi^2+A_1\xi+A_2, \tag{3.7}$$

and therefore

$$\psi^2(\xi)=\frac{-\lambda}{\lambda^2\sigma_1+\mu^2}(\phi^2(\xi)-2\mu\psi(\xi)+\lambda), \tag{3.8}$$

In the following, we present the main steps of the (G^'/G,1/G)-expansion method [17-22]:

Step 1. Assuming that the solution of Eq.(3.3) maybe articulated by a polynomial in two together variables ϕ and ψ in this manner:

$$u(\xi) = a_0 + \sum_{i=1}^{N} [a_i\phi(\xi) + b_i\psi(\xi)], \tag{3.9}$$

where $a_0, a_i$and $b_i(i = 1,2, \dots, N)$are constants expected driven later satisfying $a_N^2 + a_N^2. \neq 0$.

Step 2. Determine the positive integer number Nin Eq.(3.9) by utilizing the homogeneous balance middle from two points the highest-order derivatives and the maximal nonlinear agreements in Eq.(1.1).

Step 3. Substitute Eq.(3.9) into Eq.(2.3) in addition to (3.2) and (3.4), into the abandoned-help side of Eq.(2.3) maybe convinced into a polynomial in ϕ and ψ, at which point the grade of ψ is not more interminable than one. Equating each coefficients concerning this polynomial to nothing, yields a system of algebraic equations that may be solved by using the Maple or Mathematical to catch the values of $a_i, b_i, \omega, \mu, A_1, A_2$ and λ where $\lambda < 0$.

Step 4. Similar to step 4, substitute Eq.(3.9) into Eq.(2.3) in addition to Eq.(3.2) and Eq.(3.6) for $\lambda > 0$, (or (3.2) and (3.8) for $\lambda = 0$), we get the exact solutions of Eq. (2.3) expressed by trigonometric functions (or by rational functions) respectively.

4. Solitons and added exact wave solutions to the nonlinear foam seepage equation

In this section, we ask the generalized projective Riccati equations method and $\left(\frac{G'}{G}, \frac{1}{G}\right)$ −expansion method described in Sec. 2 to find many new solitons and different exact wave solutions of Eq.(1.1).

In order to resolve Eq.(1.1), we use the wave transformation (2.2) for converting Eq.(1.1) to the following nonlinear ODE:

$$-u'\omega + \frac{1}{2}uu''k^2 + 2u^2u'k + u^2k^2 = 0, \tag{4.1}$$

where k and ω are constants, aforementioned that ω is the velocity of the soliton, ω is the frequency of the soliton, k is the wave, and place $u(\xi) = u(kx + \omega t)$.

4.1. On resolving Eq.(1.1) using the method of section 2.

Balancing uu''accompanying u^2u'in Eq.(4.1), therefore the following relation is attained:

$$N + N + 2 = 2N + N + 1 \implies N = 1, \tag{4.1.1}$$

Since the balance number N is number, therefore we take the formal resolution:

$$u(\xi) = \alpha_0 + \alpha_1\sigma(\xi) + \beta_1\tau(\xi), \tag{4.1.2}$$

where α_0, α_1 and $\beta_1$are constants expected determined specific that $\alpha_0 \neq 0$ or $\beta_1 \neq 0$.Substituting (4.1.2) into Eq.(4.1) and using (2.5)-(2.6), the abandoned-help side of Eq.(4.1) enhances a polynomial in $\sigma(\xi)$ and $\tau(\xi)$.Setting the coefficients of these expected zero yields the following method of algebraic equations:

[illegible] : $\frac{2}{R^2}kr^2$ [illegible] $\frac{1}{R}k^2r$ [illegible] $\frac{2}{R^2}k$ [illegible] $\frac{1}{R^2}k^2r^2$ [illegible] $\frac{1}{R}k^2$ [illegible] $\frac{1}{R^2}k^2$ [illegible] $\frac{4}{R^2}kr$ [illegible]

[illegible] $\frac{6}{R}k$ [illegible] $\frac{2}{R^2}k^2r$ [illegible] $\frac{6}{R}kr$ [illegible] 0,

[illegible] : $2k^2$ [illegible] $2k$ [illegible] $\frac{1}{2}k^2$ [illegible] $\frac{8}{R}k$ [illegible] $12k$ [illegible] $\frac{3}{2R}k^2$ [illegible] $\frac{4}{R}k^2$ [illegible]

[illegible] $\frac{2}{R}k$ [illegible] $\frac{3}{2R}k^2r$ [illegible] $\frac{4}{R}k^2r$ [illegible] $\frac{2}{R}kr$ [illegible] $\frac{1}{R}k^2$ [illegible] $\frac{8}{R}kr$ [illegible] $\frac{1}{R}k^2r$ [illegible]

[illegible] $\frac{8}{R}k$ [illegible] $\frac{8}{R}kr$ [illegible] 0,

[illegible] $\frac{6}{R}k$ [illegible] $2k$ [illegible] $\frac{2}{R}k^2$ [illegible] $\frac{6}{R}kr$ [illegible] $\frac{2}{R}k^2r$ [illegible] 0,

[illegible] : k^2 [illegible] $\frac{1}{2}Rk^2$ [illegible] $4kr$ [illegible] $4k^2$ [illegible] $6k^2$ [illegible] $6k$ [illegible] Rk^2 [illegible] k^2r [illegible]

$2k^2r$ [illegible] $2kr$ [illegible] $2Rk$ [illegible] $12k$ [illegible] $\frac{1}{R}$ [illegible] $\frac{1}{2}k^2$ [illegible] $\frac{1}{R}k^2$ [illegible]

$\frac{1}{R}r$ [illegible] $6Rk$ [illegible] $2k^2$ [illegible] $\frac{1}{R}k^2r$ [illegible] $4k$ [illegible] $16k$ [illegible] $\frac{2}{R}k$ [illegible]

$\frac{2}{R}kr$ [illegible] 0,

[illegible] $4k$ [illegible] $4k$ [illegible] $2k^2$ [illegible] $12k$ [illegible] $4k^2$ [illegible] $\frac{4}{R}k$ [illegible]

$\frac{1}{R}k^2$ [illegible] $\frac{4}{R}kr$ [illegible] $\frac{1}{R}k^2r$ [illegible] 0,

[illegible] $2k^2$ [illegible] $2k$ [illegible] 2 [illegible] $2k^2$ [illegible] $6Rk$ [illegible] $\frac{1}{2}Rk^2$ [illegible] $8Rk$ [illegible]

Rk^2 [illegible] $4Rk$ [illegible] $4k$ [illegible] $\frac{7}{2}Rk^2$ [illegible] $4Rk^2$ [illegible] $8Rk$ [illegible] 0,

[illegible] 2 [illegible] k^2 [illegible] $2R$ [illegible] k^2 [illegible] $\frac{3}{2}$ [illegible] k^2 [illegible] $\frac{3}{2}R$ [illegible] k^2 [illegible] 2 [illegible] k^2 [illegible] 8 [illegible] k [illegible]

6R [illegible] k [illegible] 2 [illegible] k [illegible] 4 [illegible] k [illegible] 4R [illegible] k [illegible] 0,

[illegible] R [illegible] k^2 [illegible] R [illegible] k^2 [illegible] 2 [illegible] k^2 [illegible] 4R [illegible] k [illegible] 4R [illegible] k [illegible] 0,

[illegible] : k^2 [illegible] R [illegible] R [illegible] R^2k^2 [illegible] R^2k^2 [illegible] Rk^2 [illegible] $2R^2k$ [illegible] $2R^2k$ [illegible] 2Rk [illegible]

[illegible] 2Rk [illegible] 0.

According to Step 5, of Section 2, skilled are four cases expected considered in this manner:

Case 1. $(\varepsilon = -1, r = -1)$,we have:

$$\alpha_0 = 0, \alpha_1 = \pm\frac{1}{4}\sqrt{\frac{k(1-\mu^2)}{\omega}}, k = k, R = \frac{4\omega}{k^3}, \beta_1 = \frac{1}{2}k, \qquad (4.1.4)$$

From (2.10), (4.1.2) and (4.1.4), we understand the following exact solutions:

$$u(\xi) = \pm\frac{1}{4}k^2\sqrt{\frac{k(1-\mu^2)}{\omega}}\left(\frac{\frac{4\omega}{k^3}\operatorname{sech}\left(\sqrt{\frac{4\omega}{k^3}}\xi\right)}{\mu\operatorname{sech}\left(\sqrt{\frac{4\omega}{k^3}}\xi\right)+1}\right) + \frac{1}{2}k\left(\frac{\sqrt{\frac{4\omega}{k^3}}\tanh\left(\sqrt{\frac{4\omega}{k^3}}\xi\right)}{\mu\operatorname{sech}\left(\sqrt{\frac{4\omega}{k^3}}\xi\right)+1}\right), \qquad (4.1.5)$$

where $u(\xi) = u(kx + \omega t)$.

Case 2. $(\varepsilon = -1, r = 1)$, we have:

$$\alpha_0 = 0, \alpha_1 = \pm\frac{1}{4}\sqrt{\frac{\mu^2 k + k}{\omega}}, k = k, R = \frac{4\omega}{k^3}, \beta_1 = \frac{1}{2}k, \qquad (4.1.6)$$

From (2.11), (4.1.2) and (4.1.6), we understand the following exact solutions:

$$u(\xi) = \pm\frac{\sqrt{\omega}\left(\sqrt{\mu^2+1}\operatorname{csch}\left(2\sqrt{\frac{\omega}{k^3}}\xi\right) + \coth\left(2\sqrt{\frac{\omega}{k^3}}\xi\right)\right)}{\sqrt{k}\left(\mu\operatorname{csch}\left(\sqrt{\frac{\omega}{k^3}}\xi\right)+1\right)}, \qquad (4.1.7)$$

where $u(\xi) = u(kx + \omega t)$.

Case 3. $(\varepsilon = 1, r = -1)$, leads to the following results:

Result 1. We have From (2.12), (4.1.2) and (4.1.8), we understand the periodic solutions of Eq.(1.1) as follows:

$$u(\xi) = -\frac{\sqrt{\omega}\left(\sqrt{\mu^2-1}\sec\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right) + I\tan\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)\right)}{\sqrt{k}\left(\mu\sec\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)+1\right)}, \quad (4.1.9)$$

And

$$u(\xi) = -\frac{\sqrt{\omega}\left(\sqrt{\mu^2-1}\sec\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right) - I\cot\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)\right)}{\sqrt{k}\left(\mu\csc\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)+1\right)}, \quad (4.1.10)$$

where $u(\xi) = u(kx+\omega t)$.

Result 2. We have the same values, and from (2.13), (4.1.2) and (4.1.8), we deduce the periodic solutions of Eq. (1.1) as follows:

$$u(\xi) = \frac{\omega\sqrt{\frac{k(\mu^2-1)}{\omega}}\sec\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right) - k^2\sqrt{-\frac{\omega}{k^3}}\tan\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)}{k\left(\mu\csc\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)+1\right)}, \quad (4.1.11)$$

and

$$u(\xi) = \frac{\omega\sqrt{\frac{k(\mu^2-1)}{\omega}}\csc\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right) + k^2\sqrt{-\frac{\omega}{k^3}}\cot\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)}{k\left(\mu\csc\left(2\sqrt{-\frac{\omega}{k^3}}\xi\right)+1\right)}, \quad (4.1.12)$$

Remark 1. Note that, if $(R=\mu=0)$ therefore we have the trival solution

Case 4. $(\varepsilon=1, r=1)$, we have:

$$\alpha_0 = 0, \alpha_1 = \pm\frac{1}{4}k^2\sqrt{\frac{k(\mu^2+1)}{\omega}}, k=k, R=-\frac{4\omega}{k^3}, \beta_1 = -\frac{1}{2}k, \quad (4.1.13)$$

From (2.11), (4.1.2) and (4.1.13), we deduce the following exact solutions:

$$u(\xi) = \pm\frac{1}{4\xi}\sqrt{\frac{k(\mu^2+1)}{\omega}}k^2C - \frac{k}{2\omega}, \quad (4.1.14)$$

where $u(\xi) = u(kx+\omega t)$.

4.2 On solving Eq.(1.1) using the arrangement of section 3.

To this aim, adjust $u''u$ with u^2u' in Eq.(4.1) we have $N=1$. Therefore, (3.12) reduces to

$$u(\xi) = a_0 + a_1\phi(\xi) + b_1\psi(\xi), \tag{4.2.1}$$

where $a_i(i = 0,1)$, b_1 are constants expected determined aforementioned that $a_2 \neq 0$ or $b_2 \neq 0$. Nowe, we have two cases to solutions expected conferred as follows:

According to Step 5, to be discussed as follows:

Case 1. If $\lambda < 0$, substituting (4.2.1) and using (3.2)-(3.4) and Eq.(3.3) into Eq.(4.1), the left-hand side of Eq.(4.1) becomes a polynomial in $\sigma(\xi)$ and $\tau(\xi)$. Setting the coefficients concerning this polynomial expected zero yields the following system of algebraic equations:

[illegible] : $k^2a_1^2 - 2ka_1^3 - [illegible]\frac{k^2b_1^2 - 6ka_1b_1^2}{[illegible]} = 0,$

[illegible] : $-4ka_0a_1^2 + k^2a_0a_1 + \frac{[illegible]}{[illegible]}\left([illegible]k^2a_1b_1 - 4[illegible]ka_1^2b_1 + 4a_0kb_1^2\right) = 0,$

[illegible]: $2k^2a_1b_1 - 6ka_1^2b_1 + 2k[illegible]\frac{b_1^3}{[illegible]} = 0,$

[illegible] : $k^2a_1^2 + [illegible]a_1 - 2ka_0^2a_1 + k^2[illegible]a_1^2 - \frac{[illegible]}{[illegible]}\left([illegible]^2b_1^2 + [illegible]^2b_1^2 - 6ka_1b_1^2\right) + \frac{1}{2}k^2[illegible]b_1^2 - 2k[illegible]a_1b_1^2 -$

$\frac{1}{2}k^2[illegible]a_0b_1 + 4k[illegible]a_0a_1b_1\big) - 2k[illegible]a_1^3 = 0,$

[illegible]: $k^2a_0b_1 + [illegible]\frac{\frac{1}{2}k^2[illegible]b_1^2 - 2k[illegible]a_1b_1^2}{[illegible]} - \frac{3}{2}k^2[illegible]a_1^2 + 2k[illegible]a_1^3 - 8ka_0a_1b_1 + 2[illegible]\frac{k^2b_1^2 - 6ka_1b_1^2}{[illegible]} = 0,$

[illegible]: $2k^2a_0a_1 + \frac{[illegible]}{[illegible]}\left([illegible]k^2a_1b_1 - 4[illegible]ka_1^2b_1 + 4a_0kb_1^2\right) - 4k[illegible]a_0a_1^2 + k^2[illegible]a_0a_1 - 4k[illegible]\frac{b_1^3}{[illegible]} =$

$0,$

[illegible]: $[illegible]b_1 - 2ka_0^2b_1 + 2k^2a_1b_1 + 4k[illegible]a_0a_1^2 - \frac{3}{2}k^2[illegible]a_0a_1 - 4k[illegible]a_1^2b_1 + \frac{3}{2}k^2[illegible]a_1b_1 + 2k[illegible]\frac{b_1^3}{[illegible]} -$

$2[illegible]\frac{[illegible]}{[illegible]}\left([illegible]k^2a_1b_1 - 4[illegible]ka_1^2b_1 + 4a_0kb_1^2\right) = 0,$

[illegible]: $2k^2a_0b_1 + [illegible]\frac{\frac{1}{2}k^2[illegible]b_1^2 - 2k[illegible]a_1b_1^2}{[illegible]} - [illegible]a_1 + 2k[illegible]a_0^2a_1 + \frac{1}{2}k^2[illegible]a_0b_1 + 2[illegible]\frac{[illegible]}{[illegible]}$

$\left(k^2b_1^2 + \frac{1}{2}k^2[illegible]b_1^2 - 2k[illegible]a_1b_1^2 - \frac{1}{2}k^2[illegible]a_0b_1 + 4k[illegible]a_0a_1b_1\right) - 4k[illegible]a_0a_1b_1 = 0,$

[illegible] : $k^2a_0^2 - \frac{[illegible]}{[illegible]}\left(k^2b_1^2 + \frac{1}{2}k^2[illegible]b_1^2 - 2k[illegible]a_1b_1^2 - \frac{1}{2}k^2[illegible]a_0b_1 + 4k[illegible]a_0a_1b_1\right) + [illegible]a_1 - 2k[illegible]a_0^2a_1 -$

$2[illegible]\frac{\frac{1}{2}k^2[illegible]b_1^2 - 2k[illegible]a_1b_1^2}{[illegible]} = 0,$

We take a_0, a_1, b_1, μ and ω using Maple:

$$a_0 = 0, a_1 = \frac{1}{2}k, b_1 = \pm\frac{1}{2}k\sqrt{\frac{-\lambda^2\sigma_1 - \mu^2}{\lambda}}, \mu = \mu, \omega = -\frac{1}{4}k^3\lambda, \tag{4.2.3}$$

From (3.3), (3.4), (4.2.1) and (4.2.3), we deduce the soliton solution of Eq.(1.1) as follows:

$$u(\xi) = \frac{1}{2}\frac{k\sqrt{-\lambda}\left(A_1\cosh\left(\sqrt{-\lambda}\xi\right) + A_2\sinh\left(\sqrt{-\lambda}\xi\right)\right)}{A_1\sinh\left(\sqrt{-\lambda}\xi\right) + A_2\cosh\left(\sqrt{-\lambda}\xi\right)} + \frac{1}{2}\frac{k\sqrt{\frac{\lambda^2\sigma_1 + \mu^2}{\lambda}}}{A_1\sinh\left(\sqrt{-\lambda}\xi\right) + A_2\cosh\left(\sqrt{-\lambda}\xi\right)}, \tag{4.2.4}$$

Where $\sigma_1 = A_1^2 - A_1^2$..

Result 1. In particular, by sitting $(A_1 = 0, A_2 \neq 0, \mu = 0)$ in Eq.(4.2.4), we receive:

$$u(\xi) = \frac{1}{2}k\left[\sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\xi\right) \pm \frac{I\sqrt{\lambda A_2^2}\,\mathrm{sech}\left(\sqrt{-\lambda}\xi\right)}{A_2}\right], \tag{4.2.5}$$

Result 2. While, if $(A_1 \neq 0, A_2 = 0, \mu = 0)$, we have that that solution:

$$u(\xi) = \frac{1}{2}k\left[\sqrt{-\lambda}\coth\left(\sqrt{-\lambda}\xi\right) \pm \frac{I\sqrt{A_1^2}\csc\left(\sqrt{-\lambda}\xi\right)}{A_1}\right], \tag{4.2.6}$$

Case 2. If $\lambda > 0$, we have substituting (4.2.1) and using (3.2)-(3.4) and Eq.(3.3) into Eq.(4.1), the left-hand side of Eq.(4.1) enhances a polynomial in (ξ) and $\tau(\xi)$. Setting the coefficients concerning this polynomial expected zero yields the following system of algebraic equations:

[illegible] : $k^2a_1^2$ [illegible] $2ka_1^3$ [illegible] $\frac{k^2b_1^2 \text{[illegible]} 6ka_1b_1^2}{\text{[illegible]}}$ [illegible] 0,

[illegible] : [illegible] $4ka_0a_1^2$ [illegible] $k^2a_0a_1$ [illegible] $k^2a_1b_1$ [illegible] 4 [illegible] $ka_1^2b_1$ [illegible] $4a_0kb_1^2$ [illegible] 0,

[illegible] : $2k^2a_1b_1$ [illegible] $6ka_1^2b_1$ [illegible] $2k$ [illegible] $\frac{b_1^3}{\text{[illegible]}}$ [illegible] 0,

[illegible] : $k^2a_1^2$ [illegible] a_1 [illegible] $2ka_0^2a_1$ [illegible] k^2 [illegible] a_1^2 [illegible] $^2b_1^2$ [illegible] $^2b_1^2$ [illegible] $6ka_1b_1^2$ [illegible] $\frac{1}{2}k^2$ [illegible] b_1^2 [illegible] $2k$ [illegible] $a_1b_1^2$ [illegible]

$\frac{1}{2}k^2$ [illegible] a_0b_1 [illegible] $4k$ [illegible] $a_0a_1b_1$ [illegible] $2k$ [illegible] a_1^3 [illegible] 0,

$: k^2 a_0 b_1 \quad \frac{\frac{1}{2}k^2 b_1^2 \quad 2k a_1 b_1^2}{} \quad \frac{3}{2}k^2 a_1^2 \quad 2k a_1^3 \quad 8ka_0 a_1 b_1 \quad 2 \frac{k^2 b_1^2 \quad 6ka_1 b_1^2}{} \quad 0,$

$: \quad 2k^2 a_0 a_1 \quad \frac{}{} \quad k^2 a_1 b_1 \quad 4 ka_1^2 b_1 \quad 4a_0 kb_1^2 \quad 4k a_0 a_1^2 \quad k^2 a_0 a_1 \quad 4k \quad \frac{b_1^3}{}$

$0,$

$: \quad b_1 \quad 2ka_0^2 b_1 \quad 2k^2 a_1 b_1 \quad 4k a_0 a_1^2 \quad \frac{3}{2}k^2 a_0 a_1 \quad 4k a_1^2 b_1 \quad \frac{3}{2}k^2 a_1 b_1 \quad 2k \frac{b_1^3}{}$

$2 \frac{}{} \quad k^2 a_1 b_1 \quad 4 ka_1^2 b_1 \quad 4a_0 kb_1^2 \quad 0,$

$: 2k^2 a_0 b_1 \quad \frac{\frac{1}{2}k^2 b_1^2 \quad 2k a_1 b_1^2}{} \quad a_1 \quad 2k a_0^2 a_1 \quad \frac{1}{2}k^2 a_0 b_1$

$2 \frac{}{}\left(k^2 b_1^2 \quad \frac{1}{2}k^2 b_1^2 \quad 2k a_1 b_1^2 \quad \frac{1}{2}k^2 a_0 b_1 \quad 4k a_0 a_1 b_1\right) \quad 4k a_0 a_1 b_1 \quad 0,$

$: k^2 a_0^2 \quad \frac{}{}\left(k^2 b_1^2 \quad \frac{1}{2}k^2 b_1^2 \quad 2k a_1 b_1^2 \quad \frac{1}{2}k^2 a_0 b_1 \quad 4k a_0 a_1 b_1\right) \quad a_1 \quad 2k a_0^2 a_1$

$2 \quad \frac{\frac{1}{2}k^2 b_1^2 \quad 2k a_1 b_1^2}{} \quad 0.$

We take a_0, a_1, b_1, μ and ω using Maple:

$$a_0 = 0, a_1 = \frac{1}{2}k, b_1 = \pm\frac{1}{2}k\sqrt{\frac{-\lambda^2\sigma_2 - \mu^2}{\lambda}}, \mu = \mu, \omega = -\frac{1}{4}k^3\lambda, \tag{4.2.8}$$

From (3.5), (3.6), (4.2.1) and (4.2.8), we deduce the soliton solution of Eq.(1.1) as leads to the attends:

$$u(\xi) = \frac{k}{2}\left(\frac{\sqrt{\lambda}\left(A_1\cos(\sqrt{\lambda}\xi) - A_2\sin(\sqrt{\lambda}\xi)\right)}{A_1\sin(\sqrt{\lambda}\xi) + A_2\cos(\sqrt{\lambda}\xi)} + \frac{\sqrt{\frac{\lambda^2\sigma_2 - \mu^2}{\lambda}}}{A_1\sin(\sqrt{\lambda}\xi) + A_2\cos(\sqrt{\lambda}\xi)}\right), \tag{4.2.9}$$

Where $\sigma_1 = A_1^2 + A_1^2$.

Result 1. Putting $(A_1 = 0, A_2 \neq 0, \mu = 0)$, we receive:

$$u(\xi) = \frac{1}{2}k\sqrt{\lambda}\left[-A_2\tan(\sqrt{\lambda}\xi) \pm \frac{I\sqrt{A_2^2}\sec(\sqrt{\lambda}\xi)}{A_2}\right], \tag{4.2.10}$$

Result 2. Putting $(A_1 \neq 0, A_2 = 0, \mu = 0)$, In this result we receive:

$$u(\xi) = \frac{1}{2} k\sqrt{\lambda}\left[A_1 \cot(\sqrt{\lambda}\xi) \pm \frac{I\sqrt{A_1^2}\csc(\sqrt{\lambda}\xi)}{A_1}\right], \qquad (4.2.11)$$

Case 3. If $\lambda = 0$,, leads to the following results:

$$a_0 = 0, a_1 = a_1, b_1 = b_1, A_1 = A_1, A_2 = -\frac{1}{2}\frac{-A_1^2 a_1^2 + b_1^2}{\mu A_1^2}, \mu = \mu, \omega = 0, k = 2a_1, \quad (4.2.12)$$

Therefore, we have the trivial solution.

5. Graphical likenesses of few solutions

In this portion, we draw graphs of few exact solutions. The got soliton and periodic solutions are kink and anti-kink solitons, bell (bright) and anti-bell (dark) unsociable wave solutions and trigonometric solutions. Let us immediately try Figs. 1-3 as it represents few of our solutions got in this place item. To this aim, we select some special values of the parameters acquired, for example, in some of the resolutions (4.1.5), (4.1.7), (4.1.9), (4.1.10), (4.1.11), (4.1.12),(4.2.4) and (4.2.9) of the NLS equation with fourth-order dispersion and two-fold capacity society nonlinearity (1.1). For more convenience, the graphical likenesses of these solutions are proved in the following:

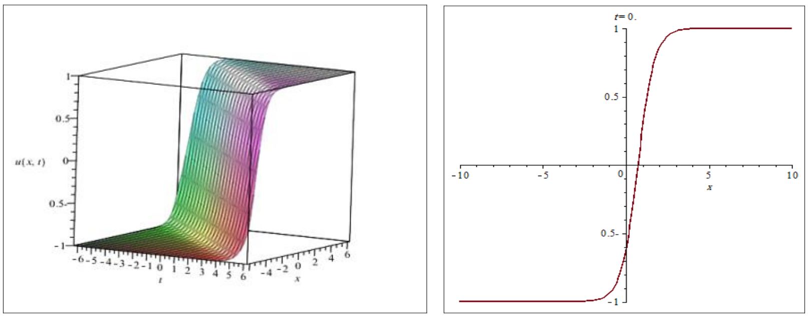

Fig. 1. Solution $|u(x,t)|$ of (4.1.5) with $k = 1, \omega = 1, \mu = 2$.

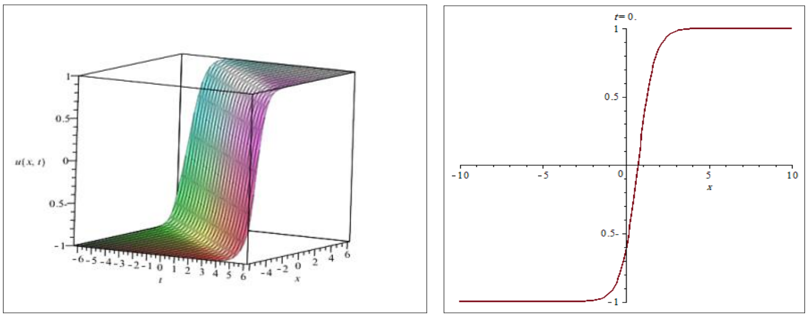

Fig. 2. Solution $|u(x,t)|$ of (4.1.7) with $k = 1, \omega = 1, \mu = 2$.

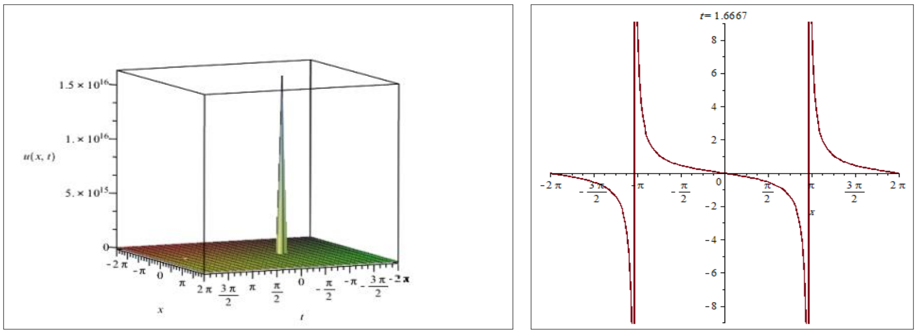

Fig. 3. Solution $|u(x,t)|$ of (4.2.5) with $k = 1, \omega = 1, \lambda = 1, \mu = 0, A_1 = 0, A_2 = 2$.

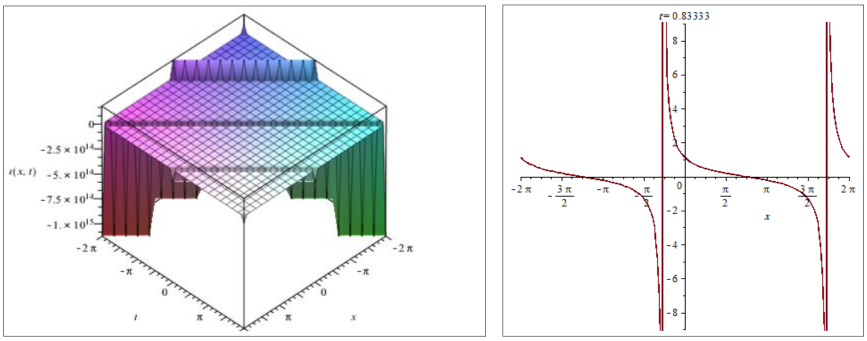

Fig.4. Solution $|u(x,t)|$ of (4.2.6) (the plus signal) with

$$k = 1, \omega = 1, \lambda = 1, \mu = 0, A_1 = 2, A_2 = 0.$$

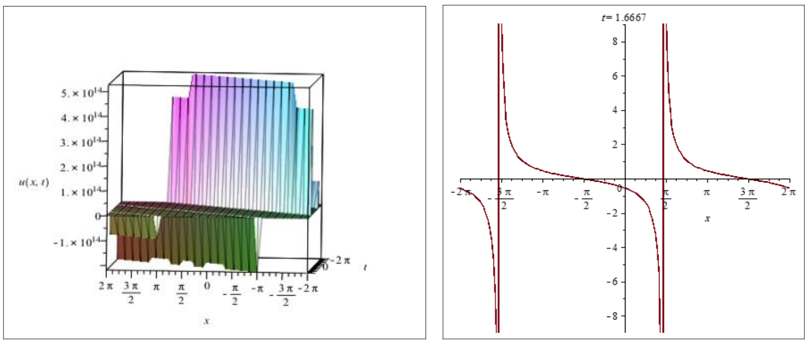

Fig. 5. Solution $|u(x,t)|$ of (4.2.6) (the mines signal) with

$$k = 1, \omega = 1, \lambda = 1, \mu = 0, A_1 = 2, A_2 = 0.$$

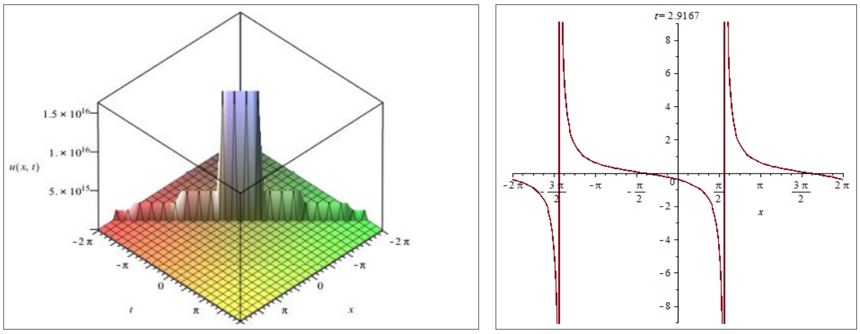

Fig. 6. Plot solution $|u(x,t)|$ of (4.2.10) (the plus signal) with

$$k = 1, \omega = 1, \lambda = 1, \mu = 0, A_1 = 0, A_2 = 2.$$

From the presented figures, it can be observed that the obtained solutions exhibit both solitary wave solutions and periodic wave solutions. Moreover, these figures illustrate the behavior of the solutions, providing readers with insights into how the behavior of these solutions is generated.

6. Conclusions

In this study, we apply two technics namely; the projective Riccati equations method and another method is $\left(\frac{G'}{G}, \frac{1}{G}\right)$ −expansion to derive exact wave solutions for the nonlinear foam drainage equation. Our findings are novel and have not been previously published. Moreover, the methods applied in this research have been demonstrated to be powerful and effective tools for determining traveling wave and solitary wave solutions of many other nonlinear equations. In section 5 we present several figures illustrating solution behavior, providing readers with insight into the dynamic characteristics of these solutions. Finally, the accuracy of our results was verified using Maple by reintroducing them into the original Equation.

Conflict of Interests

The authors reveal that skilled is no conflict of interests regarding the publication of this paper.

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Compute the Reliability of Human DNA in Forensic Science

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Abstract: Reliability in the context of DNA research refers to the consistency and accuracy of results obtained from various methodologies used in genetic analysis. As DNA technology advances,ensuring the reliability of te: chniques such as PCR, sequencing, and genotyping becomes paramount for applications in medicine, forensics, and evolutionary biology. Factors influencing reliability include sample quality, contamination, and the precision of analytical tools. Reliable DNA analysis is crucial for accurate diagnosis of genetic disorders, identification of individuals in forensic cases, and understanding genetic diversity in populations. This abstract highlights the importance of robust protocols and quality control measures to enhance the reliability of DNA-based studies, ultimately supporting scientific advancements and informed decision-making in health and environmental contexts.

Keywords: Reliability, minimal path, Series System, Parallel system .

1. Introduction

Reliability in DNA analysis is a cornerstone of modern genetics and biotechnology, playing a critical role in various fields such as medicine, forensics, and ancestry research. As our understanding of DNA and its implications for human health continues to evolve, the need for reliable methods to analyze genetic material becomes increasingly essential.[1]

DNA, or deoxyribonucleic acid, carries the genetic instructions necessary for the growth, development, and functioning of living organisms[2]. In humans, DNA analysis can reveal vital information about genetic predispositions to diseases, hereditary conditions, and individual traits[3]. However, the reliability of these analyses is influenced by numerous factors, including sample quality, the techniques employed for extraction and amplification, and the interpretation of the data (Zhang et al., 2021; Jorde et al., 2022) [4].

In clinical settings, reliable DNA testing is crucial for accurate diagnoses and personalized treatment plans. Next-generation sequencing (NGS) technologies have revolutionized the ability to detect genetic variants associated with specific diseases, allowing for tailpeutic approaches (Wang et al., 2023)[5]. In forensic science, DNA profiling is essential for the correct identification of individuals in criminal investigations, where the stakes are high, and the potential for wrongful convictions exists (Kloosterman et al., 2020). Furthermore, in genealogical research, the reliability of DNA results can significantly impact personal and familial histories, influencing how individuals perceive their ancestry (Simmons et al., 2023)[6].

Establishing robust protocols and stringent quality control measures is necessary to mitigate errors and ensure the integrity of DNA analyses. This introduction sets the stage for a deeper exploration of the methodologies, challenges, and advancements in ensuring the reliability of DNA analysis in human applications, underscoring its significance in advancing our understanding of genetics and improving health outcomes[7].

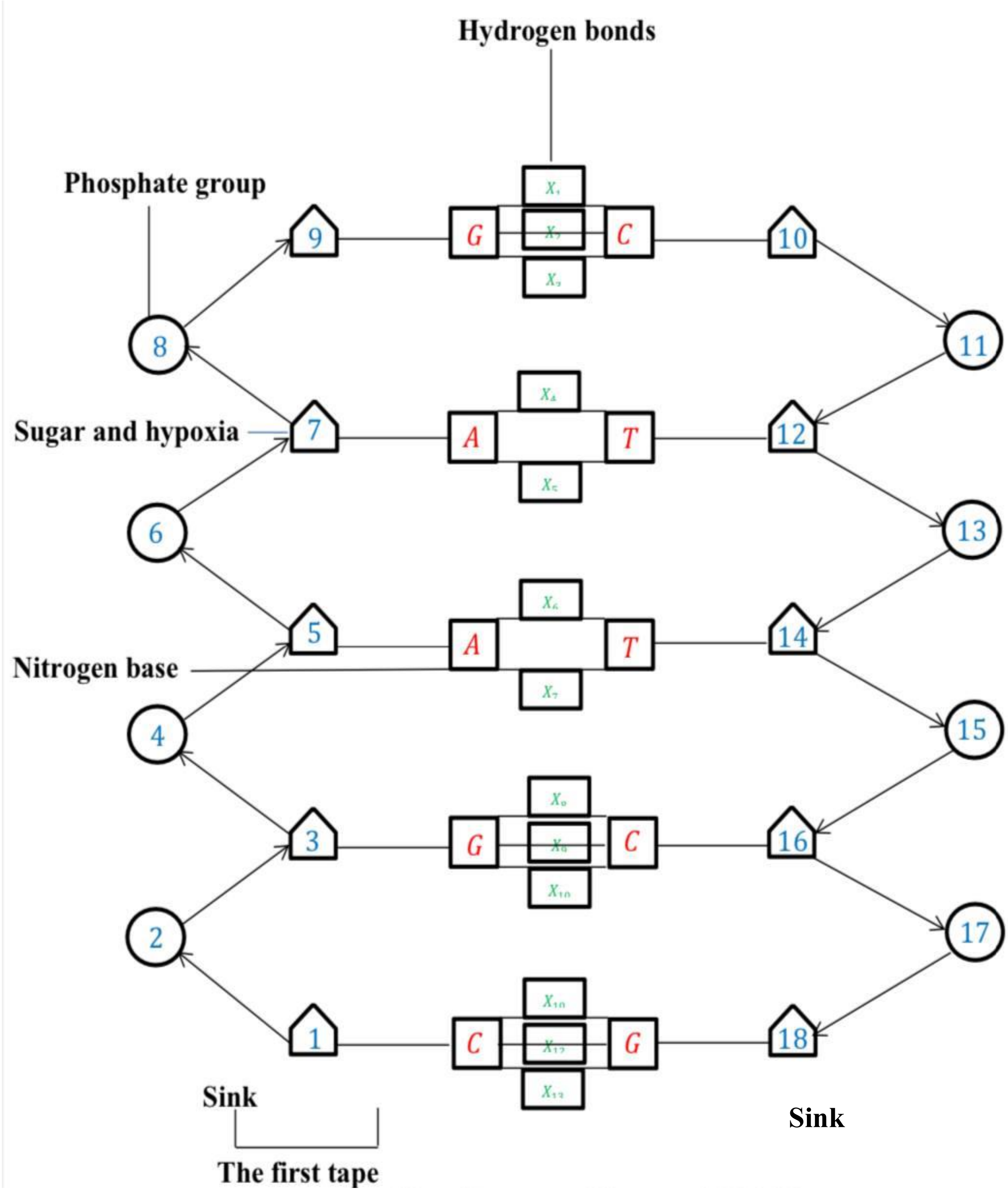

Figure 1: (Dna Structure Diagram ((DNA))

2. Basic Concept

1. Reliability system is a probability that the system will work for aperiodic time.

2. A minimal path is a collection of elements that make up a path; however, if any element is removed, the remaining set ceases to be a path.

3. Parallel system reliability: When a system requires the success of at least one component, it is referred to as a parallel system in order to determine the reliability

$$R_{sys} = 1 - \prod_{i=1}^{n} (1 - R_i) \qquad (1).$$

4. Series System

The reliability of a series system is the probability that component succeeds and component succeeds and all of the other components in the system succeed and it compute

$$R_{sys} = \prod_{i=1}^{n} R_i \qquad (2)$$

5. Series-parallel system

It is a system that consists of different subsystems connected in series. Each subsystem consists of components connected in parallel as shown in Figure.1.11. The reliability of the system is

$$R_S = \prod_{k=1}^{m} \left(1 - \prod_{i=1}^{n_k} (1 - R_{ik})\right) \qquad (3)\ [8,9].$$

3. Methods

In this method, a connected matrix is constructed to find a minimal path.

In two–terminal network, the edges are prone to failure and the nodes are perfect. In this method a connection matrix is need to be constructed to create a minimal path. We will combine n×n adjacency matrix of a simple graph with the identity matrix as the following

$$CM= \text{From node}\ \begin{matrix}1\\2\\\vdots\\n\end{matrix}\overset{\begin{matrix}To\,node\\1\quad 2\quad \cdots\quad n\end{matrix}}{\begin{bmatrix}0 & a_{12} & \cdots & a_{1n}\\ a_{21} & 0 & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{2n} & \cdots & 0\end{bmatrix}} + \begin{matrix}1\\2\\\vdots\\n\end{matrix}\overset{1\quad 2\quad \cdots\quad n}{\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix}}$$

the result after combining is combination

$$CM= \text{From node}\ \begin{matrix}1\\2\\\vdots\\n\end{matrix}\overset{\begin{matrix}To\,node\\1\quad 2\quad \cdots\quad n\end{matrix}}{\begin{bmatrix}1 & a_{12} & \cdots & a_{1n}\\ a_{21} & 1 & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{2n} & \cdots & 1\end{bmatrix}}$$

Where {1, 2, …n}, the set of nodes/sphere and a_{ij}= (i, j) the edge between node i and j. If there is a connection between node i and node j then aij=$x_{ij,}$ otherwise a_{ij}=0, therefore removes nodes that are neither the source nor the

sink from CM, one by one until the only nodes left in matrix are the source node and the sink node [10], [11], [12], [13].

When a node is removed, the entries of the connection matrix with the remaining nodes are modified using the equation:

$$a_{ij}^{1} = a_{ij} + a_{il}a_{lj} \qquad (4)$$

If node l is removed, where i≠j, i≠l, j≠l, 1≤ i < n, 1< j ≤ n for i=1, 2,…, n.

Compute the reliability of DNA is NP-Hard , as it must go through several stages. The first stage is to rely on the figure(2) andfiguare(3) based on equations(1) and (2).After that, we apply the minimal path method to find the reliability of DNA. As explained later,

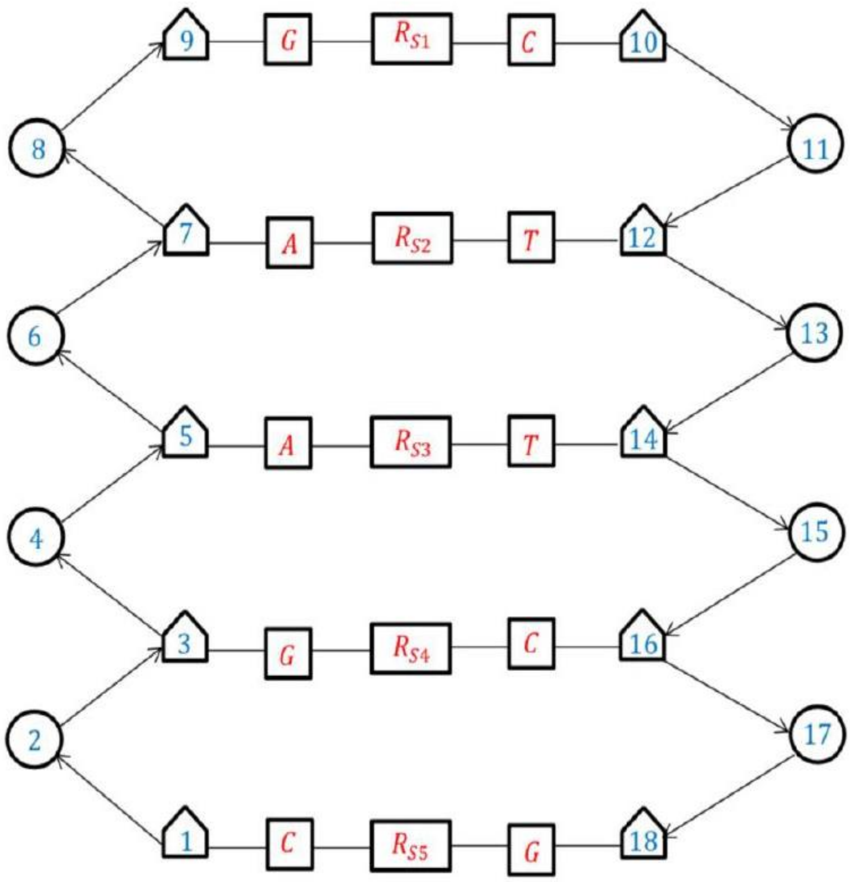

Figure 2

After applying equations(1) and (2) in figure (2), the reliability of figure(2) was calculated as

$R_{s1} = (1 - R_1)(1 - R_2)(1 - R_3)$

$R_{s2} = (1 - R_4)(1 - R_5)$

$R_{s3} = (1 - R_6)(1 - R_7)$

$R_{s4} = (1 - R_8)(1 - R_9)(1 - R_{10})$

$R_{s5} = (1 - R_{11})(1 - R_{12})(1 - R_{13})$ (5)

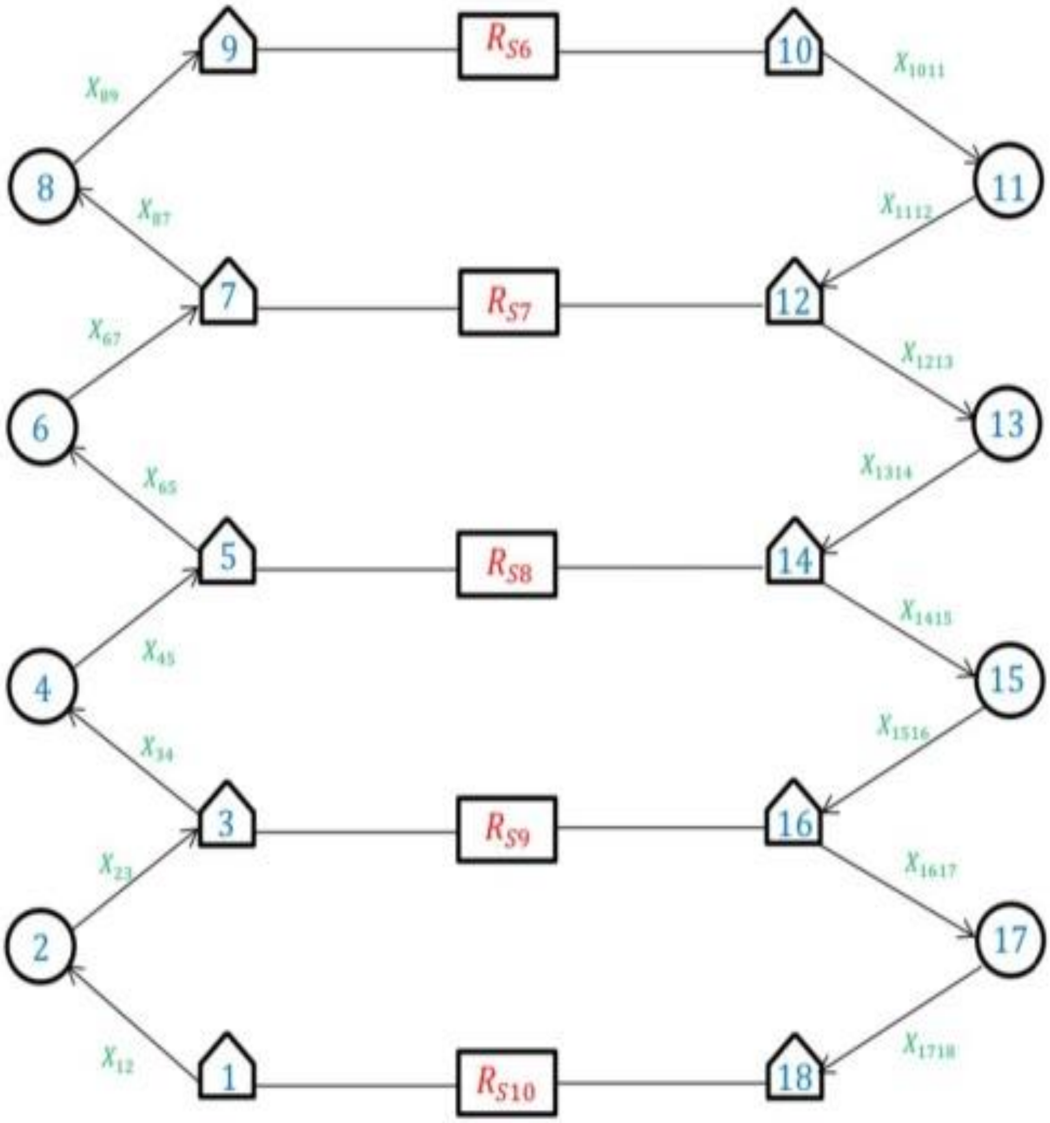

Figure 3

In the same way as calculating figure (2) and based on equations(1)and (2), we obtain the reliability of figure(3) as,

$R_{s6} = R_G R_s R_c$

$R_{s7} = R_A R_{s2} R_T$

$R_{s8} = R_A R_{s3} R_T$

$R_{s9} = R_G R_{s4} R_c$

$R_{s10} = R_c R_{c5} R_G$ (6)

At this stage, after calculating the reliability of figure(2) and (3), the minimal path method is applied and employed to find reliability.so,

4. Results

4.1 Evaluation of The Reliability of Human DNA

After applying eq.4 , get a connected matrix of DNA and all the minimal path , as shown

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	X_{12}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	R_{S10}
2	0	1	X_{23}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	1	X_{34}	0	0	0	0	0	0	0	0	0	0	0	R_{S9}	0	0
4	0	0	0	1	X_{45}	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	1	X_{56}	0	0	0	0	0	0	0	R_{S8}	0	0	0	0
6	0	0	0	0	0	1	X_{67}	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	1	X_{78}	0	0	0	R_{S7}	0	0	0	0	0	0
8	0	0	0	0	0	0	0	1	X_{89}	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	1	R_{S6}	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	R_{S6}	1	$\boldsymbol{X_{10}}$	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	1	X_{1112}	0	0	0	0	0	0
12	0	0	0	0	0	0	R_{S7}	0	0	0	0	1	X_{121}	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	1	X_{1314}	0	0	0	0
14	0	0	0	0	R_{S8}	0	0	0	0	0	0	0	0	1	X_{1415}	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	X_{151}	0	0

16	0	0	R_{S9}	0	0	0	0	0	0	0	0	0	0	0	0	1	X_{1617}	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	X_{1718}
18	R_{S10}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

$P_1 = R_{S10}$

$P_2 = X_{12}\, X_{23}\, R_{S9}\; X_{1617}\; X_{1718}$

$P_3 = X_{12}\, X_{23}\, X_{34}\, X_{45}\, R_{S8}\, X_{1415}\; X_{1516}\, X_{1617}\, X_{1718}$

$P_4 = X_{12}\, X_{23}\, X_{34}\, X_{45}\, X_{56}\, X_{67}\, R_{S7}\, X_{1213}\, X_{1314}\, X_{1415}\, X_{1516}\, X_{1617}\, X_{1718}$

$P_5 = X_{12}\, X_{23}\, X_{34}\, X_{45}\, X_{56}\, X_{67}\, X_{78}\, X_{89}\, R_{S6}\, X_{1011}\, X_{1112}\, X_{1213}\, X_{1314}\, X_{1415}$

$X_{1516}\, X_{1617}\, X_{1718}$

P1,p2, …,p5 repersent all the minimal path of DNA, by applaying eq.3, gets

$$R_{SYS} = 1 - (1 - P_1)(1 - P_2)(1 - P_3)(1 - P_4)(1 - P_5)$$

$$Rsys\; = R10\; +\; R12 * R23 * R1617 * R1718 * RC * RG\; -\; R8 * R12 * R23 * R1617 * R1718$$

$$* RC * RG\; - R9 * R12 * R23 * R1617 * R1718 * RC * RG\; -\; R10 * R12 * R23 * R1617 * R1718$$

$$* RC * RG\; + R8 * R9 * R12 * R23 * R1617 * R1718 * RC * RG\; +\; R8 * R10 * R12 * R23 * R1617$$

$* R1718 * RC * RG + R9 * R10 * R12 * R23 * R1617 * R1718 * RC * RG - R8 * R9$
$* R10 * R12$

$* R23 * R1617 * R1718 * RC * RG + R12 * R23 * R34 * R45 * R1415 * R1516 * R1617$
$* R1718$

$* RA * RT - R7 * R12 * R23 * R34 * R45 * R1415 * R1516 * R1617 * R1718 * RA * RT$
$+ R6$

$* R7 * R12 * R23 * R34 * R45 * R1415 * R1516 * R1617 * R1718 * RA * RT + R12$
$* R23 * R34$

$* R45 * R56 * R67 * R1213 * R1314 * R1415 * R1516 * R1617 * R1718 * RA * RT - R4$
$* R12$

$* R23 * R34 * R45 * R56 * R67 * R1213 * R1314 * R1415 * R1516 * R1617 * R1718$
$* RA * RT$

$-R5 * R12 * R23 * R34 * R45 * R56 * R67 * R1213 * R1314 * R1415 * R1516 * R1617$
$* R1718$

$* RA * RT + R4 * R5 * R12 * R23 * R34 * R45 * R56 * R67 * R1213 * R1314 * R1415$
$* R1516$

$* R1617 * R1718 * RA * RT + R12 * R23 * R34 * R45 * R56 * R67 * R78 * R89 * R1011$

$* R1112 * R1213 * R1314 * R1415 * R1516 * R1617 * R1718 * RG * RT - R1 * R12$
$* R23$

$* R34 * R45 * R56 * R67 * R78 * R89 * R1011 * R1112 * R1213 * R1314 * R1415$
$* R1516$

$* R1617 * R1718 * RG * RT - R2 * R12 * R23 * R34 * R45 * R56 * R67 * R78 * R89$
$* R1011$

$* R1112 * R1213 * R1314 * R1415 * R1516 * R1617 * R1718 * RG * RT - R3 * R12$
$* R23$

$* R34 * R45 * R56 * R67 * R78 * R89 * R1011 * R1112 * R1213 * R1314 * R1415$
$* R1516$

$$* R1617 * R1718 * RG * RT + R1 * R2 * R12 * R23 * R34 * R45 * R56 * R67 * R78 * R89$$

$$* R1011 * R1112 * R1213 * R1314 * R1415 * R1516 * R1617 * R1718 * RG * RT + R1 * R3$$

$$* R12 * R23 * R34 * R45 * R56 * R67 * R78 * R89 * R1011 * R1112 * R1213 * R1314 * R1415$$

$$* R1516 * R1617 * R1718 * RG * RT + R2 * R3 * R12 * R23 * R34 * R45 * R56 * R67 * R78$$

$$* R89 * R1011 * R1112 * R1213 * R1314 * R1415 * R1516 * R1617 * R1718 * RG * RT - R1$$

$$* R2 * R3 * R12 * R23 * R34 * R45 * R56 * R67 * R78 * R89 * R1011 * R1112 * R1213 * R1314$$

$$* R1415 * R1516 * R1617 * R1718 * RG * RT \qquad (7).$$

5. Conclusion

The reliability of DNA analysis is paramount across various fields, including medicine, forensics, and ancestry research. As advancements in technology continue to enhance our ability to analyze genetic material, ensuring the accuracy and consistency of these methods remains a critical challenge. Factors such as sample integrity, methodological precision, and data interpretation significantly influence the reliability of results.

Robust protocols and stringent quality control measures are essential to mitigate errors and enhance the confidence in DNA analyses. This reliability is crucial not only for accurate medical diagnoses and treatments but also for the integrity of forensic investigations and the meaningfulness of genealogical research. As the landscape of genetic research evolves, ongoing efforts to improve the reliability of DNA analysis will support scientific advancements, foster public trust, and ultimately contribute to the betterment of human health and understanding of genetic diversity.

So, in this paper, the reliability of DNA was calculated, which used to find the probability densety function of DNA . The reliability of the network shows that each part of the DNA is a coherent and strong system.

Acknowledgment

I extend my thanks and appreciation to Dr. Wissam Kamil and Haseneen Abdul Hady for providing assistance and valuable information.

Algorthem(1):

```
syms R1 R2 R3 R4 R5 R6 R7 R8 R9 R10 R11 R12 R13 RG RC RA RT R1617 R1718

syms  R23 R34 R45 RS8 R1415 R1516 R56 R67 RS7 R1314 R23 R78 R89 RS6 R1011 R1112 R1213

RS1=(1-R1)*(1-R2)*(1-R3);

RS2=(1-R4)*(1-R5);

RS3=(1-R6)*(1-R7);

RS4=(1-R8)*(1-R9)*(1-R10);

RS5=(1-R11)*(1-R12)*(1-R13);

RS6=RG*RS1*RT;

RS7=RA*RS2*RT;

RS8=RA*RS3*RT;

RS9=RG*RS4*RC;

RS10=RC*RS5*RG;

RS1=expand(RS1) ;

RS2=expand(RS2) ;

RS3=expand(RS3);

RS4=expand(RS4);

RS5=expand(RS5);

RS6=expand(RS6);
```

```
RS7=expand(RS7);
f1=R10;
f2=R12*R23*RS9*R1617*R1718;
f3=R12*R23*R34*R45*RS8*R1415*R1516*R1617*R1718;
f4=R12*R23*R34*R45*R56*R67*RS7*R1213*R1314*R1415*R1516*R1617*R17
18;
f5=R12*R23*R34*R45*R56*R67*R78*R89*RS6*R1011*R1112*R1213*R1314*
R1415*R1516*R1617*R1718;
Rsys=f1+f2+f3+f4+f5;
K=expand(Rsys)
```

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Computer Scope

Computer, Intelligent Application

Table Content

Assistive Technology Management System for Children with Autism Using Digital Board for Communication

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Abstract

Most children with autism spectrum disorder face many difficulties in verbal communication, correct pronunciation of letters, as well as communication and social interaction. Therefore, the use of alternative means that help them communicate has become a necessity, such as the digital board designed as it depends on pictures, pronunciation, sounds, and the formation of complete, understandable phrases then it is designed according to the child's needs and features can be added to it as needed, which enhances the child's confidence, the development of his thinking and independence, and reduces the embarrassment of the difficulty of expression.

The research aims to study the extent to which the use of digital boards affects the improvement of pronunciation in children with autism spectrum disorder.

Keywords: d*igital board, autism spectrum disorder, communications, pronunciation, phrases***.**

1. Introduction

Digital boards create an environment for communication and are also interactive, using images, symbols, sounds, pictures of the child's parents, and the voices of the child's parents to easily convey meaning to the child, as well as helping the child express his needs, making the learning process more enjoyable, enjoyable, and effective, and reviewing the benefits as well as mentioning the challenges associated with it. This will help improve their quality of life [1].

The technology of converting symbols and words into spoken speech is a process that relies on special software that allows the user to select images, words, or phrases and convert them into spoken speech by touching the screen to be pronounced in a better and clearer voice to facilitate the communication process [2].

2. Related Works

The search term Digital technology and communication patterns of the autistic child: case analysis by Maria da Luz Vale-Dias 2016 The presence of deficit of language and communication skills is evident in autism. The communication standards may evolve, but not normalize with age. There are obvious benefits of using technology in interventions with autistic, improving motivation, attention, learning, communication and reducing behavioural problems [4].

Digital technologies for autistic spectrum disorder students' education by Jenny A. Vlachou 2023 ICT assessment tools and ICT intervention tools are two categories for them. The evaluation describes each tool's background, functionality, and relationship to this scientific discipline while explaining specific technological features. The result is that a variety of ICT technologies can help with ASD diagnosis [5].

Autism Communication Cards | Free Printable by By Emily Parker 2024Visual aids like communication cards leverage the strength many neurodivergent kids have in processing images. According to a study by the American Speech-Language-Hearing Association, visual supports can significantly improve understanding and reduce anxiety in kids with special needs. Images provide clarity and help kids focus on the message without the distractions of verbal cues.[6]

Moreover, these tools help bridge language gaps, especially for kids who struggle with verbal expression. They create a consistent way to communicate, which is essential for building trust and reducing frustration.

3. Autism Spectrum Disorder

The word autism means self or psyche and is most often used to describe a lonely, introverted person or someone with a developmental delay that may appear in early childhood, usually the first two years of a child's life. It may include limitations in social interaction, communication, language development, and communication skills, in addition to repetitive behaviors[6].

4. Assistive Technology

Assistive technology is defined as a device, application, or system used to serve children with autism spectrum disorder (ASD) to provide an assistive technology service, such as providing hearing services at any time and providing a visual and auditory processing area. Different types of technology, from low technology to high technology, should be integrated into every aspect of life to enhance the understanding and communication of children with ASD.[7]

Simple technology includes visual support. These technologies don't require any kind of electronic equipment and are usually very simple and low-cost equipment, such as photo albums and simple computers[8].

Medium technology includes battery- or electrically powered electronic devices, such as projectors and audio devices.

High-end technology includes complex and expensive devices and strategies, such as adaptive devices and sophisticated sound output devices[9]

5. Autism Spectrum Disorder

Autism spectrum disorder (ASD) is a neurodevelopmental disorder that predominantly affects communication, behavior, and social interaction. It is characterized by unusual, repetitive, and somewhat restricted behavior patterns. Although autism can be diagnosed at any age, it is described as a "developmental disorder" because its symptoms appear before the age of three. According to the Diagnostic and Statistical Manual of Mental Disorders (DSM-5), a manual issued by the American Association for Neuropsychiatry and Psychiatry and used to diagnose mental disorders and speech difficulties, people with autism spectrum disorder experience.

- Difficulty communicating, speaking, articulating, and interacting with others
- Restricted interests and repetitive behaviors
- Symptoms that affect a person's ability to function in school, work, and other areas of life [10].

Autism is known as a "spectrum" disorder because there is a great variation in the type and severity of symptoms that people experience. No two people with autism spectrum disorder are exactly alike. Some people have difficulty speaking and expressing their desires and themselves. The severity varies depending on the type of condition, including mild cases such as Asperger's syndrome.

Technology designed as augmentative communication systems for children with autism can be used to increase or improve:

1. Their general understanding of the environment around them
2. Expressive communication skills to express themselves
3. Social interaction skills to connect with others.

4. Enhancing their ability to pay attention
5. Helping them with motivation
6. Organizational skills
7. Academic skills
8. Self-help skills and general independent daily work skills.

6. What is Assistive Technology?

With our understanding of the Modern Technology Assistance Acts for Children with Disabilities of 1988 (Public Act) No. 407-100, assistive and augmentative technology can be defined as an item, device, or production system, whether designed or purchased for use, that provides easy services to children with disabilities. These services help this group of children by making them use digital visual information more easily than auditory information only. Therefore, digital technology devices are incorporated to provide visually processed information[11].

7. Technology of all kinds

Simple technology: This is for visual support and is low-cost, easy-to-use equipment, such as three-ring binders, photo albums, simple computers, highlighters, etc.

Intermediate technology: This may be certain battery-powered devices with limited technology, such as simple projectors or simple audio devices.

High-end technology: These are complex and expensive strategies, including sophisticated audio output devices[11].

8. Visual Representation Systems

The most important step is to determine which visual representation system is most appropriate for the child based on their diagnosis. Various visual representation systems are used, such as pictures, line drawings, realistic drawings, images of the child's self, and written words, while integrating technology, assuming the child can understand the type of visual representation being used. For example, some children need visual and motor representation systems in various situations, such as when representing hunger, sleeping, or going to the bathroom. Therefore, this is an attempt to facilitate attention, facilitate the organization of thoughts, and facilitate the ability to express themselves to facilitate communication with others[7].

9. Materials and Methods

In this section include proposed system design, it works in this project and used programs html,css,Java script, sql and c#.consists of three sectionseach section has a specific performance.As infigure(1) shows the interface of the program:

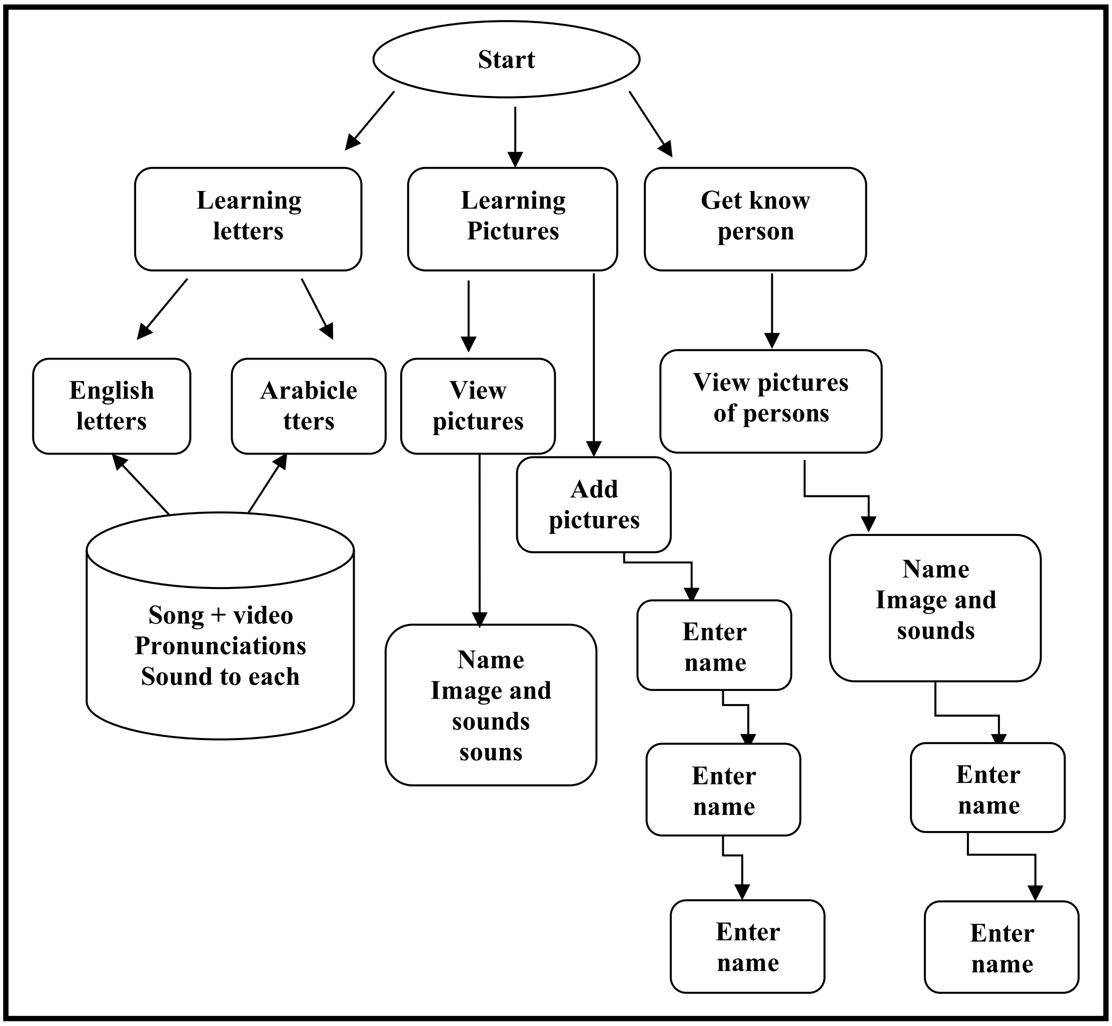

Fig. 1-. proposed digital board.

9.1 Forms Representation

Fig 2- The interface of the program.

Fig. 3-The interfacelearn Arabic.

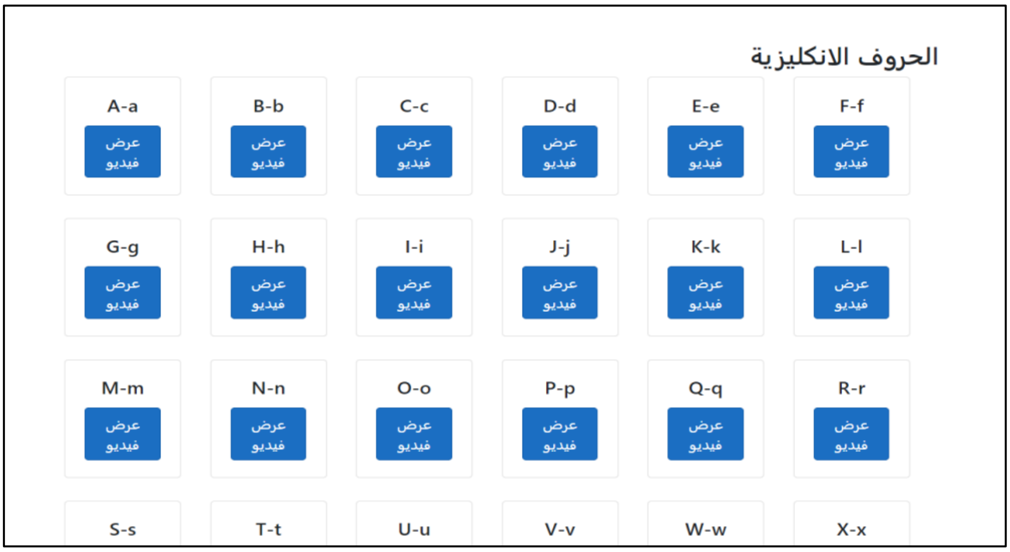

Fig. 4-. The interface learn English.

When you click on any letter, there will be a sound for the selected letter. If you click on the blue button, a video will be shown, followed by a redirect to YouTube, where the singing tutorial video for the selected letter will be shown.

Fig. 5 - A picture to go to YouTube.

We can also return to the main interface, which we named "Home Page," which we explained earlier, using functions shown in JavaScript, along with the original database that was built and used and linked to the second and third scripts for this purpose.

9.2 Learning with pictures section

In this section, we click on the image learning interface, where sounds will be added to the images, as shown in Figure 6 and Figure 7.

Fig. 6 -. Image learning interface.

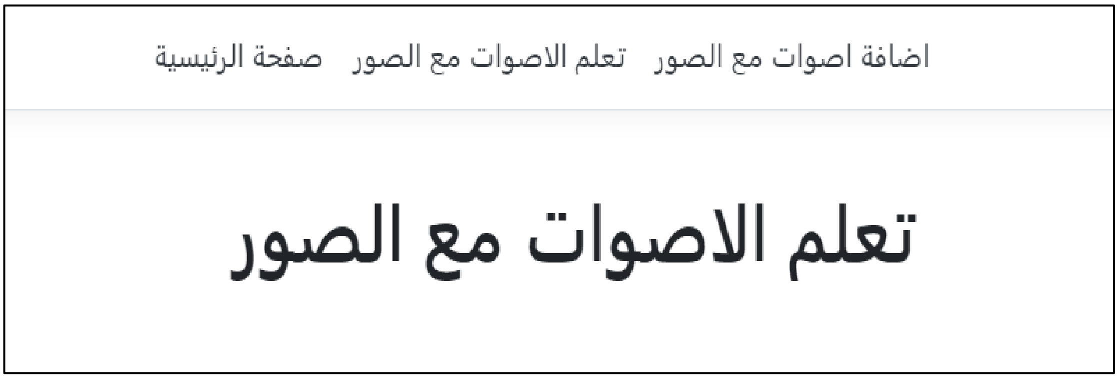

Fig. 7.-Image and sound learning interface.

Click the "Add Image with Sound" option, add images with both image and audio extensions, and type a name for the selected image

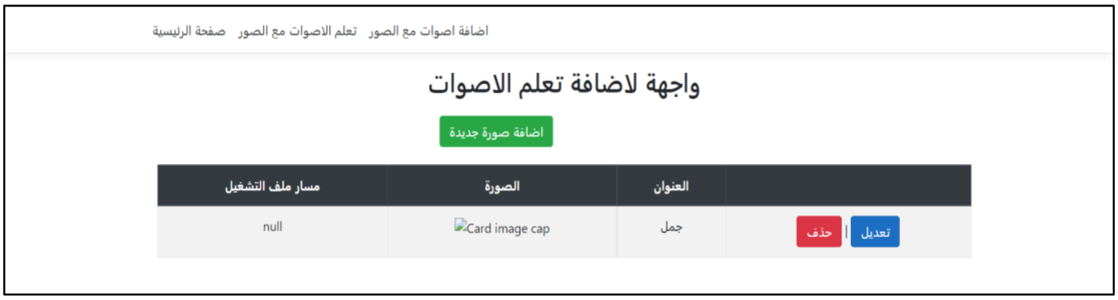

Fig. 8 -. Image and sounds with app and screen display.

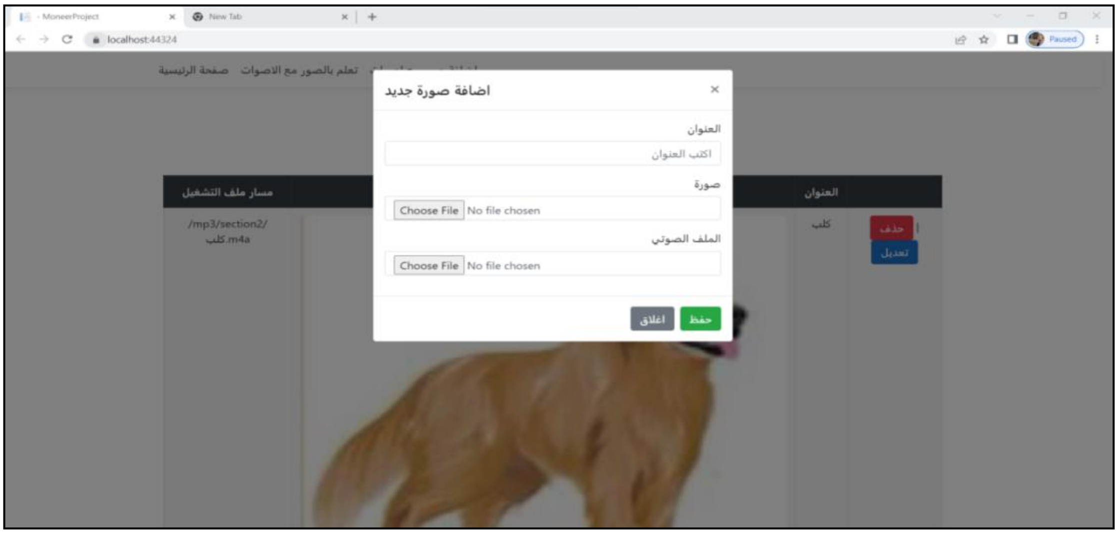

Fig. 9.-Added via this interface.

Fig.10.- view with sounds.

Then when you click on any image, a sound displays the name of the image's content. The third section displays pictures of people close to the child, such as the mother, father, and siblings, from the previously stored database of people close to the child. Therefore, it is better to use the application to contribute to the child's identification of his parents, teachers, and siblings by adding their pictures and voices to the stored database.

Fig. 11- Getting to know people.

After clicking on the main interface, we will move to another sub-interface. This interface allows authorized persons to enter the system and the program. Their data will then be included in the database.

Fig. 12- Entry of permitted persons.

Now, the commands in the third block allow us to access a field in addition to the display table, so that with the help of the main command we can return to the home page.

Fig. 13- Entry of permitted persons.

10. Challenges

- Privacy: Collecting highly confidential data from children and their parents may raise concerns.
- Cybersecurity: The system must be secured against hacking and cyber-attacks.

- Integration: Integrating the system with other systems, such as language or sign language systems, may be difficult.
- Cost: The cost of installing and maintaining the system may be high.

11. Improvements

- Integration of artificial intelligence technologies: AI can be used to automatically analyze images and identify sounds and child movements.
- Development of an interactive user interface: A user interface can be developed that allows users to easily view and analyze data.
- Enhanced integration with other systems: The system can be integrated with other systems, such as child behavior monitoring systems, to enhance attention and rapid response.

12. Added Value

- Improved Security: The system improves security and prevents unauthorized activities.
- Increased Efficiency: The collected data can be used to enhance efficiency in special cases for children with autism spectrum disorders.
- Decision Support: The collected data helps in making better decisions.

13.Author Contributions

- Conceptualization
- Data creation
- Formal analysis
- Funding acquisition
- Investigation
- Methodology
- Project administration
- Resources
- Validation
- Writing—original draft

14. Conclusion and Future Works

The concept of the role of a technological system designed for children with attention and speech difficulties is to improve children's integration into daily and social life, and to provide training services for easy pronunciation of letters, as well as the ease of expressing important life needs such as hunger, going to the bathroom, drowsiness, and anger. This is done using a new technology based on sounds, images, video, and Arabic and English letters, as well as the use of a virtual environment, or the use of a cartoon character or image that the child likes, or through paintings and drawings that attract the child with attractive colors. Future work to improve the designed system includes the possibility of training an algorithm with artificial crying, or using languages other than Arabic and English. It is also possible to use an intelligent robot that performs specific movements that express needs.

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