Journal of Iraqi Al-Khwarizmi Society (JIKhS) Year 2018 pages: 99-116

Volume:2 Special issue of the first international scientific conference of Iraqi Al-Khawarizmi Society 28-29 March 2018

On the embedding of an arc into a cubic in a finite projective plane

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Abstract :

The main aims of this research is to find the stabilizer groups of a cubic curves over a finite field of orders 2, 3 and studying the properties of their groups and then constructing the arcs of degree 2 which are embedding in a cubic curves of even size which are considering as the arcs of degree 3. Also drawing all these arcs.

Key words : Stabilizer groups, arcs, cubic curves.

1. Introduction:

The subject of this research depends on themes of

- Projective geometry over a finite field;
- Group theory;
- Field theory;
- Linear algebra.

The strategy to construct the stabilizer groups and also to embedded the arcs is given as following:

Constructing the linear transformations group $PGL(3,q)$ of $PG(2,q)$, where $q = 2,3$. Which its elements are considering the non-singular matrices $A_n = [a_{ij}]$, $a_{i,j}$ in F_q , $i,j = 1,2,3$ for some n in $\mathbb{N}$ and satisfying $K(tA_n) = K$ for all t in $F_q \setminus \{0\}$ and K be any arc. The set of all matrices A_n which construct the group, and according to the number of A_n and its order, we are doing comparison with the groups in $[6]$, so we can find which one of them similar than it. In another hand, we have found the arcs which are embedding in a cubic curves which are splitted into two sets, one of them contains the inflection points and the other does not, the set which does not contain the inflection points is considering the arc of degree two.

The summary history of this theme is shown as follows

- The ideas and definitions of this research are taken from James Hirshfeld $[1]$;
- In 2010, Najm Al-Seraji $[2]$ has been studied the cubic curves over a finite field of order 17;
- In 2011, Emad Al-Zangana $[3]$ has been shown the cubic curves over a finite field of order 19;
- In 2013, Emad Al-Zangana $[4]$ has been described the cubic curves over a finite field of orders 2, 3, 5, 7 ;
- In 2013, Emad Al-Zangana $[5]$ has been classified the cubic curves over a finite field of order 11, 13 .

Now, we introduce the definitions which are using in this research as follows:

Definition $(1.1)[1]$: Denote by S and S^* two subspaces of $P(n,K)$, A projectivity $\beta: S \to S^*$ is a bijection given by a matrix T, necessarily non-singular, where $P(X) = P(X)\beta$ if $tX^* = XT$, with $t \in K$. Write $\beta = M(T)$; then $\beta = M(\lambda T)$ for any λ in K. The group of projectivities of $PG(n,K)$ is denoted by $PGL(n+1,K)$.

Definition $(1.2)[1]$: The stabilizer of x in Λ under the action of G is the group $G_x = \{g \in G | xg = x\}$.

Definition (1.3)[1]: An $(n;r)$ arc K or arc of degree r in $PG(k,q)$ with $n \geq r+1$ is a set of points with property that every hyperplane meets K in at most r points of K and there is some hyperplane meeting K in exactly r points.

2. The classification of cubic curves over a finite field of order 2.

The polynomial of degree three $g_1(x) = x^3 + x + 1$ is primitive in $F_2 = \{0,1\}$, since $g_1(0) = 1$ and $g_1(1) = 1$, also $g_1(\gamma) = 0$, $g_1(\gamma^2) = 0$, $g_1(\gamma^4) = 0$, this means $\gamma, \gamma^2, \gamma^4$ are roots of g_1 in F_{2^3}.

The companion matrix of $g_1(x) = x^3 + x + 1$ in $F_2[x]$ generated the points and lines of $PG(2,2)$ as follows:

$$P(k) = [1,0,0]C(g)^k = [1,0,0]\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^k, \; k = 0,1,\dots,6.$$

With selecting the points in $PG(2,2)$ which are the third coordinate equal to zero, this means belong to $L_0 = v(z)$, that is $v(z) = tz = z$ for all t in $F_2 \setminus \{0\}$ and with $P(k) = k$, we obtain $L_0 = \{0,1,3\}$, that is

$$L_{k=} L_0 C(g)^k = L_0 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^k, \; k = 0,1,\dots,6.$$

The projective plane of order two $\mathrm{PG}(2,2)$ is drawn in Figure 1.

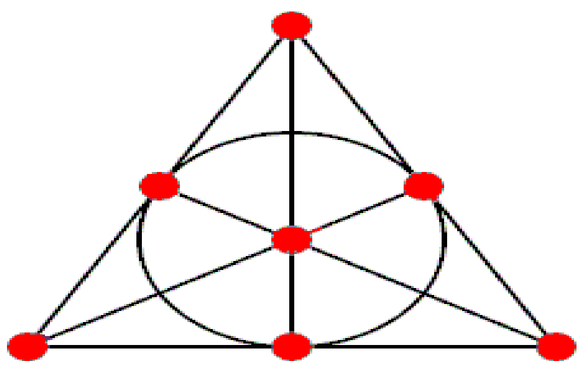

Figure 1: Drawing of $\mathbf{PG(2,2)}$

The number of distinct cubic curves is 6 see [4], one of them is given as follows:

$$f_1 = xy(x+y) + z^3 \qquad \dots(1)$$

The points of $PG(2,2)$ on f_1 in equation (1) are $[1,0,0], [0,1,0], [1,1,0]$. To find the stabilizer group of f_1 in equation (1), we are doing calculations by help the computer. Thus, the transformation matrices which stabilizing of f_1 with their orders are shown as follows:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}:4, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}:4, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}:2, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}:3,$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}:3, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}:3, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}:3, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}:1,$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}:2, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}:2, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}:2, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}:2,$$

$\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix}:4, \begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix}:2, \begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}:4, \begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}:2,$

$\begin{pmatrix}1&1&0\\0&1&0\\0&1&1\end{pmatrix}:2, \begin{pmatrix}1&1&0\\0&1&0\\1&0&1\end{pmatrix}:4, \begin{pmatrix}1&1&0\\0&1&0\\1&1&1\end{pmatrix}:4, \begin{pmatrix}1&1&0\\1&0&0\\0&0&1\end{pmatrix}:3$

$\begin{pmatrix}1&1&0\\1&0&0\\0&1&1\end{pmatrix}:3, \begin{pmatrix}1&1&0\\1&0&0\\1&0&1\end{pmatrix}:3, \begin{pmatrix}1&1&0\\1&0&0\\1&1&1\end{pmatrix}:3, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}:2.$

Therefore, the stabilizer group of f_1 in equation (1) which is denoted by G_{f_1} which contains

- 9 matrices of order 2;
- 8 matrices of order 3;
- 6 matrices of order 4;
- The identity matrix.

From $[6]$, G_{f_1} is isomorphic to S_4, that is $G_{f_1} \cong S_4$. f_1 in equation (1) is drawn in Figure 2:

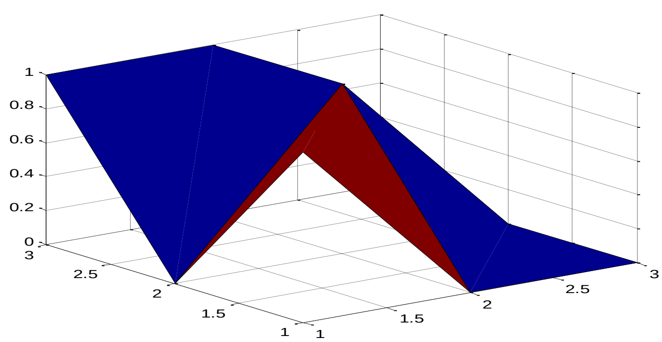

Figure 2: Drawing of f_1

Another one of cubic curve which is given in $[4]$ is:

$$\mathbf{f_2 = yz^2 + xyz + x^3 + xy^2} \qquad \ldots(2)$$

The points of $PG(2,2)$ on f_2 in equation (2) are $[0,1,0], [0,0,1], [1,1,0], [1,1,1]$. To find the stabilizer group of f_2 in equation (2), we are doing calculations by help the computer. Thus, the transformation matrices which stabilizing of f_2 with their orders are shown as follows:

$\begin{pmatrix}0&1&1\\0&1&0\\1&1&0\end{pmatrix}:2, \begin{pmatrix}0&1&1\\0&1&0\\1&1&1\end{pmatrix}:3, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}:1,$

$\begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix}:2, \begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}:2, \begin{pmatrix}1&0&1\\0&1&0\\1&1&0\end{pmatrix}:3.$

Therefore, the stabilizer group of f_2 in equation (2) which is denoted by G_{f_2} which contains

- 3 matrices of order 2;
- 2 matrices of order 3;

- The identity matrix.

From [6], G_{f_2} is isomorphic to $\boldsymbol{S_3}$, that is $G_{f_2} \cong \boldsymbol{S_3}$. f_2 in equation (2) is drawn in Figure 3:

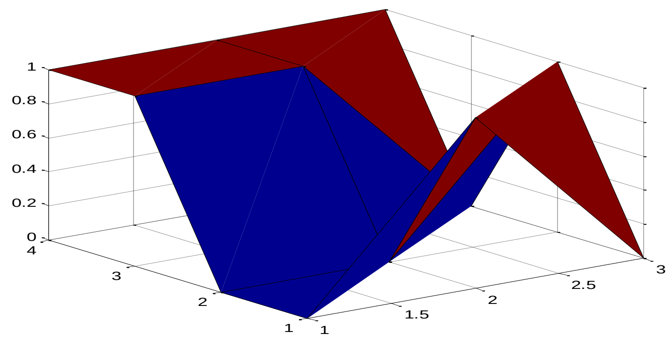

Figure 3: Drawing of $\boldsymbol{f_2}$

Let $f_2^* = \{3,5\}$ be a subset of f_2 in equation (2) which is forming by partition f_2 into two sets such that f_2^* does not contains the inflection points of f_2, so we note that f_2^* represents an arc of degree two.

Also, to find the stabilizer group of f_2^*, by some calculation

$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}0&1&0\\1&0&1\\0&0&1\end{pmatrix}:4,\begin{pmatrix}0&1&1\\1&0&0\\0&0&1\end{pmatrix}:4,\begin{pmatrix}0&1&1\\1&0&1\\0&0&1\end{pmatrix}:2,$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}:1,\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix}:2.$$

Therefore, the stabilizer group of f_2^* which is denoted by $G_{f_2^*}$ contains

- 5 matrices of order 2;
- 2 matrices of order 4;
- The identity matrix.

From $[6]$, $G_{f_2^*}$ is isomorphic to $\boldsymbol{D_4}$, that is $G_{f_2^*} \cong \boldsymbol{D_4}$. Drawing of f_2^* is given in Figure 4 as following:

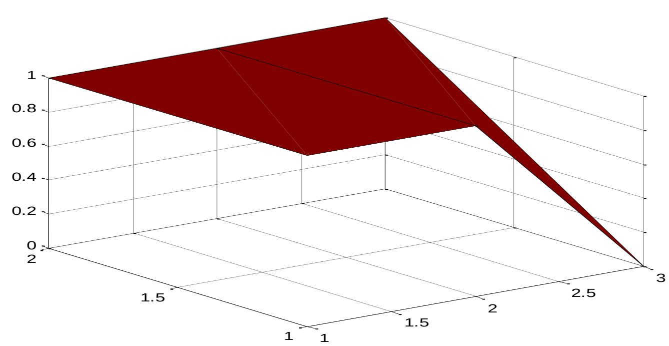

Figure 4: Drawing of $\boldsymbol{f_2^*}$

Another one of cubic curve which is given in [4] is:

$$f_3 = yz^2 + xyz + x^3 + x^2y + xy^2 \qquad \ldots(3)$$

The points of $PG(2,2)$ on f_3 in equation (3) in are $[0,1,0],[0,0,1]$. To find the stabilizer group of f_3 in equation (3), we are doing calculations by help the computer. Thus, the transformation matrices which stabilizing of f_3 with their orders are shown as follows:

$$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}:2,\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}:1,\begin{pmatrix}1&0&1\\0&0&1\\0&1&0\end{pmatrix}:4,\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}:2,$$

$$\begin{pmatrix}1&1&0\\0&0&1\\0&1&0\end{pmatrix}:4,\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&1&1\\0&0&1\\0&1&0\end{pmatrix}:2,\begin{pmatrix}1&1&1\\0&1&0\\0&0&1\end{pmatrix}:2.$$

Therefore, the stabilizer group of f_3 in equation (3) which is denoted by G_{f_3} which contains

- 5 matrices of order 2;
- 2 matrices of order 4;
- The identity matrix.

From [6], G_{f_3} is isomorphic to $\boldsymbol{D_4}$, that is $G_{f_3} \cong \boldsymbol{D_4}$.

The set of points on f_3 is constructing the arc of degree two and size 2. f_3 in equation (3) is drawn in Figure 5:

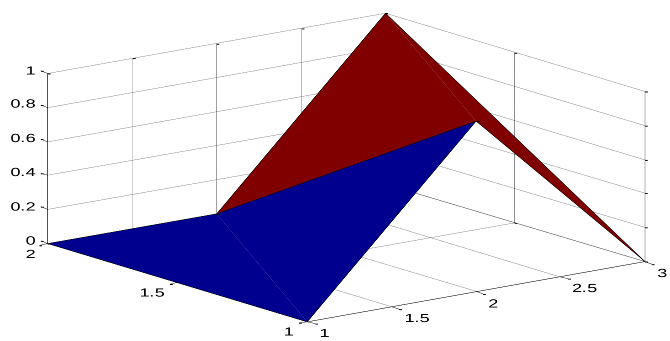

Figure 5: Drawing of $\boldsymbol{f_3}$

Another one of cubic curve which is given in [4] is:

$$f_4 = z^3y + zy^2 + x^3 + xy^2 \qquad \ldots(4)$$

The points of $PG(2,2)$ on f_4 in equation (4) are $[0,1,0],[0,0,1],[1,1,0],[0,1,1],[1,1,1]$. To find the stabilizer group of f_4 in equation (4), we are doing calculations by help the computer. Thus, the transformation matrices which stabilizing of f_4 with their orders are shown as follows:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}:1,\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&0&0\\1&1&1\\0&0&1\end{pmatrix}:2,$$

$$\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&0&1\\1&1&0\\0&0&1\end{pmatrix}:4,\begin{pmatrix}1&0&1\\1&1&1\\0&0&1\end{pmatrix}:4.$$

Journal of Iraqi Al-Khwarizmi Society (JIKhS) Year 2018 pages: 99-116
Volume:2 Special issue of the first international scientific conference of Iraqi Al-Khawarizmi Society 28-29 March 2018

Therefore, the stabilizer group of f_4 in equation (4) which is denoted by G_{f_4} which contains

- 5 matrices of order 2;
- 2 matrices of order 4;
- The identity matrix.

From $[6]$, G_{f_4} is isomorphic to $\boldsymbol{D_4}$, that is $G_{f_4} \cong \boldsymbol{D_4}$. f_4 in equation (4) is drawn in Figure 6:

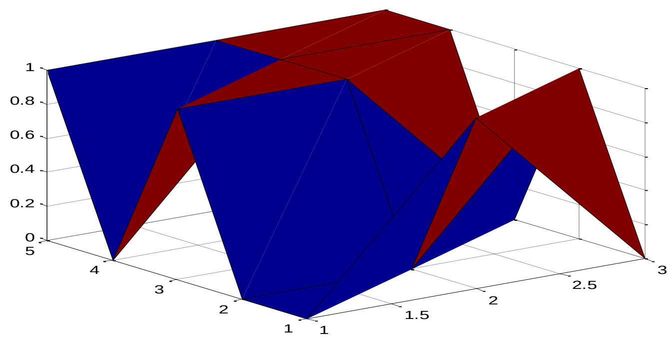

Figure 6: Drawing of $\boldsymbol{f_4}$

Another one of cubic curve which is given in $[4]$ is:

$$f_5 = z^3y + zy^2 + x^3 + xy^2 + y^3 \qquad \ldots(5)$$

The points of $PG(2,2)$ on f_5 in equation (5) are $[0,0,1]$. To find the stabilizer group of f_5 in equation (5), we are doing calculations by help the computer. Thus, the transformation matrices which stabilizing of f_5 with their orders are shown as follows:

$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}0&1&0\\1&0&1\\0&0&1\end{pmatrix}:4,\begin{pmatrix}0&1&0\\1&1&0\\0&0&1\end{pmatrix}:3,\begin{pmatrix}0&1&0\\1&1&1\\0&0&1\end{pmatrix}:3,$$

$$\begin{pmatrix}0&1&1\\1&0&0\\0&0&1\end{pmatrix}:4,\begin{pmatrix}0&1&1\\1&0&1\\0&0&1\end{pmatrix}:2,\begin{pmatrix}0&1&1\\1&1&0\\0&0&1\end{pmatrix}:3,\begin{pmatrix}0&1&1\\1&1&1\\0&0&1\end{pmatrix}:3,$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}:1,\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&0&0\\1&1&1\\0&0&1\end{pmatrix}:2,$$

$$\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&0&1\\1&1&0\\0&0&1\end{pmatrix}:4,\begin{pmatrix}1&0&1\\1&1&1\\0&0&1\end{pmatrix}:4,$$

$$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}:4,\begin{pmatrix}1&1&0\\1&0&0\\0&0&1\end{pmatrix}:3,\begin{pmatrix}1&1&0\\1&0&1\\0&0&1\end{pmatrix}:3,$$

$$\begin{pmatrix}1&1&1\\0&1&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}:4,\begin{pmatrix}1&1&1\\1&0&0\\0&0&1\end{pmatrix}:3,\begin{pmatrix}1&1&1\\1&0&1\\0&0&1\end{pmatrix}:3.$$

Therefore, the stabilizer group of f_5 in equation (5) which is denoted by G_{f_5} which contains

- 9 matrices of order 2;

- 8 matrices of order 3;
- 6 matrices of order 4;
- The identity matrix.

From $[6]$, G_{f_5} is isomorphic to S_4, that is $G_{f_5} \cong S_4$.

Another one of cubic curve which is given in $[4]$ is:

$$f_6 = z^3 - (x^3 - xy^2 + y^3) \qquad \ldots(6)$$

The points of $PG(2,2)$ on f_6 in equation (6) are $[0,1,1], [1,1,1], [1,0,1]$. To find the stabilizer group of f_6 in equation (6), we are doing calculations by help the computer. Thus, the transformation matrices which stabilizing of f_6 with their orders are shown as follows:

$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}:2, \begin{pmatrix}0&1&0\\1&1&0\\0&0&1\end{pmatrix}:3, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}:1,$$

$$\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}:2, \begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}:2, \begin{pmatrix}1&1&0\\1&0&0\\0&0&1\end{pmatrix}:3.$$

Therefore, the stabilizer group of f_6 in equation (6) which is denoted by G_{f_6} which contains

- 3 matrices of order 2;
- 2 matrices of order 3;
- The identity matrix.

From $[6]$, G_{f_6} is isomorphic to S_3, that is $G_{f_6} \cong S_3$. f_6 in equation (6) is drawn in Figure 7:

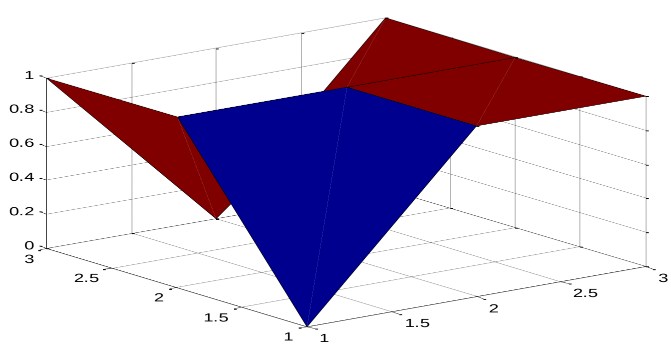

Figure 7: Drawing of f_6

3. The classification of cubic curves over a finite field of order 3.

The polynomial of degree three $g_2(x) = x^3 - x - 2$ is primitive in $F_3 = \{0,1,2\}$, since $g_2(0) = 1$, $g_2(1) = 1$ and $g_2(2) = 1$, also $g_2(\delta) = 0$, $g_2(\delta^3) = 0$, $g_2(\delta^9) = 0$, this means $\delta, \delta^3, \delta^9$ are roots of g_2 in F_{3^3}.

The companion matrix of $g_2(x) = x^3 - x - 2$ in $F_3[x]$ generated the points and lines of $PG(2,3)$ as follows:

Journal of Iraqi Al-Khwarizmi Society (JIKhS) Year 2018 pages: 99-116
Volume:2 Special issue of the first international scientific conference of Iraqi Al-Khawarizmi Society 28-29 March 2018

$$P(k) = [1,0,0]C(g)^k = [1,0,0]\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 2 & 1 & 0\end{pmatrix}^k,\ k=0,1,\dots,12\ .$$

With selecting the points in $PG(2,3)$ which are the third coordinate equal to zero, this means belong to $L_0 = v(z)$, that is $v(z) = tz = z$ for all t in $F_3\setminus\{0\}$ and with $P(k) = k$, we obtain $L_0 = \{0,1,3,9\}$, that is

$$L_{k=}L_0C(g)^k = L_0\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 2 & 1 & 0\end{pmatrix}^k,\ k=0,1,\dots,12.$$

The projective plane of order three $PG(2,3)$ is drawn in Figure 8.

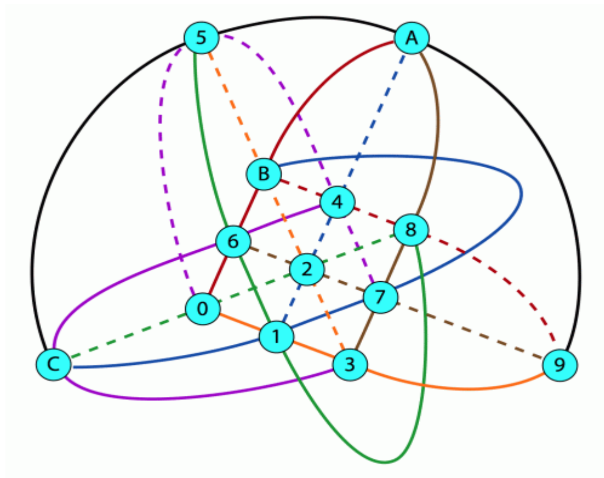

Figure 8: Drawing of $PG(2,3)$

The number of distinct cubic curves in $PG(2,3)$ is 10 see [4] , one of them is given as follows: $h_1 = xyz + (x+y+z)^3$
$\dots(7)$

The points of $PG(2,3)$ on h_1 in equation (7) are $[2,1,0], [0,2,1], [1,2,1], [2,2,1], [2,1,1], [2,0,1]$. To find the stabilizer group of h_1 in equation (7), we are doing calculations by help the computer. Thus, the transformation matrices which stabilizing of h_1 with their orders are shown as follows:

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}:2,\begin{pmatrix}0&0&1\\0&1&0\\2&2&2\end{pmatrix}:3,\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}:3,\begin{pmatrix}0&0&1\\1&0&0\\2&2&2\end{pmatrix}:4,$$

$$\begin{pmatrix}0&0&1\\2&2&2\\0&1&0\end{pmatrix}:4,\begin{pmatrix}0&0&1\\2&2&2\\1&0&0\end{pmatrix}:2,\begin{pmatrix}0&2&0\\0&0&2\\1&1&1\end{pmatrix}:4,\begin{pmatrix}0&2&0\\0&0&2\\2&0&0\end{pmatrix}:3,$$

$$\begin{pmatrix}0&2&0\\1&1&1\\0&0&2\end{pmatrix}:3,\begin{pmatrix}0&2&0\\1&1&1\\2&0&0\end{pmatrix}:4,\begin{pmatrix}0&2&0\\2&0&0\\0&0&2\end{pmatrix}:2,\begin{pmatrix}0&2&0\\2&0&0\\1&1&1\end{pmatrix}:2,$$

$$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}:2,\begin{pmatrix}1&0&0\\0&0&1\\2&2&2\end{pmatrix}:3,\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}:1,\begin{pmatrix}1&0&0\\0&1&0\\2&2&2\end{pmatrix}:2,$$

$$\begin{pmatrix}1&0&0\\2&2&2\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&0&0\\2&2&2\\0&1&0\end{pmatrix}:3,\begin{pmatrix}2&2&2\\0&0&1\\0&1&0\end{pmatrix}:2,\begin{pmatrix}2&2&2\\0&0&1\\1&0&0\end{pmatrix}:4,$$

$$\begin{pmatrix}2&2&2\\0&1&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}2&2&2\\0&1&0\\1&0&0\end{pmatrix}:3,\begin{pmatrix}2&2&2\\1&0&0\\0&0&1\end{pmatrix}:3,\begin{pmatrix}2&2&2\\1&0&0\\0&1&0\end{pmatrix}:4.$$

Therefore, the stabilizer group of h_1 in equation (7) which is denoted by G_{h_1} which contains

- 9 matrices of order 2;
- 8 matrices of order 3;
- 6 matrices of order 4;
- The identity matrix.

From $[6]$, G_{h_1} is isomorphic to $\boldsymbol{S_4}$, that is $G_{h_1} \cong \boldsymbol{S_4}$. h_1 in equation (7) is drawn in Figure 9:

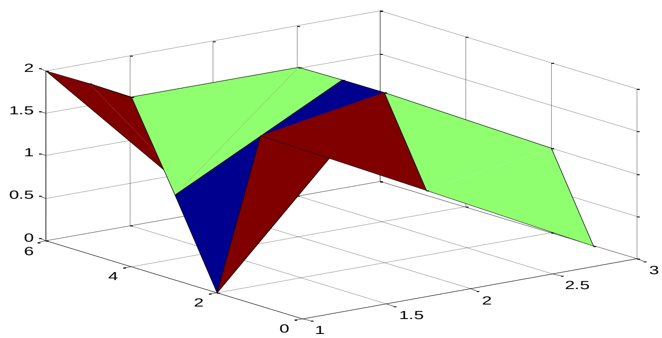

Figure 9: Drawing of h_1

Let $h_1^* = \{3,4,7\}$ be a subset of h_1 in equation (7) which is forming by partition h_1 into two sets such that h_1^* does not contains the inflection points of h_1, so we note that h_1^* represents an arc of degree two. Also, to find the stabilizer group of h_1^*, by some calculation, we obtain

$$\begin{pmatrix}0&1&2\\0&0&1\\2&2&0\end{pmatrix}:3,\begin{pmatrix}0&1&2\\0&0&2\\2&2&1\end{pmatrix}:4,\begin{pmatrix}0&1&2\\0&1&0\\2&1&0\end{pmatrix}:2,\begin{pmatrix}0&1&2\\0&1&1\\2&1&0\end{pmatrix}:4,$$

$$\begin{pmatrix}0&1&2\\0&2&0\\2&2&0\end{pmatrix}:2,\begin{pmatrix}0&1&2\\0&2&2\\2&2&1\end{pmatrix}:3,\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}:1,\begin{pmatrix}1&0&0\\0&2&0\\0&1&1\end{pmatrix}:2,$$

$$\begin{pmatrix}1&0&0\\1&1&0\\2&2&1\end{pmatrix}:3,\begin{pmatrix}1&0&0\\1&2&0\\2&2&1\end{pmatrix}:2,\begin{pmatrix}1&0&0\\2&1&0\\0&1&1\end{pmatrix}:3,\begin{pmatrix}1&0&0\\2&2&0\\0&0&1\end{pmatrix}:2,$$

$$\begin{pmatrix}1&1&1\\0&0&1\\0&2&2\end{pmatrix}:3,\begin{pmatrix}1&1&1\\0&0&2\\0&2&0\end{pmatrix}:2,\begin{pmatrix}1&1&1\\1&1&0\\2&1&0\end{pmatrix}:4,\begin{pmatrix}1&1&1\\1&1&2\\2&1&0\end{pmatrix}:2,$$

$$\begin{pmatrix}1&1&1\\2&2&0\\0&2&0\end{pmatrix}:3,\begin{pmatrix}1&1&1\\2&2&1\\0&2&2\end{pmatrix}:4,\begin{pmatrix}2&0&2\\0&1&1\\0&2&0\end{pmatrix}:3,\begin{pmatrix}2&0&2\\0&2&2\\0&0&1\end{pmatrix}:2,$$

$$\begin{pmatrix}2&0&2\\1&1&2\\0&0&1\end{pmatrix}:2,\begin{pmatrix}2&0&2\\1&2&0\\0&2&0\end{pmatrix}:4,\begin{pmatrix}2&0&2\\2&1&0\\1&1&0\end{pmatrix}:3,\begin{pmatrix}2&0&2\\2&2&1\\1&1&0\end{pmatrix}:4.$$

Therefore, the stabilizer group of h_1^* which is denoted by $G_{h_1^*}$ which contains

- 9 matrices of order 2;
- 8 matrices of order 3;
- 6 matrices of order 4;
- The identity matrix.

From [6], $G_{h_1^*}$ is isomorphic to S_4, that is $G_{h_1^*} \cong S_4$. Drawing of h_1^* is given in Figure 10 as following:

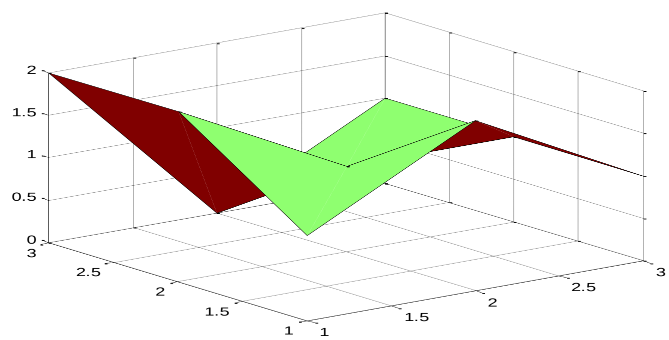

Figure 10: Drawing of h_1^*

Another one of cubic curve which is given in [4] is:

$h_2 = xyz - (x + y + z)^3$ …(8)

The points of $PG(2,3)$ on h_2 in equation (8) are $[2,1,0], [0,2,1], [2,0,1]$. After the calculation and help the computer, we are obtained that the number of matrices which are stabilizing of h_2 in equation (8) is 108, and we can not write them, because they are too much.

Therefore, the stabilizer group of h_2 which is denoted by G_{h_2} which contains

- 27 matrix of order 2;
- 26 matrix of order 3;
- 54 matrix of order 6;
- The identity matrix.

h_2 in equation (8) is drawn in Figure 11:

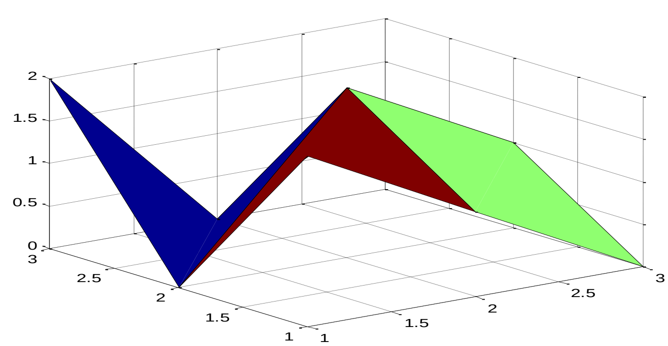

Figure 11: Drawing of h_2

Another one of cubic curve which is given in [4] is:

$$h_3 = z^2y + x^3 + yx^2 + y^3 \qquad (9)$$

The points of $PG(2,3)$ on h_3 in equation (9) are $[0,0,1], [1,1,0]$. After the calculation and help the computer, we are obtained that the number of matrices which are stabilizing of h_3 in equation (9) is 72, and we can not write them, because they are too much.

Therefore, the stabilizer group of h_4 in equation (9) which is denoted by G_{h_4} which contains

- 21 matrix of order 2;
- 8 matrices of order 3;
- 18 matrix of order 4;
- 24 matrix of order 6;
- The identity matrix.

The set of points on h_3 represents the arc of degree two and size 2.

Drawing of h_2 is given in Figure 12 as following:

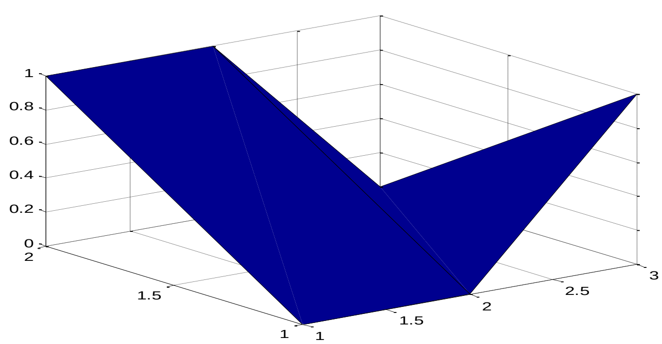

Figure 12 : Drawing of h_3

Another one of cubic curve which is given in [4] is:

$$h_4 = z^2y + x^3 + yx^2 - y^3 \qquad \ldots(10)$$

The points of $PG(2,3)$ on h_4 in equation (10) are $[0,0,1], [0,2,1], [1,2,1], [0,1,1], [2,1,1]$. To find the stabilizer group of h_4 in equation (10), we are doing calculations by help the computer. Thus, the transformation matrices which stabilizing of h_4 with their orders are shown as follows:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}:1,\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}:2,\begin{pmatrix}1&0&0\\1&2&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&0&0\\1&2&0\\0&0&2\end{pmatrix}:2,$$

$$\begin{pmatrix}2&2&0\\0&1&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}2&2&0\\0&1&0\\0&0&2\end{pmatrix}:2,\begin{pmatrix}2&2&0\\2&1&0\\0&0&1\end{pmatrix}:4,\begin{pmatrix}2&2&0\\2&1&0\\0&0&2\end{pmatrix}:4.$$

Therefore, the stabilizer group of h_4 in equation (10) which is denoted by G_{h_4} which contains

- 5 matrices of order 2;
- 2 matrices of order 4;
- The identity matrix.

From [6], G_{h_4} is isomorphic to $\boldsymbol{D_4}$, that is $G_{h_4} \cong \boldsymbol{D_4}$. h_4 in equation (10) is drawn in Figure 13:

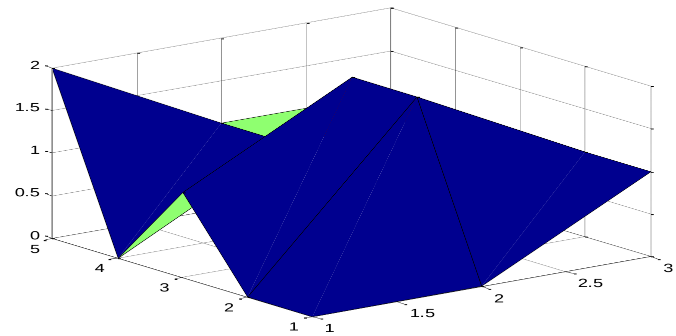

Figure 13: Drawing of h_4

Another one of cubic curve which is given in [4] is:

$$h_5 = z^2y + x^3 - xy^2 + y^3 \quad \dots(11)$$

The points of $PG(2,3)$ on h_5 in equation (11) are $[0,0,1]$. After the calculation and help the computer, we are obtained that the number of matrices which are stabilizing of h_5 in equation (11) is 432, and we can not write them, because they are too much.

Therefore, the stabilizer group of h_5 in equation (11) which is denoted by G_{h_5} which contains

- 45 matrix of order 2;
- 80 matrix of order 3;
- 54 matrix of order 4;
- 144 matrix of order 6;
- 108 matrix of order 8;
- The identity matrix.

Another one of cubic curve which is given in [4] is:

$$h_6 = z^2y + x^3 - xy^2 - y^3 \quad \dots(12)$$

The points of $PG(2,3)$ on h_6 in equation (12) are $[0,0,1], [0,2,1], [1,2,1], [1,1,1], [2,2,1], [0,0,1], [2,1,1]$. To find the stabilizer group of h_6 in equation (12), we are doing calculations by help the computer. Thus, the transformation matrices which stabilizing of h_6 with their orders are shown as follows:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}:1,\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}:2,\begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&0&0\\0&2&0\\0&0&2\end{pmatrix}:2,$$

$$\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}:3,\begin{pmatrix}1&0&0\\1&1&0\\0&0&2\end{pmatrix}:6,\begin{pmatrix}1&0&0\\1&2&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&0&0\\1&2&0\\0&0&2\end{pmatrix}:2,$$

$$\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}:3,\begin{pmatrix}1&0&0\\2&1&0\\0&0&2\end{pmatrix}:6,\begin{pmatrix}1&0&0\\2&2&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&0&0\\2&2&0\\0&0&2\end{pmatrix}:2.$$

Therefore, the stabilizer group of h_6 in equation (12) which is denoted by G_{h_6} which contains

- 7 matrices of order 2;
- 2 matrices of order 3;
- 2 matrices of order 6;
- The identity matrix.

From [6], G_{h_6} is isomorphic to $\boldsymbol{D_6}$, that is $G_{h_6} \cong \boldsymbol{D_6}$. h_6 in equation (12) is drawn in Figure 14:

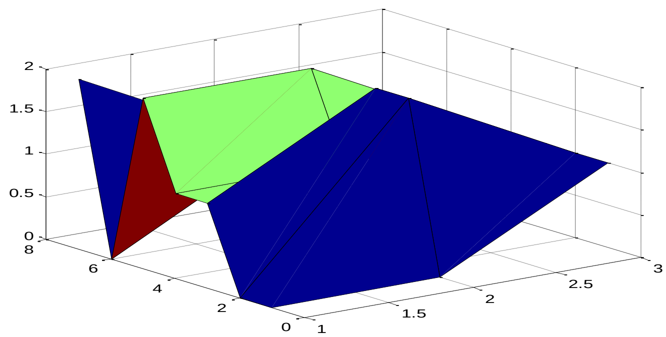

Figure 14: Drawing of h_6

Another one of cubic curve which is given in [4] is:

$$h_7 = z^2y + x^3 + xy^2 + y^3 \qquad \ldots(13)$$

The points of $PG(2,3)$ on h_7 in equation (13) are $[0,0,1], [1,2,1], [1,1,0], [2,1,1]$. To find the stabilizer group of h_7 in equation (13), we are doing calculations by help the computer. Thus, the transformation matrices which stabilizing of h_7 with their orders are shown as follows:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}:2, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}:2, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}:2, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 2 \end{pmatrix}:6,$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}:2, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 2 \end{pmatrix}:6, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}:2, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}:1,$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}:2, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 2 & 2 \end{pmatrix}:3, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}:2, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & 2 \end{pmatrix}:3.$$

Therefore, the stabilizer group of h_7 in equation (13) which is denoted by G_{h_7} which contains

- 7 matrices of order 2;
- 2 matrices of order 3;
- 2 matrices of order 6;
- The identity matrix.

From [6], G_{h_7} is isomorphic to $\boldsymbol{D_6}$, that is $G_{h_7} \cong \boldsymbol{D_6}$. h_7 in equation (13) is drawn in Figure 15:

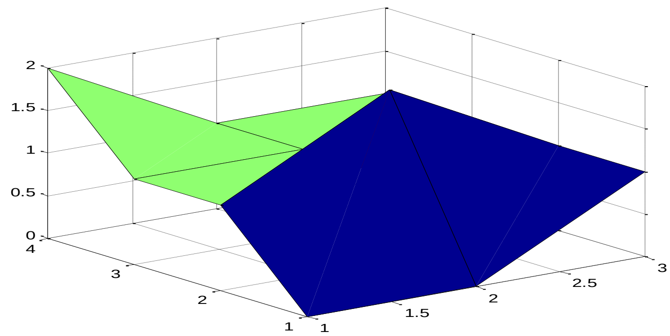

Figure 15: Drawing of h_7

Let $h_7^* = \{9,11\}$ be a subset of h_7 which is forming by partition h_7 into two sets such that h_7^* does not contains the inflection points of h_7, so we note that h_7^* represents an arc of degree two. After the calculation and help the computer, we are obtained that the number of matrices which are stabilizing of h_7^* is 72, and we can not write them, because they are too much. Moreover, the stabilizer group of h_7^* which is denoted by $G_{h_7^*}$ which contains

- 21 matrix of order 2;
- 8 matrices of order 3;
- 18 matrix of order 4;
- 24 matrix of order 6;
- The identity matrix.

Drawing of h_7^* is given in Figure 16 as following:

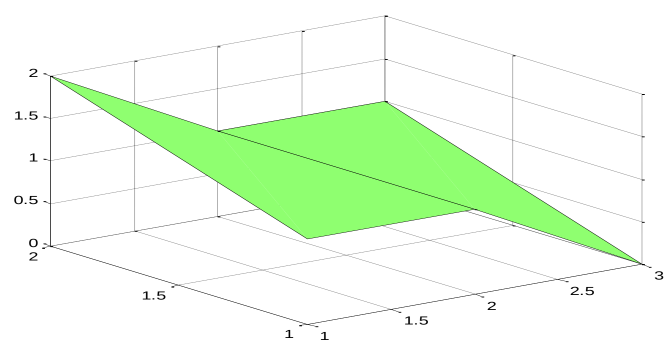

Figure 16 : Drawing of h_7^*

Another one of cubic curve which is given in $[4]$ is:

$$h_8 = z^2y + x^3 - xy^2 \qquad \ldots(14)$$

The points of $PG(2,3)$ on h_8 in equation (14) are $[0,1,0], [0,0,1], [2,1,0], [1,1,0]$. To find the stabilizer group of h_8 in equation (14), we are doing calculations by help the computer. Thus, the transformation matrices which stabilizing of h_8 with their orders are shown as follows:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}:1, \begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}:2, \begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}:2, \begin{pmatrix}1&0&0\\0&2&0\\0&0&2\end{pmatrix}:2,$$

$\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}:3, \begin{pmatrix}1&0&0\\1&1&0\\0&0&2\end{pmatrix}:6, \begin{pmatrix}1&0&0\\1&2&0\\0&0&1\end{pmatrix}:2, \begin{pmatrix}1&0&0\\1&2&0\\0&0&2\end{pmatrix}:2,$

$\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}:3, \begin{pmatrix}1&0&0\\2&1&0\\0&0&2\end{pmatrix}:6, \begin{pmatrix}1&0&0\\2&2&0\\0&0&1\end{pmatrix}:2, \begin{pmatrix}1&0&0\\2&2&0\\0&0&2\end{pmatrix}:2.$

Therefore, the stabilizer group of h_8 in equation (14) which is denoted by G_{h_8} which contains

- 7 matrices of order 2;
- 2 matrices of order 3;
- 2 matrices of order 6;
- The identity matrix.

From $[6]$, G_{h_8} is isomorphic to $\boldsymbol{D_6}$, that is $G_{h_8} \cong \boldsymbol{D_6}$. h_8 in equation (14) is drawn in Figure 17:

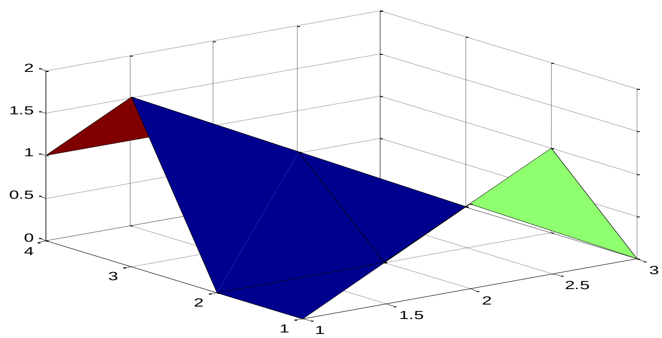

Figure 17: Drawing of h_8

Let $h_8^* = \{3,9\}$ be a subset of h_8 which is forming by partition h_8 into two sets such that h_8^* does not contains the inflection points of h_8, so we note that h_8^* represents an arc of degree two. After the calculation and help the computer, we are obtained that the number of matrices which are stabilizing of h_8^* is 72, and we can not write them, because they are too much. Moreover, the stabilizer group of h_8^* which is denoted by $G_{h_8^*}$ which contains

- 21 matrix of order 2;
- 8 matrices of order 3;
- 18 matrix of order 4;
- 24 matrix of order 6;
- The identity matrix.

Drawing of h_8^* is given in Figure 18 as following:

Journal of Iraqi Al-Khwarizmi Society (JIKhS) Year 2018 pages: 99-116

Volume:2 Special issue of the first international scientific conference of Iraqi Al-Khawarizmi Society 28-29 March 2018

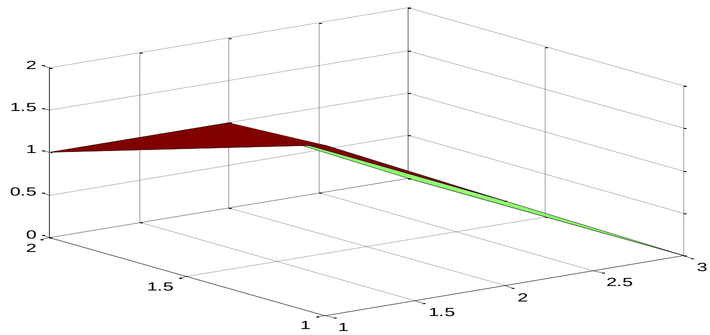

Figure 18: Drawing of h_9^*

Another one of cubic curve which is given in [4] is:

$$h_9 = x^3 + y^3 - z^3 - x^2z - xy^2 + xz^2 - yz^2 \qquad \ldots(15)$$

The points of $PG(2,3)$ on h_9 in equation (15) are $[2,2,1], [1,0,1], [2,1,1]$. To find the stabilizer group of h_9 in equation (15), we are doing calculations by help the computer. Thus the transformation matrices which stabilizing of h_9 with their orders are shown as follows:

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}:2,\begin{pmatrix}0&0&1\\0&1&1\\1&2&1\end{pmatrix}:4,\begin{pmatrix}0&0&1\\0&1&2\\1&1&1\end{pmatrix}:4,\begin{pmatrix}0&0&1\\0&2&1\\1&0&0\end{pmatrix}:2,$$

$$\begin{pmatrix}0&0&1\\0&2&1\\1&1&1\end{pmatrix}:3,\begin{pmatrix}0&0&1\\0&2&2\\1&2&1\end{pmatrix}:3,\begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}:3,\begin{pmatrix}1&1&1\\0&2&1\\0&0&1\end{pmatrix}:2,$$

$$\begin{pmatrix}1&1&1\\1&0&2\\1&2&1\end{pmatrix}:4,\begin{pmatrix}1&1&1\\1&2&0\\1&0&0\end{pmatrix}:3,\begin{pmatrix}1&1&1\\2&0&1\\1&2&1\end{pmatrix}:2,\begin{pmatrix}1&1&1\\2&1&0\\1&0&0\end{pmatrix}:4,$$

$$\begin{pmatrix}2&0&0\\0&1&0\\0&0&2\end{pmatrix}:2,\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}:1,\begin{pmatrix}2&0&0\\1&1&0\\2&1&2\end{pmatrix}:2,\begin{pmatrix}2&0&0\\1&2&0\\2&2&2\end{pmatrix}:3,$$

$$\begin{pmatrix}2&0&0\\2&1&0\\2&2&2\end{pmatrix}:2,\begin{pmatrix}2&0&0\\2&2&0\\2&1&2\end{pmatrix}:3,\begin{pmatrix}2&1&2\\0&1&1\\0&0&2\end{pmatrix}:2,\begin{pmatrix}2&1&2\\0&2&2\\0&0&2\end{pmatrix}:3,$$

$$\begin{pmatrix}2&1&2\\1&0&2\\2&2&2\end{pmatrix}:4,\begin{pmatrix}2&1&2\\1&1&0\\2&0&0\end{pmatrix}:3,\begin{pmatrix}2&1&2\\2&0&1\\2&2&2\end{pmatrix}:2,\begin{pmatrix}2&1&2\\2&2&0\\2&0&0\end{pmatrix}:4.$$

Therefore, the stabilizer group of h_9 in equation (15) which is denoted by G_{h_9} which contains

- 9 matrices of order 2;
- 8 matrices of order 3;
- 6 matrices of order 4;
- The identity matrix.

From [6], G_{h_9} is isomorphic to $\boldsymbol{S_4}$, that is $G_{h_9} \cong \boldsymbol{S_4}$. h_9 in equation (15) is drawn in Figure 19:

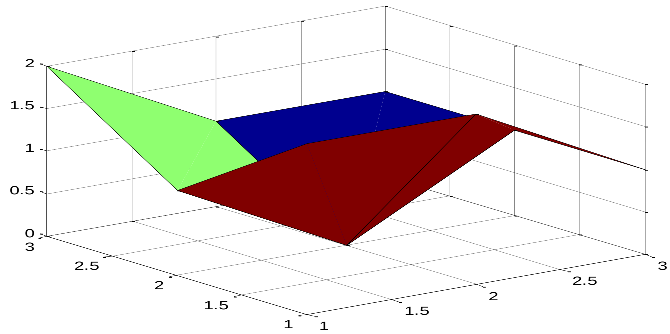

Figure 19: Drawing of h_9

Another one of cubic curve which is given in [4] is:

$$h_{10} = x^3 + y^3 - x^2z - xy^2 + xz^2 - yz^2 \qquad \ldots(16)$$

The points of $PG(2,3)$ on h_{10} in equation (16) are [0,0,1],[0,2,1],[1,2,1],[1,1.1],[0,1,1],[2,0,1]. To find the stabilizer group of h_{10} in equation (16), we are doing calculations by help the computer. Thus the transformation matrices which stabilizing of h_{10} with their orders are shown as follows:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}:1,\begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}:2,\begin{pmatrix}1&0&0\\1&1&0\\1&2&1\end{pmatrix}:3,$$

$$\begin{pmatrix}1&0&0\\1&2&0\\1&1&1\end{pmatrix}:2,\begin{pmatrix}1&0&0\\2&1&0\\1&1&1\end{pmatrix}:3,\begin{pmatrix}1&0&0\\2&2&0\\1&2&1\end{pmatrix}:2.$$

Therefore, the stabilizer group of h_{10}in equation(16) which is denoted by $G_{h_{10}}$which contains

- 3 matrices of order 2;
- 2 matrices of order 3;
- The identity matrix.

From[6], $G_{h_{10}}$is isomorphic to S_3, that is $G_{h_{10}} \cong S_3$. h_{10} in equation (16) is drawn in Figure 20:

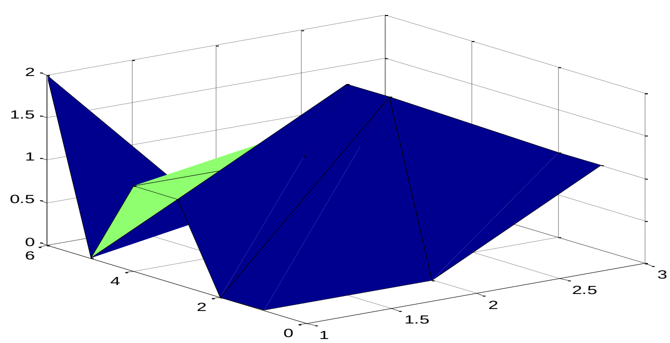

Figure 20 : Drawing of h_{10}

الخلاصة :

الاهداف الرئيسية لهذا البحث هو ايجاد الزمر المثبتة للمنحنيات المكعبة حول الحقل المنتهي من الرتب 2 و 3 ودراسة الخواص لهذه الزمر وكذلك تشكيل الاقواس من الدرجة الثانية والتي تغمر في المنحنيات المكعبة ذات الحجم الزوجي والتي نفسها تعتبر كأقواس من الدرجة الثانية. كذلك رسم كل هذه الأقواس.

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