Radon-Nikodym Theorem with Banach Algebra Valued Measure

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ABSTRACT

In a previous research, we generalized the integration by using the given measure space $(\aleph, \Gamma, \mathcal{M})$, and $\theta: \aleph \longrightarrow \mathcal{W}$ be a measurable function, where $\mathcal{W}$ is an ordered Banach algebra [15].
In this research, we have covered one of the fundamental applications for the integration theory which is Radon-Nikodym theorem which by itself is considered essential in the theory of modern probability and other parts of analysis. Several cases for this result have been considered before proceeding into the general theory and the first step in the development of the general Radon-Nikodym theorem is Jordan Hahn decomposition theorem.
In this paper, we consider the above $\mathcal{W}$ a space with identity element e.

1. Introduction

Here mathematics, Radon-Nikodym theorem is a result in the theory of measure which expresses the relation between two defined measures on the same measurable space [1,5,13]. The derivative of Radon-Nikodym theorem has an important application in the theory of probability so that it leads to the function of probability density of a random variable [7,8]. This theorem has been named after Johnn Radon, who proved the theorem of a special case when the fundamental space is $\mathbb{R}$ in 1913, and Otto Nikodym, who proved the general case in 1930 [14]. In 1936, Hans Freudenthal has further generalized the Radon-Nikodym theorem by proving the Freudenthal spectral theorem as a result in the theory of Riesz space, which contains Radon-Nikodym theorem as a special case [6,9,10]. In this research, we have generalized Radon-Nikodym theorem in Banach algebra space with taking in consideration some of necessary changes.

2. General Set Functions

We remember that Banach algebra measure is a set function $\mathcal{M} : \Gamma \longrightarrow \mathcal{W}$ that satisfies $\mathcal{M}(\Lambda) \succcurlyeq 0$ for all Λ in Γ and $\mathcal{M}(\bigcup_{n=1}^{\infty} \Lambda_n) = \sum_{n=1}^{n\infty} \mathcal{M}(\Lambda_n)$ so that $\{\Lambda_n\}$ is a sequence of disjoint sets in Γ.

Definition 2.1

Let $(\aleph, \Gamma)$ be a measurable space. A set function $\mathcal{M} : \Gamma \longrightarrow \mathcal{W}$ is called a signed Banach algebra measure on Γ, *if* $\mathcal{M}(\bigcup_{n=1}^{\infty} \Lambda_n) = \sum_{n=1}^{\infty} \mathcal{M} (\Lambda_n)$ Whenever $\{\Lambda_n\}$ is a sequence of disjoint sets in Γ.

- Every Banach algebra measure is signed Banach algebra measure and the opposite is not true.

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Definition 2.2

Let λ be a signed Banach algebra measure on the measurable space $(\aleph, \Gamma)$. A set $\Lambda \in \Gamma$ is said to be a positive set (with respect to λ) if $\lambda(B) \succcurlyeq 0$ for each measurable subset B of Λ. Similarly, a set Λ is called a negative set (with respect to λ) if $\lambda(B) \preccurlyeq 0$ for each measurable subset B of Λ. A set that is both positive and negative (with respect to λ) is called null set, i.e. a measurable set is called a null set iff each measurable subset of it has λ measure zero.

Remark 2.3

The distinction between a null set and a set whose measure is zero, is that every null set's measure must be zero and a set whose measure is zero could be a union of two sets whose measure are not zero but are negative of each ether.

Theorem 2.4

Let λ be a signed Banach algebra measure on the measurable space $(\aleph, \Gamma)$, and Λ be a measurable set.

- Λ is positive iff for every measurable set B, $\Lambda \cap B$ is measurable and $\lambda(\Lambda \cap B) \succcurlyeq 0$
- Λ is negative iff for every measurable set B, $\Lambda \cap B$ is measurable and $\lambda(\Lambda \cap B) \preccurlyeq 0$

Proof:

- Assume Λ is positive and let B is a measurable set is measurable set. Since Λ is measurable set $\Longrightarrow$ $\Lambda \cap B$ is measurable set. Since Λ is positive set, $\Lambda \cap B \subseteq \Lambda$ and $\Lambda \cap B$ measurable $\Longrightarrow \lambda(\Lambda \cap B) \succcurlyeq 0$.

  Conversely, let $\Lambda \cap B$ is measurable and $\lambda(\Lambda \cap B) \succcurlyeq 0$ for every measurable set B.

  Let C be a measurable and $C \subseteq \Lambda \Longrightarrow C = \Lambda \cap C \Longrightarrow \lambda(C) = \lambda(\Lambda \cap C) \succcurlyeq 0$ ■

Theorem 2.5

Let λ be a signed Banach algebra measure on the measurable space $(\aleph, \Gamma)$

- Each measurable subset of a positive (rsp. negative) set is positive (rsp. negative).
- The union of countable positive (rsp. negative) sets is positive (rsp. negative).

Proof:

- Let Λ be a measurable subset of a positive set B, and C be a measurable subset of $\Lambda \Longrightarrow C \subseteq B$, since B is positive $\Longrightarrow \lambda(C) \succcurlyeq 0 \Longrightarrow \Lambda$ is positive.
- Let $\{\Lambda_n\}$ be a sequence of positive sets, $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$ and B be a measurable subset of Λ.

  Put $B_n = B \cap \Lambda_n \cap \Lambda_{n-1}^c \cap \ldots \Lambda_1^c \Longrightarrow B_n$ is measurable subset of Λ_n and so $\lambda(B_n) \succcurlyeq 0$. Since the B_n are disjoint and $B = \bigcup_{n=1}^{\infty} B_n$, we have $\lambda(B) = \sum_{n=1}^{\infty} \lambda(B_n) \succcurlyeq 0 \Longrightarrow \Lambda$ is positive ■

Theorem (Hahn-Decomposition) 2.6

Let λ be a B signed Banach algebra measure on the measurable space $(\aleph, \Gamma)$. There is a positive set Λ and a negative set B with

$\Lambda \cap B = \phi$, $\Lambda \cup B = \aleph$.

Proof:

Let $\nu = \sup\{\lambda(\Lambda): \Lambda \text{ is positive set with respect } \lambda\}$. Since ϕ is positive, then $\nu \geqslant 0$.

Let $\{\Lambda_n\}$ be a sequence of positive sets such that $\nu = \lim_{\vartheta \to \infty} \lambda(\Lambda_n)$, Set $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$, by using part (2) of theorem (2.4), we have Λ is positive, also $\lambda(\Lambda) \leqslant \vartheta$.

Since $\Lambda | \Lambda_n \subset \Lambda \implies \lambda(\Lambda | \Lambda_n) \geqslant 0$ and $\lambda(\Lambda) = \lambda(\Lambda_n) + \lambda(\Lambda | \Lambda_n) \geqslant \lambda(\Lambda_n)$, so $\lambda(\Lambda) \geqslant \nu \implies 0 \leqslant \lambda(\Lambda) = \nu \implies \lambda(\Lambda) \geqslant 0$.

Let $B = \Lambda^c$, to prove B is negative, let C be a positive set and $C \subseteq B$, then $\Lambda \cap C = \phi$ and $\Lambda \cup C$ positive set

$\implies \nu \geqslant \lambda(\Lambda \cup C) = \lambda(\Lambda) + \lambda(C) = \nu + \lambda(C) \implies \lambda(C) = 0$, Since $0 \leqslant \nu$, then B does not contain positive subset with a positive measurement, and therefore, does not positively measurements subsets, so B is negative set ■

Remarks 2.7

- The Hahn decomposition is not unique.
- The Hahn decomposition Λ, B give two measures λ^+ and λ^- defined by $\lambda^+(C) = \lambda(\Lambda \cap C)$, $\lambda^-(C) = -\lambda(B \cap C)$,
  Notice that $\lambda^+(B) = 0$ and $\lambda^-(\Lambda) = 0$. Clearly $\lambda = \lambda^+ - \lambda^-$

3. Radon-Nikodym theorem

Radon-Nikodym theorem is among the most important results in real analysis. Regarding its applications, it includes the dual space of L^p, conditional expectation and the change of measure in stochastic analysis. In the beginning of proving Radon-Nikodym theorem in Banach algebra space, we will mention couple of basic definitions regarding this topic.

Definition 3.1

Let $\mathcal{M}$ and λ be two measures on a measurable space $(\aleph, \Gamma)$. We say that $\mathcal{M}$ is singular with respect to λ (written $\mathcal{M} \perp \lambda$) if there are Λ, $B \in \Gamma$ with $\Lambda \cap B = \emptyset$, $\Lambda \cup B = \aleph$ and $\mathcal{M}(\Lambda) = 0$, $\lambda(B) = 0$

Remarks 3.2

- If $\mathcal{M}$ and λ are two Banach algebra measures on a measurable space $(\aleph, \Gamma)$, then $\mathcal{M} \perp \lambda$ if there is a set $\Lambda \in \Gamma$ such that $\mathcal{M}(\Lambda) = 0$, $\lambda(\Lambda^c) = 0$.

- $\mathcal{M}$ is singular with respect to λ iff λ is singular with respect to $\mathcal{M}$, so we can say that $\mathcal{M}$ and λ are mutually singular.
- Let $\mathcal{M}$, λ be two signed Banach algebra measures on the measurable space $(\aleph, \Gamma)$, we say that $\mathcal{M}$ and λ are mutually singular iff $[\![\mathcal{M}]\!] \perp [\![\lambda]\!]$.
- If λ is a signed Banach algebra measure on the measurable space $(\aleph, \Gamma)$, then $\lambda^+ \perp \lambda^-$.

Example 3.3

Let $\aleph = \mathbb{R}, \Gamma = \beta(\mathbb{R})$, $\mathcal{M}$ is Lebesgue measure, $\lambda = \sum_{j \geqslant 1} c_j \, \delta_{q_j}, c_j$ is non–negitive real number

if $\Lambda = Q \implies \lambda(\Lambda^c = Q^c) = 0 \implies \delta_{q_j}(Q^c) = 0$, $\forall$ j $\geqslant 1 \implies \lambda(\Lambda) = 0 \implies \mathcal{M} \perp \lambda$.

Theorem (Jordan-Decomposition) 3.4

Let λ be a signed Banach algebra measure on the measurable space $(\aleph, \Gamma)$. There are two mutually singular measures λ^+ and λ^- so that $\lambda = \lambda^+ - \lambda^-$. This decomposition is unique.

Proof:

Since λ be a signed Banach algebra measure on the measurable space $(\aleph, \Gamma)$, by using Hahn-decomposition, there is a positive set Λ and a negative set B with $\Lambda \cap B = \emptyset$, $\Lambda \cup B = \aleph$, defined λ^+ and λ^- by $\lambda^+(C) = \lambda(\Lambda \cap C)$, $\lambda^-(C) = -\lambda(B \cap C)$ for all $C \in \Gamma$.

$\lambda^+(B) = \lambda(\Lambda \cap B) = \lambda(\emptyset) = 0, \lambda^-(\Lambda) = -\lambda(B \cap \Lambda) = -\lambda(\emptyset) = 0 \implies \lambda^+ \perp \lambda^-$, clearly $\lambda = \lambda^+ - \lambda^-$ ■

Definition 3.5

Let $\mathcal{M}$ and λ be two Banach algebra measures on a measurable space $(\aleph, \Gamma)$. We say that $\mathcal{M}$ is absolute continuous with respect to λ (written $\mathcal{M} \ll \lambda$) if $\lambda(\Lambda) = 0$ implies $\mathcal{M}(\Lambda) = 0$ for every $\Lambda \in \Gamma$.

Example 3.6

- Let $(\aleph, \Gamma, \mathcal{M})$ be a measure space, and $\theta \geqslant 0$ be a measurable function. Define $\lambda(\Lambda) = \int_\Lambda \theta \, d\mathcal{M}$ for all $\Lambda \in \Gamma$.Then $\lambda \ll \mathcal{M}$.
- Let $\aleph = \mathbb{N}, \Gamma = P(\mathbb{N}), \mathcal{M} = \#, \theta(n) = n^{-\alpha}$, $\lambda(\Lambda) = \sum_{n \in \Lambda} n^{-\alpha}$. Then $\lambda \ll \mathcal{M}$, also $\mathcal{M} \ll \lambda$.

Example 3.7

In the following examples, we assume that $\aleph = [0,1], \Gamma = \beta(\aleph)$

- $\mathcal{M}$ represents the length measure on ℵ, λ is a Banach algebra measure that is set for every subset Λ from ℵ that it is twice the length of Λ, then $\lambda \ll \mathcal{M}$ and $\mathcal{M} \ll \lambda$.

- $\mathcal{M}$ represents the length measure on $\aleph$, λ is a measure that is set for every subset Λ from $\aleph$ that it is the number of points of the set $\{0.1, \cdots, 0.9\}$ that's present in Λ, then $\lambda \ll \mathcal{M}, but\ \mathcal{M}$ is not an absolute continuous with respect to λ .

- $\mathcal{M} = \lambda + \delta_0$ so that λ represents the length measure on $\aleph$, and δ_0 represents Dirac measurement on 0, that is $\delta_0(\Lambda) = \begin{cases} 1, & 0 \in A \\ 0, & 0 \notin A \end{cases}$, then $\lambda \ll \mathcal{M}$.

Remark 3.8

- Let $\mathcal{M}$ and λ be two signed Banach algebra measures on the measurable space $(\aleph, \Gamma)$. We say that $\mathcal{M}$ is absolute continuous with respect to λ (written $\mathcal{M} \ll \lambda$) if for every $\Lambda \in \Gamma$ with $\lambda(\Lambda) = 0$, we have $\mathcal{M}(\Lambda) = 0$.

  (Note that $\mathcal{M} \ll \lambda \iff \mathcal{M}^+ \ll \lambda$ and $\mathcal{M}^- \preccurlyeq \lambda \iff [\![\mathcal{M}]\!] \preccurlyeq \lambda$)

- Two Banach algebra measures $\mathcal{M}$ and λ on the measurable space $(\aleph, \Gamma)$, for which $\mathcal{M} \ll \lambda$ and $\lambda \ll \mathcal{M}$ are called equivalent, in symbols $\mathcal{M} \sim \lambda$, i.e. $\mathcal{M} \sim \lambda$ iff $(\mathcal{M}(\Lambda) = 0 \iff \lambda(\Lambda) = 0$ for all $\Lambda \in \Gamma)$

- If $\mathcal{M}$ and λ are Banach algebra measures, then $\mathcal{M} \ll \mathcal{M} + \lambda$ and $\lambda \ll \mathcal{M} + \lambda$.

Theorem 3.9

Let λ be a signed Banach algebra measure and $\mathcal{M}$ be a positive measure , if $\lambda \perp \mathcal{M}$ and $\lambda \ll \mathcal{M}$, then $\lambda = 0$.

Proof:

Since $\lambda \perp \mathcal{M} \Longrightarrow \exists\ \mathrm{E}$ s.t. E is λ–null (i.e. $\lambda(\mathrm{E}) = 0$), and E^c is $\mathcal{M}$–null (i.e. $\mathcal{M}(\mathrm{E}^c) = 0$).

Since $\lambda \ll \mathcal{M}$, we know E^c is λ–null , so $\aleph = \mathrm{E} \cup \mathrm{E}^c$ is λ–null , then $\lambda = 0$ ∎

Theorem 3.10

Let $(\aleph, \Gamma)$ be a measurable space and $\mathcal{M}$ and λ be two Banach algebra measures on Γ so that $\lambda \ll \mathcal{M}$, then $\theta: \aleph \longrightarrow \mathcal{W}$ is a non-negative measurable function so that $\lambda(\Lambda) = \int_\Lambda \theta\, d\mathcal{M}$ for each $\Lambda \in \Gamma$. The function θ is unique $a.e.[\mathcal{M}]$, in other words, if η is another function that satisfies the same condition, then $\theta = \eta\ a.e.$.

Proof:

Let G be a family of the non-negative integrable θ functions with respect to $\mathcal{M}$ so that $\int_A \theta\, d\mathcal{M} \preccurlyeq \lambda(\Lambda)$ for each $\Lambda \in \Gamma$.

It's clear that $\mathrm{G} \neq \phi$ because it includes at least the zero-function. Then G is an ordered subset in the order $\theta \prec \eta$ iff $\theta \prec \eta\ a.e.[\mathcal{M}]$.

We assume that $s = \sup\{\int_{\aleph} \theta \, d\mathcal{M} : \theta \in \mathrm{G}\} \preccurlyeq \lambda(\aleph)$, we will get the supremum element in G. Now, we assume that $\theta_1, \theta_2 \in \mathrm{G}$, we have to prove that $\max\{\theta_1, \theta_2\} \in \mathrm{G}$, which means that we have to prove that $\int_{\Lambda} \max\{\theta_1, \theta_2\} \, d\mathcal{M} \preccurlyeq \lambda(\Lambda)$ for each $\Lambda \in \Gamma$.

Let $\Lambda \in \Gamma$, we define $\Lambda_1 = \{x \in \Lambda : \theta_1(x) \succcurlyeq \theta_2(x)\}$, $\Lambda_2 = \{x \in A : \theta_1(x) \prec \theta_2(x)\}$, then

$\int_{\Lambda} \max\{\theta_1, \theta_2\} \, d\mathcal{M} = \int_{\Lambda_1} \theta_1 \, d\mathcal{M} + \int_{\Lambda_2} \theta_2 \, d\mathcal{M} \preccurlyeq \lambda(\Lambda_1) + \lambda(\Lambda_2) = \lambda(\Lambda)$, therefore $\max\{\theta_1, \theta_2\} \in \mathrm{G}$.

Now, let $\{\eta_n\}$ be a sequence of functions in G so that $\eta_n \longrightarrow s$, and let $\eta_n = \max\{\theta_1, \theta_2, \cdots, \theta_n\}$, then $\eta_n \in \mathrm{G}$.

As long as $\theta_n \preccurlyeq \eta_n$ for all n values, then η_n is converge a.e. to η, that is $\eta_n \uparrow \eta \; a.e.$

By using the monotone convergence theorem, we get $\int_{\aleph} \eta \, d\mathcal{M} = s$

We have to prove that η is an upper bound for the set G; let $h \in \mathrm{G}$, if $h \preccurlyeq \eta_n \; a.e.$ for some of n values, then $h \preccurlyeq \eta \; a.e.$, and if $h \succcurlyeq \eta_n \; a.e.$ for all n values, then $h \succcurlyeq \eta \; a.e.$

Therefore, $\int_{\aleph} h \, d\mathcal{M} = \int_{\aleph} \eta \, d\mathcal{M} = s$, hence $h = \eta \; a.e.$, so η is an upper bound for the set G.

Let $\Lambda \in \Gamma$, then $0 \preccurlyeq \eta_n \mathrm{I}_{\Lambda} \uparrow \eta \mathrm{I}_{\Lambda}$, therefore $\int_{\Lambda} \eta_n \, d\mathcal{M} = \int_{\aleph} \eta_n \mathrm{I}_{\Lambda} \, d\mathcal{M} \uparrow \int_{\aleph} \eta \mathrm{I}_{\Lambda} \, d\mathcal{M} = \int_{\Lambda} \eta \, d\mathcal{M}$

As long as $\eta_n \in \mathrm{G}$, then $\int_{\Lambda} \eta_n \, d\mathcal{M} \preccurlyeq \lambda(\Lambda)$ for all n values, so $\int_{\Lambda} \eta \, d\mathcal{M} \preccurlyeq \lambda(\Lambda)$, therefore $\eta \in \mathrm{G}$, hence G is bounded from above, then by using Zorn's lemma, it possesses a maximum element as θ, which means there is a maximum element $\theta \in \mathrm{G}$.

Now, we have to prove that $\lambda(\Lambda) = \int_{\Lambda} \theta \, d\mathcal{M}$ for each $\Lambda \in \Gamma$, let $v(\Lambda) = \lambda(\Lambda) - \int_{\Lambda} \theta \, d\mathcal{M}$ for each $\Lambda \in \Gamma$, then v is a measure on Γ and $v \ll \mathcal{M}$.

If $v(\aleph) \neq 0$, then $v(\aleph) \succ 0$, therefore $\mathcal{M}(\aleph) - [\![k]\!] v(\aleph) \prec 0$ for some $k \succ 0$, by using ($corollary$ 2.1.3 in [1]), there is $D \in \Gamma$ so that $\mathcal{M}(\Lambda \cap D) - [\![k]\!] v(\Lambda \cap D) \preccurlyeq 0$ and $\mathcal{M}(\Lambda \cap D^c) - [\![k]\!] v(\Lambda \cap D^c) \succcurlyeq 0$ for each $\Lambda \in \Gamma$.

If we assume that $\mathcal{M}(D) = 0$, then $\lambda(D) = 0$ because $\lambda \ll \mathcal{M}$, therefore $v(D) = 0$

As long as $\mathcal{M}(\Lambda \cap D) - [\![k]\!] v(\Lambda \cap D) \preccurlyeq 0$ and $\mathcal{M}(\Lambda \cap D^c) - [\![k]\!] v(\Lambda \cap D^c) \succcurlyeq 0$ for each $\Lambda \in \Gamma$, then $\mathcal{M}(\aleph \cap D) - [\![k]\!] v(\aleph \cap D) \preccurlyeq 0$ and $\mathcal{M}(\aleph \cap D^c) - [\![k]\!] v(\aleph \cap D^c) \succcurlyeq 0$, therefore $\mathcal{M}(D) - [\![k]\!] v(D) \preccurlyeq 0$ and $\mathcal{M}(D^c) - [\![k]\!] v(D^c) \succcurlyeq 0$.

As long as $\mathcal{M}(D) = 0$ and $v(D) = 0$, then $\mathcal{M}(\aleph) - [\![k]\!] v(\aleph) = \mathcal{M}(D^c) - [\![k]\!] v(D^c) \succcurlyeq 0$, but $\mathcal{M}(\aleph) - [\![k]\!] v(\aleph) \prec 0$ and this is contradiction, therefore, $\mathcal{M}(D) \succ 0$.

We define $h(x) = \begin{cases} \frac{1}{[\![k]\!]} & x \in D \\ 0 & x \notin D \end{cases}$, so $\int_{A} h \, d\mathcal{M} = \frac{1}{[\![k]\!]} \mathcal{M}(\Lambda \cap D) \preccurlyeq v(\Lambda \cap D) \preccurlyeq v(\Lambda) = \lambda(\Lambda) - \int_{\Lambda} \theta \, d\mathcal{M}$, therefore $\int_{\Lambda} h \, d\mathcal{M} + \int_{\Lambda} \theta \, d\mathcal{M} \preccurlyeq \lambda(A) \Longrightarrow \int_{\Lambda} (h + \theta) \, d\mathcal{M} \preccurlyeq \lambda(\Lambda)$, but $h + \eta \succ \eta$ on the set D with $\mathcal{M}(D) \succ 0$, and this is a contradiction with θ being the maximum, so $v = 0$, and the proof is done ■

We remember that σ −Banach algebra measure is a set function $\mathcal{M} : \Gamma \longrightarrow \mathcal{W}$ so that for each Λ in Γ; there is a sequence $\{\Lambda_n\}$ of sets in Γ so that $\Lambda \subset \bigcup_{n=1}^{\infty} \Lambda_n$.

Corollary 3.11

Let $(\aleph, \Gamma)$ be a measurable space, $\mathcal{M}$ be a Banach algebra measure and λ be a $\sigma -$ Banach algebra measure on Γ, so that $\lambda \ll \mathcal{M}$. Then $\theta: \aleph \longrightarrow \mathcal{W}$ is a non-negative measurable function so that $\lambda(\Lambda) = \int_{\Lambda} \theta \, d\mathcal{M}$ for each $\Lambda \in \Gamma$. The function θ is unique $a.e.\,[\mathcal{M}]$, in other words, if η is another function that satisfies the same condition, then $\theta = \eta \; a.e.$.

Proof:

As long as λ is a $\sigma -$ Banach algebra measure on Γ and $\aleph \in \Gamma$, then $\{A_n\}$ is a partition for the set $\aleph$.

We define $\lambda_n(A) = \lambda(\Lambda \cap \Lambda_n)$ for each $\Lambda \in \Gamma$, then λ_n is a Banach algebra measure on Γ for each n.

By using the proof (3.10), there is a non-negative measurable function which is $\theta_n: \aleph \longrightarrow \mathcal{W}$ so that $\lambda(\Lambda) = \int_{\Lambda} \theta_n \, d\mathcal{M}$ for each $A \in \Gamma$.

Therefore, $\lambda(\Lambda) = \int_{\Lambda} \theta \, d\mathcal{M}$ for each $\Lambda \in \Gamma$ so that $\theta = \sum_{n=1}^{\infty} \theta_n$ ■

Corollary 3.12

Let $(\aleph, \Gamma)$ be a measurable space, $\mathcal{M}$ be a $\sigma -$ Banach algebra measure and λ be a Banach algebra measure on Γ so that $\lambda \ll \mathcal{M}$, then $\theta: \aleph \longrightarrow \mathcal{W}$ is a non-negative measurable function so that $\lambda(\Lambda) = \int_{\Lambda} \theta \, d\mathcal{M}$ for each $\Lambda \in \Gamma$. The function θ is unique $a.e.\,[\mathcal{M}]$, in other words, if η is another function that satisfies the same condition, then $\theta = \eta \; a.e.$.

Proof:

As long as $\mathcal{M}$ is a $\sigma -$ Banach algebra measure on Γ and $\aleph \in \Gamma$, then $\{\Lambda_n\}$ is a partition for the set $\aleph$. Through the using of the corollary (3.11), $\theta_n: \Lambda_n \longrightarrow \mathcal{W}$ is a non-negative measurable function with respect to Γ_{Λ_n} so that $\lambda(\Lambda \cap \Lambda_n) = \int_{\Lambda \cap \Lambda_n} \theta_n \, d\mathcal{M}$ for each $\Lambda \in \Gamma$

This could be written as $\lambda(\Lambda \cap \Lambda_n) = \int_{\Lambda} \theta_n \, d\mathcal{M}$ so that $\theta_n(\ell)$ is considered as 0 for $\ell \notin \Lambda_n$. Therefore, $\lambda(\Lambda) = \sum_{n=1}^{\infty} \lambda(\Lambda \cap \Lambda_n) = \sum_{n=1}^{\infty} \int_{\Lambda} \theta_n \, d\mathcal{M} = \int_{\Lambda} \theta \, d\mathcal{M}$ for each $\Lambda \in \Gamma$, where $\theta = \sum_{n=1}^{\infty} \theta_n$ ■

Corollary (Radon-Nikodym theorem) 3.13

Let $(\aleph, \Gamma)$ be a measurable space, $\mathcal{M}$ be a $\sigma -$ Banach algebra measure, and λ be a signed Banach algebra measure on Γ so that $\lambda \ll \mathcal{M}$, then $\theta: \aleph \longrightarrow \mathcal{W}$ is a non-negative measurable function so that $\lambda(\Lambda) = \int_{\Lambda} \theta \, d\mathcal{M}$ for each $\Lambda \in \Gamma$. The function θ is unique $a.e.[\mathcal{M}]$, in other words, if η is another function that satisfies the same condition, then $\theta = \eta \; a.e.$.

Proof:

We write $\lambda = \lambda^+ - \lambda^-$, by using the result (5.11); $\theta_1, \theta_2: \aleph \longrightarrow \mathcal{W}$ are non-negative measurable functions so that $\lambda^+(\Lambda) = \int_{\Lambda} \theta_1 \, d\mathcal{M}$, $\lambda^-(\Lambda) = \int_{\Lambda} \theta_2 \, d\mathcal{M}$ for each $\Lambda \in \Gamma$, then $\lambda(\Lambda) = \int_{\Lambda} \theta \, d\mathcal{M}$ for each $\Lambda \in \Gamma$ where $\theta = \theta_1 - \theta_2$ ■

Note:

It's necessary for $\mathcal{M}$ to be a $\sigma -$ Banach algebra in Radon-Nikodym theorem otherwise the fulfillment of the theorem will not be achieved. Below, is an example that explains the un-achievement of Radon-Nikodym theorem when $\mathcal{M}$ is not a $\sigma -$ Banach algebraLet $\mathcal{W} = [0,1]$, $\Gamma = \beta([0,1])$ and $\mathcal{M}$ is a counting measure, then $\mathcal{M}$ is not a $\sigma - finite$ measure (because $\mathcal{M}$ is a counting measure and $\aleph$ is an uncountable set).

If we assume that λ is a Lebesgue measure on $\Gamma = \beta([0,1])$, then $\lambda \ll \mathcal{M}$ because if $\mathcal{M}(\Lambda) = 0$, then $\Lambda = \emptyset$ and therefore $\lambda(\Lambda) = 0$.

Assume that Radon-Nikodym theorem is achieved, in other words, θ is a measurable function so that $\lambda(\Lambda) = \int_{\Lambda} \theta \, d\mathcal{M}$ for each $\Lambda \in \Gamma$. Use $\Lambda = \{\ell\}$ and by using the equality mentioned above, we get $\theta(\ell) = 0$ for each $\ell \in \mathcal{W}$, therefore it results in $\lambda = 0$, which means that Lebesgue measure is zero, and this is a contradiction.

Conclusion

The generalization of Jordan Hahn decomposition theorem to use it to prove the generalization of Radon-Nikodym theorem with Banach algebra valued measure.

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