Soft Banach Algebra: Theory and Applications

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ABSTRACT

Soft Banach Algebras represent a fascinating extension of traditional Banach Algebras, providing a versatile mathematical framework for studying algebraic properties in various applied contexts. This paper offers an overview of key concepts in Soft Banach Algebra theory and explores their fundamental applications.

The exposition begins by introducing the definition and general characteristics of Soft Banach Algebras, highlighting the principal distinctions from conventional Banach Algebras. The paper proceeds to delve into the essential properties of Soft Banach Algebras and demonstrates their applicability in differential and integral calculus.

Furthermore, the paper showcases practical examples and applications of Soft Banach Algebras in fields such as number theory and mathematical physics. This section emphasizes how Soft Banach Algebras can be leveraged to solve practical problems across diverse domains

1 . Introduction

In topology, compact spaces and KC-spaces are of great importance in mathematics and applied sciences. Understanding the properties of these spaces provides a strong foundation for developing theories and solving problems in a wide range of fields, from pure mathematical analysis to practical applications in engineering and sciences .The definition of $\mathcal{K}\mathfrak{c}-$ space (which every compact subset is closed)was presented by [1] and new concepts were introduced through the definition of the following topological spaces $\mathcal{K}(\mathfrak{gc})-$ spaces (which every compact subset is $\mathfrak{g}-$ closed), $\mathfrak{g}\mathcal{K}(\mathfrak{gc})-$ spaces (which every $\mathfrak{g}-$ compact subset is $\mathfrak{g}-$ closed) by S. K. Jassim and H. G. Ali[2]. In this research work, the aim was to introduce new concepts of spaces, which is named $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$spaces . New definitions were also introduced, which are On Weaker Forms of $\mathfrak{g}(\mathcal{K}\mathfrak{c})-$spaces and Co$-\mathfrak{g}-$ compact topologies.

A soft Banach algebra is a mathematical structure that combines elements of both Banach algebras and fuzzy sets. In a traditional Banach algebra, operations like addition and multiplication are defined in a precise, deterministic manner. Soft Banach algebras, on the other hand, introduce a degree of fuzziness or uncertainty in these operations.

In a soft Banach algebra, elements are associated with fuzzy sets, which assign degrees of membership to points in a given set. The operations of addition and multiplication are then extended to operate on these fuzzy sets in a way that respects the underlying algebraic structure.

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This concept finds applications in areas where uncertainty or imprecision play a significant role, such as in fuzzy mathematics, decision making, and optimization problems. Soft Banach algebras provide a framework to model and analyze situations where exact values are not always available or applicable.

2. Soft Algebras

The concept of soft set theory has been initiated by Molodtsov in 1999 as a general mathematical tool for modeling uncertainties. He also pointed out several application of this theory solving many practical problems in economics, engineering, social sciences, medical sciences etc.

Throughout the lecture, let X be an initial universe set and E be the set of parameters. $P(X)$ denote the power set of X and $A \subseteq E$.

Definition (2.1)

A pair (F, A) denoted by F_A is called a soft set over X, where F is a function given by $F : A \to P(X)$. In other words the soft set over X is a parameterized family of subsets of the universal set X. For a particular $e \in A$, $F(e)$ may be considered the set of e - approximate elements of the soft set (F, A) and if $e \notin A$, then $F(e) = \phi$, i.e. $F_A = (F, A) = \{F(e) \in P(X) : e \in A\}$

The set of all soft sets over X is denoted by $S(X)$, and called soft Power Set.

Definition (2.2)

A soft set F_A over X is said to be

1. Null soft set, denoted by ϕ if $F(e) = \phi$ for any $e \in A$.

2. Absolute soft set, denoted by X if $F(e) = X$ for any $e \in A$.

3. Non null soft set if there is at least $e \in A$ such that $A(e) \neq \phi$.

Definition (2.3)

Let $F_A, G_B \in S(X)$. we say that

1. F_A and G_B are soft equal (or F_A soft equals G_B),which we write as $F_A = G_B$, if $A = B$ and $F(e) = G(e)$ for all $e \in A$.

2. F_A is a soft subset of G_B, and denoted by $F_A \subseteq G_B$ if $A \subseteq B$ and $F(e) \subseteq G(e)$ for all $e \in A$.

Hence $F_A = G_B$ iff $F_A \subseteq G_B$ and $G_B \subseteq F_A$

3. F_A is a soft proper subset of G_B, and denoted by $F_A \subset G_B$ if $A \subset B$ and $F(e) \subset G(e)$ for all $e \in A$.

Definition (2.4)

A soft set F_A over X is called

1.A soft point and its denoted by $p_e^x = \{(e, F(e))\}$, if exactly one $e \in A$, $F(e) = \{x\}$

for some $x \in X$ and $F(y) = \phi$ for all $y \in A \mid \{e\}$.

2. A singleton soft set if there is $x \in X$ such that $F(e) = \{x\}$ for all $e \in A$.

Definition (2.5)

Let $F_A \in S(X)$. An element $x \in X$ is said to be belongs to the soft set F_A over X, denoted by $x \in F_A$ if $x \in F(e)$ for all $e \in A$. In other words, we say that $x \in F_A$ read as x belongs to the soft set F_A whenever $x \in F(e)$ for all $e \in A$.

Note that for any $x \in X$, $x \notin F_A$, if $x \notin F(e)$ for some $e \in A$.

Definition (2.6)

Let X be a nonempty set and A be a nonempty parameter set.

1. The function $\varepsilon : A \to X$ is said to be a soft element of X.

2. A soft element ε of is said to belongs to a soft set B of X, which is denoted by $\varepsilon \tilde{\in} B$, if $\varepsilon(e) \in B(e)$ for all $e \in B$ if.

Definition (2.7)

Let $\mathbb{R}$ be the set of real numbers and $B(\mathbb{R})$ be the collection of all nonempty bounded subsets of and A taken as a set of parameters. The function $F : A \to B(\mathbb{R})$ is called a soft real set. It is denoted by (F, A) or F_A

1. A soft real set F_A is said to be nonnegative soft real set if $F(e)$ is a subset of the set of nonnegative real numbers for each $e \in E$.

2. Let $\mathbb{R}(E)$ denotes the set of all soft real sets .Also $\mathbb{R}(E)^*$ denote the set of all nonnegative soft real sets.

If specifically F_A is a singleton soft set, then identifying F_A with the corresponding soft element, it will be called a soft real number and denoted $\tilde{r}$, $\tilde{s}$, $\tilde{t}$ etc. hence $\mathbb{R}(E)$ denote the set of all sort real numbers.

$\tilde{0}, \tilde{1}$ are the soft real numbers where $\tilde{0}(e) = 0$, $\tilde{1}(e) = 1$ for all $e \in E$, respectively.

Definition (2.8)

Let $F_A, G_B \in S(X)$

1. The union of F_A and G_B over X, denoted by $F_A \cup G_B$ is the soft set H_C where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e), & e \in A \mid B \\ G(e), & e \in B \mid A \\ F(e) \cup G(e), & e \in A \cap B \end{cases}.$$

and is written as $F_A \cup G_B = H_C$.

2. The intersection of F_A and G_B over X, denoted by $F_A \cap G_B$ is the soft set H_C where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$, and is written as $F_A \cap G_B = H_C$.

Definition (2.9)

Let X be a linear space over a field F and let A be a parameter set. Let F_A and G_A be two soft set over X and $\lambda \in$ F. Define

1. $(F+G)(e) = \{x + y : x \in F(e),\ y \in G(e)\}$ for all $e \in A$

2. $(\lambda F)(e) = \{\lambda x : x \in F(e)\}$ for all $e \in A$

If $F_1, F_2, \cdots, F_n$ are n soft sets over (X, A), then
$(F_1 + F_2 + \cdots, F_n)(e) = \{x_1 + x_2 + \cdots + x_n : x_i \in F(e),\quad i = 2, 3, \cdots, n\}$ for all $e \in A$

Example (2.10)

Consider the Euclidian n-dimensional space $□^n$ over □ . Let $A = \{1, 2, 3, \cdots, n\}$ be the set of parameters. Let $F : A \to P(□^n)$ be defined as follows :

$$F(i) = \{t \in □^n : \quad i\text{-th co} - \text{ordinate of } t \text{ is } 0\},\ i = 1, 2, \cdots, n.$$

Then F is a soft linear space or soft linear space of $□^n$ over □ .

Theorem (2.11)

$\lambda(F_A + G_B) = \lambda F_A + \lambda G_B$ for all soft sets F_A and G_A over X and $\lambda \in$ F.

Proof :

$$(\lambda(F+G))(e) = \{\lambda z : z \in (F+G)(e)\} = \{\lambda(x+y) : x \in F(e),\ y \in G(e)\} = \{\lambda x + \lambda y : x \in F(e),\ y \in G(e)\}$$

$(\lambda F+\lambda G))(e)=\{x'+y':x'\in(\lambda F)(e), y'\in(\lambda G)(e)\}==\{\lambda x''+\lambda y'':x''\in F(e), y''\in G(e)\}$

Hence the result follows.

Theorem (2.12)

Let F_A be a soft set over X

1. If $x\in X$, then $x+F_A$ is a soft set over X defined as follows :

$$(x+F)(e)=\{x+y:y\in F(e)\} \text{ for all } e\in A$$

2. If $M\subseteq X$, then $M+F_A$ is a soft set over X defined as follows :

$$M+F_A=\bigcup_{x\in M}(x+F_A)\text{, i.e. } (x+F)(e)=\{x+y:y\in F(e)\} \text{ for all } e\in A$$

Definition (2.13)

Let X be a linear space over a field F and let A be the parameter set . A soft set F_A over X is said to be a soft linear space or soft vector space of X over F if $F(e)$ is a subspace of X for all $e\in A$.

Definition (2.14)

Let X be a linear space over a field F . Let $x\in X$ and F_A be a

A soft set G_A over X is said to be a soft subspace of a soft linear space F_A of X over F if

1. $G_A\subseteq F_A$, i.e. $G_A(e)\subseteq F_A(e)$ for all $e\in A$.

2. G_A is a soft linear space of X over F , i.e. $G(e)$ is a subspace of X for all $e\in A$.

Theorem (2.15)

A soft subset G_A of a soft linear space F_A of X over F is a soft subspace of F_A iff $\alpha G_A+\beta G_A\subseteq G_A$ for all $\alpha,\beta\in$ F .

Proof :

Let F_A be a soft linear space of X over F

Suppose that G_A is a soft subspace of F_A, then $G(e)$ is a subspace of X for all $e\in A$.

Let $e\in A$, then $(\alpha G+\beta G)(e)=\{x'+y':x'\in\alpha G(e), y'\in\lambda G(e)\}=\{\alpha x+\beta y:x,y\in G(e)\}$

Since $x,y\in G(e)$ and $\alpha,\beta\in$ F , then $\alpha x+\beta y\in G(e)$, so $(\alpha G+\beta G)(e)\subseteq G(e)$

Hence $\alpha G_A+\beta G_A\subseteq G_A$ for all $\alpha,\beta\in$ F .

Conversely, let the given condition hold.

For $e \in A$ let $x, y \in G(e)$ and $\alpha, \beta \in \mathrm{F}$, then $(\alpha G + \beta G)(e) = \{\alpha x + \beta y : x, y \in G(e)\}$

Since $\alpha G_A + \beta G_A \subseteq G_A$ for all $\alpha, \beta \in \mathrm{F}$, i.e. $(\alpha G + \beta G)(e) \subseteq G(e)$, so $\{\alpha x + \beta y : x, y \in G(e)\} \subseteq G(e)$

Hence $\alpha x + \beta y \in G(e)$ for all $x, y \in G(e)$ and $\alpha, \beta \in \mathrm{F}$,i.e. $G(e)$ is a subspace of X for all $e \in A$.

Since G_A is a soft subset F_A, i.e. $G_A(e) \subseteq F_A(e)$ for all $e \in A$.

Therefore G_A is a soft subspace of F_A.

Corollary (2.16)

If G_A and H_A are soft subspaces of F_A of X over F, then $G_A + H_A$ and λF_A are soft subspaces of F_A of X over F.

Corollary (2.17)

If $\{G_i\}$ baa family of soft subspace of F_A of X over F, then $\bigcap_{i \in J} G_i$ is a soft subspace of F_A of X over F.

Definition (2.18)

Let X be an algebra over a field F and let A be the parameter set. A soft set F_A over X is said to be a soft algebra of X over F if $F(e)$ is a subalgebra of X for all $e \in A$.

It is very easy to see that in a soft algebra the soft elements satisfy the properties :

1. $(xy)\tilde{z} = x(y\tilde{z})$

2. $x(y + \tilde{z}) = xy + x\tilde{z}$ and $(x + y)\tilde{z} = x\tilde{z} + y\tilde{z}$

3. $\lambda(xy) = (\lambda x)y = x(\lambda y)$ where for all $x, y, \tilde{z} \in F_A$ and for any soft scalar λ ,

$$(xy)(e) = x(e)y(e) \text{ and } (\lambda x)(e) = \lambda(e)x(e) \text{ for all } e \in A$$

Definition (2.19)

Let F_A be a soft algebra of X over F

1. F_A is called a commutative soft algebra if $xy = yx$ for all $x, y \in F_A$

2. A soft element $\tilde{e} \in F_A$ is called the soft identity of F_A if $x\tilde{e} = \tilde{e}x = x$ for all $x \in F_A$

3. A soft element $x \in F_A$ is said to be invertible if it has inverse in F_A, i.e. if there exists a soft element $y \in F_A$ such that $xy = yx = \tilde{e}$ and the y is called the inverse of $\tilde{x}$, denoted by x^{-1}. Otherwise x^{-1} is said to be non-invertible soft element of F_A.

3. Soft Normed Spaces

Let X be a linear space over a field F, X is also our initial universe set and A be a nonempty set of parameters. Let X be the absolute soft linear space, i.e., $X(e) = X$, for all $e \in A$. We use the notation $x, y, \tilde{z}$ to denote soft vectors of a soft linear space and $\tilde{r}, \tilde{s}, \tilde{t}$ to denote soft real numbers whereas $\tilde{r}, \tilde{s}, \tilde{t}$ will denote a particular type of soft real numbers such that $\tilde{r}(e) = r$, for all $e \in A$ etc. For example $\tilde{0}$ is the soft real number such that $\tilde{0}(e) = 0$, for all $e \in A$. Note that, in general, $\tilde{r}$ is not related to r.

Definition (3.1)

Let X be the absolute soft linear space . The function $\|\cdot\| : SE(X) \to □(A)^*$ is said to be a soft norm on the soft linear space X, if $\|\cdot\|$ satisfies the following conditions :

1. $\|x\| \geq \tilde{0}$ for all $x \in X$

2. $\|x\| = \tilde{0}$ iff $x = \tilde{0}$

3. $\|\tilde{r} \cdot x\| = |\tilde{r}|\|x\|$ for all $x \in X$ and for every soft scalar $\tilde{r}$,

4. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

The soft linear space X with a soft norm $\|\cdot\|$ on X is said to be a soft normed linear space and is denoted by $(X, \|\cdot\|, A)$ or $(X, \|\cdot\|)$.

Example (3.2)

Let $□(A)$ be the set of all soft real numbers. Then the function $\|\cdot\| : □(A) \to □(A)^*$ which is defined by $\|x\| = |x|$, for all $x \in □(A)$, where $|x|$ denotes the modulus of soft real numbers, is a soft norm on $□(A)$ and since $SS(□(A)) = □$, thus $(□, \|\cdot\|, A)$ or $(□, \|\cdot\|)$ is a soft normed space. With the same argument $SS(□(A)) = □$ is also a soft normed space.

Example (3.3)

Let X be a normed space. In this case, for every $x_e \in SV(X)$, $\|x_e\| = |e| + \|x\|$ is a soft norm.

Proof :

1. Let $x_e \in SV(X)$, then $\|x_e\| = |e| + \|x\| \geq \tilde{0}$

2. Let $x_e \in SV(X)$, then $\|x_e\| = 0$ iff $|e| + \|x\| = \tilde{0}$, iff $e = 0$ and $x = 0$ iff $x_e = \tilde{0}$

3. Let $x_e \in SV(X)$ and for every soft scalar $\tilde{r}$, then

$$\|\tilde{r} \cdot x_e\| = \|(r \cdot x)_{re}\| = |re| + \|r \cdot x\| = |r|\,(|r| + \|x\|) = |\tilde{r}|\,\|x_e\|$$

4. Let $x_e, y_{e'} \in SV(X)$, then

$$\|x_e + y_{e'}\| = \|(x+y)_{(e+e')}\| = |e+e'| + \|x+y\| \leq |e| + |e'| + \|x\| + \|y\| = (|e| + \|x\|) + (|e'| + \|y\|) = \|x_e\| + \|y_{e'}\|.$$

Theorem (3.4)

Every parametrized family of crisp norms $\{\|\cdot\|_e : e \in A\}$ on a crisp linear space X can be considered as a soft norm on the soft linear space X.

Proof :

Let X be the absolute soft linear space over a field F, A be a nonempty set of parameters. Let $\{\|\cdot\|_e : e \in A\}$ be a family of crisp norms on the linear space X. Let $x \in X$, then $x(e) \in X$, for every $e \in A$. Let us define a function $\|\cdot\| : X \to \mathbb{R}(A)^*$ by $\|x\|(e) = \|x(e)\|_e$ for all $x \in X$, for all $e \in A$.

Then $\|\cdot\|$ is a soft norm on X.

To verify it we now verify the conditions 1,2,3 and 4 for soft norm.

1. We have $\|x\|(e) = \|x(e)\|_e \geq 0$ for all $e \in A$, for all $x \in X$, then $\|x\| \geq \tilde{0}$ for all $x \in X$

2.Let $x \in X$, then $\|x\| = \tilde{0}$ iff $\|x\|(e) = \theta$ for all $e \in A$ iff $\|x(e)\|_e = \theta$ for all $e \in A$ iff $x(e) = \theta$ for all $e \in A$ iff $x = \tilde{0}$

3. Let $x \in X$ and $\tilde{r}$ soft scalar, then $\|\tilde{r} \cdot x\|(e) = \|(\tilde{r} \cdot x)(e)\|_e = |\tilde{r}|\,\|x(e)\|_e = (|\tilde{r}|\,\|x\|)(e)$ for all $e \in A$, so

$\|\tilde{r} \cdot x\| = |\tilde{r}|\,\|x\|$ for all $x \in X$ and for every soft scalar $\tilde{r}$,

4. Let $x, y \in X$, then $(\|x\| + \|y\|)(e) = \|x\|(e) + \|y\|(e) = \|x(e)\|_e + \|y(e)\|_e \geq \|x(e) + y(e)\|_e = \|x+y\|(e)$ for all

$e \in A$, so $\|x+y\| \leq \|x\|+\|y\|$ for all $x, y \in X$

$\|\cdot\|$ is a soft norm on X and consequently $(X,\|\cdot\|)$ is a soft normed space.

Theorem (3.5)

Every crisp norm $\|\cdot\|_X$ on a crisp linear space X can be extended to a soft norm on the soft linear space X .

Proof :

First we construct the absolute soft vector space X using a nonempty set of parameters A.

Let us define a function $\|\cdot\|: SE(X) \to \mathbb{R}(A)^*$ by $\|x\|(e)=\|x(e)\|_X$ for all $x \in X$, for all $e \in A$.

Then using the same procedure as theorem (5.3), it can be easily proved that $\|\cdot\|$ is a soft norm on X .

This soft norm is generated using the crisp norm $\|\cdot\|_X$ and it is said to be the soft norm generated by $\|\cdot\|_X$.

Theorem (3.6)

Let $(X,\|\cdot\|, A)$ is a soft normed space , then

1. for any $x \in X$ and $e \in A$, then $\|x\|(e)=0$ iff $x(e)=\theta$, for any $x \in X$ and $e \in A$.

2. $\{\|x\|(e) : x(e)=x\}$ is a singleton set , for each $x \in X$ and $e \in A$

3. for each $e \in A$, define $\|\cdot\|_e : X \to \mathbb{R}^+$ be the function such that for each $x \in X$, $\|x\|_e=\|x\|(e)$,

   where $x \in X$ such that $x(e)=x$. Then for each $e \in A$, $\|\cdot\|_e$ is a norm on X .

Proof :

1. Let us consider a soft scalar λ such that $\lambda(e')=1$, if $e'=e$, $\lambda(e')=0$ if $e' \neq e$.

Then $(\lambda x)(e)=\theta$ for $e' \neq e$, $(\lambda x)(e)=x(e')$ for $e=e'$. We have $\|\lambda x\|=|\lambda|\|x\|$.

This shows that $\|x\|(e)=\tilde{0}$ iff $\lambda x=\theta$, iff $x(e)=\theta$.

2. Let $x, y \in X$, we have $\|x\|=\|x-y+y\| \leq \|x-y\|+\|y\| \quad \Rightarrow \quad \|x\|-\|y\| \leq \|x-y\|$.

Similarly $\|y\|-\|x\|\leq\|x-y\|$. So $\left|\|x\|-\|y\|\right|\leq\|x-y\|$. Now if $x,y\in X$ such that $x(e)=y(e)$ then $\left|\|x\|(e)-\|y\|(e)\right|\leq\|x-y\|(e)=0$ (by 1) since $(x-y)(e)=x(e)-y(e)=0$. i.e. $\|x\|(e)=\|y\|(e)$,

which proves (2).

3. Since for $e\in A$, $\{\|x\|(e): x(e)=x\}$ is a singleton set, the function $\|\cdot\|_e : X\to\mathbb{R}^+$ is well defined. Hence from soft norm axioms, it follows that $\|\cdot\|_e$ is a norm on X . .

Theorem (3.7) Decomposition Theorem

Let $(X,\|\cdot\|)$ is a soft normed space satisfies the following condition

N_5 : For $x\in X$ and $e\in A$, the set $\{\|x\|(e): x(e)=x\}$ is a singleton set and if for each $e\in A$, $\|\cdot\|_e : X\to\mathbb{R}^+$ be a function such that for each $x\in X$, $\|x\|_e=\|x\|(e)$, where $x\in X$ such that $x(e)=x$. Then for each $e\in A$, $\|\cdot\|_e$ is a norm on X .

Proof :

Clearly $\|\cdot\|_e : X\to\mathbb{R}^+$ is a rule that assign a vector of X to a nonnegative crisp real number for all $e\in A$. Now the well defined property of $\|\cdot\|_e$, for all $e\in A$ is follows from the condition N_5 and the soft norm axioms gives the norm conditions of $\|\cdot\|_e$, for all $e\in A$. Thus the soft norm satisfying N_5 gives a parameterized family of crisp norms.

Theorem (3.8)

Let $(X,\|\cdot\|,A)$ be a soft normed space. Let us define $d: X\times X\to\mathbb{R}(A)^*$ by $d(x,y)=\|x-y\|$, for all $x,y\in X$. Then d is a soft metric on X .

Proof :

1. Let $x,y\in X$, then $d(x,y)=\|x-y\|\geq\tilde{0}$

2. Let $x,y\in X$, then $d(x,y)=\tilde{0} \iff \|x-y\|=\tilde{0} \iff x-y=\tilde{0} \iff x=y$

3. Let $x,y\in X$, then $d(x,y)=\|x-y\|=\|y-x\|=d(y,x)$

4. Let $x,y,\tilde{z}\in X$, then $\|x-\tilde{z}\|=\|(x-y)+(y-\tilde{z})\|\leq\|x-y\|+\|y-\tilde{z}\|$, so $d(x,\tilde{z})\leq d(x,y)+d(y,\tilde{z})$

d is a soft metric on X , d is said to be the soft metric induced by the soft norm $\|\cdot\|$. From the above theorem it also follows that every soft normed space is also a soft metric space.

Theorem (3.9) Translation invariance

A soft metric d induced by a soft norm $\|\cdot\|$ on a normed linear space $(X,\|\cdot\|)$ satisfies

1. $d(x\tilde{+}z, y\tilde{+}z) = d(x,y)$, for all $x, y, \tilde{z} \in X$

2. $d(\tilde{r}\cdot x, \tilde{r}\cdot y) = |\tilde{r}|\, d(x,y)$, for all $x, y \in X$ and for every soft scalar $\tilde{r}$.

Proof :

We have,

$d(x\tilde{+}z, y\tilde{+}z) = \|(x\tilde{+}z)-(y\tilde{+}z)\| = \|x-y\| = d(x,y)$ and
$d(\tilde{r}\cdot x, \tilde{r}\cdot y) = \|\tilde{r}\cdot x - \tilde{r}\cdot y\| = \|\tilde{r}\cdot(x-y)\| = |\tilde{r}|\|x-y\| = |\tilde{r}|\, d(x,y)$

Theorem (3.10)

Let $d : X \times X \to \square(A)^*$ be a soft metric. X is a soft normed space iff the following conditions :

1. $d(x\tilde{+}z, y\tilde{+}z) = d(x,y)$, for all $x, y, \tilde{z} \in X$

2. $d(\tilde{r}\cdot x, \tilde{r}\cdot y) = |\tilde{r}|\, d(x,y)$, for all $x, y \in X$ and for every soft scalar $\tilde{r}$.

satisfied.

Proof :

If $d(x,y) = \|x-y\|$, from theorem(3.9) , we have then

$$d(x\tilde{+}z, y\tilde{+}z) = d(x,y) \text{ and } d(\tilde{r}\cdot x, \tilde{r}\cdot y) = |\tilde{r}|\, d(x,y)$$

Suppose that the conditions of the theorem are satisfied .

Taking $\|x\| = d(x,\tilde{0})$ for every $x \in X$ we have

1.Let $x \in X$, then $\|x\| = d(x,\tilde{0}) \geq \tilde{0}$

2. Let $x \in X$, then $\|x\| = \tilde{0} \Leftrightarrow d(x,\tilde{0}) = \tilde{0} \Leftrightarrow x = \tilde{0}$

3. Let $x \in X$ and for every soft scalar $\tilde{r}$, then

$$\|\tilde{r}\cdot x\| = d(\tilde{r}\cdot x,\tilde{0}) = d(\tilde{r}\cdot x, \tilde{r}\cdot\tilde{0}) = |\tilde{r}|\, d(x,\tilde{0}) = |\tilde{r}|\|x\|$$

4. Let $x, y \in X$, then

$$\|x+y\| = d(x+y,\tilde{0}) = d(x,-y) \le d(x,\tilde{0}) + d(\tilde{0},-y) = \|x\| + |-1|\|y\| = \|x\| + \|y\|$$

Definition (3.11)

Let $(X,\|\cdot\|)$ be a soft normed space and (Y,A) be a non-null member of $S(X)$. Then the function $\|\cdot\|_Y : SE(Y) \to \mathbb{R}(A)^*$ given by $\|x\|_Y = \|x\|$ for all $x \in Y$ is a soft norm on Y.

This norm $\|\cdot\|_Y$ is known as the relative norm induced on Y by $\|\cdot\|$. The soft normed space $(Y,\|\cdot\|_Y, A)$ is called a normed subspace or simply a subspace of the soft normed space $(X,\|\cdot\|,A)$.

Definition (3.12)

Let $(X,\|\cdot\|,A)$ be a soft normed space and $\tilde{r} \ge \tilde{0}$ be a soft real number. We define the followings ;

$$\beta(a,\tilde{r}) = \{x : \|x-a\| < \tilde{r}\} \subset SE(X),\ \overline{\beta}(a,\tilde{r}) = \{x : \|x-a\| \le \tilde{r}\} \subset SE(X) \text{ and}$$

$$S(a,\tilde{r}) = \{x : \|x-a\| = \tilde{r}\} \subset SE(X)$$

$\beta(a,\tilde{r})$, $\overline{\beta}(a,\tilde{r})$ and $S(a,\tilde{r})$ are respectively called an open ball, a closed ball and a sphere with centre at a and radius $\tilde{r}$. $SS(\beta(a,\tilde{r}))$, $SS(\overline{\beta}(a,\tilde{r}))$ and $SS(S(a,\tilde{r}))$ are respectively called a soft open ball, a soft closed ball and a soft sphere with centre at x and radius $\tilde{r}$.

Definition (3.13)

A sequence of soft elements $\{x_n\}$ in a soft normed space $(X,\|\cdot\|,A)$ is said to be convergent in $(X,\|\cdot\|,A)$ if there is a soft element $x \in X$ such that $\|x_n - x\| \to \tilde{0}$ as $n \to \infty$. This means for every $\varepsilon > \tilde{0}$, chosen arbitrarily, there exists a natural number $k = k(\varepsilon)$, such that $\tilde{0} \le \|x_n - x\| < \varepsilon$, whenever $n > k$. i.e., $n > k \quad x \in \beta(x,\varepsilon)$. We denote this by $x_n \to x$ as $n \to \infty$ or by

$\lim_{n\to\infty} x_n = x$, x is said to be the limit of the sequence x_n as $n \to \infty$.

Example (3.14)

Let us consider the set of all real numbers endowed with the usual norm $\|\cdot\|$. Let $(\mathbb{R},\|\cdot\|)$ or $(\mathbb{R},\|\cdot\|,A)$ be the soft norm generated by the crisp norm $\|\cdot\|$, where A is the nonempty set of parameters. Let $Y_A \subset \mathbb{R}$ such that $Y(e) = (0,1]$ in the real line, for all $e \in A$. Let us choose a sequence $\{x_n\}$ of soft elements of Y_A where $\tilde{x}_n(e) = \frac{1}{n}$ for all $n \in \mathbb{N}$, for all $e \in A$. Then there is $x \in Y_A$ such that $x_n \to x$ in $(Y,\|\cdot\|_Y, A)$. However the sequence $\{y_n\}$ of soft elements of Y_A where $y_n(e)=\frac{1}{2}$ for all $n\in\mathbb{N}$, for all $e\in A$. is convergent in $(Y,\|\cdot\|,A)$ and converges to $\frac{1}{2}$.

Theorem (3.15)

Limit of a sequence in a soft normed space, if exists is unique.

Proof :

If possible let there exists a sequence $\{x_n\}$ of soft elements in a soft normed space $(X,\|\cdot\|,A)$ such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} x_n = y$, where $x\neq y$. Then there is at least one $e\in A$ such that $\|x-y\|(e)\neq 0$. We consider a positive real number ε_e satisfying $0<\varepsilon_e<\frac{1}{2}\|x-y\|(e)$.

Let $\varepsilon>\tilde{0}$ with $\varepsilon(e)=\varepsilon_e$. Since $x_n\to x$, $x_n\to y$

Corresponding to $\varepsilon>\tilde{0}$, there exist natural numbers $k_1=k_1(\varepsilon)$, $k_2=k_2(\varepsilon)$ such that $n>k_1$
$\Rightarrow\ x_n\in\beta(x,\varepsilon)\ \Rightarrow\ \|x_n-x\|<\varepsilon\ \Rightarrow\ \|x_n-x\|(e)<\varepsilon_e$, in particular.

Also, $n>k_2\ \Rightarrow\ x_n\in\beta(y,\varepsilon)\ \Rightarrow\ \|x_n-y\|<\varepsilon\ \Rightarrow\ \|x_n-y\|(e)<\varepsilon_e$, in particular.

Hence for all $n>k=\max\{k_1,k_2\}$, $\Rightarrow\ \|x-y\|<\varepsilon\ \Rightarrow\ \|x-y\|(e)\le\|x_n-x\|(e)+\|x_n-y\|(e)<2\varepsilon_e$

So, $\varepsilon_e>\frac{1}{2}\|x-y\|(e)$. This contradicts our hypothesis. Hence the result follows.

Definition (3.16)

A sequence $\{x_n\}$ of soft elements in $(X,\|\cdot\|,A)$ is said to be bounded if the set $\{\|x_n-x_m\|:n,m\in\mathbb{N}\}$ of soft real numbers is bounded, i.e., the there exist $k>\tilde{0}$ such that $\|x_n-x_m\|\le k$ for all $n,m\in\mathbb{N}$

Definition (3.17)

A sequence $\{x_n\}$ of soft elements in a soft normed space $(X,\|\cdot\|,A)$ is said to be a Cauchy sequence in X if corresponding to every $\varepsilon>\tilde{0}$, there exist $k\in\mathbb{N}$ such that $\|x_n-x_m\|\le k$, for all $n,m\ge k$, i.e., $\|x_n-x_m\|\to\tilde{0}$ as $n,m\to\infty$

Theorem (3.18)

Every convergent sequence in a soft normed linear space is Cauchy and every Cauchy sequence is bounded.

Proof :

Let $\{x_n\}$ be a convergent sequence of soft elements with limit x (say) in $(X,\|\cdot\|)$

Then corresponding to each $\varepsilon \tilde{>} \tilde{0}$, there exists $k \in \mathbb{N}$ such that $x_n \in \beta(x,\frac{\varepsilon}{2})$ i.e., $\|x_n - x\| \le \frac{\varepsilon}{2}$ for all $n \ge k$ .Then for $n,m \ge k$, $\|x_n - x_m\| \le \|x_n - x\| + \|x - x_m\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence $\{x_n\}$ is a Cauchy sequence.

Next let $\{x_n\}$ be a Cauchy sequence of soft elements in $(X,\|\cdot\|)$. Then there exists $k \in \mathbb{N}$ such that $\|x_n - x_m\| < \tilde{1}$, for all $n,m \ge k$. Take M with $M(e) = \max\{\|x_n - y_m\|(e) : 1 \le n,m \le k\}$ for all $e \in A$. Then for $1 \le n \le k$ and $m \ge k$, $\|x_n - y_m\| \le \|x_n - y_k\| + \|x_k - y_m\| < M + \tilde{1}$.

Thus, $\|x_n - y_m\| < M + \tilde{1}$ for all $n,m \in \mathbb{N}$ and consequently the sequence is bounded.

Definition (3.19)

A soft subset Y_A with $Y(e) \ne \phi$ for all $e \in A$, in a soft normed space $(X,\|\cdot\|,A)$ is said to be bounded if there exists a soft real number k such that $\|x\| \le k$ for all $x \in Y_A$.

Definition (3.20)

A soft normed space $(X,\|\cdot\|,A)$ is said to be complete if every Cauchy sequence in X converges to a soft element of X i.e., every complete soft normed space is called a soft Banach's Space.

Theorem (3.21)

Let $(X,\|\cdot\|,A)$ be a soft normed space. Then

1. If $x_n \to x$ and $y_n \to y$ then $x_n + y_n \to x + y$.

2. If $x_n \to x$ and $\lambda_n \to \lambda$ then $\lambda_n \cdot x_n \to \lambda \cdot x$., where $\{\lambda_n\}$ is a sequence of soft scalars.

3. If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and $\{\lambda_n\}$ is a Cauchy sequence of soft scalars, then $\{x_n + y_n\}$ and $\{\lambda_n \cdot x_n\}$ are also Cauchy sequences in X .

Proof :

1. Since $x_n \to x$ and $y_n \to y$, for $\varepsilon > \tilde{0}$, there exist a positive integers k_1, k_2 such that $\|x_n - x\| < \frac{\varepsilon}{2}$ for all $n > k_1$ and $\|y_n - y\| < \frac{\varepsilon}{2}$ for all $n > k_2$. Let $k = \max\{k_1, k_2\}$, then both the above relations hold for $n \geq k$. Then

$$\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leq \|x_n - x\| + \|y_n - y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for } n \geq k .$$

$\Rightarrow \quad x_n + y_n \to x + y$.

2. Since $x_n \to x$ and $\lambda_n \to \lambda$ we get, for $\varepsilon > \tilde{0}$, , there exist a positive integers k such that $\|x_n - x\| < \varepsilon$ for all $n \geq k$

Now, $\|x_n\| = \|x_n - x + x\| \leq \|x_n - x\| + \|x\| < \varepsilon + \|x\|$, for all $n \geq k$ $\Rightarrow$ $\|x_n\| < \varepsilon + \|x\|$ for all $n \geq k$

Thus the sequence $\{\|x_n\|\}$ is bounded.

Now,

$$\|\lambda_n \cdot x_n - \lambda \cdot x\| = \|\lambda_n \cdot x_n - \lambda \cdot x_n + \lambda \cdot x_n - \lambda \cdot x\| = \|(\lambda_n - \lambda) \cdot x_n + \lambda \cdot (x_n - x)\| \leq \|(\lambda_n - \lambda) \cdot x_n\| + \|\lambda \cdot (x_n - x)\|$$
$$= |\lambda_n - \lambda| \|x_n\| + |\lambda| \|(x_n - x\|$$

$$\Rightarrow \quad \|\lambda_n \cdot x_n - \lambda \cdot x\| \leq |\lambda_n - \lambda| \|x_n\| + |\lambda| \|(x_n - x\|$$

Since $x_n \to x$ and $\lambda_n \to \lambda$ we get, $|\lambda_n - \lambda| \to \tilde{0}$ and $\|x_n - x\| \to \tilde{0}$ as $n \to \infty$.

Now using above we get, $\|\lambda_n \cdot x_n - \lambda \cdot x\| \to \tilde{0}$. Hence $\lambda_n \cdot x_n \to \lambda \cdot x$.

3. Let If $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in X ,then for $\varepsilon > \tilde{0}$, , there exist a positive integers k_1, k_2 such that $\|x_n - x_m\| < \frac{\varepsilon}{2}$ for all $n \geq k_1$ and $\|y_n - y_m\| < \frac{\varepsilon}{2}$ for all $n \geq k_2$

Let $k = \max\{k_1, k_2\}$, then both the above relations hold for $n, m \geq k$.

Now, $$\|(x_n + y_n) - (x_m + y_m)\| = \|(x_n - x_m) + (y_n - y_m)\| \leq \|x_n - x_m\| + \|y_n - y_m\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for } n, m \geq k .$$

$\Rightarrow \quad \{x_n + y_n\}$ is a Cauchy sequences in X .

Since $\{x_n\}$ is a Cauchy sequences in X , for $\varepsilon > \tilde{0}$, there exist a positive integers k such that $\|x_n - x_m\| < \varepsilon$ for all $n, m \geq k$.

Taking in particular $n = m+1$, $\|x_{m+1}\| < \varepsilon$ for all $n, m \geq k$, so $\{\|x_n\|\}$ is bounded.

Now $\{\lambda_n\}$ is bounded too.

Then, $\|\lambda_n \cdot x_n - \lambda_m \cdot x_m\| = \|\lambda_n \cdot x_n - \lambda_n \cdot x_m + \lambda_n \cdot x_m - \lambda_m \cdot x_m\| = \|\lambda_n \cdot (x_n - x_m) + (\lambda_n - \lambda_m) \cdot x_m\|$

$\|\lambda_n \cdot x_n - \lambda_m \cdot x_m\| \leq |\lambda_n| \|x_n - x_m\| + |(\lambda_n - \lambda_m)| \|x_m\| \to \tilde{0}$ as $n \to \infty$

$\Rightarrow$ $\{\lambda_n \cdot x_n\}$ are also Cauchy sequences in X.

Theorem (3.22)

If M_A is a soft subspace in a soft normed space $(X, \|\cdot\|, A)$, then the closure of M_A, $\overline{M_A}$ is also a soft subspace.

Proof :

Let $x, y \in \overline{M_A}$, we must show that any linear combination of x, y belongs to $\overline{M_A}$.

Since $x, y \in \overline{M_A}$, corresponding to $\varepsilon > \tilde{0}$, there exists soft elements $x_1, y_2 \in M_A$ such that

$$\|x - x_1\| < \varepsilon, \quad \|y - y_1\| < \varepsilon$$

For soft scalars $\alpha, \beta > \tilde{0}$,

$\|(\alpha \cdot x + \beta \cdot y) - (\alpha \cdot x_1 + \beta \cdot y_1)\| = \|\alpha \cdot (x - x_1) + \beta \cdot (y - y_1)\| \leq |\alpha| \|x - x_1\| + |\beta| \|y - y_1\| < \varepsilon(|\alpha| + |\beta|) = \varepsilon'$ (say),

The above inequality shows that $\alpha x_1 + \beta y_1$ belongs to the open ball $\beta(\alpha x + \beta y, \varepsilon')$. As $\alpha x_1 + \beta y_1$ and $\varepsilon' > \tilde{0}$ are arbitrary, it follows that $\alpha x + \beta y \in \overline{M_A}$. Hence $\overline{M_A}$ is a soft subspace of X.

Definition (3.23)

A soft linear space X is said to be of finite dimensional if there is a finite set of linearly independent soft vectors in X which also generates X.

Theorem(3.24)

Let $x_1, x_2, \cdots, x_n$ be a linearly independent set of soft vectors in a soft linear space X. Then there is a soft real number $\tilde{c} > \tilde{0}$ such that for every set of soft scalars $\lambda_1, \lambda_2, \cdots, \lambda_n$ we have

$$\|\lambda_1 \cdot x_1 + \lambda_2 \cdot x_2 + \cdots + \lambda_n \cdot x_n\| \geq \tilde{c}(|\lambda_1| + |\lambda_2| + \cdots + |\lambda_n|)$$

Proof :

The theorem will be proved if we can prove

$$\left\|\lambda_1 \cdot x_1 + \lambda_2 \cdot x_2 + \cdots + \lambda_n \cdot x_n\right\|(e) \tilde{\geq} \tilde{c}(\left|\lambda_1\right| + \left|\lambda_2\right| + \cdots + \left|\lambda_n\right|)(e) \text{ for all } e \in A$$

i.e., $\left\|\lambda_1(e) \cdot x_1(e) + \lambda_2(e) \cdot x_2(e) + \cdots + \lambda_n(e) \cdot x_n(e)\right\|_e \geq (\tilde{c}(e)(\left|\lambda_1(e)\right| + \left|\lambda_2(e)\right| + \cdots + \left|\lambda_n(e)\right|))$ for all $e \in A$.

Now, $x_1, x_2, \cdots, x_n$ being soft vectors in $\check{X}$, $x_1(e), x_2(e), \cdots, x_n(e)$ are vectors in X and $\lambda_1, \lambda_2, \cdots, \lambda_n$ being soft scalars $\lambda_1(e), \lambda_2(e), \cdots, \lambda_n(e)$ are scalars.

Then using the property of normed linear space $(X, \|\cdot\|_e)$ we get a real number c_e, such that the above relation holds for $\tilde{c}(e) = c_e$, for all $e \in A$.

Theorem (3.25)

Every Cauchy sequence in $\mathbb{R}(A)$ with finite parameter set A is convergent, i.e., the set of all soft real numbers with its usual modulus soft norm as defined in Example (5.2) with finite parameter set A, is a soft Banach space.

Proof :

Let $\{x_n\}$ be any arbitrary Cauchy sequence in $\mathbb{R}(A)$. Then corresponding to every $\varepsilon \tilde{>} \tilde{0}$, there exist $k \in \mathbb{N}$ such that $\left|x_n - x_m\right| \tilde{<} \varepsilon\,|$ for all $n, m \geq k$, i.e., $\left|x_n - x_m\right|(e) < \varepsilon(e)\,|$ for all $n, m \geq k$, i.e. $\left|x_n(e) - x_m(e)\right| < \varepsilon(e)\,|$ for all $n, m \geq k$. Then $\{x_n(e)\}$ is a Cauchy sequence of ordinary real numbers $\mathbb{R}$ for each $e \in A$. By the Completeness of $\mathbb{R}$ and finiteness of A, it follows that $\{x_n(e)\}$ is convergent for each $e \in A$. Let $x_n(e) \to x_e$, for each $e \in A$.

Consider the soft element x defined by $x(e) = x_e$, for each $e \in A$. Then x is a soft real number and it follows that the sequence $\{x_n\}$ of soft real numbers is convergent and it converges to the soft real number x. Hence $\mathbb{R}(A)$ is a soft Banach space.

Theorem (3.26)

Every finite dimensional soft normed linear space over a finite parameter set A is complete.

Proof :

Let X be a finite dimensional soft normed linear space over a finite parameter set A. Let $\{y_m\}$ be any arbitrary Cauchy sequence in X . We show that $\{y_m\}$ converges to some soft element $y \in X$. Suppose that the dimension of X is n, and let $\{x_1, x_2, \cdots, x_n\}$ be a basis for X . Then each y_m has a unique representation $y_m = \lambda_1^{(m)} \cdot x_1 + \lambda_2^{(m)} \cdot x_2 + \cdots + \lambda_n^{(m)} \cdot x_n$.

Because $\{y_m\}$ is a Cauchy sequence, for $\varepsilon > \tilde{0}$ arbitrary there exist a positive integer k such that $\|y_m - y_r\| < \varepsilon$ for $m, r > k$. From theorem(4.5.23), it follows that there exists $\tilde{c} > \tilde{0}$ such that

$$\varepsilon > \|y_m - y_r\| = \left\|\sum_{j=1}^{n}(\lambda_j^{(m)} - \lambda_j^{(r)})x_j\right\| \geq \tilde{c}\sum_{j=1}^{n}\left|\lambda_j^{(m)} - \lambda_j^{(r)}\right|, \text{ for } m, r > k .$$

Consequently, $\left|\lambda_j^{(m)} - \lambda_j^{(r)}\right| \leq \tilde{c}\sum_{j=1}^{n}\left|\lambda_j^{(m)} - \lambda_j^{(r)}\right| < \frac{\varepsilon}{c}$

shows that each of the n sequences $\lambda_j^{(m)} = \{\lambda_j^{(1)}, \lambda_j^{(2)}, \lambda_n^{(3)}, \cdots\}$, $j = 1, 2, \cdots, n$ is Cauchy in $\mathbb{R}(A)$ and A is finite, converges to λ_j, (say), $j = 1, 2, \cdots, n$.

We now define the soft element $y = \lambda_1 \cdot x_1 + \lambda_2 \cdot x_2 + \cdots + \lambda_n \cdot x_n$ which is clearly a soft element of X . Moreover, since $\lambda_j^{(m)} \to \lambda_j$ as $m \to \infty$ and $j = 1, 2, \cdots, n$; we have

$$\|y_m - y\| = \left\|\sum_{j=1}^{n}(\lambda_j^{(m)} - \lambda_j)x_j\right\| \geq \tilde{c}\sum_{j=1}^{n}\left|\lambda_j^{(m)} - \lambda_j\right|\|x_j\| \to \tilde{0} \text{ as } m \to \infty \text{. i.e. } y_m \to y \text{ as } m \to \infty .$$

4. Soft Banach Algebras

Definition (4.1)

A soft algebra F_A of X over F is called a soft Banach algebra if F_A is a soft Banach space with respect to a soft norm that satisfies the inequality $\|xy\| \leq \|x\|\|y\|$ and if F_A contains an identity $\tilde{\tilde{e}}$ such that $x\tilde{e} = \tilde{e}x = x$ with $\|\tilde{e}\| = \tilde{1}$.

Theorem (4.2)

F_A is a soft Banach algebra iff $F(e)$ is a Banach algebra for all $e \in A$.

Proof :

follows from the definition of soft algebra and the following theorem .

soft normed space $(X, \|\cdot\|)$ is soft complete iff $(X, \|\cdot\|_e)$ is complete for all $e \in A$ where $\|\cdot\|_e$ defined as $\|x\|_e = \|x\|(e)$ for each $x \in X$, where $x \in X$ such that $x(e) = x$

Theorem (4.3)

In a soft Banach algebra F_A, if $x_n \to x$ and $y_n \to \tilde{y}$ then $x_n y_n \to x\tilde{y}$.

i.e., multiplication in a soft Banach algebra is continuous.

Proof :

Since $x_n \to x$ and $y_n \to y$ in F_A . So $x_n(e) \to x(e)$ and $y_n(e) \to y(e)$ for all $e \in A$ in $(F(e), \|\cdot\|_e)$

Now since $F(e)$ is Banach algebra for all $e \in A$ (by theorem 6.2) and in Banach algebra multiplication is continuous so, $x_n(e) y_n(e) \to x(e) y(e)$ for all $e \in A$, which proves that $x_n y_n \to xy$.

Theorem (4.4)

Every parameterized family of crisp Banach algebras on a crisp linear space X can be considered as a soft Banach algebra on the soft vector space $\tilde{X}$.

Proof :

Let $\{\|\cdot\|_e : e \in A\}$ be a family of crisp norms on the linear space X such that $(X, \|\cdot\|_e)$ are Banach algebra for $e \in A$. Now let us define a function $\|\cdot\| : X \to \square(A)^*$ by $\|x\|(e) = \|x(e)\|_e$ for all $x \in X$, for all $e \in A$. Then $\|\cdot\|$ is a soft norm on X .

Now to show that $(X, \|\cdot\|)$ is a soft Banach algebra we have to show that $\|xy\| \leq \|x\| \|y\|$ for all $x, y \in X$ and $(X, \|\cdot\|)$ is complete.

Now $\|xy\|(e) = \|x(e) y(e)\|_e \leq \|x(e)\|_e \|y(e)\|_e \leq \|x\|(e) \|y\|(e)$ for all $e \in A$, which shows that $\|xy\| \leq \|x\| \|y\|$.

Now let $\{x_n\}$ be a Cauchy sequence in X . Then for any $\varepsilon > \tilde{0}$ there exists a soft natural number k such that $\|x_{n+p} - x_n\|(e) < \frac{\varepsilon}{2}$ for all $n \geq k(e)$,for all $e \in A$, then $\|x_{n+p}(e) - x_n(e)\| < \frac{\varepsilon}{2}(e)$ for all $e \in A$, i.e. $\{x_n(e)\}$ is a Cauchy sequence in $(X, \|\cdot\|_e)$ for all $e \in A$.

Since $(X, \|\cdot\|_e)$ are Banach algebra for all $e \in A$, so there exist x_e such that $x_n(e) \to x_e$ algebra for all $e \in A$. Hence there must exist some $k_e (> k(e))$ such that $\|x_n(e) - x_e\| < \frac{\varepsilon}{2}(e)$ for all $e \in A$.

Now $\|x_n - x\|(e) = \|x_n(e) - x_e\|_e < \|x_n(e) - x_{k_e}(e)\| + \|x_{k_e} - x_e(e)\|_e < \varepsilon(e)$ for all $n \geq k(e)$,for all $e \in A$,

where $x(e) = x_e$. This shows that $(X, \|\cdot\|)$ is a soft Banach space. Hence $(X, \|\cdot\|)$ is a soft Banach algebra.

Definition (4.5)

Let F_A be a soft algebra of X over F . A soft element $x \in F_A$ is said to be invertible if it has inverse in F_A, i.e. if there exists a soft element $y \in F_A$ such that $xy = yx = \tilde{e}$ and the y is called the inverse of $\tilde{x}$, denoted by x^{-1}. Otherwise x^{-1} is said to be non-invertible soft element of F_A.

Remark

Clearly $\tilde{e}$ is invertible. If x is invertible, then we can verify that the inverse is unique. because if $yx = \tilde{e} = x\tilde{z}$ Then $y = y\tilde{e} = y(x\tilde{z}) = (yx)\tilde{z} = \tilde{e}\tilde{z} = \tilde{z}$.

Further, if x and y are both invertible then xy is invertible and $(xy)^{-1} = y^{-1}x^{-1}$.

For $(xy)(y^{-1}x^{-1}) = x(yy^{-1})x^{-1} = x(\tilde{e})x^{-1} = xx^{-1} = \tilde{e}$ and similarly $(y^{-1}x^{-1})(xy) = \tilde{e}$.

Definition (4.6)

Let $(G,*)$ be a group and F_A be a soft set over G. Then F_A is said to be a soft group over G if $F(e)$ is a subgroup of $(G,*)$ for all $e \in A$.

Theorem (4.7)

Let $(G,*)$ be a group and F_A be a soft set over G. If for any $x, y \in F_A$

1. $x * y \in F_A$ 2. $x^{-1} \in F_A$,

where $x * y(e) = x(e) * y(e)$ and $x^{-1}(e) = (x(e))^{-1}$. Then F_A is a soft group over G.

Proof :

Proof is obvious.

Remark

This shows that in a soft algebra, the soft set generated by the all invertible elements is a soft group with respect to the composition defined as in theorem.

Definition (4.8)

A series $\sum_{n=1}^{\infty} x_n$ of soft elements is said to be soft convergent if the partial sum of the series $\tilde{s}_k = \sum_{n=1}^{k} x_n$ is soft convergent.

Theorem (4.9)

Let F_A be a soft Banach algebra. If $x \in F_A$ satisfies $\|x\| \tilde{\le} \tilde{1}$, then $(\tilde{e} - x)$ is invertible and

$(\tilde{e} - x)^{-1} = \tilde{e} + \sum_{n=1}^{\infty} x^n$.

Proof :

Since F_A is soft algebra, so we have $\|x^j\| \le \|x\|^j$ for any positive integer j, so that the infinite series $\sum_{n=1}^{\infty} \|x\|^n$ is soft convergent because. So the sequence of partial sum $\tilde{s}_k = \sum_{n=1}^{k} x_n$ is a soft Cauchy sequence since $\left\|\sum_{n=k}^{k+p} x^n\right\| < \sum_{n=k}^{k+p} \|x\|^n$.

Since F_A is soft complete so $\sum_{n=1}^{\infty} x^n$ is soft convergent. Now let $\tilde{s} = \tilde{e} + \sum_{n=1}^{\infty} x^n$.

Now it is only we have to show that $\tilde{s} = (\tilde{e} - x)^{-1}$.

We have

$$(\tilde{e} - x)(\tilde{e} + x + x^2 + \cdots + x^n) = (\tilde{e} + x + x^2 + \cdots + x^n)(\tilde{e} - x) = \tilde{e} - x^{n+1}$$

Now again since $\|x\| \tilde{\le} \tilde{1}$ so $x^{n+1} \to \tilde{0}$ as $n \to \infty$. Therefore letting $n \to \infty$ in and remembering that multiplication in F is continuous we get, $(\tilde{c} - x)\tilde{s} = \tilde{s}(\tilde{c} - x) = \tilde{c}$

So that $\tilde{s} = (\tilde{e} - x)^{-1}$. This proves the proposition.

Corollary (4.10)

Let F_A be a soft Banach algebra. If $x \in F_A$ and $\|\tilde{e} - x\| \tilde{<} \tilde{1}$, Then x^{-1} exists and $x^{-1} = \tilde{e} + \sum_{n=1}^{\infty} (\tilde{e} - x)^n$.

Corollary (4.11)

Let F_A be a soft Banach algebra. Let $x \in F_A$ and λ be a soft scalar such that $|\lambda| > \|x\|$.

Then $(\lambda\tilde{e} - x)^{-1}$ exists and $(\lambda\tilde{e} - x)^{-1} = \sum_{n=1}^{\infty} \lambda^{-n} x^{n-1}$ $(x^0 = \tilde{e})$

Proof :

$y \in F_A$ be such that y^{-1} exists in F_A and α be a soft scalar such that $\alpha(e) \ne 0$, for all $e \in A$. Then it is clear that $(\alpha y)^{-1} = \alpha^{-1} y^{-1}$.

Having noted this we can write $\lambda\tilde{e} - x = \lambda(\tilde{e} - \lambda^{-1} x)$ and now we show that $(\tilde{e} - \lambda^{-1} x)^{-1}$ exists.

We have $\left\| \tilde{e} - (\tilde{e} - \lambda^{-1} x) \right\| = \left\| \lambda^{-1} x \right\| = \left| \lambda^{-1} \right| \left\| x \right\| \tilde{<} \tilde{1}$ by hypothesis. So, By Corollary(6.10) $(\tilde{e} - \lambda^{-1} x)^{-1}$ exists and hence $(\lambda\tilde{e} - x)^{-1}$ exists. For the infinite series representation, using the theorem (6.9)

we have

$$(\lambda\tilde{e} - x)^{-1} = \lambda^{-1}(\tilde{e} - \lambda^{-1} x)^{-1} = \lambda^{-1}(\tilde{e} + \sum_{n=1}^{\infty}(\tilde{e} - (\tilde{e} - \lambda^{-1} x))^{n} = \lambda^{-1}(\tilde{e} + \sum_{n=1}^{\infty}(\lambda^{-1} x))^{n} = \sum_{n=1}^{\infty} \lambda^{-n} x^{n-1}$$

This proves the corollary.

Theorem (4.12)

Let F_A be a soft Banach algebra. The soft set S generated by the set of all invertible soft elements of F_A is a soft open subset in F_A.

Proof :

$x_0 \in S$. We have to show that x_0 is a soft interior point of F_A. Consider the open sphere $S(x_0, \frac{1}{\left\| x_0^{-1} \right\|})$ with centre at x_0 and radius $\frac{1}{\left\| x_0^{-1} \right\|}$. Every soft element x of this sphere satisfies the inequality $\left\| x_0 - x \right\| < \frac{1}{\left\| x_0^{-1} \right\|}$.

Let $y = x_0^{-1} x$ and $\tilde{z} = \tilde{e} - y$ then we have $\left\| \tilde{z} \right\| = \left\| y - \tilde{e} \right\| = \left\| x_0^{-1} x - x_0^{-1} x_0 \right\| \leq \left\| x_0^{-1} \right\| \left\| x - x_0 \right\| \tilde{<} \tilde{1}$.

So by theorem(6.9), $\tilde{e} - \tilde{z}$ is invertible i.e. y is invertible. Hence $y \in S$ .. Now $x_0 \in S$, $y \in S$ and so by Remark , $x_0 y \in S$. But $x_0 y = x_0 x_0^{-1} y = x$. So any x satisfying the inequality $\left\| x_0 - x \right\| < \frac{1}{\left\| x_0^{-1} \right\|}$ belongs to S.

This shows that S is a soft open subset of F_A.

Corollary (4.13)

The soft set $P = (S^c)$ of F_A is soft closed subset of F_A.

Definition (4.14)

A function T from a soft normed space F_A onto F_A is said to be continuous If for any sequence $\{x_n\}$, $x_n \to x$ implies $T(x_n) \to T(x)$.

Theorem (4.15)

In a soft Banach algebra F_A , the function $x \to x^{-1}$ of S onto S is continuous.

Proof :

Let $x_0 \in S$ and let $\{x_n\}$ be a sequence of soft elements in S such that $x_n \to x_0$.

To prove $x \to x^{-1}$ is continuous, it is enough to show that $x_n^{-1} \to x_0^{-1}$

Now : $\left\|x_n^{-1} - x_0^{-1}\right\| = \left\|x_n^{-1}(x_0 - x_n)x_0^{-1}\right\| \leq \left\|x_n^{-1}\right\|\left\|x_0 - x_n\right\|\left\|x_0^{-1}\right\|$

Since $x_n \to x_0$, for any given $\varepsilon > \tilde{0}$; there exists N such that for all $n \geq N(e)$,

$\left\|x_n - x_0\right\|(e) \leq \dfrac{\tilde{1}}{2\left\|x_0^{-1}\right\|}(e)$ where we have taken $\varepsilon = \dfrac{\tilde{1}}{2\left\|x_0^{-1}\right\|}$

Now $\left\|\tilde{e} - x_0^{-1}x_n\right\| = \left\|x_0^{-1}(x_0 - x_n)\right\| \leq \left\|x_0^{-1}\right\|\left\|x - x_n\right\|$, we get $\left\|\tilde{e} - x_0^{-1}x_n\right\| = \leq \dfrac{\tilde{1}}{2}(e) = \dfrac{1}{2}$ for all $n \geq N(e)$.

So by Corollary(4.10) , $x_0^{-1}x_n$ is invertible and its inverse is given by

$$x_n^{-1}x_0 = (x_0^{-1}x_n)^{-1} = \tilde{e} + \sum_{n=1}^{\infty}(\tilde{e} - x_0^{-1}x_n)^n$$

Thus $\left\|x_n^{-1}x_0\right\| \leq \tilde{1} + \sum_{n=1}^{\infty}\left\|\tilde{e} - x_0^{-1}x_n\right\|^n \leq \dfrac{\tilde{1}}{1 - \left\|\tilde{e} - x_0^{-1}x_n\right\|} \leq 2$ This gives $\left\|x_n^{-1}x_0\right\| \leq 2$ so that we have

$$\left\|x_n^{-1}\right\| = \left\|x_n^{-1}x_0x_0^{-1}\right\| \leq \left\|x_n^{-1}x_0\right\|\left\|x_0^{-1}\right\| \leq 2\left\|x_0^{-1}\right\|$$

we get : $\left\|x_n^{-1} - x_0^{-1}\right\|(e) \leq 2\left\|x_0^{-1}\right\|(e)\left\|x_0 - x_n\right\|(e) \to 0$ as $n \to \infty$.

This proves that $x_n^{-1} \to x_0^{-1}$. So the function $x \to x^{-1}$ of S onto S is continuous.

Corollary (4.16)

In a soft Banach algebra F_A, the function $x \to x^{-1}$ of S onto S is continuous.

Definition (4.17)

Let F_A be a soft Banach algebra. A soft element $x \in F_A$ is called a soft topological divisor of zero if there exists a sequence $\{x_n\}$ in F_A , $\|x_n\| = \tilde{1}$ for $n = 1, 2, 3, \cdots$ and such that either $x x_n \to 0$ or $x_n x \to 0$.

Theorem (4.18)

The soft set Z is a soft subset of P, where Z denotes the set of all soft topological divisors of zero.

Proof :

Let $\tilde{z} \in Z$. The there exists a sequence $\{z_n\}$ such that $\|\tilde{z}_n\| = \tilde{1}$ for $n = 1, 2, 3, \cdots$ and either $\tilde{z}\tilde{z}_n \to 0$ or $\tilde{z}_n \tilde{z} \to 0$ as $n \to \infty$. Suppose that $\tilde{z}\tilde{z}_n \to 0$.

If possible, let $\tilde{z} \in P$. Then $\tilde{z}(e)^{-1}$ exists for some e. Now as multiplication is continuous operation, we should have $\tilde{z}_n(e) = \tilde{z}(e)^{-1}(\tilde{z}\tilde{z}_n) \to \tilde{z}(e)^{-1} 0(e) = 0$ as $n \to \infty$.

This contradicts the fact that $\|\tilde{z}_n\| = \tilde{1}$ for $n = 1, 2, 3, \cdots$. Hence Z is a soft subset of P.

Definition (4.19)

Let $(X, \|\cdot\|)$ be a soft normed space and $Y \in S(X)$. A soft element $x \in X$ is called a soft boundary elements of Y if there exist two sequence $\{x_n\}$ and $\{y_n\}$ of soft elements in Y and Y^c respectively such that $x_n \to x$ and $y_n \to x$.

Theorem (4.20)

The boundary of P is a soft subset of Z.

Proof :

Let $\tilde{z}$ be a boundary point of P. So there exist two sequences of soft elements $\tilde{r}_n$ in S and $\tilde{s}_n$ in P such that $\tilde{r}_n \to \tilde{z}$ and $\tilde{s}_n \to \tilde{z}$.

Since P is soft closed so $\tilde{z} \in P$ Now let us write $\tilde{r}_n^{-1}\tilde{z} - \tilde{e} = \tilde{r}_n^{-1}(\tilde{z} - \tilde{r}_n)$. The sequence $\{\tilde{r}_n^{-1}(e)\}$ given above is unbounded for all $e \in A$. If not, then there exists some $e \in A$ and $n(e)$ such that $\|\tilde{r}_n^{-1}\tilde{z} - \tilde{e}\|(e) < 1$ for all $n \geq n(e)$, for all $e \in A$. So that by Corollary(6.11), $\{\tilde{r}_n^{-1}\tilde{z}(e)\}$ is regular and hence $\tilde{z}(e) = \tilde{r}_n(e)(\tilde{r}_n^{-1}\tilde{z})(e)$ is regular, contradicting $\tilde{z} \in P$. Hence $\{\tilde{r}_n^{-1}(e)\}$ is unbounded for all $e \in A$. so that $\|\tilde{r}_n^{-1}\| \to \infty$.

Now let us define $\tilde{z}_n = \dfrac{\tilde{r}_n^{-1}}{\left\|\tilde{r}_n^{-1}\right\|}$. From the definition of $\tilde{z}_n$, we have $\left\|\tilde{z}_n\right\| = \tilde{1}$.

Further $\tilde{z}\tilde{z}_n = \dfrac{\tilde{z}\tilde{r}_n^{-1}}{\left\|\tilde{r}_n^{-1}\right\|} = \dfrac{\tilde{e} + \tilde{z}\tilde{r}_n^{-1} - \tilde{e}}{\left\|\tilde{r}_n^{-1}\right\|} = \dfrac{\tilde{e} + (\tilde{z} - \tilde{r}_n)\tilde{r}_n^{-1}}{\left\|\tilde{r}_n^{-1}\right\|}$

But $\dfrac{\tilde{e} + (\tilde{z} - \tilde{r}_n)\tilde{r}_n^{-1}}{\left\|\tilde{r}_n^{-1}\right\|} = \dfrac{\tilde{e}}{\left\|\tilde{r}_n^{-1}\right\|} + (\tilde{z} - \tilde{r}_n)\tilde{z}_n$, we get $\tilde{z}\tilde{z}_n = \dfrac{\tilde{e}}{\left\|\tilde{r}_n^{-1}\right\|} + (\tilde{z} - \tilde{r}_n)\tilde{z}_n$

we see that $\tilde{z}\tilde{z}_n \to 0$ as $n \to \infty$. Hence $\tilde{z}$ is a topological divisor of zero.

5. Conclusion

In this paper underscores the significance of soft Banach Algebras as a powerful mathematical tool for investigating algebraic phenomena within diverse applied contexts.

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