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[0,1] Truncated Generalized Gamma – Generalized Gamma Distribution

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Abstract: In this paper, we introduce a new family of continuous distributions based on interval [0,1] truncated generalized gamma distribution. [0,1] truncated generalized gamma-generalized gamma distribution is discussed as a special case. The cumulative distribution function, the rth moment, the mean, the variance, the skewness, the kurtosis, the mode, the median, the characteristic function, the reliability function and the hazard rate function are obtained for the distribution under consideration. It is well known that an item fails when a stress to which it is subjected exceeds the corresponding strength. In this sense, strength can be viewed as "resistance to failure". Good design practice is such that the strength is always greater than the expected stress. The safety factor can be defined in terms of stress and strength as stress/strength. So, the [0,1] truncated generalized gamma-generalized gamma stress-strength model with different parameters will be derived here. The Shannon entropy and relative entropy will be derived also.

Keywords: Generalized Gamma Distribution, Shannon Entropy, Relative Entropy, Stress-Strength Model.

1. Introduction

The generalization, G , distribution have been introduced by Eugene et al. [3] when they defined the beta G distribution from any valid cumulative distribution function (cdf), say $G(x)$, as

$$F(x) = (1/\beta(a,b)) \int_0^{G(x)} w^{a-1} (1-w)^{b-1} \, dw \qquad (1)$$

where $\beta(a,b) = \int_0^1 w^{a-1} (1-w)^{b-1} \, dw$ is the beta function; a and b are two additional positive parameters whose role is to introduce skewness as well as to vary tail weight. The class of distributions (1) has an increased attention after the works by Eugene et al. [3] and Jones [5]. Application of $X = G^{-1}(y)$ to the random variable Y distributed as beta distribution with parameters a and b, yields X with cdf (1). Eugene et al. [3] defined the four parameter beta normal distribution by taking $G(x)$ to be the cdf of the normal distribution and derived some of its first moments while the general expressions for the moments of this distribution were derived by Gupta and Nadarajah [4]. Furthermore, Abid and Hassan [1], Nadarajah and Rocha [6] provided an extensive review of scientific literature on this subject.

The probability density function (pdf) corresponding to (1) is,

$$f(x) = \frac{1}{B(a.b)} (G(x))^{a-1} \left(1 - G(x)\right)^{b-1} g(x) \qquad (2)$$

where $g(x) = \frac{d}{dx}\,[G(x)]$ is the pdf of the parent distribution.

In addition, the cdf in (1) can be re-written as,

$$F(x) = I_{G(x)}(a,b) \qquad (3)$$

where, $I_y(a,b) = (1/B(a,b) \int_0^y w^{a-1} (1-w)^{b-1} \, dw$, denotes the incomplete beta function ratio. For general a and b, the cdf in (3) can express in terms of the well-known hypergeometric function defined by,

$${}_2F_1(x;\alpha,\beta,\gamma) = \sum_{i=0}^{\infty} \frac{(\alpha)_i\,(\beta)_i}{(\gamma)_i\, i!}\, x^i \qquad (4)$$

where $(\alpha)_i = \alpha(\alpha+1)\dots(\alpha+i-1)$ denotes the ascending factorial. Then, the cdf can be obtained as,

$$F(x) = \frac{(\mathrm{G}(x))^a}{a\,B(a,b)}\ {}_2F_1(a, 1-b, a+1; \mathrm{G}(x)) \qquad (5)$$

The properties of the cdf, $F(x)$ for any beta G distribution defined from a parent $G(x)$ in (1), in principle, could follow from the properties of the hypergeometric function which are well established in the literature [2].

In this paper, the generalization that motivated by the work of Eugene et al. will be our guide to propose a new class of distributions with the hope that it will find a way to apply in a variety of fields. The paper is organized as follows. In Section 2, we present the [0,1] truncated generalized gamma-G distributions. As special case, the [0,1] truncated generalized gamma-generalized gamma distribution is given in Section 3 and its further analysis including the r^{th} moment, mean, variance, skewness, kurtosis, mode, median, characteristic function, quantile function, reliability function and hazard rate function are given in Section 4. Some additional useful results including Shannon and relative entropies followed by stress-strength model are given respectively in Sections 5,6. Finally, concluding remarks are given in Section 7

2. [0,1] Truncated Generalized Gamma – G Distributions

The generalized gamma (GG) distribution was introduced by Stacy [7]. The cdf and pdf of the GG distribution are given by the following forms [6]

$$H(x;\ \alpha,\beta,\lambda) = \frac{\gamma\left[\frac{\beta}{\lambda}, \left(\frac{x}{\alpha}\right)^{\lambda}\right]}{\Gamma\left(\frac{\beta}{\lambda}\right)} \qquad (6)$$

$$h(x;\alpha,\beta,\lambda) = \frac{1}{\Gamma\left(\frac{\beta}{\lambda}\right)}\left(\frac{\lambda}{\alpha^{\beta}}\right) x^{\beta-1}\, e^{-\left(\frac{x}{\alpha}\right)^{\lambda}}\ ;\quad x>0 \qquad (7)$$

where $\Gamma(\cdot)$ is the gamma function, $\gamma(\cdot\,,\,\cdot)$ is the incomplete gamma function, β and λ are positive shape parameters and α is the positive scale parameter.

Now, suppose that $G(\cdot)$ and $g(\cdot)$ are the baseline cdf and pdf of a random variable X. By composing H with G, the proposed cdf for the new class of distributions is given by,

$$F(x) = \frac{H[G(x)] - H[0]}{H[1] - H[0]} \qquad (8)$$

Since $H(0) = 0$, the cdf in (8) can be re-written as,

$$F(x) = \frac{H[G(x)]}{H[1]} \qquad (9)$$

and,

$$f(x) = \frac{h[G(x)]g(x)}{H[1]} \qquad (10)$$

where $f(x) = \frac{d}{dx}\,[F(x)]$ is the pdf of the new distribution.

From (9) and (10) above, a new family of continuous distributions based on interval [0,1] truncated generalized gamma distribution, named [0,1] truncated generalized gamma-G distributions, have been introduced.

3. [0,1] Truncated Generalized Gamma-Generalized Gamma Distribution

In this section, [0,1] truncated generalized gamma-generalized gamma ([0,1] TGG-GG) distribution have been proposed.

Given another distribution that is distributed as $GG(a,d,p)$ with pdf and cdf are given respectively by,

$$g(x;a,d,p)=\frac{1}{\Gamma\left(\frac{d}{p}\right)}\left(\frac{p}{a^d}\right)x^{d-1}e^{-\left(\frac{x}{a}\right)^p}\ ; \quad x>0 \tag{11}$$

$$G(x;a,d,p)=\frac{\gamma\left[\frac{d}{p},\left(\frac{x}{a}\right)^p\right]}{\Gamma\left(\frac{d}{p}\right)} \tag{12}$$

Then according to (6)and (7), $H[G(x)]$, $H(1)$ and $h[G(x)]$ will be

$$H[G(x)]=\frac{\gamma\left[\frac{\beta}{\lambda},\left(\frac{G(x)}{\alpha}\right)^\lambda\right]}{\Gamma\left(\frac{\beta}{\lambda}\right)}=\frac{\gamma\left[\frac{\beta}{\lambda},\left(\frac{\gamma\left[\frac{d}{p},\left(\frac{x}{a}\right)^p\right]}{\alpha\,\Gamma\left(\frac{d}{p}\right)}\right)^\lambda\right]}{\Gamma\left(\frac{\beta}{\lambda}\right)};\quad H(1)=\frac{\gamma\left[\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^\lambda\right]}{\Gamma\left(\frac{\beta}{\lambda}\right)}$$

$$h[G(x)]=\frac{1}{\Gamma\left(\frac{\beta}{\lambda}\right)}\left(\frac{\lambda}{\alpha^\beta}\right)[G(x)]^{\beta-1}e^{-\left(\frac{G(x)}{\alpha}\right)^\lambda}=\frac{1}{\Gamma\left(\frac{\beta}{\lambda}\right)}\left(\frac{\lambda}{\alpha^\beta}\right)\left[\frac{\gamma\left[\frac{d}{p},\left(\frac{x}{a}\right)^p\right]}{\Gamma\left(\frac{d}{p}\right)}\right]^{\beta-1}e^{-\left(\frac{\gamma\left[\frac{d}{p},\left(\frac{x}{a}\right)^p\right]}{\alpha\,\Gamma\left(\frac{d}{p}\right)}\right)^\lambda}$$

by substituting $H[G(x)]$ and $H[1]$ into (9), the cdf of [0,1] TGG-GG distribution will be,

$$F(x)=\frac{\gamma\left[\frac{\beta}{\lambda},\left(\frac{\gamma\left[\frac{d}{p},\left(\frac{x}{a}\right)^p\right]}{\alpha\,\Gamma\left(\frac{d}{p}\right)}\right)^\lambda\right]}{\gamma\left[\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^\lambda\right]} \tag{13}$$

and by substituting $h[G(x)]$, $g(x)$ and $H[1]$ into (10), the associated pdf of [0,1] TGG-GG distribution with six positive parameters $(a,d,p,\alpha,\beta,\lambda)$ will be,

$$f(x)=\frac{\left(\frac{p}{a^d}\right)\left(\frac{\lambda}{\alpha^\beta}\right)x^{d-1}e^{-\left(\frac{x}{a}\right)^p}\left[\frac{\gamma\left[\frac{d}{p},\left(\frac{x}{a}\right)^p\right]}{\Gamma\left(\frac{d}{p}\right)}\right]^{\beta-1}e^{-\left(\frac{\gamma\left[\frac{d}{p},\left(\frac{x}{a}\right)^p\right]}{\alpha\,\Gamma\left(\frac{d}{p}\right)}\right)^\lambda}}{\Gamma\left(\frac{d}{p}\right)\gamma\left[\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^\lambda\right]} \tag{14}$$

The reliability, $R(x)$ and hazard rate, $A(x)$ functions are respectively given as,

$$R(x)=1-F(x)=1-\frac{\gamma\left[\frac{\beta}{\lambda},\left(\frac{\gamma\left[\frac{d}{p},\left(\frac{x}{a}\right)^p\right]}{\alpha\,\Gamma\left(\frac{d}{p}\right)}\right)^\lambda\right]}{\gamma\left[\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^\lambda\right]} \tag{15}$$

$$A(x)=\frac{f(x)}{R(x)}=\frac{\left[\dfrac{\left(\frac{p}{a^{d}}\right)\left(\frac{\lambda}{\alpha^{\beta}}\right)\ x^{d-1}e^{-\left(\frac{x}{a}\right)^{p}}\left[\dfrac{\gamma\left[\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right]}{\Gamma\left(\frac{d}{p}\right)}\right]^{\beta-1} e^{-\left(\frac{\gamma\left[\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right]}{\alpha\,\Gamma\left(\frac{d}{p}\right)}\right)^{\lambda}}}{\Gamma\left(\frac{d}{p}\right)\gamma\left[\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right]}\right]}{\left[1-\dfrac{\gamma\left[\frac{\beta}{\lambda},\left(\dfrac{\gamma\left[\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right]}{\alpha\Gamma\left(\frac{d}{p}\right)}\right)^{\lambda}\right]}{\gamma\left[\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right]}\right]} \tag{16}$$

4. Properties of the [0,1] TGG-GG Distribution

The rth raw moment can be derivative as follows,

$$E(X^{r})=\int_{0}^{\infty}x^{r}\ f(x)\,dx \tag{17}$$

In order to find $E(X^{r})$ of the [0,1] TGG-GG distribution, suppose that,

$$I\left(k+\frac{r}{t},m\right)=\int_{0}^{\infty} y^{\frac{r}{t}+k-1}\ e^{-y}\left[\frac{\gamma(k,y)}{\Gamma(k)}\right]^{m}dy \tag{18}$$

Recall $e^{x}=\sum_{k=0}^{\infty}\frac{x^{k}}{k!}$, the pdf in (14) can be re-written as,

$$f(x)=\frac{\frac{p}{a^{d}}\left(\frac{\lambda}{\alpha^{\beta}}\right)}{\Gamma\left(\frac{d}{p}\right)\gamma\left[\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right]}\sum_{k=0}^{\infty}\frac{(-1)^{k}\ \ x^{d-1}e^{-\left(\frac{x}{a}\right)^{p}}\left(\dfrac{\gamma\left[\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right]}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1}}{k!\,\alpha^{\lambda k}} \tag{19}$$

So that, (17) will be,

$$E(X^{r})=\frac{\frac{p}{a^{d}}\left(\frac{\lambda}{\alpha^{\beta}}\right)}{\Gamma\left(\frac{d}{p}\right)\gamma\left[\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right]}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!\,\alpha^{\lambda k}}\,I_{1}$$

where $I_{1}=\int_{0}^{\infty}x^{d+r-1}e^{-\left(\frac{x}{a}\right)^{p}}\left(\frac{\gamma\left[\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right]}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1}dx$

Let $u = \left(\frac{x}{a}\right)^p \rightarrow a\, u^{1/p} = x \rightarrow \frac{a}{p}\, u^{\frac{1}{p}-1}\, du = dx$

Thus,

$$I_1 = \frac{a^{d+r}}{p} \int_0^\infty u^{\frac{1}{p}(d+r)-1}\; e^{-u} \left(\frac{\gamma\left[\frac{d}{p}, u\right]}{\Gamma\left(\frac{d}{p}\right)} \right)^{\beta+\lambda k-1} du$$

According to (18),

$$I_1 = \frac{a^{d+r}}{p}\, I[\frac{1}{p}(d+r), \beta+\lambda k-1]$$

and then,

$$E(X^r) = \frac{a^r \left(\frac{\lambda}{\alpha^\beta}\right)}{\Gamma\left(\frac{d}{p}\right)\gamma\left[\frac{\beta}{\lambda}, \left(\frac{1}{\alpha}\right)^\lambda\right]} \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\alpha^{\lambda k}}\; I\left[\frac{1}{p}(d+r), \beta+\lambda k-1\right] \qquad (20)$$

The Characteristic function of the [0,1] TGG-GG distribution is,

$$\phi_x(t) = E(e^{itx}) = E\left[\sum_{r=0}^\infty \frac{(it)^r\, x^r}{r!}\right] = \sum_{r=0}^\infty \frac{(it)^r}{r!} E(X^r)$$

$$\phi_x(t) = \frac{\frac{\lambda}{\alpha^\beta}}{\Gamma\left(\frac{d}{p}\right)\,\gamma\left[\frac{\beta}{\lambda}, \left(\frac{1}{\alpha}\right)^\lambda\right]} \sum_{r=0}^\infty \sum_{k=0}^\infty \frac{(it)^r\, a^r}{r!} \frac{(-1)^k}{k!\,\alpha^{\lambda k}}\, I\left[\frac{1}{p}(d+r), \beta+\lambda k-1\right] \qquad (21)$$

The mean and variance of [0,1] TGG-GG random variable are,

$$\mu = E(X) = \frac{a\left(\frac{\lambda}{\alpha^\beta}\right)}{\Gamma\left(\frac{d}{p}\right)\,\gamma\left[\frac{\beta}{\lambda}, \left(\frac{1}{\alpha}\right)^\lambda\right]} \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\alpha^{\lambda k}}\; I\left[\frac{1}{p}(d+1), \beta+\lambda k-1\right] \qquad (22)$$

$$\sigma^2 = E(X^2) - [E(X)]^2$$
$$= \frac{a^2 \left(\frac{\lambda}{\alpha^\beta}\right)}{\Gamma\left(\frac{d}{p}\right)\,\gamma\left[\frac{\beta}{\lambda}, \left(\frac{1}{\alpha}\right)^\lambda\right]} \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\alpha^{\lambda k}}\; I\left[\frac{1}{p}(d+2), \beta+\lambda k-1\right]$$
$$- \left(\frac{a\left(\frac{\lambda}{\alpha^\beta}\right)}{\Gamma\left(\frac{d}{p}\right)\gamma\left[\frac{\beta}{\lambda}, \left(\frac{1}{\alpha}\right)^\lambda\right]} \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\alpha^{\lambda k}}\; I\left[\frac{1}{p}(d+1), \beta+\lambda k-1\right] \right)^2 \qquad (23)$$

The skewness, sk and kurtosis, kr of [0,1] TGG-GG random variable can be obtain by using,

$$sk = \frac{E(X-\mu)^3}{\sigma^3} = \frac{E(X^3) - 3\mu E(X^2) + 2\mu^3}{[\sigma^2]^{\frac{3}{2}}} \qquad (24)$$

$$kr = \frac{E(X-\mu)^4}{(\sigma^2)^2} = \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^3}{(\sigma^2)^2} - 3 \qquad (25)$$

where μ, σ^2 as in $(22), (23)$and,

$$E(X^2) = \frac{a^2 \left(\frac{\lambda}{\alpha^\beta}\right)}{\Gamma\left(\frac{d}{p}\right) \gamma\left[\frac{\beta}{\lambda}, \left(\frac{1}{\alpha}\right)^\lambda\right]} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\alpha^{\lambda k}} \; I\left[\frac{1}{p}(d+2), \beta + \lambda k - 1\right]$$

$$E(X^3) = \frac{a^3 \left(\frac{\lambda}{\alpha^\beta}\right)}{\Gamma\left(\frac{d}{p}\right) \gamma\left[\frac{\beta}{\lambda}, \left(\frac{1}{\alpha}\right)^\lambda\right]} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\alpha^{\lambda k}} \; I\left[\frac{1}{p}(d+3), \beta + \lambda k - 1\right]$$

$$E(X^4) = \frac{a^4 \left(\frac{\lambda}{\alpha^\beta}\right)}{\Gamma\left(\frac{d}{p}\right) \gamma\left[\frac{\beta}{\lambda}. \left(\frac{1}{\alpha}\right)^\lambda\right]} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\alpha^{\lambda k}} \; I\left[\frac{1}{p}(d+4). \beta + \lambda k - 1\right]$$

The mode M_o can be derived as,

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left[\frac{\frac{p}{a^d}\left(\frac{\lambda}{\alpha^\beta}\right)}{\Gamma\left(\frac{d}{p}\right)\gamma\left[\frac{d}{p}, \left(\frac{1}{a}\right)^p\right]} \sum_{k=0}^{\infty} \frac{(-1)^k x^{d-1} e^{-\left(\frac{x}{a}\right)^p} \left(\frac{\gamma\left[\frac{d}{p}, \left(\frac{x}{a}\right)^p\right]}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1}}{k!\,\alpha^{\lambda k}}\right] = 0 \quad (26)$$

The median M_e can be derived as,

$$F(x) = \frac{\gamma\left[\frac{\beta}{\lambda}, \left(\frac{\gamma\left[\frac{d}{p}, \left(\frac{x}{a}\right)^p\right]}{\alpha\,\Gamma\left(\frac{d}{p}\right)}\right)^\lambda\right]}{\gamma\left[\frac{\beta}{\lambda}, \left(\frac{1}{\alpha}\right)^\lambda\right]} = \frac{1}{2}$$

$$\gamma\left[\frac{\beta}{\lambda}, \left(\frac{\gamma\left[\frac{d}{p}, \left(\frac{x}{a}\right)^p\right]}{\alpha\,\Gamma\left(\frac{d}{p}\right)}\right)^\lambda\right] - \frac{1}{2}\gamma\left[\frac{\beta}{\lambda}, \left(\frac{1}{\alpha}\right)^\lambda\right] = 0 \qquad (27)$$

By solving the non-linear equations (26) and (27), the mode and median of X can be obtain.

Also that the quantile function x_q of [0,1] TGG-GG random variable can be obtain by solve the following equation,

$$q = P\left(x \le x_q\right) = \frac{\gamma\left[\frac{\beta}{\lambda}, \left(\frac{\gamma\left[\frac{d}{p}, \left(\frac{x_q}{a}\right)^p\right]}{\alpha\,\Gamma\left(\frac{d}{p}\right)}\right)^\lambda\right]}{\gamma\left[\frac{\beta}{\lambda}, \left(\frac{1}{\alpha}\right)^\lambda\right]} \qquad (28)$$

A random variable X has the [0,1] TGG-GG distribution can be simulated by solving the nonlinear equation,

$$U\,\gamma\left[\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right]-\gamma\left[\frac{\beta}{\lambda},\left(\frac{\gamma\left[\frac{d}{p},\left(\frac{x_q}{a}\right)^{p}\right]}{\alpha\,\Gamma\left(\frac{d}{p}\right)}\right)^{\lambda}\right]=0$$

where U has the uniform(0,1) distribution**.**

5. Shannon and Relative Entropies

The Shannon entropy is defined as a measure of uncertainty that play an important role with the information's theory. Mathematically, Shannon entropy, H define as an expected of $(-\ln f(x))$ which is equivalent to,

$$H=-\int_0^{\infty} f(x)\ln f(x)\;\;dx$$

The Shannon entropy of [0,1] TGG-GG distribution can be obtain as follows,

$$\ln f(x)=\ln\frac{\left(\frac{p}{a^d}\right)\frac{\lambda}{\alpha^{\beta}}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}+(d-1)\ln x-\left(\frac{x}{a}\right)^{p}+\ln\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\alpha^{\lambda k}}\left(\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1}\right]$$

So that

$$H=-\int_0^{\infty}\frac{\left(\frac{p}{a^d}\right)\frac{\lambda}{\alpha^{\beta}}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\alpha^{\lambda k}}\,x^{d-1}\,e^{-\left(\frac{x}{a}\right)^{p}}\left(\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1}\left[\ln\left(\frac{\left(\frac{p}{a^d}\right)\frac{\lambda}{\alpha^{\beta}}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}\right)+(d-1)\ln x-\left(\frac{x}{a}\right)^{p}+\ln\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\alpha^{\lambda k}}\left(\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right)}{\alpha\,\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1}\right)\right]\;dx$$

Suppose that,

$$I_2=(1-d)\frac{\left(\frac{p}{a^d}\right)\frac{\lambda}{\alpha^{\beta}}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\alpha^{\lambda k}}\int_0^{\infty}x^{d-1}\,e^{-\left(\frac{x}{a}\right)^{p}}\left(\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1}\ln x\;\;dx$$

and $w(x;a,p,d,\alpha,\beta,\lambda)$ new function,

$$w(x;a,p,d,\alpha,\beta,\lambda)=\int_0^{\infty}\frac{\left(\frac{p}{a^d}\right)\frac{\lambda}{\alpha^{\beta}}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\alpha^{\lambda k}}\,x^{d-1}\,e^{-\left(\frac{x}{a}\right)^{d}}\left(\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1}\ln\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\alpha^{\lambda k}}\left(\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1}\right]dx$$

Then,

$$H=-\ln\frac{\left(\frac{p}{a^d}\right)\frac{\lambda}{\alpha^{\beta}}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}+I_2+\frac{1}{a^p}\,E(X^p)-w(x;a,p,d,\alpha,\beta,\lambda)$$

By using the transformation $u=\left(\frac{x}{a}\right)^{p}$ we get,

$$I_2=(1-d)\frac{\frac{\lambda}{\alpha^{\beta}}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\alpha^{\lambda k}}\int_0^{\infty}u^{\frac{d}{p}-1}\,e^{-u}\left(\frac{\gamma\left(\frac{d}{p},u\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1}\left(\ln a+\frac{1}{p}\ln u\right)du$$

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$$I_2 = \frac{(1-d)\frac{\lambda}{\alpha^\beta}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^\lambda\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\alpha^{\lambda k}}\left[\ln a\int_0^\infty u^{\frac{d}{p}-1}e^{-u}\left(\frac{\gamma\left(\frac{d}{p},u\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1}du+\frac{1}{p}\int_0^\infty \ln u\; u^{\frac{d}{p}-1}e^{-u}\left(\frac{\gamma\left(\frac{d}{p},u\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1}du\right]$$

Since, $I\left(k+\frac{r}{t},m\right)=\int_0^\infty y^{\frac{r}{t}+k-1}e^{-y}\left[\frac{\gamma(k,y)}{\Gamma(k)}\right]^m dy$, so that,

$$I_2 = \frac{(1-d)\frac{\lambda}{\alpha^\beta}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^\lambda\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\alpha^{\lambda k}}\left[\ln a\; I\left(\frac{d}{p},\beta+\lambda k-1\right)+\frac{1}{p}\int_0^\infty \ln u\; u^{\frac{d}{p}-1}e^{-u}\left(\frac{\gamma\left(\frac{d}{p},u\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1}du\right]$$

Let $I_2^* = \int_0^\infty \ln u\; u^{\frac{d}{p}-1}e^{-u}\left(\frac{\gamma\left(\frac{d}{p},u\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1}du$

Based on using expansion series of incomplete gamma ratio function, we get

$$\left[\frac{\gamma\left(\frac{d}{p},u\right)}{\Gamma\left(\frac{d}{p}\right)}\right]^{\beta+\lambda k-1} = \left[\frac{u^{\frac{d}{p}}\sum_{i=0}^{\infty}\frac{(-1)^i u^i}{\left(\frac{d}{p}+i\right)i!}}{\Gamma\left(\frac{d}{p}\right)}\right]^{\beta+\lambda k-1} = \frac{u^{\frac{d}{p}(\beta+\lambda k-1)}}{\left[\Gamma\left(\frac{d}{p}\right)\right]^{\beta+\lambda k-1}}\sum_{i_1=0}^{\infty}\cdots\sum_{i_{\beta+\lambda k-1}=0}^{\infty}\frac{(-1)^{i_1+\cdots+i_{\beta+\lambda k-1}}\;u^{i_1+\cdots+i_{\beta+\lambda k-1}}}{\left(\frac{d}{p}+i_1\right)\cdots\left(\frac{d}{p}+i_{\beta+\lambda k-1}\right)i_1!\cdots i_{\beta+\lambda k-1}!}$$

$$I_2^* = \frac{1}{\left[\Gamma\left(\frac{d}{p}\right)\right]^{\beta+\lambda k-1}}\sum_{i_1=0}^{\infty}\cdots\sum_{i_{\beta+\lambda k-1}=0}^{\infty}\frac{(-1)^{i_1+\cdots+i_{\beta+\lambda k-1}}}{\left(\frac{d}{p}+i_1\right)\cdots\left(\frac{d}{p}+i_{\beta+\lambda k-1}\right)i_1!\cdots i_{\beta+\lambda k-1}!}\int_0^\infty ln(u)\,e^{-u}\,u^{\frac{d}{p}(\beta+\lambda k)+i_1+\cdots+i_{\beta+\lambda k-1}-1}du$$

Since, $\int_0^\infty x^{s-1}e^{-mx}ln(x)\;dx = m^{-s}\Gamma(s)\{\psi(s)-ln(m)\}$, then

$$I_2^* = \frac{1}{\left[\Gamma\left(\frac{d}{p}\right)\right]^{\beta+\lambda k-1}}\sum_{i_1=0}^{\infty}\cdots\sum_{i_{\beta+\lambda k-1}=0}^{\infty}\frac{(-1)^{i_1+\cdots+i_{\beta+\lambda k-1}}}{\left(\frac{d}{p}+i_1\right)\cdots\left(\frac{d}{p}+i_{\beta+\lambda k-1}\right)i_1!\cdots i_{\beta+\lambda k-1}!}\Gamma\left[\frac{d}{p}(\beta+\lambda k)+i_1+\cdots+i_{\beta+\lambda k-1}\right]\left\{\psi\left(\frac{d}{p}(\beta+\lambda k)+i_1+\cdots+i_{\beta+\lambda k-1}\right)-ln(1)\right\}$$

Hence,

$$I_2 = \frac{(1-d)\frac{\lambda}{\alpha^\beta}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^\lambda\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\alpha^{\lambda k}}\left\{ln(a)\,I\left(\frac{d}{p},\beta+\lambda k-1\right)+\frac{1}{p}\left[\frac{1}{\left[\Gamma\left(\frac{d}{p}\right)\right]^{\beta+\lambda k-1}}\sum_{i_1=0}^{\infty}\cdots\sum_{i_{\beta+\lambda k-1}=0}^{\infty}\frac{(-1)^{i_1+\cdots+i_{\beta+\lambda k-1}}}{\left(\frac{d}{p}+i_1\right)\cdots\left(\frac{d}{p}+i_{\beta+\lambda k-1}\right)i_1!\cdots i_{\beta+\lambda k-1}!}\Gamma\left[\frac{d}{p}(\beta+\lambda k)+i_1+\cdots+i_{\beta+\lambda k-1}\right]\psi\left(\frac{d}{p}(\beta+\lambda k)+i_1+\cdots+i_{\beta+\lambda k-1}\right)\right]\right\}$$

Therefore,

$$H=-\ln\frac{\left(\frac{p}{a^{d}}\right)\frac{\lambda}{\alpha^{\beta}}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}+\frac{(1-d)\frac{\lambda}{\alpha^{\beta}}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!\,\alpha^{\lambda k}}\left\{\ln(a)\,I\left(\frac{d}{p},\beta+\lambda k-1\right)\right.$$
$$+\frac{1}{p}\left[\frac{1}{\left[\Gamma\left(\frac{d}{p}\right)\right]^{\beta+\lambda k-1}}\sum_{i_1=0}^{\infty}\cdots\sum_{i_{\beta+\lambda k-1}=0}^{\infty}\frac{(-1)^{i_1+\cdots+i_{\beta+\lambda k-1}}}{\left(\frac{d}{p}+i_1\right)\cdots\left(\frac{d}{p}+i_{\beta+\lambda k-1}\right)i_1!\cdots i_{\beta+\lambda k-1}!}\,\Gamma\left[\frac{d}{p}(\beta+\lambda k)+i_1+\cdots\right.\right.$$
$$\left.\left.\left.+i_{\beta+\lambda k-1}\right]\psi\left(\frac{d}{p}(\beta+\lambda k)+i_1+\cdots+i_{\beta+\lambda k-1}\right)\right]\right\}+\frac{\frac{\lambda}{\alpha^{\beta}}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!\,\alpha^{\lambda k}}\,I\left[\frac{1}{p}(d+p),\beta+\lambda k-1\right]$$
$$-w(x;a,p,d,\alpha,\beta,\lambda)\qquad(29)$$

The relative entropy is define a measure of the difference between two probability distributions. In more closely, the relative entropy is the amount of information that is lost when the second distribution is used to approximate the first distribution. Mathematically, relative entropy, $DL(f_1\|f_2)$ define as,

$$DL(f_1\|f_2)=\int_0^{\infty}f_1(x)\ln\left(\frac{f_1(x)}{f_2(x)}\right)dx$$

Since,

$$f_1(x)=\frac{\left[\frac{p}{a^{d}}\right]\frac{\lambda}{\alpha^{\beta}}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!\,\alpha^{\lambda k}}\,x^{d-1}\,e^{-\left(\frac{x}{a}\right)^{p}}\left(\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1}$$

$$f_2(x)=\frac{\left[\frac{p_1}{{a_1}^{d_1}}\right]\frac{\lambda_1}{{\alpha_1}^{\beta_1}}}{\Gamma\left(\frac{d_1}{p_1}\right)\gamma\left(\frac{\beta_1}{\lambda_1},\left(\frac{1}{\alpha_1}\right)^{\lambda_1}\right)}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!\,{\alpha_1}^{\lambda_1 k}}\,x^{d_1-1}\,e^{-\left(\frac{x}{a_1}\right)^{p_1}}\left(\frac{\gamma\left(\frac{d_1}{p_1},\left(\frac{x}{a_1}\right)^{p_1}\right)}{\Gamma\left(\frac{d_1}{p_1}\right)}\right)^{\beta_1+\lambda_1 k-1}$$

So that

$$\frac{f_1(x)}{f_2(x)}=\frac{\left[\frac{p}{a^{d}}\right]\left[\frac{\lambda}{\alpha^{\beta}}\right]\Gamma\left(\frac{d_1}{p_1}\right)\gamma\left(\frac{\beta_1}{\lambda_1},\left(\frac{1}{\alpha_1}\right)^{\lambda_1}\right)}{\left[\frac{p_1}{{a_1}^{d_1}}\right]\left[\frac{\lambda_1}{{\alpha_1}^{\beta_1}}\right]\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}x^{d-d_1}\,\frac{e^{-\left(\frac{x}{a}\right)^{p}}}{e^{-\left(\frac{x}{a_1}\right)^{p_1}}}\,\frac{\left[\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right)}{\Gamma\left(\frac{d}{p}\right)}\right]^{\beta-1}}{\left[\frac{\gamma\left(\frac{d_1}{p_1},\left(\frac{x}{a_1}\right)^{p_1}\right)}{\Gamma\left(\frac{d_1}{p_1}\right)}\right]^{\beta_1-1}}\,\frac{e^{-\left(\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right)}{\alpha\,\Gamma\left(\frac{d}{p}\right)}\right)^{\lambda}}}{e^{-\left(\frac{\gamma\left(\frac{d_1}{p_1},\left(\frac{x}{a_1}\right)^{p_1}\right)}{\alpha_1\,\Gamma\left(\frac{d_1}{p_1}\right)}\right)^{\lambda_1}}}$$

and

$$\ln\frac{f_1(x)}{f_2(x)}=\ln\left[\frac{\left[\frac{p}{a^{d}}\right]\left[\frac{\lambda}{\alpha^{\beta}}\right]\Gamma\left(\frac{d_1}{p_1}\right)\gamma\left(\frac{\beta_1}{\lambda_1},\left(\frac{1}{\alpha_1}\right)^{\lambda_1}\right)}{\left[\frac{p_1}{{a_1}^{d_1}}\right]\left[\frac{\lambda_1}{{\alpha_1}^{\beta_1}}\right]\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}\right]+(d-d_1)\ln(x)-\left(\frac{x}{a}\right)^{p}+\left(\frac{x}{a_1}\right)^{p_1}+(\beta-1)\ln\left[\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right)}{\Gamma\left(\frac{d}{p}\right)}\right]-(\beta_1-1)\ln\left[\frac{\gamma\left(\frac{d_1}{p_1},\left(\frac{x}{a_1}\right)^{p_1}\right)}{\Gamma\left(\frac{d_1}{p_1}\right)}\right]-$$
$$\left(\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right)}{\alpha\,\Gamma\left(\frac{d}{p}\right)}\right)^{\lambda}+\left(\frac{\gamma\left(\frac{d_1}{p_1},\left(\frac{x}{a_1}\right)^{p_1}\right)}{\alpha_1\,\Gamma\left(\frac{d_1}{p_1}\right)}\right)^{\lambda_1}$$

Then,

$DL(f_1\|f_2) =$

$$\ln\left[\frac{\left[\frac{p}{a^d}\right]\Gamma\left(\frac{d_1}{p_1}\right)\gamma\left(\frac{\beta_1}{\lambda_1},\left(\frac{1}{\alpha_1}\right)^{\lambda_1}\right)\frac{\lambda}{\alpha^\beta}}{\left[\frac{p_1}{a_1{}^{d_1}}\right]\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)\frac{\lambda_1}{\alpha_1{}^{\beta_1}}}\right] + (d-d_1)\int_0^\infty f_1(x)\ln(x)\,dx - \frac{1}{a^p}\,E(X^p) + \frac{1}{a_1^{p_1}}\,E(X^{p_1}) + (\beta-1)\int_0^\infty f_1(x)\,\ln\left[\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^p\right)}{\Gamma\left(\frac{d}{p}\right)}\right]dx\ -$$

$$(\beta_1-1)\int_0^\infty f_1(x)\ln\left[\frac{\gamma\left(\frac{d_1}{p_1},\left(\frac{x}{a_1}\right)^{p_1}\right)}{\Gamma\left(\frac{d_1}{p_1}\right)}\right]dx - \int_0^\infty f_1(x)\left(\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^p\right)}{\alpha\,\Gamma\left(\frac{d}{p}\right)}\right)^{\lambda}dx\ + \int_0^\infty f_1(x)\left(\frac{\gamma\left(\frac{d_1}{p_1},\left(\frac{x}{a_1}\right)^{p_1}\right)}{\alpha_1\,\Gamma\left(\frac{d_1}{p_1}\right)}\right)^{\lambda_1}dx$$

Since,

$\int_0^\infty f_1(x)\,ln(x)\,dx =$

$$\frac{\frac{\lambda}{\alpha^\beta}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\alpha^{\lambda k}}\left\{ln(a)\,I\left(\frac{d}{p},\beta+\lambda k-1\right)+\frac{1}{p}\left[\frac{1}{\left[\Gamma\left(\frac{d}{p}\right)\right]^{\beta+\lambda k-1}}\sum_{i_1=0}^{\infty}\ \ldots\sum_{i_{\beta+\lambda k-1}=0}^{\infty}\frac{(-1)^{i_1+\cdots+i\ \beta+\lambda k-1}}{\left(\frac{d}{p}+i_1\right)\ldots\left(\frac{d}{p}+i_{\beta+\lambda k-1}\right)i_1!\ldots i_{\beta+\lambda k-1}!}\,\Gamma\left[\frac{d}{p}(\beta+\lambda k)+\right.\right.\right.$$

$$\left.\left.\left.i_1+\cdots+i_{\beta+\lambda k-1}\right]\left\{\psi\left(\frac{d}{p}(\beta+\lambda k)+i_1+\cdots+i_{\beta+\lambda k-1}\right)-ln(1)\right\}\right]\right\}\ E(X^p) = \frac{a^p\ \frac{\lambda}{\alpha^\beta}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\alpha^{\lambda k}}\,I\left[\frac{1}{p}(d+p),\beta+\lambda k-1\right]$$

$$E(X^{p_1}) = \frac{a^{p_1}\ \frac{\lambda}{\alpha^\beta}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\alpha^{\lambda k}}\,I\left[\frac{1}{p}(d+p_1),\beta+\lambda k-1\right]$$

$$\eta(x;a,d,p,\alpha,\beta,\lambda) = \int_0^\infty f_1(x)\ln\left[\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^p\right)}{\Gamma\left(\frac{d}{p}\right)}\right]dx$$

$$\eta_1(x;a,d,p,\alpha,\beta,\lambda) = \int_0^\infty f_1(x)\ln\left[\frac{\gamma\left(\frac{d_1}{p_1},\left(\frac{x}{a_1}\right)^{p_1}\right)}{\Gamma\left(\frac{d_1}{p_1}\right)}\right]dx$$

$$J_1 = \int_0^\infty f_1(x)\left(\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^p\right)}{\alpha\,\Gamma\left(\frac{d}{p}\right)}\right)^{\lambda}dx$$

$$J_2 = \int_0^\infty f_1(x)\left(\frac{\gamma\left(\frac{d_1}{p_1},\left(\frac{x}{a_1}\right)^{p_1}\right)}{\alpha_1\Gamma\left(\frac{d_1}{p_1}\right)}\right)^{\lambda_1}dx$$

For J_1 we get that

$$J_1 = \frac{\left[\frac{p}{a^d}\right]\lambda}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{\alpha^{\beta+\lambda\kappa+\lambda}}\int_0^\infty x^{d-1}\,e^{-\left(\frac{x}{a}\right)^p}\left(\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^p\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda(k+1)-1}dx$$

By using the transformation $u = \left(\frac{x}{a}\right)^p$, we get

$$J_1 = \frac{\lambda}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{\alpha^{\beta+\lambda(k+1)}}\int_0^\infty u^{\frac{d}{p}-1}\,e^{-u}\left(\frac{\gamma\left(\frac{d}{p},\ u\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda(k+1)-1}du$$

$$J_1 = \frac{\lambda}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{\alpha^{\beta+\lambda(k+1)}}\,I\left[\frac{d}{p},\beta+\lambda(k+1)-1\right]$$

Also that

$$J_2 = \int_0^\infty f_1(x) \left(\frac{\gamma\left(\frac{d_1}{p_1}, \left(\frac{x}{a_1}\right)^{p_1}\right)}{\alpha_1\, \Gamma\left(\frac{d_1}{p_1}\right)} \right)^{\lambda_1} dx$$

and

$$\left(\frac{\gamma\left(\frac{d_1}{p_1}, \left(\frac{x}{a_1}\right)^{p_1}\right)}{\alpha_1\, \Gamma\left(\frac{d_1}{p_1}\right)} \right)^{\lambda_1} = \frac{\left(\frac{x}{a_1}\right)^{p_1\lambda_1} \sum_{i_1=0}^{\infty} \cdots \sum_{i_{\lambda_1}=0}^{\infty} \frac{(-1)^{i_1+\cdots+i_{\lambda_1}} \left(\frac{x}{a_1}\right)^{p_1(i_1+\cdots i_{\lambda_1})}}{\left(\frac{d_1}{p_1}+i_1\right)\ldots\left(\frac{d_1}{p_1}+i_{\lambda_1}\right) i_1!\ldots i_{\lambda_1}!}}{\alpha_1{}^{\lambda_1} \left[\Gamma\left(\frac{d_1}{p_1}\right)\right]^{\lambda_1}}$$

Therefor we get,

$$J_2 = \int_0^\infty \frac{\left[\frac{p}{a^d}\right]\frac{\lambda}{\alpha\beta}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\alpha^{\lambda k}}\, x^{d-1}\, e^{-\left(\frac{x}{a}\right)^p} \left(\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^p\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1} \frac{\left(\frac{x}{a_1}\right)^{p_1\lambda_1} \sum_{i_1=0}^{\infty}\cdots \sum_{i_{\lambda_1}=0}^{\infty} \frac{(-1)^{i_1+\cdots+i_{\lambda_1}} \left(\frac{x}{a_1}\right)^{p_1(i_1+\cdots i_{\lambda_1})}}{\left(\frac{d_1}{p_1}+i_1\right)\ldots\left(\frac{d_1}{p_1}+i_{\lambda_1}\right)i_1!\ldots i_{\lambda_1}!}}{\alpha_1{}^{\lambda_1}\left[\Gamma\left(\frac{d_1}{p_1}\right)\right]^{\lambda_1}} dx$$

$$J_2 = \frac{\left[\frac{p}{a^d}\right]\frac{\lambda}{\alpha\beta}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)} \sum_{k=0}^{\infty} \sum_{i_1=0}^{\infty} \cdots \sum_{i_{\lambda_1}=0}^{\infty} \frac{(-1)^{k+i_1+\cdots+i_{\lambda_1}} \left[\int_0^\infty x^{d+p_1(\lambda_1+i_1+\cdots i_{\lambda_1})-1}\, e^{-\left(\frac{x}{a}\right)^p} \left(\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^p\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1} dx\right]}{\left(\frac{d_1}{p_1}+i_1\right)\ldots\left(\frac{d_1}{p_1}+i_{\lambda_1}\right)i_1!\ldots i_{\lambda_1}!(a_1)^{p_1(\lambda_1+i_1+\cdots i_{\lambda_1})}\ \alpha_1{}^{\lambda_1}\left[\Gamma\left(\frac{d_1}{p_1}\right)\right]^{\lambda_1} k!\alpha^{\lambda k}}$$

Now for the sub-integral we get

$$J_2^* = \int_0^\infty x^{d+p_1(\lambda_1+i_1+\cdots i_{\lambda_1})-1}\, e^{-\left(\frac{x}{a}\right)^p} \left(\frac{\gamma\left(\frac{d}{p},\left(\frac{x}{a}\right)^p\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1} dx$$

By using the transformation $u = \left(\frac{x}{a}\right)^p$, we get

$$J_2^* = \frac{a^{d+p_1(\lambda_1+i_1+\cdots i_{\lambda_1})}}{p} \int_0^\infty u^{\frac{[d+p_1(\lambda_1+i_1+\cdots i_{\lambda_1})]}{p}-1}\, e^{-u} \left(\frac{\gamma\left(\frac{d}{p},u\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^{\beta+\lambda k-1} du$$

$$J_2^* = \frac{a^d}{p}\, I \left[\frac{[d+p_1(\lambda_1+i_1+\cdots i_{\lambda_1})]}{p}, \beta+\lambda k-1\right]$$

So that

$$J_2 = \frac{\left[\frac{p}{a^d}\right]\frac{\lambda}{\alpha\beta}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)} \sum_{k=0}^{\infty} \sum_{i_1=0}^{\infty} \cdots \sum_{i_{\lambda_1}=0}^{\infty} \frac{(-1)^{k+i_1+\cdots+i_{\lambda_1}}}{\left(\frac{d_1}{p_1}+i_1\right)\ldots\left(\frac{d_1}{p_1}+i_{\lambda_1}\right)i_1!\ldots i_{\lambda_1}!(a_1)^{p_1(\lambda_1+i_1+\cdots i_{\lambda_1})}\ \alpha_1{}^{\lambda_1}\left[\Gamma\left(\frac{d_1}{p_1}\right)\right]^{\lambda_1} k!\alpha^{\lambda k}} \frac{a^d}{p}\, I \left[\frac{d}{p}, \beta+\lambda k-1\right]$$

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$$J_2 = \frac{\frac{\lambda}{\alpha^\beta}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^\lambda\right)} \sum_{k=0}^{\infty}\sum_{i_1=0}^{\infty}\cdots\sum_{i_{\lambda_1}=0}^{\infty} \frac{(-1)^{k+i_1+\cdots+i_{\lambda_1}}}{\left(\frac{d_1}{p_1}+i_1\right)\cdots\left(\frac{d_1}{p_1}+i_{\lambda_1}\right)i_1!\ldots i_{\lambda_1}!(a_1)^{p_1(\lambda_1+i_1+\cdots i_{\lambda_1})}\,\alpha_1{}^{\lambda_1}\left[\Gamma\left(\frac{d_1}{p_1}\right)\right]^{\lambda_1} k!\alpha^{\lambda k}} I\left[\frac{[d+p_1(\lambda_1+i_1+\cdots i_{\lambda_1})]}{p},\beta+\lambda k-1\right]$$

$$DL(f_1\|f_2) = \ln\left[\frac{\left[\frac{p}{a^d}\right]\Gamma\left(\frac{d_1}{p_1}\right)\gamma\left(\frac{\beta_1}{\lambda_1},\left(\frac{1}{\alpha_1}\right)^{\lambda_1}\right)\frac{\lambda}{\alpha^\beta}}{\left[\frac{p_1}{a_1{}^{d_1}}\right]\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^{\lambda}\right)\frac{\lambda_1}{\alpha_1{}^{\beta_1}}}\right]$$
$$+(d-d_1)\frac{\frac{\lambda}{\alpha^\beta}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^\lambda\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\alpha^{\lambda k}}\left\{\ln(a)\,I\left(\frac{d}{p},\beta+\lambda k-1\right)\right.$$
$$+\frac{1}{p}\left[\frac{1}{\left[\Gamma\left(\frac{d}{p}\right)\right]^{\beta+\lambda k-1}}\sum_{i_1=0}^{\infty}\cdots\sum_{i_{\beta+\lambda k-1}=0}^{\infty}\frac{(-1)^{i_1+\cdots+i_{\beta+\lambda k-1}}}{\left(\frac{d}{p}+i_1\right)\ldots\left(\frac{d}{p}+i_{\beta+\lambda k-1}\right)i_1!\ldots i_{\beta+\lambda k-1}!}\,\Gamma\left[\frac{d}{p}(\beta+\lambda k)+i_1+\cdots\right.\right.$$
$$\left.\left.\left.+i_{\beta+\lambda k-1}\right]\psi\left(\frac{d}{p}(\beta+\lambda k)+i_1+\cdots+i_{\beta+\lambda k-1}\right)\right]\right\} - \frac{\frac{\lambda}{\alpha^\beta}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^\lambda\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\alpha^{\lambda k}}\,I\left[\frac{1}{p}(d+p),\beta+\lambda k-1\right]$$
$$+\frac{1}{a_1^{p_1}}\,\frac{a^{p_1}\,\frac{\lambda}{\alpha^\beta}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^\lambda\right)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\alpha^{\lambda k}}\,I\left[\frac{1}{p}(d+p_1),\beta+\lambda k-1\right]+(\beta-1)\eta(x;\,a,d,p,\alpha,\beta,\lambda)$$
$$-(\beta_1-1)\eta_1(x;\,a,d,p,\alpha_1,\beta_1,\lambda_1)-\frac{\lambda}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^\lambda\right)}\sum_{k=0}^{\infty}\frac{\frac{(-1)^k}{k!}}{\alpha^{\beta+\lambda(k+1)}}\,I\left[\frac{d}{p},\beta+\lambda(k+1)-1\right]+$$
$$\frac{\frac{\lambda}{\alpha^\beta}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda},\left(\frac{1}{\alpha}\right)^\lambda\right)}\sum_{k=0}^{\infty}\sum_{i_1=0}^{\infty}\cdots\sum_{i_{\lambda_1}=0}^{\infty}\frac{(-1)^{k+i_1+\cdots+i_{\lambda_1}}}{\left(\frac{d_1}{p_1}+i_1\right)\cdots\left(\frac{d_1}{p_1}+i_{\lambda_1}\right)i_1!\ldots i_{\lambda_1}!(a_1)^{p_1(\lambda_1+i_1+\cdots i_{\lambda_1})}\,\alpha_1{}^{\lambda_1}\left[\Gamma\left(\frac{d_1}{p_1}\right)\right]^{\lambda_1}k!\alpha^{\lambda k}}\,.$$
$$I\left[\frac{[d+p_1(\lambda_1+i_1+\cdots i_{\lambda_1})]}{p},\beta+\lambda k-1\right] \qquad (30)$$

6. Stress - Strength Model

Suppose X (strength) and Y (stress) are two independent [0,1] TGG-GG distribution with different parameters. The stress-strength of [0,1] TGG –GG distribution is given by,

$$R = P(y<x) = \int_0^\infty f_X(x)F_Y(x)\,dx$$

$$R = \int_0^\infty \frac{\gamma\left[\frac{\beta_1}{\lambda_1},\left(\frac{\gamma\left(\frac{d_1}{p_1},\left(\frac{x}{a_1}\right)^{p_1}\right)}{\alpha\;\Gamma\left(\frac{d_1}{p_1}\right)}\right)^{\lambda_1}\right]}{\gamma\left(\frac{\beta_1}{\lambda_1},\left(\frac{1}{\alpha_1}\right)^{\lambda_1}\right)} f(x)dx$$

Since

$$\frac{\gamma\left[\frac{\beta_1}{\lambda_1},\left(\frac{\gamma\left(\frac{d_1}{p_1},\left(\frac{x}{a_1}\right)^{p_1}\right)}{\alpha_1\Gamma\left(\frac{d_1}{p_1}\right)}\right)^{\lambda_1}\right]}{\gamma\left(\frac{\beta_1}{\lambda_1},\left(\frac{1}{\alpha_1}\right)^{\lambda_1}\right)} = \frac{1}{\gamma\left(\frac{\beta_1}{\lambda_1},\left(\frac{1}{\alpha_1}\right)^{\lambda_1}\right)}\left(\frac{\gamma\left(\frac{d_1}{p_1},\left(\frac{x}{a_1}\right)^{p_1}\right)}{\alpha_1\Gamma\left(\frac{d_1}{p_1}\right)}\right)^{\beta_1}\sum_{d=0}^{\infty}\frac{\left[-\left(\frac{\gamma\left(\frac{d_1}{p_1},\left(\frac{x}{a_1}\right)^{p_1}\right)}{\alpha_1\ \Gamma\left(\frac{d_1}{p_1}\right)}\right)^{\lambda_1}\right]^d}{d!\left(d+\frac{d_1}{p_1}\right)}$$

$$= \frac{1}{\gamma\left(\frac{\beta_1}{\lambda_1},\left(\frac{1}{\alpha_1}\right)^{\lambda_1}\right)}\sum_{d=0}^{\infty}(-1)^d\frac{\left(\frac{\gamma\left(\frac{d_1}{p_1},\left(\frac{x}{a_1}\right)^{p_1}\right)}{\alpha_1\ \Gamma\left(\frac{d_1}{p_1}\right)}\right)^{\lambda_1 d+\beta_1}}{d!\left(d+\frac{d_1}{p_1}\right)}$$

$$= \frac{1}{\gamma\left(\frac{\beta_1}{\lambda_1},\left(\frac{1}{\alpha_1}\right)^{\lambda_1}\right)}\sum_{d=0}^{\infty}\frac{(-1)^{d_1}}{\left[\alpha_1\Gamma\left(\frac{d_1}{p_1}\right)\right]^{\lambda_1 d+\beta_1}}\frac{\left(\left(\frac{x}{a_1}\right)^{d_1}\sum_{s=0}^{\infty}\frac{(-1)^s\left(\frac{x}{a_1}\right)^{p_1 s}}{s!\,(s+\frac{d_1}{p_1})}\right)^{\lambda_1 d+\beta_1}}{d!\left(d+\frac{d_1}{p_1}\right)}$$

$$= \frac{1}{\gamma\left(\frac{\beta_1}{\lambda_1},\left(\frac{1}{\alpha_1}\right)^{\lambda_1}\right)}\sum_{d=0}^{\infty}\frac{(-1)^{d_1}}{\left[\alpha_1\ \Gamma\left(\frac{d_1}{p_1}\right)\right]^{\lambda_1 d+\beta_1}}\frac{\left(\left(\frac{x}{a_1}\right)^{d_1}\sum_{s=0}^{\infty}\frac{(-1)^s\left(\frac{x}{a_1}\right)^{p_1 s}}{s!\,(s+\frac{d_1}{p_1})}\right)^{\lambda_1 d+\beta_1}}{d!\left(d+\frac{d_1}{p_1}\right)}$$

$$= \frac{1}{\gamma\left(\frac{\beta_1}{\lambda_1},\left(\frac{1}{\alpha_1}\right)^{\lambda_1}\right)}\sum_{d=0}^{\infty}\frac{(-1)^{d_1}}{\left[\alpha_1\Gamma\left(\frac{d_1}{p_1}\right)\right]^{\lambda_1 d+\beta_1}}\frac{\left(\left(\frac{x}{a_1}\right)^{d_1}\sum_{s=0}^{\infty}\frac{(-1)^s\left(\frac{x}{a_1}\right)^{p_1 s}}{s!\,(s+\frac{d_1}{p_1})}\right)^{\lambda_1 d+\beta_1}}{d!\left(d+\frac{d_1}{p_1}\right)}$$

$$= \sum_{d=0}^{\infty}\frac{(-1)^{d_1}}{\gamma\left(\frac{\beta_1}{\lambda_1},\left(\frac{1}{\alpha_1}\right)^{\lambda_1}\right)\left[\alpha_1\Gamma\left(\frac{d_1}{p_1}\right)\right]^{\lambda_1 d+\beta_1} d!\left(d+\frac{d_1}{p_1}\right)}\sum_{s_1=0}^{\infty}\cdots\sum_{s_{\lambda_1 d+\beta_1}=0}^{\infty}\frac{(-1)^{s_1+\cdots+s_{\lambda_1 d+\beta_1}}\left(\frac{x}{a_1}\right)^{d_1[\lambda_1 d+\beta_1]}\left(\frac{x}{a_1}\right)^{p_1[s_1+\cdots+s_{\lambda_1 d+\beta_1}]}}{s_1!\ldots s_{\lambda_1 d+\beta_1}!\left(s_1+\frac{d_1}{p_1}\right)\ldots\left(s_1+\frac{d_1}{p_1}\right)}$$

$$= \sum_{d=0}^{\infty}\frac{(-1)^{d_1}}{\gamma\left(\frac{\beta_1}{\lambda_1},\left(\frac{1}{\alpha_1}\right)^{\lambda_1}\right)\left[\alpha_1\Gamma\left(\frac{d_1}{p_1}\right)\right]^{\lambda_1 d+\beta_1} d!\left(d+\frac{d_1}{p_1}\right)}\sum_{s_1=0}^{\infty}\cdots\sum_{s_{\lambda_1 d+\beta_1}=0}^{\infty}\frac{(-1)^{s_1+\cdots+s_{\lambda_1 d+\beta_1}}\left(\frac{x}{a_1}\right)^{d_1[\lambda_1 d+\beta_1]+p_1[s_1+\cdots+s_{\lambda_1 d+\beta_1}]}}{s_1!\ldots s_{\lambda_1 d+\beta_1}!\left(s_1+\frac{d_1}{p_1}\right)\ldots\left(s_1+\frac{d_1}{p_1}\right)}$$

Therefor the stress-strength of [0,1] TGG –GG distribution can be written as,

$$R = \sum_{d=0}^{\infty} \frac{(-1)^{d_1}}{\gamma\left(\frac{\beta_1}{\lambda_1}.\left(\frac{1}{\alpha_1}\right)^{\lambda_1}\right)\left[\alpha_1 \Gamma\left(\frac{d_1}{p_1}\right)\right]^{\lambda_1 d+\beta_1} d!\left(d+\frac{d_1}{p_1}\right)} \sum_{s_1=0}^{\infty} \cdots \sum_{s_{\lambda_1 d+\beta_1}=0}^{\infty} \frac{(-1)^{s_1+\cdots+s_{\lambda_1 d+\beta_1}}}{s_1! \ldots s_{\lambda_1 d+\beta_1}!\left(s_1+\frac{d_1}{p_1}\right) \ldots \left(s_1+\frac{d_1}{p_1}\right)}$$

$$E\left[\left(\frac{X}{a_1}\right)^{d_1[\lambda_1 d+\beta_1]+p_1[s_1+\cdots+s_{\lambda_1 d+\beta_1}]}\right] \qquad (31)$$

Such that,

$$E\left(X^{d_1[\lambda_1 d+\beta_1]+p_1[s_1+\cdots+s_{\lambda_1 d+\beta_1}]}\right) = \frac{a^{d_1[\lambda_1 d+\beta_1]+p_1[s_1+\cdots+s_{\lambda_1 d+\beta_1}]} \frac{\lambda}{\alpha^{\beta}}}{\Gamma\left(\frac{d}{p}\right)\gamma\left(\frac{\beta}{\lambda}.\left(\frac{1}{\alpha}\right)^{\lambda}\right)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\alpha^{\lambda k}} I\left[\frac{1}{p}\left(d+d_1[\lambda_1 d+\beta_1]+p_1[s_1+\cdots+s_{\lambda_1 d+\beta_1}]\right), \beta + \lambda k - 1\right]$$

7. Concluding Remarks

In statistical analysis a lot of distributions are used to represent set(s) of data. Recently, new distributions are derived to extend some of well-known families of distributions, such that the new distributions are more flexible than the others to model real data. The composing of some distributions with each other's in some way has been in the foreword of data modeling.

In this paper, we presented a new family of continuous distributions based on [0,1] truncated generalized gamma. The [0,1] truncated generalized gamma-generalized gamma ([0,1]TGG-GG) distribution is discussed as special case. Properties of [0,1] TGG-GG is derived. We provide form for characteristic function, rth raw moment, mean, variance, skewness, kurtosis, mode, median, reliability function, hazard rate function, Shannon entropy and relative entropy. This paper deals also with the determination of stress-strength $R = P(y < x)$ when X (strength) and Y (stress) are two independent [0,1] TGG-GG distribution with different parameters.

المستخلص: في هذا البحث سنقوم بتقديم عائلة جديدة من التوزيعات المستمرة بالاستناد على توزيع كاما المعمم المبتور بالفترة [1,0] . ستتم مناقشة توزيع كاما المعمم المبتور عند الفترة [1,0] - كاما المعمم كحالة خاصة . سيتم اشتقاق دالة التوزيع التجميعي ، العزم من الدرجة r ، المتوسط ، التباين ، الالتواء ، التفلطح ، المنوال ، الوسيط ، الدالة المميزة ، دالة المعولية ودالة نسبة المخاطرة للتوزيع المذكور . من المعلوم ان فشل الوحدات يحدث عندما يتجاوز الاجهاد على المرونة المناظرة له . على هذا الأساس فان المرونة يمكن اعتبارها كمقاومة للفشل . ان التصميم الجيد هو الذي تكون فيه المرونة دائما اكبر من الاجهاد المتوقع . يمكن تعريف عامل الامان بدلالة الاجهاد والمرونة كـ الاجهاد / المرونة . لذلك سيتم هنا اشتقاق نموذج الاجهاد - المرونة لتوزيع كاما المعمم - كاما المعمم المبتور عند الفترة [1,0] بمعلمات مختلفة . كما سيتم اشتقاق انتروبي شانون والانتروبي النسبي .

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