Some New Results Related with Fuzzy Soft Banach Algebra

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Publication data: 18 /6 /2024

Keywords: fuzzy soft set, fuzzy soft linear, fuzzy soft norm, fuzzy soft Banach, fuzzy soft Banach algebra.

ABSTRACT

Fuzzy Soft Banach Algebra represent a fascinating extension of Soft Banach Algebra,providing a versatile mathematical framework for studying algebraic properties in,various applied contexts .This paper offers an overview of key concepts in Fuzzy Soft Banach Algebra theory and explores their fundamental applications.

The exposition begins by introducing the definition and general characteristics of Fuzzy Soft Banach Algebra, highlighting the principal distinctions from Soft Banach Algebra. The paper proceeds to delve into the essential properties, of Fuzzy Soft Banach Algebra and demonstrates their applicability in differential and integral calculus.

Furthermore, the paper showcases practical examples and applications of Fuzzy Soft Banach Algebra in fields such as number theory and mathematical physics This section emphasizes how Fuzzy Soft Banach Algebra can be leveraged, to solve practical problems acros

1. Introduction

In actual life, the majority of problems are ambiguous or imprecise. This is due to a lack of the required information**.** Zadeh **[7]** built fuzzy sets in 1965 to address such problems. Because fuzzy set theory lacks parameterization methods, Molodostov **[8]** introduced soft set theory in 1999. Maji **[6]** examined the fusion of soft set theory ,and fuzzy set theory in 2001, introducing the notion of a fuzzy soft set. The soft set approach was expanded by this new idea, moving from crisp cases to fuzzy cases. Tanay and Kandemir initiated studies on topology and fuzzy soft set theory in **[2]**. The fuzzy soft norm, over a set was initially articulated by Zadeh **[13]** in 2013, and an analogy between the fuzzy soft norm and the fuzzy norm over a set was also drawn.

In our paper, We proposed fuzzy soft Banach algebra and investigated some of its traits in the piece we wrote. Section 2 provides initial results. The term of fuzzy soft Banach algebra is laid out in Section 3, accompanied by a breakdown of some of its basic aspects. The paper wraps up in Section 4.

2. preliminary

Over the duration of this work. Let X reflect the universe set, E the parameter set, and let I^X be the set of all the fuzzy sets over X and A⊆ E.

Definition (2.1)[4]: , If Γ is a mapping supplied by $\Gamma: A \to I^X$, so a pair Γ_A is commonly referred to as a fuzzy soft set over X. As an outcome, for every component of X in $f(e)$, e ∈ A, while $f(e)$ is a fuzzy subset of X with the membership function $f_e : X \to [0, 1]$ expressing the degree to which each member of X has the trait e ∈ E.

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$\Gamma_A(X_E)$will serve as the set of all fuzzy soft subsets of the universal set X with regard to parameter set E during the course of the research.

Definition (2.2)[4]: , (Rules of fuzzy soft set) For two fuzzy soft sets Γ_A and $\Lambda_B$over a common universe X we

have,

i. Γ_A is a fuzzy soft subset of Λ_B shown by $\Gamma_A \lessapprox \Lambda_B$ if:

1. A ⊆ B,

2. For all e in A, Γ_e(x) ≤ Λ_e(x) ∀x ∈ X.

ii. $\Gamma_A = \Lambda_B$ if $\Gamma_A \leq \Lambda_B$ and $\Lambda_B \leq \Gamma_A$.

iii. The complement of a fuzzy soft set Γ_A is denoted by ${\Gamma^c}_A$where Γ^C : A → I^X and Γ^C(e) is the complement of fuzzy set Γ (e) with membership function

${\Gamma^c}_e = 1 - \Gamma_e$ ∀e ∈ A.

iv. Γ_A = ΦA (null fuzzy soft set with respect to A), if for each e ∈ A, Γ_e (x) = 0, ∀x ∈ X.

v. $\Gamma_A = X_A$ (absolute fuzzy soft set with respect to A) , if ∀x ∈ X, Γ_e(x) = 1 for each e ∈ A.

If A = E, the null and absolute fuzzy soft set is denoted by Φ and X respectively.

vi. The union of two fuzzy soft sets Γ_A and Λ_B,is denoted by $\Gamma_A \vee \Lambda_B$, is the fuzzy soft set $(\Gamma \vee \Lambda)_C$, where C = A ∪ B and ∀e ∈ C, we have (Γ ∨ Λ)(e) = Γ(e) ∨ Λ(e) where

$$(\Gamma \vee \Lambda)_C(x) = \begin{cases} \Gamma_A(x) & \text{if } e \in A - B \\ \Lambda_B(x) & \text{if } e \in B - A \\ \max\{\Gamma_e(x), \Lambda_e(x)\} & \text{if } e \in A \cap B \end{cases}$$

for all x ∈ X.

vii, The intersection of two fuzzy soft sets Γ_A and Λ_B is denoted by $\Gamma_A \wedge \Lambda_B$, is the fuzzy soft set $(\Gamma \vee \Lambda)_C$, where C = A ∩ B ≠ ∅ and ∀e ∈ C, we have (Γ ∧ Λ)(e) = $\Gamma(e) \wedge \Lambda(e)$ where $(\Gamma \wedge \Lambda)_e(x) = \min\{\Gamma_e(x), \Lambda_e(x)\}$ for all x ∈ X.

Definition (2.3)[4] :

1. Let x^{λ_e} be a fuzzy point in X with support x ∈ X and membership degree λ_e ∈ (0, 1]. The fuzzy soft set (Px)E is called fuzzy soft point, say F.S point, whenever Px : E → I^X is a map such that for each e ∈ E,

$$(Px)e(y) = \begin{cases} \lambda_e & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

So for each e ∈ E, (Px)(e) = x^{λ_e} , or (Px)(e) = $\lambda_e \chi\{x\}$, ∀e ∈ E. In other words, the F.S point (Px)E, is a description of x ∈ X based on the decision, parameters in E. If λ_e= 1 for all e ∈ E, we call (Px)E, crisp F.S point.

2. The F.S point (Px)E belongs to F.S set Γ_E denoting by (Γx)E∈ Γ_E, whenever $0 < \lambda_e \leq \Gamma_e(x)$ for all e ∈ E.

3. The restriction of F.S point (Px)E to (Px|e)E is called, the fuzzy soft single point over E, say F.S single point,

whenever

$$(Px|e)\alpha(y) = \begin{cases} (Px)e(x) = \lambda e > 0 & \text{if } \alpha = e \text{ and } y = x \\ 0 & \text{otherwise} \end{cases}$$

The, F.S single point (Px|e)E belongs to F.S set Γ_E denoting by (Px|e)E ∈ Γ_E, whenever 0 < λe ≤ Γ_e(x).

New Notation.[4]

• The F.S point (Px)E can be denoted by x_E where (Px)(e) = x(e) =x^{λ_e} , ∀e ∈ E, i.e., the image of each parameter under map Px is a fuzzy point. Consequently(x_E)^c Can be utilized to demonstrate the complement of F.S.

point x_E such that $[(Px)(e)]^c = 1 - \lambda_e\chi\{x\}$ for all e ∈ E.

• The precise F.S point, $(Px)_E$ will be designated by x_E^1 or x_E .

• The F.S single point (Px|e, E) demonstrated by x_e .

Definition (2.4)[4]. : Let E be the parameter set and R be the set about all real numbers. R_E = {Γ: E → I^R}, where Γ(e)'s are fuzzy real numbers for any e ∈ E, specifies the set of all fuzzy soft real numbers, or F.S real numbers.

Note that each r ∈ R can be viewed as a F.S real number r_E if for all e ∈ E, Γ(e) defined by characteristic function of r, i.e., r_E = χχr. So r : E → , I^R where for each e ∈ E, r(e) defined by

$$r_e(t) = \begin{cases} 1 \text{ if } t = r \\ 0 \text{ if } t \neq r \end{cases}$$

Such an F.S. real number is commonly, referred to as crisp F.S. real number.. The F.S real number Γ_E is called non-negative F.S number if for all e ∈ E, Γ_e(t) = 0 ∀ t < 0. The set of all non-negative F.S real numbers is denoted by $R^*{}_E$. 0_E and 1_E are the crisp F.S numbers 0 and 1, for each e ∈ E, $0_e(0) = 1$, $0_e(t) = 0$ ∀t ≠ 0 and $1_e(1) = 1$, 1_e (t) = 0 ∀t ≠ 1.

In this inquiry, the F.S. numbers will be published by r_E, where r ∈ R and r: $E \to I^R$such that for each e ∈ E, r(e) is a fuzzy number.

Definition (2.5)[4]:

$[r_E]$e,α The α-level set of F.S real numbers r_E correlates to the parameter e ∈ E and is defined with the following structure: $[r_E]_{e,\alpha}$ = {t : r_E (t) ≥ α}

It can be understood as the α-level set of fuzzy real numbers r(e), whence $[r_E]$e,α = [r(e)]α.

Definition (2.6)[4]. : Let r_E and r_E be two F.S real numbers. We say that

1. $r_E \preccurlyeq r_E \Leftrightarrow [r_E]_{e,\alpha} \subseteq [r_E]_{e,\alpha}$ for all α ∈ (0, 1] and e ∈ E.

2. $r_E = r_E \Leftrightarrow [r_E]_{e,\alpha} = [r_E]_{e,\alpha}$ for all α ∈ (0, 1] and e ∈ E.

So if $[\ r(e)\ ]_\alpha = [r_\alpha^1 , r_\alpha^2]$ and $[r(e)]_\alpha = [r_\alpha^1 , r_\alpha^2]$, then we have

1. $r_E \ r_E \Leftrightarrow r_\alpha^1 = r_\alpha^1$ and r_α^2 , r_α^2 for all α ∈ (0, 1] and e ∈ E.

2. $r_E = r_E \Leftrightarrow r_\alpha^1 \leq r_\alpha^1$ and $r_\alpha^2 \leq r_\alpha^2$ for all α ∈ (0, 1] and e ∈ E

Definition (2.7)[2]:

A linear ,space is a set X over a field F together with two operations + and • satisfying the following axioms

(i) An operation called vector addition that associates a sum $u+ v \in$ X with each pair of vector u v , ∈ X such that it is associative with identity 0.

(ii) An operation ,called multiplication by a scalar that associates with each scalar a ∈F and vector u∈ X vector au∈X, called the product of a and u , such that it is distributive with identity 1.

Definition (2.8)[9]:

Suppose that X be a linear space *over* f. *A soft set* G_A *is called ,a soft* subspace of F if

(i) for each $e \in A$, $G(e)$ is a subspace of X over f and

(ii) $F(e) \supseteq G(e)$, $\forall e \in A$.

Theorem (2.9)[9]:

A soft subset $\breve{G}$ of a soft linear space X is a soft linear sub-space of X if and only if for all scalar $\alpha,\beta \in f$,$\alpha\breve{G}+\beta\breve{G} \subseteq \breve{G}$.

Definition (2.10)[3]:

Suppose that X be a linear space over f. A fuzzy set $\breve{A}$ in X is said, to be a fuzzy subspace if $\alpha\breve{A}+\beta$B $\subseteq\breve{A}$ for all $\alpha,\beta \in f$ Or equivalent $\breve{A}(\alpha x+\beta y) \geq \min\{ \breve{A}(x),B(y)\}$ for all x ,y ∈ X and all x ∈ X.

Definition (2.11):

Let X be a linear space over a field F and A be a parameter set. A fuzzy soft set Γ_A over F is called a fuzzy soft linear space, of X over F if Γ(e) is a fuzzy subspace of X, ∀e ∈ A.

Definition (2.12):[5]

Let X linear space over F ,where F is the field of real number or the field of complex number

A norm on X is function $\|.\|: X \to R$ having the following properties

(1) $\|\breve{x}\| \geq 0$, for all $\breve{x} \in X$

(2) $\|\breve{x}\| = 0$ if and only if $\breve{x}$=0

(3) $\|\hat{\lambda}\breve{x}\| = |\hat{\lambda}|\|\breve{x}\|$,for all $\breve{x} \in X$ and $\hat{\lambda} \in f$

(4) $\|\breve{x}+\breve{y}\| \leq \|\breve{x}\|+\|\breve{y}\|$,for all $\breve{x},\breve{y} \in X$

The linear space X over ƒ together with $\|.\|$ is expressed by and declared to be a normed space by $(X, \|.\|)$.

Definition (2.13):[9]

Consider X a linear space over the field ƒ (real or complex), and T a continuous t-norm. A fuzzy norm on $X \times \ddot{R}$, $\ddot{R}$ a set of all real numbers, is considered "fuzzy" if and only if $\breve{x}, \breve{y} \in X$ and $\breve{c} \in f$.

(i) for all t $\in \ddot{R}$ with $t \leq 0$, $N(\breve{x}, t) > 0$.

(ii) for all t $\in \ddot{R}$ with $t > 0$, $N(\breve{x}, t) = 1$ if and only if $\breve{x} = 0$.

(iii) for all t $\in \ddot{R}$ with $t > 0$, $N(\breve{c}\breve{x}, t) = \left(\breve{x}, \frac{t}{|\breve{c}|}\right)$ if $\breve{c} \neq 0$

.)) for all s ,t $\in \ddot{R}$, $\breve{x}, \breve{y} \in X$ ($\breve{x}+\breve{y}$,t + s) $\geq$ N ($\breve{x}$,t) *N ($\breve{y}$,s iv (

(v) N ($\breve{x}$,.) is a continuou non- decreasing function of $\ddot{R}$ and

$\lim_{x\to\infty} N(x,t) = 1$

The triplet, (X,N, T) will be dubbed a fuzzy normed linear space.

Definition (2.14):

Let X be a vector space field F and, let E be the set of parameters .The fuzzy soft norm ‖.‖ over X is defined as the map $\breve{X}$ to R^*E Such that.

i) $\tilde{x}_E = \tilde{0}_E \leftrightarrow \| \tilde{x}_E \| = \tilde{0}_E$

ii) $\| \tilde{r}.\, \tilde{x}_E \| = |r| \otimes \| \tilde{x}_E \|$.

iii) $\| \tilde{x}_E + \tilde{y}_E \| = \| \tilde{x}_E \| \oplus \| \tilde{y}_E \|$ for all $\tilde{x}_E, \tilde{y}_E \in SV(\breve{X})$

Definition (2.15): Let X be an algebra over a field, F and A be a parameter set. A fuzzy soft set Γ_A is called a fuzzy soft algebra X over F if Γ(e) is a fuzzy subalgebra of X,∀e ∈ A.

Definition(2.16) [11]:

Let $\{\breve{x}_N\}$ be a sequence of soft, elements of a soft normed space (X ,‖. ‖) such that $\{\breve{x}_N\}$ is said to be a Cauchy sequence if for every $\in \geq 0$, there is $k \in N$ such that $\|\tilde{x}_i - \tilde{x}_j\| < \in$, for all $i, j \geq k$. That is $\|\tilde{x}_i - \tilde{x}_j\| \to 0$ as $i, j \to \infty$.

Definition(2.17)[9]:A normed space is called A complete normed space , if every Cauchy sequence in X is converge to a point of X .

3. fuzzy soft Banach algebra

Definition(3.1):[1]

A complete normed algebra is called a Banach algebra. i.e. A nonempty set X is called a Banach algebra over a field if

1. X is an algebra

2. $\|.\|$ is an algebra norm on X

3. X is complete metric space induced by its norm .

Definition(3.2)[10] :

A soft normed space (X, ‖.‖) is viewed as complete once every, Cauchy sequence $\{\check{x}_N\}$ converges to a soft element in X. A complete soft normed space is commonly referred to as a soft Banach space.

Definition(3.3):

Let Γ_x turns out to be fuzzy soft, on a linear space X. A fuzzy soft subset on $\Gamma_x \times R^*$ is called fuzzy soft norm on the fuzzy soft linear space X if N fulfills the following conditions:

for all $\mathfrak{a},\mathfrak{b} \in \Gamma_x$ and $\mathfrak{t}, \tilde{\mathfrak{t}} \in R^*(E)$

(i) N ($\mathfrak{a}$, $\mathfrak{t}$) = 0, for all $\mathfrak{t} \leq 0$.

(ii) N ($\mathfrak{a}$, $\mathfrak{t}$) = 1 iff $\mathfrak{a}$ = 0, for all $\mathfrak{t} > 0$.

(iii) $N(\beta\mathfrak{a}, \mathfrak{t}) = N\left(\mathfrak{a}, \frac{\mathfrak{t}}{|\beta|}|\right)$ if $\beta \neq 0$.

(iv) N ($\mathfrak{a}\mathfrak{b}$, $\mathfrak{t}\tilde{\mathfrak{t}}$) $\geq$ N ($\mathfrak{a}$, $\mathfrak{t}$) . N ($\mathfrak{b}$, $\tilde{\mathfrak{t}}$).

(v) N(a, .) is continuous non-decreasing function of Γ_x .

That is Γ_x (a,.): $R^*(E) \rightarrow [0, 1]$ and $\lim_{t\to\infty} \Gamma_x\ (a, t) = 1$.

Then (X, N, .) is a fuzzy soft normed, space and if X is complete said that it is fuzzy soft Banach space.

Definition(3.4)[10] :

The set G_A is called a soft Banach algebra if satisfies:

(i) G_A is a soft Banach space.

(ii) G_A is a soft algebra, that is G_A satisfies: For all $\mathfrak{a},\mathfrak{b},\mathfrak{c} \in G_A$

(a) $(\mathfrak{a}\mathfrak{b})\mathfrak{c} = \mathfrak{a}(\mathfrak{b}\mathfrak{c})$.

(b) $\mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c}$, $(\mathfrak{a} + \mathfrak{b})\,\mathfrak{c} = \mathfrak{a}\mathfrak{c} + \mathfrak{b}\mathfrak{c}$.

(c) $\beta(\mathfrak{a}\mathfrak{b}) = (\beta\mathfrak{a})\mathfrak{b} = (\mathfrak{a}(\beta\mathfrak{b}))$, $\beta \geq 0$.

(iii) A soft norm satisfies the inequality $\|\mathfrak{a}\mathfrak{b}\| \leq \|\mathfrak{a}\|\|\mathfrak{b}\|$ and $\mathfrak{e}\,\mathfrak{a} = \mathfrak{a}\mathfrak{e} = \mathfrak{a}$, $\mathfrak{e}$ is a soft element with $\|e\| = 1$.

Definition(3.5)[12] :

(X, B, ∗) is called fuzzy Banach algebra space if satisfies:

(i) X is an algebra.

(ii) (X, B, ∗) is fuzzy linear space.

(iii) $N(\mathfrak{a}\mathfrak{b}, \mathfrak{t}\tilde{\mathfrak{t}}) \geq N(\mathfrak{a}, \mathfrak{t}) * N(\mathfrak{b}, \tilde{\mathfrak{t}})$, for all $\mathfrak{a}, \mathfrak{b} \in X$ and $\mathfrak{t}, \tilde{\mathfrak{t}} \geq 0$. If (X, B, ∗) is fuzzy complete space, then (X, B, ∗) is fuzzy Banach algebra.

Definition (3.6):

Let X be an algebra over F, Γ_A on X is called Fuzzy Soft Banach Algbraa if satisfies that:

(i) Fuzzy soft Banach space.

(ii) Fuzzy soft algebra.

(iii)) $N(ab, t\tilde{t}) \geq \min\{N(a, t), N(b, \tilde{t})\}$ for all a,b∈ FS, for all $t, \tilde{t} \in R^*(E)$.

(iv) If e∈ Γ_x then N(e,t) = 1.

Proposition (3.7) (Γ,A) is a fuzzy soft Banach algebra iff Γ(λ) is a fuzzy Banach algebra ∀λ ∈A.

Proof: Proof follows from the definition of fuzzy soft algebra and Proposition(A fuzzy soft normed linear space (X,$\|.\|$) is fuzzy complete iff (X,$\|.\|_{(\lambda)}$)is complete $\forall\lambda \in A$,where $\|.\|_{(\lambda)}$ defined as in proposition (Decomposition theorem).

Proposition(3.8) Since x_n ,→ x and y_n ,→ y in Γ_A. then $x_n(\lambda) \to x(\lambda)$ and $y_n(\lambda) \to y(\lambda)$ in I^X.

Proposition(3.9) In a fuzzy soft Banach algebra if x_n ,→ x and y_n ,→ y then $x_n y_n$,→ xy. i.e. multiplication in a fuzzy soft Banach algebra is continuous.

Proof: Since x_n ,→ x and y_n ,→ y in Γ_A. So $x_n(\lambda) \to x(\lambda)$ and $y_n(\lambda) \to y(\lambda)$ ∀λ∈A in (Γ(λ),$\|.\|\lambda$). Now since Γ(λ) is Banach algebra ∀λ∈A (by Proposition(1)) and in Banach algebra multiplication is continuous so, $x_n(\lambda)y_n(\lambda) \to x(\lambda)y(\lambda)$ ∀λ∈A, which proves that $x_n y_n$,→ xy (by Proposition) . Let x_n, y_n be sequences fuzzy soft converging to the fuzzy soft element x_n in (X,$\|.\|$). Take ϵ > 0, then since x ,x, so there exists a soft natural number ^{N} such that $\|x_n y_n - xy\|(\lambda) < \epsilon(\lambda) = \epsilon$,∀ n ≥ N (λ), ∀ λ ∈ A. But $\|x_n y_n - xy\|(\lambda) = \|x_n(\lambda)y_n(\lambda) - x(\lambda)y(\lambda)\|_{\lambda}$,which shows that $x_n(\lambda)y_n(\lambda) \to x(\lambda)y(\lambda)$ ∀λ ∈ A.

Proposition (3.10): Every parametrized family of soft Banach algebras on a soft linear space X can be considered as a fuzzy soft Banach algebra on the fuzzy soft linear space X .

Proof. Let $\|.\|_\lambda$: λ ∈ A be a family of soft norms on the linear space X such that (X,$\|.\|_\lambda$) are Banach algebra ∀λ ∈ A. Now let us define $\|.\| : X \to R(A)^*$ by $\|x\|(\lambda) = \|x(\lambda)\|\lambda$, ∀λ ∈ A,∀x∈X. Then (X, $\|.\|$) is a fuzzy soft normed linear space. Now to show that (X, $\|.\|$) is a fuzzy soft Banach algebra we have to show that $\|xy\| \leq \|x\|\|y\|$ ∀ x, y ∈V and (X, $\|.\|$) is complete.

Now $\|xy\|(\lambda) = \|x(\lambda)y(\lambda)\|\lambda \leq \|x(\lambda)\|\lambda\|y(\lambda)\|\lambda \leq \|x\|(\lambda)\|y\|(\lambda)$ ∀λ ∈ A, which shows that $\|xy\| \leq \|x\|\,\|y\|$.

Now let x_n be a Cauchy sequence in X Then for any ϵ>0 there exists a soft natural number N such that $\|x_{n+p} - x_n\|(\lambda) < \frac{\epsilon}{2}(\lambda)\quad \forall n \geq N(\lambda), \forall\lambda \in A \Longrightarrow \|x_{n+p}(\lambda) - x_n(\lambda)\|(\lambda) < \frac{\epsilon}{2}(\lambda)\quad \forall n \geq N(\lambda), \forall\lambda \in A$ i.e. $x_n(\lambda)$ is a Cauchy sequence in (X,$\|.\|\lambda$) ∀λ ∈ A . Since (X,$\|.\|\lambda$) are Banach algebra ∀λ ∈ A, so there exist x_λ such that $x_n(\lambda)$ converge to x_λ, ∀λ ∈ A. Hence there must exist some N_λ(> N (λ)) such that $\|x_n(\lambda) - x_n\|(\lambda) < \frac{\epsilon}{2}(\lambda)\quad \forall n \geq N(\lambda), \forall\lambda \in A.$ Now $\|x_n - x\|(\lambda) = \|x_n(\lambda) - x_\lambda\| < \|x_n(\lambda) -$

$x_{N\lambda}(\lambda)\|\lambda + \|x_{n\lambda} - x_\lambda\|\lambda < \in (\lambda) \forall n > N(\lambda), \forall \in A$, where $x(\lambda)=x_\lambda$ This shows that (X,||.||) is afuzzy soft Banach space. Hence (X,||.||) is a fuzzy soft Banach algebra.

Definition(3.11): A fuzzy soft element x∈ Γ is said to be invertible if it has an inverse in Γ i.e. if there exists a soft element y ∈Γ such that xy= yx= e and then y is called the inverse of x, denoted by x^{-1}. Otherwise x is said to be non-invertible fuzzy soft element of Γ.

Remark (3.12): Clearly e is invertible. If x is invertible, then we can verify that the inverse is unique. Because if yx=e=xz Then y = ye = y(xz) = (yx)z= ez = z. Further, if x and y are both invertible then xy is invertible and $(xy)^{-1} = y^{-1}x^{-1}$. For $(xy)(y^{-1}x^{-1}) = x(yy^{-1})x^{-1} = xex^{-1} = e$ and similarly $(y^{-1}x^{-1})(xy)= e$.

Definition (3.13): Let (G,∗) be a group and (Γ,A) be a fuzzy soft set over G. Then (Γ,A) is said to be a fuzzy soft group over G if and only if Γ(λ) is a subgroup of (G,∗) for all λ ∈ A.

Proposition(3.14) Let (G,∗) be a group and (Γ,A) be a fuzzy soft set over G. If for any x, y ∈ (Γ,A).

1- x∗y∈ (Γ,A)

2- x^{-1}∈ (Γ,A),

where x∗y(λ)=x(λ)∗y(λ) and $x^{-1}(\lambda) = (x(\lambda))^{-1}$. Then (F,A) is a soft group over G.

Proof. Proof is obvious.

Definition (3.15) A series $\sum_{n=1}^{\infty} x_n$ of fuzzy soft elements is said to be fuzzy soft convergent if the partial sum of the series $s_k = \sum_{n=1}^{k} x_n$ is fuzzy soft convergent.

Proposition (3.16) Let Γ_A be a fuzzy soft Banach algebra. If x ∈Γ satisfies ||x||<1, then (e − x) is invertible and $(e - x)^{-1} = e + \sum_{n=1}^{\infty} xn$.

Proof. Since Γ_A is fuzzy soft algebra, so we have $\|x^j\| \le \|x\|^j$ for any positive integer j, so that the infinite series $\sum_{n=1}^{\infty}\|x\|^n$ is fuzzy soft convergent because ||x||<1. So the sequence of partial sum $s_k = \sum_{n=1}^{k} x_n$ is a fuzzy soft Cauchy sequence since $\left\|\sum_{n=1}^{k+p} x^n\right\| < \sum_{n=k}^{k=p}\|x\|^n$.

Since Γ_A is fuzzy soft complete so $\sum_{n=1}^{\infty} x^n$ is fuzzy soft convergent. Now let $s = e + \sum_{n=1}^{\infty} x^n$.

Now it is only we have to show that $s = (e - x)^{-1}$.

We have

(1) $(e - x)(e + x + x^2 + \dots x^n) = (e + x + x^2 + \dots x^n)(e - x) = e - x^{n+1}$

Now again since ||x||<1 so $x^{n+1} \to \theta$ as $n \to \infty$. Therefore letting $n \to \infty$ in and remembering that multiplication in Γ is continuous we get, (e−x)s=s(e−x)=e

So that $s = (e - x)^{-1}$. This proves the proposition.

Corollary (3.17) Let Γ be a fuzzy soft Banach algebra. If x ∈ Γ and ||e− x|| < 1, then x^{-1} exists and $x^{-1} = e + \sum_{j=1}^{\infty}(e - x)^j$

Corollary (3.18) Let Γ be a fuzzy soft Banach algebra. Let x∈ Γ and μ be a fuzzy soft scalar such that |μ|>||x||. Then $(\mu e - x)^{-1}$ exists and $(\mu e - x)^{-1} = \sum_{n=1}^{\infty} \mu^{-n} x^{n-1} (x^0 = e)$.

Proof. $y \in \Gamma$ be such that y^{-1}exists in Γ and α be afuzzy soft scalar such that $\alpha(\lambda) \neq 0$, $\forall\lambda \in A$. Then it is clear that $(\alpha y)^{-1} = \alpha^{-1}y^{-1}$.

Having noted this we can write $\mu e - x = \mu(e - \mu^{-1}x)$ and now we show that $(e - \mu^{-1}x)^{-1}$ exists. We have $\|e - (e - \mu^{-1}x)\| = \|\mu^{-1}x\| = |\mu|^{-1}\|x\| < 1$ by hypothesis. So, By Corollary 10 $(e - \mu^{-1}x)^{-1}$ exists and hence $(\mu e - x)^{-1}$ exists. For the infinite series representation, using the Proposition 9 we have

$(\mu e - x)^{-1} = \mu^{-1}(e - \mu^{-1}x)^{-1}$

$= \mu^{-1}(e + \sum_{n=1}^{\infty} [e - (e - \mu^{-1}x)]^n$

$= \mu^{-1}(e + \sum_{n=1}^{\infty}(\mu^{-1}x)^n) = \sum_{n=1}^{\infty} \mu^{-n}x^{n-1}$.

This proves the corollary.

Proposition(3.19) Let Γ be a fuzzy soft Banach algebra. The fuzzy soft set Γ_A generated by the set of all invertible fuzzy soft elements of Γ is a fuzzy soft open subset in Γ.

Proof. $x_0 \in \Gamma_A$. We have to show that x_0 is a fuzzy soft interior point of Γ. Consider the open sphere $\Gamma_A(x_0, \frac{1}{\|x_0^{-1}\|}$with centre at x_0 and radius $\frac{1}{\|x_0^{-1}\|}$. Every soft element x of this sphere satisfies the inequality

$\|x_0 - x\| < \frac{1}{\|x_0^{-1}\|}$ (2)

Let $y = x_0^{-1}x$ and $z = e - y$ then we have $\|z\| = \|y - e\| = \|x_0^{-1}x - x_0^{-1}x_0\| \leq \|x_0^{-1}\|\|x - x_0\| < 1$. So by Proposition 9, $e - z$ is invertible i.e. y is invertible.

Hence $y \in \Gamma_A$. Now $x_0 \in$, $\Gamma_A y \in \Gamma_A$nd so by Remark 5, $x_0 y \in \Gamma_A$. But $x_0 y = x_0 x_0^{-1}x = x$ so any x satisfying the inequality (1) belongs toΓ_A . This shows that Γ_A is a fuzzy soft open subset of Γ.

Corollary (3.20) : The fuzzy soft set $P(= S^c)$ of Γ is fuzzy soft closed subset of Γ.

Definition (3.21): A mapping T from a fuzzy soft normed linear space Γ onto Γ is said to be continuous if for any sequence x_n, $x_n \to x$ implies $T(x_n) \to T(x)$.

Proposition (3.22): In a fuzzy soft Banach algebra Γ, the mapping $x \to x^{-1}$ of S onto S is continuous.

Proof. Let $x_0 \in S$ and let $\{x_n\}$ be a sequence of fuzzy soft elements in S such that $x_n \to x_0$ as $n \to \infty$. To prove $x \to x^{-1}$ is continuous, it is enough to show that $x_n^{-1} \to x_0^{-1}$. Now

$\|x_n^{-1} - x_0^{-1}\| = \|x_n^{-1}(x_0 - x_n)x_0^{-1}\|$

(3) $\leq \|x_n^{-1}\|\|x_0 - x_n\|\|x_0^{-1}\|$.

Since $x_n \to x_0$, for any given $\epsilon > 0$, there exists N such that for all $n \geq N(\lambda)$,

(4) $\|x_n - x_0\|(\lambda) < \frac{1}{2\|x_0^{-1}\|}(\lambda)$ we where have taken $\epsilon = \frac{1}{2\|x_0^{-1}\|}$

Now

(5) $\|e - x_0^{-1}x_n\| = \|x_0^{-1}(x_0 - x_n)\| \leq \|x_0^{-1}\|\|(x - x_n)\|$

Using (4) and (5) we get

(6) $\|e - x_0^{-1}x_n\|(\lambda)\frac{1}{2}(\lambda) = \frac{1}{2}\forall n \geq N(\lambda)$

So by Corollary $x_0^{-1}x_n$ is invertible and its inverse is given by $x_n^{-1}x_0 = (x_0^{-1}x_n)^{-1} = e +$

$\sum_{n=1}^{\infty}(e - x_0^{-1}x_n)^n$. Thus $\|x_n^{-1}x_0\| \leq 1 + \sum_{n=1}^{\infty}\|e - x_0^{-1}x_n\|^n \leq \frac{1}{1-\|e-x_0^{-1}x_n\|} \leq 2$ by (6). This gives

$\|x_n^{-1}x_0\| < 2$ so that we have

(7) $\|x_n^{-1}\| = \|x_n^{-1}x_0\ x_0^{-1}\| \leq \|x_n^{-1}x_0\|\|x_0^{-1}\| \leq 2\|x_0^{-1}\|$

From (3) and (7) we get $\|x_n^{-1} - x_0^{-1}\|(\lambda) \leq 2\|x_0^{-1}\|(\lambda)\|x_0 - x_n\|(\lambda)\|x_0^{-1}\|(\lambda) \to 0$ as $n \to \infty$.

This proves that $x_n^{-1} \to x_0^{-1}$ as $n \to \infty$. So the mapping $x \to x^{-1}$ of S onto S is continuous.

Corollary (3.23): In a fuzzy soft Banach algebra Γ, the mapping $x^{-1} \to x$ of S onto S is continuous.

Definition (3.24) : Let Γ be a fuzzy soft Banach algebra. Afuzzy soft element z ∈ Γ is called a fuzzy soft topological divisor of zero if there exists a sequence $\{z_n\}$, $z_n \in \Gamma$, $\|z_n\| = 1$ for n = 1,2,3..... and such that either $zz_n \to \Theta$ or $z_n z \to \Theta$.

Proposition(3.25): The fuzzy soft set Γ is a fuzzy soft subset of P, where Γ denotes the set of all fuzzy soft topological divisors of zero.

Proof. Let z ∈ Γ. The there exists a sequence $\{z_n\}$ such that $\|z_n\| = 1$ for n = 1,2,3..... and either $zz_n \to \Theta$ or $z_n z \to \Theta$ as $n \to \infty$. Suppose that $zz_n \to \Theta$.

If possible, let z ∉ P. Then $z(\lambda)^{-1}$ exists for some λ. Now as multiplication is continuous operation, we should have

$z_n(\lambda) = z(\lambda)^{-1}(zz_n)(\lambda) \to z(\lambda)^{-1}\Theta(\lambda) = \theta$ as $n \to \infty$.

This contradicts the fact that $\|z_n\| = 1$ for n = 1,2,3..... . Hence Γ is a fuzzy soft subset of P.

Definition (3.26): Let (X, ||.||) be a fuzzy soft normed linear space and Y ∈ Γ (X). A fuzzy soft element α ∈ X is called a fuzzy soft boundary elements of Y if there exist two sequence x_n and y_n of fuzzy soft elements in Y and Y^c respectively such that $x_n \to \alpha$ and $y_n \to \alpha$.

Proposition (3.27): The boundary of P is a fuzzy soft subset of Γ.

Proof. Let z be a boundary point of P. So there exist two sequences of fuzzy soft elements r_n in S and s_n in P such that

(8) $r_n \to z$ and $s_n \to z$.

Since P is fuzzy soft closed so z ∈ P. Now let us write $r_n^{-1}z - e = r_n^{-1}(z - r_n)$. The sequence $\{r_n^{-1}(\lambda)\}$ given above is unbounded ∀λ ∈ A. If not, then there exists some λ ∈ A and n(λ) such that $\|r_n^{-1}z - e\|(\lambda) < 1$ ∀n ≥n(λ), ∀λ ∈A. So that by Corollary 10, $r_n^{-1}z(\lambda)$ is regular and hence $z(\lambda) = r_n(\lambda)(r_n^{-1}z)(\lambda)$ is regular, contradicting z ∈ P. Hence $\{r_n^{-1}(\lambda)\}$ is unbounded ∀λ∈ A so that

(9) $\|r_n^{-1}\| \to \infty$ as $n \to \infty$.

Now let us define $z_n = \frac{r_n^{-1}}{\|r_n^{-1}\|}$ .from the definition of z_n we have

(10) $\|z_n\| = 1.$

Further

(11) $$zz_n = \frac{zr_n{}^{-1}}{\|r_n{}^{-1}\|} = \frac{e+zr_n{}^{-1}-e}{\|r_n{}^{-1}\|} = \frac{e+(z-r_n)r_n{}^{-1}}{\|r_n{}^{-1}\|}$$

But

(12) $$\frac{e+(z-r_n)r_n{}^{-1}}{\|r_n{}^{-1}\|} = \frac{e}{\|r_n{}^{-1}\|} + (z-r_n)z_n.$$

From (11) and (12), we get

(13) $$zz_n = \frac{e}{\|r_n{}^{-1}\|} + (z-r_n)z_n$$

Using (8), (9) and (10) in (13) we see that zz_n ,$\rightarrow \Theta$ as $n \rightarrow \infty$. Hence z is a topological divisor of zero.

4. Conclusion

In fuzzy soft Banach space ,we needed define soft Banach algebra[10] and fuzzy Banach agebra [12],in our paper,we a new concepts called fuzzy soft algebra , fuzzy soft Banach algebra.

References

[1] F. F.Bonsall, J. Duncan, Complete Normed Algebras,1973.

[2] Kamel, A. and Mirzavaziri, M., Closability of farthest point in fuzzy normed spaces, Bulletin of Mathematical Analysis and Applications, Vol.2,140-145, (2010).

[3] A. K. Katsaras and D. B. Liu, fuzzy vector and fuzzy topological vector space, J. Math. Anal. Appl.58(1977), 135-146.

[4] A. Z. Khameneh, A. Kilicman, A. R. Salleh, Fuzzy Soft normed space and Fuzzy Soft linear operator, arXiv:1308.4254v2 [math,FA] 21 Aug 2013.

[5] J. R. Kider, On fuzzy normed spaces,EnTech. Journal,Vol.29, no.9, (2011),1790-1795.

[6] P. K. Maji and A. R. Roy, An Application of Soft sets in a decision making problem, Computers and Mathematics with Applications, 44 (2002) 1077-1083.

[7] Maji, K., Biswas, R., and Roy, A., Fuzzy Soft Set, Journal of Fuzzy Mathematics 9,589-602, (3) (2001).

[8] D. Molodtsov, Soft set theory-first results, Computers and Mathematics with Applications, 37 (1999) 19-31.

[9].Sujoy,D.,Pinaki,M.,Syamal,On soft Linear space and soft Normed Linear space ,Ann.fuzzy Math .Inform.9,pp.91-109,(2015).

[10] Thakur, R. and Samanta, S., Soft Banach Algebra, Ann. Fuzzy Mathematics and inf.,10,pp.397-412,(2015).

[11] Tudor, B., Ftaavius, P., Sorin N., On fuzzy normed algebra, J. Nonlinear Sci. Appl. 9(2016).

[12] Walter, R., Functional Analysis, 2^{nd} University Wisconsin (1991).

[13]A. Zahedi Khameneh, A. Kilicman and A. Razak Salleh, Parameterized norm and parameterized fixed-point theorem by using fuzzy soft set theory, arXiv preprint arXiv:1309.4921, 2013.