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On Convergences in Fuzzy Soft Banach Algebra

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ABSTRACT

In this paper, we introduce the notions of s-convergence and st-convergence in fuzzy soft Banach algebra defined by Bag and Samanta. With these definitions, we investigate relations of convergence, s-convergence and st-convergence in fuzzy soft Banach algebra. We also represent convergence, s-convergence and st-convergence in fuzzy soft Banach algebra in terms of convergences with respect to their corresponding α-norms, We finally introduce the definition of s-convergence and st-convergence in fuzzy soft Banach algebra introduced by Felbin and investigate relations of convergence, s-convergence and st-convergence in fuzzy soft Banach algebra introduced by Felbin.

1. Introduction

Because of uncertainty in the real world, various problems exist in mathematics and engineering, environmental sciences and medical sciences cannot solve those problems and also there are no specific mathematical tools to describe the process arising in the areas of ambiguity and uncertainty. To deal with such problems, Zadeh [12] introduced the concept of soft set. Soft set theory is an innovative way to solve uncertainty problems, and soft set theory is a more flexible tool than usual mathematical methods through the ambiguities and uncertainties related to the real world. In 1999, Molodtsov [13] introduced the notion of soft set for the first time as a new approach to describe objects with regards to some decision parameters. In fact a soft set is a set-valued map which shows some information about objects. After presentation this notion Then the relationship between the fuzzy norm and the soft norm appeared and it became the fuzzy soft norm. In this research, through the fuzzy soft set and then the fuzzy soft standard, we took the definitions of the norm spaces based on the fuzzy soft convergence of its various types, we take definitions of fuzzy Soft Norm spaces introduced by George and Veeramani [4].

2- Preliminaries

Definition(2.1)[10] : A pair (Γ , A) denote by Γ_{A} is called a fuzzy soft set over X, where Γ is a function given by
$\Gamma: \mathrm{A} \to \mathrm{I}^{\mathrm{X}}$.

Definition (2.2)[10]: A Fuzzy soft set (Γ, A) over X is said to be an absolute fuzzy soft set, if for all $e \in E$, $F(e)$ is a fuzzy universal set $\tilde{1}$ over X and is denoted by $\tilde{\breve{E}}$.

Definition (2.3)[10]:A fuzzy soft set (Γ, E) over X is said to be a null fuzzy soft set, if for all $e \in E$, $F(e)$ is the null fuzzy set $\tilde{0}$ over X. It is denoted by $\tilde{\breve{\emptyset}}$.

Definition (2.4)[9]: For two fuzzy soft sets (Γ, A) and (Λ, B) in $\Gamma(X, E)$ we say that $(\Gamma, \mathrm{A}) \breve{\subseteq} (\Lambda , \mathrm{B})$if $\mathrm{A} \breve{\subseteq} \mathrm{B}$ and $\Gamma(\mathrm{e})(\mathrm{x}) \breve{\leq} \Lambda(\mathrm{e})(\mathrm{x})$.

Definition (2.5)[10]: Two fuzzy soft sets $(\Gamma, \mathrm{A})$and(Λ , B) in $\Gamma(\mathrm{X} , \mathrm{E})$ are equal if $\Gamma \breve{\subseteq} \Lambda$ and $\Lambda \breve{\subseteq} \Gamma$.

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Definition (2.6)[9]: The different between two fuzzy soft sets (Γ,A) and (Λ,B) in $\Gamma(X,E)$ is a fuzzy soft set (Γ/Λ, E) (say) defined by (Γ/Λ) (e)=Γ(e)/Λ(e) for each $e \in E$.$(\Gamma/\Lambda)(e): X \to I$, $(\Gamma/\Lambda)(e)(x) = \Gamma(e)(x) \,\tilde{\cap}\, \left(\Lambda(e)(x)\right)^{c} = min\{\Gamma(e)(x), \Lambda^{c}(e)(x)\}\ \forall x \in X$.

Definition (2.7)[10] : The complement of a fuzzy soft set (Γ,E) is a fuzzy soft set (Γ^{c}, E) defined by $\Gamma^{c}(e)=1/\Gamma(e)$ for each $e \in E$, $\left(\Gamma^{c}(e)\right)(x) = 1 - \Gamma(e)(x)\ \forall x \in X$.

Definition (2.8)[9]: Let (Γ, A) and (Λ, B) be two fuzzy soft sets in Γ(X,E) with $A \,\tilde{\cap}\, B \neq \Phi$, then:

a) their intersection $(\Gamma \,\tilde{\cap}\, \Lambda, C)$ is a fuzzy soft set, where $C = A \,\tilde{\cap}\, B$ and $(\Gamma \,\tilde{\cap}\, \Lambda)(e) = \Gamma(e) \,\tilde{\cap}\, \Lambda(e)$ for each $e \in C$,

$(\Gamma \,\tilde{\cap}\, \Lambda)(e)(x) = \min\{(e, \Gamma_{e}(x)), (e, \Lambda_{e}(x))$

b) their union $(\Gamma \,\tilde{\cup}\, \Lambda, C)$ is a fuzzy soft set, where

C) $A \,\tilde{\cup}\, B$ and $(\Gamma \,\tilde{\cup}\, \Lambda)e = \Gamma(e) \,\tilde{\cup}\, \Lambda(e)$for each $e \in C$

$(\Gamma \,\tilde{\cup}\, \Lambda)(e)(x) = \max\{\Gamma(e)(x), \Lambda(e)(x)\}$

Definition (2.9)[10]: A fuzzy soft set Γ_{A} over X is called a fuzzy soft point if for the element $e^{*} \in E$

$$f_{e}(x) = \begin{cases} \lambda_{x} & if\ e = e^{*} \\ 0 & otherwise \end{cases} \quad \text{for every } e \in E$$

Otherwise, for the element $x^{*} \in X$,

$$f_{e}(x) = \begin{cases} \lambda & if\ x = x^{*} \\ 0 & otherwise \end{cases} \quad \text{for every } x \in X \text{ ,where } \lambda \in (0,1].$$

The set of all fuzzy soft points is denoted as $\tilde{x}_{E}$.

Definition 2.10. Let X by a vector space over a field $K(K = R)$ and the parameter set E be the real number set R . Let (F,E) be a soft set over X . The soft set (F,E) is said to be a soft vector and denoted by $\tilde{x}_{e}$ if there is exactly one $e \in E$, such that $F(e) = \{x\}$ for some $x \in X$ and $F(e') = \varphi, \forall e' \in E/\{e\}$.

The set of all soft vectors over $\tilde{X}$ will be denoted by SV ($\tilde{X}$).

The set $SV(\tilde{X})$ is called a soft vector space.

Definition 2.11[11]: Let $SV(\tilde{X})$ be a soft vector space.

The mapping $\|.\|: SV(\tilde{X}) \to R^{+}(E)$

is called a soft norm ($\tilde{X}$), if it satisfies the following conditions

(i) $\|\tilde{x}_{e}\| \geq \tilde{0}$ for all $\tilde{x}_{e} \in SV(x)$ and $\|\tilde{x}_{e}\| = \tilde{0} \leftrightarrow \tilde{x}_{e} = \tilde{\theta}_{0}$.

ii) $\|\tilde{r}.\tilde{x}_{e}\| = |\tilde{r}|\,\|\tilde{x}_{e}\|$ for all $\tilde{x}_{e} \in SV(\tilde{X})$ for every soft scalar $\tilde{r}$.

iii) $\|\tilde{x}_{e} + \tilde{y}_{e}\| \leq \|\tilde{x}_{e}\| + \|\tilde{y}_{e}\|$ for all $\tilde{x}_{e}, \tilde{y}_{e} \in SV(\tilde{X})$.

The soft vector space $\tilde{x}_{e} \in SV(\tilde{X})$ with a soft norm $\|.\|$ on $\tilde{X}$ is said to be a soft Banach algebra and is denoted by $(\tilde{X}, \|.\|)$.

Definition 2.12. Let X be a linear space over a field $F(F = R\ or\ C)$. Let N be a fuzzy subset of $X \times R$. Then N is called a fuzzy norm on X if for all

$x, y \in X$ and $c \in F$,

$(N1)$ for all $t \leq 0, N(x.t) = 0$,

$(N2)$ for all $t > 0, N(x.t) = 1$ if and only

$(N3)$ for all $t > 0$ and $k \neq 0$, $N(kx, t) = N(x, \frac{t}{|k|})$

$(N4)$ for all $t, s \in R, N(x + y, s + t) \geq \min\{N(x,s), N(y,t)\}$,

$(N5)$ $N(x,0)$ is a non-decreasing function of R such that $\lim_{n \to \infty} N(x,t) = 1$

The pair (X, N) will be referred to as a fuzzy Banach algebra.
$(N6)$ If $N(x, 0) > 0$ for all $t > 0$, then $x = 0$.
For a fuzzy normed linear space (X, N) with $(N6)$, define a function $\|.\|_\alpha$ on X by

$$\|x\|_\alpha = \wedge\{t : N(x, t) \geq \alpha\}$$

for $\alpha \in (0,1)$. Then $\|.\|_\alpha$ is a norm on X [1]. We call these norms as α-norms on X corresponding to the fuzzy norm N on X.
$(N7)$ For $\alpha \in (0,1)$. Then $\|.\|_\alpha$ is a continuous function of R and strictly increasing on $\{t: 0 < N(x,t) < 1\}$. $t \in R$.
Definition 2.13. Let $\tilde{x}$ be an absolute soft linear space over the scalar field K. Suppose $*$ is a continuous t-norm, $R(A^*)$ is the set of all non-negative soft real numbers and $SSP(\tilde{X})$ denote the se t of all soft points on $\widetilde{X}$. A fuzzy subset Γ on $SSP(\tilde{X}) \times R(A^*)$ is called a fuzzy soft norm $\widetilde{X}$ on if and only if for $\tilde{x}_e, \tilde{y}_{\acute{e}} \in SSP(\widetilde{X})$ and $\tilde{k} \in \widetilde{K}$ (where $\tilde{k}$ is a soft scalar) the following conditions hold
1) $\Gamma(\tilde{x}_e, \tilde{t}) = 0, \forall \tilde{t} \in R(A^*)$ with $\tilde{t} \leq \tilde{0}$.
2) $\Gamma(\tilde{x}_e, \tilde{t}) = 1, \forall \tilde{t} \in R(A^*)$ with $\tilde{t} \leq \tilde{0}$ if and only if $\tilde{x}_e, \tilde{\theta}_0$.
3) $\Gamma(k\tilde{x}_e, \tilde{t}) = \Gamma(\tilde{x}_e, \frac{\tilde{t}}{|\tilde{k}|})$ if $\tilde{k} \neq \tilde{0}$ $\forall \tilde{t} \in R(A^*), \tilde{t} > \tilde{0}$.
4) $\Gamma(\tilde{x}_e \oplus \tilde{y}_e, \tilde{t} \oplus \tilde{s}) > \Gamma(\tilde{x}_e, \tilde{t}) * \Gamma(\tilde{y}_e, \tilde{s}), \forall \tilde{t}, \tilde{s} \in R(A^*), \tilde{x}_e, \tilde{y}_e \in SSP(\tilde{X})$.
5) $\Gamma(\tilde{x}_e, .)$ is a continuous non decreasing function of $R(A^*)$ and $\lim_{\tilde{x} \to \infty} \Gamma(\tilde{x}_e, \tilde{t}) = 1$.
The triplet $(\widetilde{X}, \Gamma, *)$ will be referred to as a fuzzy soft Banach algebra.

3.Relations of Convergences in Fuzzy Soft Banach algebra Introduced by Bag and Samanta
In Definition 2.12, it is stated that $N(x, .)$ is non-decreasing on R. But the following inequality from (N4) shows that $N(x, .)$ is non-decreasing:
$N(x, s) > min\{N(0, s-t), N(x,t)\} = min\{1, N(x,t)\} = N(x,t)$, for $s > t$.
We now introduce the following definitions:
Definition 3.1. Let $\{x_n\}$ be a sequence in a fuzzy soft Banach algebra $(\widetilde{X}, \Gamma, *)$ and $x \in X$. Then
(1) $\{x_n\}$ is said to be convergent to x if, for all $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_{\varepsilon,t} \in \cdot$, depending on ε and t, such that $N(x_n - x, t) > 1 - \varepsilon$, for $n \geq n_{\varepsilon,t}$.
(2) $\{x_n\}$ is said to be s-convergent to x if, for all $\varepsilon \in (0,1)$, there exists $n_\varepsilon \in \cdot$, such that $N\left(\left|x_n - x, \frac{1}{n}\right|\right) > 1 - \varepsilon$, for $n \geq n_\varepsilon$.
(3) $\{x_n\}$ is said to be st-convergent to x if, for all $\varepsilon \in (0,1)$, there exists $n_\varepsilon \in \cdot$, such that $N(|x_n - x, t|) > 1 - \varepsilon$, for $n \geq n_\varepsilon$, and for all $t > 0$.
We now investigate relations of convergence, s-convergence and st-convergence in a fuzzy soft Banach algebra introduced by Bag and Samanta.
Proposition 3.2. Let $(\widetilde{X}, \Gamma, *)$ be a fuzzy sof Banach algebra t. Then
(1) st-convergence in $(\widetilde{X}, \Gamma, *)$ implies s-convergence.
(2) s-convergence in $(\widetilde{X}, \Gamma, *)$ implies convergence.
Proof. It is easy to prove (1).

(2) Suppose that $\{x_n\}$ is s-convergent sequence to x in $(\widetilde{X}, \Gamma, *)$. Let $\varepsilon \in (0,1)$ and $t > 0$. Then there exists $n_\varepsilon \in \cdot$ such that if $n \geq n_\varepsilon$, $N\left(\left|x_n - x, \frac{1}{n}\right|\right) > 1 - \varepsilon$, Choose $n_t \in \cdot$ such that $\frac{1}{n_t} < t$. Let $n_1 = \max\{n_t, n_\varepsilon\}$. Then if $n \geq n_1, N(|x_n - x, t|) > N\left(\left|x_n - x, \frac{1}{n}\right|\right) > 1 - \varepsilon$, since $n \geq n_1 \geq n_\varepsilon$.

The converses of Proposition 3.2 do not hold. The following example is the norm form of [6, Example 4.4].

Example 3.3. Let $X = R$ and we define a fuzzy soft norm $N(x,t)$ on X by

$$N(x,t) = \begin{cases} \dfrac{t}{t + |x|} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

It is easy to see that $(\widetilde{X}, \Gamma, *)$ is a fuzzy soft Banach algebra.

Let $x_n = \frac{1}{n}$. Then

$$N\left(\left|x_n, \frac{1}{n}\right|\right) = \frac{\frac{1}{n}}{\frac{1}{n} + \frac{1}{n}} = \frac{1}{2}.$$

This implies that $\{x_n\}$ is not s-convergent to 0. We now show that $\{x_n\}$ is a convergent sequence in $(\widetilde{X}, \Gamma, *)$. Let $\varepsilon \in (0,1)$ and $t > 0$. Then there exists $n_{\varepsilon,t} \in \cdot$ such that $n_{\varepsilon,t} > \frac{1-\varepsilon}{t\varepsilon}$. If $n \geq n_{\varepsilon,t}$, $N(x,t) = \frac{t}{t+\frac{1}{n}} \geq \frac{t}{t+\frac{1}{n_{\varepsilon,t}}} > 1 - \varepsilon$.

This means that $\{x_n\}$ is convergent to 0. Therefore, $\{x_n\}$ is a convergent and non-s-convergent sequence in a fuzzy soft Banach algebra.

The converse of (2) in Proposition 3.2 does not hold.

Let $y_n = \frac{1}{n^2}$. Then

$$N\left(\left|y_n, \frac{1}{n}\right|\right) = \frac{\frac{1}{n}}{\frac{1}{n} + \frac{1}{n^2}} = \frac{1}{1 + \frac{1}{n}}.$$

This implies that $\{y_n\}$ is s-convergent to 0. We now show that $\{y_n\}$ is not st-convergent. Suppose that $\{y_n\}$ is st-convergent. Let $\varepsilon \in (0,1)$. Then there exists $n_\varepsilon \in \cdot$ such that if $n \geq n_\varepsilon$, $N\left(\left|y_n, \frac{1}{n}\right|\right) = \frac{t}{t+\frac{1}{n^2}} > 1 - \varepsilon$ for all $t > 0$.

This implies that for all $t > 0, t > \frac{1-\varepsilon}{\varepsilon n_\varepsilon^2}$. We get the contradiction.

Therefore, $\{y_n\}$ is the s-convergent sequence which is not st-convergent.

The converses of (1) in Proposition 3.2 do not hold.

By Proposition 3.2 and Example 3.3, we get the following strict implications in the fuzzy soft normed space introduced by Bag and Samanta:

st -convergence $\Rightarrow$ s -convergence $\Rightarrow$ convergence.

4. Convergences in Fuzzy Soft Banach algebra and Convergences with Respect to their Corresponding α-norms

In [1], the authors showed that in a fuzzy soft Banach algebra satisfying (N6) and (N7), every sequence is convergent if and only if it is convergent with respect to its corresponding α -norms ($0 < \alpha < 1$). We now show that in a fuzzy soft normed space satisfying (N6), every sequence is convergent if and only if it is convergent with respect to its corresponding α -norms ($0 < \alpha < 1$).

Proposition 4.1. Let $\{x_n\}$ be a sequence in a fuzzy soft Banach algebra $(\tilde{X}, \Gamma, *)$ satisfying (N6) and let $x \in X$. $\{x_n\}$ converges to x if and only if for all $\alpha \in (0,1)$ and $\varepsilon \in (0,1)$ there exists $n_{\alpha,\varepsilon} \in \cdot$ such that $n \geq n_{\alpha,\varepsilon}$, $\|x_n - x\|_\alpha < \varepsilon$.

Proof. Suppose that $x \in X$. $\{x_n\}$ converges to x. Let $\alpha \in (0,1)$. Then there exists $\delta \in (0,1)$ such that $\alpha = 1 - \delta$. Since $\{x_n\}$ converges to x, for $\varepsilon > 0$, there exists $n_{\alpha,\varepsilon} \in \cdot$, such that if $n \geq n_{\alpha,\varepsilon}$, $N\left(\left|x_n - x, \frac{\varepsilon}{2}\right|\right) > 1 - \delta = \alpha$. This implies that $\|x_n - x\|_\alpha \leq \frac{\varepsilon}{2} < \varepsilon$ for $n \geq n_{\alpha,\varepsilon}$.

Conversely, let $\alpha \in (0,1)$ and $\varepsilon > 0$. Then there exists $n_{\alpha,\varepsilon} \in \cdot$, such that $n \geq n_{\alpha,\varepsilon}$, $\|x_n - x\|_\alpha < \varepsilon$. This implies that $N(x_n - x, \varepsilon) \geq \alpha$ for $n \geq n_{\alpha,\varepsilon}$, since $N(x,\cdot)$ is non-decreasing. Since $\alpha \in (0,1)$ is arbitrary, x_n converges to x $(\tilde{X}, \Gamma, *)$.

We now consider the representation of st-convergence in fuzzy soft Banach algebra in terms of convergence with respect to its corresponding α -norms.

Theorem 4.2. Let $\{x_n\}$ be a sequence in a fuzzy soft Banach algebra $(\tilde{X}, \Gamma, *)$ satisfying (N6) and $x \in X$. $\{x_n\}$ st-converges to x if and only if for all $\alpha \in (0,1)$ there exists $n_\alpha \in \cdot$, such that $\varepsilon > 0$, $\|x_n - x\|_\alpha < \varepsilon$ for $n \geq n_\alpha$, i.e., $x_n = x$ for $n \geq n_\alpha$.

Proof. Suppose that $\{x_n\}$ st-converges to x. Let $\alpha \in (0,1)$ Then there exists $\delta \in (0,1)$ such that $\alpha = 1 - \delta$. Since $\{x_n\}$ st-converges to x, there exists $n_\delta \in \cdot$, such that if $n \geq n_\delta$, $N(x_n - x, t) > 1 - \delta = \alpha$ for all $t > 0$. Let $\varepsilon > 0$. Then $N\left(\left|x_n - x, \frac{\varepsilon}{2}\right|\right) > \alpha$ for $n \geq n_\delta$. This implies that $\|x_n - x\|_\alpha \leq \frac{\varepsilon}{2} < \varepsilon$ for $n \geq n_\delta$.

The converse is clear.

We now investigate the representation of s-convergence in fuzzy soft Banach algebra in terms of convergence with respect to its corresponding α -norms.

Theorem 4.3. Let $\{x_n\}$ be a sequence in a fuzzy sof Banach algebra t $(\tilde{X}, \Gamma, *)$ satisfying (N6) and let $x \in X$. $\{x_n\}$ s-converges to x if and only if for all $\alpha \in (0,1)$ there exists $n_\alpha \in \cdot$, such that if $n \geq n_\alpha$, $\|x_n - x\|_\alpha \leq \frac{1}{n}$.

Proof. Suppose that $x \in X$. $\{x_n\}$ s-converges to x. Let $\alpha \in (0,1)$. Then there exists $n_\alpha \in \cdot$ such that $n \geq n_\alpha$, $N\left(\left|x_n - x, \frac{1}{n}\right|\right) > \alpha$. This implies that $\|x_n - x\|_\alpha < \frac{1}{n}$ for $n \geq n_{\alpha,\varepsilon}$.

Conversely, let $0 < \varepsilon < 1$. Then $0 < \alpha = 1 - \varepsilon < 1$ and so there exists $n_\varepsilon \in \cdot$ such that if $n \geq n_\varepsilon$, $\|x_n - x\|_\alpha \leq \frac{1}{n}$. This implies that $N\left(\left|x_n - x, \frac{1}{n}\right|\right) > \alpha = 1 - \varepsilon$, for $n \geq n_\varepsilon$. This means that $\{x_n\}$ s-converges to x.

5.Relations of Convergences in Fuzzy Soft Banach algebra Introduced by Felbin[3]

We now introduce the following definitions:

Definition 5.1. Let $\{x_n\}$ be a sequence in a fuzzy soft Banach algebra $(\tilde{X}, \Gamma, *)$ introduced by Felbin and $x \in X$. Then

(1) $\{x_n\}$ is said to be convergent to x if, for all $\varepsilon > 0$ and $\alpha \in (0,1]$, there exists $n_{\alpha,\varepsilon} \in \cdot$, depending on ε and α, such that $\|x_n - x\|_\alpha^2 < \varepsilon$, for $n \geq n_{\alpha,\varepsilon}$.
(2) $\{x_n\}$ is said to be s-convergent to x if, for all $\varepsilon > 0$, there exists $n_\varepsilon \in \cdot$, such that $\|x_n - x\|_{\frac{1}{n}}^2 < \varepsilon$, for $n \geq n_\varepsilon$.
(3) $\{x_n\}$ is said to be st-convergent to x if, for all $\varepsilon > 0$, there exists $n_\varepsilon \in \cdot$, such that $\|x_n - x\|_\alpha^2 < \varepsilon$, for $n \geq n_\varepsilon$, and for all $\alpha \in (0,1]$.
We finally investigate relations of convergence, s-convergence and st-convergence in a fuzzy soft Banach algebra introduced by Felbin.

Proposition 5.2. Let $(\widetilde{X}, \Gamma, *)$ be a fuzzy soft Banach algebra introduced by Felbin. Then
(1) st-convergence in $(\widetilde{X}, \Gamma, *)$ implies s-convergence.
(2) s-convergence in $(\widetilde{X}, \Gamma, *)$ implies convergence.

Proof. It is easy to prove (1).
(2) Suppose that $\{x_n\}$ is an s-convergent sequence to x in $(\widetilde{X}, \Gamma, *)$. Let $\varepsilon > 0$. Then there exists $n_\varepsilon \in \cdot$, such that if $n \geq n_\varepsilon$, $\|x_n - x\|_{\frac{1}{n}}^2 < \varepsilon$. Let $\alpha \in (0,1]$. Then there exists $n_\varepsilon \in \cdot$, such that $\alpha \geq \frac{1}{n_\alpha}$. Let $n_{\varepsilon,\alpha} = \max\{n_\varepsilon, n_\alpha\}$. Then if $n \geq n_{\varepsilon,\alpha}$,

$$\|x_n - x\|_\alpha^2 < \|x_n - x\|_{\frac{1}{n_\alpha}}^2 < \|x_n - x\|_{\frac{1}{n}}^2 < \varepsilon,$$

Since $n \geq n_\varepsilon, \alpha \geq n_\varepsilon$. This means that (x) converges to x in $(\widetilde{X}, \Gamma, *)$.
The converses of Proposition 5.2 do not hold. The following example is found in [2]:

Example 5.3. Let $X = R$ and we define a fuzzy soft norm $\|\cdot\|$ on X by

$$\|x\| = \begin{cases} \frac{|x|}{t} & \text{if} \quad |x| \leq t, x \neq 0, \\ 1 & \text{if} \quad t = |x| = 0, \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that $(\widetilde{X}, \Gamma, *)$ is a fuzzy soft normed space and α-level set of is $(\widetilde{X}, \Gamma, *)$ given by

$$[\|x\|]_\alpha = \left[|x|, \frac{|x|}{\alpha}\right].$$

Let $x_n = \frac{1}{n}$. Let $\varepsilon > 0$ and $\alpha \in (0,1]$. Then there exists $n_{\alpha,\varepsilon} \in \cdot$ such that $n_{\alpha,\varepsilon} > \frac{1}{\alpha\varepsilon}$. If $n \geq n_{\alpha,\varepsilon}$, $\|x_n\|_\alpha^2 = \left\|\frac{1}{n}\right\|_\alpha^2 = \frac{1}{n\alpha} \leq \frac{1}{n_{\alpha,\varepsilon}\alpha} < \varepsilon$.
This means that $\{x_n\}$ is convergent in $(\widetilde{X}, \Gamma, *)$. We now show that $\{x_n\}$ is not s-convergent in $(\widetilde{X}, \Gamma, *)$. Since $\|x_n\|_{\frac{1}{n}}^2 = 1$, $\{x_n\}$ is not s-convergent in $(\widetilde{X}, \Gamma, *)$.

Let $y_n = \frac{1}{n^2}$. Let $\varepsilon > 0$. Then there exists $n_\varepsilon \in \cdot$ such that $\frac{1}{n_\varepsilon} < \varepsilon$. If $n \geq n_\varepsilon$, $\|y_n\|_\alpha^2 = \left\|\frac{1}{n}\right\|_\alpha^2 = \frac{1}{n} \leq \frac{1}{n_\varepsilon} < \varepsilon$. This means that (y_n) is s-convergent in $(\widetilde{X}, \Gamma, *)$. We now show that (y_n) is not st-convergent in $(\widetilde{X}, \Gamma, *)$. Suppose that (y_n) is st-convergent in $(\widetilde{X}, \Gamma, *)$.

Let $\varepsilon > 0$. Then there exists $n_\varepsilon \in \cdot$ such that if $n \geq n_\varepsilon$, $\|y_n\|_\alpha^2 = \frac{1}{n^2\alpha} < \varepsilon$ for all $\alpha \in (0, 1]$. This implies that $\alpha > \frac{1}{\varepsilon n_\varepsilon^2}$ for all $\alpha \in (0, 1]$. We get the contradiction. Therefore, $\{y_n\}$ is the s-convergent sequence which is not st-convergent in $(\tilde{X}, \Gamma, *)$. The converses of Proposition 5.2 do not hold.

By Proposition 5.2 and Example 5.3, we get the following strict implication in the fuzzy soft normed space introduced by Felbin:

st -convergence ⇒ s -convergence ⇒ convergence.

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