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# On fuzzy Soft Dual Space of fuzzy Soft Banach Algebra

***Authors Names***

*Qasim Ali Hatif* [a]
*Noori F. Al-Mayahi* [b]

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**ABSTRACT**

The concept of fuzzy Soft set theory was introduced by Molodtsov in the study [8]. fuzzy Soft real numbers and properties were introduced in the study [6] and fuzzy soft Banach algebra was defined in [11] . In this study, firstly we obtain a fuzzy soft Banach algebra by defining a fuzzy soft norm on (Real numbers) which is called fuzzy soft normed real space. By using this normed space we define the fuzzy soft linear functional and investigate some of its properties. Secondly, we introduce fuzzy soft dual space and fuzzy soft dual operator and investigate their properties. Finally, we state and prove the theorem about representation of fuzzy soft linear functional by inner product in fuzzy soft Hilbert spaces.

## 1. Introduction

Molodtsov [8] introduced the idea of soft set to overcome uncertainties It cannot be dealt with in classical ways in many fields such as the environment Science, economics, medical science, engineering and social sciences. This theory Applies when there is no clearly defined mathematical model. Lately, a lot Research on fuzzy soft sets has been published; See [1-7] The concept of a fuzzy soft point has been defined in different ways. Everyone introduced the concept of fuzzy soft measurement and investigated some properties of fuzzy soft measurement Metric spaces. S.Das et al introduced the concept of fuzzy soft element in [10] and defined a Fuzzy soft vector space using the fuzzy soft element concept. After that they studied Fuzzy soft modular spaces, fuzzy soft linear operators, and fuzzy soft interior product spaces and their basics Characteristics [3, 4, 9]. Later [11] defined the fuzzy soft vector space using this concept From the blurry fuzzy soft point and the fuzzy soft blurry standard spaces are presented in a new point of view. In the Study [12] identified a fuzzy soft inner product space. In this study, we advance the literature [11, 12] by defining a fuzzy soft true norm space. In this study we define a fuzzy soft inner product on a standard fuzzy soft real space. we Introducing fuzzy soft linear functions and showing the space of fuzzy soft linear functions It is a fuzzy soft vector space and this fuzzy soft vector space is the dual space of the fuzzy soft standard real space. We also studied some properties of fuzzy soft linear functions and fuzzy soft dual spaces. Finally, after defining the fuzzy soft correlation line SSV ($\tilde{R}$), we state and prove the theorem Representing fuzzy soft linear functions by the inner product in fuzzy soft Hilbert spaces.

## 2. Preliminaries

**Definition(2.1)[14] :** A pair $(\Gamma, A)$ denote by $\Gamma_A$ is called a fuzzy soft set over X, where $\Gamma$ is a function given by $\Gamma: A \to I^X$.

[a]Department of Mathematics, College of Science, University of Al-Qadisiyah,Diwaniyah-Iraq,
[a]Email:sci.math.mas.22.14@qu.end.iq, [b]E-mail: nafm60@yahoo.comi

**Definition (2.2)[14]:** A Fuzzy soft set $(\Gamma, A)$ over X is said to be an absolute fuzzy soft set, if for all $e \in E$, $F(e)$ is a fuzzy universal set $\tilde{1}$ over X and is denoted by $\tilde{\tilde{E}}$.

**Definition (2.3)[14]:**A fuzzy soft set $(\Gamma, E)$ over X is said to be a null fuzzy soft set, if for all $e \in E$, $F(e)$ is the null fuzzy set $\tilde{0}$ over X. It is denoted by $\breve{\emptyset}$.

**Definition (2.4)[13]:** For two fuzzy soft sets $(\Gamma, A)$ and $(\Lambda, B)$ in $\Gamma(X, E)$ we say that $(\Gamma, A) \breve{\subseteq} (\Lambda, B)$ if $A \breve{\subseteq} B$ and $\Gamma(e)(x) \leqq \Lambda(e)(x)$.

**Definition (2.5)[14]:** Two fuzzy soft sets $(\Gamma, A)$ and $(\Lambda, B)$ in $\Gamma(X, E)$ are equal if $\Gamma \breve{\subseteq} \Lambda$ and $\Lambda \breve{\subseteq} \Gamma$.

**Definition (2.6)[13]:** The different between two fuzzy soft sets $(\Gamma, A)$ and $(\Lambda, B)$ in $\Gamma(X, E)$ is a fuzzy soft set $(\Gamma/\Lambda, E)$ (say ) defined by $(\Gamma/\Lambda)(e) = \Gamma(e)/\Lambda(e)$ for each $e \in E$.$(\Gamma/\Lambda)(e): X \to I$, $(\Gamma/\Lambda)(e)(x) = \Gamma(e)(x) \breve{\cap} \left(\Lambda(e)(x)\right)^c = min\{\Gamma(e)(x), \Lambda^c(e)(x)\} \, \forall x \in X$.

**Definition (2.7)[14]** : The complement of a fuzzy soft set $(\Gamma, E)$ is a fuzzy soft set $(\Gamma^c, E)$ defined by $\Gamma^c(e) = 1/\Gamma(e)$ for each $e \in E$, $\left(\Gamma^c(e)\right)(x) = 1 \boxminus \Gamma(e)(x) \; \forall x \in X$.

**Definition (2.8)[13]:**Let $(\Gamma, A)$ and $(\Lambda, B)$ be two fuzzy soft sets in $\Gamma(X,E)$ with $A \breve{\cap} B \neq \Phi$, then:

a) their intersection $(\Gamma \breve{\cap} \Lambda, C)$ is a fuzzy soft set, where $C = A \breve{\cap} B$ and $(\Gamma \breve{\cap} \Lambda)(e) = \Gamma(e) \breve{\cap} \Lambda(e)$ for each $e \in C$,

$(\Gamma \breve{\cap} \Lambda)(e)(x) = \min\{\left(e, \Gamma_e(x)\right), (e, \Lambda_e(x))$

b) their union $(\Gamma \breve{\cup} \Lambda, C)$ is a fuzzy soft set, where

C) $A \breve{\cup} B$ and $(\Gamma \breve{\cup} \Lambda)e = \Gamma(e) \breve{\cup} \Lambda(e)$ for each $e \in C$

$(\Gamma \breve{\cup} \Lambda)(e)(x) = \max\{\Gamma(e)(x), \Lambda(e)(x)\}$

**Definition (2.9)[14]:** A fuzzy soft set $\Gamma_A$ over $X$ is called a fuzzy soft point if for the

element $e^* \in E$

$$f_e(x) = \begin{cases} \lambda_x & if\ e = e^* \\ 0 & otherwise \end{cases} \quad \text{for every } e \in E$$

Otherwise, for the element $x^* \in X$,

$$f_e(x) = \begin{cases} \lambda & if\ x = x^* \\ 0 & otherwise \end{cases} \quad \text{for every } x \in X \text{, where } \lambda \in (0,1].$$

The set of all fuzzy soft points is denoted as $\tilde{x}_E$.

Let $X$ be a vector space over a field $K$ ($K = \mathbb{R}$) and the parameter set $E$ be the real number set $\mathbb{R}$.

**Definition 2.10 ([11])** Let $(F,E)$ be a soft set over $X$. The soft set $(F,E)$ is said to be a soft vector and denoted by $\tilde{x}_e$ if there is exactly one $e \in E$, such that $F(e) = \{x\}$ for some $x \in X$ and $F(e') = \emptyset$, $\forall e' \in E/\{e\}$.

The set of all soft vectors over $\widetilde{X}$ will be denoted by $SV(\widetilde{X})$.

**Definition 2.11.** Let $\tilde{x}$ be an absolute soft linear space over the scalar field $K$. Suppose $*$ is a continuous t-norm, $R(A^*)$ is the set of all non-negative soft real numbers and $SSP(\tilde{X})$ denote the se t of all soft points on $\widetilde{X}$. A fuzzy subset $\Gamma$ on $SSP(\tilde{X}) \times R(A^*)$ is called a fuzzy soft norm $\widetilde{X}$ on if and only if for $\tilde{x}_e, \tilde{y}_{\acute{e}} \in SSP(\widetilde{X})$ and $\tilde{k} \in \widetilde{K}$ (where $\tilde{k}$ is a soft scalar) the following conditions hold

1) $\Gamma(\tilde{x}_e, \tilde{t}) = 0, \forall \tilde{t} \in R(A^*)$ with $\tilde{t} \leq \tilde{0}$.

2) $\Gamma(\tilde{x}_e, \tilde{t}) = 1, \forall \tilde{t} \in R(A^*)$ with $\tilde{t} \leq \tilde{0}$ if and only if $\tilde{x}_e, \tilde{\theta}_0$.

3) $\Gamma(k\tilde{x}_e, \tilde{t}) = \Gamma(\tilde{x}_e, \frac{\tilde{t}}{|\tilde{k}|})$ if $\tilde{k} \neq \tilde{0}$ $\forall \tilde{t} \in R(A^*), \tilde{t} > \tilde{0}$.

4) $\Gamma(\tilde{x}_e \oplus \tilde{y}_e, \tilde{t} \oplus \tilde{s}) > \Gamma(\tilde{x}_e, \tilde{t}) * \Gamma(\tilde{y}_e, \tilde{s}), \forall \tilde{t}, \tilde{s} \in R(A^*), \tilde{x}_e, \tilde{y}_e \in SSP(\tilde{X})$.

5) $\Gamma(\tilde{x}_e, .)$ is a continuous non decreasing function of $R(A^*)$ and $\lim_{\tilde{x}\to\infty} \Gamma(\tilde{x}_e, \tilde{t}) = 1$.

The triplet $(\widetilde{X}, \Gamma, *)$ will be referred to as a fuzzy soft Banach algebra.

**Proposition 2.1. ([2])** Every fuzzy soft set can be expressed as a union of all fuzzy soft points belonging to it. Conversely, any set of fuzzy soft points can be considered as a fuzzy soft set.

Let $SSP(\tilde{X})$ be the collection of all fuzzy soft points of $\tilde{X}$ and $\mathbb{R}(A)^*$ denote the set of all non-negative fuzzy soft real numbers.

**Definition 2.12. ([6])** A mapping $\tilde{d}: SSP(\tilde{X}) \times SSP(\tilde{X}) \to \mathbb{R}(A)^*$ is said to be a fuzzy soft metric on the fuzzy soft set $\widetilde{X}$ if $\tilde{d}$ satisfies the following conditions:
(M1) $\tilde{d}(\tilde{x}_{e1}, \tilde{y}_{e2}) \geqq \tilde{0}$ for all $\tilde{x}_{e1}, \tilde{y}_{e2} \in \tilde{X}$,
(M2) $\tilde{d}(\tilde{x}_{e1}, \tilde{y}_{e2}) = \tilde{0}$ if and only if $\tilde{x}_{e1} = \tilde{y}_{e2} \in \tilde{X}$,
(M3) $\tilde{d}(\tilde{x}_{e1}, \tilde{y}_{e2}) = \tilde{d}(\tilde{y}_{e2}, \tilde{x}_{e1})$ for all $\tilde{x}_{e1}, \tilde{y}_{e2} \in \tilde{X}$,
(M4) For all $\tilde{x}_{e1}, \tilde{y}_{e2}, \tilde{z}_{e3} \in \tilde{X}$, $\tilde{d}(\tilde{x}_{e1}, \tilde{z}_{e3}) \leqq \tilde{d}(\tilde{x}_{e1}, \tilde{y}_{e2}) + \tilde{d}(\tilde{y}_{e2}, \tilde{z}_{e3})$.

The fuzyy soft set $\widetilde{X}$ with a fuzzy soft metric $\tilde{d}$ is called a fuzzy soft metric space and denoted by $(\widetilde{X}, \tilde{d}, \Gamma_A)$.

**Proposition 2.2. ([11])** The set $SSV(\widetilde{X})$ is a vector space according to the following operations;
(1) $\tilde{x}_e + \tilde{y}_{e'} = (\widetilde{x + y})_{(e+e')}$ for every $\tilde{x}_e, \tilde{y}_{e'} \in SV(\widetilde{X})$;

(2) $\tilde{r}.\tilde{x}_e = (\widetilde{rx})_{(re)}$ for every $\tilde{x}_e \in SV(\tilde{X})$ and for every fuzzy soft real number $\tilde{r}$.

**Definition 2.13. ([11])** Let $SSV(\tilde{X})$ be a fuzzy soft vector space and $\widetilde{M} \subset SSV(\tilde{X})$ be a subset. If $\widetilde{M}$ is a fuzzy soft vector space, then $\widetilde{M}$ is said to be a fuzzy soft vector subspace of $SV(\tilde{X})$ and denoted by $SSV(\widetilde{M}) \tilde{\subseteq} SSV(\tilde{X})$.

**Definition 2.14. ([11])** A sequence of fuzzy soft vectors $\{\tilde{x}_{e_n}^n\}$ in $(\tilde{X}, \Gamma, *)$ is said to be convergent to $\tilde{x}_{e_0}^0$, if $\lim_{n\to\infty} \|\tilde{x}_{e_n}^n - \tilde{x}_{e_0}^0\| = \tilde{0}$ and denoted by $\tilde{x}_{e_n}^n \to \tilde{x}_{\lambda_0}^0$ as $n \to \infty$.

**Definition 2.15. ([11])** A sequence of fuzzy soft vectors $\{\tilde{x}_{e_n}^n\}$ in $(\tilde{X}, \Gamma, *)$ is said to be a fuzzy soft Cauchy sequence if corresponding to every $\tilde{\varepsilon} \tilde{>} \tilde{0}$, $\exists m \in N$ such that

$\left\|\tilde{x}_{e_i}^i - \tilde{x}_{e_j}^j\right\| \tilde{<} \tilde{\varepsilon}, \forall i, j \geq m$ i.e., $\left\|\tilde{x}_{e_i}^i - \tilde{x}_{e_j}^j\right\| \to \tilde{0}$ as $i, j \to \infty$.

**Proposition 2.3. ([11])** Every fuzzy soft convergent sequence is a fuzzy soft Cauchy sequence.

**Definition 2.16. ([11])** Let $(\tilde{X}, \Gamma, *)$ be a fuzzy soft Banach algebra. Then $(\tilde{X}, \Gamma, *)$ is said to be complete if every fuzzy soft Cauchy sequence in $\tilde{X}$ converges to a soft vector of $\tilde{X}$.

**Definition 2.17. ([11])** Every fuzzy soft complete fuzzy soft normed linear space is called a fuzzy soft Banach space.

**Proposition 2.4. ([11])** Every fuzzy soft normed space is a fuzzy soft metric space.

**Definition 2.18. ([11])** Let $T : SSV(\tilde{X}) \to SSV(\tilde{Y})$ be a fuzzy soft mapping. Then $T$ is said to be fuzzy soft linear operator if

(L1). $T$ is additive, i.e., $T(\tilde{x}_e, \tilde{y}_{e'}) = T(\tilde{x}_e) + T(\tilde{y}_{e'})$ for every $\tilde{x}_e, \tilde{y}_{e'} \tilde{\in} SSV(\tilde{X})$.

(L2). $T$ is homogeneous, i.e., for every fuzzy soft scalar $\tilde{r}$, $T(\tilde{r}\tilde{x}_e) = \tilde{r}.T(\tilde{x}_e)$ for every $\tilde{x}_e \tilde{\in} SSV(\tilde{X})$.

**Definition 2.19. ([11])** The fuzzy soft operator $T : SSV(\tilde{X}) \to SSV(\tilde{Y})$ is said to be fuzzy soft continuous at $\tilde{x}_e \tilde{\in} SSV(\tilde{X})$ if for every fuzzy soft sequence $\{\tilde{x}_{e_n}^n\}$ of fuzzy soft vectors of $\tilde{X}$ with $\tilde{x}_{e_n}^n \to \tilde{x}_{e_0}^0$ as $n \to \infty$, we have $T(\tilde{x}_{e_n}^n) \to T(\tilde{x}_{e_0}^0)$ as $n \to \infty$. If $T$ is soft continuous at each fuzzy soft vector of $SSV(\tilde{X})$, then $T$ is said to be fuzzy soft continuous operator.

**Definition 2.20. ([11])** The fuzzy soft operator $T : SSV(\tilde{X}) \to SSV(\tilde{Y})$ is said to be fuzzy soft bounded, if there exists a fuzzy soft real number $\widetilde{M}$ such that

$\|T(\tilde{x}_e)\| \tilde{\leq} \widetilde{M}\|\tilde{x}_e\|$, for all $\tilde{x}_e \tilde{\in} SSV(\tilde{X})$.

**Theorem 2.1. ([11])** The fuzzy soft operator $T : SSV(\tilde{X}) \to SSV(\tilde{Y})$ is fuzzy soft continuous if and only if it is fuzzy soft bounded.

**Definition 2.21. ([11])** Let $T : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$ be a fuzzy soft continuous operator .

$$\|T\| = inf\{\tilde{M}: \|T(\tilde{x}_e)\| \lesseqqgtr \tilde{M}\|\tilde{x}_e\|\}$$

is said to be a norm of $T$.

It is obvious that $\|T(\tilde{x}_e)\| \lesseqqgtr \|T\|\|\tilde{x}_e\|$.

**Theorem 2.2. ([11])** Let $T : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$ be a fuzzy soft operator. Then,

$$\|T\| = \sup_{\tilde{x}_e \neq \tilde{\theta}_0} \frac{\|T(\tilde{x}_e)\|}{\|\tilde{x}_e\|} = \sup_{\|\tilde{x}_e\| \lesseqqgtr 1} \|T(\tilde{x}_e)\|$$

**Theorem 2.3. ([11])** Let $T : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$ be a fuzzy soft operator then $\|T\|$ is a fuzzy soft norm.

**Theorem 2.4. ([11])** Let $T : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$ and $S : SSV(\tilde{Y}) \rightarrow SSV(\tilde{Z})$ be two soft operators. Then

a) $\|SoT\| \lesseqqgtr \|S\|\|T\|$,

b) If $T : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$ is a fuzzy soft operator, then

$$\|T^n\| \lesseqqgtr \|T\|^n.$$

is satisfied.

**Definition 2.22. ([12])** Let $SSV(\tilde{X})$ be a fuzzy soft vector space. The mapping

$$<\cdot> : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y}) \rightarrow \mathbb{R}(A)$$

(is called a fuzzy soft inner product on $SSV(\tilde{X})$ iff it satisfies the following conditions, for every $\tilde{x}_e, \tilde{y}_{e'}, \tilde{z}_{e''} \tilde{\in} SSV(\tilde{X})$ and for every fuzzy soft reel number $\tilde{\alpha}$ where $\mathbb{R}(A)$ is the fuzzy soft reel number set:

I1. $< \tilde{x}_e, \tilde{x}_e > \gtreqqless \tilde{0}$ and $< \tilde{x}_e, \tilde{x}_e > = \tilde{0} \Leftrightarrow \tilde{x}_e = \tilde{\theta}_0$,

I2. $< \tilde{x}_e, \tilde{y}_{e'} \geq < \tilde{y}_{e'}, \tilde{x}_e >$,

I3. $< \tilde{\alpha}\, \tilde{x}_e, \tilde{y}_{e'} > = < \tilde{x}_e, \tilde{\alpha}\tilde{y}_{e'} > = \tilde{\alpha} < \tilde{x}_e, \tilde{y}_{e'} >$,

I4. $< \tilde{x}_e + \tilde{y}_{e'}, \tilde{z}_{e''} > = < \tilde{x}_e, \tilde{z}_{e''} > + < \tilde{y}_{e'}, \tilde{z}_{e''} >$.

The triple $(\mathrm{SSV}(\tilde{\mathrm{X}}), <.>, \Gamma_{\mathrm{A}})$ is called fuzzy soft inner product space.

**Proposition 2.5. ([12])** (Parallelogram Law) Let $(\mathrm{SSV}(\tilde{\mathrm{X}}), <.>, \Gamma_{\mathrm{A}})$ be a fuzzy soft inner product space. For every $\tilde{x}_e, \tilde{y}_{e'} \tilde{\in} SSV(\tilde{X})$

$$\|\tilde{x}_e + \tilde{y}_{e'}\|^2 + \|\tilde{x}_e - \tilde{y}_{e'}\|^2 = 2(\|\tilde{x}_e\|^2 + \|\tilde{y}_{e'}\|^2).$$

is satisfied.

**Theorem 2.5. ([12])** Let $\left(\mathrm{SSV}(\tilde{\mathrm{X}}), <.>, \Gamma_{\mathrm{A}}\right)$ be a fuzzy soft inner product space. For every $\tilde{x}_e, \tilde{y}_{e'} \tilde{\in} SSV(\tilde{X})$

$$|< \tilde{x}_e, \tilde{y}_{e'} >| \tilde{\leqq} \|\tilde{x}_e\| \, \|\tilde{y}_{e'}\|$$

is hold.

**Proposition 2.6. ([12])** A fuzzy soft inner product function is continuous in a fuzzy soft inner product space. In other words, If $\{\tilde{x}^n_{e_n}\} \to \tilde{x}_e$ and $\{\tilde{y}^n_{e'_n}\} \to \tilde{y}_{e'}$, then $< \tilde{x}^n_{e_n}, \tilde{y}^n_{e'_n} > \to < \tilde{x}_e, \tilde{y}_{e'} >$.

**Proposition 2.7. ([12])** Let $\left(SSV(\tilde{X}), < . >, \Gamma_A\right)$ be a fuzzy soft inner product space and $\{\tilde{x}^n_{e_n}\}$, $\{\tilde{y}^n_{e'_n}\}$ be fuzzy soft Cauchy sequences in this space. In this case, $< \tilde{x}^n_{e_n}, \tilde{y}^n_{e'_n} >$ is also a fuzzy soft Cauchy sequence.

**Definition 2.23. ([12])** Let $\left(SSV(\tilde{X}), < . >, \Gamma_A\right)$ be a fuzzy soft inner product space If this space is complete according to the induced norm by the fuzzy soft inner product then $\left(SSV(\tilde{X}), < . >, \Gamma_A\right)$ is said to be a fuzzy soft Hilbert space.

## 3.fuzzy Soft Dual Spaces

**Proposition 3.1.** Given the fuzzy soft vector space $SSV(\widetilde{\mathbb{R}}) = \{\tilde{x}_e : x \in \mathbb{R}, e \in \mathbb{R}\}$ and for every $\tilde{x}_e, \tilde{y}_{e'} \tilde{\in} SSV(\tilde{X})$, let us define the mapping $< . > : SSV(\widetilde{\mathbb{R}}) \times SSV(\widetilde{\mathbb{R}}) \to \mathbb{R}(A)$ $(A = \mathbb{R})$ as follows

$$< \tilde{x}_e, \tilde{y}_{e'} > = x.y + e.e'$$

In this case the mapping $< . >$ is an inner product on $SSV(\widetilde{\mathbb{R}})$.

**Proof. I1.** For every $\tilde{x}_e \tilde{\in} SSV(\widetilde{\mathbb{R}})$,
$< \tilde{x}_e, \tilde{x}_e > = x.x + e.e = x^2 + e^2 \tilde{\geqq} \tilde{0}$
$< \tilde{x}_e, \tilde{x}_e > = 0 \Leftrightarrow \tilde{x}_e = 0$
$\Leftrightarrow e = 0 \; and \; x = \theta$
$\Leftrightarrow \tilde{x}_e = \tilde{\theta}_0$
**I2.** For every $\tilde{x}_e, \tilde{y}_{e'} \tilde{\in} SSV(\widetilde{\mathbb{R}})$,

$$< \tilde{x}_e, \tilde{y}_{e'} > = x.y + e.e' = y.x + e.e' = < \tilde{y}_{e'}, \tilde{x}_e >.$$

**I3.** For every $\tilde{\alpha} \tilde{\in} \mathbb{R}(A)$ and $\forall \tilde{x}_e, \tilde{y}_{e'} \tilde{\in} SSV(\widetilde{\mathbb{R}})$,

$$< \tilde{\alpha}\tilde{x}_e, \tilde{y}_{e'} > = \tilde{\alpha}\, x.y + \tilde{\alpha}\, e.e' = \tilde{\alpha} y.x + e.(\tilde{\alpha} e') = < \tilde{x}_e, \tilde{\alpha}\tilde{y}_{e'} >.$$

**I4.** For every $\tilde{x}_e, \tilde{y}_{e'}, \tilde{z}_{e''} \tilde{\in} SSV(\widetilde{\mathbb{R}})$,
$< \tilde{x}_e + \tilde{y}_{e'}, \tilde{z}_{e''} > = < (\widetilde{x+y})_{(e.e')}, \tilde{z}_{e''} >$.
$= (x + y).z + (e + e')e''$
$= xz + e'.e'' + y.z + e'.e''$
$= < \tilde{x}_e, \tilde{z}_{e''} > + < \tilde{y}_{e'}, \tilde{z}_{e''} >$.
Thus, $\left(SSV(\widetilde{\mathbb{R}}), < . >, \Gamma_A\right)$ is a fuzzy soft inner product space.

**Remark 3.1.** The induced norm from the fuzzy soft inner product defined in the preceding proposition is

$$\|\tilde{x}_e\| = \sqrt{x^2 + e^2}$$

where $\|.\| : SSV\left(\widetilde{\mathbb{R}}\right) \times SSV\left(\widetilde{\mathbb{R}}\right) \to \mathbb{R}^+(A)(A = R)$.

Also, the induced fuzzy soft metric from the fuzzy soft norm is

$$\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) = \sqrt{(x-y)^2 + (e-e')^2}$$

where $\tilde{d} : SSV\left(\widetilde{\mathbb{R}}\right) \times SSV\left(\widetilde{\mathbb{R}}\right) \to \mathbb{R}^+(A)(A = R)$.

**Proposition 3.2.** The fuzzy soft inner product space $\left(SSV\left(\widetilde{\mathbb{R}}\right), <.>, A\right)(A = R)$ is a fuzzy soft Euclidean space.

**Proof.** Let us take a fuzzy soft Cauchy sequence $\{\tilde{x}_{e_n}^n\}$ in the fuzzy soft metric space $\left(SSV\left(\widetilde{\mathbb{R}}\right), \tilde{d}\right)$. In this case, for every fuzzy soft reel number $\tilde{\varepsilon} \tilde{\geqq} \tilde{0}$ there exists a $n_0 \in N$ such that for $\forall m, n \geq n_0$

$$\tilde{d}\left(\tilde{x}_{e_n}^n, \tilde{x}_{e_m}^m\right) \tilde{\leqq} \sqrt{(x^n - x^m)^2 + (e_n - e_m)^2} \tilde{<} \tilde{\varepsilon}.$$

Thus for $\forall m, n \geq n_0$ we have $|x^n - x^m|^2 + |e_n - e_m|^2 \tilde{<} \tilde{\varepsilon}^2$. Since $|x^n - x^m| \tilde{<} \tilde{\varepsilon}$ and $|e_n - e_m| \tilde{<} \tilde{\varepsilon}$ $\{x^n\}$ and $\{e_n\}$ are Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is a complete space there exist real numbers $x$ and e such that $\{x_n\} \to x$ and $\{e_n\} \to x$. If we take the real number e as a parameter of the real number x then we have the fuzzy soft sequence $\tilde{x}_e$ and we obtain $\{\tilde{x}_{e_n}^n\} \to \tilde{x}_e$. Finally, since $x \in \mathbb{R}$ and $e \in \mathbb{R}$ we have $\tilde{x}_e \tilde{\in} SSV\left(\widetilde{\mathbb{R}}\right)$. Since the fuzzy soft Cauchy sequence $\{\tilde{x}_{e_n}^n\}$ is arbitary the fuzzy soft space SSV $\left(\widetilde{\mathbb{R}}\right)$ is a fuzzy soft Eucllidean space.

**Definition 3.1.** Let $(SSV\left(\tilde{X}\right), \|.\|, \Gamma_A)$ be a fuzzy soft normed space, $\tilde{x}_e \tilde{\in} SSV\left(\tilde{X}\right), f : X \to R$ be a linear functional and $\varphi : \mathbb{R} \to \mathbb{R}$ be a function. The mapping

$$(f, \varphi) : SSV\left(\tilde{X}\right) \to SSV\left(\widetilde{\mathbb{R}}\right)$$

which is defined as follows

$$(f, \varphi)(\tilde{x}_e) = (\widetilde{f(x)})_{\varphi(e)}$$

is called a fuzzy soft functional.

**Proposition 3.3.** The fuzzy soft functional $(f, \varphi) : SSV\left(\tilde{X}\right) \to SSV\left(\widetilde{\mathbb{R}}\right)$ is linear if and only if the function $\varphi: \mathbb{R} \to \mathbb{R}$ is linear.

**Proof.** Let $\tilde{x}_e, \tilde{y}_{e'} \tilde{\in} SSV\left(\tilde{X}\right)$ and $\lambda \tilde{\in} \widetilde{\mathbb{R}}$. In this case $(f, \varphi)$ is linear if and only if

$$(f, \varphi)\left(\tilde{\lambda}.(\tilde{x}_e) + \tilde{y}_{e'}\right) = (f, \varphi)\left((\widetilde{\lambda x})_{\lambda e} + \tilde{y}_{e'}\right) = (f, \varphi)\left((\widetilde{\lambda x + y})_{\lambda e + e'}\right)$$

$$= f(\lambda \widetilde{x + y})_{\varphi(\lambda e+e')} = (f(\lambda x)\widetilde{+ f}(y))_{\varphi(\lambda e)+\varphi(e')}$$

$$= \widetilde{f(\lambda x)}_{\varphi(\lambda e)} + \widetilde{f(y)}_{\varphi(e')} = \tilde{\lambda}.\left(\widetilde{f(x)}\right)_{\varphi(e)} + \left(\widetilde{f(y)}\right)_{\varphi(e')}$$

where$\varphi : \mathbb{R} \to \mathbb{R}$ is linear.

**Proposition 3.4.** The space of fuzzy soft linear functionals is a fuzzy soft vector space.

**Proof.** Let $(\mathrm{f}, \varphi) : \mathrm{SSV}(\tilde{\mathrm{X}}) \to \mathrm{SSV}(\tilde{\mathbb{R}})$, $(g, \psi) : SSV(\tilde{X}) \to SSV(\tilde{\mathbb{R}})$ be fuzzy soft functionals where $\mathrm{f} : \mathrm{X} \to \mathbb{R}$, $\mathrm{g} : \mathrm{X} \to \mathbb{R}$ and $\tilde{\mathrm{x}}_{\mathrm{e}} \tilde{\in} \mathrm{SSV}(\tilde{\mathrm{X}})$, $\tilde{\lambda} \tilde{\in} \tilde{\mathbb{R}}$. Then we have

$$((\mathrm{f}, \varphi) + (\mathrm{g}, \psi))(\tilde{\mathrm{x}}_{\mathrm{e}}) = \big((\mathrm{f} + \mathrm{g}), (\varphi + \psi)\big)(\tilde{\mathrm{x}}_{\mathrm{e}})$$

$$= (\mathrm{f} \widetilde{+ \mathrm{g}})(\mathrm{x})_{(\varphi + \psi)(\mathrm{e})} = (\mathrm{f}(\mathrm{x})\widetilde{+ \mathrm{g}}(\mathrm{x}))_{\varphi(\mathrm{e}) + \psi(\mathrm{e})}$$

since X is a vector space for the functionals $\mathrm{f} : \mathrm{X} \to \mathbb{R}$, $\mathrm{g} : \mathrm{X} \to \mathbb{R}$ we have $\mathrm{f}(\mathrm{x}) + \mathrm{g}(\mathrm{x}) \in \mathbb{R}$ and for the function $\varphi: \mathbb{R} \to \mathbb{R}$ we have $\varphi(\mathrm{e}) + \psi(\mathrm{e}) \in \mathbb{R}$ which means that $(\mathrm{f}(\mathrm{x})\widetilde{+ \mathrm{g}}(\mathrm{x}))_{\varphi(\mathrm{e}) + \psi(\mathrm{e})} \tilde{\in} \mathrm{SSV}(\tilde{\mathbb{R}})$ and consequently we have $(\mathrm{f}, \varphi)\mathrm{o}(\mathrm{g}, \psi) : \mathrm{SSV}(\tilde{\mathrm{X}}) \to \mathrm{SSV}(\tilde{\mathbb{R}})$.

On the other hand,

$$\tilde{\lambda}(\mathrm{A}, \varphi)(\tilde{\mathrm{x}}_{\mathrm{e}}) = \big(\tilde{\lambda}\mathrm{A}, \tilde{\lambda}\varphi\big)(\tilde{\mathrm{x}}_{\mathrm{e}}) = (\widetilde{\lambda \mathrm{A})(\mathrm{x}})_{\lambda\varphi(\mathrm{e})}.$$

**Definition 3.2.** The fuzzy soft norm of the fuzzy soft linear functional $(f, \varphi)$ is defined as follows

$$\|(\mathrm{f}, \varphi)\| = \inf\left\{\tilde{\mathrm{M}}: \|(\mathrm{f}, \varphi)(\tilde{\mathrm{x}}_{\mathrm{e}})\| \tilde{\leqq} \tilde{\mathrm{M}}\|\tilde{\mathrm{x}}_{\mathrm{e}}\|\right\}$$

is said to be a norm of $(\mathrm{f}, \varphi)$ where $\tilde{\mathrm{M}}$ is a fuzzy soft real number.

It is obvious that $\|(\mathrm{f}, \varphi)(\tilde{\mathrm{x}}_{\mathrm{e}})\| \tilde{\leqq} \|(\mathrm{f}, \varphi)\|\|\tilde{\mathrm{x}}_{\mathrm{e}}\|$.

**Remark 3.2.** The space of fuzzy soft linear functionals is a fuzzy soft normed space.

**Remark 3.3.** Let $\big(SSV(\tilde{X}), <.>, A\big)(A = \mathbb{R})$ be a fuzzy soft inner product space. For the parameter $e = 0$ the soft vector space $SSV(\tilde{X})$ is equal to the vector space $X$ and we have the inner product space $(\mathrm{X}, <.> 0) = \mathrm{SSV}(\tilde{\mathrm{X}}_0)$ which is a subspace of $\big(\mathrm{SSV}(\tilde{\mathrm{X}}), <.>, A\big)$. Here, The vector space X may not be a inner product space.

**Proposition 3.5.** Let $\big(SSV(\tilde{X}), <.>, A\big)(A = \mathbb{R})$ be a fuzzy soft inner product space. For every $\tilde{\mathrm{x}}_{\mathrm{e}} \tilde{\in} SSV(\tilde{X})$ let us define the operator $I : SSV(\tilde{X}) \to X$ where $I(\tilde{\mathrm{x}}_{\mathrm{e}}) = x \in X$ . The restriction of the operator I to $\mathrm{SSV}(\tilde{\mathrm{X}}_0)$ denoted by $I/SSV(\tilde{X}_0) :\to X$ where $\mathrm{I}(\tilde{\mathrm{x}}_0) = \mathrm{x}$ is a bijection and consequently $\mathrm{SSV}(\tilde{\mathrm{X}}_0) \backsim \mathrm{X}$.

**Remark 3.4.** For the parameter $e = 0$ the fuzzy soft vector space $\widetilde{\mathrm{X}}_0$ is equal to the vector space $X$ and thus for every $x, y \in X$ and $x_0, y_0 \tilde{\in} \widetilde{\mathrm{X}}_0$ we can write $< x, y >=< x_0, y_0 >$.

**Theorem 3.1.** Let $\left(SSV(\tilde{X}), <.>, A\right)(A = \mathbb{R})$ be a fuzzy soft inner product space and $(\mathrm{f}, \varphi) : \mathrm{SSV}(\widetilde{\mathrm{X}}) \to \mathrm{SSV}(\widetilde{\mathbb{R}})$ be a fuzzy soft functional. In this case we can state the term $(\mathrm{f}, \varphi)(\tilde{\mathrm{x}}_\mathrm{e}) = (\widetilde{f(x)})_{\varphi(e)}$ interms of soft inner paroduct as given below

$$(\mathrm{f}, \varphi)(\tilde{\mathrm{x}}_\mathrm{e}) = < x, \widetilde{u >}_{<e,v>}$$

where $f(x) =< x, u >$ and $\varphi(e) =< e, v >$. Moreover,

$$\|\mathrm{f}\| = \|\mathrm{u}\|, \|\varphi\| = \|\mathrm{v}\|.$$

**Proof.** For $(\mathrm{f}, \varphi) : \mathrm{SSV}(\widetilde{\mathrm{X}}) \to \mathrm{SSV}(\widetilde{\mathbb{R}})$ we have the functions $f : X \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$. If $f : X \to \mathbb{R}$ then for every $x_0 \tilde{\in} \widetilde{\mathrm{X}}_0$ there exist a vector $u_0 \in X_0$ such that

$$\mathrm{f}(\mathrm{x}_0) =< \mathrm{x}_0, \mathrm{u}_0 >$$

Since for the parameter $\mathrm{e} = 0$ we have $< \mathrm{x}, \mathrm{y} >=< \mathrm{x}_0, \mathrm{u}_0 >$ then $\mathrm{f}(\mathrm{x}) =< \mathrm{x}, \mathrm{u} >$

Similarly If $\varphi : \mathbb{R} \to \mathbb{R}$ then for every parameter $e \in \mathbb{R}$ there exist a real number $\mathrm{v} \in \mathbb{R}$ such that

$$\varphi(\mathrm{e}) =< \mathrm{e}, \mathrm{v} >.$$

Consequently, we have

$$(f, \varphi)(\tilde{\mathrm{x}}_\mathrm{e}) = < \widetilde{x, u} >_{<e,v>}. \qquad (3.1)$$

From the equation (3.1) we can write

$$\|(f, \varphi)(\tilde{x}_e)\| = \left\| < x, u >_{<e,v>} \right\|$$

$$= \sqrt{< x, u >^2 + < e, v >^2}$$

$$\leq \sqrt{x^2 . u^2 + e^2 . v^2}$$

$$\leq \sqrt{\mathrm{u}^2 + \mathrm{v}^2} \qquad (3.2)$$

On the other hand

$$\|(f, \varphi)(\tilde{x}_e)\| = \left\| \left(\widetilde{f(x)}\right)_{\varphi(e)} \right\|$$

$$= \sqrt{f^2(x) + \varphi^2(e)}$$

$$\leq \sqrt{\|f\|^2 \|x\|^2 + \|\varphi\|^2 \|e\|^2}$$

$$\leq \sqrt{\|f\|^2 + \|\varphi\|^2} \qquad (3.3)$$

from the inequalities (3.2) and (3.3) we have $\|f\| = \|u\|$ and $\|\varphi\| = \|v\|$

**Conclusion 3.1.** Let $(f, \varphi) : SSV(\widetilde{X}) \rightarrow SSV(\widetilde{\mathbb{R}})$ be a fuzzy soft functional. for every $\tilde{x}_e \,\widetilde{\in}\, SSV(\widetilde{X})$ there exist $u \in X$ and $v \in \mathbb{R}$ such that $\|f\| = \|u\|$ and $\|\varphi\| = \|v\|$ and consequently $\|(f, \varphi)\| = \|u_v\|$

**Definition 3.3.** The fuzzy soft vector space of fuzzy soft linear functionals is called the fuzzy soft dual space of the fuzzy soft vector space $SSV(\widetilde{X})$ and denoted by $SSV(\widetilde{X})^*$

**Definition 3.4.** Let $(SSV(\widetilde{X}), \|.\|, A)$, $(SSV(\tilde{Y}), \|.\|, A)(A = \mathbb{R})$ be fuzzy soft normed spaces, $(A, \psi) : SSV(\widetilde{X}) \rightarrow SSV(\tilde{Y})$, $(A, \psi)^* : SSV(\tilde{Y})^* \rightarrow SSV(\tilde{X})^*$ be fuzzy soft linear operators where $A : X \rightarrow Y$, $\psi : \mathbb{R} \rightarrow \mathbb{R}$, $A^* : Y^* \rightarrow X^*$, $\psi^* : \mathbb{R}^* \rightarrow \mathbb{R}^*$, $\tilde{x}_e \,\widetilde{\in}\, SSV(\widetilde{X})$ and $(f, \varphi) : SSV(\tilde{Y}) \rightarrow SSV(\widetilde{\mathbb{R}})$ be a fuzzy soft linear functional. In this case we have

$$(A, \psi)^*(f, \varphi) = (f, \varphi)o(A, \psi)$$

**Proposition 3.6.** Let $(A, \psi) : SSV(\tilde{X}) \rightarrow SSV(\tilde{Y})$ be a fuzzy soft operator. Then $(A, \psi)^* : SSV(\widetilde{Y})^* \rightarrow SSV(\widetilde{X})^*$ is a fuzzy soft linear operator.

**Proof.** Let $(f, \varphi), (g, \psi) \,\widetilde{\in}\, SSV(\widetilde{X})^*$ and $\tilde{\lambda} \,\widetilde{\in}\, \widetilde{\mathbb{R}}$. In this case, since $(A, \psi), (f, \varphi)$ and $(g, \psi)$ are linear we have

$$(A, \psi)^*\left(\tilde{\lambda}(f, \varphi) + (g, \psi)\right) = \left(\tilde{\lambda}(f, \varphi) + (g, \psi)\right)o(A, \psi)$$

$$= \tilde{\lambda}(f, \varphi)o(A, \psi) + (g, \psi)o(A, \psi)$$

$$= (A, \psi)^*\left(\tilde{\lambda}(f, \varphi)\right) + (A, \psi)^*\left((g, \psi)\right)$$

**Theorem 3.2.** If the fuzzy soft linear operator $(A, \psi) : SSV(\widetilde{X}) \rightarrow SSV(\tilde{Y})$ is continuous then $(A, \psi)^* : SSV(\widetilde{Y})^* \rightarrow SSV(\widetilde{X})^*$ is continuous.

**Proof.** Let $\tilde{x}_e \,\widetilde{\in}\, SSV(\widetilde{X})$ be any fuzzy soft vector.

$$\|(A, \psi)^*(f, \varphi)(\tilde{x}_e)\| = \|(f, \varphi)((A, \psi)(\tilde{x}_e))\|$$

$$\leq \|f, \varphi\| \|(A, \psi)(\tilde{x}_e)\|$$

$$\leq \|(A, \psi)\| \|(f, \varphi)\| \|\tilde{x}_e\| \qquad (3.4)$$

from the Inequality 3.4 we have

$$\|(A, \psi)^*(f, \varphi)\| \leq \|(A, \psi)\| \|(f, \varphi)\|$$

and consequently,

$$\|(A, \psi)^*\| \leq \|(A, \psi)\|$$

since $(A, \psi)$ is continuous it is bounded according to the Theorem 2. Since $(A, \psi)$ is bounded then $(A, \psi)^*$ is bounded and as a result it is continuous.

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