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K- *Operators*

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Abstract

In this paper, we introduce a new class of operators on a complex Hilbert space $\mathcal{H}$ which is called *K*-operators. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *K*-operators if $(TT^*)^k = k\,(TT^*)$, where $k \geq 2$ and T^* is the adjoint of the operator T.

We investigate some basic properties of such operators and study the relation between *K*-operators and some other well known classes of operators on $\mathcal{H}$.

Key word: *K*-operators , operators, adjoint operators .

1- Introduction

One of the important notions in applied mathematics and systems analysis is the operator theory investigation by obtaining a mathematical model, and then determining such properties as existence, uniqueness and regularity of solutions. Let $\mathcal{B}(\mathcal{H})$ denoted to the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called nilpotent operator if $T^n = 0$ [1], similar operator if there exists $S \in \mathcal{B}(\mathcal{H})$ such that $S = XTX^{-1}$, where X *and* X^{-1} are operators in $\mathcal{B}(\mathcal{H})$ [2], isometric operator if $T^*T = I$ [3] and unitary operator if $T^*T = TT^* = I$ [4].

2- *K*-operators

In this section, we shall study some properties which are applied of *K*-operators.

Definition (2.1): If $T \in \mathcal{B}(\mathcal{H})$, then T is called *K*-operators if $(TT^*)^k = k\,(TT^*)$ where $k \geq 2$ and T^* is the adjoint operator of T and we denoted of all the class *K*-operator by $[ok]$.

Example (2.2) : Let $T = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ operator on a Hilbert space $\mathbb{C}^2$ .Then

$$(TT^*)^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = 2(TT^*)$$

Therefore $T \in [2k]$.

Example (2.3): If $T = \begin{pmatrix} 3 & -2 \\ -1 & 0 \end{pmatrix}$ operator on a Hilbert space $\mathbb{C}^2$ .Then

$$(TT^*)^2 = \begin{pmatrix} 178 & -42 \\ -42 & 10 \end{pmatrix} \neq 2(TT^*) = \begin{pmatrix} 26 & -6 \\ -6 & 2 \end{pmatrix}$$

Thus $T \notin [2k]$.

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In the following theorem, we give some properties of this operators.

Theorem (2.4): If $T\epsilon[ok]$ and $T = T^*$,then :

(1) $T^{-1} \in [ok]$

(2) $T^* \in [ok]$

Proof: (1) since $T\epsilon[ok]$, then $(TT^*)^k = k(TT^*)$

$(T^*T)^k = k(T^*T)$ $[T = T^*]$

Taking inverse of two-sides $(T^{-1}T^{-1^*})^k = k(T^{-1}T^{-1^*})$

$\therefore\ T^{-1} \in [ok]$

(2) since $T \in [ok] \Rightarrow (TT^*)^k = k(TT^*)$

$\because\ [T = T^*] \Longrightarrow$ $(T^*T)^k = k(T^*T)$

Then $T^* \in [ok]$

The following examples show that If $S\ and\ T$ are K-operators , then not necessary $(S + T)\ and\ (S.T)$ are K-operators .

Example (2.5): If $S = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ are operators on a Hilbert space $\mathbb{C}^2$,then

$(SS^*)^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2(SS^*)$ and $(TT^*)^2 = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} = 2(TT^*)$

Then S and T are $[2k]$.

But $[(S + T)(S + T)^*]^2 = \begin{pmatrix} 29 & 12 \\ 12 & 5 \end{pmatrix} \neq 2[(S + T)(S + T)^*] = \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix}$

$\therefore\ (S + T) \notin [2k]$.

Theorem (2.6): If $S\ and\ T$ are commuting K-operators , then $(S + T)$ is K-operators .

Proof : since $S, T \in [ok]$, then there exists $k_1, k_2 \in K\ such\ that$

$(TT^*)^k = k_1(TT^*)$, $(SS^*)^k = k_2(SS^*)$ and $k_1k_2 = k$

$[(S + T)(S + T)^*]^k = [(S + T)(S^* + T^*)]^k = \left[(S^k + T^k)\left(S^{*^k} + T^{*^k}\right)\right]$

$= \left[S^k {S^*}^k + S^k\, {T^*}^k + T^k {S^*}^k + T^k {T^*}^k\right]$

$= (SS^*)^k + (ST^*)^k + (TS^*)^k + (TT^*)^k$

$= k_1 k_2\, [(SS^*)^k + (ST^*)^k + (TS^*)^k + (TT^*)^k]$

$= k\, [(S+T)(S+T)^*]$

Then $(S+T) \in [ok]$.

Example (2.7): Let $S = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ are operators on a Hilbert space $\mathbb{C}^2$ .Then S and T

are $[2k]$, but $[(ST)(ST)^*]^2 = \begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix} \neq 2\, [(ST)(ST)^*] = \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix}$

$(ST) \notin [2k]$.

Theorem (2.8): If S *and* T are K-operators such that $ST = TS$, then

(ST) is K-operators .

Proof : since $S, T \in [ok]$, then $(TT^*)^k = k(TT^*)$ and $(SS^*)^k = k(SS^*)$

and $k^2 = k$

$[(ST)(ST)^*]^k = [S(TT^*)S^*]^k = \left[S^k (TT^*)^k {S^*}^k\right]$

$=\left[k\, S^k (TT^*)\, {S^*}^k\right] = [k(\, S^k T)(T^*\, {S^*}^k)] = [k(TS^k)(\, S^k T)^*]$

$= [k(TS^k)(TS^k)^*] = \left[k\, T\left(S^k {S^*}^k\right) T^*\right] = [k\, T(SS^*)^k\, T^*]$

$= [kTk(SS^*)T^*] = [k^2(TS)(S^*T^*)] = k[ST)(ST)^*]$

Thus $(ST) \in [ok]$.

Remark (2.9): The class of $[2k]$ and $[3k]$ are independent as the following examples:

Example (2.10): If $T = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}$ is an operator on a Hilbert space $\mathbb{C}^2$ .Then

$(TT^*)^2 = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} = 2(TT^*)$ .Thus $T \in [2k]$. But

$(TT^*)^3 = \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix} \neq 3(TT^*) = \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}$ .Thus $T \notin [3k]$.

Example (2.11): If $T = \begin{pmatrix} \sqrt[4]{3} & 0 \\ 0 & 0 \end{pmatrix}$ is an operator on a Hilbert space $\mathbb{C}^2$ .Then

$(TT^*)^3 = \begin{pmatrix} 3\sqrt[2]{3} & 0 \\ 0 & 0 \end{pmatrix} = 3(TT^*)$ .Thus $T \in [3k]$. But

$(TT^*)^2 = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \neq 2(TT^*) = \begin{pmatrix} 2\sqrt[2]{3} & 0 \\ 0 & 0 \end{pmatrix}$ .Thus $T \notin [2k]$.

Proposition (2.12): If $T \in [ok]$ such that T commute with T^* , then $T^2 \in [ok]$.

Proof: $\left(T^2 T^{2^*}\right)^k = (TTT^*T^*)^k = \left(T^k T^k \, T^{*^k} \, T^{*^k}\right)$

$= T^k (TT^*)^k \, T^{*^k} = k \, T^k (TT^*) T^{*^k} = k \, T (T^k \, T^{*^k}) T^*$

$= k \, T (TT^*)^k \, T^* = k^2 \, (TTT^*T^*) = k (T^2 T^{2^*})$

Example(2.13): Let $T = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ operator on a Hilbert space $\mathbb{C}^2$.Then $T \in [2k]$,

but $(T^2 T^{2^*})^2 = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix} \neq 2\left(T^2 T^{2^*}\right) = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$

Thus $T^2 \notin [2k]$.

Remark (2.14): The class of $[ok]$ and isometric operator are independent as the following examples :

Example (2.15): Let U be the unilateral shift operator on ℓ_2 ;

(i.e. $U(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, x_4 \ \dots)$).Then

$(U^*U) = I$. Then U is isometric operator. But

$(UU^*)^k \neq k(UU^*)$. Thus $U \notin [ok]$.

In example (2.12) $T \in [2k]$ but $T^*T = 2I \neq I \Longrightarrow T$ is not isomeric operator.

proposition (2.16): If T is nilpotent operator and $T^* = T^k$, then $T \in [ok]$.

Proof: since $T^k = 0$, then $(TT^*)^k = 0 = k(TT^*)$,$T \in [ok]$.

Remark (2.17): If $T \in [ok]$ and S is similar of T, then not necessary $S \in [ok]$.

For example : If $T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ operator on a Hilbert space $\mathbb{C}^2$, then

$(TT^*)^2 = 2(TT^*) = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \Rightarrow T \in [2k]$, but if $X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ *and*

$X^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ operators on $\mathbb{C}^2 \Rightarrow S$ is similar of T (*i.e.* $S = XTX^{-1}$)

Thus $S = \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix} \Rightarrow (SS^*)^2 \neq 2(SS^*) = \begin{pmatrix} 58 & 48 \\ 48 & 40 \end{pmatrix}$ $S \notin [2k]$.

Corollary (2.18): Unitary operators and $[ok]$ are independent as we seen in example (2.11): $T \in [3k]$ but T is not unitary operator ; and if $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Operator on $\mathbb{C}^2$., then $TT^* = T^*T = I \Rightarrow T$ is unitary operator ; but

$(TT^*)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq 2(TT^*) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow T \notin [2k]$

Example (2.19): If $T \in [2k]$ when $T = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, then $T + I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Therefore $((T+I)(T+I)^*)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq 2\big((T+I)(T+I)^*\big)$

We conclude that $(T + I) \notin [2k]$.

الملخص

الهدف من هذا البحث هو تقديم نوع جديد من المؤثرات المعرفة على فضاء هلبرت الذي أطلقنا عليه اسم المؤثر -K . المؤثر $T \in \mathcal{B}(\mathcal{H})$ يسمى المؤثر – K اذا كان $(TT^*)^k = k\,(TT^*)$ حيث $k \geq 2$ و T^* هو المؤثر المرافق (المصاحب) للمؤثر T . سوف نقدم في هذا البحث بعض الخواص الأساسية لهذا المؤثر وندرس العلاقة بين المؤثر -K وبعض الأنواع الأخرى من المؤثرات.

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