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Development of Riemann Functional in a Banach Space

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ABSTRACT

In this paper, I investigate the Riemann integral when the range of the function on which the integral is defined is a Banach space and I tried to apply the known theories to other integrals when the range is a Banach space, and I found that they are fulfilled in the Riemann integral when the conditions are met in the function for us to integrate it by Riemann.

1.Introduction

The Riemann integral consider the first integral To provide an accurate definition to a function over an interval, this is what some consider to be a major weakness of this integral because it is limited to the field of the function being a closed interval, and this is what necessitated the need for the emergence of other integrals such as the Lebesgue integral and others. Riemann integration has been applied to many applications such as calculus, as it is a specific integration to calculate such applications and mathematical operations.

In [3] He introduced a detailed definition to what mean that the bounded function on a closed interval to be Riemann integral, in [7] dealt with the same definition but extensively. In [2] We note that the researcher put another definition for functions whose range is a Banach space and this new concept still maintains the properties of the classical Riemann integral.

In this paper, I show that it is possible to apply the Riemann integral to any other integral with the same properties without the need to define the measure in a single case when the function is bounded, meaning its domain is a closed interval.

2.Fundamental Concepts

In this section the important and basic concepts are given to expression all the results that need it later on.

Definition 2.1,[2]:

(1) When the norm of $\mathfrak{x}$ in a Banach space $\mathfrak{B}$ is $\|\mathfrak{x}\|$,The cloed unit ball of $\mathfrak{B}$ over a field $\mathbb{L} = \mathbb{R}\ or\ \mathbb{C}$ will denoted by $\mathcal{C}_{\mathfrak{B}} = \{\mathfrak{x} \in \mathfrak{B}: \|\mathfrak{x}\| \leq 1\}$.

(2) The dual of $\mathfrak{B}$ which denoted by $\mathfrak{B}^*: \mathfrak{B} \to \mathbb{L}$ be a continuous and Banach space and the norm of any element $\mathfrak{x}^*$in $\mathfrak{B}^*$ defined by $\|\mathfrak{x}^*\|_{\mathfrak{B}^*} = \sup_{\|\mathfrak{x}\| \leq 1} |\ \mathfrak{x}^*(\mathfrak{x})|$.

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Remark 2.2 [1]:

(1) $\mathcal{R}e\,\langle \mathfrak{x}, \mathfrak{x}^* \rangle \leq \mathfrak{c} < d \leq \mathcal{R}e\langle \mathfrak{x}, \mathcal{y}^* \rangle$, where $\mathfrak{x}$ be an elemen on a convex closed set in $\mathfrak{B}$, $\mathcal{y}$ in a convex compact set $\mathcal{T}$ in $\mathfrak{B}$ and real number $\mathfrak{c} < d$.

(2) For any element $\mathcal{y}^*$ belong to $\mathfrak{F}^*$ where $\mathfrak{F}$ be a closed in $\mathfrak{B}$ there exist $\mathfrak{x}^*$ in $\mathfrak{B}^*$ such that $\mathfrak{x}^*|_{\mathfrak{F}} = \mathcal{y}^*$ and $\|\mathfrak{x}^*\|\|\mathcal{y}^*\|$ then for any $\mathfrak{x}$ in $\mathfrak{B}$ we get $\|\mathfrak{x}\|_{\mathfrak{B}^*} = \sup_{\|\mathfrak{x}^*\| \leq 1} |\,\mathfrak{x}^*(\mathfrak{x})|$.

Definition 2.3 [7]:

The Riemann integral on the continuous then bounded function g on [c,d] is a real number under the area of graph $\mathcal{h} = g(\mathfrak{n})$ for $\mathfrak{c} \leq \mathfrak{n} \leq d$, when we integration g on $[\mathfrak{c}, \mathfrak{n}]$ where $\mathfrak{n}$ be an endpoint we will get a function on $\mathfrak{n}$

Definition 2.4, [3]:

The upper and lower Riemann integral on bounded function $g: [c, d] \to \mathfrak{B}$ which is continuous on $\mathbb{I} = [c, d]$ with supremum and infimum of g are well defned and exist then we will defined the upper and lower Riemann integral on g by $\mathcal{U}(g) = \inf_{\mathbb{I}_n} \sum_{i=1}^{n} \sup_{\mathbb{I}_i} |\mathbb{I}_i|$, where $\mathbb{I}_i$ be the endpoints of the interval $[c, d]$, $\mathcal{L}(g) = \sup_{\mathbb{I}_n} \sum_{i=1}^{n} \inf_{\mathbb{I}_i} |\mathbb{I}_i|$.

We will denoted to the upper Riemann integral by $\mathcal{U}(g) = \overline{\mathcal{R}_c^d}(g)$ and the lower Riemann integral on g as $\underline{\mathcal{R}_c^d}(g)$.

Remarks 2.5, [5]:

If $g: [c, d] \to \mathcal{B}$ be a bounded function on a compact interval $[c, d]$ be a Riemann integrable if $\overline{\mathcal{R}_c^d}(g) = \underline{\mathcal{R}_c^d}(g)$.

Theorem 2.6,[7]:

Let $g: [c, d] \to \mathfrak{B}$ be a Riemann integrable, then we have the following properties,

(1) $\mathcal{R}_c^d(\imath g) = \imath \mathcal{R}_c^d(g)$, where $\imath \in \mathfrak{B}$.

(2) If $g \leq h$ then $\mathcal{R}_c^d(g) \leq \mathcal{R}_c^d(h)$

Theorem 2.7,[4]:

Let $g: [c, d] \to \mathfrak{B}$ be a Riemann integrable and $\imath \in \mathfrak{B}$ then $\imath g$ be a Riemann integrable and $\mathcal{R}_c^d(\imath g) = \imath \mathcal{R}_c^d(g)$.

Theorem 2.8,[1]:

If $g, h: [c, d] \to \mathfrak{B}$ are Riemann integrable then $g + h$ be a Riemann integrable, and $\mathcal{R}_c^d(g + h) = \mathcal{R}_c^d(g) + \mathcal{R}_c^d(h)$.

Theorem 2.9,[2]:

Let $g: [c,d] \to \mathfrak{B}$ be a Riemann integrable, then $|g|$ be a Riemann integrable and $|\mathcal{R}_c^d(g)| \leq \mathcal{R}_c^d(|g|)$.

Theorem 2.10,[1]:

Let $g: [c,d] \to \mathfrak{B}$ be a continuous function on $[c,d]$, and $c < \imath < d$ then g be a Riemann integrable if and only if its Riemann integrable on $[c,\imath][\imath,d]$ then $\mathcal{R}_c^d(g) = \mathcal{R}_c^{\imath}(g) + \mathcal{R}_{\imath}^d(g)\, \mathcal{R}_c^d(g)$.

Remark 2.11,[6,5]:

(1) The symbol $\mathcal{C}_{[c,d]}$ denotes the indicator or characteristic function of $[c,d]$ such that

$$\mathcal{C}_{[c,d]}(\mathfrak{y}) = \begin{cases} 1, & \mathfrak{y} \in [c,d] \\ 0, & \mathfrak{y} \notin [c,d] \end{cases}$$

(2) If $g: [c,d] \to \mathfrak{B}$ be a Riemann integrable. We will denoted to the quivalence class of g by $[g]$ such that $[g] = \{h: \mathcal{R}_c^d(|g-h|) = 0\}$, where g be a continuous function.

(3) We will denoted to the norm on g by $\|g\|$ such that $\|g\| = \mathcal{R}_c^d(\left|\mathcal{C}_{[c,d]}\right|)$.

3. Main results

Definition 3.1:

A function $g: [c,d] \to \mathfrak{B}$ is a Riemann integrable on $[c,d]$ if there is a sequence of bounded functions $g_n: [c,d] \to \mathfrak{B}$ such that $\lim_{n\to\infty} \mathcal{R}_c^d(\|g_n - g\|) = 0$ if and only if $\lim_{n\to\infty} g_n = g$ a.e.

Remark 3.2:

(1) From the above definition, If g is Riemann integrable and $g = \sum_{\imath=1}^{n} \mathcal{C}_{\mathbb{I}_\imath} \varkappa_\imath$ then $\mathcal{R}_c^d(g) = \sum_{\imath=1}^{n} \mathcal{R}_c^d(\mathcal{C}_{\mathbb{I}_\imath}) \varkappa_\imath$. And the limit $\mathcal{R}_c^d(g) = \lim_{n\to\infty} \mathcal{R}_c^d(g_n)$.

(2) If $\sigma: [c,d] \to \mathfrak{B}$ be a Riemann integrable on$[c,d]$, then for all x^* in $\mathfrak{B}^*$ we conclude $\langle \mathcal{R}_c^d(\sigma), \varkappa^* \rangle = \mathcal{R}_c^d \langle \sigma, \varkappa^* \rangle$.

Theorem 3.3:

A bounded function $g: [c,d] \to \mathfrak{B}$ is Riemann integrable if and only if $\mathcal{R}_c^d(\|g\|) < \infty$ then we have $\|\mathcal{R}_c^d(g)\| \leq \mathcal{R}_c^d(\|g\|)$.

Proof:

Assume that g is a Riemann integrable then for $n > o$ we have $\mathcal{R}_c^d(\|g\|) \leq \mathcal{R}_c^d(\|g - g_n\|) + \mathcal{R}_c^d(\|g\|) < \infty$. In the other side let g be a continuous function such that $\mathcal{R}_c^d(\|g\|) < \infty$, let h_n be a continuous function such that $\lim_{n\to\infty} h_n = g$ a.e. and define $g_n = 1_{\{\|h_n\| \leq 2\|g\|\}} h_n$, since g_n is continuous then we have $\lim_{n\to\infty} g_n = g$ a.e. then $\|g_n\| \leq 2\|g\|$ pointwise, then $\lim_{n\to\infty} \mathcal{R}_c^d(\|g_n - g\|) = 0$,

Remark 3.4:

(1) If $g: [c, d] \to \mathfrak{B}$ be a Riemann integrable then for every partition $\mathfrak{N}$ of $[c, d]$ and for all $\mathfrak{z} \subseteq \mathfrak{N}$ the truncation $1_{\mathfrak{z}}g: [c, d] \to \mathfrak{B}$ is a Riemann integrable, and $g|_{\mathfrak{z}}: \mathfrak{N} \to \mathfrak{B}$ is Riemann integrable and we have $\mathcal{R}_c^d(1_{\mathfrak{z}}g) = \mathcal{R}_{\mathfrak{z}}(g|_{\mathfrak{z}})$, all inegrals denoted by $\mathcal{R}_{\mathfrak{z}}(g)$.

(2) If $\mathfrak{V} \subseteq \mathfrak{B}$ denoted $conv(\mathfrak{V})$ to the convex hull of $\mathfrak{V}$ such that $conv(\mathfrak{V}) = \{\sum_{i=1}^{n} x_i \mathfrak{H}_i : x_i \in \mathfrak{V}, \mathfrak{H}_i \geq 0, \sum_{i=1}^{n} \mathfrak{H}_i = 1\}$.

Theorem 3.5:

If $g: [c, d] \to \mathfrak{B}$ be a Riemann integrable and $\mathcal{R}_c^d(1_{\mathfrak{z}}) = 1$, then $\mathcal{R}_{\mathfrak{z}}(g) \in conv\{g(\mathfrak{y}): \mathfrak{y} \subseteq \mathfrak{f}\}$.

Proof :

Let $\mathfrak{w} \in \mathfrak{B}$ be a strictly separated from $\mathfrak{V} \subseteq \mathfrak{B}$, $\mathfrak{w}^* \in \mathfrak{B}^*$ if there is any element $\mathfrak{k}>0$ so $|\mathcal{R}e\langle \mathfrak{w}, \mathfrak{w}^* \rangle - \mathcal{R}e\langle u, \mathfrak{w}^* \rangle| \geq \mathfrak{k}$, $u \in \mathfrak{V}$ where $\mathfrak{V}$ is a strictly separated from $\mathfrak{V}$

If $\mathfrak{V}$ is convex and $\mathfrak{w} \notin \overline{\mathfrak{V}}$ then there is $\mathfrak{w}^* \in \mathfrak{B}^*$ be a strictly separates $\mathfrak{w}$ from $\mathfrak{V}$.

For $\mathfrak{w}^* \in \mathfrak{B}^*$, suppose $\mathcal{U}(\mathfrak{w}^*) = inf\{\mathcal{R}e\langle g(\mathfrak{y}), \mathfrak{w}^* \rangle : \mathfrak{y} \subseteq \mathfrak{f}\} = -\infty$, $\mathcal{L}(\mathfrak{w}^*) = sup\{\mathcal{R}e\langle g(\mathfrak{y}), \mathfrak{w}^* \rangle : \mathfrak{y} \subseteq \mathfrak{f}\} = \infty$, since $\mathcal{R}_c^d(1_{\mathfrak{z}}) = 1, \mathcal{R}e\langle \mathcal{R}_{\mathfrak{f}}(g), \mathfrak{w}^* \rangle = \mathcal{R}e\langle g, \mathfrak{w}^* \rangle \in [\mathcal{U}(\mathfrak{w}^*), \mathcal{L}(\mathfrak{w}^*)]$>

Theorem 3.6:

Let $g_n: [c, d] \to \mathfrak{B}$ be a sequences of bounded functions, each of them is a Riemann integral. Suppose that there is a functon $g: [c, d] \to \mathfrak{B}$ and a Riemann integral function h: $[c, d] \to \mathbb{R}$ such that:

(1) $\lim_{n\to\infty} g_n = g$ a.e.

(2) $\|g_n\| \leq |g|$ a.e.

then g is a Riemann integrable and we have $\lim_{n\to\infty} \mathcal{R}_c^d(\|g_n - g\|) = 0$ then we get $\lim_{n\to\infty} \mathcal{R}_c^d(g_n) = \mathcal{R}_c^d(g)$.

Proof:

We have $\|g_n - g\| \leq 2|g|$ a.e. by scalar dominated convergence theorem and by the definition of Riemann integral let $K: \mathfrak{B} \to \mathfrak{N}$ be a bounded linear operator and $\mathfrak{N}$ be a banach space which is distinct from $\mathfrak{B}$, then $kg: [a, b] \to \mathfrak{B}$ is bounded linear operator Riemann integrable and $K\mathcal{R}_c^d(g) = \mathcal{R}_c^d(Kg)$.

Define a linear operator K on a linear subspace $\mathcal{N}$(K) of $\mathfrak{B}$ which have a valued in a nother banach space $\mathfrak{O}$, if its have the graph $\mathfrak{G}$(K)= $\{(\mathfrak{v}, K\mathfrak{v}): \mathfrak{v} \in \mathcal{N}(K)\}$ then we said to be closed since $\mathfrak{G}$(K) be a closed subspace of $\mathfrak{B}\times\mathfrak{O}$, if $\mathfrak{D}$ is closed then $\mathcal{N}(K)$be a banach space with respect to $\|\mathfrak{v}\|_{\mathcal{N}(K)} = \|\mathfrak{v}\| + \|K\mathfrak{v}\|$ and K be a bounded operator and K:$\mathfrak{G}$(K)$\to\mathfrak{B}$.

Finally we conclude that if $K: \mathfrak{B} \to \mathfrak{O}$ is closed operator with domain $\mathcal{N}(K) = \mathfrak{B}$, then K is bounded.

Theorem 3.7:

Let $g: [a, b] \to \mathfrak{B}$ be a Riemann interable and K be a closed linear operator have a domain $\mathcal{N}(K)$ in $\mathfrak{B}$ with values in a banach space $\mathfrak{D}$. If the values of g in $\mathcal{N}(K)$ a.e. and a.e. defined function $Kg: [a, b] \to \mathfrak{D}$ is Riemann interable. Then $\mathcal{R}_c^d(g) \in N(K)$.

Proof:

Let $\mathfrak{B}_1, \mathfrak{B}_2$ are banach space and $g_1: [a, b] \to \mathfrak{B}_1, g_2: [a, b] \to \mathfrak{B}_2$ are Riemann interable , $g = (g_1, g_2): [a, b] \to \mathfrak{B}_1 \times \mathfrak{B}_2$ is Riemann interable and $\mathcal{R}_c^d(g) = (\mathcal{R}_c^d(g_1), \mathcal{R}_c^d(g_2))$. By the previous note the function $h: [a, b] \to \mathfrak{B} \times \mathfrak{D}$, $h(\mathfrak{w}) = (g(\mathfrak{w}), Kg(\mathfrak{w}))$, is Riemann interable, h has a values in $\mathfrak{G}$(K), we have $\mathcal{R}_c^d(h(\mathfrak{w})d\mathfrak{w}) = \left(\mathcal{R}_c^d(g(\mathfrak{w})d\mathfrak{w}), \mathcal{R}_c^d(Kg(\mathfrak{w})d\mathfrak{w})\right)$.

Theorem 3.8:

If $1 < \wp < \infty$, $\wp$ is fixed and $g: [c, d] \to \mathfrak{B}$ satisfies $\langle g, \mathfrak{s}^* \rangle \in \mathcal{L}^{\wp}([a, b])$ for all $\mathfrak{s}^* \in \mathfrak{B}^*$, there exist a unique $\mathfrak{s}_g \in \mathfrak{B}^*$ such that $\langle \mathfrak{s}_g, \mathfrak{s}^* \rangle = \mathcal{R}_c^d(\langle g, \mathfrak{s}^* \rangle)$.

Proof:

Let the linear mapping $\mathcal{M}: \mathfrak{B}^* \to \mathcal{L}^{\wp}([a, b]), \mathcal{M}\mathfrak{s}^* = \langle g, \mathfrak{s}^* \rangle$ which is bounded graph since its closed.

Put $\mathbb{I}_n = \{\|\mathfrak{x}_i\| \le n, i = 0,1,2,..,n\}$ then by 3.3, $\mathcal{R}_{\mathbb{I}_n}(g)$ exists as a Riemann interable in $\mathfrak{B}$.

For any $\mathfrak{s}^* \in \mathfrak{B}^*$ and $n \ge \mathfrak{y}$ we have,

$$\left|\langle \mathcal{R}_{\mathbb{I}_n/\mathbb{I}_\mathfrak{y}}(g(\mathfrak{s}), \mathfrak{s}^* \rangle\right| \le \left(\mathcal{R}_c^d\left(\mathcal{C}_{\mathbb{I}_n/\mathbb{I}_\mathfrak{y}}\right)\right)^{1/g} \left(\mathcal{R}_c^d(|\langle g, \mathfrak{s}^* \rangle|^{\wp})(\mathfrak{s})\right)^{1/\wp} \le \left(\mathcal{R}_c^d\left(\mathcal{C}_{\mathbb{I}_n/\mathbb{I}_\mathfrak{y}}\right)\right)^{1/g} \|\mathcal{M}\|\|\mathfrak{s}^*\|.$$

\By taking the supremum over all $\mathfrak{s}^* \in \mathfrak{B}^*$ and since $\|\mathfrak{s}^*\| \le 1$ we get

$\lim_{\mathfrak{y},n\to\infty} sup \left\|\mathcal{R}_{\mathbb{I}_n/\mathbb{I}_\mathfrak{y}}(g)\right\| \le \lim_{\mathfrak{y},n\to\infty}\left(\left(\mathcal{R}_c^d\left(\mathcal{C}_{\mathbb{I}_n/\mathbb{I}_\mathfrak{y}}\right)\right)^{1/g}\right)\|\mathcal{M}\| = 0$, the limit of $\mathfrak{s}_g = \lim_{n\to\infty}(\mathcal{R}_{\mathbb{I}_n}(g))$ exist in $\mathfrak{B}$, and $\langle \mathfrak{s}_g, \mathfrak{s}^* \rangle = \lim_{n\to\infty}(\mathcal{R}_{\mathbb{I}_n}(\langle g, \mathfrak{s}^* \rangle) = \mathcal{R}_c^d(\langle g, \mathfrak{s}^* \rangle)$ for all $\mathfrak{s}^* \in \mathfrak{B}^*$.

Definition (The Riemann space $\mathcal{L}^{\wp}([c, d];\ \mathfrak{B})$), 3.9:

(1) For $1 \le \wp < \infty$ we define $\mathcal{L}^{\wp}([c, d];\ \mathfrak{B})$ as a linear space of equivalence classes of a bounded function $g: [c, d] \to \mathfrak{B}$ such that $\mathcal{R}_c^d(\|g\|^{\mathfrak{p}}) < \infty$ having the norm $\|g\|_{\mathcal{L}^{\wp}([c,d];\ \mathfrak{B})} = \left(\mathcal{R}_c^d(\|g\|^{\mathfrak{p}})\right)^{1/\wp}$.

(2) The space $\mathcal{L}^{\infty}([c, d];\ \mathfrak{B})$ of all equivalence class of a continuous functions $g: [c, d] \to \mathfrak{B}$ such that there is an element $\mathfrak{o} \ge 0$ such that $\mathcal{R}_c^d(\mathcal{C}_{\|g\|>\mathfrak{o}}) = 0$ with the norm $\|g\|_{\mathcal{L}^{\infty}([c,d];\ \mathfrak{B})} = \inf\{\mathcal{R}_c^d(\mathcal{C}_{\|g\|>\mathfrak{o}}) = 0\}$.

Remark 3.10:

(1) The space $\mathcal{L}^{\wp}([c,d];\ \mathfrak{B})$, $\mathcal{L}^{\infty}([c,d];\ \mathfrak{B})$ be a Banach spaces.
(2) The elements of $\mathcal{L}^{1}([c,d];\ \mathfrak{B})$ be an equivalence classes of Riemann integrable functions.

4. Conclusion

The result of this paper initiated was the definition of the Riemann integral when its values in a Banach space, our results show us that the previously known Riemann properties have not changed also show that it is possible to apply the Riemann integral to any other integral defined in a Banach space with the same properties in a single case when the domain of our function is a closed interval.

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