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On F-regular, F-normal and F-connected Topological Space

Authors Names	ABSTRACT
[a]Mustafa Mohammed Turki [b]Raad Aziz Hussian **Publication date**: 22 /11 /2024 **Keywords:** FR_space, FT_3_space, FN_space, FT_4_space F_connected , F′_connected, F′′_connected , F_disconnected Spaces	In this paper, we introduce a new definitions of Separation Axiom Using definitions F_open, F_closed FT_0_space, FT_1_ space and FT_2_ space which we called R_space, FT_3_ space, FN_space , FT_4_space, and definitions F_separation, F_ connected, F′_connected, F′′_connected, F_ disconnected Spaces. Then we set the characteristics for each definition and demonstrated the interconnection between the definitions. We also demonstrated the genetic and partial characteristics in each definition.

1. Introduction

In the topological space X, a subset B of a space X is said to be a regularly-closed, called also closed domain if $B = cl(int(B))$. A subset B of X is said to be a regularly-open, called also open domain if $B = int\big(cl(B)\big)$, An open (resp., closed) subset B of a topological space (X, T) is called F_open (resp.,F_closed) set if $cl(B)\backslash B$ (resp., $B\backslash int(B)$) is finite set [6].They introduce a new type of semi-open sets which they call S_g-open sets[2]. An open (resp., closed) subset B of a topological space (X, T) is called $C-$open (resp.,C-closed) set if $cl(B)\backslash B$ (resp., $B\backslash int(B)$) is a countable set[3], they introduce a new definitions of Separation Axiom which we call FT_0 _space, FT_1 _space, FT_2_space[4], In section 3, the first paragraph We defined FR_ space, FT_3_space and we have developed theorems showing the relationship between FR_space, FT_3_space and FT_0 _space, FT_1 _space, FT_2_space, in the second part we defined FN_space, and FT_4_space, in the third part we defined F_separation, F_connected, F′_connected, F′′_connected, F_ disconnected Spaces and we have developed theorems that show the equivalence between the previous definition, We give some examples related to the separation axioms and I have proved theorems that refer to the topics that I defined in this research proved some topological and genetic characteristics.

2. Preliminaries

Definition(2.1)[6]: Let(X,τ) be topological space and A open subset of (X,τ), then the $cl(A)\backslash A$ is finite set and is denoted by F_open .

Definition(2.2)[6]: Let(X,τ) be topological space and A be closed subset of (X,τ), then the $A\backslash int(A)$ is finite set and is denoted by F_closed .

Remark(2.3)[6]: Let (X,τ) is topological space , and $U \subseteq X$.

(1)Let U is F_ open, the complement of U is F_closed .

[a,b]Department of Mathematics, College of Sciencen, University of Al-Qadisiyah, Diwaniyah, Iraq.
[a]Email: mustafa902m@gmail.com, [b]*Email: raad.hussain@qu.edu.iq*.

(2) Let U is F_closed, the complement of U is F_open .

Definition(2.4)[6]: $(X, \tau)$is a topological space, a point in X, a F_open nieghbourhood of X is a V F_open subset of X, which is containing a .

Theorem(2.5)[6]: A topological space(X, τ), then

(i) every union finite F_closed subset of X is F_closed .

(ii) every union finite F_open subset of X is F_open.

(iii) every intersection finite F_closed subset of X is F_closed.

Definition(2.6)[6]: Let (X, τ)be a topological space, and $V \subseteq X$ the intersection of all F_closed containing V is called F_closure, denoted by $\overline{V}^F$.

Theorem(2.7)[6]: Let A be a subset of the topological space, (X,τ) then $A \subseteq \overline{A} \subseteq \overline{A}^F$.

Corollary(2.8)[6]: If U is F_open set and $U \cap V = \emptyset$, then $U \cap \overline{V}^F = \emptyset$ In particular, if U and V are disjoint F _open set then, $U \cap \overline{V}^F = \emptyset = \overline{(U)}^F \cap V$.

Definition(2.9)[6]: Let (X, τ)be a topological space, and $V \subseteq X$, A point $z \in X$ is called F_limit points of V if and only if for any F_open set U containing x ,we have $(U \setminus \{z\}) \cap V \neq \emptyset$.

Remark(2.10)[6]: The set of all F_limit points of V is called the F_derived set and denoted by $d_F(K)$.

Theorem(2.11)[6]: If (X, τ) a topological space, and $H, U \subseteq X$, Then .

(i)$d(H) \subset d_F(H)$, $d(H)$ is the derived set of H .

(ii) $H \subseteq U$, then $d_F(H) \subseteq d_F(U)$.

(iii) $d_F(H) \cup d_F(U) = d_F(H \cup U)$ and $d_F(H \cap U) \subset d_F(H) \cap d_F(U)$.

Theorem(2.12)[6]: Let (X, τ)be a topological space, and $H, U \subseteq X$,Then .

(i) $\overline{(\emptyset)}^F = \emptyset$.

(ii) $H \subseteq \overline{H}^F$.

(iii) If $H \subseteq U$, then $\overline{H}^F \subseteq \overline{U}^F$.

(iv) If $\overline{(H \cup U)}^F = \overline{(H)}^F \cup \overline{(U)}^F$) .

(v) $\overline{\overline{H}^F}^F = \overline{H}^F$.

Definition(2.13)[6]: $g: (X, \tau) \longrightarrow (Y, \grave{\tau})$a function g is called F_continuous if $g^{-1}(H)$ is F_ open set in X for every open set H in Y.

Definition(2.14)[6]: $g: (X, \tau) \longrightarrow (Y, \grave{\tau})$a function g is called F_open if g(H) is a F_open set in Y for every open sets H in X.

Definition(2.15)[6]: $g: (X, \tau) \longrightarrow (Y, \grave{\tau})$ a function g is called F_closed if g(H) is a F_closed set in Y for every closed sets H in X.

Definition(2.16)[6]: $g: (X, T) \rightarrow (Y, \grave{\tau})$ a function g is called F_ hmoeomrphism if and only if h and h^{-1} are F_continuous, onto and one to one .

Theorem(2.17)[4]: Let (Y, T_y) be F_open subspace of (Y, T) if U F_open set in X then $(U \cap Y)$ F_open set in Y

Definition(2.18)[4]: If (X, τ) be a topological space, then X is called $FT_{0\_}$ space if and only for each $x, y \in X$ such that $x \neq y$ and there exists V is F_open set, $[x \in V$ and $y \notin V]$or$[x \notin V$and $y \in V]$.

Definition(2.19)[4]: Let (X, τ)be topological space is defined FT_1_space if and only if for each $x, y \in X$ such that $x \neq y$, there exists U, V is F_open set such that, $[x \in U \wedge y \notin U$ and $y \in V \wedge x \notin V]$

Definition(2.20)[4]: Let (X, τ)topological space is called a FT_2_space if for each pair distinct points $a, b \in X$, the exist F_ open sets U, V and $a \neq b$ such that $[a \in U , b \in V$, and $U \cap V = \emptyset]$.

Definition(2.21)[1]: Let (X, τ) be a topological space and $A, B \subseteq X$ such that $A \neq \emptyset$, $B \neq \emptyset$ and $E \subseteq X$, then we said that A, B form a separation for E if

1) $E = A \cup B$ 2) $\overline{A} \cap B = A \cap \overline{B} = \emptyset$

3. The Main Results

3.1 F_ Regular Space.

Definition(3.1.1): Let (X, τ) be a topological space, then the space$(X, \tau)$is called a F_regular space if and only if for each F_closed set $G \subset X$ and each point $x \notin G$, there exist F_open sets Uand V such that $x \in U, G \subset V$, and $U \cap V = \emptyset$ and denoted by FR_space.

Lemma(3.1.2): Every FR_space is not $FT_{0\_}$ space. $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\} \{b, c\}\}$, $a \neq b$, there exists $U = \{a\}$ is F_open set. $a \in U, b \notin U, a \neq c$, and $c \notin U, b \neq c$, there is not exist U is F_open set. such that $[b \in U \wedge c \notin U] \vee [b \notin U \wedge c \in]$**,** so (X, τ) not FT_0_space.[FR_space $\nRightarrow$ FT_0_space]

Theorem(3.1.3): Let (X, τ) be FR_space then for each $x \in X$ and each F_open set W containing x, there exists an F_open set U such that $x \in U \subseteq \overline{U}^F \subseteq W$.

Proof: suppose that X is FR_space. Let $x \in W$ is F_open, $x \in W \Rightarrow x \notin X - W$ X is FR_space. There exists U, V are F_open, $U \cap V = \emptyset$, $(x \in U \wedge X - W \subseteq V)$ $U \cap V = \emptyset$, $U \subseteq X - V$ we have $U \subseteq X - V$ and $X - V \subseteq W$, Then $\overline{U}^F \subseteq \overline{X - V}^F$ [since $A \subseteq B \Rightarrow \overline{A}^F \subseteq \overline{B}^F$]

then $\overline{U}^F \subseteq X - V$ since [$X - V$ F-closed , $X - V = \overline{X - V}^F$], Then $\overline{U}^F \subseteq X - V$ }, Then$\overline{U}^F \subseteq X - V \wedge X - V \subseteq W$, Then $\overline{U}^F \subseteq W$, $x \in U \subseteq \overline{U}^F \subseteq W$ (since $A \subseteq \overline{A}^F$).

Theorem(3.1.4): The property of being a FR_space is a topological property.

Proof: Let X is FR _space, since h: $(X, \tau) \rightarrow (Y, \tau`)$there exists h one to one, onto and F_ continuous, h is F_open let $y \in Y$ and G, F_closed in Y ; $y \notin G$, h onto function there exist $x \in X$; $h(x) = y$, h F_ continuous $\Rightarrow$ h^{-1} (G) isF_ closed in X; $x \in h^{-1}$ (G) since ($h(x) = y \notin G$), X is FR_space there exists U, V is F_open set $U \cap V = \emptyset$, [$x \in U \wedge h^{-1}$ (G) $\subseteq V$], h F_open, then h (U), h(V) is F_ open in W, h one to one and h onto [$h(x) \in h(U) \wedge h(h^{-1}(G)] \subseteq h(V) \Rightarrow y \in h(U) \wedge G \subseteq h(V)$(since $y = h(x) \wedge h(h^{-1}(G) = G)$, $U \cap V = \emptyset \Rightarrow$ [$h(U) \cap h(V) = h(U \cap V) = h(\emptyset) = \emptyset$, so X is FR_space.

Example(3.1.5): Let $h: (R, D) \longrightarrow (R, \tau_{cof})$; $h(x) = X$ for each $x \in R$, h is F_continuous function since the domain (R, D) is discrete topology and his onto and in general (X, D) is FR_ space, but in general $(X, \tau_{cof})$is not FR_ space.

Definition(3.1.6): Let (X, τ) be a topological space, then the space (X, τ) is called FT_3_space if and only if its F_regular and FT_1_space, FT_3_space = FT_1_space + FR_space**.**

Example(3.1.7): The space (X, D) is FT_3_space since its FT_1_space and FR_space **.**

Example(3.1.8): Let $X = \{1,2,3\}, \tau = \{\emptyset, X, \{1\}, \{2,3\}\}$, The space (X, τ) is not FT_3_space since its not FT_1_space and FR_ space.

Example(3.1.9): The usual topological space(R, τ_u) is FT_3_space since its itsFT_1_space and FR_space.

Example(3.1.10): The space (X,I) X contains more then one element is not FT_3_space since its not FT_1_space and not FR_space.

Theorem(3.1.11): The property of being a FT_3_space is a hereditary property .

Proof :since the property FT_1_space and FR_space are a hereditary property then FT_3_space is a hereditary property **.**

Theorem(3.1.12): The property of being a FT_3_space is a topological property.

Proof: Since the property FT_1_space and FR_space are a topological property, then a FT_3_space is a topological property.

Remark(3.1.13): The F_continuous image of FT_3_space is not necessarily FT_3_space if $h: (X, \tau) \rightarrow (Y, \tau')$ is F_continuous ,onto function and X is FT_3_space then Y is not necessarily FT_3_space.

Example(3.1.14): Let $h:(R,D)\rightarrow(R,I)$; $h(x)=x$ for each $x\in R$, h is F_ continuous function since the domin (R,D) is discrete topology and h is onto and in general (X,D) is FT_3_spase $(R,D)$is FT_3_spase but (X,I) not FT_3_spase.

Theorem(3.1.15): Let (X,τ) be FT_3_spase then X is a FT_2_spase.

Proof: Suppose that X is a FT_3_spase.Let $x,y\in X$; $x\neq y$, X is a FT_1spase $\Rightarrow \{y\}$ F_closed set $\Rightarrow x\notin \{y\}$ since $x\neq y$ X is a FR_spase $\Rightarrow$ there exists U, V are F_open set, $U\cap V=\emptyset$, $(x\Rightarrow U\wedge \{y\}\subseteq V)\Rightarrow (x\in U\wedge y\notin V)$so $(X,\tau)$is a FT_2_spase.

Remark (3.1.16): From above the theorem we have

FT_3_space $\overrightarrow{\not\leftarrow}$ FT_2_spase $\overrightarrow{\not\leftarrow}$ FT_1_spase $\overrightarrow{\not\leftarrow}$ FT_0_spase

3.2 F_Normal space

Definition(3.2.1): Let (X,τ)be a topological space, then the space (X,τ) is called F_Normal space and denoted by FN_space if and only if for each pair of F_closed disjoint subsets G and E of X, there exist F_open sets U and V such that $G\subseteq U$, $E\subseteq V$ and $U\cap V=\emptyset$, $(G\subseteq U\wedge E\subseteq V)$.

Example(3.2.2): Let $X=\{1,2,3\}$ and $\tau=\{X,\emptyset,\{1\}\}$. Show that (X,τ) is FN _space, the family of F_closed sets $\{X,\emptyset,\{2,3\}\}$, take every two F_ closed sets there intersection is empty as follows take, $X,\emptyset$ is F_ closed; $X\cap\emptyset=\emptyset$, there exest $U=\emptyset\wedge V=X$ is F_ open; $U\cap V=\emptyset$, $(\emptyset\subseteq U\wedge X\subseteq V)$, take $\emptyset,\{2,3\}$ is F_ closed; $\{2,3\}\cap\emptyset=\emptyset$, there exest $U=\emptyset\wedge V=X$ is F_ open; $U\cap V=\emptyset$, $(\emptyset\subseteq U\wedge \{2,3\}\subseteq V)$. So (X,τ) is FN_ space.

Notes that this space not FT_0 _space, not FT_1 _ space, not FT_2_ space, not FR _ space, andnot FT_3 _ space.

Remark(3.2.3): $FN_{space}\nRightarrow FR_{space}$, then

$(FN\_space \nRightarrow FT_1\_spase)\wedge(FN\_space \nRightarrow FT_2\_spase)$

(F R _space $\nRightarrow$ FN _space) $\wedge$ (FT_1 _space $\nRightarrow$ FN _space) $\wedge$ (T_2_space $\nRightarrow$ N _space).

Remark(3.2.4): (FT_0_spase $\nRightarrow$ FN_space)$\wedge$ (FN_space $\nRightarrow$ FT_0_spase).

Example(3.2.5): The space (R,τ_{cof}) is FT_0_ space and not FN_ space, since there is twononempty disjoint F_closed sets, but there is no two nonempty disjoint F_open set. Notes that too $(R,\tau_{cof})$is FT_1_space and not FN _space.

Example(3.2.6): The space (R,I) is not FT_0 _ space, since R is the only F_ open set contains elements and its contains all elements. But (R, I) is RN_ space since the F_closed sets are

$G=R$ and $E=\emptyset$ only, and $R\cap\emptyset=\emptyset$ and the F_open sets are R and $\emptyset$ and $R\subseteq R$ and $\emptyset\subseteq\emptyset$.

Example(3.2.7): The space (X,D) is FN _space, since every sets her is F_ open and F_closed then: If V,E is F_closed, $V\cap E=\emptyset$, then V, E is F_open; $V\subseteq V\wedge E\subseteq E$.

Theorem(3.2.8): The property of being a N_space is a topological property.

Proof: Let$(X,\tau) \cong (Y,\tau`)$and suppose that X is FN_space, to prove Y is FN_space, there exist $h:(X,\tau) \rightarrow (Y,\tau`)$ h is one to one and his F_continuous and F_open, let G, E is F _closed in Y : $G \cap E = \emptyset$, h F_continuous $h^{-1}(G)$, $h^{-1}(E)$ F_closed in X and $h^{-1}(G) \cap h^{-1}(E) = h^{-1}(G \cap E) = h^{-1}(\emptyset) = \emptyset$ (the function h is F_ continuous if and only if the inverse image of every F_closed set in codomain is F_closed in domain), X is FN_space, there exist U, V are F_open $G \cap E = \emptyset$, $(h^{-1}(G) \subseteq U \wedge h^{-1}(E) \subseteq V$ h is F_open, h(U), h(V) is F_open in Y , h is onto, $h(h^{-1}(G)) \subseteq h(U) \wedge h(h^{-1}(E)) \subseteq h(V)$, $G \subseteq h(U) \wedge E \subseteq h(v)$, $G \subseteq h(U) \wedge E \subseteq h(V)$, $U \cap V = \emptyset$ then $h(U) \cap h(V) = h(U) \cap h(V) = h(U \cap V) = h(\emptyset) = \emptyset$, so Y is N – space.

Theorem(3.2.9): The space (X,τ) is F_normal (FN_space)then for each F_closed subset $G \subseteq X$ and F_open set W containing G, there exists an F_open set U such that $G \subseteq U \subseteq \overline{U}^F$.

Proof: Suppose that X is FN – space and $G \subseteq X$, Let W is F_open; $G \subseteq W \Longrightarrow G \cap X - W = \emptyset$

X is FN – space $\Longrightarrow$ there exists U, V are F_open , $U \cap V = \emptyset$; $(G \subseteq U \wedge X - W \subseteq V)$, $\Longrightarrow$

$X - V \subseteq W,\ U \cap V = \emptyset \Longrightarrow U \subseteq X - V \Longrightarrow \overline{U}^F \subseteq \overline{X - V}^F \Longrightarrow \overline{U}^F \subseteq X - V \Longrightarrow G \subseteq U \wedge U \subseteq \overline{U}^F \subseteq X - V \wedge X - V \subseteq W \Longrightarrow G \subseteq U \subseteq \overline{U}^F \subseteq W$.

Theorem(3.2.10): A F_closed subspace of FN_space is FN_space.

Proof : Let (X,τ)be FN_space and (W,τ_W) F_closed subspace of X, to prove (W,τ_W) FN_space Let G_W, E_W are F_closed sets in W; $G_W \cap E_W = \emptyset$,there exists G, E are F_closed, $G_W = G \cap E \wedge E_W = E \cap W, G \cap E = \emptyset$, since X is FN_space there exists U, V F_open $U \cap V = \emptyset, (G \subseteq U \wedge E \subseteq V)$then $U \cap W \wedge V \cap W$ F_open in W (By theorem 2.17) $(U \cap W) \cap (V \cap W) = (U \cap V) \cap W = \emptyset \cap W = \emptyset$,since $G_W = G \cap W$ then $G_W \subseteq G \wedge G_W \subseteq W$ then $G_W \subseteq U \wedge G_W \subseteq W \Rightarrow G_W \subseteq U \cap W$ since $E_W = E \cap W\ then\ E_W \subseteq E \wedge E_W \subseteq W\ then\ E_W \subseteq V \wedge E_W \subseteq W \Rightarrow E_W \subseteq V \cap W$, so (W,τ_W) FN_space.

Definition(3.2.11): Let (X,τ)be a topological space, Then the space $(X,\tau)$is called a FT_4_space if and only if F_normal and FT_1_space.

$$FT_4\_space = FT_1\_space + FN\_space$$

Example(3.2.12): Let $X = \{1,2,3\}$ and $\tau = \{X, \tau, \{1\}, \{2,3\}\}$ Then the space (X,τ) is not FT_4_space, since its FN _ space but not FT_1 _ space.

Remark(3.2.13): If X is finite space, then (X, D) is FT_4 – space iff $\tau = D$, (because if X is finite space, then its FT_1 _ space iff $\tau = D$ and if $\tau = D$, then X is FN _ space).

Example(3.2.14): The space (X, D) is FT_4 _ space, since its FT_1 _ space and N _ space.

Example(3.2.15): The space (X, I); X F_contains more than one element is not FT_4 _ space, since its not FT_1 _space.

Remark(3.2.16): The property of being a FT_4_space is not a hereditary property, since the$F$_normality is not a hereditary property.

Example(3.2.17): The space (X, τ_{cof}) is not FT_4 _ space, since its FT_1 _ space but not FN_space.

Theorem(3.2.18): The property of being FT_4_space is a topological property.

Proof: Since the property FT_1 _ space and FN _space are a topological property, Then FT_4 _ space is a topological property.

Theorem(3.2.19): A F_closed subspace of FT_4 _ space is FT_4 _ space.

Proof : Let (X, τ) FT_4 _ space and W F_closed set in X, to prove W is FT_4_ space, X is FT_1 _ space , W is FT_1 _ space (since $FT_1$is hereditary property), W is F_closed in X and X is FN_space , W is FN _ space (by theorem3.2.10) W is FT_4 _ space.

Theorem(3.2.20): Every FT_4 _space is FR _space.

Proof: Let (X, τ)be FT_4 _ space, X is FT_1 _ space and FN _ space , Let $x \in X$ and G F_ closed set in X ; $x \notin G$, $\{x\}$is F_close (since X is FT_1_ space then $\{x\}$ F_closed for each $x \in X\}$, $\{x\} \cap G = \emptyset, X$ is FN _space, there exists U, V F_open, $X \cap V = \emptyset$, $(\{x\} \subseteq U \wedge G \subseteq V), x \in U \wedge G \subseteq V$, X is FR_space.

Corollary(3.2.21): Every FT_4 _space is FT_3 _space.

Proof: Every FT_4 _ space is FR _space, every FT_4_space is FT_1_space and FN_space we have, X is FT_1_space FR_space, so X is FT_3_ space.

Remark(3.2.22): Every FT_4_space is FT_2_ space since every FT_4_space is FT_3_space and every FT_3_space is FT_2_space so that :

FT_4 _space $\overrightarrow{\nleftarrow}$ FT_3_space $\overrightarrow{\nleftarrow}$ FT_2_space $\overrightarrow{\nleftarrow}$ FT_1_space $\overrightarrow{\nleftarrow}$ FT_0_space

Remark(3.2.23) FN_space $+FT_1$_space $\Longrightarrow FT_3$_space, and FN_space $+FT_1$_space $\Rightarrow FR$_ space.

3.3 On F_connected

Definition(3.3.1): Let (X, τ) be a topological space and $A, B \subseteq X$ such that $A \neq \emptyset$, $B \neq \emptyset$ and $E \subseteq X$, then we said that A, B form a separation for E if

1) $E = A \cup B$ 2) $\bar{A} \cap B = A \cap \bar{B} = \emptyset$

Definition(3.3.2): Let (X, τ) be a topological space, we said that X is connected if X has no separation.

Theorem (3.3.3): Let (X, τ) be a topological space, then (X, τ) is connected. $(X, \tau)$is connected space.

1) the only sets which are open and closed in X are $\emptyset, X$.

2) X is not a union of two nonempty disjoint open sets.

Definition(3.3.4): Let (X,τ) be a topological space and $A,B \subseteq X$ such that $A \neq \emptyset$, $B \neq \emptyset$ and $E \subseteq X$, then we said that A,B form a F_separation for E if 1) $E = A \cup B$ 2) $\overline{A}^F \cap B = A \cap \overline{B}^F = \emptyset$

Definition(3.3.5): A topological space (X,τ) is called F_connected if X is not a union of two nonempty disjoint F_open sets.

Theorem(3.3.6): Let (X,τ) be a topological space, Then the following are equivalent

1) (X,τ) is connected space.

2) (X,τ) is F_connected space.

Proof: 1 → 2

Let (X,τ) be connected space, Suppose (X,τ) is not F_connected.Then there exists A,B F_open sets such that $A \cap B = \emptyset$ and $X = A \cup B$**,** Then A,B are open sets such that $A \cap B = \emptyset$ and $X = A \cup B$**.** Therefore (X,τ) is not connected space which is a contradiction, Hence (X,τ) is F_counected space.

(2→ 1)Let (X,τ) is F_counected space and suppose $(X,\tau)$is not connected space, then $\exists$ A,B open sets such that $A \cap B = \emptyset$ and $A \cup B = X$. therefore A,B are F_open sets, (every open, closed set is F_open). Hence $\exists$ A,B F_open sets such that $A \cap B = \emptyset$ and $A \cup B = X$ **.**Thus $(X,\tau)$is not connected space which is a contradiction .Then (X,τ) is counected space.

Definition(3.3.7): A topological space (X,τ) is called F'_connected if the only F_open and F_closed at the same time in X are $\emptyset, X$.

Theorem(3.3.8): Let (X,τ) be a topological space. Then the following are equivalent.

1) (X,τ) is connected space.

2) (X,τ) is F'-connected space .

Proof: 1 → 2 Let (X,τ) be connected space. Suppose (X,τ) is not F'_connected. Then $\exists$ A F-open and F_closed set $\ni A \neq \emptyset \; and \; A \neq X$. Then $\exists$ A open and closed set $\ni A \neq \emptyset \; and \; A \neq X$. Then (X,τ) is not connected space space which is a contradiction. Then (X,τ) is F'_counected space.

Proof 2→ 1 Let (X,τ) is F'-counected space and suppose $(X,\tau)$is not connected space. then $\exists$ A open and closed set such that $A \neq \emptyset$ and $A \neq X$**.** let $B = X - A$. Then B is open and closed set. Hence A,B and F_open sets. Therefor A is F_open and F_closed set $\ni$ $A \neq \emptyset$ and $A \neq X$ **.**Therefor (X,τ) is not F'_connected space. which is a contradiction .Then (X,τ) is counected space.

Definition(3.3.9): A topological space (X,τ) is called F''_connected if X has no F_separation .

Theorem(3.3.10): Let (X,τ) be a topological space.Then the following are equivalent

(i)(X,τ) is connected space .

(ii)(X,τ) is F''_connected space .

Proof:1 → 2 Let (X,τ) be connected space. Suppose (X,τ) is not F''_connected space. Then $\exists A,B \ni A \neq \emptyset \; and \; B \neq \emptyset, X = A \cup B \; and \; \overline{A}^F \cap B = A \cap \overline{B}^F = \emptyset$. Then $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ Therefor (X,τ) is not connected space. which is a contradiction. Hence (X,τ) is F''_connected space .

Proof: 2→ 1 Let (X,τ) is F''_connected space .Suppose $(X,\tau)$is not connected space. then $\exists$ A open and closed set such that $A \neq \emptyset$ and $A \neq X$**.** let $B = X - A$. Then $X = A \cup B \; and \; A \neq \emptyset, B \neq \emptyset$**.** Therefor A, B are F_closed sets.Hence $\overline{A}^F = A$ and $\overline{B}^F = B$. Then $\overline{A}^F = A = X - B$. Hence $\overline{A}^F \cap B = \emptyset$. Then $\overline{B}^F = B = X - A$. Hence $\overline{B}^F \cap A = \emptyset$. Therefor A and B from F_separation space for X, Then (X,τ) is not F''_connected space. which is a contradiction**.** Therefor (X,τ) is connected space .

Theorem(3.3.11): Let A be connected sets and H, K are F_separated sets. if $A \subseteq H \cup K$, then either $A \subseteq H$.

Proof: Let A be connected set and H, K be F_separated sets. then $H \neq \emptyset, K \neq \emptyset \; and \; \overline{H}^F \cap K = H \cap \overline{K}^F = \emptyset$. Let $A \subseteq H \cup K$.Suppose $A_1 = A \cap H \neq \emptyset$, $A_2 = A \cap K \neq \emptyset$. Then $A = A_1 \cup A_2$, $A_1 \neq \emptyset, A_2 \neq \emptyset$. $A_1 \subseteq H \longrightarrow \overline{A_1}^F \subseteq \overline{H}^F \longrightarrow \overline{A_1}^F \cap A_2 \subseteq \overline{H}^F \cap A_2 \subseteq \overline{H}^F \cap K$.
Since $\overline{H}^F \cap H = \emptyset, then \; \overline{A_1}^F \cap A_2 = \emptyset$. $A_2 \subseteq K \longrightarrow \overline{A_2}^F \subseteq \overline{K}^F \longrightarrow \overline{A_2}^F \cap A_1 \subseteq \overline{K}^F \cap A_1 \subseteq \overline{K}^F \cap H$, Since $\overline{K}^F \cap H = \emptyset, then \; \overline{A_2}^F \cap A_1 = \emptyset$.Then A_2, A_1 from a $F\_$ separation for A. which is a contradiction since A connected set .Then either $A \subseteq H \; or \; A \subseteq K$.

Theorem (3.3.12): If A is connected set, then $\overline{A}^F$ is connected .

Proof : Let A be connected set. Suppose $\overline{A}^F$ is not connected. Then $\exists$ H, K from a F_separation for $\overline{A}^F$. Hence $H \neq \emptyset, K \neq \emptyset, \overline{A}^F = H \cup K$, and $\overline{H}^F \cap K = H \cap \overline{K}^F = \emptyset$, Since $A \subseteq \overline{A}^F$, Then $A \subseteq H \cup K$. Then by theorem**(3.2.11)**, either $A \subseteq H \; or A \subseteq K$. If $A \subseteq H$, then $\overline{A}^F \subseteq \overline{H}$, hence $\overline{A}^F \cap K \subseteq \overline{H}^F \cap K$. Since $\overline{H}^F \cap K = \emptyset$, then$\overline{A}^F \cap K = \emptyset$. Therefore $K = \emptyset$ which is a contradiction. By the same way get a contradiction if $A \subseteq K$.therefore $\overline{A}^F$ is connected.

Definition(3.3.13): The space (X,τ) is F_disconnected space if and only if there exist two F_open set disjoint nonempty sets A and B such that $A \cup B = X$, and $A \cap B = \emptyset, A \neq \emptyset \; and \; B \neq \emptyset$.

Example(3.3.14): Let $X = \{1,2,3\}$ and $\tau = \{X, \emptyset, \{1\}, \{2,3\}\}$, the F_open set$\{1\}, \{2,3\}$ and $X = \{1\} \cup \{2,3\}$ and $\{1\} \cap \{2,3\} = \emptyset$, $\{1\}, \{2,3\} \neq \emptyset$, So X F_disconnected.

Remark(3.3.15): In discrete topological $\tau = \{X, \emptyset\}$ X is not union of two nonempty disjoint F_open sets, then X is F_connected.

Remark(3.3.16): Let (X, T_D)be discrete topological let A be open subset of X. $b(A) = \overline{A} - A^o = A - A = \emptyset$ is finite then A is F_open set .

Remark(3.3.17): In discrete topological every open set is F-open .

Remark(3.3.18): Let (X, D) is F_disconnected if X contains more than one element, since there exists A ; $\emptyset \neq A \nsubseteq X$. $X = A \cup A^c$, A, A^c F-open sets $A \cap A^c = \emptyset$ and $A \neq \emptyset, A^c \neq \emptyset$ since $(A \neq X)$ **.**

Example(3.3.19): Let (X, τ_{cof})be is F_connected space, if X is infinite set since there are not exist nonempty disjoint open sets.

Remark(3.3.20): If (X, τ) is topological space and (W, τ_w) is a subspace of X, then the space W being F_disconnected or F_connected not directly relation by X and the open sets in X, but dependent on the F_open sets in W, its dependent on τ_w ; so that W is F_connected space if and only if there exist two F_open disjoint nonempty sets A and B in W such that $A \cup B = W$.

Remark(3.3.21): If (X, τ) is topological space and (W, τ_w) is a subspace of X, then the space W being F_disconnected or F_connected not directly relation by X and the F_open sets in X, but dependent on the F_open sets in W, its dependent on τ_w ; so that W is F_connected space if and only if there exist two F_open disjoint nonempty sets A and B in W such that $W = A \cup B$, $W$is F_disconnected $\Leftrightarrow A \cup B = W$, A, B F_open in W, $A \cap B = \emptyset; A \neq \emptyset, B \neq \emptyset$ The space (W, τ_w) is F_connected if and only if its not F_disconnected W F_connected if and only if $W \neq A \cup B; A, B$ F_open in W; $A \cap B = \emptyset; A \neq \emptyset, B \neq \emptyset$.

Remark(3.3.22): The property of being a F_connected space is not a hereditary property and the following example show that:

Example(3.3.23): Let $X = \{1,2,3\}$ and $\tau = \{X, \emptyset, \{1,2\}, \{1,3\}, \{1\}\}$ and $W \subseteq X$; $W = \{2,3\}$.Is W is F_connected space. $\tau_w = \{W \cap U; U \text{ open in } X\} = \{W, \emptyset, \{2\}, \{3\}\}$. Notes that $\tau_w = D$, then W is F_disconnected space but not F_connected space, since :$W = \{2\} \cup \{3\}$ and $\{2\}, \{3\}$ F_open in W and $\{2\} \cap \{3\} = \emptyset$ and $\{2\} \neq \emptyset, \{3\} \neq \emptyset$, Notes that X is F_connected space but not F_disconnected, while it's have F_disconnected subspace.

Remark(3.3.24): If $f: (X, \tau) \to (Y, \grave{\tau})$is F_ continuous and onto function and Y is F_connected space then, then X not necessary F_ connected space and the following example show that :

Example(3.3.25): Let $f: (R, D) \to (R, I); f(x) = x$ for each $x \in R$ clear that f is F_ continuous and onto function and (R, I) is F_ connected, but (R, D) is not F_ connected.

Theorem(3.3.26): Let $((X, \tau)$be a topological space if W is connected and F- open subsets of X and $X = A \cup B$ such that A,B F-open and $A \cap B = \emptyset$ and $A \neq \emptyset, B \neq \emptyset$ then $W \subseteq A \text{ or } W \subseteq B$.

Proof: Suppose that $W \nsubseteq A \text{ and } W \nsubseteq B \Longrightarrow W \cap A \neq \emptyset \text{ and } W \cap B \neq \emptyset$; $A, B \text{ is } F$_open in $X \Longrightarrow W \cap A, W \cap B$ is F_open in W; $W \cap A \neq \emptyset$ $(\text{since if } W \cap A = \emptyset \to W \subseteq B), W \cap B \neq \emptyset$ $(\text{since if } W \cap A = \emptyset \to W \subseteq A), (W \cap A) \cap (W \cap B) = W \cap (A \cap B) = W \cap \emptyset = \emptyset$, then $W \text{ is } F$_disconnected (C !! contradiction !!); so $W \subseteq A \vee W \subseteq B$.

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