Journal of Iraqi Al-Khwarizmi (JIKh) Volume:8 Issue:2 Year: 2024 pages: 109-125

# Dynamical Behavior of An Ecological System with Diverse Functional Response

| ***Authors Names*** | **ABSTRACT** |
|---|---|
| *Saba Noori Majeed*<br><br>Publication date: 30 / 8 /2024<br><br>***Keywords:*** *Dynamical system, Prey – Predator,* Holling type II functional response, Beddington – De Angelis functional response, diverse functional response. | The purpose of this work is to study the dynamical behavior of ecological system model of two predator-one prey with diverse functional response, the first predator functional response is Beddington –De Angelis, while the second predator functional response is Holling type II mathematical features have been studied thoroughly, the system have local and global stability when especial conditions are met had been proved respectively, the system has no saddle node bifurcation but transcritical bifurcation and Pitchfork bifurcation are satisfied while Hopf bifurcation dose not occur, numerical illustrations are performed finally in order to set the applicability of the model under consideration. |

## 1.Introduction

In most biological societies, we find that Prey can be intercepted by more than one predator terms of species, there is a predator that depends on attaining the prey by searching alone or following the method of pack hunting of the prey with its peers of the same type and probably storing the remains of the prey to hoard food for another time, and those different predator societies there is a rivalry among them to obtain food (prey) and one of those predators maybe subjected to annihilation by the other kind. Lotka–Volterra [1,2] model is the classical description interact between species which incorporate logistic growth for a prey population and the predator population. (Functional response formulates the feeding rate per predator on the prey population). Lotka–Volterra model was the key portion for researchers to investigate the kinetics of the model in a more realistic way population's functional response. Holling-type functional response describes the predation speed within a regular range, it is classified into three types Holling's type I,II and III , see [3-9]. The functional response of Beddington-DeAngelis is similar to the Holling type-II functional response but it contains a term describing the mutual interference of predators. see [10-13]. In this research, we will study the dynamic of an ecological system connect three species combined with environmental conditions compatible despite qualitative differences through, this system consist of one prey and two varying types of predators behave in diverse functional response. The first predator functional response is Beddington –De Angelis, while the second predator's functional response is Holling type II. An analytical study includes local and global stability of the dynamical system had been introduced, also the bifurcation analysis for certain equilibrium points explained. According to above the resulting system was packed with parameters, which reduced using dimensionless technique to simplify the study while preserving carefully the mathematical properties studied. A numerical demonstration is illustrated with the help of MATLAB.

## 2. Mathematical Model

In this section a Beddington–De Angelis and Holling type II prey-predator model considered is based on the ( two predator- one prey ):

$$\frac{dx_1}{dt} = r\left(1 - \frac{x_1}{x_1+k}\right) - \frac{\gamma x_1 x_2}{a+x_1+b_1 x_2} - \frac{\beta x_1 x_3}{1+b_2 x_3}$$
$$\frac{dx_2}{dt} = -\delta_1 x_2 + \frac{c_1 \gamma x_1 x_2}{a+x_1+b_1 x_2} - d_1 x_2 x_3 \qquad (1)$$

University of Baghdad, College of Education for Women, Computer Department, Saba.noori@coeduw.uobaghdad.edu.iq

$$\frac{dx_3}{dt} = -\delta_2 x_3 + \frac{c_2 \beta x_1 x_3}{1+b_2 x_3} - d_2 x_2 x_3$$

It is considered that the first and second predator species, respectively are compotation for food and other essential resources such as shelter and water sources.
Where

- $x_1(t) = x_1$ is the prey population size at time t .
- $x_2(t) = x_2, x_3(t) = x_3$, are the population size of the first and second predator species at time t ,the prey grows logistically in the absence of the predator, in the same way that the predator declines directly in the absence of the prey.
- $r, k$ are respectively the growth rate and the environmental carrying capacity of the prey species.
- $d_1, d_2$ are the predator death rates.
- $\delta_1, \delta_2$ are the rates at which the growth rate of the first predator is annihilated by the second predator and vice versa.
- $c_1, c_2$ are the conversion factor denoting the number of newly born of the first and second predator for each captured prey species respectively ($0 < c_1 ,\ c_2 < 1$).
- $\beta, \gamma$ are the maximum number of the prey that can be eaten by the first and second predator per unit time respectively, $\frac{1}{a}$ is the half saturation rate of the first predator.
- $b_1, b_2$ measure the coefficients of their mutual interference among the first and second predator respectively.
- The term $\frac{\gamma x_1 x_2}{a+x_1+b_1 x_2}$ is the Beddington –De Angelis functional response of the first predator.
- The term $\frac{c_2 \beta x_1 x_3}{1+b_2 x_3}$ is the Holling type II functional response of the second predator.
- $\frac{\gamma}{a}$ The maximum number of prey can be eaten by the first predator.
- $\frac{\beta}{b_2}$ The maximum number of prey can be eaten by the second predator.

Where $r, k, d_1, d_2, \delta_1, \delta_2, \beta, \gamma, b_1, b_2$ are all positive real numbers and $a > 0$.

The next step is number of parameters and specify the control set of parameters reduced, so in order to simplify the system, the following dimensionless variables and parameters are used:

$S = \frac{x_1}{k}$, $P_1 = \frac{\delta x_2}{rk}$, $P_2 = \frac{\beta x_3}{rk}$, $t = rt$, $\frac{dx_1}{dt} = rk\frac{dS}{dt}$, $\frac{dP_1}{dt} = \frac{\gamma}{r^2 k}\frac{dx_2}{dt}$, $\frac{dP_2}{dt} = \frac{\beta}{r^2 k}\frac{dx_3}{dt}$ , $A_1 = \frac{a}{k}$ , $\epsilon_1 = \frac{b_1 r}{\gamma}$ , $A_2 = \frac{1}{k}$ , $\epsilon_2 = \frac{b_2 r}{\beta}$ , $\theta_1 = \frac{\delta_1}{\gamma}$ , $\lambda_1 = \frac{\gamma c_1}{r}$ , $\alpha_1 = \frac{d_1 k}{\beta}$ , $\theta_2 = \frac{\delta_2}{r}$, $\lambda_2 = \frac{\beta c_2}{r}$, $\alpha_2 = \frac{d_2 k}{\gamma}$

Then the system (1) reduces the following dimensionless system:

$$\begin{aligned} \frac{dS}{dt} &= \frac{S}{S+1} - \frac{SP_1}{A_1+S+\epsilon_1 P_1} - \frac{SP_2}{A_2+\epsilon_2 P_2} \\ \frac{dP_1}{dt} &= -\theta_1 P_1 + \lambda_1 \frac{SP_1}{A_1+S+\epsilon_1 P_1} - \alpha_1 P_1 P_2 \\ \frac{dP_2}{dt} &= -\theta_2 P_2 + \lambda_2 \frac{SP_2}{A_2+\epsilon_2 P_2} - \alpha_2 P_1 P_2 \end{aligned} \qquad (2)$$

Where $S(0) \geq 0, P_1(0) \geq 0, P_2(0) \geq 0$,are the evident that the number of parameters reduced from thirteen in the system (1) to ten in the system (2).

## 3. Existence and positive invariance

In this section the local existence and uniqueness of system (2) will be demonstrate

For $t > 0$ letting $X = (S, P_1, P_2)^T$ and $F = (f_1, f_2, f_3)^T$, such that $F: \mathbb{R}^3 \to \mathbb{R}^3$, then system (2) can be written as $X' = F$, here $f_i \in C^\infty$, for $i = 1,2,3$, where

$$
\begin{aligned}
f_1 &= \frac{S}{S+1} - \frac{SP_1}{A_1+S+\epsilon_1 P_1} - \frac{SP_2}{A_2+\epsilon_2 P_2} \\
f_2 &= -\theta_1 P_1 + \lambda_1 \frac{SP_1}{A_1+S+\epsilon_1 P_1} - \alpha_1 P_1 P_2 \qquad (3) \\
f_3 &= -\theta_2 P_2 + \lambda_2 \frac{SP_2}{A_2+\epsilon_2 P_2} - \alpha_2 P_1 P_2
\end{aligned}
$$

Clearly, the interaction functions in the system (2) are continuous and have continuous partial derivative on the positive three dimensional space $\mathbb{R}_+^3 = \{(S, P_1, P_2): S(0) \geq 0, P_1(0) \geq 0, P_2(0) \geq 0\}$. Therefore these functions are Lipschitzian [13] over $\mathbb{R}_+^3$ and the system (2.3) has a unique solution see [3],[8],[11]

**Theorem1.** The solution of system (2) are uniformly bounded over $X = \{(S, P_1, P_2) \in \mathbb{R}_+^3; w(t) \leq \frac{\rho}{\mu}\}$.

**Proof:** From the first equation of the system (2), we observed that:

$$\frac{dS}{dt} \leq \frac{S}{S+1}$$

then by solving the above differential inequality we obtained that $\lim_{t\to\infty} sup\, S \leq 1$. Now assume that $W(t) = S(t) + \frac{P_1(t)}{\lambda_1} + \frac{P_2(t)}{\lambda_2}$ ,where $W$ is the total population, we get that $\frac{dW}{dt} = \frac{dS}{dt} + \frac{1}{\lambda_1}\frac{dP_1}{dt} + \frac{1}{\lambda_2}\frac{dP_2}{dt}$ which gives $\frac{dW}{dt} \leq \frac{S}{S+1} - \frac{\theta_1}{\lambda_1}P_1 - \frac{\theta_2}{\lambda_2}P_2$ ,by simplifying the last differential inequality and substituting $W$, we conclude

$$\frac{dW}{dt} \leq \frac{S}{S+1} - \mu W \qquad (4)$$

where $\mu = \min\{1, \theta_1, \theta_2\}$, yields $\frac{dw}{dt} + \mu w \leq \frac{S}{S+1}$, finally by solving the differential inequality (4) we obtain that $w(t) \leq \max\{w(t_0), \frac{\rho}{\mu}\}$ and $\lim_{t\to\infty} \sup w(t) \leq \frac{\rho}{\mu}$, hence all solutions of the system (2) are bounded over $\Omega = \{(S, P_1, P_2) \in \mathbb{R}_+^3;\ S(0) > 0, P_1(0) > 0, P_2(0) > 0\}$.

## 4. Equilibrium Points and their feasibility

The system (2) has five equilibrium points as the following:

The equilibrium points $E_0 = (0,0,0)$ , $E_1 = (1,0,0)$ are always feasible.

The first planer equilibrium point is $E_2 = (S_2, P_{12}, 0)$ , where $S_2$ is a unique positive root, see [10] for the quadratic equation

$$(1 - \frac{\lambda_1}{\theta_1})S_2^2 + \left(1 + A_1 + \lambda_1 \frac{(\epsilon_1 - 1)}{\theta_1}\right) S_2 + 1 = 0 \qquad (5)$$

while $$P_{12} = \frac{\lambda_1}{\theta_1} (\frac{S_2}{S_2+1}) \qquad (6)$$

The equilibrium point $E_2$ exists uniquely in the interior of the positive quadrant of $S_2P_{12}$_ plan provided that the following sufficient condition holds

$$\left(1 + A_1 + \lambda_1 \frac{(\epsilon_1 - 1)}{\theta_1}\right) - \sqrt{2(1 - \frac{\lambda_1}{\theta_1})} > 0 \qquad (7)$$

The second planer equilibrium point is $E_3 = (S_3, 0, P_{23})$ where $S_3$ is a unique positive root, see [10] for the quadratic equation

$$\frac{\lambda_2}{\theta_2} S_3^2 + \left(\frac{\lambda_2}{\theta_2} - \frac{\epsilon_2 \lambda_2}{\theta_2} - A_2\right) S_3 - A_2 = 0 \qquad (8)$$

while $$P_{23} = \frac{\lambda_2}{\theta_2} (\frac{S_3}{S_3+1}) \qquad (9)$$

The equilibrium point $E_3$ exists uniquely in the interior of the positive quadrant of $S_3P_{23}$ _ plan provided that the follwoing sufficient condition holds

$$\frac{\lambda_2}{\theta_2}-\frac{\epsilon_2\lambda_2}{\theta_2}-A_2-2\sqrt{\frac{A_2\lambda_2}{\theta_2}}>0. \quad (10)$$

The last equilibrium point $E_4=E^*=(S^*,P_1^*,P_2^*)$ which exists if the component ${P_2}^*$ is a positive root of the equation

$$M_1{P_2}^{*5}+M_2{P_2}^{*4}+M_3{P_2}^{*3}+M_4{P_2}^{*2}+M_5{P_2}^{*}+M_6=0 \quad (11)$$

While ${P_1}^*=\frac{1}{\alpha_2}\left(\theta_2-\lambda_2\frac{S^*}{A_2+\epsilon_2{P_2}^*}\right)$ (12)

and $S^*=\frac{(\theta_1\alpha_2\epsilon_2A_1-\alpha_1\alpha_2A_1A_2){P_2}^*-\alpha_2\alpha_1A_1\epsilon_2{P_2}^{*2}+\theta_1\alpha_2A_1A_2}{\alpha_1\alpha_2A_2{P_2}^*+\alpha_1\alpha_2\epsilon_2{P_2}^{*2}-(\theta_1+\lambda_1)\alpha_2A_2-(\theta_1+\lambda_1)\alpha_2\epsilon_2{P_2}^*}$ (13)

According to Descarte's rule, see [4] of sign equation (11) has three positive real roots in $\mathbb{R}_+^3$ under the following conditions :

$M_1>0,\ M_2<0,\ M_3>0,\ M_4>0,\ M_5>0,\ M_6<0$

and $E^*$ exist.

Where

$M_1=F_5K_4^2+F_3K_1^2-F_1K_1K_4$

$M_2=(K_1(K_2K_4-K_1K_5)+F_2K_1K_4-2F_3K_1K_2-F_4K_1^2+2K_4K_5F_5-F_6K_4^2)$

$M_3=(F_1(K_2K_5-K_1K_6+K_3K_4)+F_2(K_1K_4-K_1K_5)+F_3(K_2^2-2K_1K_3)+2F_4K_1K_2+F_5(K_5^2+2K_4K_6)-2F_6K_4K_5)$,

$M_4=(F_1(K_2K_6+K_3K_5)+F_2(K_2K_5-K_1K_6+K_3K_4)+2K_2K_3F_3-F_4(K_2^2-2K_1K_3)+2K_5K_6F_5-2F_6(K_5^2+2K_4K_6))$

$M_5=(K_3K_6F_1+F_2(K_2K_6+K_3K_5)+F_3K_3^2-2K_2K_3F_4+K_6^2F_5-2F_6K_5K_6)$

$M_6=F_2K_3K_2-K_3^2F_4-K_6^2F_6$

$K_1=\alpha_1\alpha_2A_1\epsilon_2,\ K_2=\theta_1\alpha_2\epsilon_2A_1-\alpha_1\alpha_2A_1A_2-\theta_2\epsilon_2,\ K_3=\theta_1\alpha_2A_1A_2-\theta_2A_2,\ K_4=\alpha_1\alpha_2\epsilon_2,\ K_5=\alpha_1\alpha_2A_2-(\theta_1+\lambda_1)\alpha_2\epsilon_2,\ K_6=(\theta_1+\lambda_1)\alpha_2A_2+\lambda_2,\ F_1=\frac{-\theta_2}{\alpha_2}\frac{\alpha_1}{\lambda_1}+\frac{\alpha_1}{\lambda_1}\frac{\lambda_2}{\alpha_2}+\frac{\theta_2}{\lambda_2},\ F_2=1+\frac{\theta_1}{\lambda_1}\frac{\theta_2}{\alpha_2}+\frac{\theta_1}{\lambda_1}\frac{\lambda_2}{\alpha_2}-\frac{\theta_2}{\lambda_2},\ F_3=1-\frac{\alpha_1}{\lambda_1}\frac{\lambda_2}{\alpha_2},\ F_4=\frac{\theta_1}{\lambda_1}\frac{\lambda_2}{\alpha_2},\ F_5=1+\frac{\theta_2}{\alpha_2}\frac{\alpha_1}{\lambda_1}-2\frac{\theta_2}{\lambda_2},\ F_6=-\frac{\theta_1}{\lambda_1}\frac{\theta_2}{\alpha_2}.$

## 5.Local Stability of Equilibrium points

In this section, we analyze local stability for each equilibrium point of the system (2) The Jacobian matrix of the system (2) at any point *(S, $P_1$, $P_2$)* is defined as:

$J=D_f(X)=\left[C_{ij}\right]_{3\times3}$ (14)

which is given as follows:

$c_{11}=\frac{1}{(S+1)^2}-\frac{(A_1+\epsilon_1P_1)P_1}{(A_1+S+\epsilon_1P_1)^2}-\frac{P_2}{A_2+\epsilon_2P_2}$, $c_{12}=\frac{-S(A_1+S)}{(A_1+S+\epsilon_1P_1)^2}$, $c_{13}=\frac{-A_2S}{(A_2+\epsilon_2P_2)^2}$, $c_{21}=\lambda_1\frac{P_1(A_1+\epsilon_1P_1)}{(A_1+S+\epsilon_1P_1)^2}$

$c_{22}=-\theta_1+\lambda_1\frac{S(A_1+S)}{(A_1+S+\epsilon_1P_1)^2}-\alpha_1P_2$, $c_{23}=-\alpha_1P_1$, $c_{31}=-\lambda_2\frac{P_2}{(A_2+\epsilon_2P_2)}$, $c_{32}=-\alpha_2P_2$, $c_{33}=-\theta_2+\lambda_2\frac{SA_2}{(A_2+\epsilon_2P_2)^2}-\alpha_2P_1$

**Local stability of $E_0$:** the eigenvalues of the jacobian matrix $J_0$ are *1*,$-\theta_1$ *and* $-\theta_2$. Therefore $E_0$ *is unstable* actually it is a saddle point.

$$J_0=D_f(E_0)=\begin{bmatrix}1&0&0\\0&-\theta_1&0\\0&0&-\theta_2\end{bmatrix} \quad (15)$$

**Local stability of $E_1$:** the eigenvalues of the Jacobian matrix $J_1$ are $\frac{1}{4}$, $-\theta_1+\lambda_1\frac{1}{(A_1+1)}$ and $-\theta_2+\lambda_2\frac{1}{A_2}$. Therefore $E_1$ is unstable clearly it is a saddle point

$$J_1 = D_f(E_1) = \begin{bmatrix} \frac{1}{4} & -[\frac{1}{(A_1+1)}] & -\frac{1}{A_2} \\ 0 & -\theta_1 + \lambda_1 \frac{1}{(A_1+1)} & 0 \\ 0 & 0 & -\theta_2 + \lambda_2 \frac{1}{A_2} \end{bmatrix} \quad (16)$$

**Local stability of** $E_2$: The characteristic equation of the Jacobian matrix $J_2 = D_f(X) = [a_{ij}]_{3\times3}$, where $\lambda^3 + \Omega_1\lambda^2 + \Omega_2\lambda + \Omega_3 = 0$, where $\Omega_1 = -[a_{11} + a_{22} + a_{33}]$, $\Omega_2 = a_{11}a_{22} - a_{21}a_{12} + a_{11}a_{33} + a_{22}a_{33}$ and $\Omega_3 = -a_{33}(a_{11}a_{22} - a_{21}a_{12})$, the $[a_{ij}]_{3\times3}$ elements are $a_{11} = \frac{1}{(S_2+1)^2} - \frac{P_{12}(A_1+\epsilon_1 P_{12})}{(A_1+S_2+\epsilon_1 P_{12})^2}$, $a_{12} = \frac{S_2(A_1+S_2)}{(A_1+S_2+\epsilon_1 P_{12})^2}$, $a_{13} = -\frac{S_2}{A_2}$, $a_{21} = \lambda_1 \frac{P_{12}(A_1+\epsilon_1 P_{12})}{(A_1+S_2+\epsilon_1 P_{12})^2}$, $a_{22} = -\theta_1 + \lambda_1 \frac{S_2(A_1+S_2)}{(A_1+S_2+\epsilon_1 P_{12})^2}$, $a_{23} = -\alpha_1 P_{12}$, $a_{31} = 0$, $a_{32} = 0$, $a_{33} = -\theta_2 + \lambda_2 \frac{S_2}{A_2} - \alpha_2 P_{12}$, hence by Routh-Hurwitz criterion [14] $E_2$ is locally asymptotically stable if $\Omega_1 > 0, \Omega_3 > 0$ *and* $\Delta > 0$ where $\Delta = \Omega_1\Omega_2 - \Omega_3 = -a_{11}^2(a_{22} + a_{33}) - a_{22}^2(a_{11} + a_{33}) - a_{33}^2(a_{11} + a_{22}) - 2a_{11}a_{22}a_{33} + a_{21}a_{12}(a_{11} + a_{22})$, so that $E_2$ is locally asymptotically stable point if
$a_{11} < 0$, $a_{22} < 0$, $a_{33} < 0$, that is:

$$\frac{1}{(S_2+1)^2} < \frac{P_{12}(A_1+\epsilon_1 P_{12})}{(A_1+S_2+\epsilon_1 P_{12})^2} \quad (17)$$

$$\theta_1 < \frac{S_2(A_1+S_2)}{(A_1+S_2+\epsilon_1 P_{12})^2} \quad (18)$$

$$\theta_2 < \lambda_2 \frac{S_2}{A_2} - \alpha_2 P_{12} \quad (19)$$

**Local stability of** $E_3$: The characteristic equation of the jacobian matrix $J_3 = D_f(X) = [b_{ij}]_{3\times3}$ is $\lambda^3 + \Psi_1\lambda^2 + \Psi_2\lambda + \Psi_3 = 0$, where $\Psi_1 = -[b_{11} + b_{22} + b_{33}]$, $\Psi_2 = b_{11}b_{22} - b_{31}b_{13} + b_{11}b_{33} + b_{22}b_{33}$ and $\Psi_3 = -b_{22}(b_{11}b_{33} - b_{31}b_{13})$, the $[b_{ij}]_{3\times3}$ elements are $b_{11} = \frac{1}{(S_3+1)^2} - \frac{P_{23}}{A_2+\epsilon_2 P_{23}}$, $b_{12} = \frac{-S_3}{(A_1+S_3)}$, $b_{13} = \frac{-S_3A_2}{(A_2+\epsilon_2 P_{23})^2}$, $b_{21} = 0$, $b_{22} = -\theta_1 + \lambda_1 \frac{S_3}{(A_1+S_3)} - \alpha_1 P_{23}$, $b_{23} = 0$, $b_{31} = \lambda_2 \frac{P_{23}}{A_2+\epsilon_2 P_{23}}$, $b_{32} = -\alpha_2 P_{32}$, $b_{33} = -\theta_2 + \lambda_2 \frac{S_3A_2}{(A_2+\epsilon_2 P_{23})^2}$, so by Routh-Hurwitz criterion $E_3$ is locally asymptotically stable if $\Psi_1 > 0, \Psi_3 > 0$ *and* $\Delta > 0$ where $\Delta = \Psi_1\Psi_2 - \Psi_3 = -b_{11}^2(b_{22} + b_{33}) - b_{22}^2(b_{11} + b_{33}) - b_{33}^2(b_{11} + b_{22}) - 2b_{11}b_{22}b_{33} + b_{31}b_{13}(b_{11} + b_{33})$, thus $E_2$ is locally asymptotically stable if $b_{11} < 0$,
$b_{22} < 0, b_{33} < 0$, that is

$$\frac{1}{(S_3+1)^2} < \frac{P_{23}}{A_2+\epsilon_2 P_{23}} \quad (20)$$

$$\theta_1 < \lambda_1 \frac{S_3}{(A_1+S_3)} - \alpha_1 P_{23} \quad (21)$$

$$\theta_2 < \lambda_2 \frac{S_3A_2}{(A_2+\epsilon_2 P_{23})^2} \quad (22)$$

**Local stability of E*.** Let $J^* = J = D_f(X) = D_f(E^*) = [C_{ij}]_{3\times3}$ as shown in (15) (After substituting $S$ *with* $S^*, P_1$ *with* $P_1^*$ *and* $P_2$ *with* $P_2^*$)

**Theorem 2.** The system (2) is locally asymptotically stable around the equilibrium point $E^* = (S^*, P_1^*, P_2^*) = (S, P_1, P_2)$, if the following conditions are satisfied :

$$\frac{1}{(S+1)^2} < \frac{(A_1+\epsilon_1 P_1)}{(A_1+S+\epsilon_1 P_1)^2} + \frac{P_2}{A_2+\epsilon_2 P_2} \quad (23)$$

$$\theta_1 < \lambda_1 \frac{S(A_1+S)}{(A_1+S+\epsilon_1 P_1)^2} - \alpha_1 P_2 \quad (24)$$

$$\theta_2 < \lambda_2 \frac{SA_2}{(A_2+\epsilon_2 P_2)^2} - \alpha_2 P_1 \quad (25)$$

**Proof:** Let us define the characteristic equation of the Jacobian matrix $J^* = D_f(E^*) = (c_{ij})_{3\times3} = Df\ (X)$ as $\Lambda^3 + \Theta_1\Lambda^2 + \Theta_2\Lambda + \Theta_3 = 0$ , where $\Theta_1 = -[c_{11} + c_{22} + c_{33}]$ , $\Theta_2 = c_{11}c_{22} - c_{21}c_{12} - c_{31}c_{13} + c_{11}c_{33} + c_{22}c_{33} - c_{32}c_{23}$ and $\Theta_3 = -c_{33}(c_{11}c_{22} - c_{21}c_{12}) - c_{12}c_{23}c_{31} - c_{13}c_{21}c_{32} + c_{13}c_{22}c_{31} + c_{11}c_{23}c_{32}$ , so by ( Routh-Hurwitz ) criterion $E^*$ is locally asymptotically stable if $\Theta_1 > 0, \Theta_3 > 0\ and\ \Delta > 0$
where $\Delta = \Theta_1\Theta_2 - \Theta_3 = -(c_{11} + c_{22} + c_{33})[c_{11}c_{22} - c_{21}c_{12} + c_{11}c_{33} + c_{22}c_{33} - c_{31}c_{13} - c_{32}c_{23}] + c_{33}(c_{11}c_{22} - c_{21}c_{12}) + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} - c_{13}c_{22}c_{31} - c_{11}c_{23}c_{32}$.
So $E^*$ is locally asymptotically stable if $c_{11} < 0, c_{22} < 0, c_{33} < 0$ , that is: (23) ,(24) and (25) holds. Therefore, the prove is complete.

## 6.Global Stability

In this subsection, the global stability is studied for each locally stable equilibrium point using a suitable Lyapunov function [15] that is given in the following theorems:

**Theorem 3 .** Assume that the equilibrium point $E_2 = (S_2, P_{12}, 0)$ is locally asymptotically stable in $\mathbb{R}^3$ . Then it is globally asymptotically stable that satisfy the following conditions are satisfied:

$$\frac{(P_1-P_{12})}{A_1+S+\epsilon_1 P_1} + \frac{P_2}{A_2+\epsilon_2 P_2} > \frac{1}{(S-S_2)+1} \quad (26)$$

$$\theta_1 + \alpha_1 P_2 > \frac{\lambda_1(S-S_2)}{A_1+(S-S_2)+\epsilon_1(P_1-P_{12})} \quad (27)$$

$$\theta_2 + \alpha_2(P_1 - P_{12}) > \frac{\lambda_2(S-S_2)}{A_2+\epsilon_2 P_2} \quad (28)$$

**Proof.** Applying suitable Lyapunov function at $E_2 = (S_2, P_{12}, 0)$ we get:

$$W_2 = \frac{(S-S_2)^2}{2} + \frac{(P_1-P_{12})^2}{2} + P_2 \quad (29)$$

Clearly $W_2(S, P_1, P_2) > 0$ is a continuously differentiable real valued function for all $(S, P_1, P_2) \in \mathbb{R}^3$with $(S, P_1, P_2) \neq (S_2, P_{12}, 0)$ and $W_2(S_2, P_{12}, 0) = 0$ , moreover we have that $\frac{dW_2}{dt} = (S - S_2)\frac{dS}{dt} + (P_1 - P_{12})\frac{dP_2}{dt} + \frac{dP_2}{dt}$ we get by Substituting $\frac{dS}{dt}, \frac{dP_1}{dt}$ and $\frac{dP_2}{dt}$ from (2) we get $\frac{dW_2}{dt} = (S - S_2)\left[\frac{(S-S_2)}{(S-S_2)+1} - \frac{(S-S_2)(P_1-P_{12})}{A_1+(S-S_2)+\epsilon_1(P_1-P_{12})} - \frac{(S-S_2)P_2}{A_2+\epsilon_2 P_2}\right] + (P_1 - P_{12})\left[-\theta_1(P_1 - P_{12}) + \lambda_1 \frac{(S-S_2)(P_1-P_{12})}{A_1+(S-S_2)+\epsilon_1(P_1-P_{12})} - \alpha_2 P_2(P_1 - P_{12})\right] - \theta_2 P_2 + \lambda_2 \frac{(S-S_2)P_2}{A_2+\epsilon_2 P_2} - \alpha_1(P_1 - P_{12})P_2$

Now straightforward computations give

$$\frac{dW_2}{dt} \leq -\tau_1(S - S_2)^2 - \tau_2(P_1 - P_{12}) - \tau_3 P_2$$

Where $\tau_1 = \frac{(P_1 - P_{12})}{A_1 + S + \epsilon_1 P_1} + \frac{P_2}{A_2 + \epsilon_2 P_2} - \frac{1}{(S - S_2) + 1}$

$$\tau_2 = \theta_1 + \alpha_1 P_2 - \frac{\lambda_1 (S - S_2)}{A_1 + (S - S_2) + \epsilon_1 (P_1 - P_{12})}$$

$$\tau_3 = \theta_2 + \alpha_2 (P_1 - P_{12}) > \frac{\lambda_2 (S - S_2)}{A_2 + \epsilon_2 P_2}$$

So according to conditions (26), (27) and (28) we guarantee $\frac{dW_2}{dt} < 0$

Hence $E_2$ is globally asymptotically stable

As the same we could proof that $E_3 = (S_3, 0, P_{23})$ is globally asymptotically stable.

**Theorem 4.** Assume that the equilibrium $E^* = (S^*, P_1^*, P_2^*)$ point is locally asymptotically stable in $\mathbb{R}^3$.

Then it is globally asymptotically stable if the following conditions are satisfied:

$$\frac{(P_1 - P_1^*)}{A_1 + (S - S^*) + \epsilon_1 (P_1 - P_1^*)} + \frac{(P_2 - P_2^*)}{A_2 + \epsilon_2 (P_2 - P_2^*)} > \frac{1}{(S - S^*) + 1} \qquad (30)$$

$$\theta_1 + \alpha_1 (P_2 - P_2^*) > \lambda_1 \frac{(S - S^*)}{A_1 + (S - S^*) + \epsilon_1 (P_1 - P_1^*)} \qquad (31)$$

$$\theta_2 + \alpha_2 (P_1 - P_1^*) > \lambda_2 \frac{(S - S^*)}{A_2 + \epsilon_2 (P_2 - P_2^*)} 0 \qquad (32)$$

**Proof.** Consider the following chosen Lyapunov function:

$$\mathbf{W}^* = \frac{(\mathbf{S} - \mathbf{S}^*)}{\mathbf{2}} + \frac{(\mathbf{P_1} - \mathbf{P_1^*})}{\mathbf{2}} + \frac{(\mathbf{P_2} - \mathbf{P_2^*})^2}{\mathbf{2}} \qquad (33)$$

where $W^*$ is a function of $(S^*, P_1^*, P_2^*)$ and $W^* > 0$, Now by differentiating $W^*$ with respect to time t , gives that :

$$\frac{dW^*}{dt} = (S - S^*) \frac{dS}{dt} + (P_1 - P_1^*) \frac{dP_1}{dt} + (P_2 - P_2^*) \frac{dP_2}{dt}$$

$$\frac{dW^*}{dt} = (S - S^*) \left[ \frac{(S - S^*)}{(S - S^*) + 1} - \frac{(S - S^*)(P_1 - P_1^*)}{A_1 + (S - S^*) + \epsilon_1 (P_1 - P_1^*)} - \frac{(S - S^*)(P_2 - P_2^*)}{A_2 + \epsilon_2 (P_2 - P_2^*)} \right] + (P_1 - P_1^*) \Big[ -\theta_1 (P_1 - P_1^*) + \lambda_1 \frac{(S - S^*)(P_1 - P_1^*)}{A_1 + (S - S^*) + \epsilon_1 (P_1 - P_1^*)} - \alpha_1 (P_1 - P_1^*)(P_2 - P_2^*) \Big] + (P_2 - P_2^*) \Big[ -\theta_2 (P_2 - P_2^*) + \lambda_2 \frac{(S - S^*)(P_2 - P_2^*)}{A_2 + \epsilon_2 (P_2 - P_2^*)} - \alpha_2 (P_1 - P_1^*)(P_2 - P_2^*) \Big]$$

After using the method of completing square and taking common factors of resulting algebraic terms and simplifying them, we get

$$\frac{dW^*}{dt} \leq -(S - S^*)^2 \xi_1 - (P_1 - P_1^*)^2 \xi_2 - (P_2 - P_2^*)^2 \xi_3$$

Where

$$\xi_1 = \frac{(P_1 - P_1^*)}{A_1 + (S - S^*) + \epsilon_1 (P_1 - P_1^*)} + \frac{(P_2 - P_2^*)}{A_2 + \epsilon_2 (P_2 - P_2^*)} - \frac{1}{(S - S^*) + 1}$$

$$\xi_2 = \theta_1 + \alpha_1 (P_2 - P_2^*) - \lambda_1 \frac{(S - S^*)}{A_1 + (S - S^*) + \epsilon_1 (P_1 - P_1^*)}$$

$\xi_3 = \theta_2 + \alpha_2(P_1 - P_1^*) - \lambda_2 \frac{(S-S^*)}{A_2+\epsilon_2(P_2-P_2^*)}$

So according to conditions (30), (31) and (32) we guarantee $\frac{dW^*}{dt} < 0$

Therefore $E^*$ is globally asymptotically stable.

## 7. Bifurcation Analyses

The occurrence of local bifurcation is well known that non-hyperbolic equilibrium point property is a necessary but not sufficient condition for the occurrence of bifurcation around that point. In the following theorems, the candidate bifurcation parameter is selected so that the equilibrium point under study will be a non-hyperbolic point ,we study in this section the local bifurcation for the equilibrium point by applying the Sotomayor's theorem [16] ,while $E^*$ is selected to analyze the Hopf –bifurcation[17] occurrence around certain parameter $\lambda_2$.

**Theorem 5.** The system (2) has a transcretical bifurcations and pitchfork bifurcation but not saddle node bifurcation can occur near the equilibrium point $E_2$ passes through the parameter $\theta_2^* = \lambda_2 \frac{S_2}{A_2} - \alpha_2 P_{12}$.

**Proof**. It is easy to verify that the Jacobain matrix of system (2) at $(E_2, \theta_2^*)$ can be written as

$J_2^{\theta_2^*} = \begin{bmatrix} Y_1 & -Y_3 & -\frac{S_2}{A_2} \\ Y_2 & Y_4 & -\alpha_1 P_{12} \\ 0 & 0 & 0 \end{bmatrix}$ where $Y_1 = \frac{1}{(S_2+1)^2} - \frac{P_{12}(A_1+\epsilon_1 P_{12})}{(A_1+S_2+\epsilon_1 P_{12})^2}$ , $Y_2 = \lambda_1 \frac{P_{12}(A_1+\epsilon_1 P_{12})}{(A_1+S_2+\epsilon_1 P_{12})^2}$,

$$Y_3 = \frac{S_2(A_1+S_2)}{(A_1+S_2+\epsilon_1 P_{12})^2}, Y_4 = -\theta_1 + \lambda_1 \frac{S_2(A_1+S_2)}{(A_1+S_2+\epsilon_1 P_{12})^2}$$

Clearly, the third eigenvalue $\varsigma_{3P_2}$ in the $P_2$ direction is zero while the first eigenvalue $\varsigma_1 = \frac{1}{(S_2+1)^2} - \frac{P_{12}(A_1+\epsilon_1 P_{12})}{(A_1+S_2+\epsilon_1 P_{12})^2} < 0$ and the second eigenvalue $\varsigma_2 = -\theta_1 + \lambda_1 \frac{S_2(A_1+S_2)}{(A_1+S_2+\epsilon_1 P_{12})^2} < 0$ when conditions (5.4),(5.5) are satisfied respectively, further the eigenvector $v = (v_1, v_2, v_3)^T$ corresponding to $\varsigma_{3P_2}$ satisfies the following $J_2^{\theta_2^*} v = \varsigma\, v$ then $J_2^{\theta_2^*} v = 0$ we get

$$Y_1 v_1 - Y_3 v_2 - \frac{S_2}{A_2} v_3 = 0 \qquad (34)$$

$$Y_2 v_1 + Y_4 v_2 - \alpha_1 P_{12} v_3 = 0 \qquad (35)$$

so by solving the above system of equations we get $v_1 = O_1 v_3$ and $v_2 = O_2 v_3$, where $v_3$ is a nonzero value number and $O_1 = \frac{Y_4 \frac{S_2}{A_2} + \alpha_1 P_1 Y_3}{Y_4 Y_1 + Y_3 Y_2}$ , $O_2 = \frac{Y_1 O_1 - \frac{S_2}{A_2}}{Y_3}$, thus :

$v = \begin{bmatrix} O_1 v_3 \\ O_2 v_3 \\ v_3 \end{bmatrix}$, similarly we take the eigenvector $\omega = (\omega_1, \omega_2, \omega_3)^T$ corresponding to the eigenvalue $\varsigma_{3P_2}$ of $[J_2^{\theta_2^*}]^T$ can be written as

$$\begin{bmatrix} Y_1 & Y_2 & 0 \\ -Y_3 & Y_4 & 0 \\ -\frac{S_2}{A_2} & -\alpha_1 P_{12} & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = 0, \text{ we get } \omega = (0,0,\omega_3)^T \qquad (36)$$

Here $\omega_3$is any nonzero real number.

Now rewrite the system in vector form as $\frac{dx}{dt} = f(X)$ where $X = (S, P_1, P_2)^T$ , $f = (f_1, f_2, f_3)^T$

And $\frac{\partial f}{\partial \theta_2^*} = f_{\theta_2^*}$ , we get that $f_{\theta_2^*} = [0,0,-P_2]^T$ obviously $f_{\theta_2^*}(E_1, \theta_2^*) = [0,0,0]^T$. Therefore

$$\omega^T f_{\theta_2^*}(E_2, \theta_2^*) = 0 \qquad (37)$$

Consequently, according to the Sotomayor theorem the system has no saddle-node bifurcation near $E_1$ through $\theta_2^*$, now in order to investigate the occurrence of the other types of bifurcation , the derivative of $f_{\theta_2^*}$ with respect to vector $X$ say $Df_{\theta_2^*}(E_1, \theta_2^*)$ is computed

$$Df_{\theta_2^*}(E_1, \theta_2^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ And } \quad \omega^T Df_{\theta_2^*}(E_1, \theta_2^*) v' = -v_3 w_3 \neq 0$$

Again, according to Sotomayor theorem if in addition to the above, the following holds

$$\omega^T [D^2 f_{\theta_2^*}(E_1, \theta_2^*)(v', v')] \neq 0 \qquad (38)$$

And

$$\omega^T [D^3 f_{\theta_2^*}(E_1, \theta_2^*)(v', v', v')] \neq 0 \qquad (39)$$

Then the system (2) has a transcritical bifurcation and pitchfork bifurcation at $E_2$.

## 8. Hopf-bifurcation.

**Theorem 10.** The equilibrium point $E^*$ of the system(2) has no Hopf-bifurcation around the parameter $\lambda_1$.

**Proof**. According to the local stability analysis of system (2) at $E^*$, we have that the coefficients of the characteristic equation $\Theta_i; i = 1,2,3$ are positive provided that

$$\Lambda^3 + \Theta_1 \Lambda^2 + \Theta_2 \Lambda + \Theta_3 = 0 \qquad (40)$$

However, $\Delta = \Theta_1 \Theta_2 - \Theta_3$ is positive provided that $c_{22} < 0$ in $J^*$

That is $-\theta_1 + \lambda_1 \frac{S(A_1+S)}{(A_1+S+\epsilon_1 P_1)^2} - \alpha_1 P_2 < 0$ and hence there is no Hopf- bifurcation in this case.

Now suppose that $\Delta = \Theta_1 \Theta_2 - \Theta_3 = 0$ then according to [17] there is possibility to occurrence of Hopf -bifurcation if and only if the Jacobian matrix of system (2) near $E^*$ has two complex conjugate eigenvalues , say $\kappa_i = \rho_1 \pm i\rho_2$ with the third eigenvalue is real and negative , in addition, the following two conditions are held in specific parameter say $l = l^*$ and

$$\rho_1(l^*) = 0 \qquad (41)$$

$$\frac{d\rho_1}{dl}|_{l=l^*} \neq 0 \qquad (42)$$

Now from $\Delta = \Theta_1 \Theta_2 - \Theta_3 = 0$ we obtain that

$Mc_{22}^2 + Bc_{22} + C = 0$ (43)

Where

$M = -(c_{11} + c_{33})$ is $> 0$ ,

$B = ( -(c_{11} + c_{33})^2 + c_{21}c_{12} + c_{32}c_{23})$ ,

$C = (c_{11} + c_{33})(c_{13}c_{31} + c_{11}c_{33}(c_{11} + c_{33}) + c_{11}c_{12}c_{21} + c_{33}c_{32}c_{23} + c_{13}c_{21}c_{32} + c_{12}c_{23}c_{31})$

Clearly for $C < 0$ we have two real roots of the equation (43) say

$c_{22} = \frac{-B}{2M} \pm \frac{\sqrt{B^2-4MC}}{2M}$ , since $c_{22} < 0$, then we get $c_{22} = \frac{-B}{2M} - \frac{\sqrt{B^2-4MC}}{2M}$ and hence

$-\theta_1 + \lambda_1 \frac{S(A_1+S)}{(A_1+S+\epsilon_1 P_1)^2} - \alpha_1 P_2 + \frac{B}{2M} + \frac{\sqrt{B^2-4MC}}{2M} = 0$ (44)

Which gives $f(\lambda_1^*) = 0$ *and* $\lambda_1 = \lambda_1^*$ represent a root of equation (44) consequently for $\lambda_1 = \lambda_1^*$ we get $\Theta_1\Theta_2 = \Theta_3$ from which the characteristic equation can be written as

$\rho(\Lambda) = (\Lambda + \Theta_1)(\Lambda^2 + \Theta_2) = 0$ (45)

Hence, in such case $\lambda_1 = \lambda_1^*$ the eigenvalues $\Lambda_1 = -\Theta_1 < 0$ *and* $\Lambda_{2,3} = \pm i\sqrt{\Theta_2}$ so the first condition of Hopf-bifurcation is satisfied at $\lambda_1 = \lambda_1^*$ that is $\rho_1(\lambda_1^*) = 0$ while $\rho_2 = \sqrt{\Theta_2}$ , that is $\Lambda_{2,3} = \rho_1(\lambda_1) \pm i\rho_2(\lambda_1)$ , substituting $\Lambda = \rho_1 + i\rho_2$ in equation (45) we get after some algebraic computations

$Ŋ\rho_1' - \phi\rho_2' = -\Theta$ (46)

where $\frac{d\rho_3(\Lambda)}{d\lambda_1} = \rho_3'(\Lambda)$

$\phi\rho_1' - Ŋ\rho_2' = -\Gamma$ (47)

Such that

$Ŋ = 3\rho_1^2 + 2\Theta_1\rho_1 + \Theta_2 - 3\rho_2^2$

$\phi = 6\rho_1\rho_2 + 2\Theta_1\rho_2$ (48)

$\Theta = \rho_1^2\Theta_1' + \Theta_2'\rho_1 + \Theta_3' - \Theta_1'\rho_2^2$

$\Gamma = 2\rho_1\rho_2\Theta_1' + \Theta_2'\rho_2$

Solving the linear system (46) and (47) for the unknowns $\rho_1', \rho_2'$ it is obtained that

$\rho_1' = \frac{Ŋ\Theta+\Gamma\phi}{Ŋ^2+\phi^2}$ , $\rho_2' = \frac{-\GammaŊ+\Theta\phi}{Ŋ^2+\phi^2}$ Hence, the second condition of Hopf-bifurcation will be reduced to verify that

$Ŋ\Theta + \Gamma\,\phi \neq 0$ (49)

But $\Theta_1' = -1$ , $\Theta_2' = c_{11} + c_{33}$ *and* $\Theta_3' = -\Theta_2 + \Theta_1(c_{11} + c_{33})$ thus $Ŋ = -2\Theta_2$ , $\phi = 2\Theta_1\sqrt{\Theta_2}$ , $\Theta = \Theta_1(c_{11} + c_{33})$ , $\Gamma = (c_{11} + c_{33})\sqrt{\Theta_2}$ substituting in (49). we get

$Ŋ\Theta + \Gamma\,\phi = 0$. Hence system (2) does not undergo a Hopf-bifurcation through $E^*$.

## 9. Numerical Analysis.

In this section, we studied the global dynamics of the system (2) numerically to verify the obtained analytical results and specifying the control set of parameters. For the following hypothetical set of parameters system (2) solved numerically and the obtained trajectories are drawn in the form of phase portrait and time series. First, we examine varying the value of each parameter on the dynamical behavior of the system (2). Second assure our obtained analytical results. It is spotted that, for the following set of hypothetical parameters in (50) that satisfies stability conditions of the positive equilibrium point $E^*$, system (2) has a globally asymptotically stable coexistence equilibrium point, as illustrated in figure Fig. (1.I)- below, with initial condition (0.5, 0.4, 0.5)

$$A_1 = 0.503, A_2 = 0.8, \epsilon_1 = 0.625, \epsilon_2 = 0.011, \lambda_1 = 0.0402, \lambda_2 = 0.2525, \theta_1 = 0.011, \theta_2 = 0.11,$$

$$\alpha_1 = 0.01, \alpha_2 = 0.04 \qquad (50)$$

Consequently, the following set of hypothetical parameters in (51) that satisfies stability conditions of the positive equilibrium point $E_2$ of system (2) has a globally asymptotically stable coexistence equilibrium point, as illustrated in figure Fig. (1.II)- below, with initial condition (0.5, 0.4, 0)

$$A_1 = 0.89, A_2 = 0.1, \epsilon_1 = 0.999, \epsilon_2 = 0.3, \lambda_1 = 0.245, \lambda_2 = 0.253, \theta_1 = 0.095, \theta_2 = 0.3,$$

$$\alpha_1 = 0.9, \alpha_2 = 0.041 \qquad (51)$$

However the set of parameters in (52) satisfies stability conditions of the positive equilibrium point $E_3$of system (2) has a globally asymptotically stable coexistence equilibrium point, as illustrated in figure Fig. (1.III) - below, with initial condition (0.5, 0, 0.5)

$$A_1 = 0.5, A_2 = 0.1, \epsilon_1 = 0.9, \epsilon_2 = 0.3, \lambda_1 = 0.0402, \lambda_2 = 0.253, \theta_1 = 0.11, \theta_2 = 0.3,$$

$$\alpha_1 = 0.01, \alpha_2 = 0.041 \qquad (52)$$

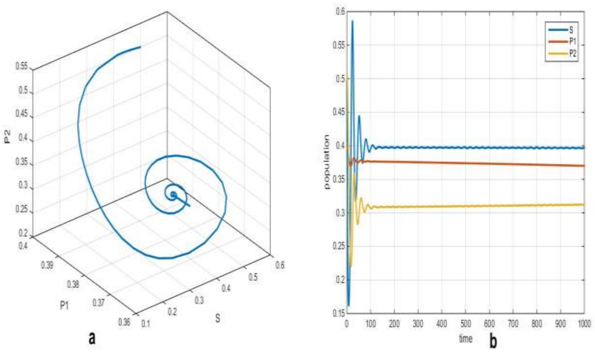


Fig. (1. I) -

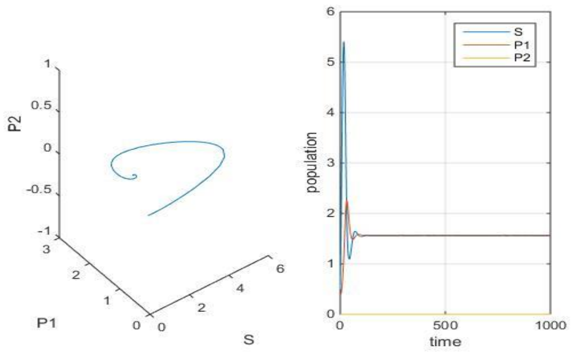


Fig. (1.II) -

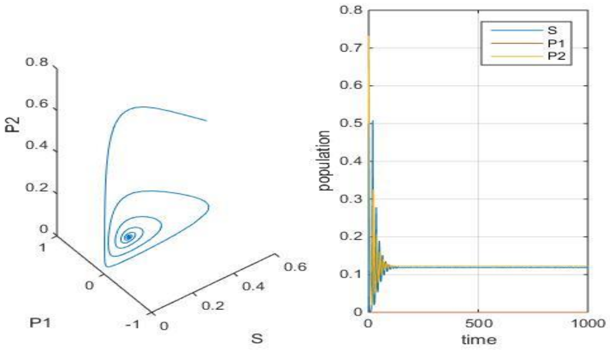


**Fig.** **(1.III)-**

**Fig.1 - Time series trajectories of system (2) of $E^*$equilibrium point**

**for the values at (50), (51) and (52) respectively.**

**Fig.1** The trajectories of system (2) for the data (50) starting from initial point (0.5, 0.4, 0.5), (a) 3D phase portrait for a globally asymptotically coexistence equilibrium $E^*$, (b) Time series for the attractor in (a) in **Fig.(1.I) -**, while in **Fig. (1.II) -**The trajectories of system (2) for the data (51) starting from initial point (0.5, 0.4, 0), (a) 3D phase portrait for a globally asymptotically coexistence equilibrium $E_2$, (b) Time series for the attractor in (a). **Fig. (1.III) -** shows the trajectories of system (2) for the data (52) starting from initial point (0.5, 0, 0.5), (a) 3D phase portrait for a globally asymptotically coexistence equilibrium $E_3$, (b) Time series for the attractor in (a).

It is clear, figure (1) ensures the obtained theoretical finding regarding the existence of globally asymptotically stable coexistence equilibrium points $E^*, E_2, E_3$ with certain conditions.

Now, by modifying one parameter at a time, the effect of changing the parameter values on the dynamics of the system (2) is explored, and the resulting trajectory is shown in figure **Fig.2 -** that as the environmental carrying capacity of the prey species $A_1$ recede from $A_1$= 0.503 to $A_1$= 0.4, the number of predator $P_2$ individuals species fades .

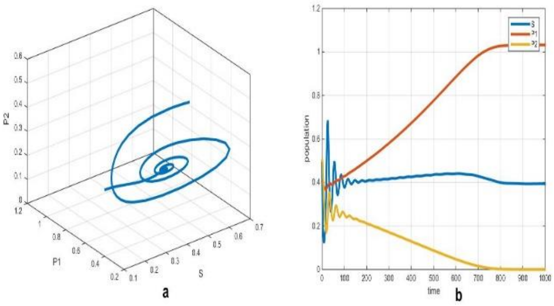


**Fig.** **(2.I) -**

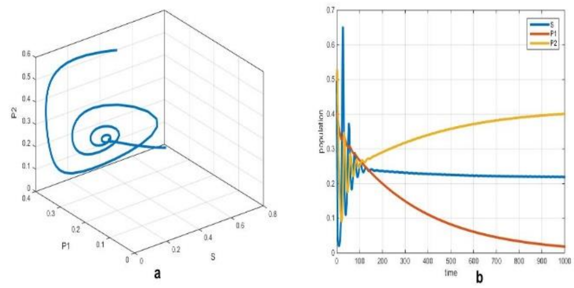


**Fig.** **(2.II)-**

**Fig.2- Time series trajectories of system (2) of $E^*$equilibrium point after recede $A_1$and $A_2$, rest of the values are at (50)**

**Fig.2 -** The trajectories of system (2) after recede $A_1$from $A_1$= 0.503 to $A_1$= 0.4 in **Fig.(2.I)** - the number of predator $P_2$ individual species fades and when $A_2$ recede from $A_2$=0.8 to $A_2$=0.5, in **Fig.(2.II)** - the number of predator $P_1$individuals species fades, with initial point (0.5, 0.4, 0.5) (a) 3D phase portrait of equilibrium $E^*$, (b) Time series for the attractor in (a).

The same way, depending on (50),if we change $\epsilon_1$ and $\epsilon_2$ (mutual interference in growth between the first and second predators, respectively, depending on eating the largest number of preys), by decreasing it from $\epsilon_1 = 0.626$ to $\epsilon_1 = 0.025$ it will cause a major disruption to the stability of the system shown in the figure **Fig.3 -** .

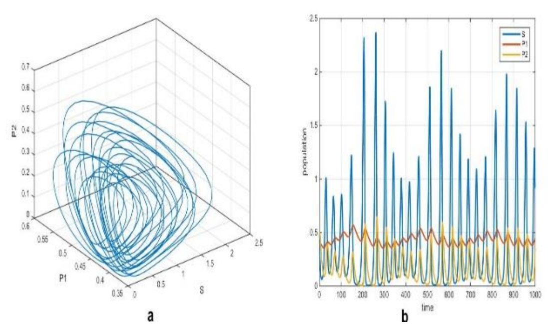


**Fig. (3.I)**

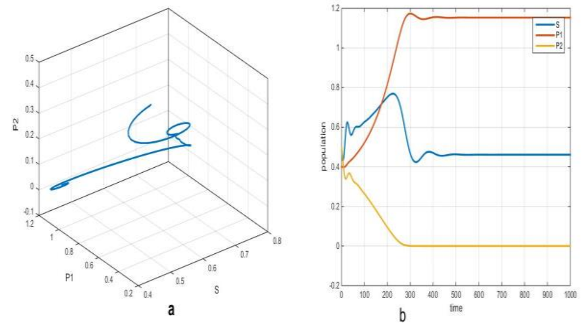


**Fig. (3.II)**

**Fig.3 - Time series trajectories of system (2) of $E^*$equilibrium point after recede $\in_1$and $\in_2$, rest of the values are at (50)**

**Fig.3-** The trajectories of system (2) after recede $\epsilon_1$from $\epsilon_1$= 0.652 to $\epsilon_1$= 0.025 a major disturbance occurs in the stability of system (2) in **Fig.** (**3.I**)- and if $\epsilon_2$ increase from $\epsilon_2$= 0.011 to $\epsilon_2$= 0.89 the predator population $P_1$ is steadily increasing to the predator population $P_2$ fades away to zero however the predator population $P_1$ is steadily increasing to $P_1$= 1.15 in **Fig.** (**3.II**)- , with initial point (0.5, 0.4, 0.5) , (a) 3D phase portrait of equilibrium $E^*$, (b) Time series for the attractor in (a).

we can summarize the effect of the parameters on system (2) stability in table (1)- for the equilibrium points $E^*, E_2$ *and* $E_3$, with initial point (0.5, 0.4, 0.5), (0.5, 0.4, 0) and (0.5, 0, 0.5) respectively as follow:

| **Parameter's value** | **Equilibrium point** | **Stability of the system** | **Figure** |
|---|---|---|---|
| $A_1 = 0.503, A_2 = 0.8, \epsilon_1 = 0.625, \epsilon_2 = 0.011, \lambda_1 = 0.0402, \lambda_2 = 0.2525, \theta_1 = 0.011, \theta_2 = 0.11, \alpha_1 = 0.01, \alpha_2 = 0.04$ | $E^*$ | ***locally asymptotically stable (LAS)*** | **Fig.1 -(1.I)** |
| $A_1 = 0.4, A_2 = 0.8, \epsilon_1 = 0.625, \epsilon_2 = 0.011, \lambda_1 = 0.0402, \lambda_2 = 0.2525, \theta_1 = 0.011, \theta_2 = 0.11, \alpha_1 = 0.01, \alpha_2 = 0.04$ | $E^*$ | ***Unstable*** (the number of predator $P_2$ individuals species fades) | **Fig.2 -(2.I)** |
| $A_1 = 0.503, A_2 = 0.4, \epsilon_1 = 0.625, \epsilon_2 = 0.011, \lambda_1 = 0.0402, \lambda_2 = 0.2525, \theta_1 = 0.011, \theta_2 = 0.11, \alpha_1 = 0.01, \alpha_2 = 0.04$ | $E^*$ | ***Unstable*** (the number of predator $P_1$ individuals species fades) | **Fig.2 -(2.II)** |
| $A_1 = 0.503, A_2 = 0.8, \epsilon_1 = 0.025, \epsilon_2 = 0.011, \lambda_1 = 0.0402, \lambda_2 = 0.2525, \theta_1 = 0.011, \theta_2 = 0.11, \alpha_1 = 0.01, \alpha_2 = 0.04$ | $E^*$ | ***Unstable*** (major disruption to the stability of the system) | **Fig.3-(3.I)** |
| $A_1 = 0.503, A_2 = 0.8, \epsilon_1 = 0.625, \epsilon_2 = 0.89, \lambda_1 = 0.0402, \lambda_2 = 0.2525, \theta_1 = 0.011, \theta_2 = 0.11, \alpha_1 = 0.01, \alpha_2 = 0.04$ | $E^*$ | ***Unstable*** (the predator population $P_2$ fades away to zero however the predator population $P_1$ is steadily increasing to $P_1$= 1.15) | **Fig.3 - (3.II)** |
| $A_1 = 0.503, A_2 = 0.8, \epsilon_1 = 0.625, \epsilon_2 = 0.011, \lambda_1 = 0.12, \lambda_2 = 0.2525, \theta_1 = 0.011, \theta_2 = 0.11, \alpha_1 = 0.01, \alpha_2 = 0.04$ | $E^*$ | ***Unstable*** (the predator population $P_2$ fades away to zero however the predator population $P_1$ is steadily increasing to $P_1$= 1.251) | **Fig.4- (4.I)** |

| | | | |
|---|---|---|---|
| $A_1 = 0.503, A_2 = 0.8, \epsilon_1 = 0.625, \epsilon_2 = 0.011,$ $\lambda_1 = 0.0402, \lambda_2 = 0.53, \theta_1 = 0.011, \theta_2 = 0.11,$ $\alpha_1 = 0.01, \alpha_2 = 0.04$ | $E^*$ | ***Unstable*** (the predator population $P_1$ fades away to zero however the predator population $P_2$ is steadily increasing to $P_2$= 0.7) | **Fig.4 - (4.II)** |
| $A_1 = 0.503, A_2 = 0.8, \epsilon_1 = 0.625, \epsilon_2 = 0.011, \lambda_1 = 0.0402, \lambda_2 = 0.2525, \theta_1 = 0.11, \theta_2 = 0.11,$ $\alpha_1 = 0.01, \alpha_2 = 0.04$ | $E^*$ | ***Unstable*** (the predator population $P_1$ fades away to zero however the predator population $P_2$ is steadily increasing to $P_2$= 0.59) | **Fig.5- (5.I)** |
| $A_1 = 0.503, A_2 = 0.8, \epsilon_1 = 0.625, \epsilon_2 = 0.011, \lambda_1 = 0.0402, \lambda_2 = 0.2525, \theta_1 = 0.011, \theta_2 = 0.11,$ $\alpha_1 = 0.01, \alpha_2 = 0.04$ | $E^*$ | ***Unstable*** (the predator population $P_1$ fades away to zero however the predator population $P_2$ is steadily increasing to $P_2$= 0.78) | **Fig.5 - (5.II)** |
| $A_1 = 0.503, A_2 = 0.8, \epsilon_1 = 0.625, \epsilon_2 = 0.011, \lambda_1 = 0.0402, \lambda_2 = 0.2525, \theta_1 = 0.011, \theta_2 = 0.11,$ $\alpha_1 = 0.4, \alpha_2 = 0.04$ | $E^*$ | ***Unstable*** (the predator population $P_1$ fades away to zero however the predator population $P_2$ is steadily increasing to $P_2$= 0.59) | **Fig.6 - (6.I)** |
| $A_1 = 0.503, A_2 = 0.8, \epsilon_1 = 0.625, \epsilon_2 = 0.011, \lambda_1 = 0.0402, \lambda_2 = 0.2525, \theta_1 = 0.011, \theta_2 = 0.11,$ $\alpha_1 = 0.01, \alpha_2 = 0.4$ | $E^*$ | ***Unstable*** (the predator population $P_2$ fades away to zero however the predator population $P_1$ is steadily increasing to $P_1$= 1.153) | **Fig.6- (6.II)** |
| $A_1 = 0.89, A_2 = 0.1, \epsilon_1 = 0.999, \epsilon_2 = 0.3, \lambda_1 = 0.245, \lambda_2 = 0.253, \theta_1 = 0.095, \theta_2 = 0.3, \alpha_1 = 0.9,$ $\alpha_2 = 0.041$ | $E_2$ | ***locally asymptotically stable (LAS)*** | **Fig.- (1.II)** |
| $A_1 = 0.5, A_2 = 0.1, \epsilon_1 = 0.9, \epsilon_2 = 0.3, \lambda_1 = 0.0402,$ $\lambda_2 = 0.253, \theta_1 = 0.11, \theta_2 = 0.3, \alpha_1 = 0.01, \alpha_2 = 0.041$ | $E_3$ | ***locally asymptotically stable (LAS)*** | **Fig. (1.III)** |

**Table 1- The stability of system (2) according to the parameters values at (50)**

The following figures are explained in the above Table (1)-

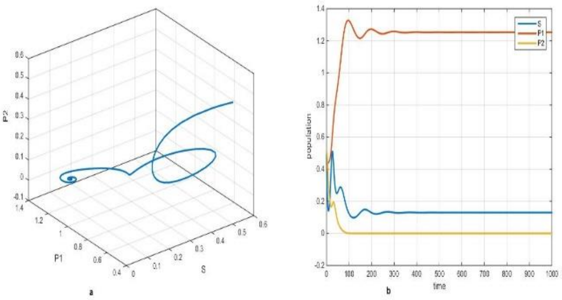

**Fig.(4.I)-**

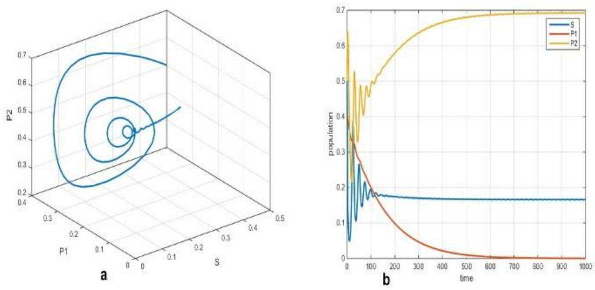

**Fig.(4.II) -**

**Fig.4 - Time series trajectories of system (2) of $E^*$equilibrium point after recede $\lambda_1$and $\lambda_2$, rest of the values are at (50)**

**Fig.4-** The trajectories of system (2) with initial point (0.5, 0.4, 0.5) by increasing $\lambda_1$from $\lambda_1$= 0.0402 to $\lambda_1$= 0.12 the predator population $P_2$ fades away to zero however the predator population $P_1$ is steadily increasing to $P_1$= 1.251in **Fig. (4.I)**-, and

increasing $\lambda_2$from $\lambda_2$= 0.253 to $\lambda_2$= 0.53 the predator population $P_1$ fades away to zero however the predator population $P_2$ is steadily increasing to $P_2$= 0.7 in **Fig.(4.II)**- (a) 3D phase portrait of equilibrium $E^*$, (b) Time series for the attractor in (a).

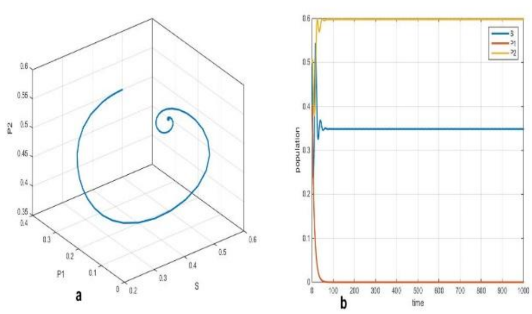


**Fig.(5.I) -**

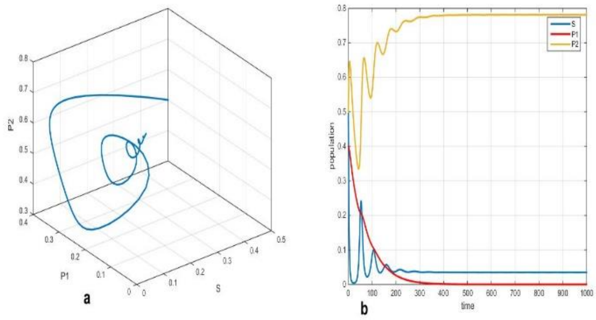


**Fig.(5.II) -**

**Fig.5 -Time series trajectories of system (2) of $E^*$equilibrium point after recede $\theta_1$and $\theta_2$, rest of the values are at (50)**

**Fig.5-** The trajectories of system (2) with initial point (0.5, 0.4, 0.5) by increasing $\theta_1$from $\theta_1$= 0.011 to $\theta_1$= 0.11 the predator population $P_1$ fades away to zero however the predator population $P_2$ is steadily increasing to $P_2$= 0.59 in **Fig.** (**5.I**), and decreasing $\theta_2$from $\theta_2$= 0.11 to $\theta_2$= 0.011 the predator population $P_1$ fades away to zero however the predator population $P_2$ is steadily increasing to $P_2$= 0.78 in **Fig.** (**5.II**)- (a) 3D phase portrait of equilibrium $E^*$, (b) Time series for the attractor in (a).

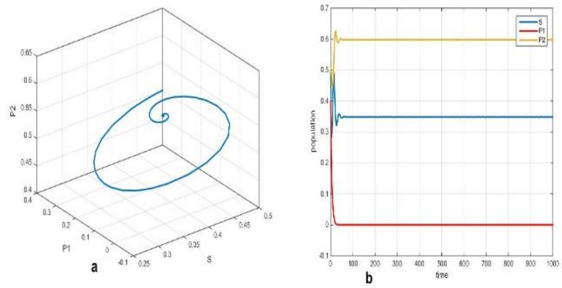


**Fig. (6.I)-**

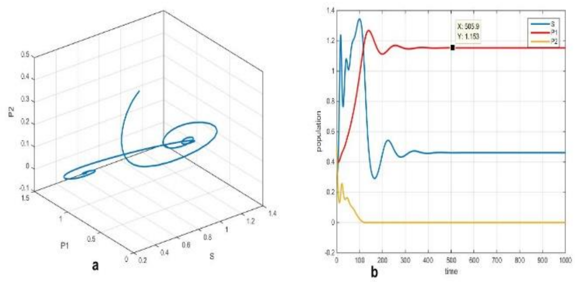


**Fig.(6.II)-**

**Fig.6 - Time series trajectories of system (2) of $E^*$equilibrium point after increasing $\alpha_1$and $\alpha_2$, rest of the values are at (50)**

**Fig.6 -** The trajectories of system (2) with initial point (0.5, 0.4, 0.5) by increasing $\alpha_1$from $\alpha_1$= 0.01 to $\alpha_1$= 0.4 the predator population $P_1$ fades away to zero however the predator population $P_2$ is steadily increasing to $P_2$= 0.59 in **Fig.(6.I)** -, and increasing $\alpha_2$from $\theta_2$= 0.04 to $\theta_2$= 0.4 the predator population $P_2$ fades away to zero however the predator population $P_2$ is steadily increasing to $P_2$= 1.153 ,in **Fig.(6.II)**- (a) 3D phase portrait of equilibrium $E^*$, (b) Time series for the attractor in (a).

## 10. Conclusion

A two predator- one prey ecological system had been studied briefly with many functional responses especially Beddington-DeAngelis and Holling type -Il and all the previous studies adopted one type of predation functional response for both predators. In this research two different types of functional response model were comprehended which reduce diverse responses from the predator for killing the prey, taking under consideration the competition between the two predators and the environment that combine the three individuals together for living, all of the solution's properties are studied. We obtained that there are only five nonnegative equilibrium points in the system. The topics of stability, feasibility, local bifurcations, and Hopf-bifurcation are all entirely scouted. The numerical simulation was used to examine global dynamics and determine the impact of changing parameters using a set of hypothetical data.

The next observation was locating:

- By modifying one parameter at a time, the effect of changing the parameter values as the environmental carrying capacity of the prey species $A_1$ recede from $A_1$= 0.503 to $A_1$= 0.4, the number of predator $P_2$ individuals species fades.
- If we change $\epsilon_1$ and $\epsilon_2$ (mutual interference in growth between the first and second predators, respectively, depending on eating the largest number of preys), by decreasing it from $\epsilon_1 = 0.626$ to $\epsilon_1 = 0.025$ it will cause a major disruption to the stability of system (2).
- Increasing $\lambda_1$from $\lambda_1$= 0.0402 to $\lambda_1$= 0.12 the predator population $P_2$ fades away to zero however the predator population $P_1$ is steadily increasing to $P_1$= 1.251, and increasing $\lambda_2$from $\lambda_2$= 0.253 to $\lambda_2$= 0.53 the predator population $P_1$ fades away to zero however the predator population $P_2$ is steadily increasing to $P_2$= 0.7. ($\lambda_1, \lambda_2$ are the maximum number of the prey that can be eaten by the first and second predator per unit time respectively)
- Increasing $\theta_1$from $\theta_1$= 0.011 to $\theta_1$= 0.11 the predator population $P_1$ fades away to zero however the predator population $P_2$ is steadily increasing to $P_2$= 0.59, and decreasing $\theta_2$from $\theta_2$= 0.11 to $\theta_2$= 0.011 the predator population $P_1$ fades away to zero however the predator population $P_2$ is steadily increasing to $P_2$= 0.78. ( $\theta_1, \theta_2$ are the rates at which the growth rate of the first predator is annihilated by the second predator and vice versa)
- Increasing $\alpha_1$from $\alpha_1$= 0.01 to $\alpha_1$= 0.4 the predator population $P_1$ fades away to zero however the predator population $P_2$ is steadily increasing to $P_2$= 0.59, and increasing $\alpha_2$from $\theta_2$= 0.04 to $\theta_2$= 0.4 the predator population $P_2$ fades away to zero however the predator population $P_2$ is steadily increasing to $P_2$= 1.153. ($\alpha_1$, $\alpha_2$ are the predators $P_1, P_2$ death rates respectively).

## References:


[1] Mira-Cristiana Anisiu, “LOTKA, VOLTERRA AND THEIR MODEL”, DIDACTICA MATHEMATICA, Vol. 32(2014), pp. 9–17

[2] R. K. Ghaziani., " Dynamics and bifurcations of a Lotka-Volterra population model", *Iranian Journal of Science & Technology, IJST*, vol. 38A3,pp. :265-279, 2014.

[3] C. S. HoLLING, the Components of Predation as Revealed by a Study of Small. Mammal Predation of the European Pine Sawfly, The Canadian Entomologist, Volume XCI, Number 5, May 1959, pp:234-261.

[4] PIERSMA, T., GILS, J.V., DE GOEIJ, P. and JAAP VAN DER MEER, Holling's functional response model as a tool to link the food-finding mechanism of a probing shorebird with its spatial distribution, Journal of Animal Ecology, vol.64, 1995, pp:493-504.

[5] Noori, Saba, An Analysis of a Partial Temporary Immunity SIR Epidemic Model with Nonlinear Treatment Rate, Baghdad Science Journal, Vol.16(3)2019, pp:639-647.

[6] Noori, Saba, Stability analysis of SIR holing type II infectious epidemic model with treatment failure rate, Mathematical Theory and Modeling, Vol.6, No.2, 2016, pp: 50-60.

[7] Noori, Saba, Local Bifurcation Analysis for a Special Type of SIR Epidemic Model, International Journal of Science and Research (IJSR), Volume 6 Issue 2, February 2017, pp:1143-1146.

[8] R. Arditi and L. Ginzburg, How Species Interact: Altering the Standard View on Trophic Ecology. Oxford, University Press USA. (2012).

[9] Zhihui Ma1, Shufan Wang, Stability analysis of prey-predator system with Holling type functional response and prey refuge, Springer, Advances in Difference Equations (2017).

[10] Noori, Saba and Naji, Raid Kamel, Dynamical Behavior of two predators one prey Ecological system with refuge and Beddington –De Angelis functional response, Iraqi Journal of Science, vol. 64, no.12, December 2023, pp: 6383-6400.

[11] Noori, Saba, Dynamical Study of an SIR Epidemic Model with Nonlinear Incidence Rate and Regress of Treatment, Ibn Al- Haitham Journal for Pure and Applied science,2017, Special Issue, pp: 384-396**.**

[12] Zhang, Shuwen, and Chen, Lansun, A study of predator-prey models with the Beddington-DeAnglis functional response and impulsive effect, Chaos, Solitons and Fractals, Volume 27, January 2006, Issue 1, p. 237-248.

[13] Raveendra Babu A.1 and Gayathri P, Mathematical Study of One Prey and Two Competing Predators Considering Beddington-DeAngelis Functional Response with Distributed Delay, Hilbert Journal of Mathematical Analysis, Vol. 2, no.1 (2023), pp: 1–19.

[14] J. Heinonen. " Lectures on Analysis on Metric Spaces", *Springer Science Business Media New York*, pp. 43-48, 2001.

[15] Henk Nijmeijer • Arjan van der Schaft , "Nonlinear Dynamical Control Systems", Springer Science and Business Media LLC New York, March 2013,p.p. 277-282.

[16] O. Katsuhiko. "Modern Control Engineering", *Englewood Cliffs, NJ: Prentice-Hall*, pp. 252–258, 1970.

[17] L. Perko . " Differential Equations and Dynamical Systems", *Springer, New York, NY, USA,* 3rd edition, pp. 23-101, 2001.

[18] T. Erneux ,P. Glorieux. " Laser Dynamics", *Cambridge University Press*, pp. 84 – 108, 2010.