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Application of the restricted fractional differential transform for Solve linear fractional integro-differential equations

Ayad R. Khudair, Sanaa L. khalaf, S.A.M. Haddad

Department of Mathematics, Faculty of Science, University of Basrah, Basrah, Iraq,

Abstract:

In 2007, Arikoglu and Ozkol developed a new semi-analytical numerical technique, fractional differential transform method (FDTM), for solving fractional differential equations (FDEs) and then developed it to solve fractional integro-differential equations (FIDEs). In fact, when the order of fractional derivative is irrational, FDTM was not achieved for solving both FDEs nor FIDEs. In 2017, Ayad et al. propose a new method, restricted fractional differential transform method (RFDTM), to be appropriate for solving rational or irrational order FDEs. In this study, we establish a new theorems to extend the RFDTM to solve linear FIDEs. Some illustrated examples are investigated to demonstrate the theoretical results.

Keywords: Fractional differential equations, two dimensional differential transformation, Fractional differential transform, restricted fractional differential transform, fractional integro-differential equations.

1-Introduction:

Fractional differential equations (FDEs) and fractional integral equations (FIDEs) play a prominent role in many disciplines including engineering, physics, economics, and biology, such as electrochemical processes [1,2], viscoelastic materials[3,4], control engineering [5,6], signal processing [7], image processing [8], bioengineering [9]. In addition, Fractional integro-differential equations are very common in the description of various physical phenomena like heat conduction in materials with memory, diffusion processes, combined conduction, convection and radiation problems [10,11,12,13]. Consequently, the application of fractional differential equations can be encountered in numerous research areas, see, e.g. Oldham and Spanier [14], Miller, B. Ross [15], and Podlubny [16], and the many references therein.

Recently, many authors have made a number of Studies for solving fractional differential equations and fractional integral equations, see, e.g. Li and Hu [17], Kumar and Singh [18], Khudair [19], Khalaf et al. [20], Li and Zhao [21], Katsikadelis [22], Zhu and Wang [23,25], Singh et al. [24], Shoja [26], and Khudair et al. [27]. However, different techniques have been suggested for finding numerical solutions of fractional integro-differential equations, such as Adomian decomposition method [28,29], collocation method [30,31,32], variational iteration method [33,34], homotopy perturbation method [34], Taylor expansion method [35], iterated Galerkin methods [36], Legendre wavelet method [37], Euler wavelet method [38], Discrete Galerkin method [39].

In recent paper Arikoglu and Ozkol [40] presented fractional differential transform method (FDTM) for solving fractional differential equations, and then extended to solve linear and nonlinear fractional integro- differential equations [41]. In fact, this method was not achieved for solving irrational order fractional differential equations . The present writers [27], proposed restricted fractional differential transform method (RFDTM) which is based on the restriction of the classical two dimensional differential transform methods to solve irrational order fractional differential equations. They showed that RFDTM is not entailed any integration or any complex manipulations even if the FDEs is content high non- linearity terms.

The goal of this paper is to employ RFDTM for solving the following linear fractional integro-differential equations

$$^{C}D^{\alpha}u(x)=f(x)+\int_0^x (b_1(x,t)u(t)+b_2(x,t)\,^{C}D^{\alpha}u(t))dt,\quad n-1<\alpha<n,\ n\in Z^{+}$$

and $u^{(k)}(0)=A_k$, $k=0,1,\dots,n-1$,

where $b_1(x,t), b_2(x,t)$ and $f(x)$ are smooth functions. The configuration of this paper is the following. Section 2 is devoted to present RFDTM. The fundamental mathematical operations performed by RFDTM are developed in Section 3. In Section 4, we present some examples. The paper ends with conclusion.

2- Restricted fractional differential transform method

There are many definitions of fractional order derivative, e.g. Riemann–Liouville derivative, Grünwald–Letnikov derivative, Caputo derivative, Sonin–Letnikov derivative, Miller–Ross derivative, Hadamard derivative, Weyl derivative, Marchaud derivative, Riesz-Miller derivative, Erdélyi–Kober derivative [15,16,41]. Caputo derivative is always used in FDEs to express of many real world physical problems since it has the advantage of defining integer order initial conditions. The Caputo derivative for any analytic function $u(x)$ is defined by [40]

$$^{C}D^{\alpha}_{x_0}u(x)=\frac{1}{\Gamma(n-\alpha)}\int_{x_0}^{x}\frac{u^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt,\quad n-1<\alpha<n,\ n\in Z^{+} \tag{3}$$

Let $f(x,y):R^2\to R$ be analytical function then it can be express as multi Taylor series about (x_0,y_0) as follows:

$$f(x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{i!\,j!}\left(\frac{\partial^{i+j}f(x,y)}{\partial x^i\partial y^j}\right)_{(x_0,y_0)}(x-x_0)^i(y-y_0)^j \tag{4}$$

By setting

$$F(i,j)=\frac{1}{i!\,j!}\left(\frac{\partial^{i+j}f(x,y)}{\partial x^i\partial y^j}\right)_{(x_0,y_0)} \tag{5}$$

Then

$$f(x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}F(i,j)(x-x_0)^i(y-y_0)^j \tag{6}$$

Clearly, $F(i, j)$ in Eq. (5) is the two dimensions differential transform of the function $f(x, y)$, while Eq. (6) represent the differential inverse transform of $F(i, j)$.

Now, If $u(x) = f(x, y)\Big|_{y=(x-x_0)^{\alpha}+y_0}$, where $\alpha > 0$, that is, the two dimensional function $f(x, y)$ is restricted to one dimensional function $u(x)$, then Eq. (5) and Eq. (6) respectively, become

$$U(i, j) = F(i, j) = \frac{1}{i!j!}\left(\frac{\partial^{i+j} f(x, y)}{\partial x^i \partial y^j}\right)_{(x_0, y_0)} \quad (7)$$

$$u(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} U(i, j)(x - x_0)^{i+\alpha j} \quad (8)$$

Eq. (7) is called the restricted fractional differential transform (RFDTM), while Eq. (8) is called inverse of RFDTM.

Now, let $u(x)$, $v(x)$ and $w(x)$ can be express as $u(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} U(i, j) x^{i+\alpha j}$, $v(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} V(i, j) x^{i+\alpha j}$ and $w(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} W(i, j) x^{i+\alpha j}$ respectively, then the fundamental mathematical operations performed by RFDTM are introduced in the following theorems.

Theorem (1): If $w(x) = u(x) + v(x)$ then $W(i, j) = U(i, j) + V(i, j)$, for $i \ge 0$, $j \ge 0$

Theorem (2): If $w(x) = u(x)v(x)$ then $W(i, j) = \sum_{k=0}^{j}\sum_{r=0}^{i} U(r, j-k) V(i-r, k)$ for $i \ge 0$, $j \ge 0$

Theorem (3): If $v(x) = x^{m+\alpha n} u(x)$, where m and n are an integer number then

$V(i, j) = 0$ for $i < m$ or $j < n$

$V(i, j) = U(i-m, j-n)$ for $i \ge m$ and $j \ge n$

Theorem (4): If $v(x) = {}^{C}D^{\alpha} u(x)$, $[\alpha] < \alpha < [\alpha]+1$ then

$U(i,0) = 0$, $i = [\alpha]+1, [\alpha]+2, [\alpha]+3, \ldots$

$$V(i, j) = U(i, j+1)\frac{\Gamma(i+\alpha j+\alpha+1)}{\Gamma(i+\alpha j+1)} \quad \text{for } i \ge 0,\ j \ge 0$$

Theorem (5): If $v(x) = \int_0^x u(t)dt$ then $V(0, j) = 0$, for $j \ge 0$ and

$$V(i, j) = \frac{U(i-1, j)}{i+\alpha j}, \quad \text{for } j \ge 0, i \ge 1.$$

Proof (5)

First rewrite $v(x)$ as follows,

$$v(x)=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}V(i,j)x^{i+\alpha j}=\sum_{j=0}^{\infty}V(0,j)x^{\alpha j}+\sum_{j=0}^{\infty}\sum_{i=1}^{\infty}V(i,j)x^{i+\alpha j} \quad .$$

Then compute,

$$\int_0^x u(t)dt=\int_0^x\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}U(i,j)t^{i+\alpha j}dt$$

$$=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}U(i,j)\int_0^x t^{i+\alpha j}dt$$

$$=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}U(i,j)\frac{x^{i+\alpha j+1}}{i+\alpha j+1}$$

$$=\sum_{j=0}^{\infty}\sum_{i=1}^{\infty}U(i-1,j)\frac{x^{i+\alpha j}}{i+\alpha j}$$

Since $v(x)=\int_0^x u(t)dt$, one can get

$$\sum_{j=0}^{\infty}V(0,j)x^{\alpha j}+\sum_{j=0}^{\infty}\sum_{i=1}^{\infty}V(i,j)x^{i+\alpha j}=\sum_{j=0}^{\infty}\sum_{i=1}^{\infty}U(i-1,j)\frac{x^{i+\alpha j}}{i+\alpha j}.$$

So, one can have

$$V(0,j)=0, \quad \text{for } j\ge 0 \text{ and } V(i,j)=\frac{U(i-1,j)}{i+\alpha j}, \quad \text{for } j\ge 0, i\ge 1.$$

Theorem (6): If $w(x)=\int_0^x h(x,t)u(t)dt$ and $h(x,t)=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}H(i,j)x^i t^j$ then

$W(i,j)=0,\ i\le j,\ j=0,1,2,\ldots$

$$W(i,j)=\sum_{r=0}^{\infty}\sum_{k=0}^{i-j-1}\frac{H(k,i-j-k-1)\,U(r,j)}{\alpha j+i-k}, \quad j<i,\ i=1,2,\ldots$$

Proof (6)

$$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}W(i,j)x^{i+\alpha j}=\int_0^x\left(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}H(i,j)x^i t^j\right)\left(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}U(i,j)t^{i+\alpha j}\right)dt=\int_0^x\left(\sum_{j=0}^{\infty}\beta_j t^j\right)\left(\sum_{j=0}^{\infty}\gamma_j t^{\alpha j}\right)$$

where $\gamma_j=\sum_{i=0}^{\infty}U(i,j)t^i$, $\beta_j=\sum_{i=0}^{\infty}H(i,j)x^i$

$$\int_0^x\left(\sum_{j=0}^{\infty}\beta_j t^j\right)\left(\sum_{j=0}^{\infty}\gamma_j t^{\alpha j}\right)dt=\int_0^x\sum_{j=0}^{\infty}\sum_{k=0}^{j}\beta_k\gamma_{j-k}t^{\alpha j-\alpha k+k}dt$$

$$=\int_0^x\sum_{j=0}^{\infty}\omega_j t^{\alpha j}dt,\quad \omega_j=\sum_{k=0}^{j}\beta_k\gamma_{j-k}t^{k-\alpha k}$$

$$\omega_j=\sum_{k=0}^{j}\left(\sum_{i=0}^{\infty}H(i,k)x^i\right)\left(\sum_{i=0}^{\infty}U(i,j-k)t^i\right)t^{k-\alpha k}$$

$$=\sum_{k=0}^{j}\sum_{i=0}^{\infty}\sum_{r=0}^{i}H(r,k)U(i-r,j-k)x^r t^{i-r}t^{k-\alpha k}$$

$$=\sum_{i=0}^{\infty}\sum_{k=0}^{j}\sum_{r=0}^{i}H(r,k)U(i-r,j-k)x^r t^{k-\alpha k+i-r}$$

So,

$$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}W(i,j)x^{i+\alpha j}=\int_0^x\sum_{j=0}^{\infty}\omega_j t^{\alpha j}dt=\int_0^x\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{j}\sum_{r=0}^{i}H(r,k)U(i-r,j-k)x^r t^{\alpha(j-k)+i+k-r}dt,$$

$$=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{j}\sum_{r=0}^{i}H(r,k)U(i-r,j-k)\frac{x^{\alpha(j-k)+i+k+1}}{\alpha(j-k)+i+k-r+1},$$

$$=\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=k}^{\infty}\sum_{i=r}^{\infty}H(r,k)\;U(i-r,j-k)\frac{x^{\alpha(j-k)+i+k+1}}{\alpha(j-k)+i+k-r+1},$$

That is,

$$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}W(i,j)x^{i+\alpha j}=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}H(r,k)U(i,j)\frac{x^{\alpha j+i+r+k+1}}{\alpha j+i+k+1},$$

By comparing the coefficients of x , one can have

$W(i,j)=0,\ \ i\le j,\ \ j=0,1,2,\ldots$

$$W(i,j)=\sum_{r=0}^{\infty}\sum_{k=0}^{i-j-1}\frac{H(k,i-j-k-1)\,U(r,j)}{\alpha j+i-k},\quad j<i,\ \ i=1,2,\ldots$$

Theorem (7): If $v(x)=\int_0^x t^{m+\alpha n}u(t)dt$, where m and n are an integer number then

$V(0,j)=0,$ for $j\ge 0$, $V(i,j)=0$ for $i<m+1$ *or* $j<n,$ and

$$V(i,j)=\frac{U(i-1-m,j-n)}{i+\alpha j},\quad \text{for}\quad i\ge m+1\ and\ \ j\ge n.$$

Proof (7)

Let the restricted fractional differential transform of $t^{m+\alpha n}u(t)$ be $F(i,j)$. Then, by using Theorem 6, one can get

$$V(0,j)=0, \quad \text{for } j\geq 0 \text{ and } V(i,j)=\frac{F(i-1,j)}{i+\alpha j}, \quad \text{for } j\geq 0, i\geq 1.$$

Using Theorem 3, we have

$$F(i,j)=0 \quad \text{for } i<m \text{ or } j<n$$
$$F(i,j)=U(i-m,j-n) \quad \text{for } i\geq m \text{ and } j\geq n$$

By utilizing these values, one can deduce

$$V(0,j)=0, \quad \text{for } j\geq 0, \quad V(i,j)=0 \quad \text{for } i<m+1 \text{ or } j<n, \quad \text{and}$$

$$V(i,j)=\frac{U(i-1-m,j-n)}{i+\alpha j}, \quad \text{for } i\geq m+1 \text{ and } j\geq n.$$

Theorem (8): If $v(x)=\int_0^x {}^{C}D^{\alpha}u(t)dt$, $[\alpha]<\alpha<[\alpha]+1$ then $V(0,j)=0$, for $j\geq 0$,

$$U(i,0)=0,\ i=[\alpha]+1,[\alpha]+2,[\alpha]+3,\ldots \text{ and } V(i,j)=\frac{\Gamma(i+\alpha j+\alpha)}{\Gamma(i+\alpha j)}U(i-1,j+1), \quad \text{for } j\geq 0, i\geq 1.$$

Proof (8) Let the restricted fractional differential transform of ${}^{C}D^{\alpha}u(t)$, $[\alpha]<\alpha<[\alpha]+1$ be $F(i,j)$. Then, by using Theorem 5, one can get

$$V(0,j)=0, \quad \text{for } j\geq 0 \text{ and } V(i,j)=\frac{F(i-1,j)}{i+\alpha j}, \quad \text{for } j\geq 0, i\geq 1.$$

Using Theorem 4, we have

$$U(i,0)=0,\ i=[\alpha]+1,[\alpha]+2,[\alpha]+3,\ldots$$
$$F(i,j)=U(i,j+1)\frac{\Gamma(i+\alpha j+\alpha+1)}{\Gamma(i+\alpha j+1)} \quad \text{for } i\geq 0,\ j\geq 0.$$

By utilizing these values, one can deduce

$$V(0,j)=0, \quad \text{for } j\geq 0, \quad U(i,0)=0,\ i=[\alpha]+1,[\alpha]+2,[\alpha]+3,\ldots \text{ and}$$
$$V(i,j)=\frac{\Gamma(i+\alpha j+\alpha)}{\Gamma(i+\alpha j)}U(i-1,j+1), \quad \text{for } j\geq 0, i\geq 1.$$

Theorem (9): If $w(x)=\int_0^x h(x,t)\,{}^CD^{\alpha}u(t)dt$, $[\alpha]<\alpha<[\alpha]+1$ and $h(x,t)=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}H(i,j)x^it^j$ then

Proof (9)

Let $v(t)={}^CD^{\alpha}u(t)$, $[\alpha]<\alpha<[\alpha]+1$. Then, by using Theorem 4, one can get

$$V(0,j)=0, \quad \text{for } j\geq 0 \text{ and } V(i,j)=\frac{U(i-1,j)}{i+\alpha j}, \quad \text{for } j\geq 0, i\geq 1.$$

From $w(x)=\int_0^x h(x,t)v(t)dt$, and theorem 6, one can have

$$W(i,j)=0, \ i\leq j, \ j=0,1,2,\ldots$$

$$W(i,j)=\sum_{r=0}^{\infty}\sum_{k=0}^{i-j-1}\frac{H(k,i-j-k-1)V(r,j)}{\alpha j+i-k}, \quad j<i, \ i=0,1,2,\ldots$$

By utilizing these values, one can deduce

3. Illustrated examples:

This section is focused to demonstrating the applicability of RFDTM to solve linear fractional integro-differential equations. Three examples are investigated to demonstrate the theoretical results. The accuracy and effectiveness of RFDTM depend on the number of terms as we will show in illustrated examples

Example 1: Consider the following linear fractional integro-differential equation

${}^CD^{\alpha}u(x)-2u(x)=-1+\int_0^x u(t)dt$, $1<\alpha\leq 2$, with initial boundary conditions $u(0)=1$, $u'(0)=-1$. when $\alpha=2$ the exact solution is $u(x)=e^{-x}$.

By applying RFDTM to the given problem, one get

$$U(0,0)=1$$

$$U(1,0)=-1$$

$$U(i,0)=0, \qquad i=2,3,4,\ldots$$

$$\frac{\Gamma(i+\alpha j+\alpha+1)}{\Gamma(i+\alpha j+1)}U(i,j+1)-2U(i,j)=\begin{cases}-1 & i=j=0\\ 0 & i=0,\ j\geq 1\\ \dfrac{U(i-1,j)}{i+\alpha j} & i\geq 1,\ j\geq 0\end{cases}$$

Now, we solve the above linear forward system by using Maple software to have

$$u(x)=1-x+\frac{x^{\alpha}}{\Gamma(1+\alpha)}-\frac{x^{1+\alpha}}{\Gamma(2+\alpha)}-\frac{x^{2+\alpha}}{\Gamma(3+\alpha)}+\frac{2x^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{2}{\Gamma(2+\alpha)}\frac{\Gamma(2+\alpha)}{\Gamma(2+2\alpha)}x^{1+2\alpha}+$$
$$\frac{1}{\Gamma(1+\alpha)(1+\alpha)}\frac{\Gamma(2+\alpha)}{\Gamma(2+2\alpha)}x^{1+2\alpha}-\frac{2}{\Gamma(3+\alpha)}\frac{\Gamma(3+\alpha)}{\Gamma(3+2\alpha)}x^{2+2\alpha}-\frac{1}{\Gamma(2+\alpha)(2+\alpha)}\frac{\Gamma(3+\alpha)}{\Gamma(3+2\alpha)}x^{2+2\alpha}+\cdots$$

If $\alpha=2$, we have

$$u(x)=1-x+\frac{1}{2}x^{2}-\frac{1}{6}x^{3}+\frac{1}{24}x^{4}-\frac{1}{120}x^{5}+\frac{1}{720}x^{6}-\frac{1}{5040}x^{7}+\cdots=e^{-x}$$

Example 2: Consider the following linear fractional integro-differential equation

$$^{C}D^{\alpha}u(x)=1+\frac{1}{2}\int_0^x (u(t)+{}^{C}D^{\alpha}u(t))dt, \qquad 1<\alpha\leq 2,$$

with initial boundary conditions $u(0)=1,\ u'(0)=1$. When $\alpha=2$ the exact solution is $u(x)=\sinh(x)+\cosh(x)$.

By applying RFDTM to the given problem, one get

$U(0,0)=1$

$U(1,0)=-1$

$U(i,0)=0, \qquad i=2,3,4,\dots$

$$\frac{\Gamma(i+\alpha j+\alpha+1)}{\Gamma(i+\alpha j+1)}U(i,j+1)=\begin{cases}-1 & i=j=0\\ 0 & i=0,\ j\geq 1\\ \dfrac{1}{2}\left(\dfrac{U(i-1,j)}{i+\alpha j}+\dfrac{U(i-1,j+1)\Gamma(i+\alpha j+\alpha)}{\Gamma(i+\alpha j+1)}\right) & i\geq 1,\ j\geq 0\end{cases}$$

$$u(x)=1+x+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{1+\alpha}}{\Gamma(2+\alpha)}+\frac{x^{2+\alpha}}{\Gamma(3+\alpha)}+\frac{1}{2}\frac{\Gamma(2+\alpha)x^{1+2\alpha}}{\Gamma(2+2\alpha)\Gamma(1+\alpha)(1+\alpha)}+$$
$$\frac{\Gamma(3+\alpha)}{2\Gamma(2+\alpha)(2+\alpha)\Gamma(3+2\alpha)}x^{2+2\alpha}+\frac{\Gamma(2+\alpha)}{4\Gamma(1+\alpha)(1+\alpha)\Gamma(3+2\alpha)}x^{2+2\alpha}$$

If $\alpha=2$, we have

$$u(x)=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\frac{1}{720}x^6+\frac{1}{5040}x^7+\frac{1}{40320}x^8+\frac{1}{362880}x^9+\cdots$$

Example 3: Consider the linear system of fractional integro-differential equations

$$\begin{aligned}{}^{C}D^{\alpha}u_1(x)&=1+x+x^2-u_1(x)-\int_0^x(u_1(t)+u_2(t))dt\\ {}^{C}D^{\alpha}u_2(x)&=-1-x+u_1(x)-\int_0^x(u_1(t)-u_2(t))dt\end{aligned}\quad,\qquad 0<\alpha\le 1$$

with initial conditions $u_1(0)=1,\ u_2(0)=-1$.

By applying RFDTM to the given problem, one get

$$U_1(0,0)=1,\quad U_2(0,0)=-1$$

$$U_1(i,0)=0,\quad U_2(i,0)=0,\qquad i=1,2,3,\dots$$

$$\frac{\Gamma(i+\alpha j+\alpha+1)}{\Gamma(i+\alpha j+1)}U_1(i,j+1)=\begin{cases}1 & i=0,j=0\\ 1-U_2(i,j)-\dfrac{U_1(i-1,j)}{i+\alpha j}-\dfrac{U_2(i-1,j)}{i+\alpha j} & i=1,2\quad j\ge 0\\ -U_2(i,j)-\dfrac{U_1(i-1,j)}{i+\alpha j}-\dfrac{U_2(i-1,j)}{i+\alpha j} & i\ge 3\quad j\ge 0\end{cases}$$

$$\frac{\Gamma(i+\alpha j+\alpha+1)}{\Gamma(i+\alpha j+1)}U_2(i,j+1)=\begin{cases}-1 & i=0,j=0\\ -1+U_1(i,j)-\dfrac{U_1(i-1,j)}{i+\alpha j}+\dfrac{U_2(i-1,j)}{i+\alpha j} & i=1\quad j\ge 0\\ U_1(i,j)-\dfrac{U_1(i-1,j)}{i+\alpha j}+\dfrac{U_2(i-1,j)}{i+\alpha j} & i\ge 2\quad j\ge 0\end{cases}$$

Now, we solve the above linear forward system by using Maple software to have

$$u_1(x)=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{1+\alpha}}{\Gamma(2+\alpha)}+\frac{2x^{2+\alpha}}{\Gamma(3+\alpha)}-\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha+1}}{\Gamma(2+2\alpha)}+\frac{2x^{2+2\alpha}}{\Gamma(3+2\alpha)}-\frac{2x^{3+2\alpha}}{\Gamma(4+2\alpha)}-\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}-\frac{x^{3\alpha+1}}{\Gamma(2+3\alpha)}+\frac{2x^{3+3\alpha}}{\Gamma(4+3\alpha)}+\cdots$$

$$u_2(x)=-1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}-\frac{3x^{1+\alpha}}{\Gamma(2+\alpha)}+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha+1}}{\Gamma(2+2\alpha)}-\frac{2x^{2+2\alpha}}{\Gamma(3+2\alpha)}-\frac{2x^{3+2\alpha}}{\Gamma(4+2\alpha)}-\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{3x^{3\alpha+1}}{\Gamma(2+3\alpha)}+\frac{2x^{2+3\alpha}}{\Gamma(3+3\alpha)}-\frac{6x^{3+3\alpha}}{\Gamma(4+3\alpha)}+\cdots$$

When $\alpha=\frac{3}{4}$ the solutions are

$$u_1(x)=1+\frac{4}{3}\frac{x^{3/4}}{\Gamma\left(\frac{3}{4}\right)}+\frac{16}{21}\frac{x^{7/4}}{\Gamma\left(\frac{3}{4}\right)}+\frac{128}{231}\frac{x^{11/4}}{\Gamma\left(\frac{3}{4}\right)}-\frac{4}{3}\frac{x^{3/2}}{\sqrt{\pi}}+\frac{8}{15}\frac{x^{5/2}}{\sqrt{\pi}}+\frac{32}{105}\frac{x^{7/2}}{\sqrt{\pi}}-\frac{64}{945}\frac{x^{9/2}}{\sqrt{\pi}}-\frac{32}{45}\frac{\sqrt{2}\Gamma\left(\frac{3}{4}\right)x^{9/4}}{\pi}-\frac{128}{585}\frac{\sqrt{2}\Gamma\left(\frac{3}{4}\right)x^{13/4}}{\pi}+\frac{4096}{208845}\frac{\sqrt{2}\Gamma\left(\frac{3}{4}\right)x^{21/4}}{\pi}+\cdots$$

$$u_2(x)=-1+\frac{4}{3}\frac{x^{3/4}}{\Gamma\left(\frac{3}{4}\right)}-\frac{16}{7}\frac{x^{7/4}}{\Gamma\left(\frac{3}{4}\right)}+\frac{4}{3}\frac{x^{3/2}}{\sqrt{\pi}}+\frac{8}{15}\frac{x^{5/2}}{\sqrt{\pi}}-\frac{32}{105}\frac{x^{7/2}}{\sqrt{\pi}}-\frac{64}{945}\frac{x^{9/2}}{\sqrt{\pi}}-\frac{32}{45}\frac{\sqrt{2}\Gamma\left(\frac{3}{4}\right)x^{9/4}}{\pi}+\frac{128}{195}\frac{\sqrt{2}\Gamma\left(\frac{3}{4}\right)x^{13/4}}{\pi}+\frac{1024}{9945}\frac{\sqrt{2}\Gamma\left(\frac{3}{4}\right)x^{17/4}}{\pi}-\frac{4096}{69615}\frac{\sqrt{2}\Gamma\left(\frac{3}{4}\right)x^{21/4}}{\pi}+\cdots$$

4. Conclusion

The DTM is an excellent tool for solving ordinary (partial) differential equations. Arikoglu and Ozkol was modified the DTM to be suitable for solving FDEs and FIDEs. This modification is called FDTM. In fact, FDTM is excellent when the order of fractional derivative in the considering FDEs and FIDEs is a specific given rational value. There are two cases FDTM cannot be applied. The first is, when the order of fractional derivative is unspecific value in a given interval, say, $0<\alpha<1$. The second is, when the order of fractional derivative is a specific irrational number, like $\alpha=\sqrt{2}$. To overcome these difficulties, Ayad et al. [27] introduced RFDTM to solve the fractional differential equation even if the order of fractional derivative is real (irrational or rational) or unspecific value. In this paper, we establish a new theorems to extend the RFDTM to solve linear FIDEs. Some illustrated examples are investigated to demonstrate the theoretical results.

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