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SOLUTION OF SOME TYPES FOR MULTI-FRACTIONAL ORDER DIFFERENTIAL EQUATIONS CORRESPONDING TO OPTIMAL CONTROL PROBLEMS

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Abstract:

This paper introduces a Multi-Order Fractional Optimal Control Problems in which the dynamic system involves integer and fractional-order derivatives are introduced in the Caputo sense. We derive the necessary optimality conditions in terms of the associated Hamiltonian, and we construct an approximation of the right Riemann−Liouville fractional derivatives and solve the fractional boundary value problems by the spectral method. Numerical methods rely on the spectral method where Chebyshev polynomials are used to approximate the unknown functions. Chebyshev polynomials are widely used in numerical computation to solve the problems are presented.

Keywords: Fractional calculus; Caputo fractional derivatives ;fractional order optima control; Chebyshev spectral method.

1. Introduction

A Fractional optimal control problems (FOCPs) is an optimal control problem in which either the performance index or the differential equations governing the dynamics of the system or both contain at least one fractional order derivative term. The formulation and solution scheme for FOCPs was first by established by Agrawal where the applied Fractional Variational Calculus (FVC) and presented a general formulation and solution scheme for FOCPs in the Riemann-Liouville (RL) sense that was based on variational virtual work

coupled with the Lagrange multiplier technique. Since the Caputo Fractional Derivative (CFD) seems more natural and allows incorporating the usual initial conditions, it becomes a popular choice for researchers. [10,16,7]

FOCPs are optimal control problems associated with fractional dynamic systems. The fractional optimal control theory is a very new topic in mathematics. FOCPs may be defined in terms of different types of fractional derivatives. The Chebyshev polynomials method was applied to solve FOCPs, Chebyshev polynomials $T_n(t)$, of degree n are important in approximation theory because the roots of the Chebyshev polynomials of the first kind, which are also called Chebyshev nodes, are used as nodes in polynomial interpolation. In Agrawal[1], Agrawal and Baleanu [2], the authors obtained necessary conditions for FOCPs with the Riemann-Liouville derivative and were able to solve the problem numerically. Agrawal [3], presented a quadratic numerical scheme for a class of fractional optimal control problems (FOCPs). In Agrawal [4], the FOCPs are formulated for a class of distributed systems where the fractional derivative is defined in the Caputo sense, and a numerical technique for FOCPs presented. Baleanu et al. [8], used a direct numerical scheme to find a solution of the FOCPs.

In [11],M. M. Khader and A. S. Hendy studied an efficient numerical scheme for solving fractional optimal control problems. In [8], Dumitru Baleanu, Ozlem Defterli and Om P. Agrawal studied a central difference numerical scheme for fractional optimal control problems.In [5],T. Akbarian and M. Keyanpour studied a new approach to the numerical solution of fractional order optimal control Problems. In [15], N.H. Sweilam, T.M.Al-Ajmi and R.H.W. Hoppe studied numerical solution of some types of fractional optimal control problems.

This paper is organized as follows: In Section 2 we present some basic notations and preliminaries as well as properties of the shifted Chebyshev polynomials are introduced. In section 3 contains the necessary optimality conditions of multi-order fractional optimal control problems model. In section 4 we present numerical approximations of the left CFD and the right RLFD using Chebyshev polynomials .In section 5, we give numerical examples to solve FOCPs and show the accuracy of the presented method. Finally, section 6, we conclude the paper.

2. Basic Notations and Preliminaries.

In this section, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

2.1 Fractional Derivatives and Integrals.

Definition (2.1.1):

Let J $=[a,b]$ $(-\infty < a < b < \infty)$ be a finite interval of R. The left and right Riemann−Liouville fractional integrals ${}_aD_t^{-\alpha}f(t)$ and ${}_tD_b^{-\alpha}f(t)$ of order $\alpha \in R^+$, are defined by

$$ {}_aD_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \qquad t > a, \quad \alpha > 0 \qquad (2.1)$$

$$ {}_tD_b^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1} f(\tau)d\tau, \qquad t < b, \quad \alpha > 0, \qquad (2.2)$$

where $\Gamma(\cdot)$ is the Gamma function defined for any complex number z as

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1} dt, \qquad (2.3)$$

The left and right Riemann-Liouvillefractional derivatives ${}_aD_t^{\alpha}f(t)$ and ${}_tD_b^{\alpha}f(t)$ of order $\alpha \in R_+$, are defined by

$$ {}_aD_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau, \qquad t > a \qquad (2.4)$$

$$ {}_tD_b^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}(-1)^n\frac{d^n}{dt^n}\int_t^b (\tau-t)^{n-\alpha-1} f(\tau)d\tau, \qquad t < b \ , \qquad (2.5)$$

In particular, when $\alpha = n \in N_0$, then

$$ {}_aD_t^{0}f(t) = {}_tD_b^{0}f(t) = f(t),$$
$$ {}_aD_t^{n}f(t) = f^{(n)}(t) \quad \text{and} \quad {}_tD_b^{n}f(t) = (-1)^n f^{(n)}(t),$$

where $f^{(n)}(t)$ is the usual derivative of $f(t)$ of order n,[7],[17].

Definition (2.1.2):

Let $\alpha > 0$ be a real number and let $n = [\alpha]+1$ for $\alpha \notin N_0$; $n = \alpha$ for $\alpha \in N_0$.If $f(t) \in AC^n[a,b]$, then the Caputo fractional derivatives ${}_a^CD_t^{\alpha}f(t)$ and ${}_t^CD_b^{\alpha}f(t)$ exist almost everywhere on $[a,b]$,[14]

i) If $\alpha \notin N_0$, ${}_a^CD_t^{\alpha}f(t)$ and ${}_t^CD_b^{\alpha}f(t)$ are represented by

$$ {}_a^CD_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau, \qquad (2.6)$$

$$ {}_t^CD_b^{\alpha}f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (\tau-t)^{n-\alpha-1} f^{(n)}(\tau)d\tau, \qquad (2.7)$$

ii) If $\alpha = n \in N_0$, then ${}_a^CD_t^{\alpha}f(t)$ and ${}_t^CD_b^{\alpha}f(t)$ are represented by

$$ {}_a^CD_t^{\alpha}f(t) = f^{n}(t) \quad \text{and} \quad {}_t^CD_b^{\alpha}f(t) = (-1)^n f^{n}(t).$$

In particular, ${}_a^C D_t^0 f(t) = {}_t^C D_b^0 f(t) = f(t)$.

2.2 **Basic Properties of Fractional Calculus of arbitrary order.**

In The following some properties of fractional calculus are presented in details which will be needed later on:

i) The relation between the RLFD and the CFD,[7]:
For $\alpha > 0$ and $n = [\alpha] + 1$, the Riemann-Liouville and Caputo fractional derivatives are related by the following formulas:

$${}_a D_t^\alpha f(t) = {}_a^C D_t^\alpha f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1-\alpha)} (t-a)^{k-\alpha} \quad (2.8)$$

$${}_t D_b^\alpha f(t) = {}_t^C D_b^\alpha f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{\Gamma(k+1-\alpha)} (b-t)^{k-\alpha}. \quad (2.9)$$

ii) The fractional operators are linear,[12]:

$P[\lambda f(t) + \mu g(t)] = \lambda\, Pf(t) + \mu\, Pg(t)$,

where P is one of ${}_a D_t^\alpha$, ${}_t D_b^\alpha$, ${}_a^C D_t^\alpha$, ${}_t^C D_b^\alpha$, ${}_a D_t^{-\alpha}$ or ${}_t D_b^{-\alpha}$ and λ, μ are real numbers.

iii) The power function and constant function of the Caputo's derivative ,is:

$${}_a^C D_t^\alpha (t-a)^\beta = \begin{cases} 0, & for \quad \beta \in N_0 \quad and \ \beta < \lceil \alpha \rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t-a)^{\beta-\alpha}, & for \quad \beta \in N_0 \quad and \ \beta \geq \lceil \alpha \rceil, \end{cases} \quad (2.10)$$

we use $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α and $N_0 = \{0, 1, 2, \ldots\}$

Recall that for $\alpha \in N$,[6].

$${}_a^C D_t^\alpha const = 0, \quad (2.11)$$

In particular , ${}_a^C D_t^\alpha 1 = 0$, and ${}_t^C D_b^\alpha 1 = 0$.

iv) For $\alpha > 0$, we have

$$\int_a^b g(t) \cdot {}_a^C D_t^\alpha f(t) dt =$$

$$\int_a^b f(t) \cdot {}_t D_b^\alpha g(t)\, dt + \sum_{j=0}^{n-1} \left[{}_t D_b^{\alpha+j-n} g(t) \cdot {}_t D_b^{n-1-j} f(t) \right]_a^b,$$

$$\int_a^b g(t) \cdot {}_t^C D_b^\alpha f(t) dt =$$

$$\int_a^b f(t) \cdot {}_a D_t^\alpha g(t)\, dt + \sum_{j=0}^{n-1} \left[(-1)^{n+j}\, {}_a D_t^{\alpha+j-n} g(t) \cdot {}_a D_t^{n-1-j} f(t) \right]_a^b$$

where ${}_a D_t^k g(t) = {}_a I_t^{-k} g(t)$ and ${}_t D_b^k g(t) = {}_t I_b^{-k} g(t)$ if $k < 0$.

Therefore, $0 < \alpha < 1$, we obtain,[7]

$$\int_a^b g(t)\cdot {}_a^C D_t^\alpha f(t)dt = \int_a^b f(t)\cdot \; {}_t D_b^\alpha g(t)\,dt + [\;\; {}_t I_b^{1-\alpha} g(t)\cdot f(t)]_a^b$$

and

$$\int_a^b g(t)\cdot {}_t^C D_b^\alpha f(t)dt = \int_a^b f(t)\cdot \; {}_a D_t^\alpha g(t)\,dt + [\;\; {}_a I_t^{1-\alpha} g(t)\cdot f(t)]_a^b.$$

Moreover,
If f is a function such that $f(a) = f(b) = 0$, we have simpler formulas:

$$\int_a^b g(t)\cdot {}_a^C D_t^\alpha f(t)dt = \int_a^b f(t)\cdot \; {}_t D_b^\alpha g(t)\,dt \qquad (2.12)$$

and

$$\int_a^b g(t)\cdot {}_t^C D_b^\alpha f(t)dt = \int_a^b f(t)\cdot \; {}_a D_t^\alpha g(t)\,dt. \qquad (2.13)$$

2.3. Shifted Chebyshev Polynomials.

The Chebyshev polynomial $T_n(t)$ of the first kind is a polynomial in t of degree n, defined by the relation

$$T_n(t) = cos(n\theta) \text{ when } t = cos(\theta). \qquad (2.14)$$

where $\in [-1,1]$, $\theta \in [0,\pi]$. $t = -1$ corresponds to $\theta = \pi$ and $t = 1$ corresponds to $\theta = 0$.

The Chebyshev polynomials can be expanded in power series as, [13]:

$$T_n(t) = \frac{n}{2}\sum_{m=0}^{\lfloor n/2 \rfloor}(-1)^m \frac{(n-m-1)!}{(m)!(n-2m)!}(2t)^{n-2m}, \qquad (2.15)$$

where $\lfloor n/2 \rfloor$ denotes the integral part of $n/2$, with $T_0(t) = 1, T_1(t) = t$.

The Chebyshev polynomials $T_n(t)$ $(n = 0,1,2,\ldots)$ are orthogonal under integration over $[-1,1]$ with the weighting function $w(t) = 1/\sqrt{1-t^2}$, with orthogonally condition :

$$\int_{-1}^{1}\frac{T_n(t)T_m(t)}{\sqrt{1-t^2}}dt = \begin{cases} 0 & \text{if} \quad n \neq m, \\ \frac{\pi}{2} & \text{if} \quad n = m \in N, \\ \pi & \text{if} \quad n = m = 0. \end{cases} \qquad (2.16)$$

In order to use these polynomials on the interval$[0, L]$ we define the so called shiftedpseudo-spectral Chebyshev polynomials by introducing the change of variable $z = \frac{2t}{L} - 1$.

The shifted Chebyshev polynomials are defined as,[7]:

$$T_n^p(t) = T_n\left(\frac{2t}{L} - 1\right), \quad \text{where} \quad T_0^p(t) = 1, T_1^p(t) = \frac{2t}{L} - 1. \qquad (2.17)$$

The analytic form of the shifted pseudo-spectral Chebyshev polynomial$T_n^p(t)$ of degree n is given by,[6]:

$$T_n^p(t) = n\sum_{k=0}^{n}(-1)^{n-k}\frac{2^{2k}(n+k-1)!}{(2k)!(n-k)!L^k}t^k, \qquad n = 1,2 \ldots, \qquad (2.18)$$

Where, $T_n^p(0) = (-1)^n$, and $T_n^p(L) = 1$.

Let the shifted Chebyshev polynomials $T_n\left(\frac{2t}{L}-1\right)$be denoted by$T_n^p(t)$, satisfying the orthogonality relation:

$$\int_0^L T_n^p(t)T_m^p(t)w^p(t)\,dt = \delta m, \quad (2.19)$$

with the weight function $w^p(t) = \frac{1}{\sqrt{Lt-t^2}}, \delta m = \frac{b_k}{2}\pi, b_0 = 2, b_k = 1$ for $k \geq 1$.
A function $x(t) \in L^2([0,L])$ i.e. (square integrable in $[0,L]$) can be expressed in terms of shifted Chebyshev polynomials as:

$$x(t) = \sum_{j=0}^{\infty} c_n T_n^p(t),$$

where the coefficients c_n are given by

$$c_n = \frac{1}{\delta n}\int_0^L x(t)T_n^p(t)w^p(t)\,dt, \quad n = 0,1,\dots \quad (2.20)$$

3. The necessary optimality conditions of multi-order fractional optimal control problems:

Let α, β be real numbers and $\alpha, \beta \in (0,1)$,and let $L, f: [a,\infty[\times \mathbb{R}^2 \to \mathbb{R}$ be two differentiable functions.

We consider a general form of Multi-Order Fractional Optimal Control Problem:

$$\text{minimize } J(x,u,T) = \int_a^T L(t, x(t), u(t))dt, \quad (3.1)$$

subject to the fractional dynamic control system

$$A\,{}_a^CD_t^{\alpha}x(t) + B\,{}_a^CD_t^{\beta}x(t) = f(t, x(t), u(t)), \quad (3.2)$$

$$x(a) = x_a, \qquad x(T) = x_T, \quad (3.3)$$

Where $A, B \neq 0, T$ is the end free time $, x_a$ and x_T are fixed real numbers. The dynamic control system (2.2) involving integer and multi-order fractional derivatives.

Theorem (3.1):

Let (x, u, T) is a minimizer of fractional optimal control problem (2.1), (2.2) and (2.3), then there exists a function $\lambda(t)$ that (x, u, T) satisfies the optimality conditions (i) and (ii):

i)
$$\begin{cases} A\ \ {}_tD_T^{\alpha}\lambda(t) + B\ \ {}_tD_T^{\beta}\lambda(t) = \frac{\partial H}{\partial x}(t, x(t), u(t), \lambda(t)) & (3.4) \\ A\,{}_a^CD_t^{\alpha}x(t) + B\,{}_a^CD_t^{\beta}x(t) = \frac{\partial H}{\partial \lambda}(t, x(t), u(t), \lambda(t)) & (3.5) \end{cases}$$

ii) $\frac{\partial H}{\partial u}(t, x(t), u(t), \lambda(t)) = 0$

for all $t \in [a,T]$; (3.6)

where the Hamiltonian $H(t,x,u,\lambda)$ is defined by

$$H(t,x,u,\lambda) = L(t,x,u) + \lambda f(t,x,u). \quad (3.7)$$

4. Numerical approximations of the fractional calculus.

We choose the Chebyshev-Gauss Lobatto points associated with the interval $[0, L]$,

$$t_r = \frac{L}{2} - \frac{L}{2}\cos(\frac{\pi r}{N}),\ r = 0, 1, \dots\ N. \tag{4.1}$$

These grids can be written as $L < x_N < x_{N-1} \dots < x_1 < x_0 = 0.$

Clenshaw and Curtis [9], introduced an approximation of the function $x(t)$, as follows

$$x_N(t) = {\sum_{n=0}^{N}}'' a_n T_n^p(t),\ a_n = \frac{2}{N}{\sum_{r=0}^{N}}'' x(t_r) T_n^p(t_r). \tag{4.2}$$

where the $('')$ on the summation means that the first and last terms are to be taken with a factor $(1/2)$.

4.1. Approximation of the Left Caputo fractional derivative.

Theorem (4.1.1),[7]

The fractional derivative of order α in the Caputo sense for thefunction $x(t)$ at the point t_s is given by

$$ {}_0^C D_t^\alpha x_N(t_s) = \sum_{r=0}^{N} d_{s,r}^{(\alpha)}\, x(t_r), \qquad \alpha > 0, \tag{4.3}$$

Such that

$$d_{s,r}^{(\alpha)} = \frac{4\theta_r}{N}\sum_{n=\lceil\alpha\rceil}^{N}\sum_{j=0}^{N}\sum_{k=\lceil\alpha\rceil}^{n}\frac{n\theta_n}{b_j}\frac{(-1)^{n-k}(n+k-1)!\,\Gamma\left(k-\alpha+\frac{1}{2}\right)T_n^p(t_r)\,T_j^p(t_s)}{L^\alpha\,\Gamma\left(k+\frac{1}{2}\right)(n-k)!\,\Gamma(k-\alpha-j+1)\,\Gamma(k-\alpha+j+1)}, \tag{4.4}$$

where

$s, r = 0, 1, 2, \dots, N$ with $\theta_0 = \theta_N = \frac{1}{2}, \theta_i = 1\ \forall\ i = 1, 2, \dots, N-1$.

Theorem (4.1.2),[7]

Let ${}_0^C D_t^\alpha x_N(t)$ be the approximation of the fractional derivative ${}_0^C D_t^\alpha$ of the function $x(t)$ as given by (4.3). Then it holds

$$\left\| {}_0^C D_t^\alpha x(t) - {}_0^C D_t^\alpha x_N(t)\right\|_2 \le {\sum_{n=0}^{N}}'' a_n\,\Omega_n\left(\frac{F\left(t^{k-\alpha}; T_0^p, \dots, T_N^p\right)}{F\left(T_0^p, \dots, T_N^p\right)}\right)^{\frac{1}{2}}, \tag{4.5}$$

where

$$\Omega_n = \sum_{k=\lceil\alpha\rceil}^{n}\frac{(-1)^{n-k}\,2n(n+k-1)!\,\Gamma\left(k-\alpha+\frac{1}{2}\right)}{b_j L^\alpha\,\Gamma\left(k+\frac{1}{2}\right)(n-k)!\,\Gamma(k-\alpha-j+1)\,\Gamma(k-\alpha+j+1)}. \tag{4.6}$$

$$F(x; y_1, y_2, \dots y_n) = \begin{vmatrix} \langle x, x\rangle & \langle x, y_1\rangle & \cdots & \langle x, y_n\rangle \\ \langle y_1, x\rangle & \langle y_1, y_1\rangle & \cdots & \langle y_1, y_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle yn;\ x\rangle & \langle yn;\ y1\rangle & \cdots & \langle yn;\ yn\rangle \end{vmatrix}.$$

4.2 Approximation of the right the Riemann-Liouville fractional derivative.

Let $0 < \alpha < 1$ and $f(t) \in AC\ [a, b]$,

$$ {}_s^C D_b^\alpha f(s) = \frac{-1}{\Gamma(1-\alpha)}\int_s^b (t-s)^{-\alpha} f'(t)\,dt, \tag{4.7}$$

The relation between the Riemann-Liouville and Caputo fractional derivatives when $0 < \alpha < 1$, (that is $n = 1$) we have

$${}_{s}^{c}D_{b}^{\alpha}f(s) = \; {}_{s}D_{b}^{\alpha}f(s) - f(b)\frac{(b-s)^{-\alpha}}{\Gamma(1-\alpha)}, \qquad (4.8)$$

We can written (4.8) as

$${}_{s}D_{b}^{\alpha}f(s) = \frac{f(b)}{\Gamma(1-\alpha)}(b-s)^{-\alpha} + {}_{s}^{c}D_{b}^{\alpha}f(s) \qquad (4.9)$$

Use (4.7) in (4.9) to obtain

$${}_{s}D_{b}^{\alpha}f(s) = \frac{f(b)}{\Gamma(1-\alpha)}(b-s)^{-\alpha} + \frac{1}{\Gamma(1-\alpha)}\int_{s}^{b}(t-s)^{-\alpha}f'(t)dt \qquad (4.10)$$

Let f be a sufficiently smooth function in $[0, b]$ and let $J(s; f)$ be defined as follows

$$J(s;f) = \int_{s}^{b}(t-s)^{-\alpha}f'(t)\,dt, 0 < s < b. \qquad (4.11)$$

Substitution (4.11) in(4.10) we deduce

$${}_{s}D_{b}^{\alpha}f(s) = \frac{f(b)}{\Gamma(1-\alpha)}(b-s)^{-\alpha} + \frac{J(s;f)}{\Gamma(1-\alpha)}. \qquad (4.12)$$

We approximate $f(t), 0 \le t \le b,$ by a sum of shifted Chebyshev polynomials $T_k(\frac{2t}{b} - 1)$ according to

$$f(t) \approx p_N(t) = {\sum_{k=0}^{N}}'' a_k\, T_k(\frac{2t}{b} - 1), a_k = \frac{2}{N}{\sum_{j=0}^{N}}'' f(t_j)T_k(\frac{2t_j}{b} - 1), \qquad (4.13)$$

where

$t_j = \frac{b}{2} - \frac{b}{2}\cos\left(\frac{\pi j}{N}\right), j = 0, \dots\dots, N,$ and obtain

$$J(s;f) \approx J(s;p_N) = \int_{s}^{b} p_N{}'(t)\,(t-s)^{-\alpha}dt. \qquad (4.14)$$

Lemma(4.2.1),[7]

Let p_N be the polynomial of degree N as given by (4.13), Then there exists a polynomial F_{N-1} of degree $N-1$ such that

$$\int_{s}^{x}\left[p_N{}'(t) - p_N{}'(s)\right](t-s)^{-\alpha}dt = \left[F_{N-1}(x) - F_{N-1}(s)\right]\,(x-s)^{1-\alpha}. \qquad (4.15)$$

Proof.

Let $p_N{}'(t) - p_N{}'(s)$ be expanded in a Taylor series at $t = s$

$$p_N{}'(t) - p_N{}'(s) = \sum_{k=1}^{N-1} A_k(s)(t-s)^k.$$

Then,

$$\int_{s}^{x}\left[p_N{}'(t) - p_N{}'(s)\right](t-s)^{-\alpha}dt = \sum_{k=1}^{N-1} A_k(s)\int_{s}^{x}(t-s)^{k-\alpha}\,dt$$

$$= \left[(t-s)^{1-\alpha}\sum_{k=1}^{N-1}\frac{A_k(s)(t-s)^k}{k-\alpha+1}\right]_{s}^{x}.$$

The assertion follows, if we choose $F_{N-1}(x) = \sum_{k=0}^{N-1}\frac{A_k(s)(x-s)^k}{k-\alpha+1},$

with an arbitrary constant $A_0(s)$.

In view of (4.15) we have

$$J(s;p_N) = \int_s^b p_N{}'(t)(t-s)^{-\alpha}dt = \left[\frac{p_N{}'(s)}{1-\alpha} + [F_{N-1}(b) - F_{N-1}(s)]\right](b-s)^{1-\alpha} \qquad (4.16)$$

Moreover, ${}_sD_b^{\alpha}f(s)$ can be approximated by means of

$${}_sD_b^{\alpha}f(s) \approx \frac{f(b)}{\Gamma(1-\alpha)}(b-s)^{-\alpha} + \frac{J(s;p_N)}{\Gamma(1-\alpha)}. \qquad (4.17)$$

We express $F_{N-1}(t)$ in (4.16) by a sum of Chebyshev polynomials and provide the recurrence relation satisfied by the Chebyshev coefficients. Differentiating both sided of (4.15) with respect to x yields

$$[p_N{}'(x) - p_N{}'(s)](x-s)^{-\alpha} =$$

$$F_{N-1}{}'(x)(x-s)^{1-\alpha} + [F_{N-1}(x) - F_{N-1}(s)](1-\alpha)(x-s)^{-\alpha},$$

when

$$p_N{}'(x) - p_N{}'(s) = F_{N-1}{}'(x)(x-s) + [F_{N-1}(x) - F_{N-1}(s)](1-\alpha). \qquad (4.18)$$

To evaluate $F_{N-1}(s)$ in (4.16). we expand $F_{N-1}{}'(x)$ in terms of the shifted chebyshev polynomials $F_{N-1}{}'(x) = \sum_{k=0}^{N-2}{}' b_k T_k\left(\frac{2x}{b} - 1\right), 0 \le x \le b,$ (4.19)

where the $(')$ on the summation means that the first term is to be taken with a factor $(1/2)$.

$$F_{N-1}(x) - F_{N-1}(s) = \frac{b}{4}\sum_{k=1}^{N-1}\frac{b_{k-1} - b_{k+1}}{k}\left(T_k\left(\frac{2x}{b} - 1\right) - T_k\left(\frac{2s}{b} - 1\right)\right), \qquad (4.20)$$

where $b_{N-1} = b_N = 0$. On the other hand , we have

$$(x-s)F_{N-1}{}'(x) = \frac{b}{2}F_{N-1}{}'(x)\left[\left(\frac{2x}{b} - 1\right) - \left(\frac{2s}{b} - 1\right)\right].$$

By using the relation $T_{k+1}(v) + T_{k-1}(v) = 2vT_k(v)$ and from (4.19), it follows that

$$(x-s)F_{N-1}{}'(x) = \frac{b}{4}\sum_{k=1}^{N-1}{}'\left(b_{k+1} - 2\left(\frac{2s}{b} - 1\right)b_k + b_{k-1}\right)T_k\left(\frac{2x}{b} - 1\right), \qquad (4.21)$$

Such that $b_{-1} = b_1$.

Let $p_N{}'(x) = \sum_{k=0}^{N-1}{}' c_k T_k\left(\frac{2x}{b} - 1\right)$.(4.22)

Inserting $F_{N-1}(x) - F_{N-1}(s)$ and $(x-s)F_{N-1}{}'(x)$ as given (4.20) and (4.21) into (4.18) and taking (4.22) into account, we get

$$\left(1 - \frac{1-\alpha}{k}\right)b_{k+1} - 2\left(\frac{2s}{b} - 1\right)b_k + \left(1 + \frac{1-\alpha}{k}\right)b_{k-1} = \frac{4}{b}c_k, \quad 1 \le k. \qquad (4.23)$$

The Chebyshev coefficients c_k of $p_N{}'(x)$ as given by (4.22) can be evaluated by integrating and comparing it with (4.13):

$$c_{k-1} = c_{k+1} + \frac{4k}{b}a_k, k = N, N-1, \dots, 1, \qquad (4.24)$$

With starting values $c_N = c_{N+1} = 0$, where a_k are the Chebyshev coefficients of $p_N(x)$.

5. Illustrative Example:

In this section, we developed the algorithm for the constraint of the general problem into multi fractional derivative ${}_0^C D_t^\alpha x(t)$ and ${}_0^C D_t^\beta x(t)$ in the optimal control problem.We consider the following linear-quadratic optimal control problem:

$$\min J(x,u) = \int_0^1 (u(t) - x(t))^2 dt, \tag{5.1}$$

subject to the multi-fractional dynamical system

$${}_0^C D_t^\alpha x(t) + {}_0^C D_t^\beta x(t) = u(t) - x(t) + \frac{6}{\Gamma(\alpha+3)} t^{\alpha+2} + \frac{6}{\Gamma(\beta+3)} t^{\beta+2}, \tag{5.2}$$

and the boundary conditions

$$x(0) = 0, x(1) = \frac{6}{\Gamma(\alpha+\beta+3)}. \tag{5.3}$$

The exact solution for $\alpha = \beta = 1$ is given by:

$$\bar{x}(t) = \bar{u}(t) = \frac{6t^{\alpha+\beta+2}}{\Gamma(\alpha+\beta+3)}. \tag{5.4}$$

Now, we develop algorithm for the solution (5.1),(5.2) and (5.3). It is based on the necessary optimality conditions of multi-order fractional optimal control from Theorem (3.1) as the following steps:

Step 1. Compute the Hamiltonian function $H(t,x,u,\lambda)$

$$H = (u(t) - x(t))^2 + \lambda(t)\left(u(t) - x(t) + \frac{6}{\Gamma(\alpha+3)} t^{\alpha+2} + \frac{6}{\Gamma(\beta+3)} t^{\beta+2}\right). \tag{5.5}$$

Step 2. Derive the Necessary Optimality Conditions of multi-order fractional optimal control problems from Theorem (3.1):

$$A \;\; {}_t D_T^\alpha \lambda(t) + B \;\; {}_t D_T^\beta \lambda(t) = \frac{\partial H}{\partial x} = -2(u(t) - x(t)) - \lambda(t), \tag{5.6}$$

$$A\, {}_a^C D_t^\alpha x(t) + B\, {}_a^C D_t^\beta x(t) = \frac{\partial H}{\partial \lambda} = u(t) - x(t) + \frac{6}{\Gamma(\alpha+3)} t^{\alpha+2} + \frac{6}{\Gamma(\beta+3)} t^{\beta+2} \tag{5.7}$$

Suppose that $A = B = 1$, and $t \in (0,1)$ then $a = 0$, $T = 1$, equations(5.6) and (5.6) become

$${}_t D_1^\alpha \lambda(t) + \;\; {}_t D_1^\beta \lambda(t) = -2(u(t) - x(t)) - \lambda(t), \tag{5.8}$$

$${}_0^C D_t^\alpha x(t) + {}_0^C D_t^\beta x(t) = u(t) - x(t) + \frac{6}{\Gamma(\alpha+3)} t^{\alpha+2} + \frac{6}{\Gamma(\beta+3)} t^{\beta+2}, \tag{5.9}$$

$$2(u(t) - x(t)) + \lambda(t) = 0. \tag{5.10}$$

$$2(u(t) - x(t)) = -\lambda(t) \quad \text{thus,} \quad u(t) - x(t) = -\frac{1}{2}\lambda(t). \tag{5.11}$$

Use (5.11) in (5.8) we get

$${}_t D_1^\alpha \lambda(t) + \;\; {}_t D_1^\beta \lambda(t) = -2\left(-\frac{1}{2}\lambda(t)\right) - \lambda(t),$$

$${}_{t}D_{1}^{\alpha}\lambda(t) + {}_{t}D_{1}^{\beta}\lambda(t) = 0. \qquad (5.12)$$

Now, from equation (5.11),we get

$2u(t) = 2x(t) - \lambda(t)$, thus, $u(t) = x(t) - \frac{\lambda(t)}{2}$ (5.13)

Also, we substitute (5.13) in (5.9)

$${}_{0}^{C}D_{t}^{\alpha}x(t) + {}_{0}^{C}D_{t}^{\beta}x(t) = -\frac{\lambda(t)}{2} + \frac{6}{\Gamma(\alpha+3)}t^{\alpha+2} + \frac{6}{\Gamma(\beta+3)}t^{\beta+2}, \qquad (5.14)$$

Step 3. The coupled system for $x(t)$ and $\lambda(t)$, we have from (5.12) and (5.14)

$$\begin{cases} {}_{t}D_{1}^{\alpha}\lambda(t) + {}_{t}D_{1}^{\beta}\lambda(t) = 0, & (5.15a) \\ {}_{0}^{C}D_{t}^{\alpha}x(t) + {}_{0}^{C}D_{t}^{\beta}x(t) = -\frac{\lambda(t)}{2} + \frac{6}{\Gamma(\alpha+3)}t^{\alpha+2} + \frac{6}{\Gamma(\beta+3)}t^{\beta+2}. & (5.15b) \end{cases}$$

Step 4. Using Chebyshev expansion, get an approximate solution of the coupled system under the boundary conditions(5.3), as follows:

i) solve approximate $(5.15a)$of ${}_{t}D_{1}^{\alpha}\lambda(t)$ and ${}_{t}D_{1}^{\beta}\lambda(t)$, by approximation of the Riemann-Liouville fractional derivatives (4.17).

$$\frac{\lambda(1)}{\Gamma(1-\alpha)}(1-t_s)^{-\alpha} + \frac{J(t_s;p_{N1})}{\Gamma(1-\alpha)} + \frac{\lambda(1)}{\Gamma(1-\beta)}(1-t_s)^{-\beta} + \frac{J(t_s;p_{N2})}{\Gamma(1-\beta)} = 0. \qquad (5.16)$$

Hence,

$$\lambda(1)\left[\frac{1}{\Gamma(1-\alpha)}(1-t_s)^{-\alpha} + \frac{1}{\Gamma(1-\beta)}(1-t_s)^{-\beta}\right] + \frac{J(t_s;p_{N1})}{\Gamma(1-\alpha)} + \frac{J(t_s;p_{N2})}{\Gamma(1-\beta)} = 0 \qquad (5.17)$$

ii) solve approximate $(5.15b)$ of $x(t)$ by theorem(4.2) of the Caputo fractional derivative.

We use (4.2) to approximate $x(t)$.A collocation scheme is defined by substituting (4.2), (4.3) andthe computed $\lambda(t)$ into $(5.15b)$and evaluating the results at the shifted Gauss-Lobattonodest_s, $s = 1, 2, \dots, N-1$. This results in:

$\sum_{r=0}^{N} d_{s,r}^{(\alpha)}\, x(t_r) + \sum_{r=0}^{N} d_{s,r}^{(\beta)}\, x(t_r) = -\frac{\lambda(t_s)}{2} + \frac{6\, t_s^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{6\, t_s^{\beta+2}}{\Gamma(\beta+3)}$ (5.18) where $d_{s,r}^{(\alpha)}$ and $d_{s,r}^{(\beta)}$ are defined in (4.4) of theorem(4.1.1).

Step 5. Evaluating the results at the shifted Gauss-Lobatto nodes t_s,from (4.1),

$s = 1,2,\dots,N-1$, and let $N = 2$and $\alpha,\beta \in (0,1)$.

Let $N = 2 \rightarrow s = 1$, we have:

$$t_s = t_1 = \frac{L}{2} - \frac{L}{2}\cos(\frac{\pi}{N}) \;\rightarrow\; = \frac{1}{2} - \frac{1}{2}\cos(\frac{\pi}{2}) \;\rightarrow t_1 = 0.5$$

Step 6. Compute approximated $x(t)$ and $u(t)$ by (4.2) and (5.2) respectively.

Let $\alpha = 0.3$ and $\beta = 0.5$ such that $\alpha, \beta \in (0,1)$.

i) We find $\lambda(t_1)$ from equation (5.17), we get

$$\lambda(1)\,[1.7464] + 0.7704\, J(t_s; p_{N1}) + 0.5642\, J(t_s; p_{N2}) = 0. \quad (5.19)$$

$J(t_s; p_{N1})$ and $J(t_s; p_{N2})$, from equation(4.16), to get

$$J(t_1; p_{N1}) = \left[\frac{p_N{'}(t_1)}{0.7} + [F_{N-1}(1) - F_{N-1}(0.5)]\right](0.5)^{0.7}, \quad (5.20)$$

$$J(t_1; p_{N2}) = \left[\frac{p_N{'}(t_1)}{0.5} + [F_{N-1}(1) - F_{N-1}(0.5)]\right](0.5)^{0.5}. \quad (5.21)$$

Use (4.22) to compute $p_N{'}(t_1)$, as follows:

$$p_N{'}(t_1) = p_N{'}(0.5) = {\sum_{k=0}^{1}}{'}\, c_k T_k\left(\frac{2(0.5)}{1} - 1\right) = \frac{1}{2}c_0 T_0(0) + c_1 T_1(0)$$

$$= \frac{1}{2}c_0(1) + c_1(-1)$$

$$= \frac{1}{2}c_0 - c_1. \quad (5.22)$$

$$c_{k-1} = c_{k+1} + \frac{4k}{b}a_k,\ k = N, N-1, \dots, 1,$$

and $c_N = c_{N+1} = 0$, then $k = 2,1$, and $c_2 = c_3 = 0$.

$$\begin{cases} \text{when } k = 2, c_1 = c_3 + \frac{8}{1}a_2 \Rightarrow c_1 = 0 + 8a_2 \Rightarrow c_1 = 8a_2, \\ \text{when } k = 1, c_0 = c_2 + \frac{4}{1}a_1 \Rightarrow c_0 = 0 + 4a_1 \Rightarrow c_0 = 4a_1. \end{cases} \quad (5.23)$$

where $a_k = a_1, a_2$. are the Chebyshev coefficients of $p_N(x)$. We can find it from (4.13):

where $t_j = \frac{b}{2} - \frac{b}{2}\cos\left(\frac{\pi j}{N}\right), j = 0,1,2.$

$$j = 0, \quad \Rightarrow t_0 = \frac{1}{2} - \frac{1}{2}\cos(0) \quad \Rightarrow t_0 = 0,$$

$$j = 1, \quad \Rightarrow t_1 = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{\pi}{2}\right) \quad \Rightarrow t_1 = 0.5,$$

$$j = 2, \quad \Rightarrow t_2 = \frac{1}{2} - \frac{1}{2}\cos(\pi) \quad \Rightarrow t_2 = 1.$$

and

$$a_1 = \frac{1}{2}\lambda(t_0)T_1\left(\frac{2t_0}{1} - 1\right) + \lambda(t_1)T_1\left(\frac{2t_1}{1} - 1\right) + \frac{1}{2}\lambda(t_2)T_1\left(\frac{2t_2}{1} - 1\right),$$

$a_1 = \frac{1}{2}\lambda(t_0)T_1(-1) + \lambda(t_1)T_1(0) + \frac{1}{2}\lambda(t_2)T_1(1),$

$a_1 = \frac{-3}{2}\lambda(t_0) - \lambda(t_1) + \frac{1}{2}\lambda(t_2),$ (5.24)

$a_2 = \frac{1}{2}\lambda(t_0)T_2\left(\frac{2t_0}{1}-1\right) + \lambda(t_1)T_2\left(\frac{2t_1}{1}-1\right) + \frac{1}{2}\lambda(t_2)T_2\left(\frac{2t_2}{1}-1\right),$

$a_2 = \frac{1}{2}\lambda(t_0)T_2(-1) + \lambda(t_1)T_2(0) + \frac{1}{2}\lambda(t_2)T_2(1),$

$a_2 = \frac{17}{2}\lambda(t_0) + \lambda(t_1) + \frac{1}{2}\lambda(t_2).$ (5.25)

Substitution (5.24) and (5.25) in (5.23), We obtain :

$c_0 = -6\lambda(t_0) - 4\lambda(t_1) + 2\lambda(t_2)$ and $c_1 = 68\lambda(t_0) + 8\lambda(t_1) + 4\lambda(t_2).$

Substitute c_0 and c_1 in equation(5.22), we get

$p_N{}'(0.5) = -71\lambda(t_0) - 10\lambda(t_1) - 3\lambda(t_2).$

Now, we calculate $[F_{N-1}(1) - F_{N-1}(0.5)]$, from (4.20), to get

$[F_{N-1}(1) - F_{N-1}(0.5)] = 1.3333(68\lambda(t_0) + 8\lambda(t_1) + 4\lambda(t_2)).$ (5.26)

After Substitution above equations in (5.17)

Then, we have $\lambda(t_1) = 0.$

To solve approximation of $x(t)$ from (5.18) ,when $r = 0, 1, 2$ and $s = 1,$ we get

$d_{1,0}^{(0.3)}x(0) + d_{1,1}^{(0.3)}x(t_1) + d_{1,2}^{(0.3)}x(1) + d_{1,0}^{(0.5)}x(0) + d_{1,1}^{(0.5)}x(t_1)$

$+d_{1,2}^{(0.5)}x(1) = \frac{6\,(0.5)^{2.3}}{\Gamma(3.3)} + \frac{6(0.5)^{2.5}}{\Gamma(3.5)}.$ (5.27)

under the boundary conditions $x(0) = 0, x(1) = \frac{6}{\Gamma(\alpha+\beta+3)}$, and use (4.4) to evaluating the results of $d_{1,1}^{(0.3)}, d_{1,2}^{(0.3)}, d_{1,1}^{(0.5)}$ and $d_{1,2}^{(0.5)}$, which is shown in Table (1), as follows:

Table (1).

Shows results of Caputo fractional derivative when $\alpha = 0.3$ and $\beta = 0.5$ in case (1) and $\alpha = \beta = 0.7$ in case (2).

n	j	k	$d_{1,1}^{(0.3)}$	$d_{1,1}^{(0.5)}$	$d_{1,2}^{(0.3)}$	$d_{1,2}^{(0.5)}$	$d_{1,1}^{(0.7)}$	$d_{1,2}^{(0.7)}$
1	0	1	0	0	0.6274	0.7183	0	0.8154
1	1	1	0	0	0	0	0	0
1	2	1	0	0	0.0287	0.0478	0	0.0572
2	0	1	0.6274	0.7183	-0.6274	-0.7183	0.8154	-0.8154
2	1	1	0	0	0	0	0	0
2	2	1	0.0287	0.0478	-0.0287	-0.0478	0.0572	-0.0572
2	0	2	-0.5210	-0.6385	0.5210	0.6385	-0.7720	0.7720
2	1	2	0	0	0	0	0	0
2	2	2	0.0620	0.0517	-0.0620	-0.0547	0.0396	-0.0396

sum	0.1971	0.1823	0.459	0.5838	0.1402	0.7324
$d_{5,r}^{(\alpha)} = 4\theta_r/N.sum$	0.3942	0.3646	0.459	0.5838	0.2804	0.7324

when

$$r = 1 \quad \rightarrow \frac{4\theta_1}{N} = \frac{4(1)}{2} = 2 \qquad\qquad r = 2 \quad \rightarrow \frac{4\theta_2}{N} = \frac{4(0.5)}{2} = 1.$$

Case 1:

When $\alpha = 0.3$ and $\beta = 0.5$

Substitute the value of Table (1) and boundary conditions above in (5.27), we get

$$d_{1,0}^{(0.3)}(0) + (0.3942)x(t_1) + (0.459)(1.2782) + d_{1,0}^{(0.5)}(0)$$

$$+(0.3646)x(t_1) + (0.5838)(1.2782) = 3.7732,$$

Now, we have:

$x(t_0) = x(0) = 0,$

$x(t_1) = x(0.5) = 0.0185,$

$x(t_2) = x(1) = 1.2781.$

Substitute $x(t_0), x(t_1)$ and $x(t_2)$ in (4.2) to find the approximation solution of $x_2(t)$

$$x_2(t) = \frac{2}{N}\sum_{n=0}^{N\,''}\sum_{r=0}^{N\,''} x(t_r)T_n^p(t_r)\,T_n^p(t),$$

Then the approximation solution of $x_2(t)$ is $x_2(t) = 2.5192t^2 - 1.2412t.$

We can use $x_2(t)$ and substitute in the constraint of the problem (5.2) to compute the control $u(t)$.

$$u(t) = {}_0^CD_t^\alpha x(t) + {}_0^CD_t^\beta x(t) + x(t) - \frac{6}{\Gamma(\alpha+3)}t^{\alpha+2} - \frac{6}{\Gamma(\beta+3)}t^{\beta+2},$$

$$u(t) = 3.2618\,t^{1.7} - 1.3660t^{0.7} + 3.7902t^{1.5} - 1.4005\,t^{0.5} + 2.5192t^2 - 1.2412t - 2.2359t^{2.3} - 1.8054t^{2.5}.$$

Case 2:

When $\alpha = \beta = 0.7$

Substitute the value of Table (1) and boundary conditions above in (5.27), we get

$$d_{1,0}^{(0.7)}x(0) + d_{1,1}^{(0.7)}x(t_1) + d_{1,2}^{(0.7)}x(1) + d_{1,0}^{(0.7)}x(0) + d_{1,1}^{(0.7)}x(t_1)$$

$$+d_{1,2}^{(0.7)}x(1) = \frac{6\,(0.5)^{2.7}}{\Gamma(3.7)} + \frac{6(0.5)^{2.7}}{\Gamma(3.7)}.$$

In the same case ,we have :

$x_2(t) = 0.6372t^2 - 0.0454t.$

and

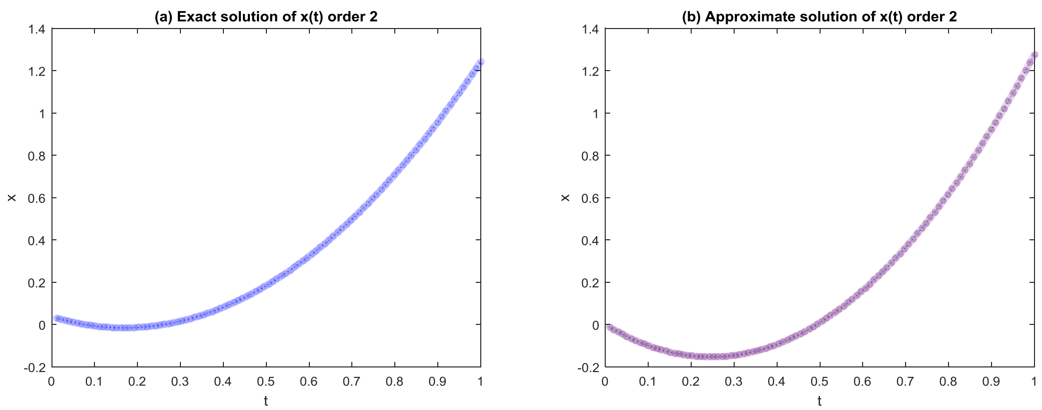
$u(t) = 2.1846\,t^{1.3} - 0.1012\,t^{0.3} + 0.6372\,t^{2} - 0.0454\,t - 2.8772\,t^{2.7}$.

Table (2).

Shows numerical results of the exact and approximate state $x(t)$ and control function $u(t)$ for $= 2$.

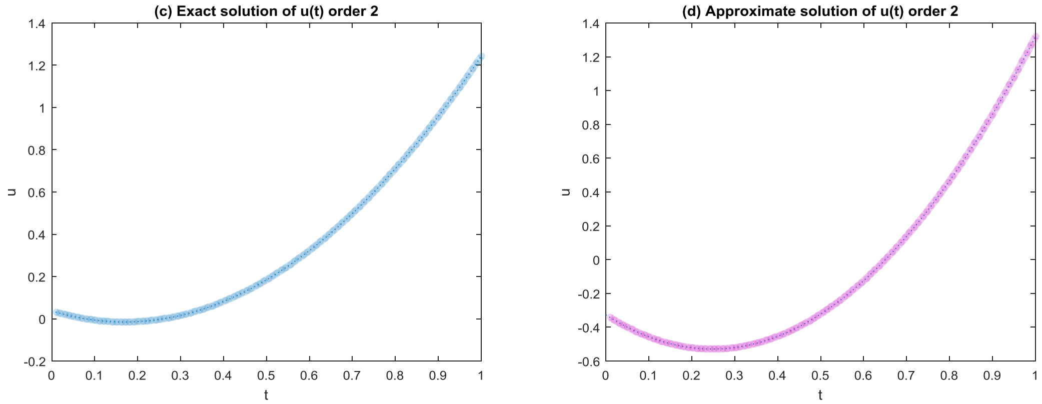
t	$x(t)$ $\alpha = 0.3$ $\beta = 0.5$	$x(t)$ $\alpha = 0.7$ $\beta = 0.7$	$\bar{x}(t)$ $\alpha = 1$ $\beta = 1$	$u(t)$ $\alpha = 0.3$ $\beta = 0.5$	$u(t)$ $\alpha = 0.7$ $\beta = 0.7$	$\bar{u}(t)$ $\alpha = 1$ $\beta = 1$
0	0	0	0	0	0	0
0.1	-0.0989	0.0018	0.0020	-0.6463	0.0054	0.0020
0.2	-0.1475	0.0164	0.0141	-0.7536	0.0142	0.014
0.3	-0.1456	0.0437	0.0439	-0.6860	0.0437	0.0439
0.4	-0.0934	0.0838	0.0983	-0.5070	0.0982	0.0983
0.5	0.0092	0.1366	0.1835	-0.2512	0.1863	0.1835
0.6	0.1622	0.2022	0.3058	0.0583	0.3184	0.3058
0.7	0.3656	0.2804	0.4708	0.4036	0.4283	0.4708
0.8	0.6193	0.3715	0.6843	0.7706	0.5155	0.6843
0.9	0.9235	0.4754	0.9516	0.1469	0.9652	0.9516
1	1.2780	0.5918	1.2782	1.5222	1.2020	1.2782

Figure 1:



(a) Exact solution of $x(t)$ for ($N = 2$) and $\alpha = \beta = 1$.

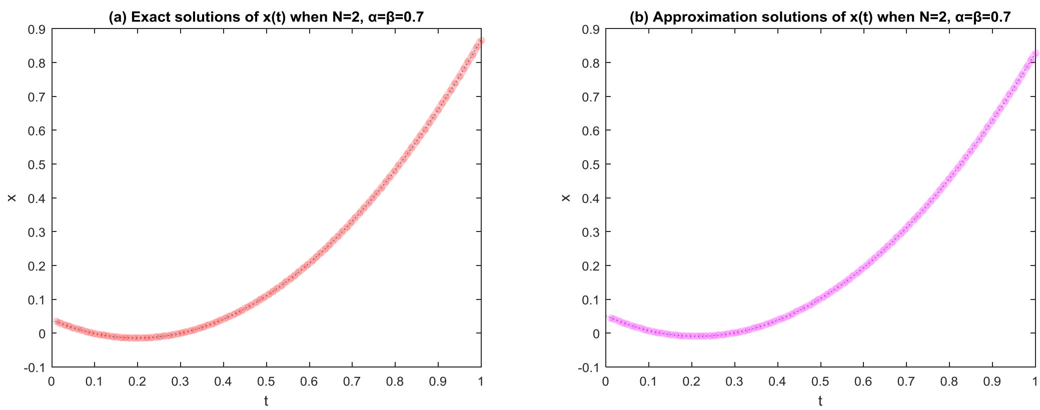
(b) Approximate solutions of $x(t)$ for ($N = 2$) and $\alpha = 0.3$ and $\beta = 0.5$.



(c) Exact solution of $u(t)$ for $(N = 2)$ and $\alpha = \beta = 1$.

(d) Approximate solutions of $u(t)$ for $(N = 2)$ and $\alpha = 0.3$ and $\beta = 0.5$.

Figure 2:



(a) Exact solution of $x(t)$ for $(N = 2)$ and $\alpha = \beta = 0.7$.

(b) Approximate solutions of $x(t)$ for $(N = 2)$ and $\alpha = \beta = 0.7$.

6. Conclusions.

In this paper, we have presented algorithm for the numerical solution of a class of multi- order fractional optimal control problems, in two cases one when $\alpha = 0.3$ and $\beta = 0.5$, and the other one when $\alpha = \beta = 0.7$. In both cases, the solution is approximated by Chebyshev series. Numerical results for illustrative example show that the algorithm converge from the exact solution when $\alpha = \beta = 1$, and we note that the convergent to the exact solution is dependent on increasing of the fractional order of derivative.

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