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Some Results on Koebe one- quarter theorem and Koebe distortion theorem

By

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Abstract.

In this research paper, we consider that $\varphi: \Omega \longrightarrow D$, be a conformal mapping from a simply connected domain Ω onto unit disk D, with a great role for each of *Koebe* one- quarter theorem and Koebe distortion theorem to prove there is a constant K depending on the modulus of z in Ω such that $1-|\varphi(z)| \leq \mathrm{K}\sqrt{|z|}$, for some $z \in \Omega$.

In addition, this result generates a sharp result on the integrability of gradient of Cauchy transform $\hat{g}(z)$ over a sequence ∂D_i which is getting larger in D, such that

$$\int_{\partial D_i} \nabla\left(\frac{|\hat{g}(z)|}{\sqrt{|z|}}\right) d\mu_i, \text{ exists and is finite on } \partial D_i.$$

Keywords: Conformal mapping, Koebe Distrotion Theorem**,** *Koebe* one-quarter theorem, gradient of Cauchy transform.

Mathematics subject classification : 32-XX

1. Introduction.

In the majority the papers which concerning the topic of a conformal mapping, we see that it is fascinating to an investigation of the boundary given domain. This influences us to inquire as to why? to reply about brought up an issue, it is vital to realize that the hypothesis of conformal mappings has been utilized more to demonstrate the conduct of limit given area through the conformal maps, as indicated by the basic idea that uses the conformal maps to transform the boundary- value problem, for the district R into a comparing one for the unit circle or half plane and afterward find the answer to solve the given problem by utilizing the opposite of conformal mapping.

In applications, if $\Omega \subset \mathbb{C}$ is a simply-connected domain, then a conformal mapping φ of the unit disk $D=\{z \in \mathbb{C}:|z|<1\}$ onto Ω exists by the Riemann mapping theorem [11] (if $\Omega \neq \mathbb{C}$), it appears difficult to compute conformal maps if the geometry of is complicated, in this way it is important to discover a connection between properties of the Riemann map the geometry of with general properties conformal maps. Subsequently, the field which unequivocally describes the conformal maps the main part of geometric function theory, and in the beginning of the century, started new skylines to this topic through works of Koebe [5].

Presents Theorem (2.1) which comes as a corollary of the Koebe one-quarter theorem and Koebe distortion theorem. This outcome together with Theorem (2.2) set up the existence and finiteness of the integrability of the derivative of conformal mapping for all $p<2$: Further more Theorem (2.1) considers a sharp result on the integrability of gradient of Cauchy change $\hat{g}(z)$ over a non-decreasing sequence ∂D_i in D, such that

$$\int_{\partial D_i} \nabla\left(\frac{|\hat{g}(z)|}{\sqrt{|z|}}\right) d\mu_i \qquad \text{exists and is finite on } \partial D_i$$

if the Cauchy transform $\hat{g}(z)$ of $g \in L^q(E, dA)$ for some $1<q \leq 2$, is an identically

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zero in $\mathbb{C}\backslash E$ and there exists a non-decreasing sequence ∂D_i in D, where E is a compact subset of the plane having connected complement, D is a connected domain $D \subset E$, to produce Theorem (2.3).

In the following serves as motivation to the main results of this work, are added to give a full support.

Definition (1.1) (Harmonic measure).

Let Ω be a bounded, open domain in n-dimentional Euclidean space $\mathbb{R}^n$, $n \geq 2$, and let $\partial\Omega$ denote the boundary of Ω. Any finite real- valued continuous function f on $\partial\Omega$, $f: \partial\Omega \longrightarrow \mathbb{R}$ corresponds to a unique function $u(x)$ on the closure $\overline{\Omega}$ of the region, is called a solution of the Dirichlet problem, if

- u is a continuous on $\overline{\Omega}$.
- u is a harmonic in Ω, that is $\Delta u \equiv 0$ in Ω.
- $u|_{\partial\Omega} = f$.

A solution of the Dirichlet Problem u corresponding to the continuous boundary function f , is called a harmonic extension of f, let us call it, $u_f = u(f)$.

If the point $x \in \Omega$ is assumed to be fixed, then by Riesz representation theorem and the maximum principle, for $u(f)$ defined on the compact set $C_c(\Omega)$ there exists a unique Borel measure $\mu(x)$ at the point x on Ω, define $u_\mu \in C_c(\Omega)^*$ by

$$u_\mu(f) = \int f \, d\mu(x, \Omega),$$

for all f in $C_c(\Omega)$, and the measure $\mu(x, \Omega)$ is called the Harmonic measure.

For much extra information about harmonic measure and other topic which are related to it, we refer to [2, 7,8] and [9].

Definition (1.2) (Co-Area Formula).

Let $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}$ be Lipschtiz continuous assume that for a.e. $r \in \mathbb{R}$, assume that the level set $\{x \in \mathbb{R}^n \mid \varphi(x) = r, r \in \mathbb{R}\}$is a smooth and $(n-1)$-dimentional hypersurface in $\mathbb{R}^n$. Also, suppose that there is $g: \mathbb{R}^n \longrightarrow \mathbb{R}$is continuous and summable.
Then

$$\int_{\mathbb{R}^n} g|\nabla u| dx = \int_{-\infty}^{\infty} \left(\int_{\{\varphi(x)=r\}} g ds \right) dr.$$

Theorem (1.1) (Distortion Koebe theorem). For each $\varphi \in S$ defined on unit disk D,

$$\frac{1-r}{(1+r)^3} \leq |\varphi'(z)| \leq \frac{1+r}{(1-r)^3}, \qquad |z| = r < 1.$$

Equality holds if and only if φ is a suitable rotation of the Koebe function.

Theorem (1.2) (Koebe One-Quarter theorem)

The range of every function of class S contains the disk $\left\{w: |w| < \frac{1}{4}\right\}$.

Lemma (1.1).[6]

Let E be a compact subset of the plane having connected complement and let $g \in L(E, dA)$ for some $q > 1$. If $\hat{g} = 0$ identically in $\mathbb{C}\setminus E$ then $\hat{g}(z_0) = 0$ at every point $z_0 \in \partial E$, where

$$\int_E \frac{|g(z)|^q}{|z - z_0|} \, dA < \infty$$

Lemma (1.2). [5,6]

Let E be a compact set with connected complement and let $g \in L(E, dA)$ for some $1 < q \leq 2$. If $\hat{g} = 0$ identically in $\mathbb{C}\setminus E$ and ξ is a point in E^0 at a distance $\delta(z) = \frac{1}{e}$ from ∂E then

$$|\hat{g}(\xi)| \leq C\left\{g^*(\xi_0)\,\delta\log\left(\frac{1}{\delta}\right) + \left[\Gamma_q(\delta)\int_{|z-\xi_0|\leq 4\delta} |g(z)|^q \, dA\right]^{\frac{1}{q}}\right\}$$

where $g^*(\xi_0) = sup_r\,(\pi r^2)^{-1}\int_{|z-\xi|<r} |g(z)|dA$ is the Hardy- Littlewood maximal function, $\Gamma_q(\delta) = \log(\frac{1}{\delta})$ or $q = 2$ or $q < 2$ and, C is a constant depending only on q and the diameter of $\{E\}$.

Lemma(1.3) .[3]

There exist positive constants K_1 and K_2, depending only on $\delta(\varphi^{-1}(0))$, such that

$$K_1\frac{1 - |\varphi(z)|}{\delta(z)} \leq |\varphi'(z)| \leq K_2\frac{1 - |\varphi(z)|}{\delta(z)}$$

where $\varphi^{-1}(0) = z$, and φ maps a simply connected domain Ω conformally onto unit disk $D(0,1)$.

Now, our starting point in this work will be with Theorem (2.1) and to indicate how this theorem can be used to produce other results.

2. Main Results.

Theorem (2.1).

Let φ be a conformal mapping of a simply connected domain Ω onto the unit disk D , then there is a constant K depending only on the modulus of z in Ω such that

$$1 - |\varphi(z)| \leq K\sqrt{|z|}, \quad \text{for some } z \in \Omega.$$

Proof.

Let Ω be a bounded simply connected domain, and φ be a conformal map defined as follows:

$\varphi : \Omega \longrightarrow \mathrm{D}(|\mathrm{w}| < 1)$.

So, the inverse function $\psi = \varphi^{-1}: D \longrightarrow \Omega$. Apply Distortion theorem (1.1) to the inverse function $\psi = \varphi^{-1}(w)$, to obtain that $\varphi^{-1}(w) \in S$, normalized by the conditions $\varphi^{-1}(0) = 0$ and $(\varphi^{-1})'(0) = 1$.

Then, for $w \in D$,

$$\frac{|w|}{(1+|w|)^2} \le |\varphi^{-1}(w)| \le \frac{|w|}{(1-|w|)^2}.$$

Fix $w_0 \in D$, we obtain

$$|\varphi^{-1}(w_0)| \le \frac{|w_0|}{(1-|w_0|)^2} \dots \dots \dots \dots \dots (1)$$

Fix $z_0 \in \Omega$ Then apply Koebe one-quarter theorem (1.2) in order to show that the range of the function $\varphi^{-1}(w) \in S$ contains the disk $\left\{\varphi^{-1}(w): |\varphi^{-1}(w)| < \frac{1}{4}\right\} \subset \Omega^o$ such that $z_0 \notin \left\{\varphi^{-1}(w): |\varphi^{-1}(w)| < \frac{1}{4}\right\}$ which implies

$$|z_0| \ge \frac{1}{4} \text{ in } \Omega \dots \dots \dots \dots (2)$$

Equation (2) can therefore be written as follows:

$|z_0| \ge \frac{1}{4}|(\varphi^{-1})'(0)|$ in Ω by condition $|(\varphi^{-1})'(0)| = 1$.

We can assume that $w_0 \in D$ is the image of -1 , that is ; $\varphi(-1) = w_0$.

By taking the inverse of both sides, we obtain

$$|\varphi^{-1}(w_0)| = 1 \dots \dots \dots \dots (3)$$

Distortion theorem which represents by equation (1) will now be applied to obtain

$$1 = |\varphi^{-1}(w_0)| \le \frac{|w_0|}{(1-|w_0|)^2} = \frac{|w_0||(\varphi^{-1})'(0)|}{(1-|w_0|)^2}$$

$$1 \le \frac{|w_0||(\varphi^{-1})'(0)|}{(1-|w_0|)^2} \le \frac{4|z_0||w_0|}{(1-|w_0|)^2}$$

See equation (2), since, w_0 and z_0 are arbitrary points, this implies that

$$1 \le \frac{4|z||w|}{(1-|w|)^2}.$$

Hence,

$$1 \le \frac{4|z||w|}{(1-|w|)^2} \le \frac{C|z|}{(1-|w|)^2} = \frac{C|z|}{(1-|\varphi(z)|)^2}$$

which gives $(1-|\varphi(z)|)^2 \le C|z|$.

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Finally, we obtain $(1-|\varphi(z)|) \leq K \leq \sqrt{|z|}$ for all $z \in \Omega$. ■

Theorem (2.2).

Let Ω be a bounded simply connected domain, whose boundary is a class C^1 Jordan curve. If φ is a conformal map of Ω to the unit disk $D\,(|w|<1)$, then

$$\iint_\Omega |\varphi'|\; dx\, dy \leq \infty \text{ for all } p < 2.$$

Proof.

We shall assume that $z_0 \in \Omega$ and $\varphi(z_0) = 0$. It can then be inferred from the co-area formula.

Let φ be a conformal mapping of Ω (simply connected domain) onto the open unit disk D ($|z|<1$), that is; $\varphi: \Omega \longrightarrow D$ with $\mathcal{A}$ be a measurable subset of D.

Let $g(z): B = \{z \in \Omega: |\varphi(z)| = r\} \subset \Omega \longrightarrow [0,\infty)$ be a measurable function defined on the measurable set B in Ω , as the following

$$g(z) = \frac{\chi_{B(z)}}{|\varphi(z)'|} : \Omega \longrightarrow [0,\infty)$$

where $\chi_{B(z)}: B \subset \Omega \longrightarrow \{0,1\}$ is the characteristic function.

$$\chi_{B(z)} = \begin{cases} 1 & if\ z \in B \\ 0 & if\ z \notin B \end{cases}$$

Because we have to calculate the integral over $B = \{z \in \Omega: |\varphi(z)| = r\}$ and we know that $0 < r \leq 1$, which implies to $\chi_{B(z)} = 1;\ z \in B$ we have then

$$\iint_\Omega \frac{\chi_{B(z)}}{|\varphi'|} |\varphi'| dx\, dy = \int_0^1 \left(\int_{B=\{z\in\Omega: |\varphi(z)|=r\}} \frac{\chi_{B(z)}}{|\varphi(z)'|} r\, d\theta \right) dr$$

As known

$$\iint_\Omega dxdy = \int_0^1 \left(\int_{B=\{z\in\Omega: |\varphi(z)|=r\}} \frac{1}{|\varphi(z)'|}\; ds \right) dr$$

Now we have

$$\iint_\Omega |\varphi'|^p dxdy = \int_0^1 \left(\int_{B=\{z\in\Omega: |\varphi(z)|=r\}} \frac{1}{|\varphi(z)'|} |\varphi'|^p\, ds \right) dr$$

$$= \int_0^1 \left(\int_{B=\{z\in\Omega: |\varphi(z)|=r\}} |\varphi'|^{p-1}\, ds \right) dr$$

$$= \int_0^1 |\varphi'|^{p-1} \left(\int_0^{2\pi r} ds \right) dr$$

$$= \int_0^1 2\pi r|\varphi'|^{p-1}\,dr$$

By Lemma (1.3) we have $|\varphi'| \le C\frac{1-|\varphi|}{\delta(z)}$, so let us assume that $\delta(z) = |z|$ such that

$$\iint_\Omega |\varphi'|^p\,dxdy = \int_0^1 2\pi r|\varphi'|^{p-1}\,dr$$

$$\le \int_0^1 2\pi r\,\frac{(1-r)^{p-1}}{|z|^{p-1}}\,dr$$

In Theorem (2.1) we deduced that $1-|\varphi| \le K_1\sqrt{|z|}$, which implies to

$$\frac{K_2}{|z|^{p-1}} \le \frac{1}{(1-r)^{2(p-1)}}$$

Hence,

$$\int_0^1 2\pi r\,\frac{(1-r)^{p-1}}{|z|^{p-1}}\,dr \le \int_0^1 2\pi r\,\frac{(1-r)^{p-1}}{(1-r)^{2(p-1)}}\,dr = \int_0^1 2\pi r(1-r)^{-(p-1)}\,dr$$

$$= 2\pi r\left[\frac{(1-r)^{-(p-1)+1}}{-(p-1)+1}\right]_0^1$$

when $-(p-1)+1 > 0 \Longrightarrow p < 2.$ ■

Theorem (2.3).

Let E be a compact subset of the plane having connected complement, $D \subset E$ be a connected domain, let $\hat{g}(z)$ be a cauchy transform of a function g, where $g \in L^p(E, dA)$ for some $1 < q \le 2$, if $\hat{g}(z)$ is an identically zero in $\mathbb{C}\backslash E$ and there exist a non-decreasing sequence ∂D_i in D then

$$\int_{\partial D_i} \nabla\left(\frac{|\hat{g}(z)|}{\sqrt{|z|}}\right) d\mu_i \qquad \text{exists and is finite on } \partial D_i$$

Proof.

Let D be a connected domain $D \subset E$. Fix arbitrary point $\xi \in D$ and assume μ be the harmonic measure on ∂D representing ξ.

Choose ∂D_i be a non-decreacing sequence such that $\xi \in \partial D_i$ for all i and $\cup D_i$ fill up D.

Given $g \in L^q(E)$, and assume the property $\int gQ\,dA = 0$ for every polynomial $Q \in H^p(E, dA)$, then $\hat{g}(z)$ vanishes identically in the unbounded complementary component of E.

We obtain $\hat{g}(\xi)$ is bounded in E at Euclidean distance $\delta(z) = dis\,(z, \partial E) < \frac{1}{\varepsilon}$

(see lemma (1.2)), that is;| $\hat{g}(\xi)$| is bounded in D.

Hence, multiply and divide $|\hat{g}(z)|$ by $\sqrt{|z|}$ where $\delta(z) = dis\,(z, \partial D_i) = |z|$ this yield the identity below:

$$log\frac{\sqrt{|z|}|\hat{g}(z)|}{\sqrt{|z|}} = \log\sqrt{|z|} + \log\frac{|\hat{g}(z)|}{\sqrt{|z|}}$$

$$\Rightarrow log|\hat{g}(z)| = \log\sqrt{|z|} + \log\frac{|\hat{g}(z)|}{\sqrt{|z|}}$$

Integrate the last quantity over ∂D_i with respect to harmonic measure $d\mu_i$ this will imply,

$$\int_{\partial D_i} log|\hat{g}(z)|\,d\mu_i = \int_{\partial D_i} log\sqrt{|z|}\,d\mu_i + \int_{\partial D_i}\log\frac{|\hat{g}(z)|}{\sqrt{|z|}}d\mu_i$$

Here, we will pay particular attention to the second integral.

As known in [6, pp. 145] that,

$$\int_{\partial D_i}\log\frac{|\hat{g}(z)|}{\sqrt{|z|}}d\mu_i < \int_{\partial D_i}\frac{|\hat{g}(z)|}{\sqrt{|z|}}d\mu_i$$

It is known in measure theory (cf.[4])

$$\sum_{j=1}^{n}\frac{d}{dx_j}\int_{\partial D_i}\frac{|\hat{g}(z)|}{\sqrt{|z|}}d\mu_i = \int_{\partial D_i}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\frac{|\hat{g}(z)|}{\sqrt{|z|}}\right)d\mu_i \ldots\ldots\ldots\ldots(4)$$

$$= \int_{\partial D_i}\nabla\left(\frac{|\hat{g}(z)|}{\sqrt{|z|}}\right)d\mu_i$$

Hence, we have

$$\int_{\partial D_i}\nabla\left(\frac{|\hat{g}(z)|}{\sqrt{|z|}}\right)d\mu_i = \int_{\partial D_i}|\hat{g}(z)|\nabla\left(\frac{1}{\sqrt{|z|}}\right)d\mu_i + \int_{\partial D_i}\frac{1}{\sqrt{|z|}}\nabla(|\hat{g}(z)|)\,d\mu_i \ldots..(5)$$

This implies that

$$\left|\int_{\partial D_i}\nabla\left(\frac{|\hat{g}(z)|}{\sqrt{|z|}}\right)d\mu_i\right| \le \underbrace{\left|\int_{\partial D_i}|\hat{g}(z)|\nabla\left(\frac{1}{\sqrt{|z|}}\right)\right||d\mu_i|}_{I_1} + \underbrace{\left|\int_{\partial D_i}\frac{1}{\sqrt{|z|}}\nabla(|\hat{g}(z)|)\right||d\mu_i|}_{I_2}\ldots\ldots(6)$$

So we may direct out efforts toward finding a bounds for I_1 and I_2. In that endeavor we have to define a *Green function* which is a harmonic function on D_i and it is defined on $D_i^{'}$ as well. In this case should be define the harmonic function $d\mu_i = \frac{\partial G_i}{\partial n}|dz|$.

- Remove a small disk $|z-\xi| \le r_0$ from D_i we obtain $D_i^{'}$ such that $|z-\xi| \le r_0$ contained in every D_i, and its boundary is smooth, this lead to, any continuous function on ∂D_i to $\mathbb{R}$ will generate harmonic function on D_i with singularity (pole) at ξ.
- Let φ_i be a conformal map of D_i onto unit disk $D = \{w: |w| < 1\}$ with $\varphi_i(\xi) = 0$; and as clear $\psi_i = \varphi_i^{-1}$, which satisfies the following:

1. $|\nabla G_i| \leq C|\varphi_i'|$ on D_i'.
2. $|\psi_i'(w)| \geq C(|1-w|)$.

An estimation on I_1 can be obtained by applying *Hölder inequality* with short calculation as follows:

$$I_1 = \left|\int_{\partial D_i} |\,\hat{g}(z)|\nabla\left(\frac{1}{\sqrt{|z|}}\right)\right| |d\mu_i|$$

$$= \int_{\partial D_i'} |\,\hat{g}(z)|\,\nabla\left(\frac{1}{\sqrt{|z|}}\right)\nabla G_i\ dA$$

$$\leq \left(\int_{\partial D_i'} |\,\hat{g}(z)|^q\ dA\right)^{\frac{1}{q}}\left(\int_{\partial D_i'}\left|\nabla\left(\frac{1}{\sqrt{|z|}}\right)\nabla G_i\right|^p dA\right)^{\frac{1}{p}}$$

$$= \|\hat{g}(z)\|_q\left(\int_{\partial D_i'}\left|\nabla\left(\frac{1}{\sqrt{|z|}}\right)\nabla G_i\right|^p dA\right)^{\frac{1}{p}}$$

And hence we obtain

$$I_1{}^p \leq C_4\int_{\partial D_i'}\left|\nabla\left(\frac{1}{\sqrt{|z|}}\right)\nabla G_i\right|^p dA$$

$$= C_4\int_{\partial D_i'}\left|\nabla\left(\frac{1}{\sqrt{|z|}}\right)\right|^p |\nabla G_i\,|^p dA$$

$$\leq C_4\int_{\partial D_i'} |\varphi_i'|^{p-2}\left|\nabla\left(\frac{1}{1-|\varphi_i|}\right)\right|^p |\varphi_i'|^2 dA$$

$$= C_4\int_{|w|<1}\frac{1}{|\psi_i'|^{p-2}}\left|\nabla\left(\frac{1}{1-r}\right)\right|^p dA$$

$$= C_4\int_{|w|<1}\frac{dA}{(1-r)^{3p-2}}$$

$$= C_4\left[\frac{(1-r)^{-3p+3}}{-3p+3}\right]_0^1$$

such that when $-3p+3>0$ this implies to $p<1$.

It is a consequence of *Hölder inequality* and *Calderón-Zygmund theorem* on the continuity of singular integral operators, (cf.[1, pp.564],[10]), that

$$I_2 = \int_{\partial D_i}\left|\frac{\nabla(|\,\hat{g}(z)|)}{\sqrt{|z|}}\right| |d\mu_i|$$

$$= \int_{\partial D_i'} |\nabla(|\hat{g}(z)|)| \left|\frac{|\nabla G_i|}{\sqrt{|z|}}\right| dA$$

$$\leq C\, \|g\|_q \left(\int_{\partial D_i'} \left|\frac{|\nabla G_i|}{\sqrt{|z|}}\right|^p dA\right)^{\frac{1}{p}}$$

$$I_2^p \leq C_1 \left(\int_{\partial D_i'} \left|\frac{|\nabla G_i|}{\sqrt{|z|}}\right|^p dA\right)$$

$$\leq C_1 \int \frac{|\varphi_i'|^{p-2}}{1-|\varphi_i|} |\varphi_i'|^2 dA$$

$$= C_1 \int_{|w|<1} \frac{1}{|\psi_i'|^{p-2}} \frac{1}{(1-|w|)}\, dA$$

$$= C_2 \int_{|w|<1} \frac{dA}{(1-|w|)^{p-1}}$$

$$= C_3 \left[\frac{(1-r)^{-p+2}}{-p+2}\right]_0^1$$

when $-p+2>0$, this implies to $p<2$.

Finally, the quantity

$$\int_{\partial D_i} \nabla\left(\frac{|\hat{g}(z)|}{\sqrt{|z|}}\right) d\mu_i \qquad \text{exists and is finite on } \partial D_i \blacksquare$$

References.

[1] A. P. Calderon and A. Zygmund . **On the existence of certain singular integrals**. Acta Math, 88(1):85–139, **1952**.

[2] C. G. Simader and H. Sohr. **The Dirichlet problem for the Laplacian in bounded and unbounded domains**. Addison Wesley Longman Ltd, United States of America, **1996**.

[3] Ch. Pommerenke. **Univalent functions**. Vandenhoeck and Ruprecht, G¨ottingen, Germany, 1975.

[4] H. Flanders. **Differential Forms with Applications to the Physical Sciences.** Academic Press, New York, **1963**.

[5] J. E. Brennan. **Invariant subspaces and weighted polynomial approximation**. J. Ark. Matt, 11:167–189, **1973**.

[6] J. E. Brennan. **Approximation in the mean by polynomials on non-caratheodory domains**. J. Ark. Matt, 15(1-2):117–168, **1977**.

[7] J. Garnett and D.E. Marshall. **Harmonic Measure**. Cambridge University Press, Cambridge, New York, **2005**.

[8] L. Evans and R. F. Gariepy. **Measure theory and fine properties of functions**. CRC Press, United States of America, **1992**.

[9] S. Axler, F.W. Gehring, and K.A. Ribet. **Harmonic function theory**, 2nd Edition. Spring- Verlag Berlin Heidelberg, New York, **2000**.

[10] V. P. Havin and V . G. Mazja. **Applications of (p; L)- capacity to some problem in the theory of exceptional sets**. Math. USSR Sbornik, 19:547–580, **1973**.

[11] Z. Nehari. **Conformal Mapping**, 1st Edition. McGraw-Hill, New York, 1952.

بعض النتائج حول نظرية $\frac{1}{4}$ ل Koebe ونظريته عن التشوية

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المستخلص:

في هذه الورقة البحثية تناولنا كون أن الدالة العقدية $\varphi: \Omega \longrightarrow \mathbb{D}$ مخططًا امتثاليًا معرفا على مجال متصل بسيط إلى قرص الوحده $\mathbb{D}$ ، لبيان الدور الكبير لكل من نظرية $\frac{1}{4}$ ل Koebe ونظريته عن التشوية لإثبات وجود ثابت K يعتمد على معامل الحد المطلق للمتغير z في Ω لغرض الحصول على العلاقه $1-|\varphi(z)| \leq K\sqrt{|z|}$ لبعض قيم $z \in \Omega$. ومن خلال تلك النتيجه قمنا بايجاد شرط تام لبرهان القابلية التكامليه لتحويل كوشي $\hat{g}(z)$ ضمن المنطقه ∂D_i الذي هو في تزايد ضمن حدود المجال D ، بحيث $\int_{\partial D_i} \nabla\left(\frac{|\hat{g}(z)|}{\sqrt{|z|}}\right) d\mu_i$, يكون موجود ومنتهي عند ∂D_i .