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Traveling wave solutions for the time fractional Zoomeron equation

Abbas H. Taqi, Muhannad A. Mahmoud*, Borhan F. Jumaa

Department of Mathematics, College of Science, University of Kirkuk, Kirkuk, Iraq.
Dr.altaqi@yahoo.com, muhannad_ker@yahoo.com, Borhan_nissan@yahoo.com

Abstract

In this paper, a traveling wave solution has been constructed using the modified extended tanh method with Riccati equation for time fractional nonlinear partial differential equation. We used the proposed method to obtain exact solution for time fractional Zoomeron equation. The equation is converted to ordinary differential equation by using fractional complex transform and the properties of modified Riemann-Liouville derivative. Our results have been plotted at different time levels.

Keywords: Fractional Zoomeron equation; Modified extended tanh method; Travelling wave; Riccati equation.

MSC: 35R11, 35G20.

1. Introduction

Fractional differential equations which are a generalization of classical integer order differential equations have been used in the modelling of many problems in different research areas. Specially in signal processing, dynamical system, mechanics, stochastic, systems identification, plasma physics, electricity, electrochemistry, economics, control theory, and engineering. Finding exact and numerical solutions to fractional differential equations is an important task. Several Powerful and reliable methods have been proposed to obtain the exact solutions of fractional differential equations, for instance, modified extended tanh method [1], exp-function method [2-6], first integral method [7-12], functional variable method [13-16], ansatz method [17-20], and Kudryashov method [21,22].

Solitary waves theory have attracted intensive interest from mathematicians and physicists. Recently, the area of fractional differential equations has been studied by a number of researchers such as, M. Kaplan *et al*. [23], K. Hosseini *et al*. [24-25], and M. Eslami [26]. The aim of this paper is to find traveling wave solution of the time fractional Zoomeron equation which is importance in mathematical physics.

This paper is organized as follows: In Sections 2, the modified Riemann– Liouville derivative is demonstrated. In Section 3, analysis of the proposed method is given to illustrate how fractional differential equation is reduced into integer-order differential equation. In Section 4, our method is applied to obtain the exact solution for the time fractional Zoomeron equation. Conclusions are introduced in Section 5.

2. Jumarie's modified Riemann-Liouville derivative and its properties

The Jumarie's modified Riemann-Liouville derivative of order α the continuous function $f : R \to R$ is defined as follows [27]

$$D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-t)^{-\alpha} \left(f(t) - f(0) \right) dt, & 0 < \alpha < 1, \\ \left(f^{(n)}(x) \right)^{(\alpha-n)}, \ n \le \alpha < n+1, \ n \ge 1. \end{cases} \tag{1}$$

where $\Gamma(x)$ is the Gamma function which is defined as

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$$

Some useful properties of the Jumarie's modified Riemann-Liouville derivative are listed below.

Property 1.

$$D_x^\alpha x^r=\frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}x^{r-\alpha}. \tag{2}$$

Property 2.

$$D_x^\alpha(a\,f(x)+b\,g(x))=aD_x^\alpha f(x)+bD_x^\alpha g(x), \tag{3}$$

where a and b are constants.

Property 3.

$$D_x^\alpha f(\xi)=\frac{df}{d\xi}D_x^\alpha(\xi), \tag{4}$$

where $\xi=g(x)$.

For other properties, see [28].

3. Analysis of the method

Consider the following nonlinear partial differential equation:

$$H\left(u,D_t^{\alpha_1}u,D_x^{\alpha_2}u,D_{tt}^{2\alpha_1}u,D_{xx}^{2\alpha_2}u,D_t^{\alpha_1}D_x^{\alpha_2}u,\ldots\right),\qquad 0<\alpha_1,\alpha_2<1, \tag{5}$$

applying fractional complex transformation:

$$u(x,y,t)=f(\xi)\quad,\qquad \xi=x-ky-\frac{c}{\Gamma(1+\alpha)}t^\alpha-x_0,$$

where k and c are nonzero constants and x_0 is arbitrary constant, converts (5) into an integer order nonlinear ordinary differential equations as follows:

$$G(f,f',f'',f''',\ldots)=0, \tag{6}$$

where the derivatives are with respect to ξ. It is assumed that the solutions of (6) is presented as a finite series, say

$$f(\xi)=a_0+\sum_{n=1}^{N}\left(a_n\phi^n(\xi)+b_n\phi^{-n}(\xi)\right), \tag{7}$$

where a_n,b_n, $n=1,2,\ldots\ldots,N$ are constants that can be computed and $\phi^n(\xi)$ satisfies the Riccati equation

$$\phi'=b+\phi^2 \tag{8}$$

where b is a constant, Eq. (8) has the following types of solutions:

(i) If $b<0$, then

$\phi=-\sqrt{-b}\tanh\left(\sqrt{-b}\xi\right)$, or $\phi=-\sqrt{-b}\coth\left(\sqrt{-b}\xi\right)$

(ii) If $b > 0$, then
$\phi = \sqrt{b}\tan\left(\sqrt{b}\xi\right)$, or $\phi = -\sqrt{b}\cot\left(\sqrt{b}\xi\right)$
(iii) If $b = 0$, then
$$\phi = \frac{-1}{\xi}.$$

The value of N is usually determined by balancing the linear and nonlinear terms of highest orders in (6). Substituting Eq. (7) and its necessary derivatives into (6) gives

$$P(\phi(\xi)) = 0, \tag{9}$$

where $P(\phi(\xi))$ is a polynomial in $\phi(\xi)$. By equating the coefficient of each power of $\phi(\xi)$ in (9) to zero, a system of algebraic equations will be obtained whose solution yields the exact solutions of (5).

4. Application

Consider the following problem: Find a function $u(x,t)$ satisfying time fractional Zoomeron equation in the form:

$$D_{tt}^{2\alpha}\left(\frac{u_{xy}}{u}\right) - \left(\frac{u_{xy}}{u}\right)_{xx} + 2D_t^{\alpha}\left(u^2\right)_x = 0, \tag{10}$$

applying the transformation:

$$u(x,t) = f(\xi), \qquad \xi = x - ky - \frac{c}{\Gamma(1+\alpha)}t^{\alpha} - x_0 \tag{11}$$

Substituting (11) into (10), we have:

$$-kc^2\left(\frac{f''}{f}\right)'' + k\left(\frac{f''}{f}\right)'' - 2c\left(f^2\right)'' = 0 \tag{12}$$

Integrating (12) twice with respect to ξ, we obtain the following nonlinear ordinary differential equation:

$$k\left(1-c^2\right)f'' - 2c f^3 - Af = 0 \tag{13}$$

Where A is a nonzero integration constant, while the first integration constant is zero.

4.1. Exact solution of the time fractional Zoomeron equation using the modified extended tanh method

Balancing f'' and f^3 in (13) results $N + 2 = 3N$, and so $N = 1$. This offers a truncated series as the following form:

$$f(\xi) = a_0 + a_1\varphi(\xi) + b_1\varphi^{-1}(\xi) \tag{14}$$

by substituting (14) into (13) and equating the coefficient of each power of $\phi(\xi)$ to zero, we derive a system of algebraic equations as follows

$$-2ca_0{}^3-12ca_0a_1b_1-Aa_0=0,$$
$$2k(1-c^2)a_1b-6c\,a_0{}^2a_1-6ca_1^2b_1-Aa_1=0,$$
$$-6ca_0a_1^2=0,$$
$$2k(1-c^2)a_1-2c\,a_1{}^3=0,$$
$$2k(1-c^2)b_1b-6c\,a_0{}^2b_1-6cb_1^2a_1-Ab_1=0,$$
$$-6ca_0b_1^2=0,$$
$$2k(1-c^2)b_1b^2-2c\,b_1{}^3=0,$$

Solving the above system yields.

Case 1.

$$a_0=a_1=0,\quad b=\frac{A}{2k(1-c^2)},\ b_1=\mp\frac{A}{2\sqrt{ck(1-c^2)}}$$

hence, the solution is formed as:

$$u_1(x,t)=\mp\frac{A}{2\sqrt{ckb(1-c^2)}}\cot(\sqrt{b}\,\xi),\ b>0$$

$$u_2(x,t)=\pm\frac{A}{2\sqrt{ckb(1-c^2)}}\tan(\sqrt{b}\,\xi),\ b>0$$

$$u_3(x,t)=\pm\frac{A}{2\sqrt{-b}\sqrt{ck(1-c^2)}}\tanh(\sqrt{-b}\,\xi),\ b<0$$

$$u_4(x,t)=\pm\frac{A}{2\sqrt{-b}\sqrt{ck(1-c^2)}}\coth(\sqrt{-b}\,\xi),\ b<0$$

where

$$\xi=x-ky-\frac{c}{\Gamma(1+\alpha)}t^{\alpha}-x_0,\ b=\frac{A}{2k(1-c^2)}$$

Case 2.

$$a_0=b_1=0,\quad b=\frac{A}{2k(1-c^2)},\quad a_1=\mp\sqrt{\frac{k(1-c^2)}{c}}.$$

thus, the solution is formed as:

$$u_5(x,t) = \mp\sqrt{\frac{kb(1-c^2)}{c}}\tan(\sqrt{b}\xi), b > 0$$

$$u_6(x,t) = \pm\sqrt{\frac{kb(1-c^2)}{c}}\cot(\sqrt{b}\xi)\ ,\ b > 0$$

$$u_7(x,t) = \pm\sqrt{-b}\sqrt{\frac{k(1-c^2)}{c}}\tanh(\sqrt{-b}\xi), b < 0$$

$$u_8(x,t) = \pm\sqrt{-b}\sqrt{\frac{k(1-c^2)}{c}}\coth(\sqrt{-b}\xi)\ , b < 0$$

$$\xi = x - ky - \frac{c}{\Gamma(1+\alpha)}t^{\alpha} - x_0, b = \frac{A}{2k(1-c^2)}$$

Case 3.

$$a_0 = 0,\quad a_1 = -\frac{\sqrt{ck(1-c^2)}}{c}, b_1 = \frac{A\sqrt{ck(1-c^2)}}{4ck(1-c^2)},\ b = -\frac{A}{4k(1-c^2)}$$

therefore, the solution is formed as:

$$u_9(x,t) = -\frac{\sqrt{ck(1-c^2)}}{c}\sqrt{b}\tan(\sqrt{b}\xi) + \frac{A\sqrt{ck(1-c^2)}}{4ck(1-c^2)\sqrt{b}}\cot(\sqrt{b}\xi), b > 0$$

$$u_{10}(x,t) = \frac{\sqrt{ck(1-c^2)}}{c}\sqrt{b}\cot(\sqrt{b}\xi) - \frac{A\sqrt{ck(1-c^2)}}{4ck(1-c^2)\sqrt{b}}\tan(\sqrt{b}\xi), b > 0$$

$$u_{11}(x,t) = \frac{\sqrt{ck(1-c^2)}}{c}\sqrt{-b}\tanh(\sqrt{-b}\xi) - \frac{A\sqrt{ck(1-c^2)}}{4ck(1-c^2)\sqrt{-b}}\coth(\sqrt{-b}\xi), b < 0$$

$$u_{12}(x,t) = \frac{\sqrt{ck(1-c^2)}}{c}\sqrt{-b}\coth(\sqrt{-b}\xi) - \frac{A\sqrt{ck(1-c^2)}}{4ck(1-c^2)\sqrt{-b}}\tanh h(\sqrt{-b}\xi), b < 0$$

Where

$$\xi = x - ky - \frac{c}{\Gamma(1+\alpha)}t^{\alpha} - x_0,\ \ b = -\frac{A}{4k(1-c^2)}$$

Case 4.

$$a_0 = 0,\quad a_1 = \frac{\sqrt{ck(1-c^2)}}{c}, b_1 = -\frac{A\sqrt{ck(1-c^2)}}{4ck(1-c^2)},\ b = -\frac{A}{4k(1-c^2)}$$

hence, the solution is formed as:

$$u_{13}(x,t)=\frac{\sqrt{ck(1-c^2)}}{c}\sqrt{b}\tan\left(\sqrt{b}\xi\right)-\frac{A\sqrt{ck(1-c^2)}}{4ck(1-c^2)\sqrt{b}}\cot\left(\sqrt{b}\xi\right), b>0$$

$$u_{14}(x,t)=\frac{\sqrt{ck(1-c^2)}}{c}\sqrt{b}\cot\left(\sqrt{b}\xi\right)+\frac{A\sqrt{ck(1-c^2)}}{4ck(1-c^2)\sqrt{b}}\tan\left(\sqrt{b}\xi\right), b>0$$

$$u_{15}(x,t)=-\frac{\sqrt{ck(1-c^2)}}{c}\sqrt{-b}\tanh\left(\sqrt{-b}\xi\right)+\frac{A\sqrt{ck(1-c^2)}}{4ck(1-c^2)\sqrt{-b}}\coth\left(\sqrt{-b}\xi\right), b<0$$

$$u_{16}(x,t)=-\frac{\sqrt{ck(1-c^2)}}{c}\sqrt{-b}\coth\left(\sqrt{-b}\xi\right)+\frac{A\sqrt{ck(1-c^2)}}{4ck(1-c^2)\sqrt{-b}}\tanh h\left(\sqrt{-b}\xi\right), b<0$$

Where

$$\xi=x-ky-\frac{c}{\Gamma(1+\alpha)}t^{\alpha}-x_0,\ \ b=-\frac{A}{4k(1-c^2)}$$

Case 5.

$$a_0=0,\ \ a_1=-\frac{\sqrt{ck(1-c^2)}}{c}, b_1=\frac{A\sqrt{ck(1-c^2)}}{8ck(1-c^2)},\ b=\frac{A}{8k(1-c^2)}$$

hence, the solution is formed as:

$$u_{17}(x,t)=-\frac{\sqrt{ck(1-c^2)}}{c}\sqrt{b}\tan\left(\sqrt{b}\xi\right)+\frac{A\sqrt{ck(1-c^2)}}{8ck(1-c^2)\sqrt{b}}\cot\left(\sqrt{b}\xi\right), b>0$$

$$u_{18}(x,t)=\frac{\sqrt{ck(1-c^2)}}{c}\sqrt{b}\cot\left(\sqrt{b}\xi\right)-\frac{A\sqrt{ck(1-c^2)}}{8ck(1-c^2)\sqrt{b}}\tan\left(\sqrt{b}\xi\right), b>0$$

$$u_{19}(x,t)=\frac{\sqrt{ck(1-c^2)}}{c}\sqrt{-b}\tanh\left(\sqrt{-b}\xi\right)-\frac{A\sqrt{ck(1-c^2)}}{8ck(1-c^2)\sqrt{-b}}\coth\left(\sqrt{-b}\xi\right), b<0$$

$$u_{20}(x,t)=\frac{\sqrt{ck(1-c^2)}}{c}\sqrt{-b}\coth\left(\sqrt{-b}\xi\right)-\frac{A\sqrt{ck(1-c^2)}}{8ck(1-c^2)\sqrt{-b}}\tanh\left(\sqrt{-b}\xi\right), b<0$$

Where

$$\xi=x-ky-\frac{c}{\Gamma(1+\alpha)}t^{\alpha}-x_0,\ \ b=\frac{A}{8k(1-c^2)}$$

Case 6.

$$a_0=0,\ \ a_1=\frac{\sqrt{ck(1-c^2)}}{c}, b_1=-\frac{A\sqrt{ck(1-c^2)}}{8ck(1-c^2)},\ b=\frac{A}{8k(1-c^2)}$$

thus, the solution is formed as:

Journal of Iraqi Al-Khwarizmi Society (JIKhS) Year 2018 pages: 168-177
Volume:2 Special issue of the first international scientific conference of Iraqi Al-Khawarizmi Society 28-29 March 2018

$$u_{21}(x,t)=\frac{\sqrt{ck(1-c^2)}}{c}\sqrt{b}\tan\left(\sqrt{b}\xi\right)-\frac{A\sqrt{ck(1-c^2)}}{8ck(1-c^2)\sqrt{b}}\cot\left(\sqrt{b}\xi\right), b>0$$

$$u_{22}(x,t)=-\frac{\sqrt{ck(1-c^2)}}{c}\sqrt{b}\cot\left(\sqrt{b}\xi\right)+\frac{A\sqrt{ck(1-c^2)}}{8ck(1-c^2)\sqrt{b}}\tan\left(\sqrt{b}\xi\right), b>0$$

$$u_{23}(x,t)=-\frac{\sqrt{ck(1-c^2)}}{c}\sqrt{-b}\tanh\left(\sqrt{-b}\xi\right)+\frac{A\sqrt{ck(1-c^2)}}{8ck(1-c^2)\sqrt{-b}}\coth\left(\sqrt{-b}\xi\right), b<0$$

$$u_{24}(x,t)=-\frac{\sqrt{ck(1-c^2)}}{c}\sqrt{-b}\coth\left(\sqrt{-b}\xi\right)+\frac{A\sqrt{ck(1-c^2)}}{8ck(1-c^2)\sqrt{-b}}\tanh\left(\sqrt{-b}\xi\right), b<0$$

Where

$$\xi=x-ky-\frac{c}{\Gamma(1+\alpha)}t^{\alpha}-x_0,\quad b=\frac{A}{8k(1-c^2)}$$

Plotting these solutions at different time levels and different values of α , shows the motion of solitary waves as shown in figures 1 and 2.

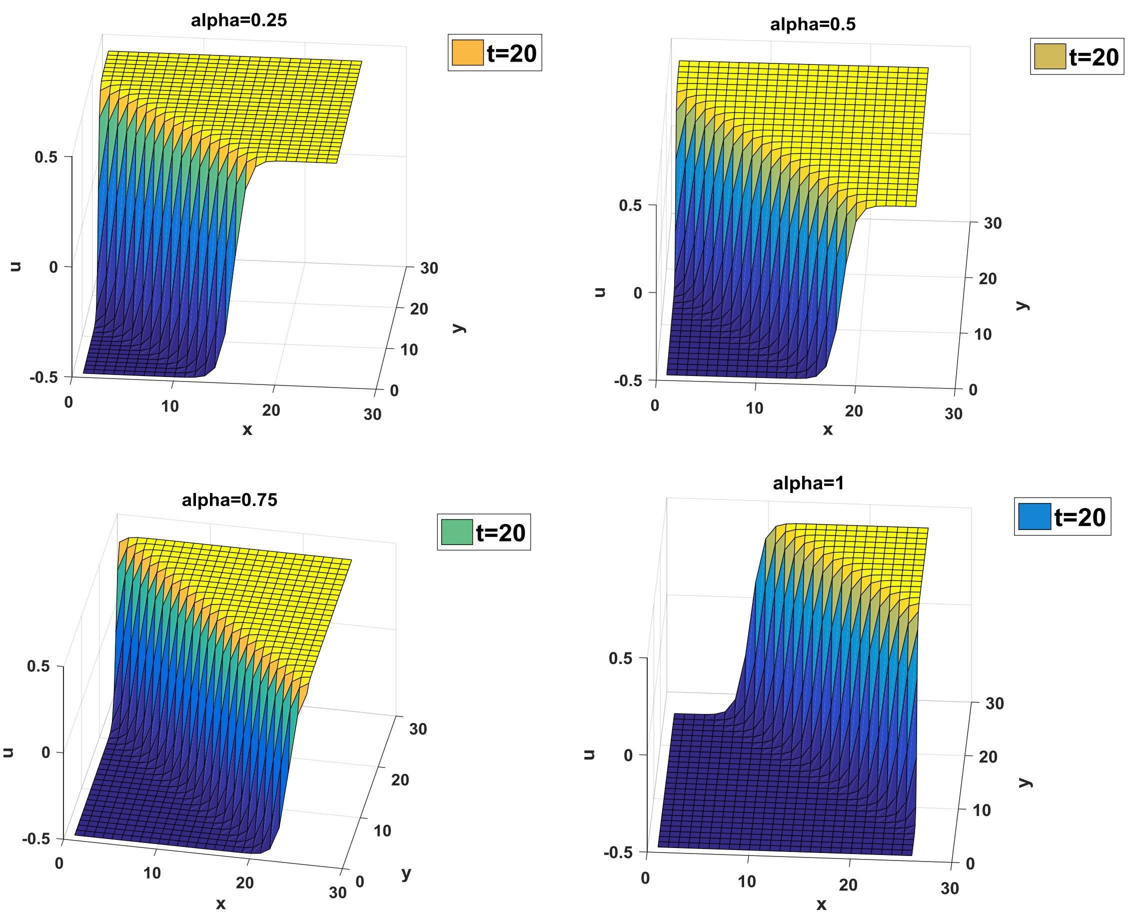

Figure 1. Analytic solutions for the time fractional Zoomeron equation with $A = k = -1,\ c = 2,\ 0 \le x, y \le 50,\ x_0 = 25$, at different time levels and $\alpha = 0.25, 0.5, 0.75, 1$.

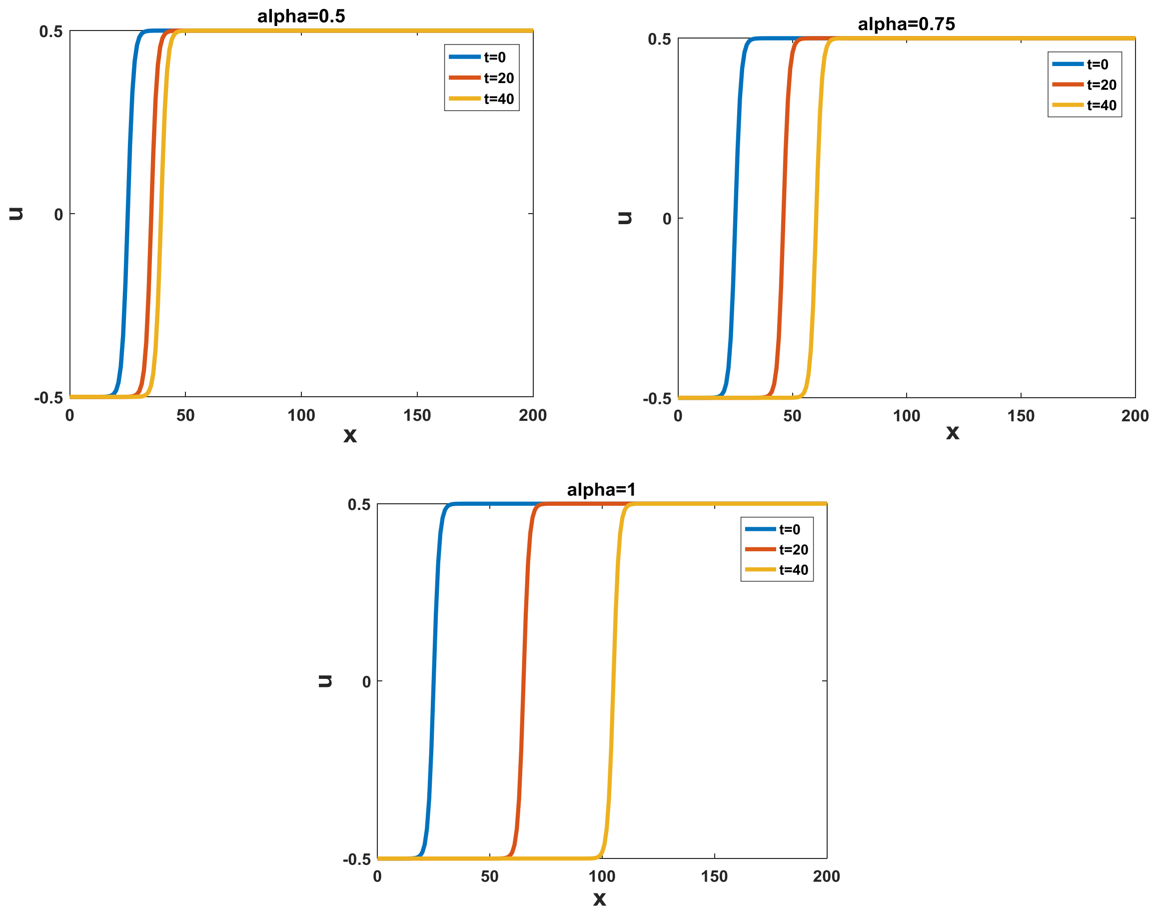

Figure 2. Analytic solutions for the time fractional Zoomeron equation with $A = k = -1,\ c = 2,\ 0 \le x \le 200,\ y = 25,\ x_0 = 50$, at different time levels and $\alpha = 0.5, 0.75, 1$.

5. Conclusion

In this paper, time fractional Zoomeron equation has been successfully solved by using modified extended tanh method with Riccati equation. Based on fractional complex transformation, original equation reduced into integer order ordinary differential equation. The proposed method was utilized to establish the exact solution of the resulted equation. Our approach is an efficient and concise technique to handle a wide range of linear and nonlinear fractional differential equations.

الخلاصة

في هذا البحث تم بناء حل الموجة المتنقلة باستخدام طريقة modified extended tanh method لحل معادلة تفاضلية جزئية غير خطية وذات رتبة كسرية في الزمن. تم استخدام الطريقة المقترحة للحصول على الحل المضبوط لمعادلة zoomeron المعادلة تم تحويلها الى معادلة تفاضلية اعتيادية باستخدام التحويل الكسري المركب وخواص مشتقات ريمان-ليوفيل المحسنة. النتائج تم رسمها بيانيا عند مستويات زمنية مختلفة.

References

[1] Taqi A. H. Soliton Solutions for Time Fractional Hamiltonian System. J. Adv. Math. and Comp. Sci., 2017; 23(6): 1-12.

[2] Guner O, Aksoy E, Bekir A, Cevikel A C. Different methods for (3 + I)-dimensional space- time fractional modified KdV-Zakharov-Kuznetsov equation. Comput. Math. Appl. doi:l0.1016/j.camwa.2016.02.004.

[3] Bekir A, Guner O, Cevikel A C. Fractional complex transform and exp-function methods for fractional differential equations. Abstr. Appl. Anal. 2013: 426462.

[4] Guner O, Bekir A, Bilgil H. A note on exp-function method combined with complex transform method applied to fractional differential equations. Adv. Nonlinear Anal. (2015);4 : 201-208.

[5] Bekir a, Guner O. Exact solutions of distinct physical structures to the fractional potential Kadomtsev-Petviashvili equation. Comput. Methods Differ. Equ. 2. 2014.

[6] Bekir A, Guner O, Bhrawy A H, Biswas A. Solving nonlinear fractional differential equations using exp-function and (G'/G) expansion methods. Romanian J. Phys. 60. 2015.

[7] Mirnl M zadeh, Eslami M, Biswas A. Solitons and periodic solutions to a couple of fractional nonlinear evolution equations. Pramana J. Phys. 82. 2014.

[8] Taghizadeh N, Najand M F, Soltani V Mohammadi. New exact solutions of the perturbed nonlinear fractional Schrodinger equation using two reliable methods. Appl. Appl. Math. 10. 2015.

[9] Lu B. The first integral method for some time fractional differential equations. J. Math. Anal. Appl. 395. 2012.

[10] Bekir A, Guner O, Unsal S. The first integral method for exact solutions of nonlinear fractional differential equations. J. Comput. Nonlinear Dyn. 10. 2015.

[11] Guner O, Bekir A, Cevikel A C. A variety of exact solutions for the time fractional Cahn-Allen equation. Eur. Phys. J. plus. 130. 2015.

[12] Younis M. The first integral method for time-space fractional differential equations. J Adv. Phys.2. 2013.

[13] Bekir A, Guner O. Analytical approach for the space-time nonlinear partial differential fractional equation. Int. J. Nonlinear Sci. Numer. Simul. 15. 2014.

[14] Matiufar M, Eslami M, Kordy M. The functional variable method for solving the fractional Korteweg-de Vries equations and the coupled Korteweg-de Vries equations. Pramana J. Phys. 85. 2015.

[15] Guner O, Eser D. Exact solutions of the space time fractional symmetric regularized long wave equation using different methods. Adv. Math. Phys. 2014:456804.

[16] Lin W, Chen K. The functional variable method for finding exact solutions of some nonlinear time-fractional differential equations. Pramana J. Phys. 2013; 81: 377-384.

[17] Guner O, Bekir A. Bright and dark soliton solutions for some nonlinear fractional differential equations. Chinese Phys. B 25. 2016.

[18] Mirzazadeh M. Topological and non-topological soliton solutions to some time-fractional differential equations. Pramana J. Phys. 85. 2015.

[19] Korkmaz A. Exact solutions of space-time fractional EW and modified EW equations. arXiv: l601.01294vl [nlin.SI] 6 Jan 2016.

[20] Guner O. Singular and non-topological Soliton solutions for nonlinear fractional differential equations. Chinese Phys. B 24. 2015.

[21] Akbari M, Taghizadeh N. Applications of He's variational principle method and the Kudryashov method to nonlinear time-fractional differential equations. Caspian J. Math. Sci. 4. 2015.

[22] Saha R S. New analytical exact solutions of time fractional KDV-KZK equation by Kudryashov methods. Chinese Phys. B 25. 2016 .

[23] Kaplan M, Bekir A. A novel analytical method for time-fractional differential equations. International Journal for Light and Electron Optics. 127. 2016.

[24] Hosseini K, Ayati Z. Exact solutions of space-time fractional EW and modified EW Equations using Kudryashov method. Nonlinear Sci. Lett. A. 7(2). 2016.

[25] Hosseini K, Mayeli P, Ansari R. Modified Kudryashov method for solving the Conformable time-fractional Klein–Gordon

equations with quadratic and cubic nonlinearities. International Journal for Light and Electron Optics. Available online 130. 2017.

[26] Eslami M. Exact traveling wave solutions to the fractional coupled nonlinear Schrodinger Equations. Applied Mathematics and Computation. 285. 2016.

[27] Jumarie G. Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results. Comput. Math. Appl. 51. 2006.

[28] Jurnarie G. Table of some basic fractional calculus formulae derived from a Modified Riemann Liouville derivative for non-differentiable functions. Appl. Math. Lett. 22. 2009.