Some Results On S-pure Submodules

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Abstract

Let R be a commutative ring with identity, S a multiplicatively closed subset of R, and M be an R-module. The goal of this work is to stuy some properties of S-pure submodules and introduce the notion of S-pure submodules relative to submodules of M as a generalization of S-pure submodules of M and prove a number of results concerning of this class of modules.

1.Introduction

Let Ṁ be a unitary module defined on commutative ring R with 1. A submodule of an R -module is called pure submodule if for every finitely generated ideal I of R[1,2] ,A subset V of R have been said a multiplicatively closed (m.c) subset of R if: 1 ∈ V and for any v_1,v_2 in V, v_1v_2 ∈ V. Let V be (m.c) of R and M is an R –module,

(1) V* a nonempty subset of M have been said V-closed if vm ∈ V^* for every v ∈ V and m ∈ V*.
(2) An V-closed subset V^* have been said saturated if the next provision is hold: where dm ∈S^* for d ∈ R also m ∈ M, then d ∈ V and m ∈ V* [3]

In [2] We say that a submodule N of an R-module M is S-pure if there exists an s ∈ S, where S is (m.c) such that s(N ∩ IM) ⊆ IN for every ideal I of R. Let Z be the ring of integers, for a prime number p, one can see that the submodule pZ of the Z-module Z is not pure. Take the multiplicatively closed subset $S = \{p^n : n \in N \cup \{0\}\}$ of Z. Then for each k ∈ N, p(pZ ∩ (kZ)) ⊆ (pZ)(kZ) implies that pZ is an S-pure submodule of Z [4].

2. **Main Results.**

Theorem 2.1. [1] Let N and K be two submodules of an R-module M such that N ⊆ K ⊆ M. Then we have the following:

(1) If N is an S-pure submodule of K and K is an S-pure submodule of M, then N is an S-pure submodule of M.

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(2) If N is an S-pure submodule of M, then N is an S-pure submodule of K.

(3) If K is an S-pure submodule of M, then K/N is an S-pure submodule of M/N. (d) If N is an S-pure submodule of M and K/N is an S-pure submodule of M/N, then K is an S-pure submodule of M.

(4) If N is an S-pure submodule of M, then there is a bijection between the S-pure submodules of M containing N and the Spure submodules of M/N.

Proposition 2.2: Assume that M is module. Then X is S-pure submodule if and only if foreach finite sets { m_i } ∈ M, { n_i } ∈ X with {r_{ij}} ∈ R and $n_j = \sum_{i=1}^{k} r_{ij} m_i$, j = 1, 2, . . . , l, there is { x_i } ∈ X which is finite set, when $n_j - \sum_{i=1}^{k} r_{ij} x_i \in X \cap K$ for each submodule K.

Proof: Let $r_{ij} \in R$ $(1 \le j \le 1, 1 \le i \le k)$. Therefore $n_j \in M \cap X$. There is $x_i \in X$ when $n_j = \sum_{i=1}^{k} r_{ij} x_i +$ w , where w ∈ K also $n_j - \sum_{i=1}^{k} r_{ij} x_i \in X \cap K$. The converse, take b' ∈ M ∩ X. Thus $b' = \sum_{i=1}^{k} a_i m_i$ when $a_i \in R$ and $m_i \in M$. Exists a finite set { x_i} ⊆ X, so that, $b' - \sum_{i=1}^{k} a_i x_i \in X \cap K$, since $b' - \sum_{i=1}^{k} a_i x_i \in M$, thus b' ∈ X. By looking carefully at the proof of the above theorem, we have, X is a submodule of module M, then X is S-pure, if and only if, X ∩ M = X.

Definition 2.3: Let M be a module, M is called have the S-pure submodule intersection property, if we have the intersection of two S-pure submodules is again S-pure submodule.

Proposition 2.4: Let M be a module.

1- If M has the S-pure submodule intersection property, then each S-pure submodule in M has the S-pure submodule intersection property.

2- Let X be an S-pure submodule in M. M has S-pure submodule intersection property, if and only if, $\frac{M}{X}$ has S-pure submodule intersection property.

Proof : 1- It is obvious

2- Assume that $\frac{N}{X}, \frac{K}{X}$ are two S-pure submodules in $\frac{M}{X}$. Let S be (m.c) Want to show that $s[\left(\frac{N}{X} \cap \frac{K}{X}\right) \cap I\left(\frac{M}{X}\right)] \subseteq I\left(\frac{N}{X} \cap \frac{K}{X}\right)$ for each ideal I of R and for some s ∈S. We claim that each one is S-pure submodule . To prove this matter, let $c \in s(N \cap IM)$. Since $\frac{N}{X}$ is S-pure submodule in $\frac{M}{X}$, hence $s(\frac{N}{X} \cap I(\frac{M}{X})) \subseteq I\frac{N}{X}$, therefore $s(\frac{N}{X} \cap \frac{IM+X}{X}) \subseteq \frac{IN+X}{X}$, then $s(\frac{N \cap (IM+X)}{X}) \subseteq \frac{IN+X}{X}$ thus, $\frac{s(N \cap IM)+X}{X} \subseteq$

$\frac{IN+X}{X}$, hence $s(N \cap IM) + X = IN + X$, and therefore $s(N \cap IM) \subseteq IN$. Since $c \in s(N \cap IM) \subseteq s(N \cap IM) + X$, thus $c \in IN + X$. Let t + w = c, $t \in IN$ and $w \in X$. Now consider w = c – t $\in s(X \cap IM) \subseteq IX \subseteq IN$ and therefore N is S-pure submodule. But M have the S-pure intersection property, therefore N ∩ K is S-pure submodule in M. Thus s[$(N \cap K) \cap IM] \subseteq I(N \cap K)$. Now, let $c \in s[\left(\frac{N}{X} \cap \frac{K}{X}\right) \cap I\left(\frac{M}{X}\right)]$, therfore t + X = c , t ∈ IM and c = x + X = y + X, when $x \in N$, $y \in K$. Therefore $c - x \in X \subseteq N, c - y \in X \subseteq K$ and therefore $t \in N \cap K$ Thus $t \in s[(N \cap K) \cap IM] \subseteq I(N \cap K)$. Thus $c = t + X \in I\left(\frac{N \cap K}{X}\right) \subseteq I\left(\frac{N}{X} \cap \frac{K}{X}\right)$. On another side, put L , F is S-pure, suppose that X is a -submodule of L and X be a submodule in F, thus $\frac{L}{X}$, $\frac{F}{X}$ are S-pure submodules of $\frac{M}{X}$. But $\frac{M}{X}$ have the S-pure intersection property, then $\frac{L}{X} \cap \frac{F}{X} \subseteq \frac{L \cap F}{X}$ is S-pure . Thus, L ∩ F is S-pure submodule in module M.

Theorem 2.6 : A module M owns the S-pure intersection property if and only if $s(IN \cap IK) = s[I(N \cap K)]$ for each ideal I ∈ R and for all S-pure submodules N , K in M, where S is (m.c) .

Proof : Put M have the S-pure submodule intersection property then, $N \cap K$ is S-pure submodules. Put I ∈ R, therefore $s[(N \cap K) \cap IM] \subseteq I(N \cap K)$ for some s ∈ S . Clearly that $s[I(N \cap K)] \subseteq s(IN \cap IK)$. But s$(IN \cap IK) \subseteq s[N \cap (K \cap IM)] \subseteq s[(N \cap K) \cap IM] \subseteq s[I(N \cap K])$. Then $s[IN \cap IK] \subseteq s[I(N \cap K)]$. On the other side , put I ∈ R, and put L , F are S- pure in M, thus s[(L ∩ F) ∩ IM] ⊆ s[L ∩ (F ∩ IM)] ⊆ s[L ∩ IF]⊆ L ∩ IF. Similarly s[(L ∩ F) ∩ IM] ⊆s[F ∩ IL] ⊆ F ∩ IL . But L, F are S-pure submodules in a module M. Then s[(L ∩ F) ∩ IM] ⊆ IL ∩ IF= I(L ∩ F).

Theorem 2-7:- Let S be (m.c) and M be a module. M have the S-pure submodule intersection property, if and only if , for each s-pure submodules, Z and X of a module M with for each homorphism g : (Z∩X) ⇾ M when s(Im g ∩ Z) +s((Im g +X) ∩ IM) = {0} for some s ∈S and Im g + Z is S-pure submodule in module M and ker g is S-pure submodule in module M.

Proof: Let S be (m.c) and M have the S-pure subhypermodule intersection property. Let N, T be S-pure submodules with g : N ∩ T ⇾ M is homomorphism where N ∩ Img = 0 and also Im g + T which is S-pure submodule in module M. Put W = {g(c) + c | c ∈ N ∩ T }. W is a submodule. To show that W is S-pure submodule. Put I∈ R and t= $\sum_{i=1}^{n}$ r_i m_i ∈ s(W ∩ IM), s∈ S, r_i ∈ R, m_i ∈ M. Hence t =$\sum_{i=1}^{n}$ r_i m_i = m + g(m) for some m ∈ (N ∩ T). So t =$\sum_{i=0}^{n}$ r_i m_i = m + g(m) ∈ N ∩ (Img + T) ⊆

Img + N and Im g + N is S-pure submodule in module M. Thus c $=\sum_{i=0}^{n}$ r_i m_i ∈ s[(Img+ N) ∩ IM] = s[I(Img +N) ∩ IM]. Therefore $\sum_{i=0}^{n}$ r_i m_i $=\sum_{i=0}^{n}$ $r_i(z_i + t_i)$ + v , β_i z_i ∈ N, t_i ∈ Img, for each i = 1,….,n. where v ∈ s((Img + N) ∩ IM). Therefore c $=\sum_{i=0}^{n}$ m_i = $\sum_{i=0}^{n}$ r_i z_i + $\sum_{i=0}^{n}$ r_i t_i + v, hence c - $\sum_{i=0}^{n}$ r_i m_i $=\sum_{i=0}^{n}$ r_i t_i - g(c) + v ∈ s(Img ∩ N) + s((Img+ N) ∩ IM) =0. Thus c = r_it_i ∈ s[(N ∩ T) ∩ IN]. While (N ∩ T) is S-pure submodule in module M, thus it is S-pure submodule in N. Therefore s[(N ∩ T) ∩ IN] = s[I(N ∩ T)] by theorem (2.6). Hence m ∈ s[I(N ∩ T)]. Let c = $r_i\sum_{i=0}^{n}$ w_i + h, w_i ∈ (N ∩ T), h ∈ s((N ∩ T) ∩IT).Then g(c) = $\sum_{i=1}^{n} r_i g(w_i)$ + g(h). Now t =c + g(c) $=\sum_{i=0}^{n}$ r_i w_i $+\sum_{i=0}^{n}$ $r_i g(w_i)$ + g(h) = $\sum_{i=0}^{n}$ r_i (w_i +g(w_i)) +g(h) ∈ IW. Thus s(W ∩ IM) ⊆ IW and W is S-pure submodule in a module M. Now we want to prove, kerg = (N ∩ T) ∩ W. Suppose that c ∈ kerg, thus c ∈ (N ∩ T) also g(c)= 0 . Therefore c ∈ W, Now let c ∈ (N ∩ T) ∩ W, thus c = t + g(t), t ∈ (N ∩ T), then c - t= g(t) ∈ s(N ∩ Img) ⊆ s(N ∩ Img) + s[((N + Img) ∩ zM)] = 0. Therefore g(c) = g(t) =0 so c ∈ ker g. Since M has the S-pure submodule intersection property, then (N ∩ T) ∩ W = kerg is S-pure in M. Reciprocally, let X, Y be two S-pure submodules in M. We define g : (X ∩ Y) ⇾ M by g(x)=0, ∀ x ∈ (X ∩ Y). So it is obvious that s(X ∩ Img) + s((Img +X) ∩ IM)= 0 for some s ∈S and Img + X = X is S-pure submodule in M, thus kerg = (X ∩ Y) is S-pure submodule in M.

Theorem 2.8: Let S be (m.c) and a module M have the S-pure submodule intersection property, if and only if, for each S-pure submodules W , a submodule X in module M, and for all homomorphism g : (W ∩ X) ⇾ T, where T is a submodule in M for this reason s(W ∩ T)+ s(W+ (T ∩ IM))= 0 and W + T are S-pure and T is S-pure submodule .

Proof: The proof by the same of proof of the previous theorem.

3- S-Pure submodules relative to submodule.

Definition 3.1: Let S be (m.c) and M be a module also L is a submodule of M. A submodule X of M is said to be S-pure relative to submodule L (resp. S-L-pure), if there is s ∈ S and for each I ∈ R, s(X ∩ IM) ⊆ IX + (L ∩ s(X ∩ IM)) .

Remark 3.2: Let S be (m.c) and M be a module also L is submodule of M

1. Let X be an S-L-pure submodule of M. If W is an S-L-pure submodule of X, then W is S-L-pure submodule in M.

2. Let X be a S-L-pure module of M. if W is a S-pure submodule of M containing X, then X is a S-L-pure in W

3. Let X be an S-L-pure submodule of M. If W is a submodule of X and so W is submodule of L, then X/W is S-L/W -pure submodule of M/W.

4. Let X and W be submodule of M, If W is S-L-pure submodule of M and X/W is S-L/W-pure submodule of M/W, then X is S-L-pure submodule of M .

Proof(1):

Put $I \in R$, because X is S-L-pure of M also W is S-L-pure in X, therefore $s(X \cap IM) \subseteq IX + (L \cap s(X \cap IM))$ and $s(W \cap IX) \subseteq IW + (L \cap s(W \cap IX))$ but W is submodule of M, therefore $s(W \cap IM) \subseteq s(X \cap IM) \subseteq IX + (L \cap s(X \cap IM))$ and hence $s(W \cap IM) \subseteq W \cap [IX + (L \cap s(X \cap IM))] \subseteq IX + [W \cap (L \cap s(X \cap IM))] \subseteq IX + [L \cap s(W \cap IM)]$ and since $W \subseteq X$, therefore $s(W \cap IM) \subseteq IW + [L \cap s(W \cap IM)]$. Hence $s(W \cap IM) \subseteq IW + [L \cap s(W \cap IM)]$

(2) Put $I \in R$, because X is S-L-pure in M, thus $s(X \cap IM) \subseteq IX + (L \cap s(X \cap IM))$. But W submodule of M, therefore, $s(X \cap IW) \subseteq s(X \cap IM) \subseteq IX + (L \cap s(X \cap IM))$ and hence $s(X \cap IW) \subseteq W \cap [IX + (L \cap s(X \cap IM))] \subseteq IX + [W \cap (L \cap s(X \cap IM))] \subseteq IX + [X \cap (L \cap s(W \cap IM))]$, since W is S-pure submodule of M, thus $IX + [X \cap (L \cap s(W \cap IM))] \subseteq IX + [L \cap (X \cap s(W \cap IM))] \subseteq IX + [L \cap s(X \cap IW)]$, hence $s(X \cap IW) \subseteq IX + [L \cap s(X \cap IW)]$ .Thus $X \cap IW \subseteq IX + [L \cap s(X \cap IW)]$.

3. Let $I \in R$, because X is S-L-pure submodule in M, therefore $s(X \cap IM) \subseteq IX + (L \cap s(X \cap IM))$.
We want to show that's $s[\frac{X}{W} \cap I(\frac{M}{W})] \subseteq I(\frac{X}{W}) + [(\frac{L}{W}) \cap s(\frac{X}{W} \cap z(\frac{M}{W})]$,

$$s[\frac{X}{W} \cap I(\frac{M}{W})] \subseteq s[\frac{X+W}{W} \cap \frac{IM+W}{W}] \subseteq \frac{s[(X+W)\cap(IM+W)]}{W} \subseteq \frac{s(X \cap zM)+W}{W} \subseteq \frac{[X+(L\cap s(X \cap zM))]+W}{W} \subseteq \frac{X+W}{W} + \frac{[L \cap s(X\cap IM)+W}{W} \subseteq \frac{X+W}{W} + \left[\frac{L+W}{W} \cap \frac{s(X\cap IM)+W}{W}\right] \subseteq I\left(\frac{X}{W}\right) + [(\frac{L}{W}) \cap s(\frac{X}{W} \cap I(\frac{M}{W})]$$

4. Clear .

Proposition 3.3 : Let S satisfying the maximal multiple condition and M be an R-module. Then we have the following.

(1) If $\{N_\lambda\}_{\lambda \in \Lambda}$ is a chain of S-L--pure submodules of M, then $\sum_{\lambda \in \Lambda} N_\lambda$ is an S-L-pure submodule of M.

(2) If N is a submodule of M, then there is a submodule K of M maximal with respect to K ⊆ N and K has S-L-pure .

Proof:

(1) Let I be an ideal of R. Then there exists an s ∈ S such that $s(N_\lambda \cap IM) \subseteq IN_\lambda$ for each $\lambda \in \Lambda$. This implies that $s(\sum_{\lambda\in\Lambda} N_\lambda \cap IM) = s\sum_{\lambda\in\Lambda}(N_\lambda \cap IM) \subseteq I\sum_{\lambda\in\Lambda} N_\lambda$, Since $\{N_\lambda\}_{\lambda\in\Lambda}$ is a chain.

(2) Let $\Sigma = \{H \leq N|H$ is a S- L− pure submodule of M$\}$. P Then $0 \in \Sigma$ 6= ∅. Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a totally ordered subset of Σ. Then $\sum_{\lambda\in\Lambda} N_\lambda \subseteq N$ and by part (a), $\sum_{\lambda\in\Lambda} N_\lambda$ is an S-pure submodule of M. Thus by using Zorn's Lemma, one can see that Σ has a maximal element, K say, as needed.

Definition 3.4: A module M is called have S-pure relative to submodule L intersection property (resp. S-L−PIP) if we have the intersection of any two S-L-pure submodule is again S-L-pure.

Proposition 3.5: Let S be (m.c) andM be a module.

1. If M have the S-L−P IP, then all S-L-pure submodule of M owns the S-L− P IP.

2. Let X be s-L-pure subhypermodule of M also X is submodule of L. M have S-L−P IP if and only if M/X have S-L/X− P IP.

Proof: It is evident.

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