Anti-hesitant Fuzzy Subgroups

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ABSTRACT

in this paper , efforts have been taken to generalize the notions of anti fuzzy subgroups and hesitant fuzzy subgroups . proposed definition was supported by graphical comparison with our result and constracted examples. So we introduce the notions of anti – hesitant fuzzy subgroups , lower level subgroup and normal anti- hesitant fuzzy subgroups and derive some properties of its .

1. Introduction

L.A.Zadah[7] in 1965 was introduce the notions of concept of fuzzy set as a function from a universal set to the unit interval [0,1] , rosenfeld in 1971 [5] study and introduce the notion of fuzzy subgrouppoid , fuzzy ideal and fuzzy homomorphism and some algebraic structure of fuzzy subgroups.

Biswas in 1990 [1] introduce the notions of anti – fuzzy subgroup , lower level subgroup which based on the notions of fuzzy subgroup. Gayen S. , JHA S. and elt in [3] study and introduce more general notion on anti-fuzzy subgroups

In 2010 , Torra [6] introduce the notions of hesitant fuzzy set as a faction from a universal st to the power set of the unit interval . Divakaran D. and John S.J. in [2] and Kim J.H. el. in [4] extended the notions of hesitanat fuzzy subgroups and introduce some properties of this concept's.

The organization of this article is in section 2 we have mentioned some preliminary aspects of fuzzy subgroup , anti- fuzzy subgroup and normal anti - fuzzy subgroups .in section 3 , a redefined version of anti-hesitant fuzzy subgroup has been given , also we have given the definitions of normal anti-hesitant fuzzy subgroup , level subgroup of anti – hesitant fuzzy subgroup and decided some

2. PRELIMINARIES

In this section , we state some essential definitions and properties of fuzzy subgroup and anti-fuzzy subgroup and the definition of hesitant fuzzy subgroups , also we re

Definition 2.1 [1,3] A fuzzy subset $\mu$of a crisp group G is termed as a fuzzy subgroup of G if for all x, y ∈ G, the subsequent conditions are satisfied: (i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$) (ii) $\mu(x^{-1}) \geq \mu(x)$.Definition 2.2 [1,3] A fuzzy subset $\mu$of a crisp group G is termed as an anti - fuzzy subgroup of G if for all x, y ∈ G, the subsequent conditions are satisfied: (i) $\mu(xy) \leq \max\{\mu(x), \mu(y)\}$) (ii) $\mu(x^{-1}) \leq \mu(x)$.

Definition 2 .3:- [3]
Let G be a group and μ a fuzzy subset of G . Then μ is called a fuzzy normal subgroup if

$$\mu(xy) = \mu(yx) \text{ for all x and y in G .}$$

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Theorem 2 . 4 [1]: Let G be a group and μ a fuzzy subset of G . Then μ is called a fuzzy subgroup of G if and only if ($x\ y^{-1}$) ≥ min { μ (x) , μ (y) }
Definition 2 . 5 [3]: Let μ be a fuzzy subgroup of a group G . For a in G , the fuzzy coset a μ of G determined by a and μ is defined by (a μ) (x) = μ (a^{-1} x) for all x in G .
Definition . 2 .6 [3]: Let λ and μ be any two fuzzy subgroups of a group G . They are said to be conjugate fuzzy subgroups of G if for some g ∈ G , λ (x) = μ (g^{-1} x g) for all x ∈ G .
Theorem 2.7 [3] : Let λ and μ be any two fuzzy subgroups of any group G . Then μ and λ are conjugate fuzzy subgroups of G if and only if μ = λ .
Definition 2.8 [2 ,4]: Let G be a group and h ∈ HFS(G) . Then h is called a hesitant fuzzy subgroup (in short, HFG) of G, if it satisfies the following conditions, for any $x, y \in G$
1. $h(xy) \supseteq h(x) \cap h(y)$.
2. $h(x^{-1}) \supseteq h(x)$.
The collection of all HFG_S in G will be referred to as HFG(G).
Theorem (2.9) [4]: Let G be a group and h ∈ HFS(G) , then h ∈ HFG(G) if and only if $h(xy^{-1}) \supseteq h(x) \cap h(y^{-1})$, for any $x, y \in G$.

3. Anti-hesitant fuzzy subgroups

In this section we investigate the definition of anti – hesitant fuzzy subgroup and give some important properties of the anti- hesitant fuzzy subgroups .

Definition 3.1 : Let G be a group and let $h \in HF(G)$ then h is called anti – hesitant fuzzy subgroup if and only if : -

1) $h(xy) \subseteq h(x) \cup h(y)$.
2) $h(x^{-1}) \subseteq h(x)$.
For all $x, y \in G$.
Example 3 . 2:
Let G = { e , a , b , c } be the klein ′s group . h : $G \longrightarrow [0,1]$ be a hesitant fuzzy set with
h (e) = [0 , 1] h (a) = [$\frac{6}{13}$, $\frac{12}{13}$] ,

h (b) = [$\frac{10}{21}$, $\frac{4}{5}$] , h (c) = [[$\frac{3}{4}$, $\frac{5}{6}$]
then easily to see that h is anti − hesitant fuzzy subgroup of G .
the following lemma give a new condition for anti – hesitant fuzzy subgroup

Lemma 3. 3: let G be a group. then h ∈ AHFG(G) if and only if $h(xy^{-1}) \subseteq h(x) \cup h(y)$ for all $x, y \in G$
Proof : - suppose that h ∈ AHFG(G) then by definition 3.1.2 we have the result directly
$h(xy^{-1}) \subseteq h(x) \cup h(y^{-1})$
$\subseteq h(x) \cup h(y)$, since $h(x^{-1}) \subseteq h(x)$
So that $h(xy^{-1}) \subseteq h(x) \cup h(y)$
Conversely suppose that $h(xy^{-1}) \subseteq h(x) \cup h(y)$,
To prove h is anti – hesitant fuzzy subgroup
$h(xy) = h(x(y^{-1})^{-1})$
$\subseteq h(x) \cup h(y^{-1})$
Also $h(x^{-1}) = h(ex^{-1})$
$= h(x\,x^{-1} . x^{-1}))$
$\subseteq h(xx^{-1}) \cup h(x)$

$\subseteq h(x) \cup h(x) \cup h(x)$
$\subseteq h(x)$
for all $x, y \in G$
So that h is an anti – hesitant fuzzy subgroup .
Proposition 3.4: Let G be a group and $h \in AHFG(G)$.
Then the set $h_* = \{ x \in G | h(x) = h(e) \}$Form a subgroup of a group G .
Proof : -
$h_* \neq \emptyset$, since $e \in h_*$
Let $x , y \in h_*$, to prove $x . y^{-1} \in h_*$.
i . e to prove $h(x . y^{-1}) = h(e)$

$$
\begin{aligned}
h(x . y^{-1}) &\subseteq h(x) \cup h(y^{-1}) \\
&\subseteq h(x) \cup h(y) \\
&\subseteq h(e) \cup h(e) \\
&\subseteq h(e) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)
\end{aligned}
$$

But $h(e) \subseteq h(x)$ ……………………………………………… (2)
From (1) and (2) we get $h(xy^{-1}) = h(e)$
So that $xy^{-1} \in h_*$and hence h_* is subgroup of G
Lemma 3. 5: Let h be an anti – hesitant fuzzy subgroup of a group G , then : -
1) $h(x) = h(x^{-1})$.
2) $h(x^n) = h(x)$
for all $x \in G$
Proof : -
(1) $h(x) = h((x^{-1})^{-1} \subseteq h(x^{-1})$,
But $h(x^{-1}) \subseteq h(x)$, by (2) of definition (3.1.1)
So that $h(x^{-1}) = h(x)$.
2) The proof is by induction on n .
If n = 1 then it is clear $h(x) \subseteq h(x)$.
If n = 2 then $h(x^2) = h(x . x) \subseteq h(x) \cup h(x)$,
So that $h(x^2) \subseteq h(x)$.
Now suppose that $h(x^{r-1}) \subseteq h(x)$,
Now to prove $h(x^r) \subseteq h(x)$

$$
\begin{aligned}
h(x^r) &= h(x^{r-1} . x) \\
&\subseteq h(x^{r-1}) \cup h(x), \\
&\subseteq h(x) \cup h(x) \\
&= h(x)
\end{aligned}
$$

So that $h(x^n) \subseteq h(x)$
Lemma 3. 6: Let G be a group and h ∈ A H F G (G) , if for all $x, y \in G$, $h(x) \subseteq h(y)$ .Then $h(xy) = h(x) \cup h(y)$
Proof : -
Suppose that h (x)) ⊆ h (y) ,

$$
\begin{aligned}
\text{So that} \quad h(y) &= h\left(x^{-1} . x . y\right) \subseteq h(x^{-1}) \cup h(x . y) \\
&= h(x) \cup h(x . y) \\
h(y) &\subseteq h(x) \cup h(x . y)
\end{aligned}
$$

and since $h(x) \subseteq h(y)$,
so that $h(y) \subseteq h(x y) \subseteq h(x) \cup h(y) = h(y)$
hence we have $h(x . y) \subseteq h(x) \cup h(y)$
and $h(x) \cup h(y) = h(y) \subseteq h(x y)$

so that $h(xy) = h(x) \cup h(y)$

Lemma 3 . 7 : Let G be a group and $h \in AHFG(G)$. then the set $H = \{ x \in G \mid h(x) = \{0\} \}$ is either empty or subgroup of G

Proof : -

suppose that $x, y \in H$,

Hence $h(xy^{-1}) \subseteq h(x) \cup h(y^{-1})$

$= h(x) \cup h(y)$

$= \{0\} \cup \{0\} = \{0\}$

Hence $xy^{-1} \in H$ and so H is subgroup of G

Lemma 3 . 8: Let G be a group and $h \in AHFG(G)$. if $h(xy^{-1}) = \emptyset$ then $h(x) = h(y)$.

Proof :-

$h(x) = h(xy^{-1}y) = h\big((xy^{-1}) . y\big)$

$\subseteq h(xy^{-1}) \cup h(y)$

$= \emptyset \cup h(y) = h(y)$

So that $h(x) \subseteq h(y)$ .... (1)

By the same way , we can prove $h(y) \subseteq h(x)$ .... (2)

From (1) and (2) , we have $h(x) = h(y)$.

Lemma 3 . 9 : -

Let G be a group and h be hesitant fuzzy set of a group G . if $h(e) = \emptyset$ and if $h(xy^{-1}) = h(x) \cup h(y)$ For all $x, y \in G$. Then $h \in AHFG(G)$

proof : - $h(y^{-1}) = h\big(e\, y^{-1}\big)$

$\subseteq h(e) \cup h(y^{-1})$

$= h(e) \cup h(y)$

$= \emptyset \cup h(y) = h(y)$

Similarly $h(y) \subseteq h\big(y^{-1}\big)$,

So that $h(y) = h(y^{-1})$

Now $h(xy) = h(x(y^{-1})^{-1})$

$\subseteq h(x) \cup h\big(y^{-1}\big)$

$= h(x) \cup h(y)$

Thus $h \in AHFG(G)$

Definition 3. 10 : Let G be a group and we define the binary operation $\circ$ and the unary operation (inverse) on the set A H F G (G)

By $(h_1 \circ h_2) = \cap \{ h_1(y) \cup h_2(z) \mid y, z \in G, yz = x \}$,

and $h_1^{-1}(x) = h_1(x^{-1})$ for all $x \in G$

The operations $\circ$ is called the product of $h_1 \circ h_2$,and $h_1^{-1}$the inverse of h_1.

Theorem 3 . 11 : - let G be a group then $h \in AHFG(G)$ if and only if h the following are holds :-

1) $h \circ h \supseteq h$.

2) $h^{-1} \supseteq h$.

Proof : -

suppose that $h \in AHFG(G)$,then

$(h \circ h)(x) = \cap \{ h(y) \cup h(z) \mid y, z \in G, yz = x \}$

$= h(y_*) \cup h(z_*)$ for some $y_* . z_* = x$ and $y_* . z_* \in G$

$\supseteq h(x)$

Since $h(x) = h(yz) \subseteq h(y) \cup h(z)$

Conversely , suppose $h \circ h \supseteq h$, Then $\cap \{h(y) \cup h(z) \mid y, z \in G,\ yz = x\} \supseteq h(x)$.

$\Rightarrow h(y) \cup h(z) \supseteq h(x)$ *for some* $y, z \in G, yz = x$.

$\Rightarrow h(y) \cup h(z) \supseteq h(yz)$,

Thus $h \in AHFG(G)$.

Theorem 3 .12: Let G be a group and $h_1, h_2 \in AHFG(G)$ then $h_1 \circ h_2 \in AHFG(G)$ if and only if $h_1 \circ h_2 = h_2 \circ h_1$

Proof : - Suppose that $h_1 \circ h_2 \in AHFG(G)$

Then $h_1 \circ h_2 = h_1^{-1} \circ h_2^{-1}$

$= (h_2 \circ h_1)^{-1} = h_2 \circ h_1$

Conversely suppose that $h_1 \circ h_2 = h_2 \circ h_1$

$h_1 \circ h_2 \subseteq h_1^{-1} \circ h_2^{-1}$

$= (h_2 \circ h_1)^{-1} = (h_1 \circ h_2)^{-1}$

So that $(h_1 \circ h_2)^{-1} \supseteq h_1 \circ h_2$

Thus $h_1 \circ h_2 \in AHFG(G)$

Theorem 3. 13: Let $h_1, h_2 \in AHFG(G)$ then $h_1 \cup h_2 \in AHFG(G)$

Proof : -

Let $x, y \in G$,

$$(h_1 \cup h_2)(xy^{-1}) = h_1(xy^{-1}) \cup h_2(xy^{-1})$$
$$\subseteq \big(h_1(x) \cup h_2(y)\big) \cup (h_2(x) \cup h_1(y))$$
$$= \big(h_1(x) \cup h_2(x)\big) \cup \big(h_1(y) \cup h_2(y)\big)$$
$$= \big((h_1 \cup h_2)(x)\big) \cup ((h_1 \cup h_2)(y))$$

Then $h_1 \cup h_2 \in AHFG(G)$

Corollary 3. 14

If $\{h_i \mid i \in I\} \subseteq AHFG(G)$ then $\cup_{i \in I} h_i \in AHFG(G)$

Example 3 .15

Let Z be a group and $h_1, h_2 : Z \longrightarrow p[0,1]$ are two anti – hesitant fuzzy subgroups defined as follows : -

$$h_1(x) = \begin{cases} [0.2, 0.8] & if \quad x \in 2z \\ \emptyset & otherwise \end{cases}$$

$$h_2(y) = \begin{cases} [0, 0.5] & if \quad y \in 3z \\ \emptyset & otherwise \end{cases}$$

Then we can easily see that $h_1 \cap h_2$ is not anti – hesitant fuzzy subgroup of Z .

Now we introduce a new definition on anti - hesitant fuzzy subgroup

Definition 3 .16: Let G be a group and let $h \in AHFG(G)$, Then h is called normal anti – hesitant fuzzy sub group (in short , $NAHFG(G)$), if $h(xy) = h(yx)$ for all $x, y \in G$.

Example 3 . 17

Consider the general linear group of a degree n , GL(n , R). Then GL (n, R) is not abelian Let I_n be the unit matrix in GL (n , R) . We define the mapping $h : GL(n, R) \rightarrow p[0,1]$ as follows : -

$$h(M) = \begin{cases} [\frac{1}{5}, \frac{2}{3}] & \text{if M is not triangular matrix.} \\ [\frac{1}{3}, \frac{1}{2}] & \text{if M is triangular matrix} \end{cases}$$

for each $I_n \neq M \in GL(n, R)$, and $h(I_n) = [0, 1]$.

Then we can easily see that $h \in NAHFG(GL(n, R))$.

Proposition 3. 18: Let G be a group , Let $h_1 \in HFS(G)$ and let $h_2 \in NAHFG(G)$, then

$h_1 \circ h_2 = h_2 \circ h_1$.

Proof :

Let $x \in G$ then $(h_1 \circ h_2)(x) = \cap_{x=yz} [\, h_1(y) \cup h_2(z) \,]$

$$= \cap_{x=yz} [\, h_1(y) \cup h_2(y^{-1}x) \,]$$

$$= \cap_{x=y(y^{-1}x)} [\, h_1(y) \cup h_2(xy^{-1}) \,]$$

[since $h_2 \in NAHF(G)$]

$$= \cap_{x=y(y^{-1}x)} [\, h_2(xy^{-1}) \cup h_1(y) \,]$$

$$= (h_2 \circ h_1)(x)$$

Thus $h_1 \circ h_2 = h_2 \circ h_1$.

Proposition 3. 19: Let G be a group and let $h \in NAHFG(G)$, then G_h is normal subgroup of G . Where $G_h = \{ x \in G : h(x) = h(e) \}$.

Proof :

From proposition (3 . 1 . 6) that G_h is a subgroup of G and $G_h \neq \emptyset$.

Let $x \in G_h$ and let $y \in G$. Then

$h(yxy^{-1}) = h((yx)y^{-1})$ [since $h \in NAHFG(G)$]

$$= h(y^{-1}(yx))$$

$$= h((y^{-1}y)x)$$

$$= h(x)$$

$$= h(e) . \quad [\text{ since } x \in G_h]$$

Thus $yxy^{-1} \in G_h$.

So G_h is a normal subgroup of G .

Remark 3 . 20: The converse of proposition (3 . 2 . 4) is need not to be true general in the following example .

Example 3 .21: Let G = { e , a , b , c } be the group in which is given by :-

.	E	a	b	C
e	E	a	b	C
a	A	e	b	C
b	B	a	e	C
c	C	a	b	E

and h be a hesitant fuzzy set of G defined by :

h (e) = h (a) = [0 , 1] ,

h (b) = (0 , 1] , h (c) = [0 , 1)

then we can easily cheek that h is a hesitant fuzzy subgroup of G .
Moreover G_h= { e , a} is normal subgroup of G, but h (a b) = h (b) = (0 , 1] ≠ [0 , 1] = h (b a) = h (a), thus h is not normal anti – hesitant fuzzy subgroup of G .

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