On Compact Set Generated by T-open Sets

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ABSTRACT

In this work, a novel definition of compactness in topological space, termed T-compact set, was introduced by relying on a new type of open set called T-open. A number of examples of the new concept were also given. Additionally, the conditions under which the concept of T-compact space were discussed. Finally, the propositions of the function through is transferred from one space to another were explained.

1. Introduction

Jingcheng Tong(Jacksonville) [1] initiate the study of t-set in 1989. The concept of T-open set was defined J. M. Saad based on a foundational t-set collection, with subsequent show that the family of T-open sets forms a topological structure, this topology is distinguished by two properties: all closed sets and all t-sets in a space are T-open sets [2].There are types of compact space known in ancient and modern times, such as: S-compact space which introduced by Travis Thompson in 1976 [3], and θ-compact space which defined by Kohli and Das in 2006 [4].

In this paper, we discuss a new types called T-compact set and generalize the characterizations of this notion in light of new open sets. The interior points of A and closure set of A by A°, ($\overline{A}$) with respectively

Definition 1.1 [1]:A is called t-set of $A^{\circ} = \overline{A}^{\circ}$. The family of t-set denoted by ts.

Definition 1.2 [2]: Let (X, τ) be a topological space, $A \subseteq X$. Then the T-closure set of A is defined by $x \in \overline{A}^{T}$ if $\forall U \in ts$, such that $U \cap A \neq \emptyset$.

Definition 1.3 [2]: The set $A \subseteq X$ is called T-closed if $A = \overline{A}^{T}$. The complement of T- closed is called T-open. The family of T-open (T-closed) set is denoted by $TO(X).(TC(X)]$.

Proposition 1.4 [2]:A family of T-open set is topology.

Definition 1.5 [2]: Let A subset a space X, a point $x \in A$ is called T-interior point of A if there exists an $U \in ts$ such that $x \in U \subseteq A$, it is denoted $A^{\circ T}$.

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Proposition 1.6 [2]: Let (X, τ) be a topological space and $A, B \subseteq X$, then A is T-open set if and only if $A = A^{\circ T}$.

Definition 1.7 [2]: A function $f: X \to Y$ is called T-continuous if the inverse image for any open set in X is T-open in Y.

Note: In fact, every open set in metrizable space is T-open.

Definition 1.8[5]: Let A be a subset of the topological space X. The open cover for A is a collection λ of open sets in X, whose union contains A.

Definition 1.9[5]: A topological space X is compact provided that every open cover of X has a finite subcover.

Definition 1.10[5]: A family A of subsets of a space X has the finite intersection property provided that every finite sub-collection of A has non-empty intersection.

2. T-Compact Spaces

Definition 2.1: Let A be a subset of the topological space X. The T-open cover for A is a collection λ of T-open sets whose union contains A. A subcover derived from the T-open cover λ is a subcollection λ' of λ whose union contains A.

Example 2.2: First of all, it is clear that every open set in any metrizable space is T-open set, therefore if $A = [1,4]$ and consider the T-open cover $\lambda = \{(z-1, z+1) | \, z \in \mathbb{Z}\}$.

Consider the sub-cover $\lambda = \{(-1,1), (0,2), (1,3), (2,4), (3,5), (4,6)\}$ is a subcover of A, and happens to be the smallest subcover of λ that covers A.

Definition 2.3: A topological space X is T-compact provided that every T-open cover of X has a finite subcover.

This says that however we write X an a union of T-open sets, there is always a finite sub-collection $\{O_i\}_{i=1}^{n}$ of these sets whose union is X. A subspace A of X is T-compact if A is a T-compact space in its subspace topology.

Since relatively T-open sets in the subspace topology are the intersections of T-open sets in X with the subspace A, the definition of T-compactness for subspaces can be restated as follows.

Alternate definition:

A subspace A of X is T-compact if and only if every T-open cover of A by T-open sets in X has a finite subcover.

Examples 2.4:

1. Every discrete space consisting of a finite number of points is T-compact.
2. The finite complement topology with real line $\mathbb{R}$ is T-compact.
3. An infinite set X with the discrete topology is not T-compact.
4. Any open interval (r_1, r_2) is not T-compact. $\lambda = \left\{\left(r_1 + \frac{1}{s}, r_2\right) \mid s = 2, \dots, \infty\right\}$ is T-open cover of (r_1, r_2). However, no finite sub-collection of these sets will cover (r_1, r_2).
5. $\mathbb{R}^n$ is not T-compact for any positive integer n, since $\lambda = \{B(0, s) \mid s = 1, \dots., \infty\}$ is T-open cover with no finite sub-cover.

Proposition 2.5: A space X is T-compact if and only if every family of T-closed sets in X with the finite intersection property has non-empty intersection.

This says that if F is a family of T-closed sets with the finite intersection property, then we must have that $\bigcap_\alpha C_\alpha \neq \emptyset$.

Proof: Assume that X is T-compact and let $F = \{C_\alpha | \alpha \in I\}$ be a family of T-closed sets with the finite intersection property. We want to show that the intersection is empty. Let $\lambda = \{O_\alpha = X \setminus C_\alpha | \alpha \in I\}$ is a collection of T-open sets in X. Then,

$$\bigcup_{\alpha \in I} O_\alpha = \bigcup_{\alpha \in I} X \setminus C_\alpha = X \setminus \bigcap_{\alpha \in I} C_\alpha = X \setminus \emptyset = X$$

Thus, λ is T-open cover for X. Since X is T-compact, it must have a finite subcover; i.e.,

$$X = \bigcup_{i=1}^{n} O_{\alpha_i} = \bigcup_{i=1}^{n} (X \setminus C_{\alpha_i}) = X \setminus \bigcap_{i=1}^{n} C_{\alpha i}$$

This means that $\bigcap_{i=1}^{n} C_{\alpha i}$ must be empty, contradicting the fact that F has the finite intersection property. Thus, if F has the finite intersection property, then the intersection of all numbers of F must be non-empty.

Proposition 2.6: Let Y be a subspace of X. Then X is T-compact space if and only if Y is T-compact.

Proof: The proof begins by considering a family representing a T-open cover of Y space. Each set in this family is an intersection of Y with T-open set in X space. Therefore, the family of T-open cover will construct a T-open cover for X possessing a finite sub-cover. Consequently, the intersection of Y with this finite cover will be a finite cover of space Y. The opposite direction of the proof is almost direct direction.

Proposition 2.7: The union of two T-compact spaces is T-compact.

Proof: To evidence that $A \cup B$ is T-compact, we will take T-open cover $\{U_y | y \in A\}$ of it, clear that is T-open cover of A and B. So, $\{U_y\}$ has finite sub-cover with respect a set A and B, thus the union of finite sub-cover is also. Hence $A \cup B$ is T-compact.

Definition 2.8: A function $f: X \to Y$ is called T-irresolute if the inverse image for any T-open set in Y is T-open in X.

Definition 2.9: A function $f: X \to Y$ is called T-open if the image for any T-open set in X is T-open in Y.

Proposition 2.10: Let X be a T-compact space and $f: X \to Y$ a T-irresolute function from X onto Y. Then Y is T-compac space.

Proof: We will outline this proof. Start with T-open cover for Y.Use the T-irresolution of f to pull it back to an T-open cover of X. Use T-compactness to extract a finite sub cover for X, and then use the fact that f is onto to reconstruct a finite subcover for Y.

Corollary 2.11: Let A be a T-compact set in a space X and $f: X \to Y$ a T-irresolute function. The image $f(A)$ in Y is a T-compact subset of Y.

Corollary 2.12: Let X be a compact space and $f: X \to Y$ a T-continuous function from X onto metrizable space Y. Then Y is T-compact.

The T-compactness is not hereditary, because $(0,1)$ is not a T-compact subset of the T-compact space $[0,1]$. It is closed hereditary.

Proposition 2.13: Each T-closed subset of a T-compact space is T-compact (i.e. A T-compactness is weak hereditary property).

Proof: Let A be T-closed subset of the T-compact space X and let λ be an T-open cover of A by T-open sets in X. Since A is T-closed, then $X \setminus A$ is T-open and $\lambda^* = \lambda \cup \{X \setminus A\}$ is T-open cover of X.

Since X is T-compact, it has a finite subcover, containing only finitely many members $O_1, \dots, O_n$ of λ and may contain $X\backslash A$.

Since $X = (X\backslash A) \cup \bigcup_{i=1}^{n} O_i$

It follows that $A \subset \bigcup_{i=1}^{n} O_i$ and A has a finite subcover.

Is the opposite implication true? Is every T-compact subset of a space is T-closed? Not necessarily. The following though is true.

Definition 2.14: A topological space X is said to be T-Hausdorff provided that every pair of distinct points of X, there exists two disjoint T-open sets each one contains one point but not the other.

Proposition 2.15: Each T-compact subset of T-Hausdorff X is T-closed.

Proof: To evidence that A is T-closed., we will evidence that its complement is T-open. Let $x \in X\backslash A$. Then for each $y \in A$ there are disjoint sets U_y and V_y with $x \in V_y$ and $y \in U_y$. The collection of T-open sets $\{U_y | y \in A\}$ forms an T-open cover of A. Since A is T-compact, this T-open cover has a finite sub-cover, $\{U_{yi} = | i = 1, \dots, n\}$. Let

$$U = \bigcup_{i=1}^{n} U_{yi} \qquad V = \bigcap_{i=1}^{n} V_{yi}$$

Now, any U_{yi} and V_{yi} are disjoint, we have U and V are disjoint. Also, $A \subset U$ and $x \in V$. Thus, for each point $x \in X\backslash A$ we have found T-open set, V containing x which is disjoint from A. Thus, $X\backslash A$ is T-open, and A is T-closed .

Corollary 2.16: A subset A of any T-compact T-Hausdorff is T-compact if and only if it is T-closed.

Proposition 2.17: Let X be T-compact space and Y be T-Hausdorff. Then T-irresolute function $f: X \to Y$ is T-closed.

Proof: Clear from Corollary 2.11.

Proposition 2.18: If A and B are disjoint T-compact subsets of T-Hausdorff X, then there exist disjoint T-open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$.

Proof: Assume A and B are T-compact sets in X such that $A \cap B = \emptyset$, that is $A \subseteq X\backslash B$ and $B \subseteq X\backslash A$. Since X is T-Hausdorff, then A and B are T-closed sets due to Proposition 2.15, so $X\backslash B$ and $X\backslash A$ are T-open sets.

Corollary 2.19: If A and B are disjoint T-closed subsets of a T-compact and T-Hausdorff X, then there exist disjoint T-open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$.

This concludes the work. In the future, a new concept, namely Compactly T-closed, will be presented, based on the concept of T-compact set.

Conclusions and future studies

The work in this research paper was focused on generalizing the concepts of compact sets according to a new concept called T-open, and in the future the concept of compactly T-closed will be discussed according to this new set.

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