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Asymptotic Sequences in Fuzzy Metric Space

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Abstract:

On the aim and properties of asymptotic sequences, in this paper we introduce the concepts asymptotic sequences in fuzzy metric space. Also, we conclude a concept convergent in space and to study relationship between uniformly continuous and asymptotic sequences. Finally, other properties were investigate in the asymptotic sequences.

Keywords: Asymptotic sequences, relationship between uniformly continuous and asymptotic sequences, convergence asymptotic sequences.

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1. Introduction:

In 1965, the concept of fuzzy sets was introduced by Zadeh [5]. With the concept of fuzzy sets, the fuzzy metric space was introduced by I. Kramosil and J. Michalek [4] in 1975. Helpern [9] in 1981first proved a fixed point theorem for fuzzy functions. Also M. Grabiec [7] in 1988 proved the contraction principle in the setting of the fuzzy metric spaces. Moreover, A. George and P. Veeramani [2] in1994 modified the notion of fuzzy metric spaces with the help of t-norm, and fuzzy matric space are studied by more authors (see[8]).

In this paper we introduce the concepts asymptotic sequences in fuzzy metric space. Also, we conclude a concept convergent in space and to study relationship between uniformly continuous and asymptotic sequences. Finally, other properties were investigated in the asymptotic sequences.

2. Preliminaries:

Definition(2.1),[1]: Let X be an arbitrary set, a fuzzy set M on X is a function from X to I Let $I^X = \{M: X \to I\}$. i.e. $M \in I^X$.

Definition(2.2),[1]: A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous (t-norm) on the set $[0,1]$,if $*$ is satisfying the following conditions :
(TN-1) $a * b = b * a$ for all $a, b \in [0,1]$ (i.e. $*$ is commutative).
(TN-2) $a * (b * c) = (a * b) * c$ for all $a, b, c \in [0,1]$,(i.e. $*$ is associative).
(TN-3) $a * 1 = a$ for all $a \in [0,1]$.
(TN-4) If $b, c \in [0,1]$ such that $b \leq c$, then $a * b \leq a * c$ for all $a \in [0,1]$, (i.e. $*$ is monotone).

Definition (2.3),[8]: Let X be a non-empty set, $*$ be a continuous t-norm on$[0,1]$.

A function $M: X \times X \times (0, \infty) \to [0, 1]$ is called a fuzzy metric function on X if it satisfies the following axioms:
for all $x, y, z \in X$ and for all $t, s > 0$

(FM-1) $M(x, y, t) > 0$.

(FM-2) $M(x, y, t) = 1 \Leftrightarrow x = y$.

(FM-3) $M(x, y, t) = M(y, x, t)$.

(FM-4) $M(x, y, t+s) \geq M(x, z, t) * M(z, y, s)$.

(FM-5) $M(x, y, \circ): (0, \infty) \to [0,1]$ is continuous.

Definition (2.4),[8]: Let $(X, M, *)$ be a fuzzy metric space. Then

A sequence $\{x_n\}$ in X is said to be convergent to x in X if for each

$r \in (0,1)$ and each $t > 0$, there exist $k \in Z^+$ such that $M(x_n, x, t) > 1 - r$ for all $n \geq k$ (or equivalent $lim_{n\to\infty} M(x_n, x, t) = 1$).

A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for each $r \in (0,1)$ and each $t > 0$, there exist $k \in Z^+$ such that $M(x_n, x_m, t) > 1 - r$ for all $n, m \geq k$ (or equivalent $lim_{n,m\to\infty} M(x_n, x_m t) = 1$).

Definition (2.5),[2] : Let $(X, M, *)$ and $(Y, M, *)$ be two fuzzy metric space . The function $f: X \to Y$ is said to be continuous at $x_0 \in X$ if for all $r \in (0,1)$ and $t > 0$ there exist $r_1 \in (0,1)$ and $s > 0$ such that for all $x \in X$

$M(x, x_0, s) > 1 - r_1$ implies $M(f(x), f(x_0), t) > 1 - r$

The function f is called a continuous function if it is fuzzy continuous at every point.

Definition (2.6), [8]: A function $\boldsymbol{f}$ from fuzzy metric space $(X, M_1, *)$ to $(Y, M_2, *)$ is said to be uniformly continuous if for each $0 < r < 1$ and $t > 0$ there exists $0 < r_1 < 1$ and $s > 0$ such that

$M_2(f(x), f(y), t) > 1 - r$ whenever $M_1(x, y, s) > 1 - r_1$.

3. Main Result:

Definition(3.1): Two sequence $\{x_n\}$ and $\{y_n\}$ in fuzzy metric space $(X, M, *)$ is said to be asymptotic sequences denoted by $(\{x_n\} \asymp \{y_n\})$, if they satisfy the following condition : for all $0 < r < 1,\ t > 0$ there exist $k \in Z^+$ such that

$M(x_n, y_n, t) > 1 - r$ for all $n > k$.

Theorem(3.2): $\{x_n\}, \{y_n\}$ is asymptotic sequences in fuzzy metric space iff $lim_{n\to\infty} M(x_n, y_n, t) = 1$

Proof: Suppose $\{x_n\} \asymp \{y_n\}$

For a given $0 < r < 1, t > 0$ there exists $k \in Z^+$ such that $M(x_n, y_n, t) > 1 - r$

Thus $1 - M(x_n, y_n, t) < r$ for all $n > k$

Therefore $lim_{n\to\infty} M(x_n, y_n, t) = 1$

Conversely : If $lim_{n\to\infty} M(x_n, y_n, t) = 1$

then for every $0 < r < 1, t > 0$ there exists $k \in Z^+$ such that

$1 - M(x_n, y_n, t) < r$ for all $n > k$.

Thus $M(x_n, y_n, t) > 1 - r$ for all $n > k$

Hence $\{x_n\} \asymp \{y_n\}$.

Example (3.3): Let $X = \{x_n : x_n$ is converge in $[0,1]\}$, let $a * b = ab$ for all $a, b \in [0,1]$ and $M(x_n, y_n, t) = \frac{t}{t + |x_n - y_n|}$

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For all $0 < r < 1$, $t > 0$

Let $x_n = \{1 + \frac{1}{n}\}_{n \in Z^+}$ and $y_n = \{1 + \frac{2}{n}\}_{n \in Z^+}$

$lim_{n \to \infty} M(x_n, y_n, t) = \frac{t}{t + \left|1 + \frac{1}{n} - 1 - \frac{2}{n}\right|} = 1$

Then $\{x_n\} \asymp \{y_n\}$.

Theorem (3.4) : Let $f: (X, M_1, *) \to (Y, M_2, *)$ be a function between two fuzzy metric space , then the following are equivalent :

(1) f is uniformly continuous.

(2) If $\{x_n\}$ and $\{y_n\}$ are asymptotic sequence in X then $\{f(x_n)\}$ and $\{f(y_n)\}$ are asymptotic sequence in Y.

Proof : (1) $\Longrightarrow$ (2)

Let $\{x_n\}$ and $\{y_n\}$ are asymptotic sequence

For all $0 < r < 1$, $t > 0$ there exist $0 < r_1 < 1$, $s > 0$ such that

$M_2(f(x), f(y), t) > 1 - r$ Whenever $M_1(x, y, s) > 1 - r_1$

Since $\{x_n\} \asymp \{y_n\}$, there exist $k \in Z^+$ such that $M_1(x_n, y_n, s) > 1 - r_1$

$\Longrightarrow$ $M_2(f(x_n), f(y_n), t) > 1 - r$

Hence $\{f(x_n)\}$ *and* $\{f(y_n)\}$ are asymptotic sequence and therefore

$\{f(x_n)\} \asymp \{f(y_n)\}$.

(2) $\Longrightarrow$ (1)

Suppose f is not uniformly continuous

There exist $0 < r < 1$, $t > 0$ for all $0 < r_1 < 1$, $s > 0$ such that

$M_1(x, y, s) > 1 - r_1$ but $M_2(f(x), f(y), t) \leq 1 - r$

Since $\{x_n\} \asymp \{y_n\}$ there exist $k \in Z^+$ such that $M_1(x_n, y_n, s) > 1 - r_1$ for all $n > k$

But $\{f(x_n)\}$ and $\{f(y_n)\}$ are not asymptotic sequence (which is contradiction). Hence f is uniformly continuous.

Theorem (3.5) : Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequence in fuzzy metric space with the same limit iff $\{x_n\} \asymp \{y_n\}$.

Proof: $(\Longrightarrow)$ Let $\{x_n\}$ and $\{y_n\}$ be a convergent sequence in $(X, M, *)$

For all $0 < r < 1$ and > 0 , there exist $0 < r_1 < 1$ such that $(1 - r_1) * (1 - r_1) > 1 - r$

Since $x_n \longrightarrow x$, there exist $k_1 \in Z^+$ such that

$M\left(x_n, x, \frac{t}{2}\right) > 1 - r_1$ for all $n > k_1$

Since $y_n \longrightarrow x$,there exist $k_2 \in Z^+$ such that $M\left(y_n, x, \frac{t}{2}\right) > 1 - r_1$ for all $n > k_2$

Taking $k = max\{k_1, k_2\}$

Now : $M(x_n, y_n, t) \geq M\left(x_n, x, \frac{t}{2}\right) * M(y_n, x, \frac{t}{2}) \geq (1 - r_1) * (1 - r_1) > 1 - r$ for all $n > k$

Hence $\{x_n\} \asymp \{y_n\}$.

Conversely: suppose $x_n \longrightarrow x$.

Let $0 < r < 1$, $t > 0$, there exist $0 < r_1 < 1$

Such that $(1 - r_1) * (1 - r_1) > 1 - r$

Since $y_n \rightarrow y$, then for all $0 < r_1 < 1, t > 0$ such that $M\left(y_n, y, \frac{t}{2}\right) > 1 - r_1$

Now: $M(x_n, y, t) \geq M\left(x_n, y_n, \frac{t}{2}\right) * M(y_n, y, \frac{t}{2}) \geq (1 - r_1) * (1 - r_1) > 1 - r$

$\Rightarrow x_n \longrightarrow y$ Which is contradiction because the converge is unique

Hence $x = y$.

Theorem (3.6): $(\asymp)$ is an equivalence relation on X .

Proof: Let $\{x_n\}, \{y_n\}, \{z_n\} \in X$

For all $0 < r < 1$ there exist $k \in \mathbb{N}$ and for all $n > k$

(i) Since $M(x_n, x_n, t) = 1 > 1 - r$

Therefore $\{x_n\} \asymp \{x_n\}$

(ii) If $\{x_n\} \asymp \{y_n\}$ we have to prove $\{y_n\} \asymp \{x_n\}$

Since $\{x_n\} \asymp \{y_n\} \Rightarrow M(x_n, y_n, t) > 1 - r$

But $M(x_n, y_n, t) = M(y_n, x_n, t)$

$\Rightarrow M(y_n, x_n, t) > 1 - r$

Hence $\{y_n\} \asymp \{x_n\}$.

(iii) Suppose $\{x_n\} \asymp \{y_n\}$ and $\{y_n\} \asymp \{z_n\}$

We have to prove $\{x_n\} \asymp \{z_n\}$

Let $0 < r < 1, t > 0$ there exist $0 < r_1 < 1$ such that

$(1 - r_1) * (1 - r_1) > 1 - r$

Since $\{x_n\} \asymp \{y_n\} \Rightarrow M(x_n, y_n, t) > 1 - r_1$

But $\{y_n\} \asymp \{z_n\} \Rightarrow M(y_n, z_n, t) > 1 - r_1$

Now $M(x_n, z_n, t) \geq M(x_n, y_n, t) * M(y_n, z_n, t) \geq (1 - r_1) * (1 - r_1) > 1 - r$

Hence $\{x_n\} \asymp \{z_n\}$

Theorem (3.7) : If $\{x_n\},\{y_n\}$ be asymptotic sequence in fuzzy metric space and let $\{x_{n_k}\},\{y_{n_k}\}$ are two subsequence of $\{x_n\},\{y_n\}$, then $\{x_{n_k}\} \asymp \{y_{n_k}\}$

Proof: Let $\{x_{n_k}\}$,$\{y_{n_k}\}$ are two subsequences of $\{x_n\},\{y_n\}$

We have to prove $\{x_{n_k}\} \asymp \{y_{n_k}\}$

Let $0 < r < 1$, $t > 0$ there exist $0 < r_1 < 1$ such that

$(1 - r_1) * (1 - r_1) * (1 - r_1) > 1 - r$

Since $\{x_n\} \asymp \{y_n\} \Longrightarrow M(x_n, y_n, t) > 1 - r_1$ for all $n > k$

Now: $M(x_{n_k}, y_{n_k}, t) \geq M\left(x_{n_k}, x_n, \frac{t}{3}\right) * M\left(x_n, y_n, \frac{t}{3}\right) * M\left(y_n, y_{n_k}, \frac{t}{3}\right)$

$\geq (1 - r_1) * (1 - r_1) * (1 - r_1) > 1 - r$ for all $n_k > k$

Thus $\{x_{n_k}\} \asymp \{y_{n_k}\}$

Theorem (3.8) : In fuzzy metric space if $\{x_n\} \asymp \{y_n\}$ and $\{w_n\} \asymp \{z_n\}$, then $\{x_n w_n\} \asymp \{y_n z_n\}$.

Proof : For all $0 < r < 1$ and $t > 0$, there exist $0 < r_1 < 1$ such that $(1 - r_1) * (1 - r_1) > 1 - r$

Since $\{x_n\} \asymp \{y_n\}$, then there exist $k_1 \in Z^+$

implies $M\left(x_n, y_n, \frac{t}{2}\right) > 1 - r_1$ for all $n > k_1$

Since $\{w_n\} \asymp \{z_n\} \Longrightarrow$ there exist $k_2 \in Z^+$

such that $M\left(w_n, z_n, \frac{t}{2}\right) > 1 - r_1$ for all $n > k_2$

Taking $k = \max\{k_1, k_2\}$

Now: $M(x_n w_n, y_n z_n, t) \geq M\left(x_n, y_n, \frac{t}{2}\right) * M\left(w_n, z_n, \frac{t}{2}\right)$

$\geq (1 - r_1) * (1 - r_1) > 1 - r$ for all $n > k$.

الملخص

حول فكرة وخصائص المتتابعات المتناقلة، في هذا البحث قدمنا مفهوم المتتابعات المتناقلة في الفضاء المتري الضبابي اضافة الى ذلك قدمنا مفهوم التقارب للمتتابعات المتناقلة في هذا الفضاء و العلاقة بين المتتابعات المتناقلة والاستمرارية المنتظمة و بالأخير تم طرح خصائص اخرى للمتتابعات المتناقلة

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