Journal of Iraqi Al-Khwarizmi (JIKh) Volume:10 Issue:1 Year: 2026 pages: 80-90

A Norm-Based Exterior Penalty Function Technique for Constrained Optimization

Authors Names	ABSTRACT
Khadija Bashir Mudhi[a] *Saad Shakir Mahmood*[b] Publication data: 6 / 4 /2026 ***Keywords:*** Constrained optimization, penalty function method, exterior penalty, norm-based penalty, quasi-Newton methods.	In this paper, we propose and investigate a norm-based exterior penalty function technique for solving constrained optimization problems. The classical quadratic penalty term is replaced by a norm formulation of the constraint violations, resulting in a flexible penalty structure in which the penalty parameter is independent of the number and form of the constraints. The resulting unconstrained optimization problems are solved using a quasi-Newton method. A collection of benchmark test problems is examined, and the numerical performance of the proposed technique is reported and discussed. The results demonstrate that the norm-based penalty approach provides an effective and practical alternative to traditional exterior penalty methods, particularly when a limited number of quasi-Newton updates is employed.

1.Introduction

Constrained optimization plays a central role in applied mathematics, engineering, economics, and the physical sciences. Many real-world problems involve minimizing or maximizing an objective function subject to equality and/or inequality constraints. [8]. Penalty function methods constitute one of the most widely used strategies for handling constraints by transforming a constrained problem into a sequence of unconstrained problems. [6]. Classical exterior penalty function methods typically employ quadratic penalty terms that heavily penalize constraint violations. [6]. Although effective, quadratic penalties often suffer from sensitivity to the penalty parameter and ill-conditioning as the parameter grows large. These issues motivate the development of alternative penalty formulations that preserve simplicity while improving numerical robustness. [5]. In this work, we proposed new penalty function method is norm-based penalty function method. The main idea is to replace the quadratic penalty term with a norm of the constraint violations, there by obtaining a simpler and more flexible penalty structure. The resulting formulation allows the penalty parameter to be selected experimentally, without requiring an explicit dependence on the constraint functions. The penalty function method converts the constrained optimization the problem into an unconstrained one by incorporating the constraint functions into the objective function through a penalty term. This penalty term depends on a parameter that controls the level of constraint violation. By adjusting this parameter, a sequence of approximate unconstrained problems can be generated, and their solutions will gradually approach the feasible region of the original constrained problem. [6]. Stefan Meili (2021) proposed the softplus and algebraic penalty functions to overcome the sensitivity of the Courant-Beltrami method to strong objective function gradients. [1].

Dharminder Singh et al. (2024) introduce the concept of constructing a technique to smooth such non differentiable functions. He being with the smoothing of the penalty function. On the basis of it, and he come up with an algorithm to find the best way to solve an optimization problem with inequality

---

[a]Al-Abour Preparatory School for Girls, Anbar, Iraq, E-Mail: Khadijeh1998@uomustansiriyah.edu.iq
[b]Al-Mustansiriya University, College of Education, Department of Mathematics, Baghdad, Iraq, E-Mail: saadshakirmahmood@uomustansiriyah.edu.iq

constraints. He also talk about how to figure out the error for a certain smooth penalty function. He also provide numerical examples to demonstrate that the suggested approach is practical and usable. [2].Meng et al. (2014) present an algorithm to solve the inequality constrained multi-objective programming (MP) by using a penalty function with objective parameters and constraint penalty parameter. First, the penalty function with objective parameters and constraint penalty parameter for MP and the corresponding unconstraint penalty optimization problem (UPOP) is defined. Under some conditions, a pareto efficient solution (or a Weakly-efficient solution) to UPOP is proved to be a pareto efficient solution (or a weakly-efficient solution) to MP. The penalty function is proved to be exact under a stable condition. Then, he design an algorithm to solve MP and prove its convergence. Finally, numerical examples show that the algorithm may help decision makers to find a satisfactory solution to MP. [3]. Logarithmic penalty approaches have also been developed for invex multi-objective fractional programming problems, demonstrating improved stability and exactness properties [4].

We consider the general constrained optimization problem

$\min_{x \in R^n} f(x)$

s.t $h_i(x) \leq 0$, $i = 1,2, \ldots, m$

$g_i(x) = 0$, $i = 1, \ldots, l$

Where the objective function $f: R^n \to R$ is assumed to be twice continuously differentiable, and the constraint functions g_i and h_i are continuously differentiable. The goal is to determine a point x^* that minimizes $f(x)$ while satisfying all constraints.

Various alternative penalty formulations have been proposed to improve stability and theoretical exactness, including logarithmic penalty methods for multi-objective programming problems [4].

2. norm-based exterior penalty function technique

2.1 Classical Exterior Penalty Method

In the classical exterior penalty framework, the constrained problem is transformed into a sequence of unconstrained problems of the form

$\Phi(x, r) = f(x) + rp(x)$ [6].

2.2 Proposed norm-based penalty function

In the proposed technique, the quadratic penalty term is replaced by a norm-based formulation. Specifically, the penalty function is defined as

$\Phi(x, r) = f(x) + r\|G(X)\|$

A key feature of this formulation is that the penalty parameter r does not explicitly depend on the number or structure of the constraints. Consequently, r can be selected using an experimental or trial-based strategy, simplifying practical implementation. [7].

2.3 Algorithmic framework

The computational procedure can be summarized as follows:

1. Choose an initial point x_0 and an initial penalty parameter $r > c$.
2. Construct the unconstrained objective function $\Phi(x, r)$.
3. Minimize $\Phi(x, r)$ using a Quasi-Newton method.
4. If the obtained solution satisfies the constraints within a prescribed tolerance, stop.
5. Otherwise, update r and repeat the process. [5], [9].

In this study, a single quasi-Newton is applied for each value of r, emphasizing computational efficiency. [10].

3. Numerical experiments

To evaluate the performance of the proposed technique, a set of benchmark constrained optimization problems of varying dimensions and constraint structures is considered. For each problem, the norm-based penalty function is constructed and minimized using a quasi-Newton method.

In this technique we change the quadratic form of the exterior point method with a norm function and hence the penalty function take another form as follows:

Consider the constrained optimization problem

$\min f(x)\,, \quad x \in R^n$

s.t $\qquad h_i(x) \leq 0\,, i = 1, \dots, m$

$g_j(x) = 0\,, i = 1, \dots, l$.

The penalty function method is as follows:

$$\min. p = f(x) + r\sum \|\max(0, h_i(x)\| + r\sum \left\|g_j(x)\right\|$$

$$\frac{\partial p}{\partial x_i} = \frac{\partial f}{\partial x_i} + r\sum\frac{\partial h_i(x)}{\partial x_i} + r\sum\frac{\partial g_j(x)}{\partial x_i} = 0$$

$$r = -\frac{\partial f}{\partial x_i}\Big/\sum\frac{\partial h_i(x)}{\partial x_i} + \sum\frac{\partial g_j(x)}{\partial x_i}$$

Clear that r is not depending on any constraints that means the value of r can be determine by method of experimental method.

Example:

$\min f(x) = (x_1 - 2)^2 + (x_2 - 1)^2$

s.t $\quad 0.25x_1^2 + x_2^2 - 1 \leq 0$

$x_1 - 2x_2 + 1 = 0$

to solve this example, we can construct the penalty function as follows:

$\min p = (x_1 - 2)^2 + (x_2 - 1)^2 + r\|\max(0,0.25x_1^2 + x_2^2 - 1)\| + r\|x_1 - 2x_2 + 1\|$

We can take the value of r as a try value and then solve it by BFGS method.

Let beginning with $r = 1$ the problem is as follows:

$\min p = (x_1 - 2)^2 + (x_2 - 1)^2 + \|\max(0,0.25x_1^2 + x_2^2 - 1)\| + \|x_1 - 2x_2 + 1\|$

Starting point $(2;2)$, $x^0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $B^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\epsilon = 0.05$

$\nabla p(x^0) = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$,

$d_o = -B^{0^{-1}} \nabla p(x^0) = \begin{bmatrix} 0 \\ -8 \end{bmatrix}$

$x^1 = x^0 + \alpha_0 d_o = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \alpha_0 \begin{bmatrix} 0 \\ -8 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 - 8\alpha_0 \end{bmatrix}$

By line search technique we have $\alpha_0 = 1/8$ then $x^1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$,

$\nabla p(x^1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$,

$S_0 = x^1 - x^0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$y_0 = \nabla p(x^1) - \nabla p(x^0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

$$B^1 = B^0 + \frac{y_0 {y_0}^T}{{y_0}^T S_0} - \frac{B_0 S_0 {S_0}^T B_0}{{S_0}^T B_0 S_0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\begin{bmatrix} 2 \\ -6 \end{bmatrix} \begin{bmatrix} 2 & -6 \end{bmatrix}}{\begin{bmatrix} 2 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}} - \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$
$$= \begin{bmatrix} 0.3333 & 2.0000 \\ 2.0000 & -6.0000 \end{bmatrix}$$

$B^1 = \begin{bmatrix} 0.3333 & 2 \\ 2 & -6 \end{bmatrix}$, since $\|\nabla p(x^1)\| = 2.8284 > \epsilon$

The solution must be continue .by letting $x^0 = x^1$ and $B^0 = B^1$.

By using MATLAB programing we have the final results of the above problem is as follows :

$x^* = \begin{bmatrix} 1.2 \\ 1 \end{bmatrix}$ and $f^* = 0.6400$ and the iteration $itr = 7$

Note that the optimal solution is $x^* = \begin{bmatrix} 0.8229 \\ 0.9114 \end{bmatrix}$ and $f^* = 1.3935$

So the value of $r = 1$ can't give the optimal solution ,and hence we can take the next value of r that is $r = 10$ and repeat the above method for $r = 10$ until we find the final solution. Since theirs no relation between the value of r and the constrainers so we can find the value of r by experimental method and the equivalent problem can be construct as follows:

Problem 1

$min. f(x) = (x1 - 2)^2 + (x2 - 1)^2$

Constraints:

$-0.25x_1^2 - x_2^2 + 1 \geq 0$

$x_1 - 2x_2 + 1 = 0$

Equivalent to the unconstrained optimization problem:

$$min. P(x) = (x1 - 2)^2 + (x2 - 1)^2 + 16\|max(0, 0.25x_1^2 + x_2^2 - 1)\| + 16\|x_1 - 2x_2 + 1\|$$

Problem 2

$min. f(x) = (x1 - 2)^2 + (x2 - 1)^2$

Constraints:

$-x_1 - x_2 + 2 \geq 0$

$-x_1^2 + x_2 \geq 0$

Equivalent to the unconstrained optimization problem:

$min. P(x) = (x1 - 2)^2 + (x2 - 1)^2 + \|\max(0, x_1 + x_2 - 2)\| + \|\max(0, x_1^2 - x_2)\|$

Problem 3

$min. f(x) =$
$-75.196 + 3.8112x_1 + 0.0020567x_1^3 - 0.000010345x_1^4 + 6.8306x_2 - 0.030234x_1x_2 + 0.00128134x_2x_1^2 + 0.0000002266x_1^4x_2 - 0.25645x_2^2 + 0.0034604x_2^3 - 0.000013514x_2^4 + \frac{28.106}{(x_2+1)} + 0.0000052375x_1^2x_2^2 + 0.000000063x_1^3x_2^2 - 0.0000000007x_1^3x_2^3 - 0.0003405x_1x_2^2 + 0.0000016638x_1x_2^3 + 2.8673\exp(0.0005x_1x_2) - 0.000035256x_1^3x_2$

Constraints:

$x_1x_2 - 700 \geq 0,$

$x_2 - x_1^2/125 \geq 0,$

$(x_2 - 50)^2 - 5(x_1 - 55) \geq 0,$

$0 \leq x_1 \leq 75,$

$0 \leq x_2 \leq 65\,.$

Equivalent to the unconstrained optimization problem:

$$min.\,P(x) = -75.196 + 3.8112x_1 + 0.0020567x_1^3 - 0.000010345x_1^4 + 6.8306x_2 - 0.030234x_1x_2 + 0.00128134x_2x_1^2 + 0.0000002266x_1^4x_2 - 0.25645x_2^2 + 0.0034604x_2^3 - 0.000013514x_2^4 + \frac{28.106}{(x_2+1)} + 0.0000052375x_1^2x_2^2 + 0.000000063x_1^3x_2^2 - 0.0000000007x_1^3x_2^3 - 0.0003405x_1x_2^2 + 0.0000016638x_1x_2^3 + 2.8673\exp(0.0005x_1x_2) - 0.000035256x_1^3x_2 + 50\|\max(0, -x_1x_2 + 700)\| + 50\left\|\max\left(0, -x_2 + \frac{x_1^2}{125}\right)\right\| + 50\|\max(0, -(x_2 - 50)^2 + 5(x_1 - 55))\| + 50\|\max(0, -x_1)\| + 50\|\max(0, x_1 - 75)\| + 50\|\max(0, -x_2)\| + 50\|\max(0, x_2 - 65)\|$$

Problem 4

$min.\,f(x) = 1000 - x_1^2 - 2x_2^2 - x_3^2 - x_1x_2 - x_1x_3$

Constraints:

$8x_1 + 14x_2 + 7x_3 - 56 = 0,$

$x_1^2 + x_2^2 + x_3^2 - 25 = 0\,,$

$0 \leq x_i,\; i = 1,2,3\,.$

Equivalent to the unconstrained optimization problem:

$$min.\,p(x) = 1000 - x_1^2 - 2x_2^2 - x_3^2 - x_1x_2 - x_1x_3 + 500\|(0{,}8x_1 + 14x_2 + 7x_3 - 56)\| + 500\|(0, x_1^2 + x_2^2 + x_3^2 - 25)\| + 500\|\max(0, -x_1)\| + 500\|\max(0, -x_2)\| + 500\|\max(0, -x_3)\|$$

Problem 5

$min.\,f(x) = \sum_{i=1}^{99}(f_i(x))^2$

Where

$f_i(x) = -0.01i + \exp(-1/x_1(u_i - x_2)^{x_3})$

$u_i = 25 + (-50\ln(0.01i))^{2/3}, i = 1, \dots, 99$.

Constraints:

$0.1 \le x_1 \le 100$

$0 \le x_2 \le 25.6$

$0 \le x_3 \le 5$

Equivalent to the unconstrained optimization problem:

$$min.\, p(x) = \sum_{i=1}^{99}(f_i(x))^2 + 100\|\max(0, -x_1 + 0.1)\| + 100\|\max(0, x_1 - 100)\| + 100\|\max(0, -x_2)\| + 100\|\max(0, x_2 + 25.6)\| + 100\|\max(0, -x_3)\| + 100\|\max(0, x_3 - 5)\|$$

Problem 6

$$min.\, f(x) = 9 - 8x_1 - 6x_2 - 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3$$

Constraints:

$3 - x_1 - x_2 - 2x_3 \ge 0$

$0 \le x_i, i = 1,2,3.$

Equivalent to the unconstrained optimization problem:

$$min.\, p(x) = 9 - 8x_1 - 6x_2 - 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + \|\max(0, -3 + x_1 + x_2 + 2x_3)\| + \|\max(0, -x_1)\| + \|\max(0, -x_2)\| + \|\max(0, -x_3)\|$$

Problem 7

$$min.\, f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 + 90(x_4 - x_3^2)^2 + (1 - x_3)^2 + 10.1((x_2 - 1)^2 + (x_4 - 1)^2) + 19.8(x_2 - 1)(x_4 - 1)$$

Constraints:

$-10 \le x_i \le 10, i = 1, \dots, 4$

Equivalent to the unconstrained optimization problem:

$$min.\, p(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 + 90(x_4 - x_3^2)^2 + (1 - x_3)^2 + 10.1((x_2 - 1)^2 + (x_4 - 1)^2) + 19.8(x_2 - 1)(x_4 - 1) + 0.01\|\max(0, -x_1 - 10)\| + 0.01\|\max(0, -x_2 - 10)\| + 0.01\|\max(0, -x_3 - 10)\| + 0.01\|\max(0, -x_4 - 10)\| + 0.01\|\max(0, x_1 - 10)\| + 0.01\|\max(0, x_2 - 10)\| + 0.01\|\max(0, x_3 - 10)\| + 0.01\|\max(0, x_4 - 10)\|$$

Problem 8

$min. f(x) = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4$

Constraints:

$8 - x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_1 + x_2 - x_3 + x_4 \geq 0$

$10 - x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 + x_1 + x_4 \geq 0$

$5 - 2x_1^2 - x_2^2 - x_3^2 - 2x_1 + x_2 + x_4 \geq 0$

Equivalent to the unconstrained optimization problem:

$min. p(x) = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4 + 1.83485\|\max(0, -8 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3 - x_4)\| + 1.83485\|\max(0, -10 + x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4)\| + 1.83485\|\max(0, -5 + 2x_1^2 + x_2^2 + x_3^2 + 2x_1 - x_2 - x_4)\|$

Problem 9

$min. f(x) = 24.55x_1 + 26.75x_2 + 39x_3 + 40.50x_4$

Constraints:

$2.3x_1 + 5.6x_2 + 11.1x_3 + 1.3x_4 - 5 \geq 0$

$12x_1 + 11.9x_2 + 41.8x_3 + 52.1x_4 - 21 - 1.645(0.28x_1^2 + 0.19x_2^2 + 20.5x_3^2 + 0.62x_4^2)^{1/2} \geq 0$

$x_1 + x_2 + x_3 + x_4 - 1 = 0$

$0 \leq x_i, i = 1, \ldots, 4$.

Equivalent to the unconstrained optimization problem:

$min. p(x) = 24.55x_1 + 26.75x_2 + 39x_3 + 40.50x_4 + 3.53\|\max(0, -2.3x_1 - 5.6x_2 - 11.1x_3 - 1.3x_4 + 5)\| + 3.53\left\|\max\left(0, -12x_1 - 11.9x_2 - 41.8x_3 - 52.1x_4 + 21 + 1.645(0.28x_1^2 + 0.19x_2^2 + 20.5x_3^2 + 0.62x_4^2)^{1/2}\right)\right\| + 3.53\|(0, x_1 + x_2 + x_3 + x_4 - 1)\| + 3.53\|\max(0, -x_1)\| + 3.53\|\max(0, -x_2)\| + 3.53\|\max(0, -x_3)\| + 3.53\|\max(0, -x_4)\|$

Problem 10

$min. f(x) = 5.3578547x_3^2 + 0.8356891x_1x_5 + 37.293239x_1 - 40792.141$

Constraints:

$92 \geq a_1 + a_2x_2x_5 + a_3x_1x_4 - a_4x_3x_5 \geq 0$

$20 \geq a_5 + a_6 x_2 x_5 + a_7 x_1 x_2 + a_8 x_3^2 - 90 \geq 0$

$5 \geq a_9 + a_{10} x_3 x_5 + a_{11} x_1 x_3 + a_{12} x_3 x_4 - 20 \geq 0$

$78 \leq x_1 \leq 102$

$33 \leq x_2 \leq 45$

$27 \leq x_i \leq 45 \, , i = 3,4,5 \, .$

Where

$$a_1 = 85.334407 \, , a_2 = 0.0056858 \, , a_3 = 0.0006262 \, ,$$

$$a_4 = 0.0022053 \, , a_5 = 80.51249 \, , a_6 = 0.0071317 \, ,$$

$$a_7 = 0.0029955 \, , a_8 = 0.0021813 \, , a_9 = 9.300961,$$

$$a_{10} = 0.0047026 \, , a_{11} = 0.0012547, a_{12} = 0.0019085$$

Equivalent to the unconstrained optimization problem:

$min. p(x) =$
$5.3578547 x_3^2 + 0.8356891 x_1 x_5 + 37.293239 x_1 - 40792.141 + 157.35\|\max(0, -a_1 - a_2 x_2 x_5 - a_3 x_1 x_4 + a_4 x_3 x_5)\| + 157.35\|max(0, a_1 + a_2 x_2 x_5 + a_3 x_1 x_4 - a_4 x_3 x_5 - 92)\| + 157.35\|\max(0, -a_5 - a_6 x_2 x_5 - a_7 x_1 x_2 - a_8 x_3^2 + 90)\| + 157.35\|\max(0, a_5 + a_6 x_2 x_5 + a_7 x_1 x_2 + a_8 x_3^2 - 90 - 20)\| + 157.35\|\max(0, -a_9 - a_{10} x_3 x_5 - a_{11} x_1 x_3 - a_{12} x_3 x_4 + 20)\| + 157.35\|\max(0, a_9 + a_{10} x_3 x_5 + a_{11} x_1 x_3 + a_{12} x_3 x_4 - 20 - 5)\| + 157.35\|\max(0, -x_1 + 78)\| + 157.35\|\max(0, x_1 - 102)\| + 157.35\|\max(0, -x_2 + 33)\| + 157.35\|\max(0, x_2 - 45)\| + 157.35\|\max(0, -x_3 + 27)\| + 157.35\|\max(0, x_3 - 45)\| + 157.35\|\max(0, -x_4 + 27)\| + 157.35\|\max(0, x_4 - 45)\| + 157.35\|\max(0, -x_5 + 27)\| + 157.35\|\max(0, x_5 - 45)\|$

Problem 11

$min. f(x) = 0.0204 x_1 x_4 (x_1 + x_2 + x_3) + 0.0187 x_2 x_3 (x_1 + 1.57 x_2 + x_4) + 0.0607 x_1 x_4 x_5^2 (x_1 + x_2 + x_3) + 0.0437 x_2 x_3 x_6^2 (x_1 + 1.57 x_2 + x_4)$

Constraints:

$0.001 x_1 x_2 x_3 x_4 x_5 x_6 - 2.07 \geq 0 \, ,$

$1 - 0.00062 x_1 x_4 x_5^2 (x_1 + x_2 + x_3) - 0.00058 x_2 x_3 x_6^2 (x_1 + 1.57 x_2 + x_4) \geq 0,$

$0 \leq x_i \, , i = 1, \ldots, 6 \, .$

Equivalent to the unconstrained optimization problem:

$min.\, p(x) = 0.0204x_1x_4(x_1 + x_2 + x_3) + 0.0187x_2x_3(x_1 + 1.57x_2 + x_4) + 0.0607x_1x_4x_5^2(x_1 + x_2 + x_3) + 0.0437x_2x_3x_6^2(x_1 + 1.57x_2 + x_4) + 1000\|\max(0, -0.001x_1x_2x_3x_4x_5x_6 + 2.07)\| + 1000\|\max(0, -1 + 0.00062x_1x_4x_5^2(x_1 + x_2 + x_3) + 0.00058x_2x_3x_6^2(x_1 + 1.57x_2 + x_4))\| + 1000\|\max(0, -x_1)\| + 1000\|\max(0, -x_2)\| + 1000\|\max(0, -x_3)\| + 1000\|\max(0, -x_4)\| + 1000\|\max(0, -x_5)\| + 1000\|\max(0, -x_6)\|$

The numerical results are summarized in Table 1, which reports the number of variables, initial points, penalty parameters, obtained solutions, and iteration counts.

Table 1. numerical results for the norm-based exterior penalty function technique.

(Results correspond to the numerical data reported in the original thesis, Table 1)

Table 1

N.o	R	x^*	$f(x^*)$	x_0	Iter
1	16	(0.8229;0.9114)	1.3935	(2;2)	10
2	1	(1;1)	1	(2;2)	8
3	50	(13.4859;51.906)	-867.5944	(90;10)	2
4	500	(3.5121;0.2169;3.5524)	961.7136	(2;2;2)	36
5	100	(49.9997;25;1.5)	4.1437×10^{-12}	(100;12.5;-3)	44
6	1	(1.3308;0.7787;0.4455)	0.1111	(0.5;0.5;0.5)	10
7	0.01	(1.0003;1.0006;1.0001;1.0001)	5.8514×10^{-6}	(-3;-1;-3;-1)	35
8	1.83485	(0.0026;1.0026;2.0138;-0.9750)	-44.0742	(0;0;0;0)	11
9	3.53	(0.562;-0.000;0.4379;-0.0037)	89.6011	(1;1;1;1)	9
10	157.35	(78.0002;33;3.5343;27.0423;27)	-30665	(78;33;27;27;27)	12
11	10000	(1.5270;1.3190;2.7191;3.9655;4.9974;5.5133)	88.3102	(5.54;4.4;12.02;11.82;0.702;0.852)	3

The results demonstrate that the proposed method successfully transforms constrained problems into unconstrained ones and produces solutions that are close to the true constrained optima. In several cases, the method achieves convergence with a relatively small number of iterations, highlighting its computational efficiency. [5].

However, the experiments also reveal that the quality of the solution depends on the choice of the penalty parameter. In some problems, small values of r are insufficient to enforce feasibility, requiring further adjustment of the parameter, the choice of the penalty parameter the depends on the try method because there is no relation between the penalty parameter and the constraint. [6].

4. Conclusions

In this paper, a norm-based exterior penalty function technique for constrained optimization has been presented and analyzed. By replacing the quadratic penalty term with a norm of the constraint violations, the proposed approach provides a flexible and practical alternative to traditional exterior penalty methods.

Numerical experiments confirm that the method is capable of producing accurate solutions with reduced computational effort, particularly when combined with a limited number of quasi-Newton updates. The technique can be viewed as a compact and efficient substitute for multiple penalty tables or ready-made optimization software.

Future research may focus on extending this approach by incorporating alternative quasi-newton updates or adaptive strategies for penalty parameter selection.

References

[1] Meili S., (2021), "Softplus Penalty Functions for Constrained Optimization", arXiv preprint arXiv:2107 .04451v1, math. OC.
[2] Singh D., Singh A., Singh A. P., Garg r., Prashar D., Khan A. A. and Yimam F. A., (2024), "A Smooth penalty Function Algorithm Constraint Optimization Problem", Mathematical Problems in Engineering, Vol. 2024, Article ID 1224408.
[3] Meng Z., Shen R. and Jiang M., (2014), "A Penalty Function Algorithm with Objective Parameters and Constraint Penalty Parameter for Multi-Objective Programming", American Journal of Operations Research, Vol. 4, No.6, pp. 458-466.
[4] Hassan M., Baharum A. and Ali M. K. M.,(2020), "Logarithmic Penalty Function Method for Invex Multi-Objective Fractional Programming Problems", Journal of Taibah University for Science, Vol. 14, No. 1,pp. 211-216.
[5] Dennis J. E. and Schnabel R. B., (1996), "Numerical Methods for Unconstrained Optimization and Nonlinear Equations",SIAM.
[6] Fiacco A. V. and McCormick G. P., (1968), "Nonlinear Programming: Sequential Unconstrained Minimization Techniques", Wiley, New York.
[7] Han S. P. and Mangasarian O.L., (1979), "Exact Penalty functions in nonlinear programming", Mathematical Programming, Vol. 17, pp. 251-269.
[8] Bertsekas D. P., (1999), "Nonlinear Programming", Athena Scientific.
[9] Nocedal J. and Wright S. J., (2006), "Numerical Optimization", Springer, Second Edition.
[10] Ortega J. M. and Rheinboldt W.C., (1970), "Iterative Solution of Nonlinear Equations in Several Variables", Academic Press.