

Hahn-Banach Theorem in Fuzzy Soft Dual Space

| Authors Names | ABSTRACT |
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| <p><i>Dunia.A.Al-hikany</i>^a <i>Dhuha.T.Al-dawoodi</i>^b <i>Haneen .Z.Hassan</i>^{a,b,*}</p> <p>Publication date: 2/4 /2026 Keywords: fuzzy soft set, fuzzy soft Banach algebra, fuzzy soft linear functional, fuzzy soft dual space.</p> | <p>Molodtsov in [8] discussed the theory of soft fuzzy sets, and a study of soft fuzzy real numbers and their properties was presented in [6]. In [11], the notion of fuzzy soft Banach algebra was presented. This paper first reviews the definitions of the fuzzy soft linear function and the fuzzy soft dual space. Secondly, the concept of the Hahn-Banach theorem in the fuzzy soft dual space and some of its applications are presented.</p> |

1. Introduction

The Hahn–Banach theorem is one of the fundamental pillars in functional analysis, as it represents a pivotal tool in the study of normed spaces and dual spaces, particularly in the problems of extending linear functions and constructing dual spaces. This theorem has made a substantial contribution to the development of many branches of applied and theoretical mathematics, prompting researchers to attempt to generalize it in more general mathematical frameworks that accommodate ambiguity and uncertainty. In this context, Molodtsov [8] discussed the notion of soft sets as a useful way to deal with kinds of uncertainty that are not amenable to conventional mathematical methods, especially in cases where a precise mathematical model is unavailable. The interaction between fuzzy set theory and soft set theory has led to the emergence of the notion of soft fuzzy sets, which provides a more flexible mathematical structure for representing uncertain information, see[1-7]. Over the past few years, significant developments have occurred in the study of fuzzy soft algebraic and analytical structures. S. Das et al presented the soft fuzzy element notion in [10], soft fuzzy points, fuzzy soft vector spaces, as well as dual space and soft fuzzy normed spaces have been introduced. These works have provided the necessary theoretical foundation for progressing to the study of classical functional analysis theorems in this new context. Although the notion of fuzzy soft double spaces has been presented in some previous studies, the generalization of the Hahn–Banach theorem to this type of space, and the study of their properties within the fuzzy soft framework, still represents an active research area that requires further depth and precision. This research aims to define and study the Hahn–Banach theorem in fuzzy soft double spaces, with a focus on its appropriate formulation in this context, and to prove some of the fundamental properties that distinguish it from its classical counterpart. The study aims to clarify the conditions under which the theorem holds in fuzzy soft double spaces and to highlight the similarities and differences between it and the Hahn–Banach theorem in conventional normed spaces. The results obtained in this research contribute to enriching the theory of fuzzy soft set analysis and provide a theoretical foundation for subsequent studies that address broader applications of the extension and separation theorems in mathematical environments characterized by ambiguity and uncertainty.

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2.1 Preliminaries

Definition(2.1.1)[14]: A pair (Γ, A) is a fuzzy soft set over X is represented by Γ_A , where Γ denotes a function defined as $\Gamma: A \rightarrow I^X$.

Definition(2.1.2)[11]: A soft set (P, G) over X . if there is only one $a \in G$, such that $P(a) = \{x\}$ for some $x \in X$ and $P(\acute{a}) = \emptyset, \forall \acute{a} \in G \setminus \{a\}$, In that case, the soft set (P, G) is referred to as a soft vector and denoted by \check{x}_a .

The set of all soft vectors over \check{X} is denoted by $SV(\check{X})$.

Definition (2.1.3)[14]: A soft fuzzy point is described as a soft fuzzy set Γ_A over X if for element $a^* \in G$

$$f_a(x) = \left\{ \begin{array}{ll} \lambda_x & \text{if } a = a^* \\ 0 & \text{otherwise} \end{array} \right. \quad \text{for every } a \in G$$

Otherwise, for the element $x^* \in X$,

$$f_a(x) = \left\{ \begin{array}{ll} \lambda & \text{if } x = x^* \\ 0 & \text{otherwise} \end{array} \right. \quad \text{for every } x \in X, \text{ where } \lambda \in (0,1]$$

The notation \widetilde{x}_G represents the set of all fuzzy soft points.

Let X be a vector space over a field K ($K = \mathbb{R}$) and the parameter set G be the real number set \mathbb{R} .

Definition(2.1.4)[5]: Over the scalar field K , assume that \check{x} is an absolute soft linear space. Let $R(A^*)$ be the set of all non-negative soft real numbers, $*$ be a continuous t-norm, and $SSP(\check{X})$ be the set of all soft points on X . A fuzzy soft norm \check{X} is defined as a fuzzy subset Γ on $SSP(\check{X}) \times R(A^*)$, $\check{x}_e, \check{y}_e \in SSP(\check{X})$, and $\check{k} \in \check{K}$ (where \check{k} is a soft scalar).

i) $\Gamma(\check{x}_e, \check{t}) = \check{0}, \forall \check{t} \in R(A^*)$, with $\check{t} \leq \check{0}$.

ii) $\Gamma(\check{x}_e, \check{t}) = 1, \forall \check{t} \in R(A^*)$ with $\check{t} \leq \check{0}$ if and only if $\check{x}_e = \check{t}$

iii) $\Gamma(\check{k}\check{x}_e, \check{t}) = \Gamma(\check{x}_e, \frac{\check{t}}{\check{k}})$ if $\check{k} \neq (0) \sim \forall \check{t} \in R(A^*) \sim, \check{t} > \check{0}$.

iv) $\Gamma(\check{x}_e \oplus \check{y}_e, \check{t} \oplus \check{s}) > \Gamma(\check{x}_e, \check{t}) * \Gamma(\check{y}_e, \check{s}), \forall \check{t}, \check{s} \in R(A^*), \check{x}_e, \check{y}_e \in SSP(\check{X})$.

v) $\Gamma(\check{x}_e, \cdot)$ is a Continuous non decreasing Function of $R(A^*)$, and $\lim_{\check{x} \rightarrow \infty} \Gamma(\check{x}_e, \check{t}) = 1$.

We shall refer to the triplet $(\check{X}, \Gamma, *)$ as a soft fuzzy Banach algebra.

Definition(2.1.5)[11]: Suppose $\widetilde{M} \subseteq SSV(\check{X})$ is a subset of $SSV(\check{X})$ is a fuzzy soft vector space. Then, \widetilde{M} called a fuzzy soft vector subspace of $SSV(\check{X})$. If \widetilde{M} is a fuzzy soft vector space, and denoted by $SSV(\widetilde{M})$.

Definition(2.1.6)[11]: suppose $G = SSV(\check{X}) \rightarrow SSV(\check{Y})$ a Fuzzy Soft mapping. Then G is called a linear fuzzy soft operator if

(1) G additive, $G(\check{x}_e + \check{y}_e) = G(\check{x}_e) + G(\check{y}_e)$ for all $\check{x}_e + \check{y}_e \in SSV(\check{X})$.

(2) G homogeneous, for each fuzzy soft \check{k} , $G(\check{k}\check{x}_e) = \check{k} G(\check{x}_e)$ for each $\check{x}_e \in SSV(\check{X})$.

Definition (2.1.7)[5]: Suppose $(SSV(\check{X}), \|\cdot\|, \Gamma_A)$ is a soft fuzzy Normed space, $\check{x}_e \in SSV(\check{X})$, $G: \check{X} \rightarrow \check{R}$ Linear function, and $\psi: \check{R} \rightarrow \check{R}$ is a function. The mapping

$(G, \psi) : SSV(\tilde{X}) \rightarrow SSV(R)$ is described as follows: $(G, \psi)_{\tilde{x}_e} = (G(\tilde{x}))\psi_{(e)}$

It is described as a fuzzy soft functional.

Proposition(2.1.8)[5]: If $\psi: \tilde{R} \rightarrow \tilde{R}$ is linear function then the Fuzzy soft function $(G, \psi) : SSV(\tilde{X}) \rightarrow SSV(\tilde{R})$ is linear.

Proposition(2.1.9)[5]: A fuzzy soft functional space is soft fuzzy vector space .

Definition(2.1.9)[5]: The fuzzy soft norm of the fuzzy soft linear functional (G, ψ) is defined as

$$\|(G, \psi)\| = \inf\{\tilde{M}: \|(G, \psi)_{\tilde{x}_e}\| \leq \tilde{M}\|\tilde{x}_e\|\}$$

It is called a norm of (G, ψ) such that \tilde{M} is a fuzzy soft real number. It is clear that

$$\|(G, \psi)_{\tilde{x}_e}\| \leq \|(G, \psi)\|\|\tilde{x}_e\|.$$

3.1 Soft Fuzzy dual space

Definition(3.1.1)[5]: The fuzzy Soft dual space, represented by $SSV(\tilde{X})^*$, of the fuzzy Soft Vector space $SSV(\tilde{X})$, consists of its fuzzy soft linear functionals in the fuzzy soft vector space.

Theorem(3.1.2) :Suppose a subspace $SSV(\tilde{M})^*$ of fuzzy soft dual space $SSV(\tilde{X})^*$, let $(g, \psi) \in SSV(\tilde{M})^*$. Then it exists $(f, \varphi) \in SSV(\tilde{X})^*$ Such that $(f, \varphi) = (g, \psi)$ for all $\tilde{x}_e \in SSV(\tilde{M})^*$ Moreover, if $SSV(\tilde{X})^*$ fuzzy Soft dual.space, then $\|(f, \varphi)\| = \|(g, \psi)\|$.

Proof: let G be a collection of all ordered pairs $((f, \varphi), SSV(\tilde{M})^*)$ such that

(i) $SSV(\tilde{M}_y)^*$ sub space of $SSV(\tilde{X})^*$ and $SSV(\tilde{M})^* \cong SSV(\tilde{M}_y)^*$.

(ii) $(f_y, \varphi_y) \in SSV(\tilde{M}_y)^*$ such that $(f_y, \varphi_y)_{\tilde{x}_e} = (g, \psi)_{\tilde{x}_e}$ for all $\tilde{x}_e \in SSV(\tilde{M})^*$

\Rightarrow is non-empty and partially ordered by

$$(f_y, \varphi_y) \leq (f_\alpha, \varphi_\alpha) \Leftrightarrow SSV(\tilde{M}_y)^* \subseteq SSV(\tilde{M}_\alpha)^*,$$

$$(f_y, \varphi_y)_{\tilde{x}_e} = (f_\alpha, \varphi_\alpha)_{\tilde{x}_e} \quad \forall \tilde{x}_e \in SSV(\tilde{M}_y)^*.$$

Let $\Phi = \{ (f_y, \varphi_y), SSV(\tilde{M}_y)^* \}$ be a totally ordered set in G , then it is easy to see that Φ has an upper bound $((\Psi, \vartheta), \cup SSV(\tilde{M}_y)^*)$ where $(\Psi, \vartheta)_{\tilde{x}_e} = (f_\alpha, \varphi_\alpha)_{\tilde{x}_e} \quad \forall \tilde{x}_e \in SSV(\tilde{M}_\alpha)^*$.

By using the Zorn lemma, there exists a maximal element $((f, h), SSV(\tilde{H})^*)$ in G , to complete the proof; we must show that $SSV(\tilde{H})^* = SSV(\tilde{X})^*$. suppose that $SSV(\tilde{H})^* \neq SSV(\tilde{X})^*$, then it exists $(f, \varphi) \in SSV(\tilde{X})^*$ Such that $(f, \varphi) \notin SSV(\tilde{H})^*$. put $SSV(\tilde{H}_0)^* = [SSV(\tilde{H})^* \cup (f, \varphi)]$ by using the first part in this proof, we have $(f_\gamma, \varphi_\gamma) \in SSV(\tilde{H}_0)^*$, Such that $(f_\gamma, \varphi_\gamma)_{\tilde{x}_e} = (f, h)_{\tilde{x}_e}$ for every $\tilde{x}_e \in SSV(\tilde{H}_0)^*$, but this contradicts the maximality of $((f, h), SSV(\tilde{H})^*)$. Consequently, we must have $SSV(\tilde{H})^* = SSV(\tilde{X})^*$, the proof is finished.

Theorem(3.1.3): If a proper subspace of fuzzy soft dual space $SSV(\tilde{X})^*$ is $SSV(\tilde{M})^*$, and $\tilde{x}_{e0} \notin SSV(\tilde{M})^*$, then it exist $(f, \varphi) \in SSV(\tilde{X})^*$ in a way that $(f, \varphi)_{\tilde{x}_{e0}} = 1$ and $(f, \varphi)_{\tilde{x}_e} = 0$ for all $\tilde{x}_e \in SSV(\tilde{M})^*$.

Proof: let $SSV(\tilde{M}_0) = [SSV(\tilde{M}) \cup \{ \tilde{x}_{e0} \}] = [m + \lambda \tilde{x}_{e0} : m \in SSV(\tilde{M}), \lambda \in R]$

Then a subspace of \tilde{x}_e is $SSV(\tilde{M}_0)$. Define $(g, \Psi) : SSV(\tilde{M}_0) \rightarrow R$,

$$(g, \Psi)_{\tilde{x}_e} = (g, \Psi)[m + \lambda \tilde{x}_{e0}] = \lambda \quad \text{for all } \tilde{x}_e \in SSV(\tilde{M}_0)$$

We must prove

1. (g, Ψ) is linear

$$\begin{aligned} \text{Let } \tilde{x}_e, \tilde{y}_e \in SSV(\widetilde{M}_0) \text{ and } \alpha, \beta \in R, \tilde{x}_e &= m_1 + \lambda_1 \tilde{x}_{e0} \text{ and } \tilde{y}_e = m_2 + \lambda_2 \tilde{x}_{e0} \\ \Rightarrow \alpha \tilde{x}_e + \beta \tilde{y}_e &= \alpha(m_1 + \lambda_1 \tilde{x}_{e0}) + \beta(m_2 + \lambda_2 \tilde{x}_{e0}) \\ &= (\alpha m_1 + \beta m_2) + (\alpha \lambda_1 + \beta \lambda_2) \tilde{x}_{e0} \\ \Rightarrow (g, \Psi)_{(\alpha \tilde{x}_e + \beta \tilde{y}_e)} &= \alpha \lambda_1 + \beta \lambda_2 = \alpha (g, \Psi)_{\tilde{x}_e} + \beta (g, \Psi)_{\tilde{y}_e} \end{aligned}$$

$$(g, \Psi) \tilde{\in} SSV(\widetilde{M}_0)^*.$$

2. $(g, \Psi)_{\tilde{x}_{e0}} = 1$ and $(g, \Psi)_{\tilde{x}_e} = 0$ for all $\tilde{x}_e \in SSV(\widetilde{M})$

$$\text{Since } \tilde{x}_{e0} = 0 + 1(\tilde{x}_{e0}) \Rightarrow (g, \Psi)_{\tilde{x}_{e0}} = 1$$

$$\text{And let } \tilde{x}_e \in SSV(\widetilde{M}) \Rightarrow \tilde{x}_e = \tilde{x}_e + 0 \tilde{x}_{e0} = 0 \Rightarrow (g, \Psi)_{\tilde{x}_e} = 0.$$

If $SSV(\widetilde{M}_0)^* = SSV(\widetilde{X})^*$, then we finish, $SSV(\widetilde{M}_0)^* \neq SSV(\widetilde{X})^*$, then a proper subspace $SSV(\widetilde{M}_0)^*$ of the dual space $SSV(\widetilde{X})^*$, and $(g, \Psi) \tilde{\in} SSV(\widetilde{M}_0)^*$. By using the above theorem, there exists $(f, \varphi)_{\tilde{x}_e} \tilde{\in} SSV(\widetilde{X})^*$ such that $(f, \varphi)_{\tilde{x}_e} = (g, \Psi)_{\tilde{x}_e}$ for all $\tilde{x}_e \in SSV(\widetilde{M})$

$$\text{Hence } (f, \varphi)_{\tilde{x}_{e0}} = 1 \text{ and } (f, \varphi)_{\tilde{x}_e} = 0 \text{ for all } \tilde{x}_e \in SSV(\widetilde{M})$$

Corollary(3.1.4): Let $SSV(\widetilde{X})^*$ a fuzzy soft Dual space if $\tilde{x}_{e0} \in SSV(\widetilde{X})$ such that $(f, \varphi)_{\tilde{x}_{e0}} = 0$ for all $(f, \varphi) \tilde{\in} SSV(\widetilde{X})^*$, then $\tilde{x}_{e0} = 0$.

Proof: put $SSV(\widetilde{M}) = \{0\}$, then $SSV(\widetilde{M})$ is sub space of $SSV(\widetilde{X})$ and $\tilde{x}_{e0} \notin SSV(\widetilde{M})$. By using last theorem, there exists $(f, \varphi) \in SSV(\widetilde{X})^*$ such that $(f, \varphi)_{\tilde{x}_{e0}} = 1$. This is a contradiction $\Rightarrow \tilde{x}_{e0} = 0$.

Theorem(3.4): Generalized Hahn-Banach theorem in fuzzy soft Dual space

Suppose

1. $SSV(\widetilde{M})^*$ is a subspace of fuzzy soft dual space $SSV(\widetilde{X})^*$
2. (p, θ) is fuzzy soft linear functional on $SSV(\widetilde{X})^*$
3. $(g, \Psi) \tilde{\in} SSV(\widetilde{M})^*$ such that $(g, \Psi)_{\tilde{x}_e} \leq (p, \theta)_{\tilde{x}_e}$ for all $\tilde{x}_e \in SSV(\widetilde{M})$, there exist $(f, \varphi)_{\tilde{x}_e} \in SSV(\widetilde{X})^*$ so as to

$$\text{i- } (f, \varphi)_{\tilde{x}_e} = (g, \Psi)_{\tilde{x}_e} \quad \forall \tilde{x}_e \in SSV(\widetilde{M}).$$

$$\text{ii- } (f, \varphi)_{\tilde{x}_e} \leq (p, \theta)_{\tilde{x}_e} \quad \forall \tilde{x}_e \in SSV(\widetilde{X}).$$

proof:

$$\text{let } \tilde{x}_0 \notin SSV(\widetilde{M}) \text{ and } SSV(\widetilde{M}_0) = [SSV(\widetilde{M}) \cup \{\tilde{x}_0\}] = \{m + \lambda \tilde{x}_0 : m \in SSV(\widetilde{M}), \lambda \in R\}$$

Then $SSV(\widetilde{M}_0)$ is subspace of $SSV(\widetilde{X})$

$$\text{Define } (f_0, \varphi_0): SSV(\widetilde{M}_0) \rightarrow R \text{ by } (f_0, \varphi_0)_{\tilde{x}_e} = (f_0, \varphi_0)_{m + \lambda \tilde{x}_0} = (g, \Psi)_m + \lambda \tilde{r}_0, \tilde{r}_0 \in R.$$

It's easy to see that (f_0, φ_0) is a linear functional and $(f_0, \varphi_0)_{\tilde{x}_e} = (g, \Psi)_{\tilde{x}_e}$ for all $\tilde{x}_e \in SSV(\widetilde{M})$.

Now to prove $(f_0, \varphi_0)_{\tilde{x}_e} \leq (p, \theta)_{\tilde{x}_e}$ for all $\tilde{x}_e \in SSV(\widetilde{M}_0)$

$$\text{Let } \tilde{x}_{e1}, \tilde{x}_{e2} \in SSV(\widetilde{M})$$

$$\begin{aligned} (g, \Psi)_{\tilde{x}_{e2}} - (g, \Psi)_{\tilde{x}_{e1}} &= (g, \Psi)_{(\tilde{x}_{e2} - \tilde{x}_{e1})} \leq (p, \theta)_{(\tilde{x}_{e2} - \tilde{x}_{e1})} = (p, \theta)_{(\tilde{x}_{e2} + \tilde{x}_e) + (-\tilde{x}_{e1} - \tilde{x}_e)} \leq \\ &= (p, \theta)_{(\tilde{x}_{e2} + \tilde{x}_e)} + (p, \theta)_{(-\tilde{x}_{e1} - \tilde{x}_e)} - (g, \Psi)_{\tilde{x}_{e1}} - (p, \theta)_{(-\tilde{x}_{e1} - \tilde{x}_e)} \leq - (g, \Psi)_{\tilde{x}_{e2}} + (p, \theta)_{(\tilde{x}_{e2} + \tilde{x}_e)} \end{aligned} \quad \text{for}$$

$$\text{all } \tilde{x}_{e1}, \tilde{x}_{e2} \in SSV(\widetilde{M}).$$

So that

$$\text{Sup}_{\tilde{y}_e \in SSV(\widetilde{M})} \{-(g, \Psi)_{\tilde{y}_e} - (p, \theta)_{(-\tilde{y}_e - \tilde{x}_e)}\} \leq \text{inf}_{\tilde{y}_e \in SSV(\widetilde{M})} \{-(g, \Psi)_{\tilde{y}_e} + (p, \theta)_{(\tilde{y}_e + \tilde{x}_e)}\}$$

Choose r_0 such that

$$\text{Sup}_{\tilde{y}_e \in SSV(\widetilde{M})} \{-(g, \Psi)_{\tilde{y}_e} - (p, \theta)_{(-\tilde{y}_e - \tilde{x}_e)}\} \leq r_0 \leq \text{inf}_{\tilde{y}_e \in SSV(\widetilde{M})} \{-(g, \Psi)_{\tilde{y}_e} + (p, \theta)_{(\tilde{y}_e + \tilde{x}_e)}\}$$

It follows that

$$-(g, \Psi)_{\tilde{y}_e} - (p, \theta)_{(-\tilde{y}_e - \tilde{x}_e)} \leq r_0 \leq -(g, \Psi)_{\tilde{y}_e} + (p, \theta)_{(\tilde{y}_e + \tilde{x}_e)} \quad (*) \quad \forall \tilde{y}_e \in SSV(\widetilde{M})$$

let $\tilde{x}_e \in SSV(\widetilde{M}_0) \Rightarrow \tilde{x}_e = m + \lambda \tilde{x}_0$

if $\lambda = 0 \Rightarrow \tilde{x}_e = m$ then $(f_0, \varphi_0)_m = (g, \Psi)_m \leq (P, \theta)_m = (P, \theta)_{\tilde{x}_e}$

if $\lambda \neq 0 \Rightarrow$ put $\tilde{y}_e = \frac{m}{\lambda} \Rightarrow \tilde{y}_e \in m$ in (*) to obtain

$$-(g, \Psi)_{\frac{m}{\lambda}} - (p, \theta)_{(-\frac{m}{\lambda} - \tilde{x}_e)} \leq r_0 \leq -(g, \Psi)_{\frac{m}{\lambda}} + (p, \theta)_{(\frac{m}{\lambda} + \tilde{x}_e)} \quad (**) \quad \forall m \in SSV(\widetilde{M})$$

If $\lambda > 0$, then the right-hand inequality in (**) gives

$$r_0 \leq -(g, \Psi)_{\frac{1}{\lambda}(m)} + (p, \theta)_{\frac{1}{\lambda}(m + \lambda \tilde{x}_e)}$$

$$\lambda r_0 \leq -(g, \Psi)_{(m)} + (p, \theta)_{(m + \lambda \tilde{x}_e)} \Rightarrow (g, \Psi)_{(m)} + \lambda r_0 \leq (p, \theta)_{(m + \lambda \tilde{x}_e)}$$

$$(f_0, \varphi_0)_{\tilde{x}_e} \leq (P, \theta)_{\tilde{x}_e}$$

If $\lambda > 0, \frac{-1}{\lambda} (g, \Psi)_{(m)} + \frac{1}{\lambda} (p, \theta)_{(m + \lambda \tilde{x}_e)} \leq r_0$

Since $\lambda < 0, -(g, \Psi)_{(m)} + (p, \theta)_{(m + \lambda \tilde{x}_e)} \geq \lambda r_0 \Rightarrow (g, \Psi)_{(m)} + \lambda r_0 \leq (p, \theta)_{(m + \lambda \tilde{x}_e)}$

Thus, when $\lambda \neq 0$, then obtain

$(f_0, \varphi_0)_{\tilde{x}_e} \leq (P, \theta)_{\tilde{x}_e}$ for all $\tilde{x}_e \in SSV(\widetilde{M}), (f_0, \varphi_0) \in SSV(\widetilde{M}_0)^*$ and

$(f_0, \varphi_0)_{\tilde{x}_e} = (g, \Psi)_{(\tilde{x}_e)}$ for all $\tilde{x}_e \in SSV(\widetilde{M})$ hence $(f_0, \varphi_0)_{\tilde{x}_e} \leq (P, \theta)_{\tilde{x}_e}$ for all $\tilde{x}_e \in SSV(\widetilde{M}_0)$

If $SSV(\widetilde{M}_0)^* = SSV(\widetilde{X})^*$ complete prove, either if $SSV(\widetilde{M}_0)^* \neq SSV(\widetilde{X})^*$

Let G be the collection of all ordered pairs $((f_\gamma, \varphi_\gamma), SSV(\widetilde{M}_\gamma)^*)$

Such that

i- $SSV(\widetilde{M}_\gamma)^*$ is subspace of fuzzy soft dual space $SSV(\widetilde{X})^*$ and $SSV(\widetilde{M}) \subset SSV(\widetilde{M}_\gamma)$

ii- $(f_\gamma, \varphi_\gamma) \in SSV(\widetilde{M}_\gamma)^*$ such that $(f_\gamma, \varphi_\gamma)_{\tilde{x}_e} = (g, \Psi)_{(\tilde{x}_e)}$ for all $\tilde{x}_e \in SSV(\widetilde{M})$

iii- $(f_\gamma, \varphi_\gamma)_{\tilde{x}_e} \leq (P, \theta)_{\tilde{x}_e}$ for all $\tilde{x}_e \in SSV(\widetilde{M}_\gamma) \Rightarrow G$ is non empty and partially ordered by

$$((f_\gamma, \varphi_\gamma), SSV(\widetilde{M}_\gamma)^*) \leq ((f_\alpha, \varphi_\alpha), SSV(\widetilde{M}_\alpha)^*) \text{ , } SSV(\widetilde{M}_\gamma)^* \subset SSV(\widetilde{M}_\alpha)^* \Leftrightarrow$$

that $(f_\gamma, \varphi_\gamma)_{\tilde{x}_e} = (f_\alpha, \varphi_\alpha)_{(\tilde{x}_e)}$ for all $\tilde{x}_e \in SSV(\widetilde{M}_\gamma)$.

Let $\Phi = \{((f_\gamma, \varphi_\gamma), SSV(\widetilde{M}_\gamma)^*)\}$ be a set totally ordered in G , then it is easy to prove upper bound in

Φ $(\psi, \cup SSV(\widetilde{M}_\alpha)^*)$ in which $\psi(\tilde{x}_e) = (f_\alpha, \varphi_\alpha)_{(\tilde{x}_e)}$ for all $\tilde{x}_e \in SSV(\widetilde{M}_\alpha)$.

By Zorn's lemma, there exists a maximal element $((f, Q), SSV(\widetilde{H})^*)$ in G , M to complete the proof,

we must show that $SSV(\widetilde{H})^* = SSV(\widetilde{X})^*$. Suppose that $SSV(\widetilde{H})^* \neq SSV(\widetilde{X})^*$, then there exist

$\tilde{x}_e \in SSV(\widetilde{X})$ such that $\tilde{x}_e \notin SSV(\widetilde{H})$, Put $SSV(\widetilde{H}_0) = \{SSV(\widetilde{H}) \cup \{\tilde{x}_e\}\}$ by using first part in this

proof we have $(g, h) \in SSV(\widetilde{H}_0)^*$ such that $(g, h)_{\tilde{x}_e} = (f_\gamma, \varphi_\gamma)_{\tilde{x}_e}$ for all $\tilde{x}_e \in SSV(\widetilde{H}_0)$, but this

contradiction the maximally of $((f, Q), SSV(\widetilde{H})^*)$, consequently we must $SSV(\widetilde{H})^* = SSV(\widetilde{X})^*$ and

the proof is complete .

4. Conclusion

In this research, the Hahn–Banach theorem was addressed within the framework of fuzzy soft dual spaces, where an appropriate formulation of the theorem in this context was presented and proved

under assumptions compatible with the fuzzy soft structure of the space. Some properties related to the Hahn–Banach theorem in the fuzzy soft dual space were also studied, highlighting the similarities and differences between it and the classical formulation of the theorem in traditional normed spaces.

The obtained results contribute to expanding the scope of application of the Hahn–Banach theorem to a broader class of spaces that deal with vagueness and uncertainty, providing a theoretical basis that can be built upon in subsequent studies addressing other functional analysis theorems in the fuzzy soft framework, or potential applications in fields that rely on modeling imprecise information.

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