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On Almost Finitely Generated S-acts

Authors Names	**ABSTRACT**
Zainab Ahmed Abdullah [a] *, Tagreed Hussain abd* [b] *,* *Muna Jasime Mohammed Alir* [*] Publication data: 25 / 5 /2026 ***Keywords:*** . Pure Submodules, S-pure Submodules, S-pure submodule relative to submodules,	Let S be a monoid, and M be act over monoid U. The goal of this work is to study some properties of almost finitely generated acts over monoid .

1. Introduction

Let Ṁ be a unitary S-act defined on commutative Monoid S. A Commutative Monoid is semigroup S that is commutative (st = ts) and contains a multiplicative identity element 1 such that su = 1u = u for all u ∈ S. An S-act A is said to be cyclic that is generated by a single element a ∈ A, denoted by A = aS [1] A monoid S is called group if in which every element has an inverse. The monoid $End_{S(A)}$ of all U-homomorphisms (act maps) from A to itself, with the binary operation being function composition [2] . In [3] an S-act A is aaid to be torsion free, where for all a, b ∈ A and a right cancellable element c ∈ S the equality a c = b c implies a = b. Following [4] subact Z(A), which is related to annihilators or essential congruences. An act is Non-Singular if Z(A) = Ø. An S-act A is called injective S-act such that for every monomorphism f: B → C and S-homomorphism g: B → A, there exists an S-homomorphism h : C → A extending g (i.e., h ° f = g). An injective Hull is injective S-act E(A) containing A as an essential subact (i.e., A intersects non-trivially with every non-zero subact of E(A) .Quasi-Injective Act, an S-act A such that every S-homomorphism f: B →A (where B is a subact of A) can be extended to an endomorphism g: A → A .Regular act where every u ∈ S has x ∈ S such that s = sxs) [3].

In the study of algebraic structures, the transition from ring theory to the theory of semigroups provides a fertile ground for exploring general properties of modules, translated here into the language of S-acts. Central to this exploration is the classification of acts based on finiteness conditions and structural integrity. A Noetherian semigroup provides the foundational environment where every ascending chain of congruences terminates, ensuring a level of stability that extends to the acts defined over it.

Within this framework, we examine Noetherian S-acts and Artinian S-acts, which represent the chain conditions for subacts. While Noetherian acts satisfy the ascending chain condition (ACC), Artinian

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acts satisfy the descending chain condition (DCC), both of which are essential for understanding the decomposition and finiteness of algebraic systems.

The relationship between the semigroup and its acts is further deepened by the concept of Faithful S-acts, where the action of the semigroup is injective, meaning distinct elements of S act as distinct transformations. Moreover, we delve into the internal structure of these acts through Pure Subacts, which maintain a certain "algebraic closedness" relative to the parent act, and Uniform S-acts, where any two non-zero subacts have a non-empty intersection, providing a measure of structural indecomposability. The discussion is grounded in the specific case where the Semigroup is an integral domain. In this context, the absence of zero-divisors within S imposes rigorous constraints on the behavior of its acts, bridging the gap between classical commutative algebra and modern semigroup theory. A semigroup S is called Noetherian if it satisfies the Ascending Chain Condition (ACC) on its congruences [4]. That is, every strictly ascending chain of congruences on S is finite. Noetherian S-act: A is Noetherian if every ascending chain of subacts $A_1 \subseteq A_2 \subseteq \ldots.$ terminates. Equivalently, every subact of A is finitely generated.. Artinian S-act: A is Artinian if every descending chain of subacts $A_1 \supseteq A_2 \supseteq \ldots.$.terminates. An S-act A is called Faithful if for any two distinct elements s, t $\in$S, there exists an element a $\in$ A such that as $\neq$ at. In other words, the representation Ø: S $\rightarrow$\{End\}(A) is injective [3]. A subact B of an S-act A is called Pure if for every b$\in$ B and s$\in$ S, the equation as = b having a solution a $\in$ A implies that there exists a solution b' $\in$ B such that b's = b. This is a concept often related to the preservation of tensor products [5]. An S-act A is called Uniform if the intersection of any two non-empty subacts of A is non-empty. This means A cannot be decomposed into a disjoint union of subacts, and any two "parts" of the act are fundamentally linked [6].

In the context of semigroups (often monoids), S is considered an Integral Domain (or a domain) if it is a commutative semigroup with a zero element 0, and for any x, y $\in$ S, xy = 0 implies x = 0 or y = 0. In many semigroup contexts, this is simply referred to as a Zero-divisor free semigroup.

2. Main Results.

Definition 2.1: [8]

An S-act A is called almost finitely generated act (AFGA), if that is not finitely generated as an act, but every proper subact of A is finitely generated.

Example 2.2: [8]

Let p be a prime number and let Q_p = \{ $\frac{a}{p^i} \Big| \; a \in Z , i = 0,1,2, \ldots..$\}. The set of rational numbers whose denominators are powers of p. It is clear from the above that Qp is a subgroup of the group of rational numbers Q, and the group of integers Z is a proper subact of Q_p.. We denote the group of fractions $\frac{Q_p}{Z}$ by Z_{P^∞}. Considering Z_{P^∞} is an act over Z. We can prove that every proper subact generated by $\frac{1}{Pi} + Z$

where i is a non-negative integer. And all subacts form ascending chain $0 \subsetneq \left(\frac{1}{P}+Z\right) \subsetneq \left(\frac{1}{P^2}+Z\right) \subsetneq \cdots$, therefore $Z_{P^\infty} = \cup_{i=1}^{\infty}\left(\frac{1}{P^i}+Z\right)$

Definition 2.3 : Let M be S-act. A prime ideal P in S is said to be associated with M, if there exists an element x in M such that P = $ann_S(x)$. The set of associated ideals is denoted by $Ass_S(M)$

Proposition 2.4 : Let S be notherian semigroup and M be non-zero S-act then $Ass_S(M)$ is non-empty.

Proof : Let U = { $ann_{S(}(x)$; $0 \neq x \in M$ }. Since S is notherian semigroup, then U have maximal element say P. this mean that there is y belongs to M such that P= $ann_S(y)$. We prove that P is prime ideal, let $ab \in P$ and $a \notin P$. But aby = 0 therefore $b \in ann_S(ay)$. Since P = $ann_S(y) = ann_S(ay)$ and $ann_S(y)$ maximal element, then $b \in P$, thus P is prime ideal, this mean that $Ass(M) \neq \emptyset$**.**

Proposition 2.5 : Let M be AFG faithful, Artinian S-act, then M does not contain a non-zero proper divisible subact.

Proof :

Let N be a non-zero proper subact in M**.** M is torsion by proposition 2.6. However, N is FN, therefore, there exists a non-zero element $s \in S$, such that sN = 0. This implies that N cannot be divisible S-act..

Proposition 2.6 : Let M be AFG faithful, Artinian S-act, then M does not contain a non-zero proper pure subact.

Proof :

Suppose that N is a non-zero proper subact in M, and let x be a non-zero element in S. Thus N = (x)M = (x)N ⋂ M. But N is AFG, therefore either xM = 0 or xM = M.

- If xM = 0, this means that $x \in ann_S(M)$ which is contradiction.
- Hence xM = M, which means that N = N ⋂ (x)M = N ∩M = N. This contradicts the proposition 2.7. therefore, there is no pure proper subact in M.

Proposition 2.7 : Let M be AFG faithful Artinian S-act, then M is uniform S-act.

Let N be a non-zero proper subact in M . M is Artinian and AFG. Thus M = $\cup_{i=0}^{\infty} Sx_i$, $x_i \in M$ for all i (proposition 2,4). Since N is FG, there exists a positive integer j such that $N \subset Rx_j$.

Now let K be a non-zero subact in M. Using the same previous, there exists a positive integer k such that $K \subset Sx_k$, Accordingly, $N \cap K \subset Sx_j \cap Sx_k = Sx_w$, where w – min{j, k}. Since N and K are non-zero subacts, there exist $s_j \neq 0$ in K, that is $s_1 x_w = s_1$ and $s_2 x_w = s_2$. Thus $s_1 s_2 x_w$ will be a non-zero element in N ∩ K because S is integral domain. This leads to N being essential, and consequently, M is uniform.

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