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On Fuzzy Quotient Banach Algebra

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Abstract

This paper studies the concepts of fuzzy Quotient Banach Algebra and introduces definitions to the fuzzy convergence, fuzzy normed space, fuzzy quotient algebra . It also proves some theorems in this subject.

Keywords: algebra**,** fuzzy convergent sequence, fuzzy Cauchy sequence, fuzzy completeness in fuzzy space, quotient algebra**.**

1. Introduction

Many mathematicians have studied fuzzy normed spaces from several angles . The concepts of fuzzy norm was introduced by Katsaras in 1984. This paper introduced some theorems related to this concept as fuzzy quotient algebra and fuzzy continuity .In this paper [5] which includes fuzzy Banach space and we circulated it on fuzzy Banach algebra.

2. Preliminaries

This section deals with the basic concepts of algebra, fuzzy Banach algebra and some of their properties.

Definition (2.1): [8]

Let X be a non-empty set . X is called an algebra if :

(1) $(X,+,\cdot)$ is vector space over a field $\mathbb{F}$.

(2) $(X,+,\circ)$ is a ring .

(3) $(\lambda x)\circ y = \lambda(x\circ y) = x\circ(\lambda y)$ for every $x,y\in X$ for every $\lambda\in\mathbb{F}$.

Definition (2.2): [1]

A binary operation $*:[0,1]\longrightarrow[0,1]\times[0,1]$ is a t-norm if $*$ is satisfies the following condition :

(1) $*$ Is commutative and associative .
(2) $a*1=a$ for all $a\in[0,1]$.
(3) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Definition (2.3): [7]

Let X be vector space over a field $\mathbb{F}$. A function $N: X\times(0,\infty)\longrightarrow[0,1]$ is said to be a fuzzy norm on X if for all $x,y\in X$ and $t,s>0$:

(1) $N(x,t)=0$ if $t\le 0$.

(2) $N(x,t)=1$ iff $x=0$ for all $t>0$.

(3) $N(\lambda x,t)=N\left(x,\frac{t}{|\lambda|}\right)$ if $\lambda\neq 0$.

(4) $N(x+y, t+s) \geq N(x,t) * N(y,s)$.

(5) $N(x,\cdot) : (0,\infty) \to [0,1]$ is continuous .

(6) $\lim_{t\to\infty} N(x,t) = 1$.

The tuple $(X, N, *)$ is called a fuzzy normed vector space .

Definition (2.5): [3]

Let $(X, N, *)$ be a fuzzy normed vector space , a sequence $\{x_n\}$ in X said to

(1) Fuzzy convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$, i.e. for each $\varepsilon > 0$ and each $t > 0$,then exists $K \in \mathbb{Z}^+$ such that $N(x_n - x, t) > 1 - \varepsilon$. In this case, x is the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.

(2) Fuzzy Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists $K \in \mathbb{Z}^+$ such that for all $n \geq K$ and $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

(3) Fuzzy normed space in which every fuzzy Cauchy sequence is convergent is said to be complete . A complete fuzzy normed space is called a fuzzy Banach space .

Definition (2.6): [2]

A nonempty set X is called Fuzzy Banach Algebra if:

(1) X is algebra .

(2) X is a complete fuzzy normed vector space .

(3) $N(xy, ts) \geq N(x,t) * N(y,s)$ for every $x, y \in X$, $t, s > 0$.

Example (2.7): [4]

Let $(X, \|\cdot\|)$ be a Banach algebra , then

$$N(x,t) = \begin{cases} 0 & if\ t \leq \| x \| \\ 1 & if\ t > \| x \| \end{cases}$$

is a fuzzy norm algebra and so $(X, N, *)$ is a fuzzy Banach algebra .

3. **fuzzy Quotient Algebra** .

In this section discusses concepts of fuzzy quotient algebra and relationship between fuzzy quotient algebra and fuzzy Banach algebra and also proves some theorems in this subject .

Let M be any subalgebra of an algebra X over $\mathbb{F}$. Let x be any element of X. The set $x + M = \{x + m : m \in M\}$ is called a left coset of M in X generated by x . Let X/M deote the set all coset of M in X/M in X, i.e $X/M = \{x + m : x \in X\}$.

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Theorem (3.1): [6]

If M is a sub algebra of algebra X over $\mathbb{F}$, the X/M is an algebra over $\mathbb{F}$ for the a addition and multiplication and scalar multiplication compositions defined as follows :

(1) $(x + M) \oplus (y + M) = (x + y) + M$ for all $x, y \in X$.

(2) $(x + M) \otimes (y + M) = xy + M$ for all $x, y \in X$.

(3) $\lambda \odot (x + M) = \lambda x + M$ for all $\lambda \in \mathbb{F}$, for all $x \in X$.

Proof :

Let $x, y \in X \Rightarrow x + y \in X$ and $x, y \in X$.

Also $x \in X$ and $\lambda \in \mathbb{F} \Rightarrow \lambda x \in X$.

Therefore $(x + y) + M \in X/M$ and $xy + M \in X/M$ also $\lambda x + M \in X/M$.

Thus X/M is closed with respect to addition and multiplication of coset and scalar multiplication as defined above .

Now first of all we shall these three composition are will defined .

Let $x + M = x' + M$ $x, x' \in M$ and $y + M = y' + M$ $y, y' \in M$

$x + M = x' + M \Rightarrow x - x' \in M$ and $y + M = y' + M \Rightarrow y - y' \in M$.

Since M is algebra of X, we have $(x - x') + (y - y') \in M \Rightarrow (x + y) - (x' + y') \in M$

$$\Rightarrow (x + y) + M = (x' + y') + M.$$

Therefore addition in X/M is well defined.

Also let $x + M = x' + M$, $x, x' \in X$,and let $y + M = x' + M$

$y, y' \in M \Rightarrow x - x' \in M$, $y - y' \in M$.

But $x \cdot y - x' \cdot y' = x \cdot y - x \cdot y' + x \cdot y' - x' \cdot y' = x \cdot (y - y') + (x - x') \cdot y'$, since $x \cdot (y - y'), (x - x') \cdot y' \in M$

$\Rightarrow x \cdot (y - y') + (x - x') \cdot y' \in M \Rightarrow x \cdot y - x' \cdot y' \in M$.

Hence $(x \cdot y) + M = (x' \cdot y') + M$.

Therefore a multiplication in X/M is well define .

Again let $x - x' \in M, \lambda \in \mathbb{F} \Rightarrow \lambda(x - x') \in M \Rightarrow \lambda x - \lambda x' \in M \Rightarrow \lambda x + M = \lambda x' + M \Rightarrow$ scalar multiplication in X/M. Satisfies the conditions of an algebra .

Theorem (3.2):

If M is a closed sub algebra of fuzzy normed algebra X and $N(x + M, t)$ is defined by ;
$N(x + M, t) = \sup\{N(x + m, t): m \in M\}$ then :

(1) N is a fuzzy norm algebra on X/M.

(2) If $(X, N, *)$ is a fuzzy Banach algebra, the so is $(X/M, N, *)$.

Proof:

To prove if $t \leq 0$.

Then $N(x+M,t) = N(x+M,0) = 0$.

Thus $N(x+M,t) = 0$ for all $t \leq 0$.

Also whenever $x = 0$ if and only if $t > 0 \Rightarrow N(x+M,t) = N(0+M,t) = 1$.

Then $N(x+M,t) = 1$ for every $t > 0$ and if and only if $x = 0$, and

by definition there is a sequence $\{x_n\}$ in M such that $N(x+x_n) \to 1$.

So $x + x_n \longrightarrow 0 \Rightarrow x_n \longrightarrow (-x)$ and by theorem [let $x \in \overline{M}$ iff $\exists \{x_n\}$ in M such that

$lim_{n\to\infty} N(x-y,t) = 1$].

Since M is a closed $\Rightarrow \overline{M} = M \Rightarrow x \in M$ and $x + M = M,$ the zero element of X/M .

On the other hand we have .

$N\big((x+M)+(y+M),t+s\big) = N((x+y)+M,t+s)$

by defined above we get

$= \sup\{N((x+y)+m,t+s): m \in M\}$

$\geq N\big((x+m)+(y+m),t+s\big)$

by (4) in definition (2.3) we get

$N\big((x+M)+(y+M),t+s\big) \geq N(x+m,t) * N(y+m,s)$

for $m \in M$, $x,y \in X$ and $t,s > 0$.

Now if we take sup on both sides , we have

$$N\big((x+M)+(y+M),t+s\big) \geq N(x+M,t) * N(y+M,s) .$$

Also $N\big((x+M)*(y+M),ts\big) = N((xy)+M,ts)$

$=\sup\{N((xy)+m,ts) : m \in M\}$

by (3) in definition (2.2) we get

$\geq N((x+m)*(y+m),ts)$

$\geq N(x+m,t) * (y+m,s)$

for $m \in M$, $x,y \in X$ and $t,s > 0$.

Now if we take $\sup$ on both sides, we have

$$N\big((x+M)*(y+M),ts\big) \geq N(x+M,t) * N(y+M,s).$$

Also we have $N(\lambda(x+M,t) \geq N(\lambda x+M,t)$

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$=\sup\{N(\lambda x+\lambda m,t): m\in M\}$

$=\sup\{N(x+m,\frac{t}{|\lambda|}): m\in M\}$

$=N(x+M,\frac{t}{|\lambda|})$, $\lambda\neq 0$.

Therefore $(X,N,*)$ is a fuzzy normed algebra .

Let $\{x_n+M\}$ be a fuzzy Cauchy sequence in X/M . then there exists $\varepsilon_n>0$ such that $\varepsilon_n\to 0$ and, $N((x_n+M)-(x_{n+1}+M),t)\geq 1-\varepsilon_n$.

If we take $n=1$ and let $y_2=0$ we choose $y_2\in M$ such that

$N(x_1-(x_2-y_2),t)\geq N((x_1-x_2)+M,t)*(1-\varepsilon_1)$.

But $N((x_1-x_2)+M,t)\geq(1-\varepsilon_1)$.

Therefore, $N(x_1-(x_2-y_2),t)\geq(1-\varepsilon_1)(1-\varepsilon_1)$.

Now suppose y_{n-1} has been chosen, $y_n\in M$ can be chosen such that

$N((x_{n-1}+y_{n-1})-(x_n+y_n),t)\geq N((x_{n-1}-(x_n)+M,t)*(1-\varepsilon_{n-1})$.

And therefore

$N((x_{n-1}+y_{n-1})-(x_n+y_n),t)\geq(1-\varepsilon_{n-1})*(1-\varepsilon_{n-1})$.

Then, $\{x_n+y_n\}$ is a fuzzy Cauchy sequence in X. since X is complete there is an in x_0 such that X in $x_n+y_n\longrightarrow x_0$ in X .

On the other hand $x_n+M=(x_n+y_n)+M=x_0+M$.

Therefore every fuzzy Cauchy sequence $\{x_n+M\}$ is a fuzzy convergent in X/M.

Thus X/M is complete and $(X/M,N,*)$ is a fuzzy Banach algebra .

Theorem (3.3):

Let M be a closed sub algebra of a fuzzy normed algebra X. If M and X/M are fuzzy Banach algebra, then X is fuzzy Banach algebra .

Proof:

Suppose X is a fuzzy normed algebra, and $M, X/M$ are fuzzy Banach algebra .

Let $\{x_n\}$ be a fuzzy Cauchy sequence in X, then $\{x_n+M\}$ is a Cauchy in X/M, since X/M is fuzzy Bnanch algebra, then $\{x_n+M\}$ is a converges in X/M .

Now to prove $x_n\to x$, let $t>0$ for all $\varepsilon_1\in(0,1)$ there exists $\varepsilon\in(0,1)$ such that $(1-\varepsilon_1)*(1-\varepsilon_1)\geq(1-\varepsilon)$.

Since $\{x_n\}$ is a fuzzy Cauchy sequence in X, then for all $t>0$ and $\varepsilon_1\in(0,1)$ there exists $K_1\in\mathbb{Z}^+$ such that

$N((x_n-x_l)+m,\frac{t}{2})>1-\varepsilon_1$ for all $n,l\geq K_1$ and $m\in M$.

Since $\{x_n+M\}$ is a fuzzy converges to $x+M$, there exists $K_2\in\mathbb{Z}^+$ such that

$N((x_n - x) + m, \frac{t}{2}) > 1 - \varepsilon_1$ for all $n \geq K_2$ and $m \in M$.

Take $K = \min\{K_1, K_2\}$

$$N\big((x_n - x) + M, t\big) = N((x_n - x_l) + (x_l - x) + m, \tfrac{t}{2} + \tfrac{t}{2})$$

$$\geq N((x_n - x_l) + m, \tfrac{t}{2}) * N((x_l - x) + m, \tfrac{t}{2})$$

$$\geq (1 - \varepsilon_1) * (1 - \varepsilon_1)$$

$$\geq 1 - \varepsilon \quad \text{for all } n \geq K .$$

By theorem (3.2) we get

$N\big((x_n - x) + M, t\big) \geq N\big((x_n - x) + m, t\big)$ by theorem (3.2) we get

$$\geq N((x_n - x), t)$$

$$\geq 1 - \varepsilon .$$

Whenever $m = 0$ such that $m \in M$.

Therefor $x_n \to x, \{x_n\}$ is a fuzzy converges to x .

Hence $(X, N, *)$ is a complete fuzzy normed algebra .

Then $(X, N, *)$ is a fuzzy Banach algebra .

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المستخلص

في هذا البحث درسنا مفاهيم جبر بناخ القسمة الضبابي . وقدمنا تعاريف التقارب الضابي ،فضاء المعياري الضبابي وجبر القسمة الضبابي . وكذلك برهنا بعض المبرهنات حول هذا الموضوع .