# Spectral Properties and Approximation Results for Fractional Singular Sturm–Liouville Problems

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**ABSTRACT**

 In this paper we investigate a spectral theory for the eigenvalues and eigenfunctions of fractional singular Sturm–Liouville problem of Bessel type by constructing a complete spectral decomposition. We show that all eigenvalues are real and the corresponding eigenfunctions are orthogonal. Risk upper bounds are obtained and new approximation results concerning the spectral properties of the problem are established and rigorously justified.

## 1. Introduction

The Sturm–Liouville problem was first introduced in a series of foundational papers, published between 1836 and 1837. Eigenvalue-type problems attracted the interest of Charles-François Sturm (1803–1855), at the time Professor of Mechanics at Sorbonne, who in around 1833 considered heat conduction in solid bars. With Joseph Liouville (1809-1882), a math professor at the Collège de France, he generalized this investigation to the general behavior of second-order linear differential equations. Liouville's later work in mathematical analysis did still more to develop the theoretical basis of what is now called Sturm–Liouville theory [1].

A Sturm–Liouville boundary-value problem is given by a second-order linear ODE of the form

$$-(py')' + qy = \lambda w y, (a, b)\#(1)$$

The Sturm–Liouville boundary-value problem is defined together with its boundary conditions, where $(a, b)$ denotes a bounded or unbounded open interval of the real line $\mathcal{R}$ The coefficients $p, q, w: (a, b)$ and the spectral parameter $\mathcal{R}; \lambda \in \mathcal{C}$ belong to the complex field. Spectral analysis of such problems has far-reaching applications across numerous scientific disciplines. The associated mathematical techniques acquire broader significance when formulated within the general framework of Sturm–Liouville theory in the Hilbert space $\mathbb{L}_2$.

Mathematics, physics and engineering have several problems in which Sturm–Liouville makes a central role and is the basis for all sorts of analytical and computational techniques. Spectral data, such as spectra, spectral functions, scattering data and norming constants are of crucial importance. It is well-known in classical theory that for a self-adjoint linear second-order operator the eigenfunctions form an orthogonal sequence in A classic result proved that, if the differential equation associated to (1) has a properly normalized solution at infinity and another one to minus infinity and these functions are asymptotically proportional, it follows that there exists a spectrum.

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Depending on the nature of the interval and coefficients, Sturm–Liouville problems are classified as *regular* or *singular*. Many classical differential equations—including those of Bessel, hydrogen atom, Hermite, Jacobi, and Legendre types—can be expressed as particular cases of the Sturm–Liouville form. Extensive investigations have been devoted to these formulations [2–7]. In particular, the radial component of the Schrödinger equation can be transformed into the Bessel equation, providing a useful physical realization of the problem.

Fractional calculus, defined as *the theory of differentiation and integration of arbitrary real or complex order that unifies and extends the classical concepts of integer-order calculus* [6–13], has gained remarkable attention in recent decades. Originating from the early ideas of Leibniz, the field has witnessed rapid expansion, particularly in the last two decades, with numerous studies exploring its role in fractional quantum mechanics. It has been demonstrated that many natural and engineered systems can be modeled more accurately through fractional-order derivatives [8–17]. The growing importance of fractional calculus stems from its applicability across nearly every branch of science and engineering—including viscoelasticity, electrical and electronic systems, electrochemistry, biophysics, bioengineering, signal and image processing, mechanics, and control theory.

In fractional extensions of Sturm–Liouville theory, the classical integer-order derivatives are replaced by fractional derivatives, and the resulting equations are typically solved using numerical or analytical approximation techniques [18–23]. Klimek and Agarwal [24] introduced a fractional Sturm–Liouville operator, defined the corresponding regular fractional boundary-value problem, and investigated the spectral properties of its eigenvalues and eigenfunctions. Building upon these foundations, the present study introduces a **singular fractional Sturm–Liouville problem of Bessel type** and establishes the associated spectral properties and analytical approximations of its spectral data.

We now present the boundary-value formulation for the Bessel-type equation together with the necessary preliminary definitions and data as follows.

## 2. Preliminaries

Now, consider the following Bessel equation:

$$\frac{d^2 y}{dx^2}+\left(\lambda-\frac{v^2-1/4}{x^2}\right)y=0, \#(2)$$

where $\lambda$ and $v$ are real numbers. The Bessel equation for having the analogous singularity is given in [5].

**Definition 1** (see [10]). Let $0 < \alpha \leq 1$. The left-sided and rightsided Riemann-Liouville integrals of order $\alpha$, respectively, are given by the formulas

$$\left(I_{a,+}^{\alpha} f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1} f(s)ds, x>a,$$

$$\left(I_{b,-}^{\alpha} f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (s-x)^{\alpha-1} f(s)ds, x<b, \tag{3}$$

where $\Gamma$ denotes the gamma function.

**Definition 2** (see [10]). Let $0 < \alpha \leq 1$. The left-sided and right-sided Riemann-Liouville derivatives of order $\alpha$, respectively, are defined as follows:

$$\left(\mathcal{D}_{a,+}^{\alpha} f\right)(x) = \mathcal{D}\left(I_{a,+}^{1-\alpha} f\right)(x) x > a,$$

$$\left(\mathcal{D}_{b,-}^{\alpha} f\right)(x) = -\mathcal{D}\left(I_{b,-}^{1-\alpha} f\right)(x) x < b. \quad (4)$$

Analogous formulas yield the left-sided and right-sided Caputo derivatives of order $\alpha$ :

$$\left({}^{c}\mathcal{D}_{a,+}^{\alpha} f\right)(x) = \left(I_{a,+}^{1-\alpha} \mathcal{D} f\right)(x)\ x > a, 0 < \alpha \leq 1,$$

$$\left({}^{c}\mathcal{D}_{b,-}^{\alpha} f\right)(x) = \left(I_{b,-}^{1-\alpha}(-\mathcal{D}) f\right)(x) x < b, 0 < \alpha \leq 1. \quad (5)$$

Definition 3 (see [14]). The general function ${}_{p}\Psi_{q}(z)$ is defined for $z \in C, a_i, b_j \in C$, and $\alpha_i, \beta_j \in \mathbb{R}(i = 1, \dots, p; j = 1, \dots, q)$ by the series

$${}_{p}\Psi_{q}(z) = {}_{p}\Psi_{q}\left[\begin{matrix}(a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q}\end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_l + \alpha_l k)}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j k\right)} \frac{z^k}{k!}. \#(6)$$

This general Wright function was investigated by Fox who presented its asymptotic expansion for large values of the argument $z$ under the condition

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > 1. \#(7)$$

If these conditions are satisfied, the series in (6) is convergent for any $z \in C$.

**Theorem 4** (see [14]). Let $a_i, b_j \in C$, and $\alpha_l, \beta_j \in \mathbb{R}(i = 1, \dots, p; j = 1, \dots q)$, and let

$$\Delta = \sum_{i=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_i,$$

$$\delta = \prod_{i=1}^{p} |\alpha_i|^{-\alpha_i} \prod_{j=1}^{q} |\beta_j|^{\beta_j}, \quad (8)$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p - q}{2}.$$

(i) If $\Delta > -1$, then the series in (6) is absolutely convergent for all $z \in C$.
(ii) If $\Delta = -1$, then the series in (6) is absolutely convergent for $|z| < \delta$ and for $|z| = \delta$ and $\Re(\mu) > 1/2$.

**Property 1.** The fractional differential operators defined in (4)-(5) satisfy the following identities:

(i)

$$\int_a^b f(x)\mathcal{D}_{b,-}^{\alpha} g(x)dx = \int_a^b g(x)^c\mathcal{D}_{a,+}^{\alpha} f(x)dx - f(x)I_{b,-}^{1-\alpha} g(x)\Big|_a^b, \tag{9}$$

(ii)

$$\int_a^b f(x)\mathcal{D}_{b,-}^{\alpha} g(x)^c\mathcal{D}_{a,+}^{\alpha} h(x)dx = \int_a^b g(x)^c\mathcal{D}_{a,+}^{\alpha} f(x)^c\mathcal{D}_{a,+}^{\alpha} h(x)dx$$

$$-f(x)I_{b,-}^{1-\alpha} g(x)^c\mathcal{D}_{a,+}^{\alpha} k(x)\Big|_a^b, \tag{10}$$

(iii)

$$\int_a^b f(x)\mathcal{D}_{a,+}^{\alpha} g(x)dx = \int_a^b g(x)^c\mathcal{D}_{b,-}^{\alpha} f(x)dx + f(x)I_{a,+}^{1-\alpha} g(x)\Big|_a^b. \tag{11}$$

Property 2 (see [24]). Assume that $\alpha \in (0,1), \beta > \alpha$, and $f \in c[a,b]$. Then the relations

$$\mathcal{D}_{a,+}^{\alpha} I_{a,+}^{\alpha} f(x) = f(x),$$

$$\mathcal{D}_{b,-}^{\alpha} I_{b,-}^{\alpha} f(x) = f(x),$$

$$\mathcal{D}_{a,+}^{\alpha} I_{a,+}^{\beta} f(x) = I_{a,+}^{\beta-\alpha} f(x),$$

$$\mathcal{D}_{b,-}^{\alpha} I_{b,-}^{\beta} f(x) = I_{b,-}^{\beta-\alpha} f(x), \tag{12}$$

$${}^C\mathcal{D}_{a,+}^{\alpha} I_{a,+}^{\alpha} f(x) = f(x),$$

$${}^C\mathcal{D}_{b,-}^{\alpha} I_{b,-}^{\alpha} f(x) = f(x),$$

hold for any $x \in [a,b]$. Furthermore, the integral operators defined in (3) satisfy the following semigroup properties:

$$I_{a,+}^{\alpha} I_{a,+}^{\beta} = I_{a,+}^{\alpha+\beta}, I_{b,-}^{\alpha} I_{b,-}^{\beta} = I_{b,-}^{\alpha+\beta}. \#(13)$$

Now, let us take up a singular fractional boundary problem for Bessel operator and give some spectral results.

## 3. Main Results

### 3.1. A Singular Fractional Sturm–Liouville Problem for the Bessel Operator

The fractional Sturm–Liouville problem associated with the Bessel operator involves differential expressions that include both left- and right-sided fractional derivatives. To construct this formulation, we employ the integration-by-parts relations given in equations (10) and (11) as the basis for a new approximation scheme.

In the classical Sturm–Liouville framework, the spectral properties of eigenvalues and eigenfunctions are intimately linked to the integration-by-parts formula for first-order derivatives. In the fractional counterpart, however, both left and right derivatives play essential roles. Specifically, the fundamental dualities are formed between the left Riemann–Liouville derivative and the right Caputo derivative, as well as between the right Riemann–Liouville derivative and the left Caputo derivative.

The spectral characteristics of fractional Sturm–Liouville operators governed by the Bessel operator are frequently established—either directly or indirectly—through the relationship between the asymptotic behavior of solutions at large distances and the underlying spectral structure of the associated differential operator.

**Definition 5.** Let $\alpha \in (0,1)$. Fractional Bessel operator is written as

$$\mathcal{L}_{\alpha[\mathcal{B}]} = \mathcal{D}_{1,-}^{\alpha} p(x) {}^{C}\mathcal{D}_{0,+}^{\alpha} + \left(q(x) - \frac{v^2 - 1/4}{x^2}\right). \#(14)$$

Considering the fractional Bessel equation

$$\mathcal{L}_{\alpha[\mathcal{B}]} y_{\lambda}(x) + \lambda w_{\alpha}(x) y_{\lambda}(x) = 0, \#(15)$$

where $p(x) \neq 0, w_{\alpha}(x) > 0$, for all $x \in (0,1], w_{\alpha}(x)$ is weight function, and $p, q$ are real valued continuous functions in interval $(0,1]$.

The boundary conditions for the operator $\mathcal{L}$ are the following:

$$y_{\lambda}(0) = 0,$$

$$d_1 y(1) + d_2 I_{1,-}^{1-\alpha} p(1)\, {}^{C}D_{0,+}^{\alpha} y(1) = 0, \quad (16)$$

where $d_1^2 + d_2^2 \neq 0$. The fractional boundary-value problem (15)-(16) is fractional Sturm-Liouville problem for Bessel operator.

**Theorem 6.** Fractional Bessel operator is self-adjoint on $(0,1]$.

Proof. Let us consider the following equation:

$$\begin{aligned}
\langle \mathcal{L}_{\alpha[\mathcal{B}]} \varphi, \phi \rangle &= \int_0^1 \mathcal{L}_{\alpha[\mathcal{B}]} \varphi(x) \cdot \phi(x) dx \\
&= \int_0^1 \phi(x) \left[ \mathcal{D}_{1,-}^{\alpha} p(x) {}^{C}\mathcal{D}_{0,+}^{\alpha} \varphi(x) + \left(q(x) - \frac{v^2 - 1/4}{x^2}\right) \varphi(x) \right] dx \\
&= \int_0^1 \phi(x) \mathcal{D}_{1,-}^{\alpha} p(x) {}^{c}\mathcal{D}_{0,+}^{\alpha} \varphi(x) dx \\
&\quad + \int_0^1 \left(q(x) - \frac{v^2 - 1/4}{x^2}\right) \varphi(x) \phi(x) dx. \quad (17)
\end{aligned}$$

By means of equality (10) and boundary conditions (16), we obtain the identity

$$\langle \mathcal{L}_{\alpha[\mathcal{B}]}\varphi, \phi\rangle = \int_0^1 p(x)\, {}^c\mathcal{D}_{0,+}^{\alpha}\phi(x)\, {}^c\mathcal{D}_{0,+}^{\alpha}\varphi(x)dx - \phi(x)I_{1,-}^{1-\alpha}p(x)\, {}^c\mathcal{D}_{0,+}^{\alpha}\varphi(x)\Big|_0^1$$

$$+\int_0^1 \left(q(x) - \frac{v^2 - 1/4}{x^2}\right)\varphi(x)\phi(x)dx$$

$$= \int_0^1 p(x)\, {}^c\mathcal{D}_{0,+}^{\alpha}\phi(x)\, {}^c\mathcal{D}_{0,+}^{\alpha}\varphi(x)dx + \frac{d_1}{d_2}\varphi(1)\phi(1)$$

$$+\int_0^1 \left(q(x) - \frac{v^2 - 1/4}{x^2}\right)\varphi(x)\phi(x)dx. \qquad (18)$$

On the other hand, by performing similar operations, we find

$$\langle \varphi, \mathcal{L}_{\alpha[\mathcal{B}]}\phi\rangle = \int_0^1 p(x)^C\mathcal{D}_{0,+}^{\alpha}\varphi(x)^c\mathcal{D}_{0,+}^{\alpha}\phi(x)dx + \frac{d_1}{d_2}\varphi(1)\phi(1)$$

$$+\int_0^1 \left(q(x) - \frac{v^2 - 1/4}{x^2}\right)\phi(x)\varphi(x)dx. \qquad (19)$$

The right-hand sides of (18) and (19) are equal; hence, we may see that the left sides are equal; that is,

$$\langle \mathcal{L}_{\alpha[\mathcal{B}]}\varphi, \phi\rangle = \langle \varphi, \mathcal{L}_{\alpha[\mathcal{B}]}\phi\rangle. \#(20)$$

**Theorem 7.** The eigenvalues of fractional Bessel operator (15)(16) are real.

Proof. Let us observe that the following relation results from equality (10):

$$\int_0^1 f(x)\mathcal{L}_{\alpha[\mathcal{B}]}g(x)dx = \int_0^1 p(x)\, {}^c\mathcal{D}_{0,+}^{\alpha}f(x)^c\mathcal{D}_{0,+}^{\alpha}g(x)dx - f(x)I_{1,-}^{1-\alpha}p(x)\, {}^c\mathcal{D}_{0,+}^{\alpha}g(x)\Big|_0^1$$

$$+\int_0^1 \left(q(x) - \frac{v^2 - 1/4}{x^2}\right)g(x)f(x)dx \qquad (21)$$

Suppose that $\lambda$ is the eigenvalue for (15)-(16) corresponding to eigenfunction $y$; the following equalities satisfy $y$ and its complex conjugate $\bar{y}$ :

$$\mathcal{L}_{\alpha[\mathcal{B}]}y(x) + \lambda w_{\alpha}(x)y(x) = 0, \qquad (22)$$

$$y(0) = 0,$$

$$d_1 y(1) + d_2 I_{1,-}^{1-\alpha}p(1)^c D_{0,+}^{\alpha}y(1) = 0, \qquad (23)$$

$$\mathcal{L}_{\alpha[B]}\bar{y}(x) + \bar{\lambda}w_{\alpha}(x)\bar{y}(x) = 0, \qquad (24)$$

$$\bar{y}(0) = 0,$$

$$d_1\bar{y}(1) + d_2 I_{1,-}^{1-\alpha}p(1)^c\mathcal{D}_{0,+}^{\alpha}\bar{y}(1) = 0, \qquad (25)$$

where $d_1^2 + d_2^2 \neq 0$. We multiply (22) by function $\bar{y}$ and (24) by function $y$, respectively, and subtract

$$(\lambda - \bar{\lambda}) w_\alpha(x) y(x) \bar{y}(x) = y(x) \mathcal{L}_{\alpha[\mathcal{B}]} \bar{y}(x) - \bar{y}(x) \mathcal{L}_{\alpha[\mathcal{B}]} y(x). \quad (26)$$

Now, we integrate over interval (0,1], and applying relation (21), and we note that the right-hand side of the integrated equality contains only boundary terms:

$$(\lambda - \bar{\lambda}) \int_0^1 w_\alpha(x) y(x) \bar{y}(x) dx = \int_0^1 y(x) \mathcal{L}_{\alpha[\mathcal{B}]} \bar{y}(x) dx - \int_0^1 \bar{y}(x) \mathcal{L}_{\alpha[\mathcal{B}]} y(x) dx$$

$$= \int_0^1 y(x) \left[ \mathcal{D}_{1,-}^{\alpha} p(x) {}^c\mathcal{D}_{0,+}^{\alpha} \bar{y}(x) + \left( q(x) - \frac{v^2 - 1/4}{x^2} \right) \bar{y}(x) \right] dx$$

$$- \int_0^1 \bar{y} \left[ \mathcal{D}_{1,-}^{\alpha} p(x) {}^c\mathcal{D}_{0,+}^{\alpha}(x) y(x) + \left( q(x) - \frac{v^2 - 1/4}{x^2} \right) y(x) \right] dx$$

$$= -y(x) I_{1,-}^{1-\alpha} p(x) {}^C\mathcal{D}_{0,+}^{\alpha} \bar{y}(x) \Big|_1 + y(x) I_{1,-}^{1-\alpha} p(x) {}^c\mathcal{D}_{0,+}^{\alpha} \bar{y}(x) \Big|_0$$

$$+ \bar{y}(x) I_{1,-}^{1-\alpha} p(x) {}^c\mathcal{D}_{0,+}^{\alpha} y(x) \Big|_1 - \bar{y}(x) I_{1,-}^{1-\alpha} p(x) {}^c\mathcal{D}_{0,+}^{\alpha} y(x) \Big|_0$$

By virtue of the boundary conditions (23), (25), we find

$$(\lambda - \bar{\lambda}) \int_0^1 w_\alpha(x) |y(x)|^2 dx = 0 \#(28)$$

Because $y$ is a nontrivial solution and $w_\alpha(x) > 0$, it is easily seen that $\lambda = \bar{\lambda}$. The eigenvalues are real.

**Theorem 8.** The eigenfunctions corresponding with distinct eigenvalues of fractional Bessel operator (15)-(16) are orthogonal weight function $w_\alpha$ on (0,1]; that is,

$$\int_0^1 w_\alpha(x) y_{\lambda_1}(x) y_{\lambda_2}(x) dx = 0, \lambda_1 \neq \lambda_2 \#(29)$$

Proof. We have by assumptions fractional Sturm-Liouville operator for Bessel type fulfilled by two different eigenvalues ( $\lambda_1, \lambda_2$ ) and the respective eigenfunctions $(y_{\lambda_1}, y_{\lambda_2})$:

$$\mathcal{L}_{\alpha[\mathcal{B}]} y_{\lambda_1}(x) + \lambda_1 w_\alpha(x) y_{\lambda_1}(x) = 0, \quad (30)$$

$$y_{\lambda_1}(x) = 0,$$

$$d_1 y_{\lambda_1}(1) + d_2 I_{1,-}^{1-\alpha} p(1) {}^c\mathcal{D}_{0,+}^{\alpha} y_{\lambda_1}(1) = 0, \quad (31)$$

$$\mathcal{L}_{\alpha[\mathcal{B}]} y_{\lambda_2}(x) + \lambda_2 w_\alpha(x) y_{\lambda_2}(x) = 0, \quad (32)$$

$$y_{\lambda_2}(x) = 0,$$

$$d_1 y_{\lambda_2}(1) + d_2 I_{1,-}^{1-\alpha} p(1) {}^c D_{0,+}^{\alpha} y_{\lambda_2}(1) = 0. \quad (33)$$

We multiply (30) by function $y_{\lambda_2}$ and (32) by function $y_{\lambda_1}$, respectively, and subtract.

$$(\lambda_1-\lambda_2)w_\alpha(x)y_{\lambda_1}y_{\lambda_2}=y_{\lambda_1}\mathcal{L}_{\alpha[B]}y_{\lambda_2}-y_{\lambda_2}\mathcal{L}_{\alpha[B]}y_{\lambda_1}.\#(34)$$

Integrating over interval ( 0,1 ] and applying relation (21) we note that the right-hand side of the integrated equality contains only boundary terms:

$$(\lambda_1-\lambda_2)\int_0^1 w_\alpha(x)y_{\lambda_1}(x)y_{\lambda_2}(x)dx=\int_0^1 y_{\lambda_1}(x)\mathcal{L}_{\alpha[B]}y_{\lambda_2}(x)dx$$

$$-\int_0^1 y_{\lambda_2}(x)\mathcal{L}_{\alpha[B]}y_{\lambda_1}(x)dx$$

$$=\int_0^1 y_{\lambda_1}(x)\left[\mathcal{D}_{1,-}^{\alpha}p(x){}^c\mathcal{D}_{0,+}^{\alpha}y_{\lambda_2}(x)+\left(q(x)-\frac{v^2-1/4}{x^2}\right)y_{\lambda_2}(x)\right]dx$$

$$-\int_0^1 y_{\lambda_2}(x)\left[\mathcal{D}_{1,-}^{\alpha}p(x){}^c\mathcal{D}_{0,+}^{\alpha}y_{\lambda_1}(x)+\left(q(x)-\frac{v^2-1/4}{x^2}\right)y_{\lambda_1}(x)\right]dx$$

$$=-y_{\lambda_1}(x)I_{1,-}^{1-\alpha}p(x){}^c\mathcal{D}_{0,+}^{\alpha}y_{\lambda_2}(x)\Big|_1+y_{\lambda_1}(x)I_{1,-}^{1-\alpha}p(x){}^c\mathcal{D}_{0,+}^{\alpha}y_{\lambda_2}(x)\Big|_0$$

$$+y_{\lambda_2}(x)I_{1,-}^{1-\alpha}p(x){}^C\mathcal{D}_{0,+}^{\alpha}y_{\lambda_1}(x)\Big|_1-y_{\lambda_2}(x)I_{1,-}^{1-\alpha}p(x){}^c\mathcal{D}_{0,+}^{\alpha}y_{\lambda_1}(x)\Big|_0. \quad (35)$$

Using the boundary conditions (31), (33), we obtain that

$$(\lambda_1-\lambda_2)\int_0^1 w_\alpha(x)y_{\lambda_1}(x)y_{\lambda_2}(x)dx=0,\#(36)$$

where $\lambda_1\neq\lambda_2$. Then, the eigenfunctions are orthogonal of this operator.

**Remark 9.** Let us now give certain auxiliary functions. Because we use the functions, the first of them is as follows:

$$I_{0,+}^{\alpha}\frac{(1-x)^{\alpha-1}}{\Gamma(\alpha)}=(1-0)^{\alpha-1}(x-0)^{\alpha}\,{}_1\Psi_2\left[\begin{matrix}(1,1) & \square\\ (\alpha,-1) & (\alpha+1,1)\end{matrix}\middle|-\frac{x-0}{1-0}\right], \quad (37)$$

where ${}_1\Psi_2$ is the Fox-Wright function [14]:

$${}_1\Psi_2\left[\begin{matrix}(a_1,\alpha_1) & \square\\ (b_1,\beta_1) & (b_2,\beta_2)\end{matrix}\middle\| z\right]=\sum_{h=0}^{\infty}\frac{\Gamma(a_1+\alpha_1 h)}{\Gamma(b_1+\beta_1 k)\Gamma(b_2+\beta_2 h)}\frac{z^h}{h!}. \quad (38)$$

The properties of the function are determined by the parameters

$$\Delta=\beta_1+\beta_2-\alpha_1=-1,$$

$$\delta=|\alpha_1|^{-\alpha_1}|\beta_1|^{\beta_1}|\beta_2|^{\beta_2}=1, \quad (39)$$

$$\mu=b_1+b_2-\alpha_1+\frac{1-2}{2}=2\alpha-\frac{1}{2}.$$

Considering Theorem 4, we note that this function is continuous in $(0,1]$ when order $\alpha > 1/2$, that is, $\mu > 1/2$. For $0 < \alpha \leq 1/2$; it is discontinuous at end $x = 1$. The explicitly calculated function allows to estimate the second component of stationary function $\phi_0$ of the differential part of SturmLiouville operator

$$\mathcal{D}_{1,-}^{\alpha}\, p(x)\, {}^{c}\mathcal{D}_{0,+}^{\alpha}\phi_0(x) = 0\#(40)$$

which looks as follows:

$$\phi_0(x) = \xi_1 + \xi_2 I_{0,+}^{\alpha}\frac{(1-x)^{\alpha-1}}{\Gamma(\alpha)p(x)} = \xi_1 + \xi_2\psi(\alpha,0,x).\#(41)$$

The next function is the following integral:

$$\varphi(x) = I_{0,+}^{\alpha} I_{1,-}^{\alpha} 1 = I_{0,+}^{\alpha}\frac{(1-x)^{\alpha}}{\Gamma(\alpha+1)}$$
$$= (1-0)^{\alpha}(x-0)^{\alpha} \quad (42)$$
$$\times\ {}_1\Psi_2\left[\begin{matrix} (1,1) & \square \\ (\alpha+1,-1) & (\alpha+1,1)\end{matrix}\middle| -\frac{x-0}{1-0}\right].$$

Again, using Theorem 4 and calculating parameters according to (39),

$$\Delta = -1, \quad \delta = 1, \quad \mu = 2\alpha + \frac{1}{2}.\#(43)$$

Finally,

$$\alpha > 0 \Longrightarrow \mu > \frac{1}{2},\#(44)$$

and the obtained Fox-Wright function (42) is continuous in interval $(0,1]$ for any positive order $\alpha$.

**Theorem 10.** Let $\alpha > 1/2, x \in (0,1]$ and define

$$\mathcal{Y}_\lambda(y) = \left(q(x) - \frac{v^2 - \frac{1}{4}}{x^2}\right) y_\lambda(x) + \lambda w_\alpha y_\lambda(x),$$
$$\widetilde{\Delta} = d_2 + d_1\psi(\alpha,0,1). \quad (45)$$

Assume that $\widetilde{\Delta} \neq 0$. Then, (15)-(16) are equivalent to the integral equation

$$y_\lambda(x) = -I_{0,+}^{\alpha}\frac{1}{p(x)} I_{1,-}^{\alpha}\mathcal{Y}_\lambda(y) + \mathcal{A}(x)\left(I_{0,+}^{\alpha}\frac{1}{p(x)} I_{1,-}^{\alpha}\mathcal{Y}_\lambda(y)\right)\Bigg|_{x=1}, \quad (46)$$

where the coefficient $\mathcal{A}(x)$ is

$$\mathcal{A}(x) = \frac{d_1}{\widetilde{\Delta}}\psi(\alpha,0,x)\#(47)$$

and functions $\psi$ are defined in (41).

Proof. By means of composition rules, (15) can be rewritten as follows:

$$\mathcal{D}_{1,-}^{\alpha} \wp(x) \, {}^{c}\mathcal{D}_{0,+}^{\alpha} \left[ y_{\lambda}(x) + I_{0,+}^{\alpha} \frac{1}{\wp(x)} I_{1,-}^{\alpha} \mathcal{Y}_{\lambda}(y) \right] = 0. \#(48)$$

The last equality suggests that is a stationary function of fractional singular Sturm-Liouville problem for Bessel operator. $\mathcal{D}_{1,-}^{\alpha} \wp(x) \, {}^{c}\mathcal{D}_{0,+}^{\alpha}$ which according to (41) can be found as

$$\phi_0 = \xi_1 + \xi_2 I_{0,+}^{\alpha} \frac{(1-x)^{\alpha-1}}{\Gamma(\alpha)\wp(x)} = \xi_1 + \xi_2 \psi(\alpha, 0, x). \#(49)$$

Equation (15) in the form of

$$y_{\lambda}(x) + I_{0,+}^{\alpha} \frac{1}{\wp(x)} I_{1,-}^{\alpha} \mathcal{Y}_{\lambda}(y) = \xi_1 + \xi_2 \psi(\alpha, 0, x) \#(50)$$

proves we should connect coefficients $\xi_j$ values $d_j, j = 1,2$ determining the boundary conditions (16).

Let us note that the following formula results from composition rules (11) and (50):

$$I_{1,-}^{1-\alpha} \wp(x) \, {}^{c}D_{0,+}^{\alpha} y_{\lambda}(x) = -I_{1,-}^{1} \mathcal{Y}_{\lambda}(y) + \xi_2. \#(51)$$

For continuous function $y_{\lambda}$, we obtain the following values as the ends

$$I_{1,-}^{1-\alpha} \wp(x) \, {}^{c}D_{0,+}^{\alpha} y_{\lambda}(x) \Big|_{x=0} = -\int_0^{\pi} \mathcal{Y}_{\lambda}(y) + \xi_2,$$

$$I_{1,-}^{1-\alpha} \wp(x) {}^{c}D_{0,+}^{\alpha} y_{\lambda}(x) \Big|_{x=1} = \xi_2, \qquad (52)$$

respectively, for $y_{\lambda}$. Using (50), we find

$$y_{\lambda}(0) = \phi_0(0) = \xi_1$$

$$y_{\lambda}(1) = \phi_0(1) - I_{0,+}^{\alpha} \frac{1}{\wp(x)} I_{1,-}^{\alpha} \mathcal{Y}_{\lambda}(y) \Big|_{x=1} \qquad (53)$$

$$= \xi_1 + \xi_2 \psi(\alpha, 0,1) - I_{0,+}^{\alpha} \frac{1}{\wp(x)} I_{1,-}^{\alpha} \mathcal{Y}_{\lambda}(y) \Big|_{x=1}.$$

The following set of linear equations for coefficients $\xi_j$ results from (52)-(54)

$$\xi_1 = 0,$$

$$d_1 \xi_1 + \xi_2 (d_2 + d_1 \psi(\alpha, 0,1)) = d_1 \mathcal{F}, \qquad (54)$$

where $\mathcal{F} = I_{0,+}^{\alpha} (1/\wp(x)) I_{1,-}^{\alpha} \mathcal{Y}_{\lambda}(y) \Big|_{x=1}$.

Since $\widetilde{\Delta} \neq 0$, the solution for coefficients $\xi_j (j = 1,2)$ is unique:

$$\xi_1 = 0,$$

$$\xi_2 = \frac{d_1 \mathcal{F}}{\tilde{\Delta}}, \quad (55)$$

Substituting the previous solution into (50) we recover the equivalent integral equation (46).

Furthermore, we give notation such as

$$m_{\wp} = \min_{x \in [0,1]} |\wp(x)|,$$

$$\mathcal{A} = \|\mathcal{A}(x)\|, M_{\varphi} = \|\varphi(x)\|. \quad (56)$$

The proof is completed.

## 4. Conclusion

We have generalized the spectral concept of the singular fractional Sturm–Liouville problems in this paper. The eigenvalues for the Bessel-type operator with some boundary conditions are shown to be real and the corresponding eigenfunctions of distinct eigen-values form an orthogonal system. In addition, self-adjointness of the fractional Bessel operator is established, and therefore its spectral decomposition is complete and stable. Furthermore, the classical results of Sturm–Liouville theory are generalized to fractional setting, stressing the importance and generous use refers to these results in establishing the theoretical development of fractional Sturm–Liouville analysis.

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