Semi-Analytical Treatment of a Coupled Advection–Diffusion–Reaction System for Spatiotemporal Drug Therapy Modeling via the Triple Laplace Transform and Adomian Decomposition

Authors Names	**ABSTRACT**
Azhar Mohamed Hajo [a] ***Ahmed Farooq Qasim*** [b] *Publication date: 6 / 6/2026* *Keywords: Triple Laplace transform; Adomian Decomposition Method; Nonlinear PDEs; Semi-analytical method*	This research presents an approximate method for solving a medical application system using the concept of load, diffusion, and interaction, where the system depicts the concentrations of the drug, diseased cells, and healthy cells in a given tissue. Inspired by biology, this approach utilizes the triple Laplace transform to convert the problem into an algebraic form within the transformation domain. Nonlinear limits are handled using the Adomian formula, an iterative method that allows for the construction of a solution sequence. After obtaining the general load formula, three arbitrary cases of functions that satisfy the system without residues are selected for numerical testing. The results demonstrate the validity of the method as a tool for generating solution formulas for medical systems, provided a precise solution is available.

1.Introduction

This research deals with a medical system of four equations of the type of load, diffusion, and interaction that describe the evolution of the concentrations of two drugs d1 d2 and their effect on two types of cells, one diseased p and the other healthy q. The model includes the transport limit $\nabla \cdot (u_i d_i)$ and diffusion $(D_i \nabla d_i)$and the time-dependent source of drug injection $B(t)$ as well as the growth limits Fp Fq and other limits representing the effect of the two drugs represented by the symbols Cp Cq [7-10]. On the other hand, the solution method depends on two consistent paths, one of which is an Adomian approach to deal with the nonlinear terms, where the solution is represented as a series $\sum_{n\geq 0} u_n$ [3-5], and the other path is the triple Laplace transform, which in turn works to transform the differential terms into algebraic terms in the transformation domain for ease of handling [1,2,6].

Motivated by these considerations, this work proposes a semi-analytical approach for solving the aforementioned four-equation system by combining the triple Laplace transform with ADM to produce explicit recursive sequences for the solution components. To systematically verify the correctness of the implementation, we employed three manufactured-solution test cases embedded in Maple codes, for which the equation residuals vanish symbolically upon substitution, together with pointwise numerical tests demonstrating the decay of truncation error for finite approximations (e.g., ., $d_{1.0} + d_{1,1} + d_{1,2}$and $p_0 + p_1 + p_2$). Accordingly, the proposed framework offers a reproducible setting for generating reference solutions for verification and for supporting sensitivity and calibration studies in spatiotemporal pharmacotherapy models.

[a] *Department of Mathematics, faculty of education for women, University of Kufa, 540011 Najaf, Iraq.*
E-Mail: iqbals.alattabi@student.uokufa.edu.iq
[b] Department of Mathematics, faculty of education for women, University of Kufa, 540011 Najaf, Iraq.
E-Mail: asmhanf.alzuhairy@uokufa.edu.iq

2. The model is:[16]

$$\begin{aligned}
&\frac{\partial d_1}{\partial t} + \nabla(u_1 d_1) = D_1(\Delta\, d_1) + \Gamma_1(\, d_1 B(t) - d_1) - \lambda_1 d_1\\
&\frac{\partial d_2}{\partial t} + \nabla(u_2 d_2) = D_2(\Delta\, d_2) + \Gamma_2(d_2 B(t) - d_2) - \lambda_2 d_2\\
&\frac{\partial P}{\partial t} + \nabla\big((u_1 + u_2)p\big) = M_p\, \Delta\, p + F_p(p) - C_p(d_1, d_2, p)\\
&\frac{\partial q}{\partial t} + \nabla\big((u_1 + u_2)q\big) = M_q\, \Delta\, q + F_q(q) - C_q(d_1, d_2, q)
\end{aligned} \quad (1)$$

Variables: $d1\,, d2$ Concentration of two drugs; p Density of tumour cells; q Density of sensitive healthy cells. Fields $u1, u2$ represent load/flow; $Di, \Gamma i, \lambda i$ represent diffusion, blood-tissue exchange, and decay, respectively. The functions , Mp,Mq: cell motility $F\ and\ C$ describe self-replication and drug inhibition.

3. Accelerated Adomian decomposition method (AADM):

The Adomian method is a systematic analytical framework for solving nonlinear differential equations by decomposing both the solution and the nonlinearity into two convergent sequences. We write the solution [11]

$$u = \sum_{n=0}^{\infty} u_n \quad (2)$$

and the nonlinearity

$$N(u) = \sum_{n=0}^{\infty} A_n \quad (3)$$

where A_n are Adomian polynomials. These polynomials are generated by a precise generator formula [12]:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N\left(\sum_{k=o}^{\infty} \lambda^k u_k\right)\Big|_{\lambda=0} \quad (4)$$

By reformulating the linear operator equation

$$Lu = N(u) - Ru - g$$

the limits of the series u_n are determined iteratively through successive linear problems that ensure consistency with the nonlinear decomposition above.

To speed up convergence and reduce computational cost, Kalle uses partial sums for the solution

$$s_n = \sum_{i=0}^{n} u_i \quad (5)$$

The polynomials are calculated directly from these sums. The recursive formula attached to the image is given as [13,14]:

$$A_n = N(s_n) - \sum_{i=0}^{n-1} A_i \quad (6)$$

This formula preserves the identity $N(u) = \sum_{n=0}^{\infty} A_n$ and improves convergence.

This approach reduces the number of steps and calculations, thus accelerating convergence. Using the Adomian method alone and the Olplace method alone demonstrates a difference in efficiency, while combining them enhances the efficiency and stability of the solution, easily and efficiently handling nonlinear terms. This makes this combination the best option for solving nonlinear differential equations.

4. Triple Laplace transform

The Laplace transform is applied to three-dimensional partial differential equations (one time dimension and two spatial dimensions), where the transformation scope is extended to three variables: s, p, and q.. It is defined for a function $f(x,y,t)$ by the formula [15]:

$$F(p,q,s) = \iiint_0^\infty e^{-px}e^{-qy}e^{-st}\, f(x,y,t)dxdydt \quad (7)$$

By shifting this transformation from differential representation to algebraic preference, finding solutions becomes easier. Therefore, the triplet transformation of the APLAS is a powerful and efficient tool for solving complex differential equations and initial and boundary value problems, and combining it with the ADOMEAN transformation for handling nonlinear boundary conditions accelerates the search for solutions.

Derivatives for the Triple Laplace Transform Starting with the one-dimensional Laplace transform in time only, treating x, y as constant during integration:

$$\mathcal{L}_t\{u_t(x,y,t)\} = \int_0^\infty e^{-st}\ \frac{\partial u}{\partial t}(x,y,t)\, dt \quad (8)$$

Use integration by parts for equation (10):

$$\mathcal{L}_t\{u_t\} = s\,\mathcal{L}_t\{u\} - u(x,y,0) \quad (9)$$

Now apply Laplace transform in x, y to both sides (linearity):

$$\mathcal{L}_{x,y,t}\{u_t\} = s\,U(p,q,s) - \mathcal{L}_{x,y}\{u(x,y,0)\} \quad (10)$$

Let:

$$U_0(p,q) := \mathcal{L}_{x,y}\{u(x,y,0)\} \quad (11)$$

So, the final rule is:

$$\mathcal{L}_{x,y,t}\{u_t\} = s\,U(p,q,s) - U_0(p,q) \quad (12)$$

Repeat the same idea, or use the known rule:

$$\mathcal{L}_t\{u_{tt}\} = s^2\mathcal{L}_t\{u\} - s\,u(x,y,0) - u_t(x,y,0) \quad (13)$$

Then apply Laplace transform in x, y:

$$\mathcal{L}_{x,y,t}\{u_{tt}\} = s^2U(p,q,s) - s\,U_0(p,q) - U_1(p,q) \quad (14)$$

were

$$U_1(p,q) \coloneqq \mathcal{L}_{x,y}\{u_t(x,y,0)\} \quad (15)$$

The general rule (fundamental in all Laplace applications) is:

$$\mathcal{L}_t\left\{\frac{\partial^n u}{\partial t^n}\right\} = s^n\mathcal{L}_t\{u\} - \sum_{k=0}^{n-1} s^{\,n-1-k}\ \frac{\partial^k u}{\partial t^k}(x,y,0) \tag{16}$$

Including Laplace transform in x, y:

$$\mathcal{L}_{x,y,t}\left\{\frac{\partial^n u}{\partial t^n}\right\} = s^n U(p,q,s) - \sum_{k=0}^{n-1} s^{\,n-1-k}\ U_k(p,q) \tag{17}$$

where

$$U_k(p,q) := \mathcal{L}_{x,y}\left\{\frac{\partial^k u}{\partial t^k}(x,y,0)\right\} \tag{18}$$

5.Verification for Special-Case Tests and Triple Laplace Transform Outputs

The aim of this research is to find solutions specific to the hypotheses of functions that satisfy the four equations without remainders used to verify the mathematical formulation of the solution for the medical system and to ensure the soundness of the applied mathematical work. The selected special cases provide a precise criterion for this.

5.1 Derivation of the special-case construction and verification

We expand (rewrite) the system (1) to obtain the following form:

$$\frac{\partial d1}{\partial t} + u_1\left(\frac{\partial}{\partial x}(d_1) + \frac{\partial}{\partial y}(d_1)\right) + d_1\left(\frac{\partial}{\partial x}(u_1) + \frac{\partial}{\partial y}(u_1)\right) = D_1\left(\frac{\partial^2 d_1}{\partial x^2} + \frac{\partial^2 d_1}{\partial y^2}\right) + \frac{\partial}{\partial x}D_1\frac{\partial}{\partial x}(d_1) + \frac{\partial}{\partial y}D_1\frac{\partial}{\partial y}(d_1) + \Gamma_1(d_{1B}(\mathrm{t}) - d_1) - \lambda_1 d_1$$

$$\frac{\partial d2}{\partial t} + u_1\left(\frac{\partial}{\partial x}(d_2) + \frac{\partial}{\partial y}(d_2)\right) + d_2\left(\frac{\partial}{\partial x}(u_2) + \frac{\partial}{\partial y}(u_2)\right) = D_2\left(\frac{\partial^2 d_2}{\partial x^2} + \frac{\partial^2 d_2}{\partial y^2}\right) + \frac{\partial}{\partial x}D_2\frac{\partial}{\partial x}(d_2) + \frac{\partial}{\partial y}D_2\frac{\partial}{\partial y}(d_2) + \Gamma_2(d_{2B}(\mathrm{t}) - d_2) - \lambda_2 d_2$$

$$\frac{\partial P}{\partial t} + (u_1 + u_2)\left(\frac{\partial}{\partial x}(p) + \frac{\partial}{\partial y}(p)\right) + p\left(\frac{\partial}{\partial x}\big((u_1 + u_2)\big) + \frac{\partial}{\partial y}\big((u_1 + u_2)\big)\right) = M_P\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) + F_p(p) - (c_{p,1} + c_{p,2})$$

$$\frac{\partial q}{\partial t} + (u_1 + u_2)\left(\frac{\partial}{\partial x}(q) + \frac{\partial}{\partial y}(q)\right) + q\left(\frac{\partial}{\partial x}\big((u_1 + u_2)\big) + \frac{\partial}{\partial y}\big((u_1 + u_2)\big)\right) = M_q\left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2}\right) + F_q(q) - (c_{q,1} + c_{q,2}) \tag{19}$$

Solve the model using the triple Laplace transform and the improved Adomian method by take triple laplace to both sides of each equation to get system:

$$L_{x,y,t}\left\{\frac{\partial d1}{\partial t} + u_1\left(\frac{\partial}{\partial x}(d_1) + \frac{\partial}{\partial y}(d_1)\right) + d_1\left(\frac{\partial}{\partial x}(u_1) + \frac{\partial}{\partial y}(u_1)\right)\right\} = L_{x,y,t}\{D_1\left(\frac{\partial^2 d_1}{\partial x^2} + \frac{\partial^2 d_1}{\partial y^2}\right) + \frac{\partial}{\partial x}D_1\frac{\partial}{\partial x}(d_1) + \frac{\partial}{\partial y}D_1\frac{\partial}{\partial y}(d_1) + \Gamma_1(d_{1B}(\mathrm{t}) - d_1) - \lambda_1 d_1\}$$

$$L_{x,y,t}\left\{\frac{\partial d2}{\partial t}+u_1\left(\frac{\partial}{\partial x}(d_2)+\frac{\partial}{\partial y}(d_2)\right)+d_2\left(\frac{\partial}{\partial x}(u_2)+\frac{\partial}{\partial y}(u_2)\right)\right\}=L_{x,y,t}\{D_2\left(\frac{\partial^2 d_2}{\partial x^2}+\frac{\partial^2 d_2}{\partial y^2}\right)+\frac{\partial}{\partial x}D_2\frac{\partial}{\partial x}(d_2)+\frac{\partial}{\partial y}D_2\frac{\partial}{\partial y}(d_2)+\Gamma_2(d_{2B}(\mathrm{t})-d_2)-\lambda_2 d_2\}$$

$$L_{x,y,t}\left\{\frac{\partial P}{\partial t}+(u_1+u_2)\left(\frac{\partial}{\partial x}(p)+\frac{\partial}{\partial y}(p)\right)+p\left(\frac{\partial}{\partial x}((u_1+u_2))+\frac{\partial}{\partial y}((u_1+u_2))\right)\right\}=L_{x,y,t}\{M_P(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2})+F_p(p)-(c_{p,1}+c_{p,2})\}$$

$$L_{x,y,t}\{\frac{\partial q}{\partial t}+(u_1+u_2)\left(\frac{\partial}{\partial x}(q)+\frac{\partial}{\partial y}(q)\right)+q\left(\frac{\partial}{\partial x}((u_1+u_2))+\frac{\partial}{\partial y}((u_1+u_2))\right)\}=L_{x,y,t}\{M_q(\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial y^2})+F_q(q)-(c_{q,1}+c_{q,2})\} \quad (20)$$

Using (12) to get system (21)

$$s_3\, d1(s_1,s_2,s_3)-d1(s_1,s_2,0)=L_{x,y,t}\left\{-\{u_1\left(\frac{\partial}{\partial x}(d_1)+\frac{\partial}{\partial y}(d_1)\right)+d_1\left(\frac{\partial}{\partial x}(u_1)+\frac{\partial}{\partial y}(u_1)\right)\}\right\}+L_{x,y,t}\{D_1\left(\frac{\partial^2 d_1}{\partial x^2}+\frac{\partial^2 d_1}{\partial y^2}\right)+\frac{\partial}{\partial x}D_1\frac{\partial}{\partial x}(d_1)+\frac{\partial}{\partial y}D_1\frac{\partial}{\partial y}(d_1)+\Gamma_1(d_{1B}(\mathrm{t})-d_1)-\lambda_1 d_1\}$$

$$s_3\, d2(s_1,s_2,s_3)-d2(s_1,s_2,0)=L_{x,y,t}\left\{-\{u_1\left(\frac{\partial}{\partial x}(d_2)+\frac{\partial}{\partial y}(d_2)\right)+d_2\left(\frac{\partial}{\partial x}(u_2)+\frac{\partial}{\partial y}(u_2)\right)\}\right\}+L_{x,y,t}\{D_2\left(\frac{\partial^2 d_2}{\partial x^2}+\frac{\partial^2 d_2}{\partial y^2}\right)+\frac{\partial}{\partial x}D_2\frac{\partial}{\partial x}(d_2)+\frac{\partial}{\partial y}D_2\frac{\partial}{\partial y}(d_2)+\Gamma_2(d_{2B}(\mathrm{t})-d_2)-\lambda_2 d_2\}$$

$$s_3\, p(s_1,s_2,s_3)-p(s_1,s_2,0)=L_{x,y,t}\left\{-\{(u_1+u_2)\left(\frac{\partial}{\partial x}(p)+\frac{\partial}{\partial y}(p)\right)+p\left(\frac{\partial}{\partial x}((u_1+u_2))+\frac{\partial}{\partial y}((u_1+u_2))\right)\}-\frac{\partial}{\partial y}\left((u_{1,y}+u_{2,y})p\right)\right\}+L_{x,y,t}\{M_P(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2})+F_p(p)-(c_{p,1}+c_{p,2})\}$$

$$s_3\, q(s_1,s_2,s_3)-q(s_1,s_2,0)=L_{x,y,t}\left\{-\{(u_1+u_2)\left(\frac{\partial}{\partial x}(q)+\frac{\partial}{\partial y}(q)\right)+q\left(\frac{\partial}{\partial x}((u_1+u_2))+\frac{\partial}{\partial y}((u_1+u_2))\right)\}-\frac{\partial}{\partial y}\left((u_{1,y}+u_{2,y})q\right)\right\}+L_{x,y,t}\{M_q(\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial y^2})+F_q(q)-(c_{q,1}+c_{q,2})\} \quad (21)$$

Take triple laplace inverse for system (21) and solve for d1 ,d2,p,q to get system:

$$d1(x,y,t)=L^{-1}{}_{x,y,t}\{\frac{1}{s_3}\{d1(s_1,s_2,0)+L_{x,y,t}\left\{-\left\{\sum_{n=0}^{\infty}(u_1\left(\frac{\partial}{\partial x}(d_{1,n})+\frac{\partial}{\partial y}(d_{1,n})\right)+d_{1,n}\left(\frac{\partial}{\partial x}(u_1)+\frac{\partial}{\partial y}(u_1)\right))\right\}\right\}+L_{x,y,t}\{\sum_{n=0}^{\infty}(D_1\left(\frac{\partial^2 d_{1,n}}{\partial x^2}+\frac{\partial^2 d_{1,n}}{\partial y^2}\right)+\frac{\partial}{\partial x}D_1\frac{\partial}{\partial x}(d_{1,n}))+\frac{\partial}{\partial y}D_1\frac{\partial}{\partial y}(d_{1,n})+\Gamma_1(d_{1B}(\mathrm{t})-d_{1,n})-\lambda_1 d_{1,n})\ \}\}$$

$$d2(x,y,t) = L^{-1}{}_{x,y,t}\{\frac{1}{s_3}\{d2(s_1,s_2,0) + L_{x,y,t}\Big\{-\Big\{\sum_{n=0}^{\infty}(u_1\left(\frac{\partial}{\partial x}(d_{2,n}) + \frac{\partial}{\partial y}(d_{2,n})\right) + d_{2,n}\left(\frac{\partial}{\partial x}(u_2) + \frac{\partial}{\partial y}(u_2)\right))\Big\}\Big\} + L_{x,y,t}\{\sum_{n=0}^{\infty}(D_2\left(\frac{\partial^2 d_{2,n}}{\partial x^2} + \frac{\partial^2 d_{2,n}}{\partial y^2}\right) + \frac{\partial}{\partial x}D_2\frac{\partial}{\partial x}(d_{2,n})) + \frac{\partial}{\partial y}D_2\frac{\partial}{\partial y}(d_{2,n}) + \Gamma_2(d_{2B}(\mathrm{t}) - d_{2,n}) - \lambda_2 d_{2,n})\}\}\}$$

$$p(x,y,t) = L^{-1}{}_{x,y,t}\{\frac{1}{s_3}\Big\{p(s_1,s_2,0) + L_{x,y,t}\Big\{-\{\sum_{n=0}^{\infty}((u_1 + u_2)\left(\frac{\partial}{\partial x}(p_n) + \frac{\partial}{\partial y}(p_n)\right) + p_n\left(\frac{\partial}{\partial x}((u_1 + u_2)) + \frac{\partial}{\partial y}((u_1 + u_2))\right))\Big\} + L_{x,y,t}\{\sum_{n=0}^{\infty}(M_P(\frac{\partial^2 p_n}{\partial x^2} + \frac{\partial^2 p_n}{\partial y^2}) + F_p(p) - (c_{p,1} + c_{p,2}))\}\Big\}\}$$

$$q(x,y,t) = L^{-1}{}_{x,y,t}\{\frac{1}{s_3}\{q(s_1,s_2,0) + L_{x,y,t}\Big\{-\{\sum_{n=0}^{\infty}((u_1 + u_2)\left(\frac{\partial}{\partial x}(q_n) + \frac{\partial}{\partial y}(q_n)\right) + q_n\left(\frac{\partial}{\partial x}((u_1 + u_2)) + \frac{\partial}{\partial y}((u_1 + u_2))\right))\}\Big\} + L_{x,y,t}\{\sum_{n=0}^{\infty}(M_q(\frac{\partial^2 q_n}{\partial x^2} + \frac{\partial^2 q_n}{\partial y^2}) + F_q(q) - (c_{q,1} + c_{q,2}))\}\}\} \quad (22)$$

Now defining the following formula in a recursive manner:

$$d_{1,0}(x,y,t) = L^{-1}{}_{x,y,t}\{\frac{1}{s_3}\{d1(s_1,s_2,0)\}\} \quad (23)$$

$$d_{1,n+1}(x,y,t) = L^{-1}{}_{x,y,t}\{\frac{1}{s_3}\{L_{x,y,t}\Big\{-\Big\{\sum_{n=0}^{\infty}(u_1\left(\frac{\partial}{\partial x}(d_{1,n}) + \frac{\partial}{\partial y}(d_{1,n})\right) + d_{1,n}\left(\frac{\partial}{\partial x}(u_1) + \frac{\partial}{\partial y}(u_1)\right))\Big\}\Big\} + L_{x,y,t}\{\sum_{n=0}^{\infty}(D_1\left(\frac{\partial^2 d_{1,n}}{\partial x^2} + \frac{\partial^2 d_{1,n}}{\partial y^2}\right) + \frac{\partial}{\partial x}D_1\frac{\partial}{\partial x}(d_{1,n}) + \frac{\partial}{\partial y}D_1\frac{\partial}{\partial y}(d_{1,n})) + \Gamma_1(d_{1B}(\mathrm{t}) - d_{1,n}) - \lambda_1 d_{1,n})\}\} \quad (24)$$

$$d_{2,0}(x,y,t) = L^{-1}{}_{x,y,t}\{\frac{1}{s_3}\{d2(s_1,s_2,0)\}\} \quad (25)$$

$$d_{2,n+1}(x,y,t) = L^{-1}{}_{x,y,t}\{\frac{1}{s_3}\{L_{x,y,t}\Big\{-\Big\{\sum_{n=0}^{\infty}(u_1\left(\frac{\partial}{\partial x}(d_{2,n}) + \frac{\partial}{\partial y}(d_{2,n})\right) + d_{2,n}\left(\frac{\partial}{\partial x}(u_2) + \frac{\partial}{\partial y}(u_2)\right))\Big\}\Big\} + L_{x,y,t}\{\sum_{n=0}^{\infty}(D_2\left(\frac{\partial^2 d_{2,n}}{\partial x^2} + \frac{\partial^2 d_{2,n}}{\partial y^2}\right) + \frac{\partial}{\partial x}D_2\frac{\partial}{\partial x}(d_{2,n}) + \frac{\partial}{\partial y}D_2\frac{\partial}{\partial y}(d_{2,n})) + \Gamma_2(d_{2B}(\mathrm{t}) - d_{2,n}) - \lambda_2 d_{2,n})\}\}\} \quad (26)$$

$$p_0(x,y,t) = L^{-1}{}_{x,y,t}\{\frac{1}{s_3}\{p(s_1,s_2,0)\}\} \quad (27)$$

$$p_{n+1}(x,y,t) = L^{-1}{}_{x,y,t}\{\frac{1}{s_3}\left\{L_{x,y,t}\left\{-\{\sum_{n=0}^{\infty}((u_1+u_2)\left(\frac{\partial}{\partial x}(p_n)+\frac{\partial}{\partial y}(p_n)\right)+p_n\left(\frac{\partial}{\partial x}((u_1+u_2))+\frac{\partial}{\partial y}((u_1+u_2))\right)\right)\right\}+L_{x,y,t}\{\sum_{n=0}^{\infty}(M_P(\frac{\partial^2 p_n}{\partial x^2}+\frac{\partial^2 p_n}{\partial y^2})+F_p(p)-(c_{p,1}+c_{p,2}))\}\right\}\} \quad (28)$$

$$q_0(x,y,t) = L^{-1}{}_{x,y,t}\{\frac{1}{s_3}\{q(s_1,s_2,0)\}\} \quad (29)$$

$$q_{n+1}(x,y,t) = L^{-1}{}_{x,y,t}\{\frac{1}{s_3}\{L_{x,y,t}\left\{-\{\sum_{n=0}^{\infty}((u_1+u_2)\left(\frac{\partial}{\partial x}(q_n)+\frac{\partial}{\partial y}(q_n)\right)+q_n\left(\frac{\partial}{\partial x}((u_1+u_2))+\frac{\partial}{\partial y}((u_1+u_2))\right)\right)\}\right\}+L_{x,y,t}\{\sum_{n=0}^{\infty}(M_q(\frac{\partial^2 q_n}{\partial x^2}+\frac{\partial^2 q_n}{\partial y^2})+F_q(q)-(c_{q,1}+c_{q,2}))\}\}\} \quad (30)$$

When

$$A_n = C_{p1} = d_1 p\ , B_n = C_{p2} = d_2 p\ ,\ G_n = C_{q1} = d_1 q\ , H_n = C_{q2} = d_2 q \quad (31)$$

$$A_0 = d_{1,0}p_0\ ,\ A_1 = d_{1,1}p_0 + d_{1,0}p_1 + d_{1,1}p_1\ ,\ A_2 = d_{1,2}p_0 + d_{1,2}p_1 + d_{1,0}p_2 + d_{1,1}p_2 + d_{1,2}p_2$$

$$B_0 = d_{2,0}p_0\ , B_1 = d_{2,1}p_0 + d_{2,0}p_1 + d_{2,1}p_1\ ,\ B_2 = d_{2,2}p_0 + d_{2,2}p_1 + d_{2,0}p_2 + d_{2,1}p_2 + d_{2,2}p_2$$

$$G_0 = d_{1,0}q_0\ ,\ G_1 = d_{1,1}q_0 + d_{1,0}q_1 + d_{1,1}q_1\ ,\ G_2 = d_{1,2}q_0 + d_{1,2}q_1 + d_{1,0}q_2 + d_{1,1}q_2 + d_{1,2}q_2$$

$$H_0 = d_{2,0}q_0\ , H_1 = d_{2,1}q_0 + d_{2,0}q_1 + d_{2,1}q_1\ ,\ H_2 = d_{2,2}q_0 + d_{2,2}q_1 + d_{2,0}q_2 + d_{2,1}q_2 + d_{2,2}q_2$$

5.2 General notation and unified setup for the cases

To unify the presentation across all cases, we adopt the following shorthand notation:

$S = x + y, E = e^{x+y}$

Each case is constructed by specifying closed-form expressions for the variables (d_1, d_2, p, q), and by defining the transport/advection coefficients (u_1, u_2), diffusion coefficients (D_1, D_2), and additional parameters $(k_1, k_2, \lambda_1, \lambda_2, M_p, M_q)$, as well as auxiliary coupling terms $(c_{p1}, c_{p2}, F_p, c_{q1}, c_{q2}, F_q)$, in a manner consistent with the adopted formulation and the implemented code.

5.3 Symbolic residual verification criterion

After substituting each special case into the system, residual expressions are formed for each equation and then simplified symbolically. A case is accepted if and only if the residuals of all equations are identically equal to zero (a symbolic identity), thereby confirming the correct assembly of transport and diffusion terms, coupling terms, and the definitions of the composite terms.

5.4 Documentation of the first components from the triple Laplace transform

After passing the symbolic verification, the case is used as a reference to validate the correct application of the triple Laplace transform and its inversion within the iterative scheme. This is done by comparing the first components produced by the symbolic execution (such as $d_{1,0}$, $d_{1,1}$, $d_{1,2}$, p_0, p_1, p_2) with the reference expressions recorded.Note that equations 1 and 2 will give the same results, as will equations 3 and 4; therefore, we will only present the results of equations 1 and 3 from the system.

5.5 Tables of special cases and outputs

This section provides two tables: (1) for defining the special cases, and (2) for documenting the first components produced by the triple Laplace scheme.

Table 1: Definition of the special-case solutions used for verification

Item	Case I	Case II	Case III
Definition of S, E	$S = x + y,\ E = e^{x+y}$	$S = x + y,\ E = e^{x+y}$	$S = x + y,\ E = e^{x+y}$
$d_1(x, y, t)$	$e^t \sin(s)$	$e^t E$	$e^t \cos(s)$
$d_2(x, y, t)$	$e^t \sin(s)$	$e^t E$	$e^t \cos(s)$
$p(x, y, t)$	$e^t \mathrm{E}$	$2e^t E$	$e^t E$
$q(x, y, t)$	$e^t E$	$-2e^t E$	$e^t \mathrm{E}$
$u_1(x, y), u_2(x, y)$	$u_1 = 2E,\ u_2 = 2E$	$u_1 = 3E,\ u_2 = 3E$	$u_1 = 2E,\ u_2 = 2E$
$D_1(x, y), D_2(x, y)$	$D_1 = 2E,\ D_2 = 2E$	$D_1 = 2E,\ D_2 = 2E$	$D_1 = 2E,\ D_2 = 2E$
$k_1(x, y), k_2(x, y)$	$k_1 = E,\ k_2 = E$	$k_1 = 0,\ k_2 = 0$	$k_1 = E,\ k_2 = E$
$\lambda_1(x, y), \lambda_2(x, y)$	$\lambda_1 = -(8E + 1),\ \lambda_2 = \lambda_1$	$\lambda_1 = -(4E + 1),\ \lambda_2 = \lambda_1$	$\lambda_1 = -(8E + 1),\ \lambda_2 = \lambda_1$
$M_p(x, y), M_q(x, y)$	$M_p = 7E + \frac{1}{2},\ M_q = M_p$	$M_p = 12E,\ M_q = 12E$	$M_p = 7E + \frac{1}{2},\ M_q = M_p$
d_{1B}, d_{2B}	$d_{1B} = d_1,\ d_{2B} = d_2$	$d_{1B} = d_1,\ d_{2B} = d_2$	$d_{1B} = d_1,\ d_{2B} = d_2$
c_{p1}, c_{p2}	$c_{p1} = E\,p,\ c_{p2} = E\sin(s)\,p$	$c_{p1} = E\,p,\ c_{p2} = E\,p$	$c_{p1} = E\,p,\ c_{p2} = E\cos(s)\,p$
F_p	$F_p = 2E\,p + c_{p1} + c_{p2}$	$F_p = c_{p1} + c_{p2} + p$	$F_p = 2E\,p + c_{p1} + c_{p2}$
c_{q1}, c_{q2}	$c_{q1} = E\,q,\ c_{q2} = E\sin(s)\,q$	$c_{q1} = E\,q,\ c_{q2} = E\,q$	$c_{q1} = E\,q,\ c_{q2} = E\cos(s)\,q$
F_q	$F_q = 2E\,q + c_{q1} + c_{q2}$	$F_q = c_{q1} + c_{q2} + q$	$F_q = 2E\,q + c_{q1} + c_{q2}$

Table 2: First components produced by the triple Laplace scheme

Case	$d_{1,0}(x,y)$	$d_{1,1}(x,y,t)$	$d_{1,2}(x,y,t)$	$p_0(x,y)$	$p_1(x,y,t)$	$p_2(x,y,t)$
Case I (sin)	$\sin(x+y)$	$t\sin(x+y)$	$\frac{1}{2}t^2\sin(x+y)$	e^{x+y}	te^{x+y}	$\frac{t^2e^{x+y}}{2}$
Case II (exp)	e^{x+y}	te^{x+y}	$\frac{1}{2}t^2e^{x+y}$	$2e^{x+y}$	$2te^{x+y}$	t^2e^{x+y}
Case III (cos)	$\cos(x+y)$	$t\cos(x+y)$	$\frac{1}{2}t^2\cos(x+y)$	e^{x+y}	te^{x+y}	$\frac{t^2e^{x+y}}{2}$

5.6 Numerical results for the special cases:

Table 3 (Case I (sin)): Abs error of d1 and p at t=0.1.

x	**y**	**Abs error d1**	**Abs error p**
0	0	0	0.000170918
0.1	0.1	0.0000339562	0.000208759
0.2	0.2	0.0000665586	0.000254980
0.3	0.3	0.0000965076	0.000311433

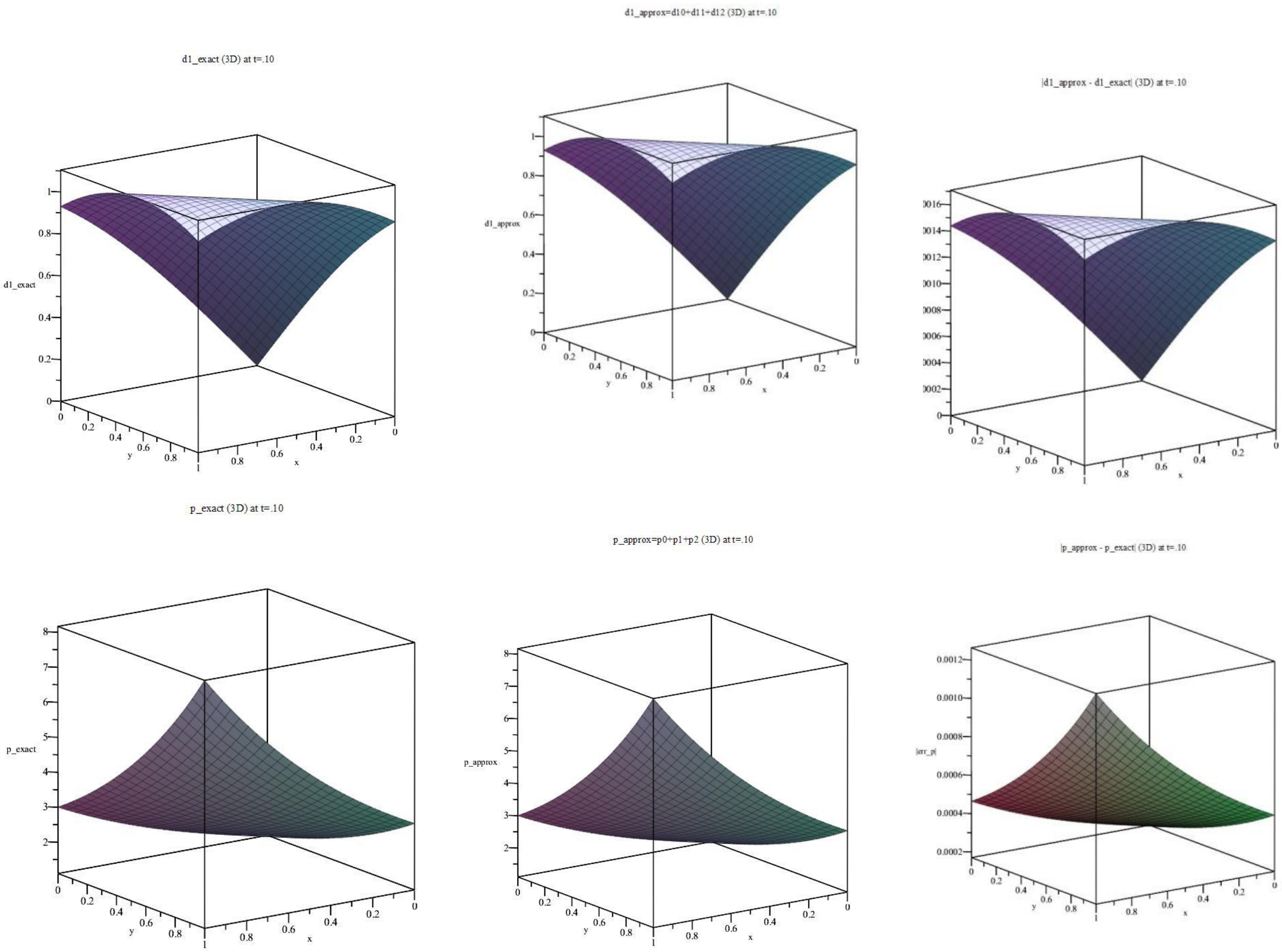
Table 4 (Case II (exp)): Abs error of d1 and p at t=0.1.

x	**y**	**Abs error d1**	**Abs error p**
0	0	0.000170918	0.000341836
0.1	0.1	0.000208759	0.000417521
0.2	0.2	0.000254980	0.000509959
0.3	0.3	0.000311433	0.000622866

Table 5(Case III (cos)): Abs error of d1 and p at t=0.1.

x	y	Abs error d1	Abs error p
0	0	0.000170918	0.000170918
0.1	0.1	0.000167511	0.000208759
0.2	0.2	0.000157426	0.000254980
0.3	0.3	0.0001410647	0.000311433

5.7 Graphical representation of numerical results



Case I :

Figure1: Three-dimensional diagrams showing both approximate and exact solutions and the magnitude of error for the two variables d1 and p for the first case.

Case II :

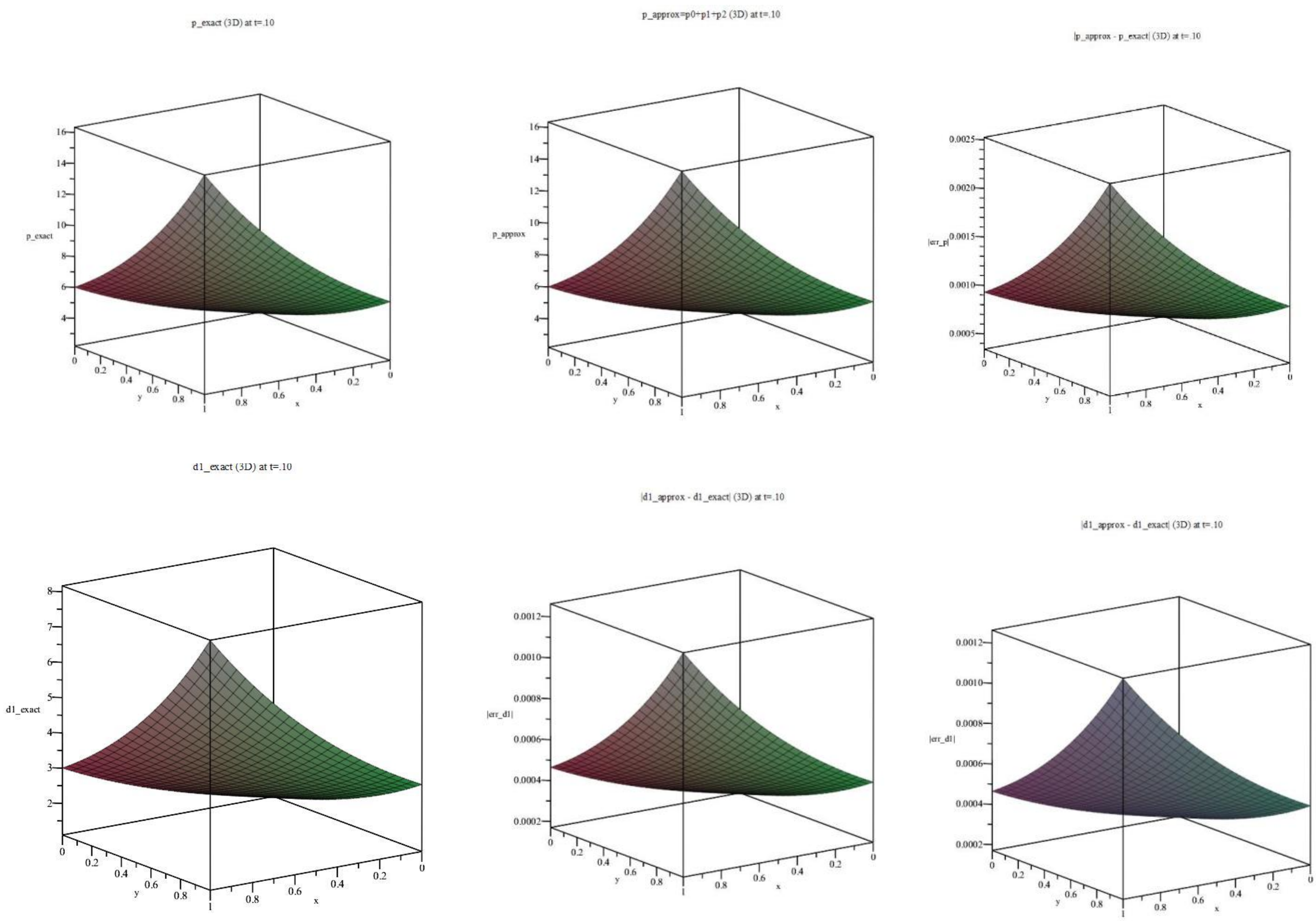


Figure2: Three-dimensional diagrams showing both approximate and exact solutions and the magnitude of error for the two variables d1 and p for the second case.

Case III :

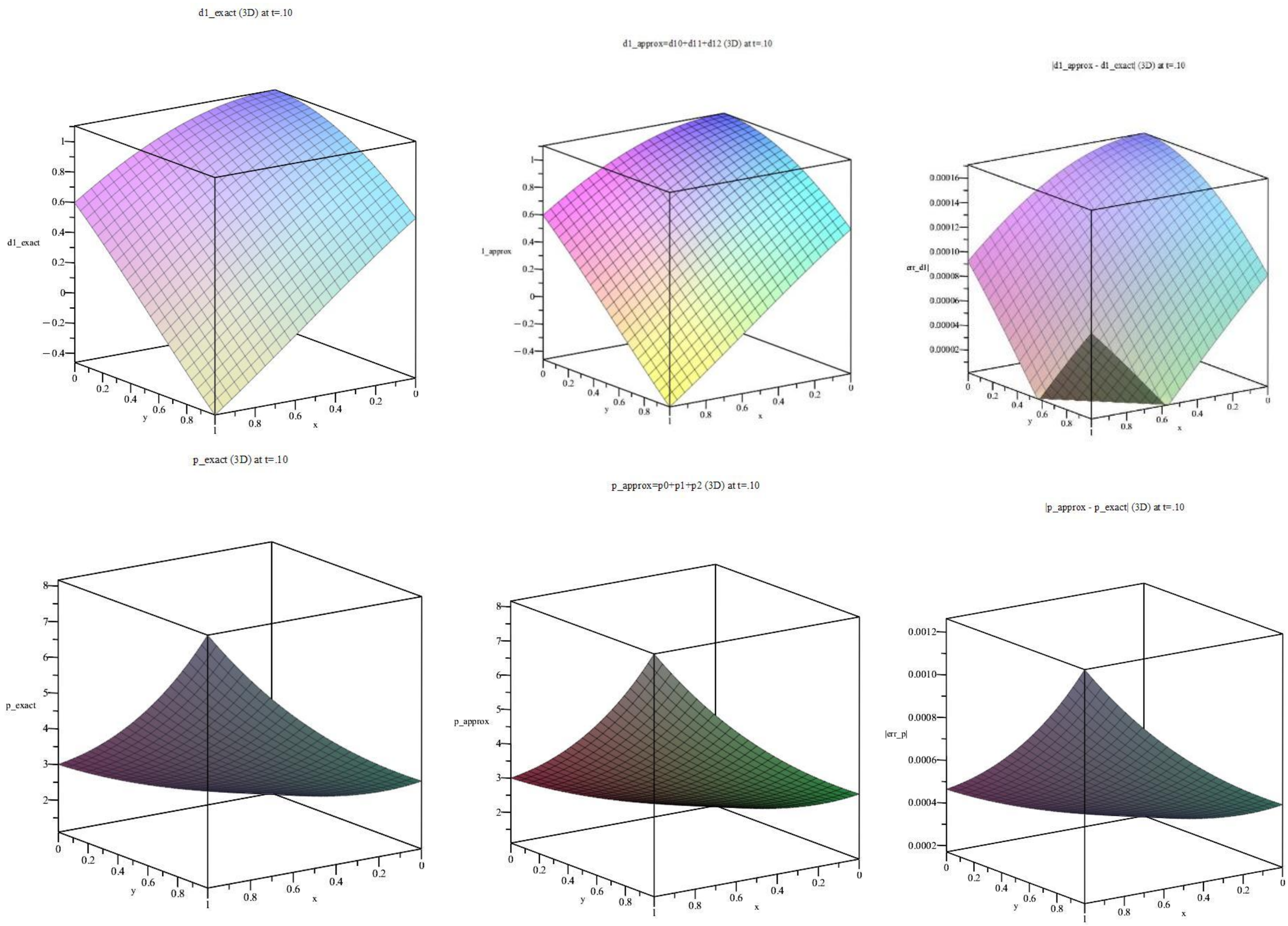


Figure3: Three-dimensional diagrams showing both approximate and exact solutions and the magnitude of error for the two variables d1 and p for the third case.

6.Conclusions:

Our numerical results for the three special cases at t=0.1 show that the Triple Laplace transform Adomean transformation is a powerful method and tool for solving this type of equation and medical system. The absolute errors of the dependent variables remained within 10^(-4) at the test points, reflecting the regular behavior of the solution, as in the case of exponential choice. On the other hand, the comparison between the sine and cosine functions reveals expected differences in errors depending on the spatial structure of the solution. The errors in p are higher at certain points due to the complexity of the interaction/source boundaries. Finally, our results confirm the validity of the method as a semi-analytical tool capable of handling load, diffusion, and interaction systems in mathematical medical modeling.

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