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On Artin Cokernel of The Quaternion Group Q_{2m} When $m=2^h$,h any positive integers Number

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Abstract

In this article we find the cyclic decomposition of the finite abelian factor group AC(G)= $\overline{R}$ (G)/T(G) where G = Q_{2m} and $m = 2^h$, h any positive integers and Q_{2m} is the Quaternion group of order 4m.

(the group of all Z-valued generalized characters of G over the group of induced unit characters from all cyclic subgroups of G) .

We find that the cyclic decomposition AC(Q_{2m}) depends on the elementary divisor of m

If $m = 2^h$, h any positive integers , then :

$$AC(Q_{2m}) = \mathop{\oplus}_{i=1}^{h+1} C_2$$

Moreover, we have also found the general form of Artin characters table Ar(Q_{2m}) when m is an even number .

1. Introduction

Moreover , representation and characters theory provide applications , not only in other branches of mathematics but also in physics and chemistry .

Fore a finite group G , the factor group $\overline{R}(G)/T(G)$ is called the Artin cokernel of G denoted AC(G) , $\overline{R}(G)$ denotes the abelian group generated by Z-valued characters of G under the operation of pointwise addition , $T(G)$ is a subgroup of $\overline{R}(G)$ which is generated by Artin characters .

A well-known theorem which is due to Artin asserted that $T(G)$ has a finite index in $\overline{R}(G)$ i.e, $[\overline{R}(G):T(G)]$ is finite so AC(G) is a finite abelian group.

The exponent of AC(G) is called Artin exponent of G and denoted by A(G).

In 1967, T.Y.Lam [12] proves a sharp form of Artin theorem and he determines the least positive integer A(G) such that $[\overline{R}(G):T(G)]$=A(G).

In 1970 , K. Yamacchi studies the 2 – part of A(G). In 1976 , G. David [6] studies A(G) of arbitrary characters of cyclic subgroups.

In 1995, N. R. Mahmood [10] studied the cyclic decomposition of the factor group cf(Q_{2m},Z)/ $\overline{R}$ (Q_{2m}) and he found the rational valued characters table of the Quaternion group Q_{2m}.

In 1996 , K. Knwabuez [9] studied A(G) of p-groups and in 2000, H. R. Yassein [7] found AC(G) for the group $\bigoplus_{i=1}^{n} Z_p$.

In 2001 A. M. Ibraheem [3] studied A(G) of alternating group. In 2002, K. Sekieguchi [8] studied the irreducible Artin characters of p-group.

In 2006, A.S. Abed [1] found in her thesis the Artin characters table of dihedral group D_n when n is an odd number. In 2007 , R.N. Mirza [12] found in her thesis Artin cokernel of the dihedral group D_n when n is an odd number.

In 2007 A.H. Mohammed [5] found in her thesis Artin cokernel of the dihedral group D_n when n is an even number and in 2008 A.H. Abdul-Mun'em [4] found in her thesis Artin cokernel of the quaternion group Q_{2m} when m is an odd number .

Proposition (1.1): [11]

If p is a prime number and s is a positive integer , then

$$M(C_{p^s}) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

which is of order (s+1)×(s+1) .

Example (1.2):

Consider the matrix M(C_{64}) , we can find it by proposition (1.1)

$$M(C_{64}) = M(C_{2^6}) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

It is 7×7 square matrix .

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Proposition (1.3): [11]

The general form of the matrices $P(C_{p^s})$ and $W(C_{p^s})$ are :

$$P(C_{p^s}) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & & & \ddots & & \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

which is (s+1)×(s+1) square matrix .

$W(C_{p^s}) = I_{s+1}$,where I_{s+1} is an identity matrix and

$$D(C_{p^s}) = diag\{\underbrace{1,1,1,\dots,1}_{S+1}\}.$$

Remarks (1.4): [2]

If $m = 2^h, h$ is any positive integer ,then we can write $M(C_m)$ as the following :

$$M(C_m) = \begin{bmatrix} & & & & 1 & 1 \\ & & & & 1 & 1 \\ & & R_1(C_m) & & \vdots & \vdots \\ & & & & & \\ & & & & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}$$

Which is (h+1)×(h+1) square matrix , $R_1(C_m)$ is the matrix obtained by omitting the last two rows {0,0,...,1,1} and {0,0,...,0,0,1} and the last two columns {1,1,...,1,0} and {1,1,...,1,1} from the matrix $M(C_{2^h})$ in proposition (1.1).

Proposition (1.5): [10]

The general form of the rational valued character table of the Quaternion group Q_{2m} when m= 2^h , h is any positive integer and it is given by :

$\overset{*}{\equiv}(Q_{2m}) = \overset{*}{\equiv}(Q_{2\cdot 2^h}) =$

Γ-CLASSES	$[1]$	$[x^{2^h}]$	$[x^{2^{h-1}}]$	$[x^{2^{h-2}}]$	$\cdots$	$[x^2]$	$[x]$	$[y]$	$[xy]$
θ_1	2^h	-2^h	0	0	$\cdots$	0	0	0	0
θ_2	2^{h-1}	2^{h-1}	-2^{h-1}	0	$\cdots$	0	0	0	0
θ_3	2^{h-2}	2^{h-2}	2^{h-2}	-2^{h-2}	$\cdots$	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
θ_{l-2}	2	2	2	2	$\cdots$	-2	0	0	0
θ_{l-1}	1	1	1	1	$\cdots$	1	-1	-1	1
θ_l	1	1	1	1	$\cdots$	1	1	-1	-1
θ_{l+1}	1	1	1	1	$\cdots$	1	-1	1	-1
θ_{l+2}	1	1	1	1	$\cdots$	1	1	1	1

Where l is the number of Γ-classes of C_m .

Theorem (1.6): **[9]**

Let M be an n×n matrix with entries in a principal ideal domain R ,then there exists matrices P and W such that :

1 - P and W are invertible .

2 - P M W = D .

3 - D is a diagonal matrix .

4 -If we denote D_{ii} by d_i then there exists a natural number m ; $0 \le m \le n$

such that $j > m$ implies $d_j = 0$ and $j \le m$ implies $d_j \ne 0$ and $1 \le j \le m$

implies $d_j \mid d_{j+1}$.

2. The Main Results

Theorem(2.1):

The Artin characters table of the Quaternion group Q_{2m} when m is an even number is given as follows :

$Ar(Q_{2m})$ =

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Γ –CLASSES	- CLASSES OF $\Gamma\ C_{2m}$				$[y]$	$[xy]$
	$[1]$	$[x^m]$				
$\lvert CL_\alpha\rvert$	1	1	2 2 ….	2	m	M
$\lvert C_{Q_{2m}}(CL_\alpha)\rvert$	4m	4m	2m 2m ….	2m	4	4
Φ_1	2Ar(C_{2m})				0	0
Φ_2					0	0
$\vdots$					$\vdots$	$\vdots$
Φ_l					0	0
Φ_{l+1}	M	m	0 … 0	0	2	0
Φ_{l+2}	M	m	0 … 0	0	0	2

where l is the number of Γ-classes of C_{2m} and Φ_j , $1\le j\le l+2$ are the Artin characters of the quaternion group Q_{2m} .

Proof:-

Let $g\in Q_{2m}$

Case (I):

If H is a subgroup of $C_{2m}=<x>$, $1\le j\le l$ and φ the principal character of H , then by using theorem (2.1.5)

$$\Phi_j(g)=\begin{cases}\dfrac{|C_G(g)|}{|C_H(g)|}\displaystyle\sum_{i=1}^{n}\varphi(h_i) & if \quad h_i\in H\cap CL(g)\\ 0 & if \quad H\cap CL(g)=\phi\end{cases}$$

(i) If g =1

$$\Phi_j(1)=\frac{|C_{Q_{2m}}(1)|}{|C_H(1)|}\cdot\varphi(1)=\frac{4m}{|C_H(1)|}\cdot 1=\frac{2\cdot 2m}{|C_H(1)|}\cdot 1=\frac{2|C_{C_{2m}}(1)|}{|C_H(1)|}\cdot 1=2.\varphi'_j(1)$$

$$=2\varphi'_j(1) \quad \text{since } H\cap CL(1)=\{1\}$$

and φ is the principal character where φ'_j is the Artin characters of C_{2m} .

(ii) If g= x^m and $g\in H$

$\Phi_j(g)= \frac{|C_{Q_{2m}}(g)|}{|C_H(g)|}\cdot\varphi(g) = \frac{4m}{|C_H(g)|}\cdot 1$ since $H \cap CL(g) = \{g\}$ and $\varphi(g)=1$

$$= \frac{2\cdot 2m}{|C_H(g)|}\cdot\varphi(g) = \frac{2|C_{C_{2m}}(g)|}{|C_H(g)|}\cdot\varphi(g) = 2.\varphi'_j(g)$$

(iii) If $g \neq x^m$ and $g \in H$

$\Phi_j(g)= \frac{|C_{Q_{2m}}(g)|}{|C_H(g)|}\left(\varphi(g)+\varphi(g^{-1})\right)$

$= \frac{2m}{|C_H(g)|}(1+1)$ since $H \cap CL(g)=\{g,g^{-1}\}$ and $\varphi(g)=\varphi(g^{1})=1$

$= \frac{2|C_{C_{2m}}(g)|}{|C_H(g)|} = 2.\varphi'_j(g)$

(iv) If $g \notin H$

$\Phi_j(g)= 2.\ 0 = 2.\varphi'_j(g)$ since $H \cap CL(g) = \phi$

<u>Case (II)</u>:

If $H = \langle y \rangle = \{1, y, y^2, y^3\}$

(i) If g = 1

$\Phi_{l+1}(1)= \frac{|C_{Q_{2m}}(1)|}{|C_H(1)|}\cdot\varphi(1) = \frac{4m}{4}\cdot 1 = m$ since $H \cap CL(1) = \{1\}$

(ii) If $g= x^m = y^2$ and $g \in H$

$\Phi_{l+1}(g)= \frac{|C_{Q_{2m}}(g)|}{|C_H(g)|}\cdot\varphi(g) = \frac{4m}{4}\cdot 1 = m$ since $H \cap CL(g)=\{g\}$ and $\varphi(g)=1$

(iii) If $g \neq x^m$ and $g \in H$, i.e.$\{g = y$ or $g = y^3\}$

$\Phi_{l+1}(g) = \dfrac{|C_{Q_{2m}}(g)|}{|C_H(g)|}\left(\varphi(g)+\varphi(g^{-1})\right)$

$= \dfrac{4}{4}(1+1) = 2$ since $H \cap CL(g)=\{g,g^{-1}\}$ and $\varphi(g)=\varphi(g^{-1})=1$

Otherwise

$\Phi_{l+1}(g)= 0$ since $H \cap CL(g) = \phi$

Case (III):

If $H = \langle xy \rangle = \{1, xy, (xy)^2 = y^2 = x^m, (xy)^3 = xy^3\}$

(i) If g = 1

$\Phi_{l+2}(g)= \dfrac{|C_{Q_{2m}}(1)|}{|C_H(1)|}\cdot\varphi(1) = \dfrac{4m}{4}\cdot 1 = m$ since $H \cap CL(1) = \{1\}$

(ii) If $g= (xy)^2 = y^2 = x^m$ and $g \in H$

$\Phi_{l+2}(g)= \dfrac{|C_{Q_{2m}}(g)|}{|C_H(g)|}\cdot\varphi(g) =$ since $H \cap CL(g)=\{g\}$ and $\varphi(g) =1$

(iii) If $g \neq (xy)^2 = y^2 = x^m$ and $g \in H$, i.e.$\{g = xy$ or $g = (xy)^3\}$

$\Phi_{l+2}(g) = \dfrac{|C_{Q_{2m}}(g)|}{|C_H(g)|}\left(\varphi(g)+\varphi(g^{-1})\right)$

$= \dfrac{4}{4}(1+1) = 2$ since $H \cap CL(g)=\{g,g^{-1}\}$ and $\varphi(g)=\varphi(g^{-1})=1$

Otherwise

$\Phi_{l+2}(g)= 0$ since $H \cap CL(g) = \phi$

Example (2.2):

To construct $Ar(Q_{256})$ by using theorem (2.1) we get the following table:

$Ar(Q_{256}) = Ar(Q_{2^8}) =$

Γ-CLASSES	[1]	[x^{128}]	[x^{64}]	[x^{32}]	[x^{16}]	[x^{8}]	[x^{4}]	[x^{2}]	[x]	[y]	[xy]
$\|CL_\alpha\|$	1	1	2	2	2	2	2	2	2	128	128
$\|C_{Q_{2m}}(CL_\alpha)\|$	512	512	256	256	256	256	256	256	256	4	4
Φ_1	512	0	0	0	0	0	0	0	0	0	0
Φ_2	256	256	0	0	0	0	0	0	0	0	0
Φ_3	128	128	128	0	0	0	0	0	0	0	0
Φ_4	64	64	64	64	0	0	0	0	0	0	0
Φ_5	32	32	32	32	32	0	0	0	0	0	0
Φ_6	16	16	16	16	16	16	0	0	0	0	0
Φ_7	8	8	8	8	8	8	8	0	0	0	0
Φ_8	4	4	4	4	4	4	4	4	0	0	0
Φ_9	2	2	2	2	2	2	2	2	2	0	0
Φ_{10}	128	128	0	0	0	0	0	0	0	2	0
Φ_{11}	128	128	0	0	0	0	0	0	0	0	2

<u>Proposition (2.3)</u>:

If $m = 2^h$, h any positive integers, then the matrix $M(Q_{2m})$ of the quaternion group Q_{2m} is :

$$M(Q_{2m})=\left[\begin{array}{cccccc|cccc} & & & & & & 1 & 1 & 1 & 1 \\ & 2R_1(C_{2m}) & & & & & 1 & 1 & 1 & 1 \\ & & & & & & 1 & 1 & 1 & 1 \\ & & & & & & \vdots & \vdots & \vdots & \vdots \\ & & & & & & \vdots & \vdots & \vdots & \vdots \\ & & & & & & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

Which is (h+4)× (h+4) square matrix, $R_1(C_{2m})$ is similar to the matrix in remarks (1.4).

<u>Proof :</u>

By theorem (2.1) we obtain the Artin characters table $Ar(Q_{2m})$ of the quaternion group, and from the proposition (1.5) we get the rational valued characters ($\overset{*}{\equiv}(Q_{2m})$) table of the quaternion group .

Thus , by the definition of M(G) we can find the matrix $M(Q_{2m})$.

$M(Q_{2m}) = Ar(Q_{2m}).\,(\overset{*}{\equiv}(Q_{2m}))^{-1}$

$$= \begin{bmatrix} 2 & 2 & 2 & \cdots & \cdots & 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & \cdots & \cdots & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & \cdots & \cdots & 2 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} & & & & & & 1 & 1 & 1 & 1 \\ & & & 2R_1(C_{2m}) & & & 1 & 1 & 1 & 1 \\ & & & & & & 1 & 1 & 1 & 1 \\ & & & & & & \vdots & \vdots & \vdots & \vdots \\ & & & & & & \vdots & \vdots & \vdots & \vdots \\ & & & & & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Which is $(h+4)\times(h+4)$ square matrix .

Example (2.4):

Consider the quaternion group Q_{256} , we can find the matrix $M(Q_{256})$

by using two ways :

First : by the definition of M(G)

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$M(Q_{256}) = M(Q_{2^8}) = Ar(Q_{2^8}) \,.\, (\overset{*}{\equiv}(Q_{2^8}))^{-1}$

$$
= \begin{bmatrix}
512 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
256 & 256 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
128 & 128 & 128 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
64 & 64 & 64 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
32 & 32 & 32 & 32 & 32 & 0 & 0 & 0 & 0 & 0 & 0 \\
16 & 16 & 16 & 16 & 16 & 16 & 0 & 0 & 0 & 0 & 0 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 0 & 0 & 0 & 0 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 \\
128 & 128 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
128 & 128 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}
\bullet
\begin{bmatrix}
\frac{1}{256} & \frac{1}{256} & \frac{1}{256} & \frac{1}{256} & \frac{1}{256} & \frac{1}{256} & \frac{1}{384} & \frac{1}{384} & \frac{1}{384} & \frac{1}{384} & \frac{1}{384} \\
-\frac{1}{256} & \frac{1}{256} & \frac{1}{256} & \frac{1}{256} & \frac{1}{256} & \frac{1}{256} & \frac{1}{384} & \frac{1}{384} & \frac{1}{384} & \frac{1}{384} & \frac{1}{384} \\
0 & -\frac{1}{128} & \frac{1}{128} & \frac{1}{128} & \frac{1}{128} & \frac{1}{128} & \frac{1}{192} & \frac{1}{192} & \frac{1}{192} & \frac{1}{192} & \frac{1}{192} \\
0 & 0 & -\frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{96} & \frac{1}{96} & \frac{1}{96} & \frac{1}{96} & \frac{1}{96} \\
0 & 0 & 0 & -\frac{1}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{48} & \frac{1}{48} & \frac{1}{48} & \frac{1}{48} & \frac{1}{48} \\
0 & 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{16} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{8} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{6}
\end{bmatrix}
=
\begin{bmatrix}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
$$

Which is 11×11 square matrix .

<u>Second :</u> By proposition (2.3)

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$$R_1\left(C_{2^8}\right)=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Which is 7 × 7 square matrix .

Then

$$M(Q_{2^8}) = \begin{bmatrix} & & & & & & & 1 & 1 & 1 & 1 \\ & & & 2R_1\left(C_{2^8}\right) & & & & 1 & 1 & 1 & 1 \\ & & & & & & & 1 & 1 & 1 & 1 \\ & & & & & & & 1 & 1 & 1 & 1 \\ & & & & & & & 1 & 1 & 1 & 1 \\ & & & & & & & 1 & 1 & 1 & 1 \\ & & & & & & & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Which is 11×11 square matrix .

<u>Proposition (2.5)</u>:

If $m = 2^h$, h any positive integers then the matrices $P(Q_{2m})$ and $W(Q_{2m})$ are taking the forms :

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$$P(Q_{2m})=\left[\begin{array}{ccccccc} & & & & & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & \vdots & \vdots \\ & & P(C_{2m}) & & & \vdots & \vdots \\ & & & & & 0 & 0 \\ & & & & & -1 & 1 \\ & & & & & 0 & -1 \\ 0 & 0 & \ldots & \cdots & 0 & 1 & -1 \\ 0 & 0 & \ldots & \cdots & 0 & 0 & 1 \end{array}\right]$$

And

$$(Q_{2m})=\left[\begin{array}{ccccccccc} & & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 0 \\ & & I_{h+1} & & & & \vdots & \vdots & \vdots \\ & & & & & & \vdots & \vdots & \vdots \\ & & & & & & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & \cdots & -1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Where I_{h+1} is the identity matrix, they are $(h+4)\times(h+4)$ square matrix

<u>Proof</u> :

By using theorem (1.6) and taking the matrix $M(Q_{2m})$ from proposition (2.3) and the above forms of $P(Q_{2m})$ and $W(Q_{2m})$ then we have :

$P(Q_{2m}) . M(Q_{2m}) . W(Q_{2m}) =$

$$\begin{bmatrix} 2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}$$

$= D(Q_{2m}) = diag\{2,2,\ldots,2,1,1,1\}$

Which is $(h+4)\times(h+4)$ square matrix .

Example(2.7):

Consider the quaternion group Q_{512} ,by proposition (2.3) we can find the matrix $M(Q_{512})$ and from proposition (2.5) , we find the matrices $P(Q_{512})$ and $W(Q_{512})$:

Where $Q_{512} = Q_{2^9}$

$P(Q_{2^9}) . M(Q_{2^9}) . W(Q_{2^9}) =$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \bullet$$

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&1&1&1&1&1&1&1&1&1&0&0\\ 0&0&0&0&0&0&0&0&1&0&1&0\\ 0&-1&-1&-1&-1&-1&-1&-1&-1&0&0&1 \end{bmatrix} = \begin{bmatrix} 2&0&0&0&0&0&0&0&0&0&0&0\\ 0&2&0&0&0&0&0&0&0&0&0&0\\ 0&0&2&0&0&0&0&0&0&0&0&0\\ 0&0&0&2&0&0&0&0&0&0&0&0\\ 0&0&0&0&2&0&0&0&0&0&0&0\\ 0&0&0&0&0&2&0&0&0&0&0&0\\ 0&0&0&0&0&0&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&2&0&0&0&0\\ 0&0&0&0&0&0&0&0&2&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

= diag{2,2,2,2,2,2,2,2,2,1,1,1}

Which is 12×12 square matrix .

Theorem (2.8):

If $m = 2^h$, h any positive integers then the cyclic decomposition of $AC(Q_{2m})$ is :

$$AC(Q_{2m}) = \bigoplus_{i=1}^{h+1} C_2$$

Proof :

By using proposition(4.2.4) we find $M(Q_{2m})$ and by proposition(4.2.6) we have $P(Q_{2m})$ and $W(Q_{2m})$

Hence

$P(Q_{2m}) . M(Q_{2m}) . W(Q_{2m})$ =diag{2,2,…,2,2,1,1,1}

=diag$\{d_1, d_2, d_3, \dots, d_h, d_{h+1}, d_{h+2}, d_{h+3}, d_{h+4}\}$

Then by theorem(3.4.10) we have

$$AC(Q_{2m}) = \bigoplus_{i=1}^{h+1} C_2$$

Example(2.9):

Consider the groups Q_{32768} , Q_{131072}, $Q_{1048576}$, $Q_{67108869}$, $Q_{268435456}$,then :

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$$1- \quad AC(Q_{32768}) = AC(Q_{2^{15}}) = \bigoplus_{i=1}^{15} C_2$$

$$2- \quad AC(Q_{131072}) = AC(Q_{2^{17}}) = \bigoplus_{i=1}^{17} C_2$$

$$3- \quad AC(Q_{1048576}) = AC(Q_{2^{20}}) = \bigoplus_{i=1}^{20} C_2$$

$$4- \quad AC(Q_{67108869}) = AC(Q_{2^{26}}) = \bigoplus_{i=1}^{26} C_2$$

$$5- \quad AC(Q_{268435456}) = AC(Q_{2^{28}}) = \bigoplus_{i=1}^{28} C_2$$

المستخلص :

في هذا البحث قمنا بايجاد التجزئة الدائرية للزمرة الابيلية الكسرية المنتهية $AC(G)=\overline{R}(G)/T(G)$ عندما $G = Q_{2m}$ و m عدد زوجي ، الزمرة الرباعية العمومية ذات الرتبة 4m (زمرة كل الشواخص العمومية ذات القيم الصحيحة للزمرة G على زمرة الشواخص المحتثة من الشواخص الاحادية للزمرة الجزئية الدائرية) .

وجدنا ان التجزئة الدائرية للزمرة $AC(Q_{2m})$ تعتمد على القواسم الاولية للعدد m حيث وجدنا انه :

اذا كان $m = 2^h$ ، h عدد صحيح موجب فان: $AC(Q_{2m})= \bigoplus_{i=1}^{h+1} C_2$ كذلك وجدنا الصيغة العامة لجدول شواخص ارتن $Ar(Q_{2m})$ عندما m عدد زوجي.

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