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Bayesian Reliability Estimation for Series System Based on The Inverse Ramos – Louzada Distribution Under Different Loss Functions

Authors Names	ABSTRACT
Abdul Salam Q. Al-Hadithi [a] *Feras. S. M. Batah* [b] *Publication data: 11 / 7/2026* *Keywords:* Series System Reliability, Inverse Ramos–Louzada Distribution, Bayesian Estimation, Doubly Censored Data, Huber Loss Function, LINEX Loss Function, Log-Cosh Loss Function.	In this paper, we apply Bayesian estimation of a series system based on the Inverse Ramos–Louzada (IRL) distribution on doubly censored data. The three prior distributions, namely Flat, Gamma, and Jeffrey's priors, are taken with the three loss functions, namely LINEX, Huber, and Log-Cosh. Mathematical representation of the reliability function of the series system is made and the estimators are obtained using Bayesian methods. A Monte Carlo simulation with sixteen cases is performed to estimate the estimator based on Mean Squared Error (MSE), Pitman Closeness (PC), and Relative Efficiency (RE). In actuality, the Bayes-Gamma estimator yields reliability values that are closer to the true system reliability, and the best estimates can be found using accuracy criteria. Such results support that Bayesian estimation is efficient for modelling reliability with doubly censored samples.

1. Introduction

Reliability approach is known as one of the basic elements in the performance assessment of engineering and industrial systems. “System reliability” is the possibility that a system is able to deliver its intended function for an entire period of time when operating in particular conditions. Of all the system architectures, the series system is considered one of the most sensitive configurations due to the complete system life being completely attributed to all its parts working normally. The total failure of the system occurs if one component fails, the concept known graphically as 'the weakest link'.

The mathematical basis for examination of such systems was introduced by Barlow and Proschan (1996), who defined the reliability function of a series system as [1]:

$$R_{sys}(t) = \prod_{i=1}^{n} P(X_i > t)$$

Thus, failure of one component is enough to cause the system failure. The sensitivity of considering models such as is usually determined to rely on the choice of probability distribution to estimate failure time.

In this regard, the Inverse Ramos–Louzada (IRL) distribution has recently developed into a flexible model that can capture reliability measurements, especially those that are observed from an upside-down bathtub hazard rate, as it is often the case with fine engineering systems. The data distribution was brought forward by Hassan, et al. (2025) [2], being an appealing parameter to represent such systems. Application failure time data is frequently limited by time and cost constraints and, because of time and cost limitations to the first place, the failure time data is often missing or censored.

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The framework used for this study is based on doubly censored data, that is the type-I and type-II censors. The Type-I censoring model terminated before a predetermined period and, consequently, not all units fail in that time period. In Type-II censoring, the experiment is stopped once a specified number of failures are observed. Data of the form also increases the complexity of the likelihood function and decreases the effectiveness of regular estimation techniques like Maximum Likelihood Estimation (MLE) especially when analyzing small samples.

The Bayesian framework is utilized to solve these problems as it can integrate both known facts and observed data to obtain the best estimate accuracy. Other previous studies including Dey and Maiti (2010) [3], Krishna & Malik (2012) [4], Haddad and Batah (2021) [5], Yadav, et al. (2018) [6], have provided important contributions in the area of Bayesian estimation under different censoring schemes. Finally, this proposed work extends the current research trend for the reliability function of a series of distributed IRL data with doubly censored data under different priors such as Flat, Gamma, and Jeffreys priors according to the IRL distribution. Choosing the loss function is, of course, an important part of Bayesian estimation, in particular for asymmetric estimation errors whose costs of estimation errors are frequently, this is the case in engineering applications. Hence, three different loss functions are taken into account in this study as the LINEX loss function to account for asymmetry, Huber loss function to lessen the impact of outliers, and the Log-Cosh loss function for a contemporary alternative between smoothness and robustness.

The novelty here is the combination of the recently developed IRL distribution with series-based models and doubly censored data (such as those proposed for the recently proposed system reliability function) and the original mathematical derivations of the system reliability function (in this case). Additionally, a broad analysis approach is considered from the beginning to the end by going further into the Mean Squared Error (MSE) and incorporating the Relative Efficiency (RE) and Pitman Closeness Criterion (PC), providing a more complete statistical analysis for systems with high precision that may be valuable.

2. The Inverse Ramos-Louzada (IRL) distribution

In 2025, Hassan, et al. introduced the Inverse Ramos–Louzada (IRL) distribution. Assume we denote X a random variable that follows the IRL distribution with parameter δ, denoted by $X \sim IRL(\theta)$. The cumulative distribution function (CDF) and the probability density function (PDF) are given, respectively, by [2]:

$$F_{IRL}(x) = \frac{\theta + \frac{1}{\theta x} - 1}{(\theta - 1)} e^{-\frac{1}{\theta x}} \quad , x \geq 0, \theta > 2 \tag{1}$$

$$f_{IRL}(x) = \frac{(\theta - 2)\theta x + 1}{\theta^2 x^3 (\theta - 1)} e^{-\frac{1}{\theta x}} \quad , x \geq 0, \theta > 2 \tag{2}$$

The survival function expressed as [7]:

$$S_{IRL}(x) = 1 - F_{IRL}(x)$$

The corresponding survival function of the IRL distribution is expressed as:

$$S_{IRL}\,(x) = 1 - \frac{\theta + \frac{1}{\theta x} - 1}{(\theta - 1)} e^{-\frac{1}{\theta x}} \qquad (3)$$

The hazard rate function is given by [7]:

$$h_{IRL}\,(X) = \frac{f_{IRL}(x)}{S_{IRL}\,(x)}$$

Moreover, the hazard rate function of the IRL distribution is given by:

$$h_{IRL}\,(X) = \frac{\frac{(\theta-2)\theta x+1}{\theta^2 x^3(\theta-1)} e^{-\frac{1}{\theta x}}}{1 - \frac{\theta+\frac{1}{\theta x}-1}{(\theta-1)} e^{-\frac{1}{\theta x}}} \qquad (4)$$

3. Series System Reliability

The reliability of a series system represents the probability that the whole system continues to operate successfully when its components are connected in a dependent failure structure through the minimum lifetime. In this study, the lifetime of the strength variable X and the component lifetimes $Y_1, Y_2, \dots, Y_K$ are assumed to follow the IRL distribution with different parameters. Since the failure of any component affects the performance of the whole series system, the minimum lifetime $Z = \min(Y_1, Y_2, \dots, Y_K)$ plays a central role in the derivation. The main objective is to obtain an explicit analytical expression for the reliability probability $R_{series} = P(X < Z)$.This derivation provides a useful mathematical basis for estimating and comparing the reliability of series systems under the IRL model.

Let $X \sim \mathrm{IRLD}(\theta_1)$, and let $Y_1, Y_2, \dots, Y_K \sim \mathrm{IRLD}(\theta_2)$ be independent random variables. Define $Z = \min(Y_1, Y_2, \dots, Y_K)$. The reliability of the series system is

$$R_{\text{series}} = P(X < Z)$$

Hence

$$R_{\text{series}} = \int_0^\infty \int_0^z f_X\,(x) f_Z(z)\, dx\, dz$$

Since

$$\int_0^z f_X\,(x)\, dx = F_X(z)$$

then

$$R_{\text{series}} = \int_0^\infty F_X\,(z) f_Z(z)\, dz$$

For each component Y_i, the survival function is

$$S_Y(z) = P(Y > z) = 1 - F_Y(z)$$

Because $Y_1, Y_2, \dots, Y_K$ are independent,
$P(Z > z) = P(Y_1 > z, Y_2 > z, \dots, Y_K > z)$. Therefore

$$P(Z > z) = [S_Y(z)]^K = [1 - F_Y(z)]^K$$

Thus

$$F_Z(z) = P(Z \leq z) = 1 - P(Z > z)$$

and hence

$$F_Z(z) = 1 - [1 - F_Y(z)]^K$$

Differentiating with respect to z gives

$$f_Z(z) = \frac{d}{dz}\{1 - [1 - F_Y(z)]^K\}$$

Thus

$$f_Z(z) = K[1 - F_Y(z)]^{K-1} f_Y(z).$$

Substituting this density into the reliability integral gives

$$R_{\text{series}} = K \int_0^{\infty} F_X\,(z)[1 - F_Y(z)]^{K-1} f_Y(z)\,dz$$

For the IRL distribution, the probability density function is

$$f(z;\theta) = \frac{(\theta - 2)\theta z + 1}{\theta^2(\theta - 1)z^3} \exp\left(-\frac{1}{\theta z}\right), \qquad z > 0,\ \theta > 2.$$

The cumulative distribution function is

$$F(z;\theta) = \frac{\theta + \frac{1}{\theta z} - 1}{\theta - 1} \exp\left(-\frac{1}{\theta z}\right), \qquad z > 0,\ \theta > 1.$$

Hence

$$F_X(z) = \frac{\theta_1 + \frac{1}{\theta_1 z} - 1}{\theta_1 - 1} \exp\left(-\frac{1}{\theta_1 z}\right),$$

$$f_Y(z) = \frac{(\theta_2 - 2)\theta_2 z + 1}{\theta_2^2(\theta_2 - 1)z^3} \exp\left(-\frac{1}{\theta_2 z}\right),$$

and

$$F_Y(z) = \frac{\theta_2 + \frac{1}{\theta_2 z} - 1}{\theta_2 - 1} \exp\left(-\frac{1}{\theta_2 z}\right).$$

After substitution, the reliability integral becomes

$$R_{\text{series}} = K \int_0^{\infty} G_1\,(z) G_2(z) G_3(z)\,dz,$$

where

$$G_1(z) = \frac{\theta_1 + \frac{1}{\theta_1 z} - 1}{\theta_1 - 1} \exp\left(-\frac{1}{\theta_1 z}\right),$$

$$G_2(z) = \left[1 - \frac{\theta_2 + \frac{1}{\theta_2 z} - 1}{\theta_2 - 1} \exp\left(-\frac{1}{\theta_2 z}\right)\right]^{K-1},$$

and

$$G_3(z) = \frac{(\theta_2 - 2)\theta_2 z + 1}{\theta_2^2(\theta_2 - 1)z^3} \exp\left(-\frac{1}{\theta_2 z}\right).$$

Let

$$A(z) = \frac{\theta_2 + \frac{1}{\theta_2 z} - 1}{\theta_2 - 1}\exp\left(-\frac{1}{\theta_2 z}\right).$$

Using the binomial expansion

$$(1 - A)^{K-1} = \sum_{j=0}^{K-1}\binom{K-1}{j}(-1)^j A^j.$$

Therefore

$$G_2(z) = \sum_{j=0}^{K-1}\binom{K-1}{j}(-1)^j\left[\frac{\theta_2 + \frac{1}{\theta_2 z} - 1}{\theta_2 - 1}\exp\left(-\frac{1}{\theta_2 z}\right)\right]^j$$

After simplification, the reliability can be written as

$$R_{\text{series}} = \frac{K}{(\theta_1 - 1)(\theta_2 - 1)\theta_2^2}\sum_{j=0}^{K-1}\binom{K-1}{j}\frac{(-1)^j}{(\theta_2 - 1)^j}I_j.$$

The integral I_j is

$$I_j = \int_0^\infty Q_j\,(z)\,dz,$$

where

$$Q_j(z) = \frac{(\theta_2 - 2)\theta_2 z + 1}{z^3}\left(\theta_1 + \frac{1}{\theta_1 z} - 1\right)\left(\theta_2 + \frac{1}{\theta_2 z} - 1\right)^j\exp\left(-\frac{\alpha_j}{z}\right).$$

Here

$$\alpha_j = \frac{1}{\theta_1} + \frac{j+1}{\theta_2} = \frac{\theta_2 + (j+1)\theta_1}{\theta_1\theta_2}.$$

Also

$$\left(\theta_2 + \frac{1}{\theta_2 z} - 1\right)^j = \left[(\theta_2 - 1) + \frac{1}{\theta_2 z}\right]^j.$$

Using the binomial expansion again gives

$$\left(\theta_2 + \frac{1}{\theta_2 z} - 1\right)^j = \sum_{m=0}^{j}\binom{j}{m}\frac{(\theta_2 - 1)^{j-m}}{\theta_2^m z^m}.$$

Hence

$$I_j = \sum_{m=0}^{j}\binom{j}{m}\frac{(\theta_2 - 1)^{j-m}}{\theta_2^m}J_{jm},$$

where

$$J_{jm} = \int_0^\infty \frac{(\theta_2 - 2)\theta_2 z + 1}{z^{3+m}}\left(\theta_1 - 1 + \frac{1}{\theta_1 z}\right)\exp\left(-\frac{\alpha_j}{z}\right)dz.$$

Use the transformation

$$u = \frac{1}{z}, \qquad z = \frac{1}{u}, \qquad dz = -\frac{1}{u^2}du.$$

When $z \to 0$, $u \to \infty$. When $z \to \infty$, $u \to 0$. Thus

$$J_{jm} = \int_0^\infty [(\theta_2 - 2)\theta_2 u^m + u^{m+1}]\left(\theta_1 - 1 + \frac{u}{\theta_1}\right)\exp(-\alpha_j u)\,du.$$

Expanding the integrand gives

$$J_{jm} = \int_0^\infty [C_0 u^m + C_1 u^{m+1} + C_2 u^{m+2}]\exp(-\alpha_j u)\,du,$$

where

$$C_0 = (\theta_2 - 2)\theta_2(\theta_1 - 1),$$
$$C_1 = \frac{(\theta_2 - 2)\theta_2}{\theta_1} + \theta_1 - 1,$$

and

$$C_2 = \frac{1}{\theta_1}.$$

Using the gamma integral

$$\int_0^\infty u^n\, e^{-\alpha u}\, du = \frac{\Gamma(n+1)}{\alpha^{n+1}}, \qquad \alpha > 0,$$

we obtain

$$J_{jm} = C_0 \frac{\Gamma(m+1)}{\alpha_j^{m+1}} + C_1 \frac{\Gamma(m+2)}{\alpha_j^{m+2}} + C_2 \frac{\Gamma(m+3)}{\alpha_j^{m+3}}.$$

Therefore

$$I_j = \sum_{m=0}^{j} \binom{j}{m} \frac{(\theta_2 - 1)^{j-m}}{\theta_2^m} \left[C_0 \frac{\Gamma(m+1)}{\alpha_j^{m+1}} + C_1 \frac{\Gamma(m+2)}{\alpha_j^{m+2}} + C_2 \frac{\Gamma(m+3)}{\alpha_j^{m+3}} \right].$$

Finally, the closed form of the series system reliability is

$$R_{\text{series}} = \frac{K \sum_{j=0}^{K-1} \binom{K-1}{j} \frac{(-1)^j}{(\theta_2 - 1)^j}}{(\theta_1 - 1)(\theta_2 - 1)\theta_2^2} I_j \qquad (5)$$

where

$$I_j = \sum_{m=0}^{j} \binom{j}{m} \frac{(\theta_2 - 1)^{j-m}}{\theta_2^m} \left[C_0 \frac{\Gamma(m+1)}{\alpha_j^{m+1}} + C_1 \frac{\Gamma(m+2)}{\alpha_j^{m+2}} + C_2 \frac{\Gamma(m+3)}{\alpha_j^{m+3}} \right],$$

with

$$\alpha_j = \frac{\theta_2 + (j+1)\theta_1}{\theta_1 \theta_2},$$
$$C_0 = (\theta_2 - 2)\theta_2(\theta_1 - 1),$$
$$C_1 = \frac{(\theta_2 - 2)\theta_2}{\theta_1} + \theta_1 - 1,$$

and

$$C_2 = \frac{1}{\theta_1}.$$

4. Estimation

To verify the efficiency of the estimation of R_{series} for IRL distribution, 3 estimation methods Bayesian estimations are derived, as follows:

4.1 Flat Bayesian

Under the assumption of a flat prior distribution, the posterior density is proportional to the likelihood function. Therefore, the posterior distribution can be expressed as:

$$\pi(\theta \mid X) \propto L(\theta)\pi(\theta) = L(\theta).$$

Under the squared error loss function, the Bayes estimator is given by the posterior mean. Hence, the Bayes estimator of θ is obtained as [8]:

$$\hat{\theta}_{Flat} = E(\theta \mid X) = \frac{\int_0^\infty \theta \ \pi(\theta \mid X)\, d\theta}{\int_0^\infty \pi\,(\theta \mid X)\, d\theta}.$$

For each parameter, we obtain

$$\hat{\theta}_1 = E(\theta_1 \mid X) = \frac{\int_0^\infty \theta_1 \ \pi(\theta_1 \mid X)\, d\theta_1}{\int_0^\infty \pi\,(\theta_1 \mid X)\, d\theta_1},$$
$$\hat{\theta}_2 = E(\theta_2 \mid X) = \frac{\int_0^\infty \theta_2 \ \pi(\theta_2 \mid X)\, d\theta_2}{\int_0^\infty \pi\,(\theta_2 \mid X)\, d\theta_2},$$

Hence, the two Bayes estimators are denoted by

$$\hat{\theta}_1^{(F)}, \quad \hat{\theta}_2^{(F)}.$$

Therefore, the reliability under the flat prior is expressed as

$$\hat{R}_{\text{series}} = \frac{K \sum_{j=0}^{K-1} \binom{K-1}{j} \frac{(-1)^j}{\left(\hat{\theta}_2^{(F)}-1\right)^j}}{\left(\hat{\theta}_1^{(F)}-1\right)\left(\hat{\theta}_2^{(F)}-1\right)\left(\hat{\theta}_2^{(F)}\right)^2} I_{j(F)} \tag{6}$$

where

$$I_{j(F)} = \sum_{m=0}^{j} \binom{j}{m} \frac{\left(\hat{\theta}_2^{(F)} - 1\right)^{j-m}}{\left(\hat{\theta}_2^{(F)}\right)^m} \left[C_{0(F)} \frac{\Gamma(m+1)}{\left(\alpha_{j(Flat)}\right)^{m+1}} + C_{1(F)} \frac{\Gamma(m+2)}{\left(\alpha_{j(Flat)}\right)^{m+2}} + C_{2(F)} \frac{\Gamma(m+3)}{\left(\alpha_{j(Flat)}\right)^{m+3}} \right],$$

with

$$\alpha_{j(F)} = \frac{\hat{\theta}_2^{(F)} + (j+1)\hat{\theta}_2^{(F)}}{\hat{\theta}_1^{(F)}\hat{\theta}_2^{(F)}},$$
$$C_{0(F)} = \left(\hat{\theta}_2^{(F)} - 2\right)\hat{\theta}_2^{(F)}\left(\hat{\theta}_1^{(F)} - 1\right),$$
$$C_{1(F)} = \frac{\left(\hat{\theta}_2^{(F)} - 2\right)\hat{\theta}_2^{(F)}}{\hat{\theta}_1^{(F)}} + \hat{\theta}_1^{(F)} - 1,$$

and

$$C_{2(F)} = \frac{1}{\hat{\theta}_1^{(F)}}.$$

4.2 Bayesian under Gamma prior

Under the assumption of a Gamma prior distribution, the likelihood function of θ is given by [9]:

$$L(\theta) = \prod_{i=1}^{n} f\,(x_i \mid \theta) \propto \frac{\prod_{i=1}^{n}\left((\theta-2)\theta x_i + 1\right)}{\theta^{2n}(\theta-1)^n} \exp\left(-\sum_{i=1}^{n} \frac{1}{\theta x_i}\right).$$

Let the prior distribution of θ be Gamma with parameters a and b, that is,

$$\theta \sim \text{Gamma}(a, b),$$

with prior density

$$\pi(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}.$$

By combining the likelihood function with the Gamma prior, the posterior distribution of θ can be written as

$$\pi(\theta \mid X) \propto L(\theta)\pi(\theta) \propto \theta^{a-1-2n}(\theta-1)^{-n}\prod_{i=1}^{n}\left((\theta-2)\theta x_i+1\right)\exp\left(-b\theta-\sum_{i=1}^{n}\frac{1}{\theta x_i}\right).$$

Under the squared error loss function, the Bayes estimator is given by the posterior mean. Hence, the Bayes estimator of θ under the Gamma prior is

$$\hat{\theta}_{Bayes-Gamma} = E(\theta \mid X) = \frac{\int_0^\infty \theta\ \pi(\theta \mid X)\, d\theta}{\int_0^\infty \pi\,(\theta \mid X)\, d\theta}.$$

Accordingly, for the two model parameters, the Bayes estimators are obtained as

$$\hat{\theta}_1 = E(\theta_1 \mid X) = \frac{\int_0^\infty \theta_1\ \pi(\theta_1 \mid X)\, d\theta_1}{\int_0^\infty \pi\,(\theta_1 \mid X)\, d\theta_1},$$
$$\hat{\theta}_2 = E(\theta_2 \mid X) = \frac{\int_0^\infty \theta_2\ \pi(\theta_2 \mid X)\, d\theta_2}{\int_0^\infty \pi\,(\theta_2 \mid X)\, d\theta_2},$$

The Bayes estimate of the reliability function is then obtained by substituting these posterior mean estimates into the reliability expression. Thus,

$$\hat{R}_{Bayes} = R\left(\hat{\theta}_1^{(B)}, \hat{\theta}_2^{(B)}\right).$$

More explicitly, the Bayes estimate of reliability can be written as

$$\hat{R}_{\text{series}} = \frac{K\sum_{j=0}^{K-1}\binom{K-1}{j}\frac{(-1)^j}{\left(\hat{\theta}_2^{(B)}-1\right)^j}}{\left(\hat{\theta}_1^{(B)}-1\right)\left(\hat{\theta}_2^{(B)}-1\right)\left(\hat{\theta}_2^{(B)}\right)^2} I_{j(B)} \qquad (7)$$

where

$$I_{j(B)} = \sum_{m=0}^{j}\binom{j}{m}\frac{\left(\hat{\theta}_2^{(B)}-1\right)^{j-m}}{\left(\hat{\theta}_2^{(B)}\right)^m}\left[C_{0(B)}\frac{\Gamma(m+1)}{\left(\alpha_{j(B)}\right)^{m+1}} + C_{1(B)}\frac{\Gamma(m+2)}{\left(\alpha_{j(B)}\right)^{m+2}} + C_{2(B)}\frac{\Gamma(m+3)}{\left(\alpha_{j(B)}\right)^{m+3}}\right],$$

with

$$\alpha_{j(B)} = \frac{\hat{\theta}_2^{(B)} + (j+1)\hat{\theta}_2^{(B)}}{\hat{\theta}_1^{(B)}\hat{\theta}_2^{(B)}},$$
$$C_{0(B)} = \left(\hat{\theta}_2^{(B)}-2\right)\hat{\theta}_2^{(B)}\left(\hat{\theta}_1^{(B)}-1\right),$$
$$C_{1(B)} = \frac{\left(\hat{\theta}_2^{(B)}-2\right)\hat{\theta}_2^{(B)}}{\hat{\theta}_1^{(B)}} + \hat{\theta}_1^{(B)} - 1,$$

and

$$C_{2(B)} = \frac{1}{\hat{\theta}_1^{(B)}}.$$

This formulation provides the Bayes estimator of the reliability measure under the Gamma prior distribution, where the unknown parameters are replaced by their corresponding posterior mean estimates.

4.3 Bayesian under Jeffrey's prior

Under Jeffrey's prior, the posterior density is proportional to the product of the likelihood function and Jeffrey's prior.

$$\pi(\theta \mid X) \propto L(\theta)\pi(\theta) \propto \frac{\prod_{i=1}^{n}\left((\theta-2)\theta x_i+1\right)}{\theta^{2n+1}(\theta-1)^n} \exp\left(-\sum_{i=1}^{n}\frac{1}{\theta x_i}\right)$$

Therefore, the posterior distribution of the parameter θ can be expressed as [10]:

$$\hat{\theta}_J = E(\theta \mid X) = \frac{\int_0^\infty \theta\ \pi(\theta \mid X)\,d\theta}{\int_0^\infty \pi\,(\theta \mid X)\,d\theta}$$

Under the squared error loss function, the Bayes estimator is given by the posterior mean. Thus, the Bayes estimators of the parameters θ_1, θ_2 are obtained from their corresponding posterior expectations and are denoted by $\hat{\theta}_1, \hat{\theta}_2$ respectively:

$$\hat{\theta}_1^{(JE)} = E(\theta_1 \mid X) = \frac{\int_0^\infty \theta_1\ \pi(\theta_1 \mid X)\,d\theta_1}{\int_0^\infty \pi\,(\theta_1 \mid X)\,d\theta_1},$$

$$\hat{\theta}_2^{(JE)} = E(\theta_2 \mid X) = \frac{\int_0^\infty \theta_2\ \pi(\theta_2 \mid X)\,d\theta_2}{\int_0^\infty \pi\,(\theta_2 \mid X)\,d\theta_2},$$

Substituting these Bayes estimates into the reliability function yields the Bayes estimate of reliability under Jeffrey's prior, denoted by $\hat{R}_{series}$:

$$\hat{R}_{\text{series}} = \frac{K\sum_{j=0}^{K-1}\binom{K-1}{j}\frac{(-1)^j}{\left(\hat{\theta}_2^{(JE)}-1\right)^j}}{\left(\hat{\theta}_1^{(JE)}-1\right)\left(\hat{\theta}_2^{(JE)}-1\right)\left(\hat{\theta}_2^{(JE)}\right)^2} I_{j(JE)} \tag{8}$$

where

$$I_{j(JE)} = \sum_{m=0}^{j}\binom{j}{m}\frac{\left(\hat{\theta}_2^{(JE)}-1\right)^{j-m}}{\left(\hat{\theta}_2^{(JE)}\right)^m}\left[C_{0(JE)}\frac{\Gamma(m+1)}{\left(\alpha_{j(JE)}\right)^{m+1}} + C_{1(JE)}\frac{\Gamma(m+2)}{\left(\alpha_{j(JE)}\right)^{m+2}} + C_{2(JE)}\frac{\Gamma(m+3)}{\left(\alpha_{j(JE)}\right)^{m+3}}\right],$$

with

$$\alpha_{j(JE)} = \frac{\hat{\theta}_2^{(JE)} + (j+1)\hat{\theta}_2^{(JE)}}{\hat{\theta}_1^{(JE)}\hat{\theta}_2^{(JE)}},$$

$$C_{0(JE)} = \left(\hat{\theta}_2^{(JE)}-2\right)\hat{\theta}_2^{(JE)}\left(\hat{\theta}_1^{(JE)}-1\right),$$

$$C_{1(JE)} = \frac{\left(\hat{\theta}_2^{(JE)}-2\right)\hat{\theta}_2^{(JE)}}{\hat{\theta}_1^{(JE)}} + \hat{\theta}_1^{(JE)} - 1,$$

and

$$C_{2(JE)} = \frac{1}{\hat{\theta}_1^{(JE)}}.$$

5. Reliability Estimation under Three Loss Functions for the Series IRL Model

The reliability function of the series system is first expressed in terms of the unknown parameters θ_1 and θ_2. After deriving the closed analytical form, different estimators of these parameters can be substituted into the same reliability expression. This approach keeps the mathematical structure of the reliability function unchanged and allows direct comparison between the estimators obtained under different loss functions. In this section, the reliability estimator is derived under the Log-Cosh, Huber, and LINEX loss functions.

5.1 Reliability Estimation under the Log-Cosh Loss Function

The Log-Cosh loss function is a smooth and continuously differentiable loss function. It behaves approximately like the squared error loss for small estimation errors and becomes close to the absolute error loss for large estimation errors. Therefore, it reduces the influence of extreme deviations while preserving numerical stability during estimation. For a parameter θ_i, the estimation error is defined as [11]:

$$\Delta_i = d_i - \theta_i, \qquad i = 1,2,$$

where d_i is a candidate estimator of θ_i. The Log-Cosh loss function is defined by

$$L_{\mathrm{LC}}(d_i, \theta_i) = \log\{\cosh(d_i - \theta_i)\}.$$

The estimator under the Log-Cosh loss function is obtained by minimizing the posterior expected loss. Hence,

$$\hat{\theta}_i^{(\mathrm{LC})} = \underset{d_i}{\operatorname{argmin}} E[\log\{\cosh(d_i - \theta_i)\} \mid X], i = 1,2.$$

Thus, the two parameter estimators are denoted by

$$\hat{\theta}_1^{(\mathrm{LC})}, \qquad \hat{\theta}_2^{(\mathrm{LC})}.$$

By substituting the Log-Cosh parameter estimators into equation (5), the reliability estimator under the Log-Cosh loss function is expressed as

$$\hat{R}_{\mathrm{series}}^{(\mathrm{LC})} = \frac{K \sum_{j=0}^{K-1} \binom{K-1}{j} (-1)^j I_j^{(\mathrm{LC})}}{\left(\hat{\theta}_1^{(\mathrm{LC})} - 1\right)\left(\hat{\theta}_2^{(\mathrm{LC})} - 1\right)\left(\hat{\theta}_2^{(\mathrm{LC})}\right)^2 \left(\hat{\theta}_2^{(\mathrm{LC})} - 1\right)^j},$$

where

$$I_j^{(\mathrm{LC})} = \sum_{m=0}^{j} \binom{j}{m} \frac{\left(\hat{\theta}_2^{(\mathrm{LC})} - 1\right)^{j-m}}{\left(\hat{\theta}_2^{(\mathrm{LC})}\right)^m} \left[C_0^{(\mathrm{LC})} \frac{\Gamma(m+1)}{\left(\alpha_j^{(\mathrm{LC})}\right)^{m+1}} + C_1^{(\mathrm{LC})} \frac{\Gamma(m+2)}{\left(\alpha_j^{(\mathrm{LC})}\right)^{m+2}} + C_2^{(\mathrm{LC})} \frac{\Gamma(m+3)}{\left(\alpha_j^{(\mathrm{LC})}\right)^{m+3}}\right],$$

with

$$\alpha_j^{(\mathrm{LC})} = \frac{\hat{\theta}_2^{(\mathrm{LC})} + [j+1]\hat{\theta}_1^{(\mathrm{LC})}}{\hat{\theta}_1^{(\mathrm{LC})}\hat{\theta}_2^{(\mathrm{LC})}},$$

$$C_0^{(\mathrm{LC})} = \left(\hat{\theta}_2^{(\mathrm{LC})} - 2\right)\hat{\theta}_2^{(\mathrm{LC})}\left(\hat{\theta}_1^{(\mathrm{LC})} - 1\right),$$

$$C_1^{(\mathrm{LC})} = \frac{\left(\hat{\theta}_2^{(\mathrm{LC})} - 2\right)\hat{\theta}_2^{(\mathrm{LC})}}{\hat{\theta}_1^{(\mathrm{LC})}} + \hat{\theta}_1^{(\mathrm{LC})} - 1,$$

$$C_2^{(\mathrm{LC})} = \frac{1}{\hat{\theta}_1^{(\mathrm{LC})}}.$$

5.2 Reliability Estimation under the Huber Loss Function

The Huber loss function combines the advantages of the squared error loss and the absolute error loss. It assigns a quadratic penalty to small errors and a linear penalty to large errors. This property makes it useful when the estimator may be affected by outlying observations or heavy-tailed lifetime behaviour. For a cut-off constant $c_H > 0$, the Huber loss function is defined as [12]:

$$L_{\mathrm{H}}(d_i,\theta_i)=\begin{cases}\dfrac{1}{2}(d_i-\theta_i)^2, & |d_i-\theta_i|\le c_H\\ c_H\left(|d_i-\theta_i|-\dfrac{1}{2}c_H\right), & |d_i-\theta_i|>c_H\end{cases}$$

The estimator under the Huber loss function is obtained by minimizing the posterior expected Huber loss. Therefore,

$$\hat{\theta}_i^{(\mathrm{H})}=\underset{d_i}{\operatorname{argmin}}E[L_{\mathrm{H}}(d_i,\theta_i)\mid X],\qquad i=1,2.$$

Thus, the two parameter estimators are denoted by

$$\hat{\theta}_1^{(\mathrm{H})},\qquad \hat{\theta}_2^{(\mathrm{H})}.$$

By substituting the Huber parameter estimators into equation (5), the reliability estimator under the Huber loss function is expressed as

$$\hat{R}_{\text{series}}^{(\mathrm{H})}=\frac{K\sum_{j=0}^{K-1}\binom{K-1}{j}\frac{(-1)^j}{\left(\hat{\theta}_2^{(\mathrm{H})}-1\right)^j}}{\left(\hat{\theta}_1^{(\mathrm{H})}-1\right)\left(\hat{\theta}_2^{(\mathrm{H})}-1\right)\left(\hat{\theta}_2^{(\mathrm{H})}\right)^2}I_j^{(\mathrm{H})}$$

where

$$I_j^{(\mathrm{H})}=\sum_{m=0}^{j}\binom{j}{m}\frac{\left(\hat{\theta}_2^{(\mathrm{H})}-1\right)^{j-m}}{\left(\hat{\theta}_2^{(\mathrm{H})}\right)^m}\left[C_0^{(\mathrm{H})}\frac{\Gamma(m+1)}{\left(\alpha_j^{(\mathrm{H})}\right)^{m+1}}+C_1^{(\mathrm{H})}\frac{\Gamma(m+2)}{\left(\alpha_j^{(\mathrm{H})}\right)^{m+2}}+C_2^{(\mathrm{H})}\frac{\Gamma(m+3)}{\left(\alpha_j^{(\mathrm{H})}\right)^{m+3}}\right],$$

with

$$\alpha_j^{(\mathrm{H})}=\frac{\hat{\theta}_2^{(\mathrm{H})}+[j+1]\hat{\theta}_1^{(\mathrm{H})}}{\hat{\theta}_1^{(\mathrm{H})}\hat{\theta}_2^{(\mathrm{H})}},$$

$$C_0^{(\mathrm{H})}=\left(\hat{\theta}_2^{(\mathrm{H})}-2\right)\hat{\theta}_2^{(\mathrm{H})}\left(\hat{\theta}_1^{(\mathrm{H})}-1\right),$$

$$C_1^{(\mathrm{H})}=\frac{\left(\hat{\theta}_2^{(\mathrm{H})}-2\right)\hat{\theta}_2^{(\mathrm{H})}}{\hat{\theta}_1^{(\mathrm{H})}}+\hat{\theta}_1^{(\mathrm{H})}-1,$$

$$C_2^{(\mathrm{H})}=\frac{1}{\hat{\theta}_1^{(\mathrm{H})}}.$$

5.3 Reliability Estimation under the LINEX Loss Function

The LINEX loss function is asymmetric and is used when overestimation and underestimation do not have the same practical impact. This feature is important in reliability analysis because an overestimated reliability value may lead to an unrealistic evaluation of system performance. Let $c_L \neq 0$ be the LINEX shape parameter. The LINEX loss function is defined as [13]:

$L_{\text{LINEX}}(d_i,\theta_i)=\exp\{c_L(d_i-\theta_i)\}-c_L(d_i-\theta_i)-1.$

Taking the posterior expectation and minimizing it with respect to d_i gives the Bayes estimator under the LINEX loss function as

$$\hat{\theta}_i^{(\mathrm{L})} = -\frac{1}{c_L}\ln\left[E\left(e^{-c_L\theta_i} \mid X\right)\right], \qquad i = 1,2.$$

Using the posterior density proportional to the product of the likelihood function and the prior density, the estimator can be written as

$$\hat{\theta}_i^{(\mathrm{L})} = -\frac{1}{c_L}\ln\left[\frac{\int_1^\infty e^{-c_L\theta_i}\, L(\theta_i)\pi(\theta_i)\, d\theta_i}{\int_1^\infty L\,(\theta_i)\pi(\theta_i)\, d\theta_i}\right], i = 1,2.$$

Accordingly, the two LINEX estimators are denoted by

$$\hat{\theta}_1^{(\mathrm{L})}, \qquad \hat{\theta}_2^{(\mathrm{L})}.$$

By substituting the LINEX parameter estimators into equation (5), the reliability estimator under the LINEX loss function is expressed as

$$\hat{R}_{\mathrm{series}}^{(\mathrm{L})} = \frac{K\sum_{j=0}^{K-1}\binom{K-1}{j}\frac{(-1)^j}{\left(\hat{\theta}_2^{(\mathrm{L})}-1\right)^j}}{\left(\hat{\theta}_1^{(\mathrm{L})}-1\right)\left(\hat{\theta}_2^{(\mathrm{L})}-1\right)\left(\hat{\theta}_2^{(\mathrm{L})}\right)^2} I_j^{(\mathrm{L})},$$

where

$$I_j^{(\mathrm{L})} = \sum_{m=0}^{j}\binom{j}{m}\frac{\left(\hat{\theta}_2^{(\mathrm{L})}-1\right)^{j-m}}{\left(\hat{\theta}_2^{(\mathrm{L})}\right)^m}\left[C_0^{(\mathrm{L})}\frac{\Gamma(m+1)}{\left(\alpha_j^{(\mathrm{L})}\right)^{m+1}} + C_1^{(\mathrm{L})}\frac{\Gamma(m+2)}{\left(\alpha_j^{(\mathrm{L})}\right)^{m+2}} + C_2^{(\mathrm{L})}\frac{\Gamma(m+3)}{\left(\alpha_j^{(\mathrm{L})}\right)^{m+3}}\right],$$

with

$$\alpha_j^{(\mathrm{L})} = \frac{\hat{\theta}_2^{(\mathrm{L})} + [j+1]\hat{\theta}_1^{(\mathrm{L})}}{\hat{\theta}_1^{(\mathrm{L})}\hat{\theta}_2^{(\mathrm{L})}},$$

$$C_0^{(\mathrm{L})} = \left(\hat{\theta}_2^{(\mathrm{L})}-2\right)\hat{\theta}_2^{(\mathrm{L})}\left(\hat{\theta}_1^{(\mathrm{L})}-1\right),$$

$$C_1^{(\mathrm{L})} = \frac{\left(\hat{\theta}_2^{(\mathrm{L})}-2\right)\hat{\theta}_2^{(\mathrm{L})}}{\hat{\theta}_1^{(\mathrm{L})}} + \hat{\theta}_1^{(\mathrm{L})} - 1,$$

$$C_2^{(\mathrm{L})} = \frac{1}{\hat{\theta}_1^{(\mathrm{L})}}.$$

The three reliability estimators preserve the same analytical form of the derived series system reliability. The only difference is the method used to estimate θ_1 and θ_2. This makes the comparison between Log-Cosh, Huber, and LINEX estimators direct and mathematically consistent.

6. Simulation Study

Monte Carlo Simulation of $\widehat{\mathrm{R}}_{\mathrm{B}}$ for the IRL Distribution (Doubly Censored Data)

6.1 Data Generation Scheme

In this study we consider only one type of censored data: Doubly Censored Data (Double Censoring) scheme. This type combines both Type-I censoring and Type-II censoring and thus is more realistic to practical reliability cases where observations are limited by both time constraints and fixed numbers of failures.

Based on the IRL distribution, the simulation procedure was carried out in order to estimate the bounded reliability of the series system under this censoring mechanism.

6.2 Estimation Procedures and Sample Configurations

Three Bayesian prior distributions were considered: Flat Prior, Gamma Prior, and Jeffreys Prior. In addition, three different loss functions were applied: LINEX Loss Function, Huber Loss Function, and Log-Cosh Loss Function

This produced six main Bayesian estimation methods used for estimating the bounded reliability function. To study the effect of sample size (n) and Monte Carlo replications (m) sixteen simulation cases were constructed from all possible combinations of: $(n) = \{10,30,50,100\}, m = \{10,30,50,100\}$). The simulation cases are given as follows:

- Case 1: $n = 10, m = 10$
- Case 2: $n = 30, m = 10$
- Case 3: $n = 50, m = 10$
- Case 4: $n = 100, m = 10$
- Case 5: $n = 10, m = 30$
- Case 6: $n = 30, m = 30$
- Case 7: $n = 50, m = 30$
- Case 8: $n = 100, m = 30$
- Case 9: $n = 10, m = 50$
- Case 10: $n = 30, m = 50$
- Case 11: $n = 50, m = 50$
- Case 12: $n = 100, m = 50$
- Case 13: $n = 10, m = 100$
- Case 14: $n = 30, m = 100$
- Case 15: $n = 50, m = 100$
- Case 16: $n = 100, m = 100$

For each case, the estimation procedure was repeated mmm times, and the bounded reliability estimator

$\hat{R}_{series}$ was calculated using all Bayesian methods under the doubly censored scheme only.

6.3 Performance Evaluation Criteria

The performance of the estimators was evaluated using three statistical criteria:

1- **Mean Squared Error (MSE)** [14]**:** It is defined by:

$$MSE(R_B) = E\left[\left(\hat{R}_B - R_B\right)^2\right].$$

Smaller MSE values indicate better estimation accuracy.

2- **Pitman Closeness (PC)** [15]**:** for $\hat{R}_{series1}$ relative to $\hat{R}_{series2}$ is defined by:

$$PC\left(\hat{R}_{series1}, \hat{R}_{series2}\right) = P\left(\left|\hat{R}_{series1} - R_{series}\right| < \left|\hat{R}_{series2} - \hat{R}_{series}\right|\right).$$

Higher PC values indicate a better estimator.

3- **Relative Efficiency (RE)** [16]: It is defined as:

$$RE(\hat{R}_{series i}) = \frac{MSE(\hat{R}_{ref})}{MSE(\hat{R}_{series i})}.$$

Where $\hat{R}_{ref}$ is the reference estimator (typically the one with the minimum MSE). Larger values indicate better efficiency.

6.4 Simulation Results and Discussion

Table 1: Reliability Estimation value of $R = 0.306918259$ for Double data simulation when $\theta_1 = 5, \theta_2 = 4$

Case no.	Bayes_Gamma	Bayes_Jeffreys	Bayes_Flat	LogCosh	Huber	LINEX	Best
1	0.30091921	0.42716557	0.44644899	0.17994985	0.18119114	0.17970517	Bayes_Gamma
2	0.33317919	0.41694504	0.43005360	0.17977998	0.18094273	0.17951724	Bayes_Gamma
3	0.35075486	0.42416244	0.43573967	0.17982089	0.18094716	0.17955846	Bayes_Gamma
4	0.35166638	0.41606048	0.42663302	0.17979549	0.18091772	0.17952799	Bayes_Gamma
5	0.33616968	0.42924718	0.44340404	0.17974011	0.18085968	0.17947206	Bayes_Gamma
6	0.36727760	0.41796567	0.42548373	0.17956962	0.18060802	0.17928589	Bayes_Gamma
7	0.37486165	0.41593241	0.42221107	0.17952361	0.18053486	0.17923632	Bayes_Gamma
8	0.38306488	0.41653599	0.42183908	0.17952582	0.18052630	0.17923779	Bayes_Gamma
9	0.34349174	0.42520800	0.43756702	0.17962831	0.18072496	0.17935366	Bayes_Gamma
10	0.37835821	0.41981264	0.42575371	0.17954176	0.18053171	0.17925572	Bayes_Gamma
11	0.38659970	0.41856272	0.42321734	0.17949454	0.18046261	0.17920442	Bayes_Gamma
12	0.39479448	0.41932356	0.42304610	0.17950178	0.18046606	0.17920959	Bayes_Gamma
13	0.35728043	0.43371967	0.44477645	0.17965187	0.18072952	0.17937640	Bayes_Gamma
14	0.38884787	0.42267423	0.42726374	0.17952132	0.18050848	0.17923127	Bayes_Gamma
15	0.39298783	0.41703040	0.42034808	0.17942610	0.18039144	0.17913244	Bayes_Gamma
16	0.40216239	0.41879259	0.42116697	0.17942082	0.18038445	0.17912475	Bayes_Gamma

Table 2: Accuracy criteria value of $R = 0.306918259$ for Double data simulation when $\theta_1 = 5, \theta_2 = 4$

Case no.	Accuracy	Bayes_Gamma	Bayes_Jeffreys	Bayes_Flat	LogCosh	Huber	LINEX	Best
1	MSE	0.00602548	0.03186898	0.03699214	0.01612365	0.01580913	0.01618599	Bayes_Gamma
	PC	0.65153846	0.45461538	0.38284615	0.40484615	0.48330769	0.25092308	Bayes_Gamma
	RE	3.20640176	0.54385625	0.45788258	1.15006998	1.17448103	1.14534427	Bayes_Gamma
2	MSE	0.0058849	0.02314597	0.02626169	0.01616635	0.01587136	0.01623334	Bayes_Gamma
	PC	0.6650000	0.47400000	0.40853846	0.38730769	0.46484615	0.23292308	Bayes_Gamma
	RE	2.68741484	0.62591451	0.54252902	0.92935629	0.94805894	0.92520348	Bayes_Gamma
3	MSE	0.00696131	0.02364695	0.02633443	0.01615609	0.01587038	0.01622297	Bayes_Gamma
	PC	0.65576923	0.46369231	0.39392308	0.39753846	0.47561538	0.24330769	Bayes_Gamma
	RE	2.27086923	0.61423231	0.5436985	0.93468997	0.95290181	0.93051957	Bayes_Gamma
4	MSE	0.00624593	0.02087897	0.02331978	0.01616206	0.01587743	0.01623023	Bayes_Gamma
	PC	0.65792308	0.48300000	0.41476923	0.38253846	0.46061538	0.22838462	Bayes_Gamma
	RE	2.37663944	0.65705898	0.58023519	0.87127236	0.88827004	0.86728937	Bayes_Gamma
5	MSE	0.00597705	0.02814849	0.03210497	0.01617579	0.01589181	0.01624413	Bayes_Gamma
	PC	0.66961538	0.47376923	0.39100000	0.39730769	0.47530769	0.24323077	Bayes_Gamma
	RE	2.96039965	0.56802191	0.48854158	1.04538581	1.06544042	1.04066394	Bayes_Gamma
6	MSE	0.00859218	0.01935311	0.02108708	0.01621875	0.01595506	0.01629116	Bayes_Gamma
	PC	0.64846154	0.48584615	0.40630769	0.38961538	0.46853846	0.23523077	Bayes_Gamma
	RE	1.64250376	0.68644914	0.62367793	0.83397506	0.8490293	0.82992617	Bayes_Gamma
7	MSE	0.00908566	0.01783164	0.01922996	0.0162303	0.01597343	0.01630364	Bayes_Gamma

	PC	0.64369231	0.50600000	0.42623077	0.37853846	0.45776923	0.22423077	Bayes_Gamma
	RE	1.48768964	0.72028529	0.6623157	0.79894085	0.81302588	0.79500107	Bayes_Gamma
8	MSE	0.00971525	0.01708837	0.01826903	0.01622964	0.01597553	0.01630316	Bayes_Gamma
	PC	0.64692308	0.50953846	0.42930769	0.37769231	0.45630769	0.22353846	Bayes_Gamma
	RE	1.35889789	0.73938474	0.68662941	0.78257608	0.79624776	0.77870034	Bayes_Gamma
9	MSE	0.00564056	0.02599526	0.02938642	0.01620396	0.01592558	0.01627403	Bayes_Gamma
	PC	0.68415385	0.49861538	0.41584615	0.38100000	0.45961538	0.22676923	Bayes_Gamma
	RE	3.48181601	0.69526696	0.60615715	1.16186589	1.18351931	1.1565316	Bayes_Gamma
10	MSE	0.00873117	0.01762413	0.01901751	0.01622544	0.01597406	0.01629844	Bayes_Gamma
	PC	0.66646154	0.49592308	0.41300000	0.38538462	0.46400000	0.23092308	Bayes_Gamma
	RE	1.54042488	0.72432761	0.66562141	0.79339786	0.80709419	0.78950011	Bayes_Gamma
11	MSE	0.00982811	0.01676757	0.01782819	0.01623744	0.0159915	0.0163115	Bayes_Gamma
	PC	0.65123077	0.50076923	0.41507692	0.38361538	0.46361538	0.22946154	Bayes_Gamma
	RE	1.32894619	0.74710995	0.69808721	0.77401357	0.78710054	0.77014997	Bayes_Gamma
12	MSE	0.01054221	0.01594381	0.01678348	0.01623547	0.01599054	0.01631004	Bayes_Gamma
	PC	0.65961538	0.51100000	0.42500000	0.37676923	0.45584615	0.22207692	Bayes_Gamma
	RE	1.20666632	0.77179941	0.72933838	0.7565528	0.76931916	0.75274188	Bayes_Gamma
13	MSE	0.00561342	0.02715571	0.03041688	0.01619772	0.01592425	0.01626798	Bayes_Gamma
	PC	0.70415385	0.48461538	0.40161538	0.38769231	0.46569231	0.23330769	Bayes_Gamma
	RE	3.04900485	0.56924498	0.49996559	1.00638698	1.02499047	1.00170829	Bayes_Gamma
14	MSE	0.00935605	0.01711671	0.01822938	0.01623047	0.01597979	0.01630449	Bayes_Gamma
	PC	0.66769231	0.51161538	0.42446154	0.37961538	0.45907692	0.22476923	Bayes_Gamma
	RE	1.4095469	0.73558589	0.68599285	0.77995192	0.79339406	0.77606158	Bayes_Gamma
15	MS	0.0099685	0.0152481	0.0160003	0.0162546	0.0160093	0.0163296	Bayes_Gamm

	E	5	2	1	7	5	6	a
	PC	0.66661538	0.52384615	0.43700000	0.36761538	0.44753846	0.21300000	Bayes_Gamma
	RE	1.2522562	0.79203639	0.7511858	0.7382269	0.75071818	0.73448344	Bayes_Gamma
16	MSE	0.01121666	0.01490296	0.01543864	0.01625593	0.01601105	0.01633153	Bayes_Gamma
	PC	0.65830769	0.51969231	0.43661538	0.36969231	0.44884615	0.21538462	Bayes_Gamma
	RE	1.09608135	0.80593471	0.77530199	0.73245518	0.74483404	0.72870821	Bayes_Gamma

Table 1 presents the estimated reliability values for the true reliability: $R = 0.306918259$ when $\theta_1 = 5, \theta_2 = 4$ under doubly censored data simulation. The results clearly show that the **Bayes-Gamma** estimator consistently produced the closest estimated values to the true system reliability across all sixteen simulation cases. This indicates that the **Bayes-Gamma** loss function provides strong robustness and stability in reliability estimation, especially under doubly censored samples.

On the other hand, Table 2 shows the accuracy criteria results based on MSE, PC, and RE. It can be observed that the Bayes-Gamma estimator achieved the best performance in most cases according to all three criteria. It provided the smallest MSE values, the highest Pitman Closeness probabilities, and the largest Relative Efficiency values. Therefore, the simulation results recommend that:

- **Bayes-Gamma** is the best method for obtaining reliability estimates closest to the true value.
- **Bayes-Gamma** is the best estimator according to statistical accuracy criteria.

These findings confirm that Bayesian estimation under doubly censored data provides highly reliable and efficient results for series system reliability modelling.

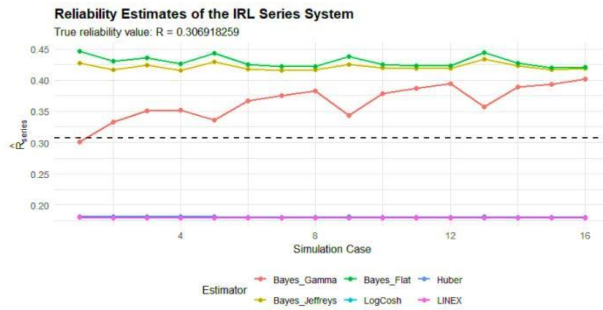

Figure 1. Reliability Estimates of the IRL Series System under Doubly Censored Data

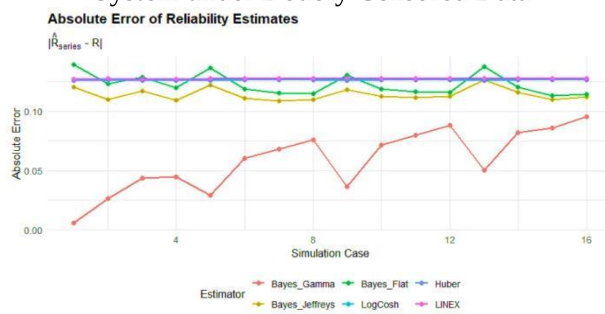

Figure 2. Absolute Error of Reliability Estimates

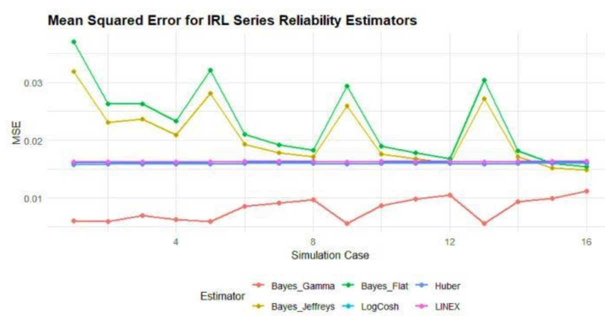

Figure 3. Mean Squared Error of Series Reliability Estimators

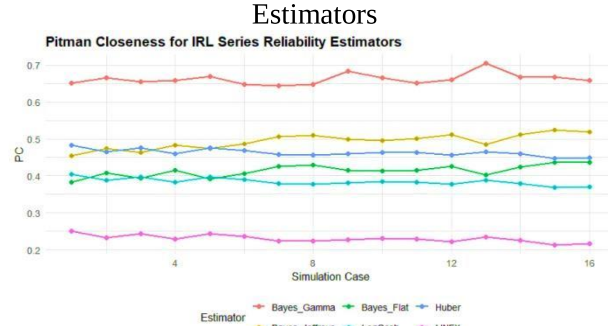

Figure 4. Pitman Closeness of Series Reliability Estimators

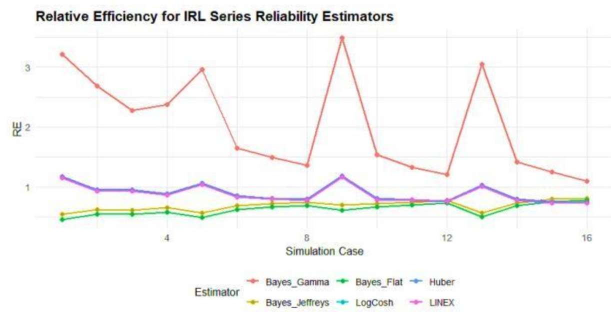

Figure 5. Relative Efficiency of Series Reliability Estimators

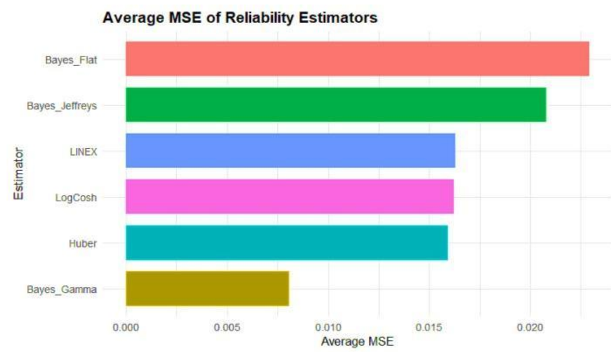

Figure 6. Average MSE of Reliability Estimators

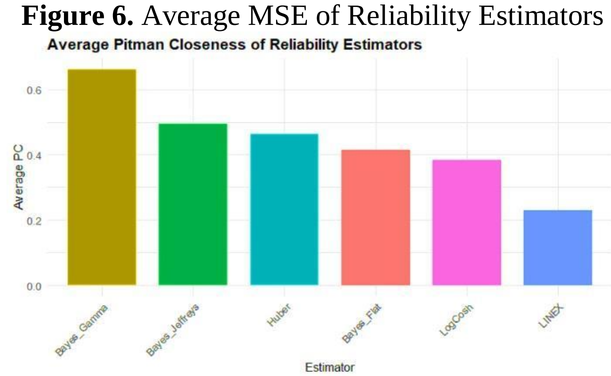

Figure 7. Average Pitman Closeness of Reliability Estimators

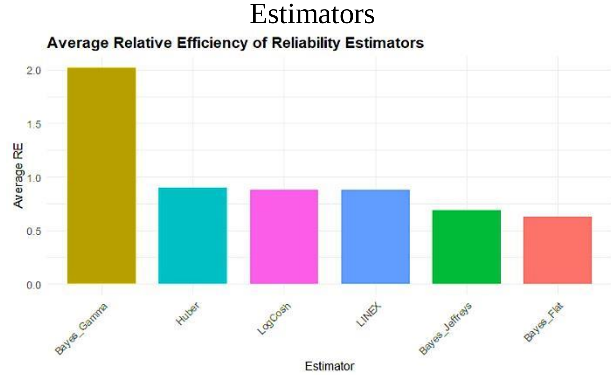

Figure 8. Average Relative Efficiency of Reliability Estimators

Figure 1 compares the estimated reliability values obtained from the six estimation methods across the sixteen simulation cases. The true reliability value is $R = 0.306918259$. The Huber, Log-Cosh, and LINEX estimators remain very close to the true value, while Bayes-Gamma, Bayes-Jeffreys, and Bayes-Flat give larger estimates. The Huber estimator shows the closest overall behaviour to the true reliability value. Figure 2 presents the absolute error between each estimated reliability value and the true reliability value. The smallest errors are obtained by the Huber, Log-Cosh, and LINEX estimators. Among them, Huber remains the most stable and closest in most simulation cases. The Bayesian prior-based estimators show larger deviations from the true value. Figure 3 shows the MSE values for all estimation methods across the sixteen cases. Bayes-Gamma gives the lowest MSE in most cases, which indicates stronger numerical accuracy according to this criterion. Huber, Log-Cosh, and LINEX produce close MSE values with small differences among them. Bayes-Flat and Bayes-Jeffrey's show relatively higher MSE values in several cases. Figure 4 displays the Pitman Closeness values for the competing estimators. Bayes-Gamma achieves the highest PC values in nearly all simulation cases. This means that Bayes-Gamma has the highest probability of being closer to the true reliability value compared with the other estimators under this criterion. LINEX records the lowest PC values among the methods. The comparative efficiency values of the 6 estimators are given in Figure 5. Further, Bayes-Gamma proves to be the best, particularly in small and moderate simulation scenarios. We confirm its efficiency criterion-based superiority with this result. The rest of the approaches exhibit lower and more enduring RE values, in several cases Huber is slightly better than Log-Cosh and LINEX. The average MSE for all simulation cases is presented in Figure 6. Bayes-Gamma appears to achieve the lowest average MSE, and so has the best overall accuracy based on the mean squared error criterion. Bayes-Flat and Bayes-Jeffrey's have the highest average MSE values by far. Huber, Log-Cosh, and LINEX are all close, with a slight advantage of Huber. Average PC values for all estimators are shown in Figure 7. Bayes-Gamma achieves the highest average Pitman Closeness, indicating that Bayes-Gamma performance is better in comparative closeness. After that comes Bayes-Jeffrey's, followed by Huber and Bayes-Flat. LINEX has the lowest average PC value, suggesting it performs worse under this criterion. Average relative efficiency for all the estimators are expressed in Figure 8. Bayes-Gamma has the highest average RE, indicating its better efficiency for all the simulation scenarios. Huber, Log-Cosh, and LINEX exhibit intermediate and similar efficiencies. Bayes-Flat and Bayes-Jeffreys are significantly less efficient than Bayes-Gamma.

7. Conclusion

In this paper, a Bayesian framework was developed to estimate the bounded reliability of a series system based on the IRL distribution, under doubly censored data. The reliability function was derived in closed form, and Bayesian estimators were obtained under Flat, Gamma, and Jeffrey's priors using different loss functions. In simulation we observed that the **Bayes-Gamma** method obtained the most stable reliability estimates closer to the true value R = 0.306918259. The Bayes-Gamma performed the best based on MSE, PC, and RE criteria.

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