Endo Classical 2-Absorbing Second Submodules and Related Concepts

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ABSTRACT

In this article, we introduce the concept of endo classical 2-Absorbing second submodule as a new generalization of endo 2-Absorbing second submodule over commutative ring with identity, where a non-zero submodule $\mathbb{P}$ of $\check{\mathbb{V}}$ as a $\mathbb{D}$-module is called endo classical 2-Absorbing second submodule if $\forall \ell, g, \hbar \in End(\check{\mathbb{V}})$ and $\mathcal{B}$ is completely irreducible of $\check{\mathbb{V}}$ if whenever $(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$, implies either $\ell \circ g(\mathbb{P}) \subseteq \mathcal{B}$ or $\ell \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$ or $g \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$. These concepts are defined by combining generalized absorbing conditions on second submodule with the action of module endomorphisms. We develop several characterizations and inclusion relations among these classes and examine a number of their algebraic characteristics.

There is a thorough discussion of the relationships between endo classical 2-Absorbing second submodule and other related ideas like endo-second submodule, 2-Absorbing second submodule, endo 2-absorbing second submodule, and endo quasi prime second submodule. Furthermore, we investigate the preservation of these qualities under factor modules, sum of submodules, module homomorphic image, and inverse homomorphic image. Conditions under which these concepts coincide are also found.

1. Introduction

In this paper, all rings are assumed to be rings with identity, and every module is considered as a unitary right module. The concept of second modules was first introduced by S. Yassemi in [6], where a $\mathbb{D}$-module $\check{\mathbb{V}}$ is referred to as the second module if $\check{\mathbb{V}} \neq 0$ and for each $t \in$ D, either $\check{\mathbb{V}}t = (0)$ or $\check{\mathbb{V}}t = \check{\mathbb{V}}$. Equivalently $\check{\mathbb{V}}$ is second module if for each ideal $\check{\mathbb{V}}$ of $\mathbb{D}$, either $H\check{\mathbb{V}} = (0)$ or $H\check{\mathbb{V}} = \check{\mathbb{V}}$. A non-zero submodule is referred to as $\mathbb{P}$ of a $\mathbb{D}$-module $\check{\mathbb{V}}$ is a second submodule that is generalized 2-Absorbing Anytime $a, b \in$ D, U is a submodule that is completely irreducible of $\check{\mathbb{V}}$ and $abJ \subseteq U$, then $a\mathbb{P} \in U$ or $b\mathbb{P} \in U$ or $ab \in Ann\,\mathrm{D}(\mathbb{P})$. The term "generalized 2-Absorbing second module" refers to a module that is itself a generalized 2-Absorbing second submodule. [3], [4].[8] A submodule $\mathbb{P}$ of $\check{\mathbb{V}}$ is completely irreducible, if $\mathbb{P} = \cap_{i \in I} B_i$, where $\{B_i\}_{i \in I}$ is a family of submodules of $\check{\mathbb{V}}$, implies $\mathbb{P} = B_i$for some i ∈ I . In [7], the authors introduced the notion of classical 2-Absorbing submodules as a generalization of 2-Absorbing submodules and studied some properties of this class of modules. A proper submodule N of M is called classical 2-Absorbing submodule if whenever $a, b, c \in R$ and $m \in M$ with $abcm \in N$, then $abm \in N$ or $acm \in N$ or $bcm \in N$ [7]. This study's main goal is to provide and explore the ideas of endo classical 2-Absorbing second subm as new generalized classes in module theory. These ideas extend a number of traditional ideas about second subm by combining the action of module endomorphisms with generalized absorption conditions.

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2. Endo Classical 2-Absorbing Second Submodules

Definition 2.1 Let $\mathbb{P} \neq 0$ be a submodule of $\mathbb{D}$-module $\mathbb{V}$, $\mathbb{P}$ is named an endo classical 2-Absorbing second submodule (simple ECTAB) second submodule of V whenever $\ell, g, \hbar \in End(\mathbb{V})$ and $\mathcal{B}$ is completely

irreducible submodule of $\mathbb{V}$ such that $(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$ implies either $\ell \circ g(\mathbb{P}) \subseteq \mathcal{B}$ or $\ell \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$ or $g \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$.

A $\mathbb{D}$-module $\mathbb{V}$ is called an ECTAB second module if $\mathbb{V}$ is an ECTAB second submodule of itself.

Example 2.2

Think of Z_{24} as Z-module and $\ell, g, \hbar \in End(Z_{24})$ such that $\ell(\bar{x}) = 5\bar{x}$, $g(\bar{x}) = \bar{x}$, $\hbar(\bar{x}) = 2(\bar{x})$, $\forall \bar{x} \in Z_{24}$ where $(\bar{2})$ is completely irreducible submodule and $(\bar{3})$ is ECTAB second submodule since $(\ell \circ g \circ \hbar)((\bar{3})) = (\bar{6}) \subseteq (\bar{2})$, then $\ell \circ g((\bar{3})) = (\bar{6}) \subseteq (\bar{2})$, $(\ell \circ \hbar)((\bar{3})) = (\bar{6}) \subseteq (\bar{2})$and $(g \circ \hbar)((\bar{3})) = (\bar{6}) \subseteq (\bar{2})$.

Proposition 2.3

Let $\mathbb{V}$ be a $\mathbb{D}$ -module and $\mathbb{P} \neq 0$ submodule of $\mathbb{V}$,Then the following statements are equivalent :

1. $\mathbb{P}$ is an ECTAB second submodule of $\mathbb{V}$.
2. For $\ell, g \in End(\mathbb{V})$ and $\mathcal{B}$ is completely irreducible submodule of $\mathbb{V}$ with $(\ell \circ g)(\mathbb{P}) \nsubseteq \mathcal{B}$,then $(\mathcal{B}:_{\mathrm{D}} (\ell \circ g)(\mathbb{P})) = (\mathcal{B}:_{\mathrm{D}} \ell(\mathbb{P})) \cup (\mathcal{B}:_{\mathrm{D}} g(\mathbb{P}))$;
3. For $\ell, g \in End(\mathbb{V})$ and $\mathcal{B}$ is completely irreducible submodule of $\mathbb{V}$ with$(\ell \circ g)(\mathbb{P}) \nsubseteq \mathcal{B}$,then $(\mathcal{B}:_{\mathrm{D}} (\ell \circ g)(\mathbb{P})) = (\mathcal{B}:_{\mathrm{D}} \ell(\mathbb{P}))\ or\ (\mathcal{B}:_{\mathrm{D}} g(\mathbb{P}))$

Proof:

(1)$\Longrightarrow$ (2) Let $\hbar \in (\mathcal{B}:_{\mathrm{D}} (\ell \circ g)(\mathbb{P}))$, then $(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$, but $(\ell \circ g)(\mathbb{P}) \nsubseteq \mathcal{B}$and $\mathbb{P}$ is an ECTAB second submodule of $\mathbb{V}$, so we have $(\hbar \circ \ell)(\mathbb{P}) \subseteq \mathcal{B}$ or $(\hbar \circ g)(\mathbb{P}) \subseteq \mathcal{B}$, hence $\hbar \in (\mathcal{B}:_{\mathrm{D}} \ell(\mathbb{P}))\ or\ (\mathcal{B}:_{\mathrm{D}} g(\mathbb{P}))$, so that $\hbar \in (\mathcal{B}:_{\mathrm{D}} \ell(\mathbb{P})) \cup (\mathcal{B}:_{\mathrm{D}} g(\mathbb{P}))$. Thus,$(\mathcal{B}:_{\mathrm{D}} (\ell \circ g)(\mathbb{P})) \subseteq (\mathcal{B}:_{\mathrm{D}} \ell(\mathbb{P})) \cup (\mathcal{B}:_{\mathrm{D}} g(\mathbb{P}))$.

Now, let $\hbar \in (\mathcal{B}:_{\mathrm{D}} \ell(\mathbb{P})) \cup (\mathcal{B}:_{\mathrm{D}} g(\mathbb{P}))$, then $\hbar \in (\mathcal{B}:_{\mathrm{D}} \ell(\mathbb{P}))$ or $\hbar \in (\mathcal{B}:_{\mathrm{D}} g(\mathbb{P}))$, hence $(\hbar \circ \ell)(\mathbb{P}) \subseteq \mathcal{B}$ or $(\hbar \circ g)(\mathbb{P}) \subseteq \mathcal{B}$ since $\mathbb{P}$ is an ECTAB second submodule of $\mathbb{V}$, so that $(\hbar \circ \ell \circ g)(\mathbb{P}) \subseteq \mathcal{B}$, then $\hbar \in (\mathcal{B}:_{\mathrm{D}} (\ell \circ g)(\mathbb{P}))$,then $(\mathcal{B}:_{\mathrm{D}} \ell(\mathbb{P})) \cup (\mathcal{B}:_{\mathrm{D}} g(\mathbb{P})) \subseteq (\mathcal{B}:_{\mathrm{D}} (\ell \circ g)(\mathbb{P}))$. Thus, $(\mathcal{B}:_{\mathrm{D}} (\ell \circ g)(\mathbb{P})) = (\mathcal{B}:_{\mathrm{D}} \ell(\mathbb{P})) \cup (\mathcal{B}:_{\mathrm{D}} g(\mathbb{P}))$.

(2)$\Longrightarrow$ (3) This following from the fact that if an ideal is the union of two ideals, then it is equal to one of them .

(3)$\Longrightarrow$ (1) Let $\ell, g, \hbar \in End(\mathbb{V})$ and $\mathcal{B}$ is completely irreducible submodule of $\mathbb{V}$ such that $(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$ with$(\ell \circ g)(\mathbb{P}) \nsubseteq \mathcal{B}$, then $\hbar \in (\mathcal{B}:_{\mathrm{D}} (\ell \circ g)(\mathbb{P}))$ from (3), we have $\hbar \in (\mathcal{B}:_{\mathrm{D}} \ell(\mathbb{P}))$ or $\hbar \in (\mathcal{B}:_{\mathrm{D}} g(\mathbb{P}))$, hence $(\hbar \circ \ell)(\mathbb{P}) \subseteq \mathcal{B}$ or $(\hbar \circ g)(\mathbb{P}) \subseteq \mathcal{B}$.Thus, $\mathbb{P}$ is ECTAB second submodule.

Definition 2.4

Let $\mathbb{P} \neq 0$ be a submodule of $\mathbb{D}$-module $\check{\mathbb{V}}$ $\ell, g, \hbar \in End(\check{\mathbb{V}})$ and $\mathcal{B}$ is completely irreducible of $\check{\mathbb{V}}$. Then $\mathbb{P}$ is endo quasi prime second submodule (simply EQP second) If whenever $(\ell \circ g)(\mathbb{P}) \subseteq \mathcal{B}$ implies either $\ell(\mathbb{P}) \subseteq \mathcal{B}$ or $g(\mathbb{P}) \subseteq \mathcal{B}$.
A $\mathbb{D}$-module $\check{\mathbb{V}}$ is EQP second module if it itself EQP second submodule.

Example 2.5

Think of Z_{12} as Z-module and $\ell, g \in End(Z_{12})$ such that $\ell(\bar{x}) = 2\bar{x}$, $g(\bar{x}) = \bar{x}$, $\forall \bar{x} \in Z_{12}$ where $(\bar{2})$ is completely irreducible and $(\bar{3})$ is EQP second submodule since $(\ell \circ g)\big((\bar{3})\big) = (\bar{6}) \subseteq (\bar{2})$, then $\ell\big((\bar{3})\big) = (\bar{6}) \subseteq (\bar{2})$.

Proposition 2.6

Every EQP second submodule of a $\mathbb{D}$-module $\check{\mathbb{V}}$ is ECTAB second submodule.

Proof

Let $\ell, g, \hbar \in End(\check{\mathbb{V}})$ and let $\mathcal{B}$ be a completely irreducible submodule of $\check{\mathbb{V}}$ such that $(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$,then$(\ell \circ g)(\hbar(\mathbb{P})) \subseteq \mathcal{B}$, but P $\mathbb{P}$ is EQP second submodule with $(\ell \circ g)(\mathbb{P}) \nsubseteq \mathcal{B}$, hence $\ell(\hbar(\mathbb{P})) \subseteq \mathcal{B}$ or $g\big(\hbar(\mathbb{P})\big) \subseteq \mathcal{B}$, then $\ell \circ g(\mathbb{P}) \subseteq \mathcal{B}$ or $g \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$.Thus, $\mathbb{P}$ is ECTAB second submodule.

Remark 2.7

The converse of Proposition 2.6 incorrect in general, for example: consider $Z_3 \oplus Z_3$ as Z_3-module , $\langle(\bar{1}, \bar{2})\rangle$ is completely irreducible submodule where $\ell(\bar{x}, \bar{y}) = (\bar{x}, \bar{0})$, $g(\bar{x}, \bar{y}) = (\bar{0}, \bar{y})$ and $\hbar(\bar{x}, \bar{y}) = (0, \overline{2y}), \forall\, \ell, g, \hbar \in End(Z_3 \oplus Z_3)$, then $\langle(\bar{1}, \bar{2})\rangle = \{(\bar{0},\bar{0}), (\bar{1}, \bar{2}), (\bar{2}, \bar{1})\}$ so $\langle(\bar{1}, \bar{2})\rangle$ is ECTAB second submodule in $(Z_3 \oplus Z_3)$ since $(\ell \circ g \circ \hbar)\langle(\bar{1}, \bar{2})\rangle = (\bar{0}, \bar{0}) \in \langle(\bar{1}, \bar{2})\rangle$, then $(\ell \circ g)\langle(\bar{1}, \bar{2})\rangle = (\bar{0}, \bar{0}) \in \langle(\bar{1}, \bar{2})\rangle$ and $(\ell \circ \hbar)\langle(\bar{1}, \bar{2})\rangle = (\bar{0}, \bar{0}) \in \langle(\bar{1}, \bar{2})\rangle$, but $\langle(\bar{1}, \bar{2})\rangle$ is not EQP second submodule in $(Z_3 \oplus Z_3)$ since$(\ell \circ g)\langle(\bar{1}, \bar{2})\rangle = (\bar{0}, \bar{0}) \in \langle(\bar{1}, \bar{2})\rangle$, but $\ell\langle(\bar{1}, \bar{2})\rangle = \langle(\bar{1}, \bar{0})\rangle \nsubseteq \langle(\bar{1}, \bar{2})\rangle$ and $g(\langle(\bar{1}, \bar{2})\rangle) = \langle(\bar{0}, \bar{2})\rangle \nsubseteq \langle(\bar{1}, \bar{2})\rangle$.

Definition 2.8

Let $\mathbb{P} \neq 0$ be a sub-module of $\mathbb{D}$-module $\check{\mathbb{V}}$, $\ell, g \in End(\check{\mathbb{V}})$ and $\mathcal{B}$ is completely irreducible of $\check{\mathbb{V}}$, then $\mathbb{P}$ is endo 2-Absorbing second submodule (simply ETAB) second submodule if when ever $(\ell \circ g)(\mathbb{P}) \subseteq \mathcal{B}$ implies either $\ell(\mathbb{P}) \subseteq \mathcal{B}$ or $g(\mathbb{P}) \subseteq \mathcal{B}$ or$(\ell \circ g)(\mathbb{P}) = 0$.

Example 2.9

See Example 2.5

Proposition 2.10

Every ETAB second submodule of a $\mathbb{D}$-module $\check{\mathbb{V}}$ is ECTAB second submodule.

Proof

Let $\ell, g, \hbar \in End(\check{\mathbb{V}})$ and let $\mathcal{B}$ be a completely irreducible submodule of $\check{\mathbb{V}}$ such that $(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$, then$(\ell \circ g)(\hbar(\mathbb{P})) \subseteq \mathcal{B}$ and let $(\ell \circ g)(\hbar(\mathbb{P})) \neq 0$, but $\mathbb{P}$ is ETAB second submodule, then either $\ell(\hbar(\mathbb{P})) \subseteq \mathcal{B}$ or $g\big(\hbar(\mathbb{P})\big) \subseteq \mathcal{B}$, so that $\ell \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$ or $g \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$.Thus , $\mathbb{P}$ is ECTAB second submodule of $\check{\mathbb{V}}$.

Remark 2.11

The converse of Proposition 2.10 incorrect in general, for example: consider $Z_7 \oplus Z_7$ as Z_7-module , $\langle(\bar{1},\bar{6})\rangle$ is completely irreducible where $\ell(\bar{x},\bar{y}) = (\bar{x},\bar{0})$, $g(\bar{x},\bar{y}) = (\bar{0},\bar{y})$ and $\hbar(\bar{x},\bar{y}) = (0,\overline{2y}), \forall\, \ell, g, \hbar \in End(Z_7 \oplus Z_7)$, then $\langle(\bar{1},\bar{6})\rangle=\{(\bar{0},\bar{0}), (\bar{1},\bar{6}), (\bar{2},\bar{5}), (\bar{3},\bar{4}), (\bar{4},\bar{3}), (\bar{5},\bar{2}), (\bar{6},\bar{1})\}$, so $\langle(\bar{1},\bar{2})\rangle$ is ECTAB second submodule in $(Z_7 \oplus Z_7)$ since $(\ell \circ g \circ \hbar)(\mathbb{P})\langle(\bar{1},\bar{2})\rangle = (\bar{0},\bar{0}) \in \langle(\bar{1},\bar{6})\rangle$, then $(\ell \circ g)\langle(\bar{1},\bar{2})\rangle = (\bar{0},\bar{0}) \in \langle(\bar{1},\bar{6})\rangle$ and $(\ell \circ \hbar)\langle(\bar{1},\bar{2})\rangle = (\bar{0},\bar{0}) \in \langle(\bar{1},\bar{6})\rangle$, but $\langle(\bar{1},\bar{2})\rangle$ is not ETAB second submodule in $(Z_7 \oplus Z_7)$ since $(\ell \circ g)\langle(\bar{1},\bar{2})\rangle = (\bar{0},\bar{0}) \in \langle(\bar{1},\bar{6})\rangle$, but $\ell(\langle(\bar{1},\bar{2})\rangle) = \langle(\bar{1},\bar{0})\rangle \nsubseteq \langle(\bar{1},\bar{6})\rangle$, $g(\langle(\bar{1},\bar{2})\rangle) = \langle(\bar{0},\bar{2})\rangle \nsubseteq \langle(\bar{1},\bar{6})\rangle$.

Definition 2.12

Give $\mathbb{P} \neq 0$ be a sub-module of $\mathbb{D}$-module $\mathbb{V}$, $\mathbb{P}$ is referred to as the endo second submodule of $\mathbb{V}$, if $\forall\, \ell \in End(\mathbb{V})$then either$\ell(\mathbb{P}) = \mathbb{P}$ *or* $\ell(\mathbb{P}) = 0$.

A $\mathbb{D}$-module $\mathbb{V}$ is endo second module if it itself endo second submodule.

Examples 2.13

1) Consider Z_6as Z-module then $(\bar{2})$,$(\bar{3})$ are endo second submodule of Z_6 where $\ell \in End(Z_6)$ such that $\ell(\bar{x}) = 3\bar{x}$, $\forall \bar{x} \in Z_6$, $\ell((\bar{2})) = 0$ *and* $\ell((\bar{3})) = (\bar{3})$.
2) Consider Z_{12} as Z-module, then $(\bar{3})$ is not Endo Second submodule of Z_{12} where $\ell(\bar{x}) = 2\bar{x}$ there for $\ell((\bar{3})) = (\bar{6}) \neq (\bar{3})$ and $\ell((\bar{3})) \neq 0$.

Proposition 2.14

Let $\mathbb{P} \neq 0$ be a submodule of $\mathbb{D}$-module $\mathbb{V}$. If $\mathbb{P}$ is endo second submodule of $\mathbb{V}$, then $\mathbb{P}$ is ECTAB second submodule of $\mathbb{V}$.

Proof:

Let $\ell, g, \hbar \in End(\mathbb{V})$ and let $\mathcal{B}$ be a completely irreducible of $\mathbb{V}$ such that $(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$,then $f(g \circ \hbar(\mathbb{P})) \subseteq \mathcal{B}$, with $(\ell \circ g)(\mathbb{P}) \nsubseteq \mathcal{B}$ and $(\ell \circ \hbar)(\mathbb{P}) \nsubseteq \mathcal{B}$, but $\mathbb{P}$ is endo second submodule, then $(g \circ \hbar)(\mathbb{P}) = \ell(g \circ \hbar(\mathbb{P})) \subseteq \mathcal{B}$ hence $g \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$. Thus, $\mathbb{P}$ is ECTAB second submodule of $\mathbb{V}$.

Remark 2.15

The converse of Proposition 2.14 is incorrect in general, for example: consider Z_{24} a Z-module, and $\ell, g, \hbar \in End(Z_{24})$ such that $\ell(\bar{x}) = 2\bar{x}$, $g(\bar{x}) = 4\bar{x}$, $\hbar(\bar{x}) = 6\bar{x}$, $\forall \bar{x} \in Z_{24}$ where $(\bar{2})$ is completely irreducible then $(\ell \circ g \circ \hbar)\big((\bar{3})\big) = (\ell \circ g)\big(\hbar(\bar{3})\big) = \ell\big(g(\overline{18})\big) = \ell\big((\bar{0})\big) = (\bar{0}) \subseteq (\bar{2})$, then $(\ell \circ g)(\bar{3}) = (\bar{0}) \subseteq (\bar{2})$, $(\ell \circ \hbar)(\bar{3}) = (\overline{12}) \subseteq (\bar{2})$ and $(g \circ \hbar)(\bar{3}) = (\bar{0}) \subseteq (\bar{2})$.. Hence $(\bar{3})$ is ECTAB second submodule, but $(\bar{3})$ is not endo second submodule since $\ell\big((\bar{3})\big) = (\bar{6}) \neq (\bar{3})$ and $\ell\big((\bar{3})\big) \neq (\bar{0})$, $g\big((\bar{3})\big) = (\overline{12}) \neq (\bar{3})$ and $g\big((\bar{3})\big) \neq (\bar{0})$, also $\hbar\big((\bar{3})\big) = (\overline{18}) \neq (\bar{3})$ and $\hbar\big((\bar{3})\big) \neq (\bar{0})$.

Remark 2.16 [1]

Let $\mathbb{P}$ and S be two submodules of a $\mathbb{D}$-module $\mathbb{V}$. To prove $\mathbb{P} \subseteq$ S, it is enough to show that if $\mathcal{B}$ is a completely irreducible submodule of $\mathbb{V}$ such that S $\subseteq \mathcal{B}$, then $\mathbb{P} \subseteq \mathcal{B}$.

Proposition 2.17

If $\mathbb{P} \neq 0$ is fully invariant submodule of $\mathbb{D}$-module $\check{\mathbb{V}}$, then statements are comparable :

a) $\mathbb{P}$ is ECTAB second submodule of $\check{\mathbb{V}}$;

b) $(\ell \circ g \circ \hbar)(\mathbb{P}) = (\ell \circ g)(\mathbb{P})$ or $(\ell \circ g \circ \hbar)(\mathbb{P}) = (\ell \circ \hbar)(\mathbb{P})$ or $(\ell \circ g \circ \hbar)(\mathbb{P}) = (g \circ \hbar)(\mathbb{P})$

c) $(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3$ where $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ are completely irreducible submodule of $\check{\mathbb{V}}$, implies either $(\ell \circ g)(\mathbb{P}) \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3$ or $(\ell \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3$, $(g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3$

Proof:

(a) $\Rightarrow$ (b) Assume that $(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$, where $\mathcal{B}$ are completely irreducible submodule of $\check{\mathbb{V}}$, so by part (a), implies either $(\ell \circ g)(\mathbb{P}) \subseteq \mathcal{B}$ or $(\ell \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$ or $(g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$, then by Remark 2.10, we have $(\ell \circ g)(\mathbb{P}) \subseteq (\ell \circ g \circ \hbar)(\mathbb{P})$ or $(\ell \circ \hbar)(\mathbb{P}) \subseteq (\ell \circ g \circ \hbar)(\mathbb{P})$ or $(g \circ \hbar)(\mathbb{P}) \subseteq (\ell \circ g \circ \hbar)(\mathbb{P})$, but $\mathbb{P}$ is fully invariant submodule, so that $(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq (\ell \circ g)(\mathbb{P})$ or$(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq (\ell \circ \hbar)(\mathbb{P})$ or $(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq (g \circ \hbar)(\mathbb{P})$. Thus, $(\ell \circ g \circ \hbar)(\mathbb{P}) = (\ell \circ g)(\mathbb{P})$ or $(\ell \circ g \circ \hbar)(\mathbb{P}) = (\ell \circ \hbar)(\mathbb{P})$ or $(\ell \circ g \circ \hbar)(\mathbb{P}) = (g \circ \hbar)(\mathbb{P})$.

(b) $\Rightarrow$ (a) Let $\mathcal{B}$ be completely irreducible submodule with $(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$. By part (b), we have $(\ell \circ g \circ \hbar)(\mathbb{P}) = (\ell \circ g)(\mathbb{P})$ or $(\ell \circ g \circ \hbar)(\mathbb{P}) = (\ell \circ \hbar)(\mathbb{P})$ or $(\ell \circ g \circ \hbar)(\mathbb{P}) = (g \circ \hbar)(\mathbb{P})$, hence either$(\ell \circ g)(\mathbb{P}) = (\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$ or $(\ell \circ \hbar)(\mathbb{P}) = (\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$ or $(g \circ \hbar)(P) = (\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$. Thus, $\mathbb{P}$ is ECTAB second submodule of $\check{\mathbb{V}}$.

(a) $\Rightarrow$ (c) Assume that $(\ell \circ g)(\mathbb{P}) \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3$, where $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ are completely irreducible submodules of $\check{\mathbb{V}}$, hence $(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3 \subseteq \mathcal{B}_1$,$(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3 \subseteq \mathcal{B}_2$ and $(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3 \subseteq \mathcal{B}_3$, by part (c), implies either $(\ell \circ g)(\mathbb{P}) \subseteq \mathcal{B}_1$ or $(\ell \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}_1$ or $(g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}_1$, and $(\ell \circ g)(\mathbb{P}) \subseteq \mathcal{B}_2$ or $(\ell \circ \hbar)(P) \subseteq \mathcal{B}_2$ or $(g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}_2$, also $(\ell \circ g)(\mathbb{P}) \subseteq \mathcal{B}_3$ or $(\ell \circ \hbar)(P) \subseteq \mathcal{B}_3$ or $(g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}_3$, so that $(\ell \circ g)(\mathbb{P}) \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3$ or $(\ell \circ g)(\mathbb{P}) \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3$ or $(g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3$.

(c) $\Rightarrow$ (b) similarly by (a) $\Rightarrow$ (b), we get the result.

Lemma 2.18.[2] Let $\ell : \check{\mathbb{V}} \rightarrow \check{\mathbb{V}}'$ be a monomorphism of $\mathbb{D}$ -modules. If $\mathcal{B}'$ is a completely irreducible submodule of $\ell(\check{\mathbb{V}})$, then ℓ^{-1} $(\mathcal{B}')$ is a completely irreducible submodule of $\check{\mathbb{V}}$.

Lemma 2.19.[2] Let $\ell : \check{\mathbb{V}} \rightarrow \check{\mathbb{V}}'$ be a monomorphism of $\mathbb{D}$-modules. If $\mathcal{B}$ is a completely irreducible submodule of $\check{\mathbb{V}}$, then $\ell(\mathcal{B})$ is a completely irreducible submodule of $\ell(\check{\mathbb{V}})$.

Proposition 2.20

Let $\ell : \check{\mathbb{V}} \rightarrow \check{\mathbb{V}}'$ be a isomorphism of $\mathbb{D}$-module, where End($\check{\mathbb{V}}$) is commutative ring. Then we have the following.

1. If $\mathbb{P}$ is fully invariant and ECTAB second submodule of $\check{\mathbb{V}}$ such that $\ker\ell \subseteq \mathbb{P}$, then $\ell(\mathbb{P})$ is ECTAB second submodule of $\check{\mathbb{V}}'$.
2. If $\acute{\mathbb{P}}$ is ECTAB second submodule of $\check{\mathbb{V}}'$ and $\mathbb{P}' \subseteq \ell(\check{\mathbb{V}})$, then ℓ^{-1} $(\mathbb{P}')$ is ECTAB second submodule of $\check{\mathbb{V}}$.

Proof: (1) If $\ell(\check{\mathbb{V}}) = 0$,then $\ell^{-1}\ell(\mathbb{P}) = 0$,so that $\mathbb{P} = 0$ that is contradiction, then $\ell(\mathbb{P}) \neq 0$, let $\mathcal{L}, g, \hbar \in End(\check{\mathbb{V}})$ and $\mathcal{B}$ is completely irreducible submodule of $\check{\mathbb{V}}$ such that $(\mathcal{L} \circ g \circ \hbar)\big(\ell(\mathbb{P})\big) \subseteq \mathcal{B}$,then$(\mathcal{L} \circ g \circ \hbar)(\ell^{-1}\ell(\mathbb{P})) \subseteq \ell^{-1}(\mathcal{B})$, Hence

$(\mathcal{L} \circ g \circ \hbar)(\mathbb{P}) \subseteq \ell^{-1}(\mathcal{B})$, Since $\mathbb{P}$ is ECTAB second submodule of $\mathbb{V}$ and $\ell^{-1}(\mathcal{B})$ is completely irreducible submodule of $\mathbb{V}$, by Lemma 2.18, either $(\mathcal{L} \circ g)(\mathbb{P}) \subseteq \ell^{-1}(\mathcal{B})$ or $(\mathcal{L} \circ \hbar)(\mathbb{P}) \subseteq \ell^{-1}(\mathcal{B})$ or $(g \circ \hbar)(\mathbb{P}) \subseteq \ell^{-1}(\mathcal{B})$, then$(\mathcal{L} \circ g)(\ell(\mathbb{P})) \subseteq \ell\ell^{-1}(\mathcal{B})$ or $(\mathcal{L} \circ \hbar)(\ell(\mathbb{P})) \subseteq \ell\ell^{-1}(\mathcal{B})$ or $(g \circ \hbar)(\ell(\mathbb{P})) \subseteq \ell\ell^{-1}(\mathcal{B})$,since $End(\mathbb{V})$ is commutative ring, so that $(\mathcal{L} \circ g)(\ell(\mathbb{P})) \subseteq B$ or $(\mathcal{L} \circ \hbar)(\ell(\mathbb{P})) \subseteq \mathcal{B}$ or $(g \circ \hbar)(\ell(\mathbb{P})) \subseteq \mathcal{B}$. Therefore, $\ell(\mathbb{P})$ is ECTAB second submodule of $\mathbb{V}'$.

(2) If $\ell^{-1}(\mathbb{P}') = 0$,then $\ell(\mathbb{V}) \cap \mathbb{P} = \ell\ell^{-1}(\mathbb{P}') = \ell(0) = 0$, so that $\mathbb{P}' = 0$ that is contradiction, then $\ell^{-1}(\mathbb{P}') \neq 0$, let $\mathcal{L}, g, \hbar \in End(\mathbb{V})$,and $\mathcal{B}$ is completely irreducible submodule of $\mathbb{V}$ such that $(\mathcal{L} \circ g \circ \hbar)\big(\ell^{-1}(\acute{\mathbb{P}})\big) \subseteq \mathcal{B}$, then $(\mathcal{L} \circ g \circ \hbar)\big(\ell\ell^{-1}(\acute{\mathbb{P}})\big) \subseteq \ell(\mathcal{B})$, hence $(\mathcal{L} \circ g \circ \hbar)(\mathbb{P}') \subseteq \ell(\mathcal{B})$, Since $\acute{\mathbb{P}}$ is ECTAB second submodule of $\mathbb{V}'$ and $\ell(\mathcal{B})$ is completely irreducible submodule of $\mathbb{V}'$ by Lemma 2.19, then either $(\mathcal{L} \circ g)(\mathbb{P}') \subseteq \ell(\mathcal{B})$ or $(\mathcal{L} \circ \hbar)(\mathbb{P}') \subseteq \ell(\mathcal{B})$ or $(g \circ \hbar)(\mathbb{P}') \subseteq \ell(\mathcal{B})$. So that either $(\mathcal{L} \circ g)(\ell^{-1}(\mathbb{P}')) \subseteq \ell^{-1}\ell(\mathcal{B})$ or $(\mathcal{L} \circ \hbar)(\ell^{-1}(\mathbb{P}')) \subseteq \ell^{-1}\ell(\mathcal{B})$ or $(g \circ \hbar)(\ell^{-1}(\mathbb{P}')) \subseteq \ell^{-1}\ell(\mathcal{B})$ since $End(\mathbb{V})$ is commutative ring, hence either $(\mathcal{L} \circ g)(\ell^{-1}(\mathbb{P}')) \subseteq (\mathcal{B})$ or $(\mathcal{L} \circ \hbar)(\ell^{-1}(\mathbb{P}')) \subseteq (\mathcal{B})$ or $(g \circ \hbar)(\ell^{-1}(\mathbb{P}')) \subseteq (\mathcal{B})$. Therefore, $\ell^{-1}(\mathbb{P}')$ is ECTAB second submodule of $\mathbb{V}$.

Proposition 2.21

Let $\mathbb{V}$ be a $\mathbb{D}$-module and let $\{K_i\}_{i\in I}$ be a chain of ECTAB second subs of $\mathbb{V}$, then $\cup_{i\in I} K_i$ is ECTAB second submodule of $\mathbb{V}$.

Proof : Let $\ell, g, \hbar \in End(\mathbb{V})$, $\mathcal{B}$ is completely irreducible submodule of $\mathbb{V}$ and $(\ell \circ g \circ \hbar)\big((\cup_{i\in I} K)\big) \subseteq \mathcal{B}$, Assume that $(\ell \circ g)\big((\cup_{i\in I} K)\big) \nsubseteq \mathcal{B}$ and $(\ell \circ \hbar)\big((\cup_{i\in I} K)\big) \nsubseteq \mathcal{B}$ then there are $m, n \in I$, where $\ell(K_n) \nsubseteq \mathcal{B}$ and $g(K_m) \nsubseteq \mathcal{B}$, hence for every $K_n \subseteq K_s$ and $K_m \subseteq K_d$ we have $(\ell \circ g)(K_s) \nsubseteq \mathcal{B}$ and $(\ell \circ \hbar)(K_d) \nsubseteq \mathcal{B}$. Therefore, for each ECTAB second submodule K_h such that $K_n \subseteq K_h$ and $K_m \subseteq K_h$ we have $(g \circ \hbar)(K_h) \subseteq \mathcal{B}$. Hence $(g \circ \hbar)(\cup_{i\in I} K_i) \subseteq \mathcal{B}$, as needed.

Definition 2.22

An ECTAB second submodule $\mathbb{P}$ of a $\mathbb{D}$-module $\mathbb{V}$ is named maximal ECTAB second (simple MECTAB) second submodule of a sub-module K of $\mathbb{D}$-module $\mathbb{V}$, if $\mathbb{P} \subseteq K$ and not there exist an ECTAB second submodule S of a $\mathbb{D}$-module $\mathbb{V}$ such that $\mathbb{P} \subset S \subset K$.

Example 2.23

Think of Z_{24} as Z-module and $\ell, g, \hbar \in End(Z_{24})$ such that $\ell(\bar{x}) = 5\bar{x}$, $g(\bar{x}) = \bar{x}$, $\hbar(\bar{x}) = 2(\bar{x})$, $\forall \bar{x} \in Z_{24}$ where $(\bar{2})$ is completely irreducible and $(\bar{2})$ is MECTAB second submodule since $(\ell \circ g \circ \hbar)((\bar{2})) = (\bar{4}) \subseteq (\bar{2})$, then $(\ell \circ g)\big((\bar{2})\big) = (\bar{2}) \subseteq (\bar{2})$, $(\ell \circ \hbar)\big((\bar{2})\big) = (\bar{4}) \subseteq (\bar{2})$ and $(g \circ \hbar)\big((\bar{2})\big) = (\bar{4}) \subseteq (\bar{2})$ and no there exist any an ETCAB second submodule contains $(\bar{2})$.

Proposition 2.24

Every ECTAB second submodule of a $\mathbb{D}$-module $\mathbb{V}$ is contained in a MECTAB second submodule of $\mathbb{V}$.

Proof:

This is proved easily by using Zorn's lemma and Proposition 2.21.

Proposition 2.25
Every EQP second submodule of a $\mathbb{D}$-module $\breve{\mathbb{V}}$ is contained in a MECTAB second submodule.
Proof:
Since every EQP second submodule is ECTAB second submodule by Proposition 2.6 and by Proposition 2.24, we get result.

Proposition 2.26
Let $\mathbb{P}$ and S be a non-zero submodule of $\mathbb{D}$-module $\breve{\mathbb{V}}$, and $\mathbb{P} \subset S$. If $\mathbb{P}$ is ECTAB second submodule of $\breve{\mathbb{V}}$, then $\mathbb{P}$ is ECTAB second submodule of S.
Proof:
If $S = \breve{\mathbb{V}}$,then do not need to be proved.
Now, let $\ell, \mathscr{g}, \hbar \in End(\breve{\mathbb{V}})$ and $\mathcal{B}$ be completely irreducible submodule of S since S is submodule of $\breve{\mathbb{V}}$, then $\mathcal{B}$ is completely irreducible of $\breve{\mathbb{V}}$, so that $(\ell \circ \mathscr{g} \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$ implies either $\ell \circ \mathscr{g}(\mathbb{P}) \subseteq \mathcal{B}$ or $\ell \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$ or $\mathscr{g} \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$. Thus, $\mathbb{P}$ is ECTAB second submodule of S.

Proposition 2.27
Give $\mathbb{P} \neq 0$ and $S \neq 0$ be two submodules of a $\mathbb{D}$-module $\breve{\mathbb{V}}$ such that $S \subset \mathbb{P}$, and $\mathbb{P}$ is ECTAB second submodule, then $\mathbb{P}/S$ is ECTAB second submodule of $\breve{\mathbb{V}}/S$.

Proof:
Let $\ell, \mathscr{g}, \hbar \in End(\breve{\mathbb{V}})$ and $\mathcal{B}$ be completely irreducible submodule of $\breve{\mathbb{V}}$ such that $(\ell \circ \mathscr{g} \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$, Define $L, K,$ H: $\frac{W}{S} \to \frac{W}{S}$ by $\mathrm{L}(\mathbb{P}/S) = (\ell \circ \mathscr{g})(\mathbb{P}) + S$, $\mathrm{K}(\mathbb{P}/S) = \ell \circ \hbar(\mathbb{P}) + S$, $\mathrm{H}(\mathbb{P}/S) = \mathscr{g} \circ \hbar(\mathbb{P}) + S$ It is clear that L, K and H are well defined, since S is a fully invariant.
Now, $(L \circ K \circ H)(P/S) = (\ell \circ \mathscr{g} \circ \hbar)(\mathbb{P}) + \mathrm{S} \subseteq \frac{\mathcal{B}}{S}$ where $\frac{\mathcal{B}}{S}$ is completely irreducible submodule of $\breve{\mathbb{V}}/S$, but $\mathbb{P}/S$ is ECTAB second submodule of $\breve{\mathbb{V}}/S$, then either $L(\mathbb{P}/S) = \ell \circ \mathscr{g}(\mathbb{P}) + \mathrm{S} \subseteq \frac{\mathcal{B}}{S}$ or $K(\mathbb{P}/S) = \ell \circ \hbar(\mathbb{P}) + \mathrm{S} \subseteq \frac{\mathcal{B}}{S}$ or $H(\mathbb{P}/S) = \mathscr{g} \circ \hbar(\mathbb{P}) + \mathrm{S} \subseteq \frac{\mathcal{B}}{S}$, hence $\ell \circ \mathscr{g}(\mathbb{P}) \subseteq \mathcal{B}$ or $\ell \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$ or $\mathscr{g} \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$. Thus, $\mathbb{P}$ is ECTAB second submodule of $\breve{\mathbb{V}}$.

Proposition 2.28
Let $\mathbb{P} \neq 0$ be a submodule of $\mathbb{D}$-module $\breve{\mathbb{V}}$. If $\mathbb{P}$ is ECTAB second submodule of $\breve{\mathbb{V}}$, then $S^{-1}\,\mathbb{P}$ is ECTAB second submodule of $S^{-1}\,\breve{\mathbb{V}}$.

Proof:
Let $\frac{\ell}{s_1}, \frac{\mathscr{g}}{s_2}, \frac{\hbar}{s_3} \in End(S^{-1}\breve{\mathbb{V}})$, such that $\frac{(\ell \circ \mathscr{g} \circ \hbar)}{s_1 s_2 s_3}(S^{-1}\mathbb{P}) \subseteq S^{-1}(\mathcal{B})$ where $\mathcal{B}$ is completely irreducible submodule of $\breve{\mathbb{V}}$ then there exists $r \in S$, such that $r(\ell \circ \mathscr{g} \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$, so that $(\ell \circ \mathscr{g} \circ \hbar)(r\mathbb{P}) \subseteq \mathcal{B}$, but $\mathbb{P}$ is ECTAB second submodule of $\breve{\mathbb{V}}$, then $\ell \circ \mathscr{g}(r\mathbb{P}) \subseteq \mathcal{B}$ or $\ell \circ \hbar(r\mathbb{P}) \subseteq \mathcal{B}$ or $\mathscr{g} \circ \hbar(r\mathbb{P}) \subseteq \mathcal{B}$, so that $\frac{r\ell \circ \mathscr{g}(\mathbb{P})}{r s_1 s_2} = \frac{\ell \circ \mathscr{g}(S^{-1}\mathbb{P})}{s_1 s_2} \subseteq S^{-1}\mathcal{B}$ or $\frac{r\,\ell \circ \hbar(\mathbb{P})}{r s_2 s_3} = \frac{\ell \circ \hbar\,(S^{-1}\mathbb{P})}{s_2 s_3} \subseteq S^{-1}\mathcal{B}$ or $\frac{r\,\mathscr{g} \circ \hbar(\mathbb{P})}{r s_2 s_3} = \frac{\mathscr{g} \circ \hbar(S^{-1}\mathbb{P})}{s_2 s_3} \subseteq S^{-1}\mathcal{B}$. Thus, $S^{-1}\mathbb{P}$ is ECTAB second submodule of $S^{-1}\breve{\mathbb{V}}$.

Proposition 2.29

Let $\mathbb{P} \neq 0$ be a submodule of $\mathbb{D}$-module $\check{\mathbb{V}}$. If $\mathbb{P}$ is ECTAB second submodule, then $\mathbb{P}$ is CTAB second submodule.

Proof:

Suppose $\ell, g, \hbar \in End(\check{\mathbb{V}})$ such that $\ell(x) = r \cdot x$, $g(x) = s \cdot x$ and $\hbar(x) = u \cdot x, \forall x \in \check{\mathbb{V}}$, $r, s, u \in$ D and $\mathbb{P} \neq 0$ is a submodule of $\check{\mathbb{V}}$,but $\mathbb{P}$ is ECTAB second submodule, so that $(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$ implies either $\ell \circ g(\mathbb{P}) \subseteq \mathcal{B}$ or $\ell \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$ or $g \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$ then $r.s.\mathbb{P} = \ell \circ g(\mathbb{P}) \subseteq \mathcal{B}$ or $r \cdot u \cdot \mathbb{P} = \ell \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$ or $s \cdot u \cdot \mathbb{P} = g \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$. Thus, $\mathbb{P}$ is CTAB second submodule.

A $\mathbb{D}$-module $\check{\mathbb{V}}$ is called a Scalar module if for each $\ell \in End(\check{\mathbb{V}})$, $there\ exists\ r \in$ D $such\ that\ \ell(m) = rm$, for $m \in \check{\mathbb{V}}$.[9]

The converse of Proposition 2.29 is true under condition scalar module as in the following proposition:

Proposition 2.30

Let $\mathbb{P} \neq 0$ be a submodule of a scalar $\mathbb{D}$-module $\check{\mathbb{V}}$. I$\mathbb{P}$ is ECTAB second submodule, if and only if $\mathbb{P}$ is CTAB second submodule.

Proof:

$\Rightarrow) By$ Proposition 2.29

$\Leftarrow)$ Let $\ell, g, \hbar \in End(\check{\mathbb{V}}), \ell(x) = ax$, $g(x) = bx$, $\hbar(x) = cx$ $\forall a, b, c \notin Ann\mathbb{P}$, a, b, c $\in$ D, $\forall x \in \check{\mathbb{V}}$ and $\mathcal{B}$ is completely irreducible of $\mathbb{D}$-module $\check{\mathbb{V}}$, such that $(\ell \circ g \circ \hbar)(\mathbb{P}) \subseteq \mathcal{B}$, but $\check{\mathbb{V}}$ is scalar module , So that $(\ell \circ g \circ \hbar)(\mathbb{P}) = abc\mathbb{P} \subseteq \mathcal{B}$, but $\mathbb{P}$ is CTAB second submodule of $\check{\mathbb{V}}$, then we have $ab\mathbb{P} \subseteq \mathcal{B}$ or $ac\mathbb{P} \subseteq \mathcal{B}$ or $bc\mathbb{P} \subseteq \mathcal{B}$. Hence $\ell \circ g(\mathbb{P}) \subseteq \mathcal{B}$ or $\ell \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$ or $g \circ \hbar(\mathbb{P}) \subseteq \mathcal{B}$. Thus, $\mathbb{P}$ is ECTAB second submodule of $\check{\mathbb{V}}$.

A module $\check{\mathbb{V}}$ is called finitely generated if it has a finite generating set, say X, that is $\check{\mathbb{V}} = \langle X \rangle$. [10]

Let $\mathbb{D}$ be a commutative ring with unity and $\mathbb{D}$-module $\check{\mathbb{V}}$ be an until $\mathbb{D}$-module. Then $\check{\mathbb{V}}$ is called a multiplication module if for each submodule $\mathbb{P}$ of $\check{\mathbb{V}}$, there exists an ideal I of $\mathbb{D}$ such that $\mathbb{P}$=I $\check{\mathbb{V}}$.[11]

Corollary 2.31 [5]

Every finitely generated multiplication $\mathbb{D}$-module $\check{\mathbb{V}}$ is scalar module.

Proposition 2.32

Let $\mathbb{P} \neq 0$ be a submodule of finitely generated multiplication $\mathbb{D}$-module $\check{\mathbb{V}}$, $\mathbb{P}$ is ECTAB second submodule of $\check{\mathbb{V}}$ if and only if $\mathbb{P}$ is CTAB second submodule of $\check{\mathbb{V}}$.

Proof

From Corollary 2.31 and Proposition 2.30, we get result.

3. Conclusions

In this article, we conclude that every ETAB second submodule is an ECTAB second submodule, but the opposite is not generally true. Similarly, if $\mathbb{P}$ is an EQP second submodule, then $\mathbb{P}$ is an ECTAB second submodule, but the opposite is not generally true. Additionally, if $\mathbb{P} \neq 0$ and $S \neq 0$ be two submodules of a $\mathbb{D}$-module $\check{\mathbb{V}}$, such that $S \subset \mathbb{P}$, and $\mathbb{P}$ is ECTAB second submodule, then $\mathbb{P}/S$ is ECTAB second submodule of $\check{\mathbb{V}}/S$.; additionally, the homomorphic image and inverse homomorphic image of ECTAB second submodule are ECTAB second submodule. Lastly, $S^{-1}\mathbb{P}$ is ECTAB second subm of $S^{-1}\check{\mathbb{V}}$ if $\mathbb{P}$ is ECTAB second subm of $\check{\mathbb{V}}$.

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