Oscillatory property for third order differential equation with three delay

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ABSTRACT

This paper, the 3rd-order differential equations with Three delay (NETHD) was investigated we investigated the oscillatory natura of the solution around the equilibrium state. To achieve this goal, several lemmas are established to facilitate finding key results that ensure the oscillation of all solutions around the of equilibrium point. An illustrative example of these results was presented.

1. Introduction

The oscillations in all solutions of 3^{rd}-order (NETHD)are studied:

$$\frac{d^2}{dt^2}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right]+\delta_2(t)\big(G(y(\vartheta(t)))\big)+\delta_3(t)\big(G(y(\mu(t)))\big)=0 \qquad (1)$$

Where $\mathfrak{H}(t) equal\ to\ \big(y(t)+\delta_4(t)(y(\omega(t)))\big)$ and the delays $\vartheta(t)\&\ \mu(t)\ and\ \omega(t)\in C^1[t_0,\infty)$, $\vartheta^{-1}(t)$and , $\mu^{-1}(t)\ also\ \omega^{-1}(t)$ are the inverse of delays and $\vartheta(t),\mu(t),\omega(t)$

We always suppose that:

(D1) $\delta_1(t)$ and , $\delta_2(t)$ also $\delta_3(t)$ are continues positive functions

(D2) $G(y)>\alpha y,$, where $\alpha>0$

(D3) $0\le\delta_4(t)\le\delta<\infty$.

There were a lot of emphasis in the oscillation and non-oscillation of

Differential equation of different order and dissimilar delays, refer here by way of example [1-7] and the

bibliography. this article examines, we seek to deliver new conditions that ensure every solution of (1)is an oscillating.

2. Key findings

Lemma 1: presume $y(t)$ is positive general solution of the equation (1) then we have two cases :

i) $\frac{d}{dt}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right]$,greater than zero , $\delta_1(t)\frac{d}{dt}\big(\mathfrak{H}(t)\big)$ greater than zero where $\mathfrak{H}(t)>0$,

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ii) $\frac{d}{dt}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right]$, greater than zero , $\delta_1(t)\frac{d}{dt}\big(\mathfrak{H}(t)\big)$ less than zero when , $\mathfrak{H}(t)$ is positive .

proof : let $y(t)$ is positive general solution of (1) .

from (1) we get

$$\frac{d^2}{dt^2}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right] = -(\delta_2(t)G(y\big(\vartheta(t)\big) + \delta_3(t)G(y\big(\mu(t)\big)) \leq 0 \quad (2)$$

Hence $\frac{d^2}{dt^2}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right]$ less than zero so $\frac{d}{dt}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right]$ and $\frac{d}{dt}(\mathfrak{H}(t))$ also $\mathfrak{H}(t)$ are monotone ,in such away that $\frac{d}{dt}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right]$ where $\frac{d}{dt}(\mathfrak{H}(t))$ also $\mathfrak{H}(t)$,of costant singn .

We claim that $\frac{d}{dt}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right]$ greater than zero ,other wise $\frac{d}{dt}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right]$ less than zero then the consequently $\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))$ less than zero where$\mathfrak{H}(t)$, is strictly negative which a contradiction.

Accordingly $\frac{d}{dt}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right] \geq 0 ,\forall t \geq t_0$,

Now $\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right] \geq 0$ or $\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right]$ less than zero if $\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right]$greater than zero consequently $\frac{d}{dt}(\mathfrak{H}(t)) \geq 0$ also we have $\mathfrak{H}(t) \geq 0$,furthermore $\lim_{t\to\infty}\mathfrak{H}(t) = 0$,also if $\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right]$ less than zero ,then $\frac{d}{dt}(\mathfrak{H}(t)) \leq 0$ and $\mathfrak{H}(t) \geq 0$.

Theorem 1: suppose that (D1-D3) are hold and , $\omega\big(\vartheta(t)\big) = \vartheta\big(\omega(t)\big)$, $\omega\big(\mu(t)\big) = \mu\big(\omega(t)\big)$, $\mu(\vartheta^{-1}(t)) < t$, Since if ,

(a) $\frac{d}{dt}[\mathfrak{S}(t)] + \frac{d}{dt}\delta[\mathfrak{S}(\omega(t))] + \alpha\mathfrak{S}(t)\Big(\mathcal{H}_1\aleph(\vartheta(t)) + \mathcal{H}_2\aleph\big(\mu(t)\big)\Big) \leq 0$ (3)

Has no positive solution ,and

(b) $\int_{t^*}^{t}\int_{t}^{\vartheta^{-1}(t)} \alpha\mathcal{H}_3 ds dr = \int_{t^*}^{t}\int_{t}^{\vartheta^{-1}(t)} \alpha\mathcal{H}_4 ds\, dr = \infty$ (4)

then every solution of (1) are oscillation .

Proof : presume eq.(1) has a positive general solution $y(t)$, let $y(\vartheta(t))$, $y\big(\mu(t)\big))$

$y(\omega(t))$ are positive , According to Lemma 1, there are only the following cases for discussion

1) $\frac{d}{dt}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right] \geq 0$, $\delta_1(t)\frac{d}{dt}\big(\mathfrak{H}(t)\big) \geq 0$, $\mathfrak{H}(t) > 0$,

2) $\frac{d}{dt}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right] \geq 0$, $\delta_1(t)\frac{d}{dt}\big(\mathfrak{H}(t)\big) \leq 0$, $\mathfrak{H}(t) > 0$,

Now

Case 1 : by replacing $\omega(t)$ of t in eq (1) we obtain

$$\frac{d^2}{dt^2}\left[\delta_1(\omega(t)\ )\frac{d}{dt}(\mathfrak{H}(\omega(t)\ ))\right] + \delta_2(\omega(t))G(y(\vartheta(\omega(t)))) + \delta_3(t)G(y(\mu(\omega(t)))) = 0$$

By using (D2) we get

$$\frac{d^2}{dt^2}\left[\delta_1(\omega(t)\ )\frac{d}{dt}(\mathfrak{H}(\omega(t)\ ))\right] + \alpha\delta_2(\omega(t))y\big(\vartheta(\omega(t))\big) + \alpha\delta_3(\omega(t))y\big(\mu(\omega(t))\big) \le 0 \qquad (5)$$

Multiply (5) by δ and using (D3) we have :

$$\frac{d^2}{dt^2}\delta\left[\delta_1(\omega(t)\ )\frac{d}{dt}(\mathfrak{H}(\omega(t)\ ))\right] + \alpha\delta\delta_2(\omega(t))y\big(\vartheta(\omega(t))\big) + \alpha\delta\delta_3(\omega(t))y\big(\mu(\omega(t))\big) \le 0 \quad (6)$$

Summing equation (1) and (6) we have

$$\frac{d^2}{dt^2}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right] + \alpha\delta_2(t)y\big(\vartheta(t)\big) + \alpha\delta_3(t)y\big(\mu(t)\big) + \frac{d^2}{dt^2}\delta\left[\delta_1(\omega(t)\,)\frac{d}{dt}\big(z(\omega(t)\,)\big)\right]$$
$$+ \alpha\delta\delta_2(\omega(t))y\Big(\vartheta\big(\omega(t)\big)\Big) + \alpha\delta\delta_3(\omega(t))y\Big(\mu\big(\omega(t)\big)\Big) \le 0 \qquad (7)$$

So

$$\frac{d^2}{dt^2}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right] + \frac{d^2}{dt^2}\delta\left[\delta_1(\omega(t)\,)\frac{d}{dt}\big(\mathfrak{H}(\omega(t)\,)\big)\right] + \alpha\mathcal{H}_1\big[y\big(\vartheta(t)\big) + \delta y\big(\vartheta(\omega(t))\big)\big]$$
$$+ \alpha\mathcal{H}_2\big[y\big(\mu(t)\big) + \delta y\big(\mu(\omega(t))\big)\big] \le 0$$

Where $\mathcal{H}_1 = min\{\delta_2(t), \delta_2(\omega(t))\}$ and $\mathcal{H}_2 = min\{\delta_3(t), \delta_3(\omega(t))\}$

So

$$\frac{d^2}{dt^2}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right] + \frac{d^2}{dt^2}\delta\left[\delta_1(\omega(t)\,)\frac{d}{dt}\big(\mathfrak{H}(\omega(t)\,)\big)\right] + \alpha\mathcal{H}_1\mathfrak{H}\big(\vartheta(t)\big) + \alpha\mathcal{H}_2\mathfrak{H}\big(\mu(t)\big) \le 0 \ (8)$$

Let $$\mathfrak{S}(t) = \frac{d}{dt}\left(\delta_1(t)\frac{d}{dt}\big(\mathfrak{H}(t)\big)\right) \qquad (9)$$

Integrating of both members of (9) from t_1 to t we get

$$\int_{t_1}^{t}\frac{d}{ds}\left(\delta_1(s)\frac{d}{ds}\big(\mathfrak{H}(s)\big)\right) = \int_{t_1}^{t}\mathfrak{S}(s)\,ds$$

$$\delta_1(t)\frac{d}{dt}\big(\mathfrak{H}(t)\big) - \delta_1(t_1)\frac{d}{dt}\big(\mathfrak{H}(t)\big) = \int_{t_1}^{t}\mathfrak{S}(s)\,ds$$

Since $\mathfrak{S}(t)$ is decreasing then

$\delta_1(t)\frac{d}{dt}\big(\mathfrak{H}(t)\big) \ge \mathfrak{S}(t)\int_{t_1}^{t}ds$ so $\quad \frac{d}{dt}\big(\mathfrak{H}(t)\big) \ge \frac{\mathfrak{S}(t)}{\delta_1(t)}(t - t_1)$ (10)

Integratimg of both sided of (10) over the interval (t_2, t) yields

$$\mathfrak{H}(t) - \mathfrak{H}(t_2) \ge \int_{t_2}^{t}\frac{\mathfrak{S}(s)}{\delta_1(s)}(s - s_1)\,ds$$

Then $\mathfrak{H}(t) \geq \mathfrak{S}(s)\aleph(t)$ where $\aleph(t) = \int_{t_2}^{t} \frac{(s-s_1)}{\delta_1(s)}\, ds$

Substation in (8) we obtain

$$\frac{d}{dt}[\mathfrak{S}(t)] + \frac{d}{dt}\delta[\mathfrak{S}(\omega(t)\,)] + \alpha\mathcal{H}_1\mathfrak{S}(\vartheta(t))\aleph(\vartheta(t)) + \alpha\mathcal{H}_2\mathfrak{S}(\mu(t))\aleph\big(\mu(t)\big)\big) \leq 0$$

$$\frac{d}{dt}[\mathfrak{S}(t)] + \frac{d}{dt}\delta[\mathfrak{S}(\omega(t)\,)] + \alpha\mathfrak{S}(t)\left(\mathcal{H}_1\aleph(\vartheta(t)) + \mathcal{H}_2\aleph\big(\mu(t)\big)\big)\right) \leq 0$$

since $\mathfrak{S}(t)$ is positive then is contradiction with (3) ,

2) $\frac{d}{dt}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right] \geq 0, \delta_1(t)\frac{d}{dt}\big(\mathfrak{H}(t)\big) \leq 0\,, \mathfrak{H}(t) > 0\,,$

We replace $\vartheta^{-1}(\omega\big(\vartheta(t)\big))$ is slead of t in (5) once and $\mu^{-1}\left(\omega\big(\mu(t)\big)\right)$ instead of t in (5) second time .

$$\frac{d^2}{dt^2}\left[\delta_1(\vartheta^{-1}(\omega\big(\vartheta(t)\big)))\frac{d}{dt}\big(\mathfrak{H}(\vartheta^{-1}(\omega\big(\vartheta(t)\big)))\big)\right] + \delta_2(\vartheta^{-1}(\omega\big(\vartheta(t)\big)))G(y\big(\omega\big(\vartheta(t)\big)\big)\big) + \delta_3(\vartheta^{-1}(\omega\big(\vartheta(t)\big)))G(y\big(\mu\big(\vartheta^{-1}(\omega\big(\vartheta(t)\big)\big)\big)\big)\big) = 0 \qquad (11)$$

$$\frac{d^2}{dt^2}\left[\delta_1(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\frac{d}{dt}\left(\mathfrak{H}(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\right)\right] + \delta_2(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))G(y\left(\vartheta\left(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)\right)\right)) + \delta_3(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))G(y\big(\omega\big(\mu(t)\big)\big)\big) = 0 \quad (12)$$

Summing (1),(11)and (12) ,and using (D2-D3)

$$\frac{d^2}{dt^2}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right] + \delta_2(t)y\big(\vartheta(t)\big) + \delta_3(t)y(\mu(t))) +$$
$$\delta\frac{d^2}{dt^2}\left[\delta_1(\vartheta^{-1}(\omega\big(\vartheta(t)\big)))\frac{d}{dt}\big(\mathfrak{H}(\vartheta^{-1}(\omega\big(\vartheta(t)\big)))\big)\right] + \alpha\delta\delta_2(\vartheta^{-1}(\omega\big(\vartheta(t)\big)))y\left(\omega\big(\vartheta(t)\big)\right) +$$
$$\alpha\delta\delta_3(\vartheta^{-1}(\omega\big(\vartheta(t)\big)))y\left(\mu\left(\vartheta^{-1}\left(\omega\big(\vartheta(t)\big)\right)\right)\right) +$$
$$\delta\frac{d^2}{dt^2}\left[\delta_1(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\frac{d}{dt}\left(\mathfrak{H}(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\right)\right] +$$
$$\alpha\delta\delta_2(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))y\left(\vartheta\left(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)\right)\right) + \alpha\delta\delta_3(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))y\left(\omega\big(\mu(t)\big)\right) \leq 0 \qquad (13)$$

Or

$$\frac{d^2}{dt^2}\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right] + \delta\frac{d^2}{dt^2}\left[\delta_1(\vartheta^{-1}(\omega\big(\vartheta(t)\big)))\frac{d}{dt}\big(\mathfrak{H}(\vartheta^{-1}(\omega\big(\vartheta(t)\big)))\big)\right] +$$
$$\delta\frac{d^2}{dt^2}\left[\delta_1(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\frac{d}{dt}\left(\mathfrak{H}(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\right)\right] + \alpha\mathcal{H}_3\mathfrak{H}\big(\vartheta(t)\big) + \alpha\mathcal{H}_4\mathfrak{H}\big(\mu(t)\big) \leq 0$$
(14)

Wher $\mathcal{H}_3 = min\left\{\delta_2(t), \delta_2(\vartheta^{-1}\left(\omega\big(\vartheta(t)\big)\right), \delta_2(\mu^{-1}\left(\omega\big(\mu(t)\big)\right))\right\}$

$\mathcal{H}_4 = min\left\{\delta_3(t), \delta_3(\vartheta^{-1}\left(\omega\big(\vartheta(t)\big)\right), \delta_3(\mu^{-1}\left(\omega\big(\mu(t)\big)\right))\right\}$

Since eq(14) can be written as the from

$$\alpha\mathcal{H}_3\mathfrak{H}\big(\vartheta(t)\big) + \alpha\mathcal{H}_4\mathfrak{H}\big(\mu(t)\big) \leq$$
$$-\frac{d^2}{dt^2}\left[\delta_1(t)\frac{d}{dt}\big(\mathfrak{H}(t)\big)\right] - \delta\frac{d^2}{dt^2}\left[\delta_1\left(\vartheta^{-1}\left(\omega\big(\vartheta(t)\big)\right)\right)\frac{d}{dt}\left(\mathfrak{H}\left(\vartheta^{-1}\left(\omega\big(\vartheta(t)\big)\right)\right)\right)\right] -$$
$$\delta\frac{d^2}{dt^2}\left[\delta_1(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\frac{d}{dt}\left(\mathfrak{H}(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\right)\right] \qquad (15)$$

Integration (15) from t to $\vartheta^{-1}(t)$ we get

$$\int_t^{\vartheta^{-1}(t)} \alpha\mathcal{H}_3\mathfrak{H}\left(\omega\big(\vartheta(s)\big)\right)ds + \int_t^{\vartheta^{-1}(t)} \alpha\mathcal{H}_4\mathfrak{H}\left(\omega\big(\mu(s)\big)\right)ds$$
$$\leq -\frac{d}{dt}\left[\delta_1(\vartheta^{-1}(t))\frac{d}{dt}\big(\mathfrak{H}(\vartheta^{-1}(t))\big)\right] + \frac{d}{dt}\left[\delta_1(t)\frac{d}{dt}\big(\mathfrak{H}(t)\big)\right]$$
$$-\delta\frac{d}{dt}\left[\delta_1\left(\vartheta^{-1}\left(\omega\big(\vartheta(\vartheta^{-1}(t))\big)\right)\right)\frac{d}{dt}\left(\mathfrak{H}\left(\vartheta^{-1}\left(\omega\big(\vartheta(\vartheta^{-1}(t))\big)\right)\right)\right)\right]$$
$$+\delta\frac{d}{dt}\left[\delta_1\left(\vartheta^{-1}\left(\omega\big(\vartheta(t)\big)\right)\right)\frac{d}{dt}\left(\mathfrak{H}\left(\vartheta^{-1}\left(\omega\big(\vartheta(t)\big)\right)\right)\right)\right]$$
$$-\delta\frac{d^2}{dt}\left[\delta_1(\mu^{-1}\left(\omega\big(\mu(\vartheta^{-1}(t))\big)\right)))\frac{d}{dt^*}\left(\mathfrak{H}(\mu^{-1}\left(\omega\big(\mu(\vartheta^{-1}(t))\big)\right)))\right)\right]$$
$$+\delta\frac{d}{dt}\left[\delta_1(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\frac{d}{dt}\left(\mathfrak{H}(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\right)\right]$$

Since $\mathfrak{H}(t)$ is descreasing then

$$\mathfrak{H}(t)\int_t^{\vartheta^{-1}(t)}\alpha\mathcal{H}_3 ds + \mathfrak{H}\big(\mu(\vartheta^{-1}(t))\big)\int_t^{\vartheta^{-1}(t)}\alpha\mathcal{H}_4 ds \leq$$
$$\frac{d}{dt}\left[\delta_1(t)\frac{d}{dt}\big(\mathfrak{H}(t)\big)\right] + \delta\frac{d}{dt}\left[\delta_1\left(\vartheta^{-1}\left(\omega\big(\vartheta(t)\big)\right)\right)\frac{d}{dt}\left(\mathfrak{H}\left(\vartheta^{-1}\left(\omega\big(\vartheta(t)\big)\right)\right)\right)\right] +$$
$$\delta\frac{d}{dt}\left[\delta_1(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\frac{d}{dt}\left(\mathfrak{H}(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\right)\right] \qquad (16)$$

Integrating (16) from t_1 to t we get

$$\int_{t1}^t \mathfrak{H}(r)\int_t^{\vartheta^{-1}(t)}\alpha\mathcal{H}_3 ds dr + \int_{t_1}^t \mathfrak{H}\big(\mu(\vartheta^{-1}(r))\big)\int_t^{\vartheta^{-1}(t)}\alpha\mathcal{H}_4 ds\, dr \leq$$
$$\left[\delta_1(t)\frac{d}{dt}\big(\mathfrak{H}(t)\big)\right] + \left[\delta_1(t_1)\frac{d}{dt}\big(\mathfrak{H}(t_1)\big)\right] + \delta\left[\delta_1\left(\vartheta^{-1}\left(\omega\big(\vartheta(t)\big)\right)\right)\frac{d}{dt}\left(\mathfrak{H}\left(\vartheta^{-1}\left(\omega\big(\vartheta(t)\big)\right)\right)\right)\right] -$$
$$\delta\left[\delta_1\left(\vartheta^{-1}\left(\omega\big(\vartheta(t_1)\big)\right)\right)\frac{d}{dt}\left(\mathfrak{H}\left(\vartheta^{-1}\left(\omega\big(\vartheta(t_1)\big)\right)\right)\right)\right] +$$
$$\left[\delta_1(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\frac{d}{dt}\left(\mathfrak{H}(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\right)\right] -$$
$$\left[\delta_1(\mu^{-1}\left(\omega\big(\mu(t_1)\big)\right)))\frac{d}{dt}\left(\mathfrak{H}(\mu^{-1}\left(\omega\big(\mu(t_1)\big)\right)))\right)\right] \leq$$

$\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}\mathfrak{H}(t))\right]+\delta\left[\delta_1\left(\vartheta^{-1}\left(\omega\big(\vartheta(t)\big)\right)\right)\frac{d}{dt}\left(\mathfrak{H}\left(\vartheta^{-1}\left(\omega\big(\vartheta(t)\big)\right)\right)\right)\right]+$
$\left[\delta_1(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\frac{d}{dt}\left(\mathfrak{H}(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\right)\right]$

So $\mathfrak{H}(t)$ is decreasing and by using $\mu(\vartheta^{-1}(t))<t$ then

$$\mathfrak{H}(t)\int_{t^*}^{t}\int_{t}^{\vartheta^{-1}(t)}\alpha\mathcal{H}_3 ds dr+\mathfrak{H}(t)\int_{t^*}^{t}\int_{t}^{\vartheta^{-1}(t)}\alpha\mathcal{H}_4 ds\, dr$$
$$\leq\left[\delta_1(t)\frac{d}{dt}(\mathfrak{H}(t))\right]+\delta\left[\delta_1\left(\vartheta^{-1}\left(\omega\big(\vartheta(t)\big)\right)\right)\frac{d}{dt}\left(\mathfrak{H}\left(\vartheta^{-1}\left(\omega\big(\vartheta(t)\big)\right)\right)\right)\right]$$
$$+\left[\delta_1(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\frac{d}{dt}\left(\mathfrak{H}(\mu^{-1}\left(\omega\big(\mu(t)\big)\right)))\right)\right]$$

Since $\frac{d}{dt}\big(\mathfrak{H}(t)\big)<0$,then

$\int_{t^*}^{t}\int_{t}^{\vartheta^{-1}(t)}\alpha\mathcal{H}_3 ds dr\to-\infty$,and $\int_{t^*}^{t}\int_{t}^{\vartheta^{-1}(t)}\alpha\mathcal{H}_4 ds\, dr\to-\infty$,which is contraducation of (4).

Example :let the following equation be given .

$\frac{d^2}{dt^2}\left[t^2\frac{d}{dt}(\cos t+\delta_4(t)\cos(t-4\pi))\right]+(7t^2+8t-2\,)G(y(\vartheta(t)))+(t^3+2t^2-10t-$
$4)G(y(\mu(t)))=0$ (17)

$\delta_1(t)=t^2,\delta_4(t)=t+1,\delta_2(t)=7t^2+8t-2\,,\delta_3(t)=t^3+2t^2-10t-4$

$,\vartheta(t)=t-2\pi,\mu(t)=t-\frac{5\pi}{2},\omega(t)=t-4\pi,\alpha=1.$

$\mathcal{H}_1=min\{7t^2+8t-2\,,7(t-4\pi)^2+8(t-4\pi)-2\,)\}=7t^2+8t-2$

$\mathcal{H}_2=t^3+2t^2-10t-4,\mathcal{H}_3=7t^2+8t-2,\mathcal{H}_4=t^3+2t^2-10t-4,$

$$\frac{d}{dt}[(-2t-t^3)\sin t+t^2\cos t]$$
$$+\frac{d}{dt}\delta[(-2(t-4\pi)-(t-4\pi)^3)\sin(t-4\pi)+(t-4\pi)^2\cos(t-4\pi))]$$
$$+\big((-2t-t^3)\sin t+t^2\cos t\big)\Big((7t^2+8t-2)\aleph(\vartheta(t))+\mathcal{H}_2\aleph\big(\mu(t)\big)\Big)\leq 0$$

Since $\int_{t^*}^{t}\int_{t}^{t+2}(7s^2+8s-2)dsdr=\infty$ as $t\to\infty$,

$\int_{t^*}^{t}\int_{t}^{t+\frac{1}{2}}(s^3+2s^2-10s-4\,)dsdr=\infty$ as $t\to\infty$,

Then every conditions of th1 is satisfies where $y=\cos t$.

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