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Analytical Study of Convergence Filters in Topological Vector Spaces via Proximit Structure

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ABSTRACT

This paper is investigates the concept of convergence in Di-proximit topological vector space and examines several results in this new context. We introduce the notions of the terms cluster point, Di-Cauchy filter, Di-convergent filter, Di-complete in Di-proximit topological vector space and the relationship between Di-Cauchy filter and Di-convergent filter in Di-proximit topological vector space.

1.Introduction

The study of convergence in generalized structures has played a fundamental role in the development of modern topology and functional analysis. Classical works, such as Kelley[1] laid the foundations of general topology, introducing essential notions of convergence, continuity and separation axioms that later influenced the development of more abstract the frameworks. In particular, the classical book by Dugundji [2] provides a systematic and rigorous treatment of general topology, including topological and metric spaces, convergence and continuous mappings .In addition, Bourbaki [3] contributions to general topology provided further refinement of convergence concepts and their connection with algebraic and geometric structures. In the decades that followed, the classical theory of convergence was significantly expanded by the introduction of approach spaces, a powerful framework that unifies and generalizes metric and topological structures. Approach theory provides a flexible framework to describe convergence through distance-like functions, allowing the characterization of Cauchy sequences, completeness and continuity settings more general than metric spaces. More recently, Hussein and her students have made substantial contributions to this area. Furthermore, Chu[4] explored the theory of proximity spaces, presenting early attempts to generalize metric convergence using nearness relation . In this direction ,Abbas and Hussein [5] introduced new results in the theory of normed an Approach Space, where the classical normed space structure is generalized through the approach framework. Their work investigates convergence, Cauchy sequences and completeness. In another direction, Neamah and Hussein [6,7] introduced t^{ω}-

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Normed approach space, presenting notions of t^{ω}- convergence and t^{ω}- Cauchy sequences and establishing the concept of t^{ω}- Banach Spaces. These contributions expanded the analytical tools for studying convergence in generalized metric frameworks. Furthermore, Kadhim and Hussein [8,9] offered new results of Banach Algebra via Proximit Structure and Approach Structure, introducing several new algebraic convergence concepts and demonstrating their applications with in functional analysis. Similarly, Saeed and Hussein [10] developed fundamental results in normed approach spaces via β- Approach Structure, showing that every β-Normed Approach Structure induces a normed structure but not necessarily vice versa. Their work also provided examples of β-Banach spaces and explored completeness under the β-Approach. The works of Kream and Hussein [11] introduced random approach normed spaces and random approach spaces, establishing new types of convergence such as δ_R- Cauchy and δ_R- convergent sequences, and linking these structures with classical Banach spaces. Abed and Hussein [12] describes of Ş-proximit space for nonempty sets. The relationship between approach space and Ş-proximit space has been clarified , we also discussed the definitions of Cauchy sequence, cluster point, Ş-convergent sequence, and Ş-complete. In this work [13] Lowen adopts this viewpoint of convergence and focuses on filter convergence.

2. Preliminaries

Definition 2.1([3]). Let Ꭓ be a non-empty set, and $\mathcal{F}$ is a collection of subsets of Ꭓ is called a filter on Ꭓ if it possesses the following properties:

(1) $\emptyset \notin \mathcal{F}$.

(2) If $Á, É \in \mathcal{F}$, then $Á \cap É \in \mathcal{F}$.

(3)If $Á \in \mathcal{F}$ and $Á \subseteq É$, then $É \in \mathcal{F}$.

Example 2.2:

Let $Ꭓ = \{1,2,3\}$ then (1) $\mathcal{F}_1 = \{Ꭓ, \{1,2\}\}$ is a filter on 1.

(2) $\mathcal{F}_2 = \{Ꭓ, \{1\}, \{1,2\}, \{1,3\}\}$ is a filter on 2.

(3) $\mathcal{F}_3 = \{Ꭓ, \{1,2\}, \{2,3\}\}$ is not filter on 3.

Definition 2.3([2]). In metric space $(Ꭓ, đ)$, a filter $\mathcal{F}$ is Cauchy filter, if for each $\epsilon > 0$ there exists $ß \in \mathcal{F}$, $đ(n, m) < \epsilon$. For all $n, m \in ß$.

3. Convergence in Di-proximit Topological Vector Space

Definition 3.1: Let Ӿ be a non-empty set. A function $\hat{S} : 2^{Ӿ} \times 2^{Ӿ} \rightarrow [0, \infty]$ define on Ӿ is said to be Di-functional if for all $\mathcal{A}, \mathcal{D}, \mathcal{C} \in 2^{Ӿ}$ satisfy the following properties :-

P_1: If $\hat{S}(\mathcal{A}, \mathcal{D}) = 0 \leftrightarrow \mathcal{A} \cap \mathcal{D} \neq \emptyset$.

P_2: If $\mathcal{D} = \emptyset$, then $\hat{S}(\mathcal{A}, \mathcal{D}) = \infty$.

P_3: $\hat{S}(\mathcal{A}, \mathcal{D} \cup \mathrm{C}) = \min\{\hat{S}(\mathcal{A}, \mathcal{D}), \hat{S}(\mathcal{A}, \mathrm{C})\}$

P_4: If $\mathcal{D} \subseteq \mathcal{C}$, then $\hat{S}(\mathcal{A}, \mathcal{C}) \leq \hat{S}(\mathcal{A}, \mathcal{D})$

P_5: $\hat{S}(\mathcal{A}, \mathcal{D}) \leq \hat{S}(\mathcal{A}^{\epsilon}, \mathcal{D}^{\gamma}) + \epsilon + \gamma$.

The pair (Ӿ, $\hat{S}$) is called Di-proximit space . For every $\mathcal{A} \in 2^{Ӿ}$ and $\epsilon, \gamma \in [0, \infty]$, we write:

$\mathfrak{I}_{\epsilon}(\mathcal{A}) = \{ n \in Ӿ \mid \hat{S}(\{n\}, \mathcal{A}) \leq \epsilon \}$.

Definition 3.2: A quadruple (Ӿ, $\hat{S}$, $\mathfrak{I}_{\epsilon}$, +) is called a Di-proximit group if satisfy the following :

(1) (Ӿ, $\hat{S}$, $\mathfrak{I}_{\epsilon}$) is Di- proximit space .

(2) (Ӿ, +) is group .

(3) +:Ӿ ⊕ Ӿ→Ӿ such that (n ,b) → n +b is Di-contraction.

(4) − : Ӿ→Ӿ such that n→ - n is Di- contraction.

Definition 3.3: Let (Ӿ, $\hat{S}$, $\mathfrak{I}_{\epsilon}$)is Di-proximit space with two operation (addition and scalar multiplication) is Di-proximit vector space if following condition satisfy :-

For all n , m ∈ Ӿ and ζ, Ϋ ∈ Ƒ

(1) (Ӿ, $\hat{S}$, $\mathfrak{I}_{\epsilon}$, +) is Di-proximit group .

(2) Ϋ · n ∈ Ӿ.

(3) Ϋ · (n + b) = Ϋ. n + Ϋ. b

(4) (n + b)Ϋ = n . Ϋ + b . Ϋ

(5) (ζ · Ϋ) · n = ζ · (Ϋ . n).

(6) 1 · n = n .

Definition 3.4: A function ψ:(Ӿ,$\hat{S}_{Ӿ}$, $\mathfrak{I}_{\epsilon}'$) → (Ꞧ,$\hat{S}_{Ꞧ}$, $\mathfrak{I}_{\epsilon}''$) such that (Ӿ,$\hat{S}_{Ӿ}$, $\mathfrak{I}_{\epsilon}'$) and (Ꞧ,$\hat{S}_{Ꞧ}$, $\mathfrak{I}_{\epsilon}''$) are Di-proximit space is called Di-contraction if $\psi(\mathfrak{I}_{\epsilon}'(\mathcal{Y})) \subseteq \mathfrak{I}_{\epsilon}''(\psi(\mathcal{Y}))$ for all $\mathcal{Y} \subseteq$ Ӿ and $\epsilon \in \mathcal{R}^{+}$.

Definition 3.5: A Di-proximit topological vector space (Di-p.t.v.s for short) is Di-proximit vector space Ꭓ (over Ƒ) together proximit topology τ on Ꭓ such that the maps (n,m) → n+m and (ζ,n) → ζ n are contraction from Ꭓ × Ꭓ → Ꭓ and Ƒ× Ꭓ → Ꭓ.

Definition 3.6: A set $É \in 2^{Ꭓ}$ is claimed to be cluster point in Di-p.t.v.s ($Ꭓ, \hat{S}_\tau, \mathfrak{I}_\epsilon$,+ , .). If a disjoint filter exists $\{\mathcal{F}_i\}_{i=1}^{\infty}$ of Ꭓ for all $Á_i \in \{\mathcal{F}_i\}_{i=1}^{\infty}$ such that $\inf_{i\epsilon Z^+} \hat{S}_\tau(Á_i, É) = 0$ and $\inf_{i\epsilon Z^+, \epsilon \in \mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) \subseteq É$,which is expressed by $\{Á_i\}_{i=1}^{\infty} \to É$. We repersented the set of all cluster points in Di-p.t.v.s by ʹΩ(Ꭓ).

Definition 3.7: A filter $\{\mathcal{F}_i\}_{i=1}^{\infty}$ of Ꭓ is referred to as a Cauchy filter in Di-p.t.v.s (Di- Cauchy), if each of the cluster point $É_k$, $\lim_{i,k\to\infty} \inf_{Á_i, É_k \in \{\mathcal{F}_i\}_{i=1}^{\infty}} \hat{S}_\tau(Á_i, É_k) = 0$ and $\lim_{i,k\to\infty} \inf_{\epsilon \in \mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) \subseteq É_k$, for all $Á_i \in \{\mathcal{F}_i\}_{i=1}^{\infty}$. A filter $\{\mathcal{F}_i\}_{i=1}^{\infty}$in Ꭓ is claimed to be Di-convergent filter in Di-p.t.v.s if for each $É \in$ ʹΩ(Ꭓ)like that $\hat{S}_\tau(Á_i, É) = 0$ and $\mathfrak{I}_\epsilon(Á_i) \subseteq É$, for all $Á_i \in \{\mathcal{F}_i\}_{i=1}^{\infty}$ and $i\epsilon Z^+$.

Proposition 3.8: Let's ($Ꭓ, \hat{S}_\tau, \mathfrak{I}_\epsilon$,+ , .)be Di-p.t.v.s, hence the following are equivalents :

1. Di-convergent filter in Di-p.t.v.s.

2. $\lim_{i\to\infty} \inf_{Á_i, É \in \{\mathcal{F}_i\}_{i=1}^{\infty}} \hat{S}_\tau(Á_i, É) = 0$, $\lim_{i\to\infty} \inf_{\epsilon \in \mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) \subseteq É$, $\lim_{i\to\infty} \sup_{Á_i, É \in \{\mathcal{F}_i\}_{i=1}^{\infty}} \hat{S}_\tau(Á_i, É) = 0$ *and* $\lim_{i\to\infty} \sup \mathfrak{I}_\epsilon(Á_i) \subseteq É$.

Proof:

Let's $\{\mathcal{F}_i\}_{i=1}^{\infty}$ be Di-convergent filter in Di-p.t.v.s.

Hence for all $É \in$ ʹΩ(Ꭓ) and for all $Á_i \subseteq \{\mathcal{F}_i\}_{i=1}^{\infty}$ like that $\hat{S}_\tau(Á_i, É) = 0$ and $\mathfrak{I}_\epsilon(Á_i) \subseteq É$, so $\inf_{i\epsilon Z^+} \hat{S}_\tau(Á_i, É) = 0$, $\inf_{i\epsilon Z^+, \epsilon \in \mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) \subseteq É$ *and* $\sup_{i\epsilon Z^+} \hat{S}_\tau(Á_i, É) = 0$, $\sup_{i\epsilon Z^+, \epsilon \in \mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) \subseteq É$. Thus $\lim_{i\to\infty} \inf_{Á_i, É \in \{\mathcal{F}_i\}_{i=1}^{\infty}} \hat{S}_\tau(Á_i, É) = 0$, $\lim_{i\to\infty} \inf_{\epsilon \in \mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) \subseteq É$, $\lim_{i\to\infty} \sup_{Á_i, É \in \{\mathcal{F}_i\}_{i=1}^{\infty}} \hat{S}_\tau(Á_i, É) = 0$ and $\lim_{i\to\infty} \sup_{\epsilon \in \mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) \subseteq É$ .For all $Á_i \in \{\mathcal{F}_i\}_{i=1}^{\infty}$.

On the other hand , if condition(2)is met that's $\lim_{i\to\infty} \inf_{Á_i, É \in \{\mathcal{F}_i\}_{i=1}^{\infty}} \hat{S}_\tau(Á_i, É) = 0$, $\lim_{i\to\infty} \inf_{\epsilon \in \mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) \subseteq É$, $\lim_{i\to\infty} \sup_{Á_i, É \in \{\mathcal{F}_i\}_{i=1}^{\infty}} \hat{S}_\tau(Á_i, É) = 0$ and $\lim_{i\to\infty} \sup_{\epsilon \in \mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) \subseteq É$, for all $Á_i \in \{\mathcal{F}_i\}_{i=1}^{\infty}$.Afterward É is cluster point, that's $\inf_{i\epsilon Z^+} \hat{S}_\tau(Á_i, É) = 0$ and $\inf_{i\epsilon Z^+, \epsilon \in \mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) \subseteq É$. So that $\hat{S}_\tau(Á_i, É) = 0$ and $\mathfrak{I}_\epsilon(Á_i) \subseteq É$. So $\{\mathcal{F}_i\}_{i=1}^{\infty}$ is Di-convergent filter in Di-p.t.v.s.

Remark 3.9: All Di-convergent filter are Di-Cauchy filter (Cauchy filter Di-p.t.v.s), but the reverse not true .

Example 3.10: Let's Ӿ =$\Re/\{0\}$ and $\hat{S}_\tau: 2^{Ӿ} \times 2^{Ӿ} \rightarrow [0,\infty]$ defined as

$$\hat{S}_\tau(Á_{\dot{\imath}}, É)=\begin{cases} \infty & Á_{\dot{\imath}} \neq \emptyset \text{ and } É = \emptyset \\ \inf_{n\in Á_{\dot{\imath}}, m\in É}|n-m| & Á_{\dot{\imath}} \neq \emptyset \text{ and } É \neq \emptyset \end{cases}$$

$\mathcal{F}_{\dot{\imath}} = \{$ Á$\subseteq \Re/\{0\}$: $\exists$n$\in Z^+$,$(\frac{-1}{n},\frac{1}{n}) \subseteq$ Á $\}$

be Di- Cauchy filter proximit topological vector space not Di-convergent filter in Di-p.t.v.s.

Proposition 3.11: The following are equivalents if Ӿ is a Di-proximit metrizable topological vector space:

(1)$\{\mathcal{F}_{\dot{\imath}}\}_{\dot{\imath}=1}^{\infty}$ is Di-convergent filter in Di-p.t.v.s.

(2) $\sup_{É\in Ω(Ӿ)}\hat{S}_{đ}(Á_{\dot{\imath}}, É) = 0.$ For all $Á_{\dot{\imath}} \in \{\mathcal{F}_{\dot{\imath}}\}_{\dot{\imath}=1}^{\infty}$.

Proof:

Suppose that $\{\mathcal{F}_{\dot{\imath}}\}_{\dot{\imath}=1}^{\infty}$ is Di-convergent filter in Di-p.t.v.s.

Then for each É ∈ ′Ω(Ӿ) like that $\hat{S}_\tau(Á_{\dot{\imath}}, É) = 0$ and $\mathfrak{I}_\epsilon(Á_{\dot{\imath}}) \subseteq$ É.

$$\hat{S}_\tau(Á_{\dot{\imath}}, É) = \inf_{m_{\dot{\imath}}\in Á_{\dot{\imath}}} \inf_{m\in É} đ(m_{\dot{\imath}}, m) = 0$$

$$= \sup_{É\in′Ω(Ӿ)} \inf_{m_{\dot{\imath}}\in Á_{\dot{\imath}}} \inf_{m\in É} đ(m_{\dot{\imath}}, m) = 0$$

$$= \sup_{É\in′Ω(Ӿ)}\hat{S}_{đ}(Á_{\dot{\imath}}, É) = 0.$$

On the other hand , if condition(2) is met that's

$$\sup_{É\in′Ω(Ӿ)}\hat{S}_{đ}(Á_{\dot{\imath}}, É) = \sup_{É\in′Ω(Ӿ)}\inf_{m_{\dot{\imath}}\in Á_{\dot{\imath}}} \inf_{m\in É} đ(m_{\dot{\imath}}, m) = 0$$

For $m_{\dot{\imath}} \in Á_{\dot{\imath}}$, $m \in$ É then $\hat{S}_{đ}(Á_{\dot{\imath}}, É) = \hat{S}_\tau(Á_{\dot{\imath}}, É) = 0$, and $\mathfrak{I}_\epsilon(Á_{\dot{\imath}}) \subseteq$ É.

Proposition 3.12: If $\{\mathcal{F}_{\dot{\imath}}\}_{\dot{\imath}=1}^{\infty}$ is filter in Ӿ and (Ӿ, $\hat{S}_\tau$,$\mathfrak{I}_\epsilon$,+, .) is Di-p.t.v.s, then the filter in(Ӿ, đ) is Cauchy filter if and only if it is Di-Cauchy filter in(Ӿ, $\hat{S}_\tau$,$\mathfrak{I}_\epsilon$,+, .)

Proof:

Let's $\{\mathcal{F}_{\dot{\imath}}\}_{\dot{\imath}=1}^{\infty}$ is Cauchy filter in (Ӿ, đ).

Then for each $\epsilon > 0$ there may ß ∈ $\mathcal{F}$, đ(n, m) $< \epsilon$. For all n, m ∈ ß.

$$\hat{S}_\tau(Á_{\dot{\imath}}, É) = \inf_{n_{\dot{\imath}}\in Á_{\dot{\imath}}}\inf_{m\in É} đ(n_{\dot{\imath}}, m) = 0. \text{ For all } Á_{\dot{\imath}} \in \{\mathcal{F}_{\dot{\imath}}\}_{\dot{\imath}=1}^{\infty}.$$

Since $\{É \in \mathcal{F} : \hat{S}_\tau(Á_i, É) = 0 < \epsilon\} = \mathfrak{I}_\epsilon(Á_i)$, such that $\mathfrak{I}_\epsilon(Á_i) \subseteq É$.

Then $\{\mathcal{F}_i\}_{i=1}^{\infty}$ is Di-Cauchy filter in ($Ӿ$, $\hat{S}_\tau$,$\mathfrak{I}_\epsilon$,+, .).

Reversely ,Let's $\{\mathcal{F}_i\}_{i=1}^{\infty}$ be Di-Cauchy filter in ($Ӿ$, $\hat{S}_\tau$,$\mathfrak{I}_\epsilon$,+, .)

For each of the cluster point $É_k$, $\lim_{i,k\to\infty} \inf_{Á_i,É_k \in \{\mathcal{F}_i\}_{i=1}^{\infty}} \hat{S}_\tau(Á_i, É_k) = 0$ and $\lim_{i,k\to\infty} \inf_{\epsilon \in \mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) \subseteq É_k$

Then $\inf_{Á_i,É_k \in \{\mathcal{F}_i\}_{i=1}^{\infty}} \hat{S}_\tau(Á_i, É_k) = 0$ and $\inf_{\epsilon \in \mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) \subseteq É_k$

$$\hat{S}_\tau(Á_i, É_k) = \inf_{n_i \in Á_i} \inf_{m_k \in É_k} đ(n_i, m_k) = 0, \text{then}$$

Then for each $\epsilon > 0$ there may $ß \in \mathcal{F}$, $đ(n, m) < \epsilon$. For all $n, m \in ß$.

Consequently $\{\mathcal{F}_i\}_{i=1}^{\infty}$ is Cauchy filter in ($Ӿ$, đ).

Theorem 3.13: Let's ($Ӿ$, $\hat{S}_\tau$,$\mathfrak{I}_\epsilon$,+, .) be Di-p.t.v.s and let $\{\mathcal{F}_i\}_{i=1}^{\infty}$and $\{\mathcal{J}_i\}_{i=1}^{\infty}$be Di-convergent filter in ($Ӿ$, $\hat{S}_\tau$,$\mathfrak{I}_\epsilon$,+, .) between É and Ś , in that order. After that,

(1) $\{\mathcal{F}_i + \mathcal{J}_i\}_{i=1}^{\infty}$ is Di-convergence filter to $É + Ś$.

(2) $\{\zeta\, \mathcal{F}_i\}_{i=1}^{\infty}$ is Di-convergence filter to $\zeta É$.

(3) $\{\mathcal{F}_i \mathcal{J}_i\}_{i=1}^{\infty}$ is Di-convergence filter to ÉŚ.

Proof:

(1) Let's $\{\mathcal{F}_i\}_{i=1}^{\infty}$, $\{\mathcal{J}_i\}_{i=1}^{\infty}$ is Di-convergence to É, Ś such that

$$\{\mathcal{F}_i + \mathcal{J}_i\}_{i=1}^{\infty} = \{\forall\, Ṕ = n_i + m_i : n_i \in \mathcal{F}_i, m_i \in \mathcal{J}_i\}.$$

$$\{É + Ś\} = \{n + m : n \in É, m \in Ś\}.$$

Since $\{\mathcal{F}_i\}_{i=1}^{\infty}$is Di $-$ convergence to É. For all $Á_i \in \{\mathcal{F}_i\}_{i=1}^{\infty}$ and

$$\lim_{i\to\infty} \inf_{Á_i,É \in \{\mathcal{F}_i\}_{i=1}^{\infty}} \hat{S}_\tau(Á_i, É) = 0, \lim_{i\to\infty} \inf_{\epsilon \in \mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) \subseteq É\ , \lim_{i\to\infty} \sup_{Á_i,É \in \{\mathcal{F}_i\}_{i=1}^{\infty}} \hat{S}_\tau(Á_i, É) = 0,$$

$$\text{and} \lim_{i\to\infty} \sup_{\epsilon \in \mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) \subseteq É$$

And since $\{\mathcal{J}_i\}_{i=1}^{\infty}$ is Di-convergence to Ś. For all $Ĺ_i \in \{\mathcal{J}_i\}_{i=1}^{\infty}$

$$\lim_{i\to\infty} \inf_{Ĺ_i,Ś \in \{\mathcal{F}_i\}_{i=1}^{\infty}} \hat{S}_\tau(Ĺ_i, Ś) = 0, \lim_{i\to\infty} \inf_{\epsilon \in \mathfrak{R}^+} \mathfrak{I}_\epsilon(Ĺ_i) \subseteq Ś\ , \lim_{i\to\infty} \sup_{Ĺ_i,Ś \in \{\mathcal{F}_i\}_{i=1}^{\infty}} \hat{S}_\tau(Ĺ_i, Ś) = 0 \text{ and}$$

$$\lim_{i\to\infty} \sup_{\epsilon \in \mathfrak{R}^+} \mathfrak{I}_\epsilon(Ĺ_i) \subseteq Ś.$$

$$\lim_{i\to\infty} \inf_{Á_i,É_k \in \{\mathcal{F}_i\}_{i=1}^{\infty}} \inf_{n_i \in Á_i} \inf_{n \in É} đ(n_i, n) = 0 \ \ and \ \lim_{i\to\infty} \inf_{Ĺ_i,Ś \in \{\mathcal{F}_i\}_{i=1}^{\infty}} \inf_{m_i \in Ĺ_i} \inf_{m \in Ś} đ(m_i, m) = 0$$

Consequently

$$\lim_{i\to\infty} \inf \hat{S}_\tau(Á_i + Ĺ_i, É + Ś) = \lim_{i\to\infty} \inf \min\{\inf_{n_i + m_i \in Á_i + Ĺ_i} \inf_{n+m \in É + Ś} đ(n_i + m_i, n + m)\}$$

$$\leq \lim_{i\to\infty} \inf \ \min\{\inf_{n_i+m_i\in Á_i+Ĺ_i}\inf_{n\in É} đ(n_i+m_i,n), \inf_{n_i+m_i\in Á_i+Ĺ_i}\inf_{m\in Ś} đ(n_i+m_i,m)\}$$

$$\leq \lim_{i\to\infty} \inf \ \min\{\inf_{n_i\in Á_i}\inf_{n\in É}đ(n_i,n)+\inf_{m_i\in Ś}\inf_{n\in É} đ(m_i,n), \inf_{n_i\in Á_i}\inf_{m\in Ś} đ(n_i,m)$$

$$+ \inf_{m_i\in Ĺ_i}\inf_{m\in Ś} đ(m_i,m)\} = 0.$$

Thus $\lim_{i\to\infty} \inf Ŝ_\tau(Á_i+Ĺ_i, É+Ś) = 0$ and $\lim_{i\to\infty} \sup Ŝ_\tau(Á_i+Ĺ_i, É+Ś) = 0$ and $\lim_{i\to\infty} \sup_{\epsilon\in\mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i+Ĺ_i) = \lim_{i\to\infty} \sup_{\epsilon\in\mathfrak{R}^+} \{ É+Ś \in \mathcal{F}: Ŝ_\tau(É+Ś, Á_i+Ĺ_i) < \epsilon\}$.

$$= \lim_{i\to\infty} \sup_{\epsilon\in\mathfrak{R}^+}\{ \inf_{n+m\in É+Ś}\inf_{n_i+m_i\in Á_i+Ĺ_i} đ(n+m, n_i+m_i)\} < \epsilon$$

$$\leq \lim_{i\to\infty} \sup_{\epsilon\in\mathfrak{R}^+}\{ \inf_{n\in É}\inf_{n_i\in Á_i} đ(n,n_i) + \inf_{m\in Ś}\inf_{m_i\in Ĺ_i} đ(m,m_i)\} < \epsilon$$

$$\leq \lim_{i\to\infty} \sup_{\epsilon\in\mathfrak{R}^+}[\{ \inf_{n\in É}\inf_{n_i\in Á_i} đ(n,n_i)\} < \epsilon + \inf_{m\in Ś}\inf_{m_i\in Ĺ_i} đ(m,m_i)\} < \epsilon].$$

$$\leq \lim_{i\to\infty} \sup_{\epsilon\in\mathfrak{R}^+}[\mathfrak{I}_\epsilon(Á_i) + \mathfrak{I}_\epsilon(Ĺ_i)].$$

$$\leq \lim_{i\to\infty} \sup_{\epsilon\in\mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) + \lim_{i\to\infty} \sup_{\epsilon\in\mathfrak{R}^+} \mathfrak{I}_\epsilon(Ĺ_i) \subseteq É + Ś.$$

And $\lim_{i\to\infty} \inf_{\epsilon\in\mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i+Ĺ_i) \subseteq É + Ś$.

Then $\{\mathcal{F}_i + \mathcal{J}_i\}_{i=1}^{\infty}$ is Di-convergence filter to $É + Ś$.

(2) Let's $\{\mathcal{F}_i\}$ is Di-convergent filter to É and $\zeta \in Ғ$ like that $\{\zeta\mathcal{F}_i\} = \{ \zeta n_i : i \in Z^+\}$.

So For all $Á_i \in \{\mathcal{F}_i\}_{i=1}^{\infty}$

$\lim_{i\to\infty} \inf_{Á_i, É_k\in\{\mathcal{F}_i\}_{i=1}^{\infty}} Ŝ_\tau(Á_i, É) = 0$, $\lim_{i\to\infty} \inf_{\epsilon\in\mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) \subseteq É$, $\lim_{i\to\infty} \sup_{Á_i, É_k\in\{\mathcal{F}_i\}_{i=1}^{\infty}} Ŝ_\tau(Á_i, É)$ and

$$\lim_{i\to\infty} \sup_{\epsilon\in\mathfrak{R}^+} \mathfrak{I}_\epsilon(Á_i) \subseteq É$$

$$\lim_{i\to\infty} \inf_{Á_i, É_k\in\{\mathcal{F}_i\}_{i=1}^{\infty}} Ŝ_\tau(Á_i, É) = \lim_{i\to\infty} \inf_{Á_i, É_k\in\{\mathcal{F}_i\}_{i=1}^{\infty}} \inf_{n_i\in Á_i} \inf_{n\in É} đ(n_i, n) = 0,$$

Then $\lim_{i\to\infty} \inf_{\zeta Á_i, \zeta É_k\in\{\mathcal{F}_i\}_{i=1}^{\infty}} Ŝ_\tau(\zeta Á_i, \zeta É) = \lim_{i\to\infty} \inf_{\zeta Á_i, \zeta É_k\in\{\mathcal{F}_i\}_{i=1}^{\infty}} \inf_{\zeta n_i\in\zeta Á_i} \inf_{\zeta n\in\zeta É} đ(\zeta n_i, \zeta n) = 0.$

Consequently $\lim_{i\to\infty} \inf_{\zeta Á_i, \zeta É_k\in\{\mathcal{F}_i\}_{i=1}^{\infty}} Ŝ_\tau(\zeta Á_i, \zeta É) = 0$ and $\lim_{i\to\infty} \sup_{\zeta Á_i, \zeta É_k\in\{\mathcal{F}_i\}_{i=1}^{\infty}} Ŝ_\tau(\zeta Á_i, \zeta É) = 0$

$$\lim_{i\to\infty} \inf_{\epsilon\in\mathfrak{R}^+} \mathfrak{I}_\epsilon(\zeta Á_i) \subseteq \zeta É \text{ and } \lim_{i\to\infty} \sup_{\epsilon\in\mathfrak{R}^+} \mathfrak{I}_\epsilon(\zeta Á_i) \subseteq \zeta É$$

Then $\{\zeta\mathcal{F}_i\}$ Di-convergence filter to ζ É.

(3) Let's $\{\mathcal{F}_{i}\}_{i=1}^{\infty}, \{\mathcal{J}_{i}\}_{i=1}^{\infty}$ be Di-convergent filter to $\acute{E}, \acute{S}$, like that $\{\mathcal{F}_{i}.\mathcal{J}_{i}\} = \{\forall\ \hbar = n_{i}.m_{i} : i \in z^{+}\}$. Since $\{\mathcal{F}_{i}\}_{i=1}^{\infty}$ is Di $-$ convergence to $\acute{E}$ then for each $\acute{E} \in \Omega(X)$ such that $\hat{S}_{\tau}(\acute{A}_{i}, \acute{E}) = 0$, $\mathfrak{I}_{\epsilon}(\acute{A}_{i}) \subseteq \acute{E}$.

Since $\{\mathcal{J}_{i}\}_{i=1}^{\infty}$ is Di $-$ convergence to $\acute{S}$ then for each $\acute{S} \in \Omega(X)$, $\hat{S}_{\tau}(\acute{L}_{i}, \acute{S}) = 0$, $\mathfrak{I}_{\epsilon}(\acute{L}_{i}) \subseteq \acute{S}$. Put $\acute{P} = \acute{E}\acute{S}$ and $n, m \in \acute{P}$ such that

$$\hat{S}_{\tau}(\acute{A}_{i}\acute{L}_{i}, \acute{P}) = \inf_{n_{i}m_{i} \in \acute{A}_{i}\acute{L}_{i}} \inf_{nm \in \acute{P}} đ(n_{i}m_{i}, nm) = 0.$$

Thus $\hat{S}_{\tau}(\acute{A}_{i}\acute{L}_{i}, \acute{P}) = 0$ and $\mathfrak{I}_{\epsilon}(\acute{A}_{i}\acute{L}_{i}) \subseteq \acute{P}$.

Then $\{\mathcal{F}_{i}\mathcal{J}_{i}\}$ is Di-convergence filter to $\acute{E}\acute{S}$.

Definition 3.14: A Di-p.t.v.s is known to Di-complete if each Di-Cauchy filter is Di-convergent in $(X, \hat{S}_{\tau}, \mathfrak{I}_{\epsilon}, +, .)$.

Theorem 3.15: A Di-p.t.v.s $(X, \hat{S}_{\tau}, \mathfrak{I}_{\epsilon}, +, .)$ is Di-complete space if and only if $(X, \hat{S}_{đ}, \mathfrak{I}_{\epsilon}, +, .)$ is complete.

Proof:

Let's $\{\mathcal{F}_{i}\}_{i=1}^{\infty}$ be Di-Cauchy filter in $(X, \hat{S}_{\tau}, \mathfrak{I}_{\epsilon}, +, .)$, then it is Di-Cauchy filter in $(X, \hat{S}_{đ}, \mathfrak{I}_{\epsilon}, +, .)$ by Proposition 3.12.

Given that $(X, \hat{S}_{\tau}, \mathfrak{I}_{\epsilon}, +, .)$ is Di-complete, so for everyone $\acute{E} \in \Omega(X)$ like that $\hat{S}_{\tau}(\acute{A}_{i}, \acute{E}) = 0$ and $\mathfrak{I}_{\epsilon}(\acute{A}_{i}) \subseteq \acute{E}$. So $\hat{S}_{\tau}(\acute{A}_{i}, \acute{E}) = \hat{S}_{đ}(\acute{A}_{i}, \acute{E}) = 0$, then $(X, \hat{S}_{đ}, \mathfrak{I}_{\epsilon}, +, .)$ is complete.

Reversely, Let's $\{\mathcal{F}_{i}\}_{i=1}^{\infty}$ be Di-Cauchy filter in $(X, \hat{S}_{đ}, \mathfrak{I}_{\epsilon}, +, .)$.

Since $(X, \hat{S}_{đ}, \mathfrak{I}_{\epsilon}, +, .)$ is complete. For all $\acute{A}_{i} \in \{\mathcal{F}_{i}\}_{i=1}^{\infty}$, $\hat{S}_{đ}(\acute{A}_{i}, \acute{E}) = 0$ and $\mathfrak{I}_{\epsilon}(\acute{A}_{i}) \subseteq \acute{E}$. Consequently

$$\sup_{\acute{E} \in \Omega(X)} \hat{S}_{đ}(\acute{A}_{i}, \acute{E}) = 0 \text{ . For all } \acute{A}_{i} \in \{\mathcal{F}_{i}\}_{i=1}^{\infty}$$

Then $\{\mathcal{F}_{i}\}_{i=1}^{\infty}$ is Di-convergent filter in Di-p.t.v.s $(X, \hat{S}_{\tau}, \mathfrak{I}_{\epsilon}, +, .)$.

Then X is Di-complete.

Example 3.16: Let's $\mathfrak{R}$ be a set of real number with usual topology, defined by

$\hat{S}_{\tau}: 2^{\mathfrak{R}} \times 2^{\mathfrak{R}} \longrightarrow [0, \infty]$

$$\hat{S}_\tau(\acute{A},\acute{E}) = \begin{cases} 0 & \acute{A},\acute{E}\ are\ unbounded \\ \infty & \acute{A} \neq \emptyset, \acute{E} = \emptyset \\ \inf_{n\in\acute{A}}\inf_{m\in\acute{E}}|n-m| & \acute{A},\acute{E}\ are\ bounded \end{cases}$$

and

$$\mathfrak{I}_\epsilon(\acute{A}) = \begin{cases} \mathfrak{R} & \acute{A}\ \ is\ unbounded \\ \emptyset & \acute{A} \neq \emptyset \\ \inf_{n\in\{x\}}\inf_{m\in\acute{A}}|n-m| & \acute{A}\ is\ \ bounded \end{cases}$$

Then ($\mathfrak{R}$, $\hat{S}_{đ}$, $\mathfrak{I}_\epsilon$, $+,.$) is Di-complete proximit toplogical vector space.

Solve:- For $ị \in Z^+$, for all $\acute{A}_ị \in \{\mathcal{F}_ị\}_{ị=1}^{\infty}$

$$\hat{S}_\tau(\acute{A}_ị,\acute{E}_ķ) = \begin{cases} 0 & \acute{A}_ị,\acute{E}_ķ\ unbounded \\ \infty & \acute{A}_ị \neq \emptyset, \acute{E}_ķ = \emptyset \\ \inf_{n_ị\in\acute{A}_ị}\inf_{m_ị\in\acute{E}_ķ}|n_ị-m_ị| & \acute{A}_ị,\acute{E}_ķ are\ bounded \end{cases}$$

and

$$\mathfrak{I}_\epsilon(\acute{A}_ị) = \begin{cases} \mathfrak{R} & \acute{A}_ị\ \ is\ unbounded \\ \emptyset & \acute{A}_ị \neq \emptyset \\ \inf_{n\in\acute{A}_ị}\inf_{m\in\acute{E}_ķ}|n-m| & \acute{A}_ị\ is\ \ bounded \end{cases}$$

Let's $\{\mathcal{F}_ị\}_{ị=1}^{\infty}$ be Di-Cauchy filter , for each of the cluster point $\acute{E}_ķ$ $\lim_{ị,ķ\to\infty} \inf_{\acute{A}_ị,\acute{E}_ķ\in\{\mathcal{F}_ị\}_{ị=1}^{\infty}} \hat{S}_\tau(\acute{A}_ị,\acute{E}_ķ) = 0$ and $\lim_{ị,ķ\to\infty} \inf_{\epsilon\in\mathfrak{R}^+} \mathfrak{I}_\epsilon(\acute{A}_ị) \subseteq \acute{E}_ķ$.

(i) If $\acute{E} \subseteq \mathfrak{R}$ is unbounded, $\lim_{ị\to\infty} \inf_{\acute{A}_ị,\acute{E}_ķ\in\{\mathcal{F}_ị\}_{ị=1}^{\infty}} \hat{S}_\tau(\acute{A}_ị,\acute{E}_ķ)= 0$ and $\lim_{ị\to\infty} \inf_{\epsilon\in\mathfrak{R}^+} \mathfrak{I}_\epsilon(\acute{A}_ị) \subseteq \acute{E}_ķ$.

$$\inf_{\acute{A}_ị,\acute{E}_ķ\in\{\mathcal{F}_ị\}_{ị=1}^{\infty}} \hat{S}_\tau(\acute{A}_ị,\acute{E}_ķ)= 0 \text{ and } \inf_{ị\in Z^+,\epsilon\in\mathfrak{R}^+} \mathfrak{I}_\epsilon(\acute{A}_ị) \subseteq \acute{E}_ķ.$$

$$\hat{S}_\tau(\acute{A}_ị,\acute{E}) = 0 \text{ and } \mathfrak{I}_\epsilon(\acute{A}_ị) \subseteq \acute{E}$$

(ii) If $\acute{A}_ị,\acute{E}_ķ$ *are bounded* ,

$$\lim_{ị\to\infty} \inf_{\acute{A}_ị,\acute{E}_ķ\in\{\mathcal{F}_ị\}_{ị=1}^{\infty}} \hat{S}_\tau(\acute{A}_ị,\acute{E}_ķ) = \lim_{ị\to\infty} \inf_{\acute{A}_ị,\acute{E}_ķ\in\{\mathcal{F}_ị\}_{ị=1}^{\infty}} \inf_{n_ị\in\acute{A}_ị}\inf_{m_ị\in\acute{E}_ķ}|n_ị-m_ị| = 0 \text{ and}$$

$$\lim_{ị\to\infty} \inf_{\epsilon\in\mathfrak{R}^+}\mathfrak{I}_\epsilon(\acute{A}_ị) = \lim_{ị\to\infty} \inf_{\epsilon\in\mathfrak{R}^+}\{\acute{E}_ķ \in \mathcal{F} : \hat{S}_\tau(\acute{A}_ị,\acute{E}_ķ) < \epsilon\}$$

$$= \lim_{ị\to\infty} \inf_{\epsilon\in\mathfrak{R}^+} \inf_{n\in\acute{A}_ị}\inf_{m\in\acute{E}_ķ}|n-m|$$

$|n_ị - m_ị| <\in$, for all $ị, ķ \in Z^+, \in > 0$ and $\mathfrak{I}_\epsilon(\acute{A}_ị) \subseteq \acute{E}_ķ$. For all $\acute{A}_ị \in \{\mathcal{F}_ị\}_{ị=1}^{\infty}$.

Then $\{\mathcal{F}_{i}\}_{i=1}^{\infty}$is Cauchy filter in($\mathfrak{R}$,đ),but $\mathfrak{R}$ is complete. Then $\{\mathcal{F}_{i}\}_{i=1}^{\infty}$is Di-convergent filter in $\mathfrak{R}$, for everyone É ∈ Ώ($\mathfrak{R}$) like that $|n_{i} - n| = 0$. Then $\inf_{n_{i} \in Á_{i}} \inf_{n \in É} |n_{i} - n| = 0$, $\hat{S}_{đ}(Á_{i}, É)$=0 and $\mathfrak{I}_{\epsilon}(Á_{i}) \subseteq É$.

Thus $\{\mathcal{F}_{i}\}_{i=1}^{\infty}$ is convergent on (Ӿ, $\hat{S}_{đ}, \mathfrak{I}_{\epsilon}, +, .$) then (Ӿ, $\hat{S}_{đ}, \mathfrak{I}_{\epsilon}, +, .$)is Di-complete proximit topological vector space.

4. Conclusions

In this paper , we proved every Di-convergence filter are Di-Cauchy filter of Di-proximit topological vector space. Moreover , we deduced that Di-complete of Di-proximit topological vector space and his relationship with (Ӿ, $\hat{S}_{đ}, \mathfrak{I}_{\epsilon}, +, .$) .

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