AN ADVANCED STUDY RELATED TO THE CONCEPT OF COMPACTNESS MODULO AN IDEAL

Authors Names

Hasanain Ali Sadeq[1]
Yieze Al-Talkany[2]
Publication data: 19 /7 /2026
Keywords: Ideal; I-compact space; compactness modulo an ideal; ideal topological space.

ABSTRACT

This paper includes a study related to the topic of compactness in ideal topological space, where the research includes providing further proofs for some theorems in addition to studying other cases in these spaces.

1. Introduction

Maurice Fréchet (1906) [10] was the first to define the notion of compactness as we refer to it now by sequential compactness, and he defined it as follows: "A set A is compact if for every sequence $X_n \subseteq A$, there exists a subsequence X_{nk} that is convergent in A". In 1914, Felix Hausdorff [17] generalized Fréchet's concept of compactness by stating that "A set A is compact if there exists a limit point in A for every infinite subset of A". Alexandroff & Urysohn (1923) [4] introduced the modern definition of compactness: "X is compact if for any collection of sets in the topology τ covering X there is a finite sub-collection that covers X", which was known as bi-compact at the time. From the 1940s to the present day, the concept of topological compactness has undergone a profound structural evolution, which can be summarized briefly through the following stages: In the 1940s, Jean Dieudonné [8], in his 1944 paper, laid the foundation for paracompact spaces, while Edwin Hewitt [19], in his 1948 study, shifted the focus towards the algebra of continuous real functions by formulating realcompact spaces. In the 1950s, John Kelly [26], in his 1950 paper, established a logical and historical connection by proving that the Tychonoff Product Theorem is equivalent to the axiom of choice, while Sibe Mardeši´c & Pavle Papi´c [33], in 1955, formulated the first methodological seeds of what is known as feebly compact spaces. In the 1960s, the study of quasi-closed spaces emerged. The QHC, which focused on open-H-closures by Jürgen Porter in 1967 [37], was followed by the fundamental shift led by Robert Newcomb [35] in his 1967 dissertation, formally establishing the concept of compactness (I-compactness). In the 1970s, Dr.Ranˇcin [38], in his 1972 paper, deepened the structural properties of ideal compactness, while S. Maheshwari and O. Tapi [31], in 1978, introduced the concept of "feebly open sets" as a new operational tool. In the 1980s, S. Maheshwari and O. Tapi employed these tools in their paper to define and study "feebly compact spaces" [32] built upon doubly open covers (or α-open sets). In the 1990s, Dragan Jankovi´c and T.R. Hamlett [16], in their seminal work, revolutionized idealist topology by introducing the "local operator" A^* to construct new derived topologies ($\tau *$) closely related to ideal compactness. At the turn of the millennium (2000), Eli Glasner & Michael Megrelishvili [11] focused on algebraic structures by studying compact extensions of topological quasi-groups and operator spaces. Up to the current decade (2020–2025), research has focused on refining these concepts, where S. Roy and colleagues [39] presented a study on delta-compact spaces, while F. Issaka and M. Özkoç [22] developed mechanisms for deriving topologies via idealisms, in conjunction with the recent migration of these weak and idealist concepts to fuzzy structures.

[a]*Department of Mathematics, faculty of education for women, University of Kufa, 540011 Najaf, Iraq., E-Mail: iqbals.alattabi@student.uokufa.edu.iq*
[b]Department of Mathematics, faculty of education for women, University of Kufa, 540011 Najaf, Iraq., E-Mail: asmhanf.alzuhairy@uokufa.edu.iq

2. Related works

In (1933), Kuratowski [27] brought the notion of the ideal. He defined it as a non-void family of sets that verifies the hereditary and finitely additive properties. An extended study was conducted by Vaidyanathaswamy (1946) [41, 42]. in addition to the contributions of Hamlett and Jankovic [15, 16, 23, 24] and Hayashi in (1964) [18] and others. A new type of local function was defined by Al-fahham, Afeefa, and Altakany Yiezi [1]. Some types of continuous functions are defined by Noor, Entessar, and Altakany, Y. K., in ideal topological spaces [36], in addition to L.A.A Al Swidi's papers that study several topological concepts related to the ideal topological space [3,5–7].
The concept of I−compactness, or what is known topologically as (Compactness modulo an ideal), has evolved through successive historical stages that have changed the rigidity of classical open covers. The compactness modulo an ideal, or Ideal compactness (I−compact), was introduced as a new concept of compactness by Newcomb in (1967) [35], introducing the idea of excluding "small" or neglected sets that belong to the ideal from the finite subcover. More properties were obtained by the investigations of other mathematicians.
In his independent study of 1972, Dr. Ranˇcin [38] deepened the structural characteristics of these spaces, demonstrating their convergent behavior under the influence of open covers modulo an ideal.
In their seminal paper of 1990, Dragan Jankovi´c and T.R. Hamlett [24] made a qualitative leap by linking I−compactness to the properties of quasi-closed spaces (QHC) via the "Local Function" to construct a supertopology τ *. In their joint research in 2006, M. K. Gupta and T. Noiri [13] expanded the mechanisms of topological classification by incorporating the concepts of C-type compactness modulo an ideal. Contemporary research has focused on deciphering derived spaces, as F. Issaka and M. Özkoç [22], in their study in 2024, presented sophisticated mechanisms for deriving new topologies from old topologies through ideals that incubate ideal compactness.

3. Preliminaries

Fundamental concepts will be given in this section. That is a crucial cornerstone in this paper work.

Definition 3.1. [9,20] The function f: (M, τ) → (N, σ) is called a continuous at $p \in M$ if and only if $\forall f(p) \in V \subseteq \sigma, \exists p \in G \subseteq \tau \; where \; f(G) \subseteq V.$

Theorem 3.1. [21, 40] If $f^{-1}(Y) \in \tau, \forall Y \in \sigma$, where f: (M, τ) → (N, σ), then f is called a continuous function.

Remark 3.1. [21] A continuous function f: (M, τ) → (N, σ) is called mapping.

Definition 3.2. [2, 29] A collection $\{Gi \in \tau : i \in \Delta\} \subseteq X$ of open sets, that covers X such that $X = \cup \{Gi : i \in \Delta\}$, is called an open cover of X.

Definition 3.3. [2,29] A finite collection $\{Gi \in \tau: i \in \Delta_0\} \subseteq$X of open sets that covers X is a finite sub-cover of X if and only if $X = \cup \{Gi : i \in \Delta_0\}$.
Definition 3.4. [2,29,44] If for any cover $\{G_\lambda : \lambda \in \Delta\} \subseteq \tau$ such that $X \subseteq \cup \{G_\lambda : \lambda \in \Delta\}$, there exist finite sub-cover $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Delta$ implies that $X = \cup_{\lambda=1}^{n} G_\lambda$, then (X, τ) is a compact space.
Proposition 3.1. [9,43] Any finite space is compact.
Definition 3.5. [27–29] A family of non-void sets $I \subseteq P(M)$ is called ideal on M if these properties hold:

1. Hereditary property: $D \in I \,\&\, C \subseteq D, then\ C \in I$.
2. Finite additivity: $D \in I \,\&\, C \in I, then\ D \cup C \in I$.

An ideal I called proper ideal if $M \notin$, and (M, τ, I) called an ideal space.

Example 3.1. These collections are examples of an ideal in (X, τ):

1. $I = \{\phi\}, I = P(X)(power\ set\ of\ X)$.
2. I_{fin} (collection of finite sets in X).
3. $I_C(collection\ of\ countable\ sets\ in\ X)$.
4. $I_n(collection\ of\ nowhere\ dense\ sets\ in\ X)$.

Definition 3.6. [25,29,41] A local function (M^* (τ, I)) of the set M w.r.t τ & I in (N, τ, I) is an operator" $(.)^*$: P(N) → P(N)" that:

$$M^*\ (\tau, I) = \{\, p \in X : M \cap G \notin I \mid \forall\ G \in N_\tau(p)\}$$

and $N_\tau(p)$ is the open neighborhood system of p.

Remark 3.2. [24] 1. $\tau^*\ (I) = \{G \subseteq X : Cl^*\ (X \backslash G) = X \backslash G\}, \tau^*(I)$ or in short $\tau^*$is the topological space generated by Cl^* which is greater than τ (the original topological on X), i.e., $\{\tau \subseteq \tau^*\}$.

2. $\forall\ M \subseteq X$, define $Cl^*(M) = M \cup M^*$ is closure operator of Kuratowski of the topology τ^*.

Definition 3.7. [24] A family $\beta(I, \tau) = \{G \setminus \mu \mid G \in \tau, \mu \in I\}$ called a basis of τ^*, and the union of subsets of the basis generate τ^* i. e. $\tau^* = \{\cup \{Bi : i \in \Lambda\} : Bi \subseteq \beta(I, \tau)\}$.
Definition 3.8. Let (M, τ, I) be an ideal topological space. Then for any non-void set $Y \subseteq M$, the triple (Y, τ_Y, I_Y) is called an ideal subspace such that:

1. The topology: $\tau_Y = \{G \cap Y : G \in \tau\}$ [17,40].

2. The Ideal: $I_Y = \{I \cap Y : I \in I\}$ [24].

Remark 3.3. Let (M, τ, I) be an ideal topology, and $N \subseteq M$ is a subspace of M, then $I_N \subseteq I$ is an ideal on N, so since for any $N_0 \in I_N$, then by definition 3.8 we have:

$$N_0 = \{N \cap J : J \in I\}$$

And since $N \cap J \subseteq J \in I$.

Proposition 3.2. [24] Let (M, τ, I) be an ideal topological space, and (N, τ_N, I_N) be an ideal subspace. For any $B \subseteq N$, the following identities hold:

1. Local function: the local function of B subset of N is: $B^*(N, \tau_N, I_N) = B^*(M, \tau, I) \cap N$.
2. Kuratowski closure operator: the $Cl_Y^*(A)$ in a subspace Y is: $Cl_N^*(B) = Cl^*(B) \cap N$.
3. Subspace *-topology: the topology generated in the subspace Y: $(\tau^*)_N = (\tau_N)^*$.

Corollary 3.1. [30] Let $B \subseteq M$ in the ideal topological space (M, τ, I) the following holds:

1. If $I = \{\phi\}$, then $B^* = Cl^*(B) = Cl(B)$ and $\tau = \tau^*(I)$.
2. If $I = P(M)$, then $B^* = \{\phi\}, \forall B \subseteq M$, and $\tau^* = P(M)$.

Theorem 3.2. [24] For any C & $D \subseteq M$ of the ideal space (M, τ, I) then:

1. $D^* = Cl^*(D) \subseteq Cl(D)$ in general (D^* is a closed set of $Cl(D)$).
2. $(C \cup D)^* = C^* \cup D^*$.
3. $(C \cap D)^* \subseteq C^* \cap D^*$.

Remark 3.4. For any set $D \subseteq M$ in the space (M, τ, I), then:
1. A set D called "τ^*-dense in itself", if $D \subseteq D^*$[18].
2. A set D called "τ^*-closed set", if $D^* \subseteq D$ [24].
3. A set D called "τ^*-perfect set", if $D = D^*$ [18].

Definition 3.9. [35] The space (M, τ, I) called ideal-compact(I-compact) if and only if any collection $\{G_\lambda : \lambda \in \Delta\}$ of open sets covers M, there exists ($\Delta_0 \subseteq \Delta$) which is finite $\{G_\lambda : \lambda \in \Delta_0\}$ satisfies the following:

$$M \setminus \bigcup \{G_\lambda : \lambda \in \Delta_0\} \in I$$

Remark 3.5. [35] "(M, τ) is compact if and only if (M, τ, ϕ) is ϕ-compact."

Example 3.2. The space (R, U) is not compact. but if $I = P(R)$, then for any $G_0 \subseteq G$ we obtain: $R \setminus \cup G_0 \subseteq R \in P(R) = I$, thus R is P(R)-compact.

Example 3.3. The topological space (R, τ_I) where τ_I is the Indiscrete topology defined on R, is an I-compact space, since for any cover $H = \{R : R \in \tau_I\}$ there is finite sub-collection $H_0 = \{R \subseteq R\}$, then we get: $R = \cup H_0$, then by definition (3.9) $R \setminus \cup H0 = \phi \in I$ for any ideal on R, then R is I-compact space.

Remark 3.6. Any ideal topological space (M, τ, I) is a P(X)-compact.

The following theory has been presented in many papers where the researchers based the theory of Newcomb [35] and Hamlet & Jankovic [14] and is referred to as follows:

Lemma 3.1. [14] Let $\Phi : (M, \tau, I) \to (N, \sigma)$ be a function. Then $\Phi(I) = \{\Phi(E): E \in I\}$ is an ideal on N.
Lemma 3.2. [14] Let $\Psi : (M, \tau) \to (N, \sigma, J)$ be a function. Then $\Psi^{-1}(J) = \{\Psi^{-1}(A) : A \in J\}$.

However, in source [45], this theory was presented as follows:
Lemma 3.3. [45] The following properties hold:
1. f(S) is an ideal on N, if $f : (M, \tau, S) \to (N, \sigma)$ surjective map, where $f(S) = \{f(A) | A \in S\}$
2. $f^{-1}(S) = \{f^{-1}(B) : B \in S\}$ is an ideal on M, if f: $(M, \tau) \to (N, \sigma, S)$ were injective map.

Remark 3.7. A map $f : (M, \tau, \S) \longrightarrow (N, \sigma)$, where § is any proper ideal on M, then $f(\S)$ is not necessarily is a proper ideal on N as shown in the example.

Example 3.4. Let $f : (M, \tau, \S) \longrightarrow (N, \sigma)$, and $M = \{a, b, c, d, e\}$, and $\S = \{\phi, \{a\}, \{c\}, \{a, c\}\}$ be a proper ideal on M, and $N = \{1, 2\}$, where $f(a) = f(b) = 1, f(c) = f(d) = f(e) = 2\; f(\phi) = \phi$, then $f(\S) = \{\phi, \{1\}, \{2\}, \{1, 2\}\} = P(N)$, since $N \in f(\S)$ hence f(§) is not a proper ideal on N.

Remark 3.8. Let $\Psi : (M, \tau) \to (N, \sigma, J)$ be a function where J is an ideal defined on N, then $\Psi^{-1}(J)$ is not necessarily an ideal on M, as in the following example.

Example 3.5. Let $\Psi : (M, \tau) \to (N, \sigma, J)$, where $M = \{a, b, c\}, N = \{1, 2, 3\}$, and $J = \{\phi, \{1\}\}$, and $\Psi(a) = \Psi(b) = 1, \Psi(c) = 2$, then $\Psi - 1(J) = \{\phi, \{a, b\}\}$ is not an ideal on M.

The following theory, which we have proven, shows that an ideal image is only ideal if the condition of injectivity is met, and all the details that illustrate the importance of this condition have been mentioned.

Theorem 3.3. Let Ψ: $(\mathrm{M}, \tau, \mathrm{I}) \to (\mathrm{N}, \Omega)$ be an injective map, then $\Psi(I)$ is an ideal on N.

• Let $\Psi(A) \in \Psi(I)$ and $\Psi(B) \subseteq \Psi(A)$, then $\Psi^{-1}\Psi(B) \subseteq \Psi^{-1}\Psi(A)$, given that Ψ is injective, then by [46] [theorem 8, p.27], we get: $\Psi^{-1}\Psi(B) = B$ and $\Psi^{-1}\Psi(A) = A$, then we obtain: $B \subseteq A$, given that I is an ideal on M, then $B \in I$, therefore, $\Psi(B) \in \Psi(I)$.

• Let $\Psi(A), \Psi(B) \in \Psi(I)$, given that Ψ is injective, then we get $A, B \in I$, since I is an ideal on M, then $A \cup B \in I$ which implies that: $\Psi(A \cup B) \in \Psi(I)$, then by [46] [theorem 8, p.27], we get:

$$\Psi(A \cup B) = \Psi(A) \cup \Psi(B) \in \Psi(I)$$

Therefore, $\Psi(I)$ is an ideal on N.

The properties presented in the source [12] were proven to confirm their validity with respect to the ideal subspaces.

Proposition 3.3. [12] The closed set C of an I-compact space (M, τ, I) is I_C-compact, where I_C is the induced sub ideal defined on C.

proof: If the collection $V = \{vi : i \in \Lambda\}$ of open sets cover of a closed subset C, then the open sets $V \cup (M \setminus C)$ cover of M. Given M is I-compact, then by definition (3.9) we get that:

$$M \setminus \left(\cup\{vi : i \in \Lambda_0\} \cup (M \setminus C) \right) \in I$$

But

$$M \setminus \left(\cup\{vi : i \in \Lambda_0\} \cup (M \setminus C) \right) = M \setminus \cup\{vi : i \in \Lambda_0\} \cap \left(M \setminus (M \setminus C)\right)$$
$$= (M \setminus \cup\{vi : i \in \Lambda_0\}) \cap C$$
$$= C \cap M \setminus \cup\{vi : i \in \Lambda_0\}$$
$$= C \setminus \cup\{vi : i \in \Lambda_0\} \in I$$

and by definition 3.8 we have:

$$C \setminus \cup\{vi : i \in \Lambda_0\} \cap C \in I_C$$

Hence, C is an I_C-compact set.

Proposition 3.4. [12] If $C \subseteq D$, where D is I-compact and C is open, then D \ C is $I_{D\setminus C}$-compact for any $D\&C \subseteq M$ in the space (M, τ, I).

proof: If the collection $U = \{ui : i \in \Delta\}$ of open sets is cover of (D \ C). Then the collection $U \cup \{C\}$ of open sets is cover of D, given that $C \subseteq D$ and D is I-compact by definition (3.9) implies that:

$$D \setminus (\cup\{ui : i \in \Delta_0\} \cup C) \in I$$

Now we obtain that:

$$D \setminus (\cup\{ui : i \in \Delta_0\} \cup C) = D \cap (M \setminus \cup \{ui : i \in \Delta_0\} \cup C)$$
$$= D \cap (M \setminus \cup\{ui : i \in \Delta_0\}) \cap (M \setminus C)$$
$$= D \cap (M \setminus C) \cap (M \setminus \cup\{ui : i \in \Delta 0\})$$
$$= (D \setminus C) \cap M \setminus \cup \{ui : i \in \Delta_0\} \in I$$

Since by definition (3.8)

$$(\mathrm{D} \setminus \mathrm{C}) \setminus \cup \{\mathrm{ui} : \mathrm{i} \in \Delta_0\} \cap (\mathrm{D} \setminus \mathrm{C}) \in \mathrm{I}_{\mathrm{D}\setminus\mathrm{C}}$$

Then, (D \ C) is $I_{D\setminus C}$-compact.

Proposition 3.5. *[12] For any $C \subseteq D \subseteq M$ in the space (M, τ, I), Then, if* D *open and* I-*compact set, then* C *is* I_C-*compact.*

proof: For any collection $\{Gi \in \tau : i \in \Lambda\}$ cover of C, *since* D *is open then the collection $\{Gi : i \in \Lambda\} \cup \{D\}$ is cover of D, given that* D *is* I-*compact, then by definition (3.9) we obtain:*

$$C \setminus \cup\{Gi : i \in \Lambda_0\} \in I$$

since $C \subseteq D$, then

$$C \setminus \cup\{Gi : i \in \Lambda_0\} \subseteq D \setminus \cup\{Gi : i \in \Lambda_0\} \in I$$

then

$$C \setminus \cup\{Gi : i \in \Lambda_0\} \in I$$

Then by definition 3.8 and remark 3.3 we get:

$$C \setminus \cup\{Gi : i \in \Lambda_0\} \cap C \in I_C \subseteq I$$

Then C *is* I_C-*compact.*

Proposition 3.6. *The intersection $M \cap N$ of any two sets is* $I_{M\cap N}$-*compact, if M or N is an* I-*compact & open in* (X, τ, I).

proof: Let N be an I-*compact & open set and let $C = \{ci : i \in \Lambda\}$ cover of open set of $(M \cap N)$, then $C \cup N$ cover of N, then by definition (3.9) there exists $C0 = \{ci : i \in \Lambda_0\}$, where $(\Lambda 0 \subseteq \Lambda)$, that:*

$$N \setminus \cup\{ci : i \in \Lambda_0\} \in I$$

but $(M \cap N) \subseteq N$, then

$$(M \cap N) \setminus \cup\{ci : i \in \Lambda_0\} \subseteq N \setminus \cup\{ci : i \in \Lambda_0\} \in I$$

and then we get

$$(M \cap N) \setminus \cup\{ci : i \in \Lambda 0\} \in I$$

Then by definition 3.8 and remark 3.3 we get:

$$(M \cap N) \setminus \cup\{ci : i \in \Lambda_0\} \cap N \in I_{\{M\cap N\}} \subseteq I$$

thus $(M \cap N)$ *is* $I_{M\cap N}$-*compact.*

Remark 3.9. The opposite of proposition (3.6) is not necessarily true in general, as shown in the example.

Example 3.6. The topology (R, U), with an ideal I = {ϕ} defined on R, let $M = P \cap Q = [1,2]$, where $P = (0,3), Q = [1,2]$, is compact and I-compact, since M is closed and bounded set, then by Heine-Borel Theorem [34] M is a compact set, then by remark 3.5 M is I-compact. But the set P is neither compact nor I-compact.

Proposition 3.7. The intersection (C ∩ J) of a closed set C and I-compact set J is $I_{C\cap J}$-compact for any $C \& J \subseteq M$ in the space (M, τ, I).

proof: If the collection $U = \{ui : i \in \Lambda\}$ of open sets is cover of $(C \cap J)$, then the collection $U \cup (M \setminus C)$ of open sets is cover of J Since {for each $x \in J : if\ x \notin U\ then\ x \notin (C \cap J), thus\ x \in M \setminus (C \cap J), now\ x \in (M \setminus C) \cup (M \setminus J)\ hence\ x \in (M \setminus C)$}, as given, J is an I-compact set and by definition (3.9) then we have:

$$J \setminus (\cup\{ui : i \in \Lambda_0\} \cup (M \setminus C)) \in I$$

But

$$J \setminus (\cup\{ui : i \in \Lambda_0\} \cup (M \setminus C)) = J \cap (M \setminus (\cup\{ui : i \in \Lambda_0\} \cup (M \setminus C))$$
$$= J \cap (M \setminus \cup\{ui : i \in \Lambda_0\} \cap (M \setminus (M \setminus C))$$
$$= J \cap (M \setminus \cup\{ui : i \in \Lambda_0\} \cap C$$
$$= J \cap C \cap (M \setminus \cup\{ui : i \in \Lambda_0\}$$
$$= (J \cap C) \setminus \cup\{ui : i \in \Lambda_0\}$$

Since by definition 3.8

$$= (J \cap C) \setminus \cup\{ui : i \in \Lambda_0\} \cap (J \cap C) \in I_{\{C\cap J\}}$$

Then by remark 3.3 (J ∩ C) is $I_{C\cap J}$-compact.

4. Results

Proposition 4.1. If (M ∪ N) is an I-compact set and N is open in (X, τ, I) then M & N are I_M & I_N-compact.

proof: If $\{Ui : i \in \Delta\}$ cover of open sets of the set M, then $\{Ui : i \in \Delta\} \cup \{N\}$ cover of $(M \cup N)$. But (M∪N) is I-compact as given and by definition (3.9) we get $\{Ui : i \in \Delta_0\} \subseteq \{Ui : i \in \Delta\}$ that:

$$(M \cup N) \setminus \cup\{Ui : i \in \Delta_0\} \in I$$

Since $M \subseteq M \cup N$, then

$$M \setminus \cup \{Ui : i \in \Delta_0\} \subseteq (M \cup N) \setminus \cup \{Ui : i \in \Delta_0\} \in I$$

and $N \subseteq (M \cup N)$ then

$$N \setminus \cup \{Ui : i \in \Delta_0\} \subseteq (M \cup N) \setminus \cup \{Ui : i \in \Delta_0\} \in I$$

so, by definition (3.9)

$$M \setminus \cup \{Ui : i \in \Delta_0\} \in I, \qquad N \setminus \cup \{Ui : i \in \Delta_0\} \in I$$

by definition 3.8 and remark 3.3 we get:

$$M \setminus \cup \{Ui : i \in \Delta_0\} \cap M \in I_M \subseteq I$$
$$N \setminus \cup \{Ui : i \in \Delta_0\} \cap N \in I_N \subseteq I$$

Thus, M is I_M-compact and N is I_N-compact.

Theorem 4.1. If $f : (M, \tau, I) \to (N, \sigma)$ bijective map of an I-compact space (M, τ, I), then N is f(I)-compact.

proof: If the collection $V = \{vi : i \in \Delta\}$ of open sets is cover of N, given that f is a map, then the collection f^{-1}(V) of open sets is cover of M, since M is I-compact so by definition (3.9) we have $(\Delta 0 \subseteq \Delta)$ applies that:

$$M \setminus \cup f^{-1}(\{vi : i \in \Delta_0\}) \in I$$

now:

$$f(M \setminus \cup f^{-1}(\{vi : i \in \Delta_0\}) \in f(I)$$

then

$$f(M) \setminus f(\cup f^{-1}(\{vi : i \in \Delta_0\}) \in f(I)$$

and then

$$N \setminus \cup f f^{-1}(\{vi : i \in \Delta_0\}) \in f(I)$$

since f is bijective, then by theorem 8 in [40] now we get

$$N \setminus \cup (\{vi : i \in \Delta_0\}) \in f(I)$$

Thus, N is f(I)-compact space.

Theorem 4.2. Any compact space (M, τ, I) is I-compact.

proof: Given that M is compact, then by definition (3.4) we obtain: $M = \cup \{Gi : i \in \Delta_0\}$ where the finite collection of open sets $\{Gi : i \in \Delta_0\}$ is finite covers of M, then $M \setminus \cup \{Gi : i \in \Delta_0\} = \phi \in I$, hence $M \cup \{Gi : i \in \Delta_0\} \in I$, then by definition (3.9) M is an I-compact space.

Remark 4.1. The opposite of Theorem 4.2 is not necessarily true, as shown in the example.

Example 4.1. (R, U) is not compact, where U is the standard topology, but (R, U, I), where I = P(R) defined as an ideal on R, is I-compact. Since for any open cover C = {(−n, n): n ∈ N} take any finite sub-collection $C0 = \{(-n,n)_i : i \in \Delta\}$, then $\cup \{(-n,n)i : i \in \Delta\} = (-N,N)$, then we get:

$$R\setminus (-N,N) \in P(R) = I$$

Theorem 4.3. (M, τ^*) is {ϕ}-compact if and only if (M, τ) is {ϕ}-compact.

proof: If the collection $G = \{G_j \in \tau : j \in \Lambda\}$ is cover of M, since $I = \{\phi\}$, then by corollary (3.1) $\tau^* = \tau$, hence $G = \{G_j \in \tau^* : j \in \Lambda\}$, As given (M, τ) is {ϕ}-compact then by definition (3.9), it implies that for any open cover we get: $\{G_j : j \in \Lambda_0\} \subseteq G$ where ($\Lambda_0 \subseteq \Lambda$) that:

$$M \setminus \cup \{Gj : j \in \Lambda_0\} \in I$$

Thus (M, τ^*) is {ϕ}-compact.

Theorem 4.4. Let $I_1 \subseteq I_2$ are two ideals define on (M, τ), then if (M, τ, I_1) is I_1-compact space, then (M, τ, I_2) is I_2-compact.

proof: If $H = \{hi : i \in \Delta\}$ cover of open sets for M, given that M is I_1-compact, then by definition (3.9) we obtain: ($\Delta_0 \subseteq \Delta$) that:

$$M \setminus \cup \{hi : i \in \Delta_0\} \in I_1 \subseteq I_2$$

hence

$$M \setminus \cup \{hi : i \in \Delta_0\} \in I_2$$

Thus (M, τ, I_2) is I_2-compact.

Proposition 4.2. If $M \in I$, for any $M \subseteq X$, then M is an I-compact set.

proof: If $G = \{Gi : i \in \Delta\}$ cover of open sets for M, then for any finite collection $\{Gi : i \in \Delta_0\}$ where ($\Delta_0 \subseteq \Delta$) we have: $M \setminus \cup \{Gi : i \in \Delta_0\} \subseteq M \in I$, and then by definition (3.9) $M \setminus \cup \{Gi : i \in \Delta_0\} \in I$ thus M is I-compact.

Proposition 4.3. If M is I-compact and τ^*-closed set in (X, τ, I), then M^* is I-compact.

proof: If $G = \{Gi : i \in \Lambda\}$ cover of open sets of M^*, since in theorem (3.2) M^* is closed, then $G \cup (X \setminus M^*)$ cover of M, and as given, M is I-compact, then by definition (3.9):

$$M \setminus \cup \{Gi : i \in \Lambda_0\} \in I$$

then by remark (3.4) $M^* \subseteq M$ and definition (3.9):

$$M^* \setminus \cup \{Gi : i \in \Lambda_0\} \subseteq M \setminus \cup \{Gi : i \in \Lambda_0\} \in I$$

now we get

$$M^* \setminus \cup \{Gi : i \in \Lambda_0\} \in I$$

hence M^* is I-compact.

Proposition 4.4. If M^* is I-compact and M is τ^*-dense in itself in (X, τ, I), then M is I-compact.

proof: If $C = \{Ci : i \in \Lambda\}$ cover of open sets of M, and by theorem (3.2) M^* is closed, then $C \cup (X \setminus M^*)$ is cover of M^*, given that M^* is I-compact so by definition (3.9) we get that:

$$M^* \setminus \cup \{Ci : i \in \Lambda_0\} \in I$$

where ($\Lambda_0 \subseteq \Lambda$), since M is τ^*-dense in itself by remark (3.4) and definition (3.9) we get:

$$M \setminus \cup \{Ci : i \in \Lambda_0\} \subseteq M^* \setminus \cup \{Ci : i \in \Lambda_0\} \in I$$

hence

$$M \setminus \cup \{Ci : i \in \Lambda_0\} \in I$$

and M is I-compact.

Proposition 4.5. If M is "τ^*-perfect". then M is I-compact if and only if M^* is I-compact.

proof: The statement holds by propositions (4.3) and (4.4).

Proposition 4.6. if (X, τ, I) is compact space, then for each M subset of X, M^* is compact.

proof: If the collection $C = \{ci : i \in \Lambda\}$ is cover of the set M^*. Since M^* is closed by the theorem (3.2), then $C \cup (X \setminus M^*)$ cover of X, and as given X is a compact space, then by the definition (3.4): $X = \cup \{ci : i \in \Lambda_0\}$, since $M^* \subseteq X = \cup \{ci : i \in \Lambda_0\}$, thus $M^* = \cup \{ci : i \in \Lambda_0\}$, then M^* is compact.

Proposition 4.7. The set A^* is an I-compact set for each subset A of a compact space (X, τ, I).

proof: If the collection $C = \{Ci \in \tau : i \in \Lambda\}$ is cover of A^* & Given that A^* is closed by Theorem (3.2), then the collection $C \cup (X \setminus A^*)$ is cover of X, given that X is compact then by theorem (4.2) we get that X is I-compact, i.e.,

$$X \setminus \cup \{Ci : i \in \Lambda_0\} \in I$$

since $A^* \subseteq X$ then

$$A^* \setminus \cup \{Ci : i \in \Lambda_0\} \subseteq X \setminus \cup \{Ci : i \in \Lambda_0\} \in I$$

hence

$$A^* \setminus \cup \{Ci : i \in \Lambda_0\} \in I$$

hence, A^* is I-compact.

Proposition 4.8. If (X, τ) is a compact space, then (X, τ^*) is a compact space.

proof: If the collection $C = \{Ci \in \tau^* : i \in \Delta\}$ is cover of X, hence by the definition (3.7) we get that:

$$Ci = \{G_i \setminus j_i | G_i \in \tau, j_i \in I\}$$

hence

$$Ci \subseteq Gi$$

and the collection $\{Gi \in \tau : i \in \Lambda\}$ is cover of X, given that X is compact, hence we obtain:

$$X = \cup \{Gi \in \tau : i \in \Lambda 0\}$$

where ($\Lambda_0 \subseteq \Lambda$), by remark (3.2) we have: ($\tau \subseteq \tau^*$),i.e$\cup \{Gi : i \in \Lambda_0\} \in \tau \subseteq \tau^*$, hence $\cup \{Gi : i \in \Lambda_0\} \in \tau^*$, hence

$$X = \cup \{Gi \in \tau^* : i \in \Lambda_0\}$$

thus (X, τ^*) is compact.

Proposition 4.9. If (X, τ) is a compact space, then (X, τ^*) is an I-compact space.

proof: If the collection $C = \{Ci \in \tau^* : i \in \Lambda\}$ cover of X, hence by definition (3.7):

$$Ci = \{G_i \setminus j_i : G_i \in \tau, j_i \in I\}$$

hence

$$Ci \subseteq Gi$$

then the collection $\{Gi \in \tau : i \in \Delta\}$ is cover of X, given that X is compact then (X, τ) is I-compact by theorem (4.2), then:

$$X \setminus \cup \{Gi \in \tau : i \in \Delta_0\} \in I$$

by remark (3.2), we get that: $(\tau \subseteq \tau^*)$, i. e., $\{Gi : i \in \Delta_0\} \in \tau \subseteq \tau^*$, hence $\{Gi : i \in \Delta_0\} \in \tau^*$, thus we get:

$$X \setminus \cup \{Gi \in \tau^* : i \in \Delta_0\} \in I$$

then (X, τ^*) is I-compact.

Remark 4.2. The opposite of Proposition (4.9) need not be true, as shown in the example.
Example 4.2. The space (R,U) is not compact, but the triple(R, τ^*, I), where τ^*-topology on R such that by remark (3.2): $\tau^* = \{C \subseteq R : Cl^*(R \setminus C) = (R \setminus C)\}$ with an ideal $I = P(R)$ defined on R, is I-compact, since for any collection $\{Gi \in \tau^* : i \in \Lambda\}$ cover of R, there exists finite sub-collection $\{Gi \in \tau^* : i \in \Lambda_0\}$, where $\Lambda_0 \subseteq \Lambda$ we have:

$$R \setminus \cup \{Gi \in \tau^* : i \in \Lambda_0\} \in P(R)$$

Hence

$$R \setminus \cup \{Gi \in \tau^* : i \in \Lambda_0\} \in I$$

Then (R, τ^*, I) is I-compact but not compact.
Proposition 4.10. Any finite space (M, τ, I) is I-compact.
proof: The statement holds by proposition (3.1) and theorem (4.2)

5. Conclusions:

This research included a study of compact spaces in ideal topological spaces. Through our examination of this topic, we concluded that most theorems that hold in topological spaces do not generally hold in ideal topological spaces unless a set of additional conditions is satisfied, as stated in Theorem (4.1). We also observed that some topological spaces, such as (R, U), are not compact, but they become compact in ideal topological spaces with respect to I = P(X), as explained in example (3.2).

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