A New Iterative Method for Nonexpansive Mappings in Banach Space

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ABSTRACT

This paper proposes a new iterative method within a Banach space framework for approximating fixed points (FPs) of nonexpansive mappings (NE-maps) and mappings that satisfy the E-condition (EC-maps). We get some convergence findings with appropriate assumptions. Furthermore, the proposed iterative method offers a faster convergence rate than a variety of current methods, as evidenced by numerical calculations and graphical representations..

1. Introduction

In mathematics, fixed point (FP) theory is crucial and has a big impact on both the theoretical and applied fields. Since FP theory is a powerful area of modern mathematics with several applications, many authors have explored it on a variety of topics [1-6].

Nonlinear functional analysis relies heavily on FP theory. Numerous nonlinear issues are challenging to resolve analytically. FP approximation techniques are employed to get around these issues. The FP strategy may be used to solve a large number of nonlinear problems[7-10]. In many applied disciplines, FP approximation is essential to obtaining the desired solution when an analytical solution is not feasible. Such issues are solved by transforming the desired problem into the FP problem of a certain operator, whose solution is identical to that of a particular nonlinear problem. The convergence of the most basic iterative procedure serves as the foundation for the demonstration of the Banach contraction principle (BCP) [11]. It deals with an FP problem for a contraction mapping defined on a complete metric space. In some situations, it is apparent that a solution to the FP issue exists, but it is impossible to pinpoint the precise answer. A solution approximation is highly needed in these circumstances, leading to the creation of several iterative techniques. Picard [12] offered one of the first iterative approaches for this purpose, with an emphasis on identifying FPs in specific nonlinear mappings. Banach [13] showed that Picard's technique works well for contraction maps. Other academics expanded the applicability of Picard's technique to various types of mappings. For example, see[14-16]. In [17], Mann introduces the following iterative method.

$$\breve{g}_{n+1} = (1-\omega_n)\breve{g}_n + \omega_n \mathbb{\Gamma}\, \breve{g}_n \qquad \text{for all } n \geq 0 \qquad (1)$$

where $\omega_n \in (0,1)$ and $\breve{g}_\circ \in G$ ($G \subseteq \mathrm{H}$ where H is Banach space).

Ishikawa [18] devised the iterative approach that follows:

$$\hbar_n = (1-\gamma_n)\breve{g}_n + \gamma_n \mathbb{\Gamma}\, \breve{g}_n$$

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$$\breve{g}_{n+1} = (1-\omega_n)\breve{g}_n + \omega_n \mathbb{\Gamma}\, \hbar_n \quad (2)$$

In 2000, Noor [19] proposed a new iterative method.

$$\nu_n = (1-\mu_n)\breve{g}_n + \mu_n \mathbb{\Gamma}\, \breve{g}_n$$

$$\hbar_n = (1-\gamma_n)\breve{g}_n + \gamma_n \mathbb{\Gamma}\, \nu_n$$

$$\breve{g}_{n+1} = (1-\omega_n)\breve{g}_n + \omega_n \mathbb{\Gamma}\, \hbar_n \quad (3)$$

for all $n \geq 0$ where $\omega_n, \gamma_n, \mu_n \in (0,1)$.

Agarwal et al. [20] presented an iterative approach as follows.

$$\hbar_n = (1-\gamma_n)\breve{g}_n + \gamma_n \mathbb{\Gamma}\, \breve{g}_n$$

$$\breve{g}_{n+1} = (1-\omega_n)\mathbb{\Gamma}\breve{g}_n + \omega_n \mathbb{\Gamma}\, \hbar_n \quad (4)$$

Abbas and Nazir [21] offered the following type of iterative method:

$$\nu_n = (1-\mu_n)\breve{g}_n + \mu_n \mathbb{\Gamma}\, \breve{g}_n$$

$$\hbar_n = (1-\gamma_n)\mathbb{\Gamma}\breve{g}_n + \gamma_n \mathbb{\Gamma}\, \nu_n$$

$$\breve{g}_{n+1} = (1-\omega_n)\mathbb{\Gamma}\, \hbar_n + \omega_n \mathbb{\Gamma}\nu_n \quad (5)$$

However, Thakur et al. [22] developed the following iterative method.

$$\nu_n = (1-\mu_n)\breve{g}_n + \mu_n \mathbb{\Gamma}\, \breve{g}_n$$

$$\hbar_n = (1-\gamma_n)\nu_n + \gamma_n \mathbb{\Gamma}\nu_n$$

$$\breve{g}_{n+1} = (1-\omega_n)\mathbb{\Gamma}\, \nu_n + \omega_n \mathbb{\Gamma}\hbar_n \quad (6)$$

Several more iterative strategies may be observed in[23–27]. New iterative techniques for estimating the FPs of several different mappings types have been offered by researchers; for instance in research publications[28-31] , FP of Suzuki mappings has been estimated, while the approximation of FPs for nonexpansive mappings was investigated in [32-35]. The goal of this study is to propose a new iterative method, known as the AR* iteration technique. We compare our results with those of other iterative approaches and demonstrate that ours have faster iterations, greater convergence efficiency, and higher convergence rates. We provide examples of the iterative method's convergence findings for the NE-map, which may help demonstrate the superiority of our results over the earlier techniques. In order to compare the results, we also provide some examples.

2. Preliminaries

Definition 2.1 ([33]). Let $G \subseteq H$. A mapping $\mathbb{\Gamma}: G \to G$. is termed as nonexpansive (NE-map) if for any $\nu, \hbar \in G$,

$$\| \mathbb{\Gamma}(\nu) - \mathbb{\Gamma}(\hbar) \| \leq \| \nu - \hbar \| \tag{7}$$

Definition 2.2.([34]) A map $\mathbb{\Gamma}: G \to G$ where $G \subseteq H$ is said to be satisfy the E-condition (EC-map) if for each pair of elements $\nu, \hbar \in G$, there is some real constant $\varphi \geq 1$, such that

$$||\nu - \mathbb{\Gamma}\hbar|| \leq \varphi||\nu - \mathbb{\Gamma}\nu|| + ||\nu - \hbar|| \tag{8}$$

Lemma 2.3 ([34]) Let G be a nonempty subset of a Banach space H and let $\mathbb{\Gamma}: G \to G$ satisfy (E)-condition. Subsequently, for all $\nu \in \Im(\mathbb{\Gamma})$ and $\hbar \in G$, we have $||\mathbb{\Gamma}\nu - \mathbb{\Gamma}\hbar|| \leq ||\nu - \hbar||$.

3. Main Results

A new iteration technique called the AR* iteration approach is presented, and some convergence results are found. Following describes the way the new iteration technique is defined:

$$\boldsymbol{\nu_n} = \frac{\boldsymbol{\eta}\breve{g}_n + \mathbb{\Gamma}\breve{g}_n}{\boldsymbol{\eta}+1}$$

$$\boldsymbol{\hbar_n} = \mathbb{\Gamma}\left(\frac{\boldsymbol{\eta\nu_n} + \mathbb{\Gamma}\boldsymbol{\nu_n}}{\boldsymbol{\eta}+1}\right)$$

$$\breve{g}_{\boldsymbol{n+1}} = \mathbb{\Gamma}(\mathbb{\Gamma}\boldsymbol{\hbar_n}) \tag{9}$$

where $\boldsymbol{\eta} \geq \boldsymbol{0}$.

In this section, we approximated the FP for NE-map and EC-map using the iterative technique (9). We prove a few convergence results. The speed of the suggested method is compared with some popular iterative methods using a numerical example.

Theorem 3.1: Let $G \subseteq H$ (G non-empty closed convex) and $\mathbb{\Gamma}: G \to G$ is NE-map with $\Im(\mathbb{\Gamma}) \neq \emptyset$. For $\breve{g}_0 \in G$, the sequence $\{\breve{g}_0\}$ generated by eq. (9) . Then $\lim_{n\to\infty} \| \breve{g}_n - \vartheta \|$ exists for all $\vartheta \in \Im(\mathbb{\Gamma})$.
Proof: Consider $\vartheta \in \Im(\mathbb{\Gamma})$. From definition 2.1, we have

$$\begin{aligned}
\| \nu_n - \vartheta \| &= \left\| \frac{\eta\breve{g}_n + \mathbb{\Gamma}\breve{g}_n}{\eta+1} - \vartheta \right\| \\
&= \left\| \frac{\eta}{\eta+1}(\breve{g}_n - \vartheta) + \frac{1}{\eta+1}(\mathbb{\Gamma}\breve{g}_n - \vartheta) \right\| \\
&\leq \frac{\eta}{\eta+1}\|\breve{g}_n - \vartheta\| + \frac{1}{\eta+1}\|\mathbb{\Gamma}\breve{g}_n - \vartheta\| \\
&\leq \frac{\eta}{\eta+1}\|\breve{g}_n - \vartheta\| + \frac{1}{\eta+1}\|\breve{g}_n - \vartheta\| \\
&= \|\breve{g}_n - \vartheta\|
\end{aligned} \tag{10}$$

We acquire something comparable

$$\| \hbar_n - \vartheta \| = \| \mathbb{\Gamma}(\frac{\eta \nu_n + \mathbb{\Gamma} \nu_n}{\eta+1}) - \vartheta \|$$

$$\leq \left\| \frac{\eta \nu_n + \mathbb{\Gamma} \nu_n}{\eta+1} - \vartheta \right\|$$
$$\leq \frac{\eta}{\eta+1} \|\nu_n - \vartheta\| + \frac{1}{\eta+1} \|\mathbb{\Gamma}\nu_n - \vartheta\|$$
$$\leq \frac{\eta}{\eta+1} \|\nu_n - \vartheta\| + \frac{1}{\eta+1} \|\nu_n - \vartheta\|$$
$$= \|\nu_n - \vartheta\| \tag{11}$$

Now, from (10) we obtain $\| \hbar_n - \vartheta \| \leq \|\breve{g}_n - \vartheta\|$.
Using (10) and (11), we obtain

$$\| \breve{g}_{n+1} - \vartheta \| = \| \mathbb{\Gamma}(\mathbb{\Gamma}\hbar_n) - \vartheta \|$$
$$\leq \|\mathbb{\Gamma}\hbar_n - \vartheta\|$$
$$\leq \|\hbar_n - \vartheta\|$$
$$\leq \|\breve{g}_n - \vartheta\|$$

Hence $\lim_{n\to\infty} \| \breve{g}_n - \vartheta \|$ exists for all $\vartheta \in \mathfrak{I}(\mathbb{\Gamma})$.

Theorem 3.2: Let $G \neq \emptyset$ (G non-empty closed convex), where $G \subseteq$ H . Let $\mathbb{\Gamma}: G \to G$ be a NE-map and consider$\{\breve{g}_n\}$ is generated by (9) with $\mathfrak{I}(\mathbb{\Gamma}) \neq \emptyset$ .Then $\{\breve{g}_n\}$ converges to $\vartheta \in \mathfrak{I}$ ($\mathbb{\Gamma}$) if and only if $\lim_{n\to\infty} \inf \sigma(\breve{g}_n, \mathfrak{I}(\mathbb{\Gamma})) = 0$, where $\sigma(\breve{g}_n, \mathfrak{I}(\mathbb{\Gamma})) = inf\,\{\|\breve{g}_n - \vartheta\| : \vartheta \in \mathfrak{I}(\mathbb{\Gamma})\}$.

Proof: Firstly, suppose $\{\breve{g}_n\}$ converges to $\vartheta \in \mathfrak{I}$ ($\mathbb{\Gamma}$). Then $\lim_{n\to\infty} inf\ \sigma\left(\breve{g}_n, \mathfrak{I}(\mathbb{\Gamma})\right) = 0$.
Conversely, assume that $\lim_{n\to\infty} \inf\ \sigma\left(\breve{g}_n, \mathfrak{I}(\mathbb{\Gamma})\right) = 0$. (12)
From Theorem 3.1, we obtain that $\lim_{n\to\infty} ||\breve{g}_n - \breve{g}||$ exists for every $\breve{g} \in \mathfrak{I}(\mathbb{\Gamma})$, so $\lim_{n\to\infty} \sigma\left(\breve{g}_n, \mathfrak{I}(\mathbb{\Gamma})\right) = 0$. exists. Now, from (12), we conclude $\lim_{n\to\infty} \sigma\left(\breve{g}_n, \mathfrak{I}(\mathbb{\Gamma})\right) = 0$.
For $\xi > 0$, there exists $N_1 \in$ N such that for every n $\geq N_1$ we have $\sigma(\breve{g}_n, \mathfrak{I}(\mathbb{\Gamma})) < \frac{\xi}{2}$
Particularly, $\inf\{||\breve{g}_{N1} - \vartheta^*|| < \frac{\xi}{2}$
Therefore, exists $\vartheta^* \in \mathfrak{I}(\mathbb{\Gamma})$, such that $||\breve{g}_{N1} - \vartheta^*|| < \frac{\xi}{2}$

Now for all ρ, n $\geq N_1$, we obtain

$$\left\|\breve{g}_{\rho+n} - \breve{g}_n\right\| \leq \left\|\breve{g}_{\rho+n} - \vartheta^*\right\| + \left\|\breve{g}_n - \vartheta^*\right\| \leq 2\left\|\breve{g}_{N1} - \vartheta^*\right\| < \xi.$$

It demonstrates that $\{\breve{g}_n\}$ is a Cauchy sequence in G. Additionally, since G is a closed set, we may deduce that there exists $\vartheta \in G$ such that $\lim_{n\to\infty} \breve{g}_n = \vartheta$. This implies $\lim_{n\to\infty} \sigma\left(\breve{g}_n, \mathfrak{I}(\mathbb{\Gamma})\right) = 0$. So, $\vartheta \in \mathfrak{I}$ ($\mathbb{\Gamma}$).

Some examples are now provided to validate the convergence result that was discussed in the earlier findings. The iterative technique (AR*) provided in Equation (9) converges to FP more quickly than the iterations of Mann [11], Ishikawa [12], Noor [13], Agarwal [14], Abbas [15], and Thakur [16], according to a numerical and graphical analysis of the cases.

Example 3.3: Let $H = \mathbb{R}$ and consider $G = [1,40]$. Set a map $\mathbb{\Gamma}: G \to G$ as follows:
$\mathbb{\Gamma}(u) = \sqrt{u^2 - 9u + 54}$ for all $u \in G$. For $u_\circ = 10$, $\eta = 2$ and $\omega_n, \gamma_n, \mu_n = 0.8$, n = 1, 2, . . .

Table 1: Comparison of convergence rates for various iteration approaches

step	Mann	Ishikawa	Noor	Agarwal	Abbas	Thakur	AR*
1	10.00000	10.00000	10.00000	10.00000	10.00000	10.00000	10.00000
2	8.400000	7.597713	7.267004	7.1977138	6.878770	6.550937	6.113302
3	7.277713	6.514150	6.342142	6.2131813	6.096858	6.033254	6.0010292
4	6.607051	6.150237	6.088738	6.0294632	6.008612	6.001745	6.0000090
5	6.265233	6.042582	6.022792	6.0038633	6.000746	6.000090	6.0000000
6	6.110440	6.011964	6.005839	6.0005028	6.000064	6.000004	6.0000000
7	6.044934	6.003353	6.001495	6.0000653	6.000005	6.000000	
8	6.018099	6.000939	6.000382	6.0000085	6.000000	6.000000	
9	6.007260	6.000262	6.000098	6.0000011	6.000000	6.000000	
10	6.002907	6.000073	6.000025	6.0000001	6.000000		
11	6.001163	6.000020	6.000006	6.0000000			
12	6.000465	6.000005	6.000001	6.0000000			
13	6.000186	6.000001	6.000000	6.0000000			
14	6.000074	6.000000	6.000000				
15	6.000029	6.000000	6.000000				
16	6.000011	6.000000	6.000000				
17	6.000004	6.000000	6.000000				
18	6.000001	6.000000					
19	6.000000						
20	6.000000						
21	6.000000						
22	6.000000						
23	6.000000						
24	6.000000						

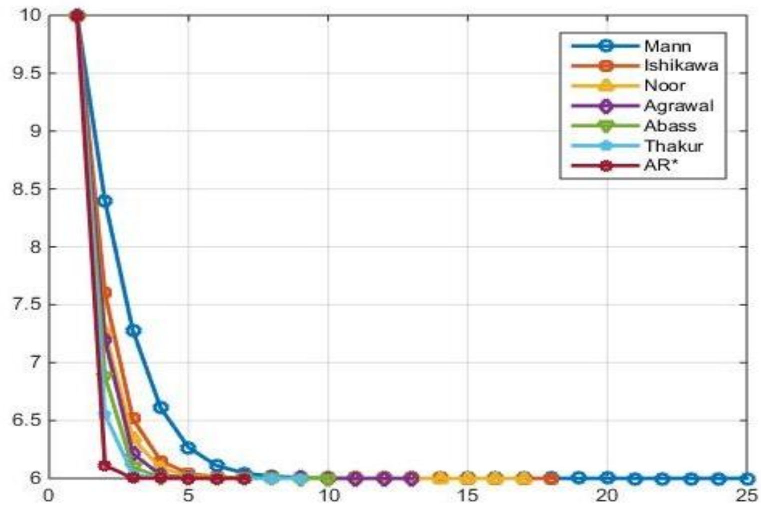

Fig. 1 - Graphic illustration of the convergence of iterative approaches.

Table 2: Comparison of the number of iterations to get FP under different initial points.

Start points: u_0=5, $\eta = 1$ and ω_n=γ_n=μ_n=0.9.							
Iterative approaches:	Mann	Ishikawa	Noor	Agarwal	Abbas	Thakur	AR*
Number of iterations:	18	12	11	9	8	7	5
Start points u_0=9, $\eta = 5$ and ω_n=γ_n=μ_n=0.25.							
Iterative approaches:	Mann	Ishikawa	Noor	Agarwal	Abbas	Thakur	AR*
Number of iterations:	39	36	32	16	10	9	6
Start points u_0=20, $\eta = 9$ and ω_n=γ_n=μ_n=0.75.							
Iterative approaches:	Mann	Ishikawa	Noor	Agarwal	Abbas	Thakur	AR*
Number of iterations:	31	23	19	14	12	10	8

Example 3.4: Let $H = \mathbb{R}$ and consider $G = [1,30]$. Set a map $\Gamma: G \to G$ as follows:

$$\Gamma(u) = \begin{cases} 4 - u \; if \; u \in [1, \frac{9}{7}) \\ \frac{u+12}{7} \; if \; u \in \left[\frac{9}{7}, 3\right] \end{cases}$$ for all $u \in G$. For $u_0 = 1.5$, $\eta = 2$ and $\omega_n, \gamma_n, \mu_n = 0.8$, n = 1, 2, . . .

Table 3 : Comparison of convergence rates for various iteration approaches

step	Mann	Ishikawa	Noor	Agarwal	Abbas	Thakur	New iteration
0	1.5	1.5	1.5	1.5	1.5	1.5	1.5
1	1.84285714	1.88204082	1.88651895	1.96775510	1.98111953	1.98986589	1.99925626
2	1.95061224	1.97217126	1.97424410	1.99792053	1.99928706	1.99979460	1.99999889
3	1.98447813	1.99343469	1.99415439	1.99986590	1.99997308	1.99999584	2.00000000
4	1.99512170	1.99845112	1.99867327	1.99999135	1.99999898	1.99999992	
5	1.99846682	1.99963459	1.99969888	1.99999944	1.99999996	2.00000000	
6	1.99951814	1.99991379	1.99993166	1.99999996	2.00000000		
7	1.99984856	1.99997966	1.99998449	2.00000000			
8	1.99995240	1.99999520	1.99999648				
9	1.99998504	1.99999887	1.99999920				
10	1.99999530	1.99999973	1.99999982				
11	1.99999852	1.99999994	1.99999996				
12	1.99999954	1.99999999	1.99999999				
13	1.99999985	2.00000000	2.00000000				
14	1.99999995						
15	1.99999999						
16	2.00000000						

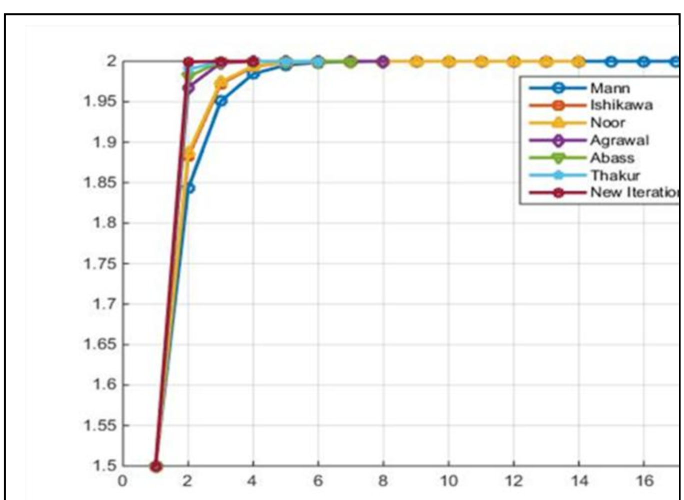

Fig.2: Graphic illustration of the convergence of iterative approaches

It is clear from the data in Tables 1, 2, and 3 as well as the graphs in Figures 1 and 2 for each of the previously given examples that the proposed novel method gets the FP faster than different approaches.

Now, we discuss the approximate FP for the mapping that satisfies E- condition

Theorem 3.5: Let $G \neq \emptyset$ (G *is* closed and convex) where $G \subseteq H$. Let $\mathbb{\Gamma}: G \to G$ be a map satisfying (E)-condition with $\Im(\mathbb{\Gamma}) \neq \emptyset$. If $\{\breve{g}_n\}$ is generated by eq. (9) . Then$\lim_{n\to\infty} \| \breve{g}_n - \vartheta \|$ exists for all $\vartheta \in \Im(\mathbb{\Gamma})$.

Proof: Let $\vartheta \in \Im(\mathbb{\Gamma})$. Then

$$\begin{aligned}\| \nu_n - \vartheta \| &= \| \frac{\eta\breve{g}_n+\mathbb{\Gamma}\breve{g}_n}{\eta+1} - \vartheta \| \\ &= \left\| \frac{\eta}{\eta+1}(\breve{g}_n - \vartheta) + \frac{1}{\eta+1}(\mathbb{\Gamma}\breve{g}_n - \vartheta) \right\| \\ &\leq \frac{\eta}{\eta+1}\|\breve{g}_n - \vartheta\| + \frac{1}{\eta+1}\|\mathbb{\Gamma}\breve{g}_n - \vartheta\|\end{aligned}$$

Now, by Lemma 2.3, we have

$$\| \nu_n - \vartheta \| \leq \frac{\eta}{\eta+1}\|\breve{g}_n - \vartheta\| + \frac{1}{\eta+1}\|\breve{g}_n - \vartheta\| = \|\breve{g}_n - \vartheta\|$$

and

$$\begin{aligned}\| \hbar_n - \vartheta \| &= \| \mathbb{\Gamma}(\frac{\eta\nu_n+\mathbb{\Gamma}\nu_n}{\eta+1}) - \vartheta \| \\ &\leq \left\| \frac{\eta\nu_n+\mathbb{\Gamma}\nu_n}{\eta+1} - \vartheta \right\| \\ &\leq \frac{\eta}{\eta+1}\|\nu_n - \vartheta\| + \frac{1}{\eta+1}\|\mathbb{\Gamma}\nu_n - \vartheta\| \\ &\leq \frac{\eta}{\eta+1}\|\nu_n - \vartheta\| + \frac{1}{\eta+1}\|\nu_n - \vartheta\| = \|\nu_n - \vartheta\|\end{aligned}$$

While using the above inequities, we have

$\| \breve{g}_{n+1} - \vartheta \| = \| \mathbb{\Gamma}(\mathbb{\Gamma}\hbar_n) - \vartheta \| \leq \|\mathbb{\Gamma}\hbar_n - \vartheta\| \leq \|\hbar_n - \vartheta\| \leq \|\breve{g}_n - \vartheta\|$. Hence $\lim_{n\to\infty} \| \breve{g}_n - \vartheta \|$ exists for all $\vartheta \in \Im(\mathbb{\Gamma})$. Thus,$\{||\breve{g}_{n+1} - \vartheta||\}$ is bounded and non-increasing, which implies that $\lim_{n\to\infty}\|\breve{g}_n - \vartheta\|$ exists for each $\vartheta \in \Im(\mathbb{\Gamma})$.

Example 3.6: Let $H = \mathbb{R}$ and consider $G = [0,30]$. Set a map $\mathbb{\Gamma}: G \to G$ as follows:
$\mathbb{\Gamma}(u) = 1 + \frac{1}{2}u$ for all $u \in G$.For $u_0 = 4$, $\eta = 2$ and $\omega_n, \gamma_n, \mu_n = 0.8$,n = 1, 2, ...

Table 4: Comparison of convergence rates for various iteration approaches

Step	Mann	Ishikawa	Noor	Agarwal	Abbas	Thakur	AR*
1	3.20000000	2.88000000	2.75200000	2.68000000	2.54800000	2.40800000	2.17361111
2	2.72000000	2.38720000	2.28275200	2.23120000	2.15015200	2.08323200	2.01507041
3	2.43200000	2.17036800	2.10631475	2.07860800	2.04114165	2.01697933	2.00130820
4	2.25920000	2.07496192	2.03997435	2.02672672	2.01127281	2.00346378	2.00011356
5	2.15552000	2.03298324	2.01503035	2.00908708	2.00308875	2.00070661	2.00000986

6	2.09331200	2.01451263	2.00565141	2.00308961	2.00084632	2.00014415	2.00000086
7	2.05598720	2.00638556	2.00212493	2.00105047	2.00023189	2.00002941	2.00000007
8	2.03359232	2.00280964	2.00079897	2.00035716	2.00006354	2.00000600	2.00000001
9	2.02015539	2.00123624	2.00030041	2.00012143	2.00001741	2.00000122	2.00000000
10	2.01209324	2.00054395	2.00011296	2.00004129	2.00000477	2.00000025	
11	2.00725594	2.00023934	2.00004247	2.00001404	2.00000131	2.00000005	
12	2.00435356	2.00010531	2.00001597	2.00000477	2.00000036	2.00000001	
13	2.00261214	2.00004634	2.00000600	2.00000162	2.00000010	2.00000000	
14	2.00156728	2.00002039	2.00000226	2.00000055	2.00000003		
15	2.00094037	2.00000897	2.00000085	2.00000019	2.00000001		
16	2.00056422	2.00000395	2.00000032	2.00000006	2.00000000		
17	2.00033853	2.00000174	2.00000012	2.00000002			
18	2.00020312	2.00000076	2.00000005	2.00000001			
19	2.00012187	2.00000034	2.00000002	2.00000000			
20	2.00007312	2.00000015	2.00000001				
21	2.00004387	2.00000007	2.00000000				
22	2.00002632	2.00000003					
23	2.00001579	2.00000001					
24	2.00000948	2.00000001					
25	2.00000569	2.00000000					
26	2.00000341						
27	2.00000205						
28	2.00000123						
29	2.00000074						
30	2.00000044						
31	2.00000027						
32	2.00000016						
33	2.00000010						
34	2.00000006						
35	2.00000003						
36	2.00000002						
37	2.00000001						

38	2.00000001						
39	2.00000000						

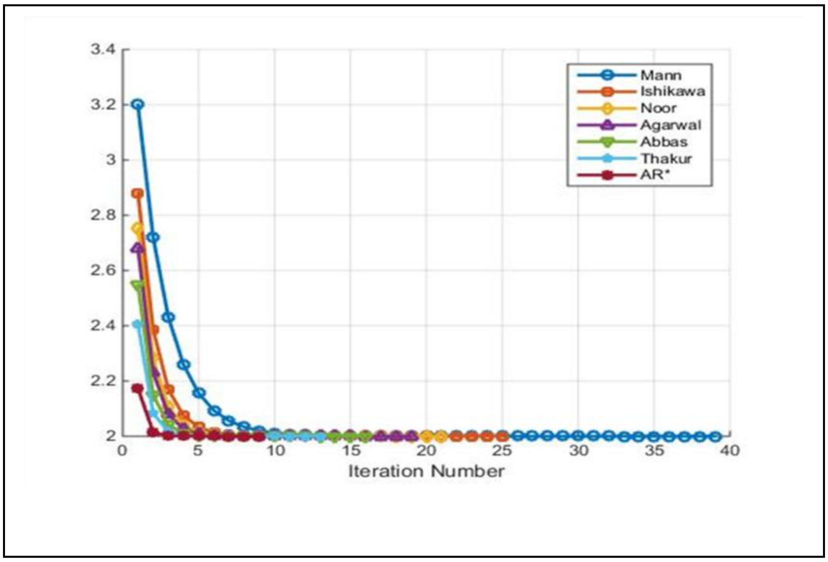

Fig.3: Graphic illustration of the convergence of iterative approaches

The previous example clearly shows that the proposed new method reaches the FP($u^* = 2$) faster than other ways, indicating that it is more efficient.

4. Conclusions

In order to estimate the FPs of nonexpansive maps and maps that meet the E-condition, we present a novel iterative technique in this study. We have numerically shown that our novel iterative technique converges faster than previous standard iterative methods. Furthermore, we established certain convergence results in a Banach space.

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