Extension of Conformal Fractional Derivatives

Authors Names	**ABSTRACT**
Ameer Hussein Fadhil [a] *Methaq Hamza Geem* [b] *Publication date: 7 / 8 /2026* *Keywords: Conformal fractional Derivative, Extended Conformal Operator,* g_{mn}^{α}*-Transform;Integro-Differential Equation, Viscoelastic Damping Model.*	In this work , We have introduced a new conformable fractional derivative operator, named as the $T_{\alpha,\beta}$ operator, which depends on two parameters α, β in (0,1] with $\alpha \neq \beta$, and based on this operator we propose an extended conformable fractional integrating operator which is called $T_{\alpha,\beta}$. We investigate some essential properties involving the commutators, non-commutativity relations when iterating the extended operator, and have represented the extended operator as a combination of ordinary differential operators and single-parameter conformal derivatives. To show that this methodology has some real utility we employ the g_{mn}^{α}-integral transform on a Conformal Fractional Integro-Differential Equation (CFIDE) which describes a viscoelastic damped system. The novel extended fractional method provides an effective analytical and series-based technique for the treatment of challenging fractional order integro-differential and boundary-value problems.

1. Introduction

Fractional calculus has proven to be a valuable tool for modelling of memory-dependent and non-local dynamic systems in real physical applications including viscoelastic material behaviours, signal processing, anomalous diffusion processes. The standard definitions of fractional derivatives, like that of Riemann-Liouville and Caputo derivatives, have also non-local features and the fractional derivatives of constants are not equal to zero making, that on many occasions the treatment of analytical problems becomes increasingly more difficult.

In order to overcome these drawbacks, local fractional derivatives were proposed. Khalil et al. [7] introduced a limit-based conformal fractional derivative of order α in (0,1] for a function $f: [0, \infty) \to R$ as:

$$T_{\alpha}\big(f(x)\big) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}$$

This definition satisfies usual properties of binomial, quotients as products rules and identity $T_{\alpha}(k) = = 0$ for any constant k. This groundwork was also generalized and extended to fractional calculus problems, by Abdeljawad [1].

Integral transforms are crucial to the solutions of differential, integral equations. Let us mention also classical transforms such as the Fourier transform (Bracewell [3]) and certain generalized integral transforms (Davies [5]). These include the Sumudu transform defined by Watugala [9] and the Sumudu transform of a distribution space studied by Fethibin and Ahmed [6], the Elzaki transform defined by Tarig [8], and duality studies among integral transforms by Chauhan et al. [4]. Recently, Fadhil and Geem [10] the generalized g_{mn} -transform to a function f(x) as:

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$$\mathrm{g}_{\mathrm{mn}}^{\alpha}\big(f(x)\big) = s^{m}\int_{0}^{\infty} e^{-\frac{s^{n}x^{\alpha}}{\alpha}} f(x)\, x^{\alpha-1}dx = F_{mn}^{\alpha}(s) \ \ ,\alpha \in (0,1]$$

where s is a positive constant. This integral transform has operational properties which facilitate the study of conformal fractional operator forms.In this article, we define a new two-parameter extension of the conformal fractional derivative operator

$$T_{\alpha,\beta}\big(f(x)\big) = T_{\alpha}(T_{\beta}\big(f(x)\big)$$

$\alpha, \beta \in (0,1]$ be a two parameters with $\alpha \neq \beta$

and we prove some properties of $T_{\alpha,\beta}$ We prove its basic properties, non-commutative nature and operational form. At last, we apply the extended operator $T_{\alpha,\beta}$ and $\mathrm{g}_{\mathrm{mn}}^{\alpha}$ -transform to solve explicitly series solution of a Conformal Fractional Integro-Differential Equation (CFIDE), which is motivated by viscoelastic damping systems.

2. Fundamental concepts

2.1 Definition [1]:

The conformal fractional of order one is defined as following:

$$T_{\alpha}\big(f(x)\big) = \lim_{\varepsilon\to 0}\frac{f(x+\varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}$$

2.2 Proposition [1]:

1. $T_{\alpha}(k) = 0$
2. $T_{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g)$
3. $T_{\alpha}(gh) = gT_{\alpha}(h) + hT_{\alpha}(g)$
4. $T_{\alpha}\left(\frac{f}{g}\right) = \frac{gT_{\alpha}(f) - fT_{\alpha}(g)}{g^{2}}$
5. $T_{2\alpha}(f(x)) = T_{\alpha}\,(T_{\alpha}(f(x))$
6. $T\alpha(f\,) = x^{1-\alpha}\,\frac{df}{dx}$

2.3 Definition [10]:

We define $\mathrm{g}_{\mathrm{mn}}^{\alpha}$ −transform for a function f(x) for all $x \in [0, \infty)$, as following formula:

$$\mathrm{g}_{\mathrm{mn}}^{\alpha}\big(f(x)\big) = s^{m}\int_{0}^{\infty} e^{-\frac{s^{n}x^{\alpha}}{\alpha}} f(x)\, x^{\alpha-1}dx = F_{mn}^{\alpha}(s) \ \ ,\alpha \in (0,1] \qquad (1)$$

Where the integral exists and s is positive constant.

2.4 Example [10]:

1) $\mathrm{g}_{\mathrm{mn}}^{\alpha}(k) = s^m \int_0^\infty e^{-\frac{s^n x^\alpha}{\alpha}} k\, x^{\alpha-1} dx = k s^m \int_0^\infty e^{-\frac{s^n x^\alpha}{\alpha}}\, x^{\alpha-1} dx$

$$= -k \frac{s^m}{s^n} e^{-\frac{s^n x^\alpha}{\alpha}} \Big|_0^\infty = k s^{m-n}$$

2) $\mathrm{g}_{\mathrm{mn}}^{\alpha}\left(e^{bx^\alpha}\right) = s^m \int_0^\infty e^{-\frac{s^n x^\alpha}{\alpha}} e^{bx^\alpha}\, x^{\alpha-1} dx = s^m \int_0^\infty e^{-\frac{(s^n-\alpha b)x^\alpha}{\alpha}}\, x^{\alpha-1} dx$

$$= -\frac{s^m}{(s^n-\alpha b)} e^{-\frac{s^n x^\alpha}{\alpha}} \Big|_0^\infty = \frac{s^m}{(s^n-\alpha b)}$$

3) $\mathrm{g}_{\mathrm{mn}}^{\alpha}((x^\alpha)^k) = s^m \int_0^\infty e^{-\frac{s^n x^\alpha}{\alpha}} (x^\alpha)^k\, x^{\alpha-1} dx$

Let $v = \frac{x^\alpha}{\alpha} \to dv = x^{\alpha-1} dx$, thus

$$\mathrm{g}_{\mathrm{mn}}^{\alpha}((x^\alpha)^k) = s^m \int_0^\infty e^{-s^n v} (\alpha v)^k\, dv = \alpha^k s^m \int_0^\infty e^{-s^n v} v^k\, dv$$

$$= \alpha^k \Gamma(k+1) s^{m-n(k+1)}$$

2.5 Definition [10]:

Let f,h be a function then the conformal convolution of them is given as following:

$$f *_\alpha h = \int_0^x f\left((x^\alpha - u^\alpha)^{\frac{1}{\alpha}}\right) h(u)\; u^{\alpha-1}\; du$$

2.6 Proposition [10]:

Let f,g be a real valued functions and $a, b \in R$, $\alpha \in (0,1]$ then

1-$\mathrm{g}_{\mathrm{mn}}^{\alpha}(af+bg) = a\mathrm{g}_{\mathrm{mn}}^{\alpha}(f) + b\mathrm{g}_{\mathrm{mn}}^{\alpha}(g)$

2-$\mathrm{g}_{\mathrm{mn}}^{\alpha}(f *_\alpha h) = s^{-m} F_{mn}^{\alpha}(s). H_{mn}^{\alpha}(s)$

$3 - \mathrm{g}_{\mathrm{mn}}^{\alpha}\left(\mathrm{T}\alpha h(x)\right) = s^n \mathrm{g}_{\mathrm{mn}}^{\alpha}\left(h(x)\right) - \mathrm{s}^m \mathrm{h}(0)$

$4 - \mathrm{g}_{\mathrm{mn}}^{\alpha}(T_\alpha(T_\alpha h(x))) = s^{2n} \mathrm{g}_{\mathrm{mn}}^{\alpha}\left(h(x)\right) - \mathrm{s}^m T_\alpha h(0) - s^{n+m} \mathrm{h}(0)$

$5 - \mathrm{g}_{\mathrm{mn}}^{\alpha}\left(\left((T_\alpha)^k h(x)\right)\right) = s^{kn} \mathrm{g}_{\mathrm{mn}}^{\alpha}\left(h(x)\right) - \mathrm{s}^m \sum_{i=0}^{k-1} \mathrm{S}^{n(k-i-1)} (T_\alpha)^i h(0)$

3. Extension Conformal Fractional Derivatives

3.1 Definition:

Let $\alpha, \beta \in (0,1]$ be a two parameters with $\alpha \neq \beta$and let $f: [0, \infty) \to R$ be a twice differentiable function then we define extension conformal fractional operator $T_{\alpha,\beta}$ as following:

$$T_{\alpha,\beta}\big(f(x)\big) = T_\alpha(T_\beta\big(f(x)\big)$$

3.2 Proposition:

let $f: [0, \infty) \to R$ be a twice differentiable function then

$$T_{\alpha,\beta}\big(f(x)\big) = x^{2-\alpha-\beta} f''(x) + (1-\beta)x^{1-\alpha-\beta} f'(x) \quad , \quad \alpha, \beta \in (0,1] \qquad (1)$$

Proof:

$$T_{\alpha,\beta}\big(f(x)\big) = T_\alpha(T_\beta\big(f(x)\big) = T_\alpha\left(x^{1-\beta} f'(x)\right) = x^{1-\alpha}\big[x^{1-\beta} f''(x) + (1-\beta)x^{-\beta} f'(x)\big]$$

$$= x^{2-\alpha-\beta} f''(x) + (1-\beta)x^{1-\alpha-\beta} f'(x)$$

3.3 Remark:

We note that $T_{\alpha,\beta}\big(f(x)\big) \neq T_{\beta,\alpha}\big(f(x)\big)$ also

$$\begin{aligned} &T_{\alpha,\beta}\big(f(x)\big) - T_{\beta,\alpha}\big(f(x)\big) \\ &\qquad = x^{2-\alpha-\beta} f''(x) + (1-\beta)x^{1-\alpha-\beta} f'(x) - x^{2-\beta-\alpha} f''(x) - (1-\alpha)x^{1-\beta-\alpha} f'(x) \\ &\qquad = (\alpha-\beta)x^{1-\alpha-\beta} f'(x) \neq 0 \end{aligned}$$

3.4 Proposition:

Let f be a twice differentiable function and $\alpha, \beta \in (0,1]$ then

$$T_{\alpha,\beta}\big(f(x)\big) = x^{\beta-\alpha} T_\beta(T_\beta\big(f(x)\big)$$

Proof:

$$T_{\alpha,\beta}\big(f(x)\big) = T_\alpha(T_\beta\big(f(x)\big)$$

Since $T_\alpha\big(h(x)\big) = x^{1-\alpha} h'(x)$ and $T_\beta\big(h(x)\big) = x^{1-\beta} \frac{dh(x)}{dx} \to \frac{dh(x)}{dx} = x^{\beta-1} T_\beta\big(h(x)\big)$ thus

$$T_{\alpha,\beta}\big(f(x)\big) = T_\alpha\left(T_\beta\big(f(x)\big)\right) = x^{1-\alpha} \frac{d}{dx}\left(T_\beta\big(f(x)\big)\right)$$

$$= x^{1-\alpha} x^{\beta-1} T_\beta\left(T_\beta\big(f(x)\big)\right) = x^{\beta-\alpha} T_\beta(T_\beta\big(f(x)\big)$$

3.5 Application:

Conformal Fractional Integro-Differential Equation (CFIDE)

Consider the following equation:

$$T_{\alpha,\beta}\big(u(x)\big) + CT_{\beta}\big(u(x)\big) + Du(x) + \gamma \int_{0}^{x} K\left(\left(x^{\beta} - t^{\beta}\right)^{\frac{1}{\beta}}\right) u(t)\, t^{\beta-1}\, dt = h(x) \qquad (2)$$

With conditions

$$T_{\beta}\big(u(0)\big) = u_1, \;\; u(0) = u_0 \qquad (3)$$

Where u, K are twice differentiable functions and $\alpha, \beta \in (0,1]$

To solve Eqs.(2,3) , we suppose that $v(x) = T_{\beta}\big(u(x)\big)$ to get the following system

$$x^{\beta-\alpha}T_{\beta}\big(v(x)\big) + Cv(x) + Du(x) + \gamma \int_{0}^{x} K\left(\left(x^{\beta} - t^{\beta}\right)^{\frac{1}{\beta}}\right) u(t)\, t^{\beta-1}\, dt = h(x) \qquad (4)$$

$$T_{\beta}\big(u(x)\big) = v(x) \qquad (5)$$

Now we take $-\mathrm{g}_{\mathrm{mn}}^{\beta}$-transform for both sides of Eq.(4) and Eq.(5) to obtain:

$$\mathrm{g}_{\mathrm{mn}}^{\beta}\left(x^{\beta-\alpha}T_{\beta}\big(v(x)\big)\right) + CV_{mn}^{\beta} + DU_{mn}^{\beta} + \gamma \mathrm{g}_{\mathrm{mn}}^{\beta}\big(K(x) *_{\alpha} u(x)\big) = H_{mn}^{\beta}$$

$$\mathrm{g}_{\mathrm{mn}}^{\beta}\left(x^{\beta-\alpha}T_{\beta}\big(v(x)\big)\right) + CV_{mn}^{\beta} + DU_{mn}^{\beta} + \gamma K_{mn}^{\beta}U_{mn}^{\beta} = H_{mn}^{\beta} \qquad (6)$$

Let $R_{mn}^{\alpha,\beta}\left(\mathrm{g}_{\mathrm{mn}}^{\beta}\big(f(x)\big)\right) = R_{mn}^{\alpha,\beta}\left(\mathrm{F}_{\mathrm{mn}}^{\beta}\right) = \mathrm{g}_{\mathrm{mn}}^{\beta}\big(x^{\beta-\alpha}f(x)\big)$, thus Eq.(6) becomes

$$R_{mn}^{\alpha,\beta}\left(\mathrm{g}_{\mathrm{mn}}^{\beta}\left(T_{\beta}\big(v(x)\big)\right)\right) + CV_{mn}^{\beta} + DU_{mn}^{\beta} + \gamma K_{mn}^{\beta}U_{mn}^{\beta} = H_{mn}^{\beta}$$

$$R_{mn}^{\alpha,\beta}\left(s^{n}\mathrm{V}_{\mathrm{mn}}^{\beta} - \mathrm{s}^{m}\mathrm{v}(0)\right) + CV_{mn}^{\beta} + DU_{mn}^{\beta} + \gamma K_{mn}^{\beta}U_{mn}^{\beta} = H_{mn}^{\beta}$$

$$R_{mn}^{\alpha,\beta}\left(\mathrm{s}^{n}\mathrm{V}_{\mathrm{mn}}^{\beta} - \mathrm{s}^{m}T_{\beta}\big(u(0)\big)\right) + CV_{mn}^{\beta} + DU_{mn}^{\beta} + \gamma K_{mn}^{\beta}U_{mn}^{\beta} = H_{mn}^{\beta}$$

$$R_{mn}^{\alpha,\beta}\left(\mathrm{s}^{n}\mathrm{V}_{\mathrm{mn}}^{\beta} - \mathrm{s}^{m}u_1\right) + CV_{mn}^{\beta} + DU_{mn}^{\beta} + \gamma K_{mn}^{\beta}U_{mn}^{\beta} = H_{mn}^{\beta} \qquad (7)$$

Also, we get from the second equation in system:

$$s^n U_{mn}^{\beta} - s^m u_0 = V_{mn}^{\beta} \quad \rightarrow U_{mn}^{\beta} = \frac{V_{mn}^{\beta} + s^m u_0}{s^n} \tag{8}$$

By substitution Eq.(8) in Eq.(7) we get

$$R_{mn}^{\alpha,\beta}\left(s^n V_{mn}^{\beta} - s^m u_1\right) + C V_{mn}^{\beta} + D\left(\frac{V_{mn}^{\beta} + s^m u_0}{s^n}\right) + \gamma K_{mn}^{\beta}\left(\frac{V_{mn}^{\beta} + s^m u_0}{s^n}\right) = H_{mn}^{\beta}$$

By solve the last equation for V_{mn}^{β} we obtain

$$V_{mn}^{\beta} = \emptyset\left(H_{mn}^{\beta}, \gamma K_{mn}^{\beta}, C, D, u_0, u_1, s\right) \tag{9}$$

By substitution Eq.(9) in Eq.(8) we get

$$U_{mn}^{\beta} = \frac{1}{s^n} V_{mn}^{\beta} + s^{m-n} u_0 = \frac{1}{s^n} \emptyset\left(H_{mn}^{\beta}, \gamma K_{mn}^{\beta}, C, D, u_0, u_1, s\right) + s^{m-n} u_0 \tag{10}$$

By taking inverse g_{mn}^{β}-transform of both sides Eq.(10):

$$u(x) = \left(g_{mn}^{\beta}\right)^{-1}\left(\frac{1}{s^n} \emptyset\left(H_{mn}^{\beta}, \gamma K_{mn}^{\beta}, C, D, u_0, u_1, s\right) + s^{m-n} u_0\right)$$

3.6 Example:

Consider The conformal integro-differential equation

$$T_{1,0.5}\big(u(x)\big) + 2T_{0.5}\big(u(x)\big) + \int_0^x e^{x^{0.5} - t^{0.5}} u(t)\, t^{0.5}\, dt = 0 \quad , \quad u(0) = 1 \; , \qquad T_{0.5}\big(u(0)\big) = 0$$

Which it is viscoelastic damping model.

We recall that $T_{0.5}(u) = x^{0.5} \frac{du}{dx}$, $T_1(u) = \frac{du}{dx}$

Hence $T_1(u) = x^{-0.5} T_{0.5}(u)$

We suppose that $v = T_{0.5}(u)$, $u(0) = 1$ (11)

Also by substitution in original equation we get

$$x^{-0.5} T_{0.5}(v) + 2v + e^{x^{0.5}} *_{\alpha} u(x) = 0 \quad , \qquad v(0) = 0 \tag{12}$$

By taking $g_{mn}^{0.5}$-transform of both sides Eq.(12):

$$\mathrm{g}_{\mathrm{mn}}^{0.5}\left(x^{-0.5}\mathrm{T}_{0.5}(v)\right) + 2V_{mn}^{0.5} + \frac{s^m}{(s^n - \alpha)} U_{mn}^{0.5} = 0$$

$$R_{mn}^{1,0.5}(\mathrm{V}_{\mathrm{mn}}^{0.5}) + 2V_{mn}^{0.5} + \frac{s^m}{(s^n - \alpha)} U_{mn}^{0.5} = 0 \qquad (13)$$

Now we compute $R_{mn}^{1,0.5}(\mathrm{V}_{\mathrm{mn}}^{0.5})$ as following:

First we use the property $\mathrm{g}_{0,1}\left(xf(x)\right) = -\frac{dF}{ds}$, thus if $f(x) = \frac{h(x)}{x}$ $, then\ h(x) = xf(x)$

Therefore

$$H(s) = \mathrm{g}_{0,1}^{0.5}(h(x)) = \mathrm{g}_{0,1}^{0.5}\left(xf(x)\right) = -\frac{dF}{ds}$$

By integration both sides with respect to s from s to ∞, we get

$$\int_s^{\infty} H(r)dr = -\int_s^{\infty} \frac{dF}{dr} dr = -F(r)\,|_s^{\infty} = F(s), since\ F(\infty) = \lim_{s\to\infty} \int_0^{\infty} e^{-sz} f(z)dz = 0$$

Hence

$$\mathrm{g}_{0,1}^{0.5}\left(\frac{h(x)}{x}\right) = \int_s^{\infty} H(r)dr$$

Suppose that $y = \frac{x^{0.5}}{0.5} = 2\sqrt{x}$

$$R_{mn}^{1,0.5}(\mathrm{V}_{\mathrm{mn}}^{0.5}) = \mathrm{g}_{\mathrm{mn}}^{0.5}\left(\frac{2}{y}\mathrm{T}_{0.5}\left(v(y)\right)\right) = 2\int_s^{\infty} \mathrm{g}_{\mathrm{mn}}^{0.5}\left(\mathrm{T}_{0.5}\left(v(y)\right)\right) dr$$

$$= 2\int_s^{\infty} r^n \mathrm{V}_{\mathrm{mn}}^{0.5}(r)\, dr$$

By substitution $R_{mn}^{1,0.5}(\mathrm{V}_{\mathrm{mn}}^{0.5})$ in Eq.(13) we get

$$2\int_s^{\infty} r^n \mathrm{V}_{\mathrm{mn}}^{0.5}(r)\, dr + 2V_{mn}^{0.5} + \frac{s^m}{(s^n - \alpha)} U_{mn}^{0.5} = 0 \qquad (14)$$

By taking $\mathrm{g}_{\mathrm{mn}}^{0.5}$-transform of both sides Eq.(11):

$$s^n U_{mn}^{0.5} - \mathrm{s}^m \mathrm{u}(0) = V_{mn}^{0.5}$$

By simply this equation we obtain

$$U_{mn}^{0.5} = \frac{V_{mn}^{0.5} + s^m}{s^n} \tag{15}$$

By substitution Eq.(15) in Eq.(14) to get:

$$2\int_s^\infty r^n \mathrm{V}_{\mathrm{mn}}^{0.5}(r)\,dr + 2V_{mn}^{0.5} + \frac{s^m}{(s^n - \alpha)}\left(\frac{V_{mn}^{0.5} + s^m}{s^n}\right) = 0 \tag{16}$$

We recall that m=0 , n=1 , thus Eq.(16) becomes

$$2\int_s^\infty r\mathrm{V}_{01}^{0.5}(r)\,dr + \left(\frac{2s^2 - 2s + 1}{s(s-1)}\right)V_{01}^{0.5} = \frac{-1}{s(s-1)} \tag{17}$$

By differentiate Eq.(17) with respect to s

$$2s\mathrm{V}_{01}^{0.5}(s) + \left(\frac{2s^2 + 2s + 1}{s(s-1)}\right)\frac{d\mathrm{V}_{01}^{0.5}(s)}{ds} + \frac{-1}{s^2(s-1)^2}V_{01}^{0.5} = \frac{2s-1}{s^2(s-1)^2}$$

By rearrange the sides we get

$$\left(\frac{2s^2 + 2s + 1}{s(s-1)}\right)\frac{d\mathrm{V}_{01}^{0.5}(s)}{ds} + \left(2s - \frac{1}{s^2(s-1)^2}\right)V_{01}^{0.5} = \frac{2s-1}{s^2(s-1)^2}$$

$$\frac{d\mathrm{V}_{01}^{0.5}(s)}{ds} + \left(\frac{2s}{\frac{2s^2+2s+1}{s(s-1)}} - \frac{1}{s^2(s-1)^2\left(\frac{2s^2+2s+1}{s(s-1)}\right)}\right)V_{01}^{0.5} = \frac{2s-1}{s^2(s-1)^2\left(\frac{2s^2+2s+1}{s(s-1)}\right)}$$

We can approximation of these variables of equation

$$\frac{d\mathrm{V}_{01}^{0.5}(s)}{ds} + \left(\frac{1}{s} - O\left(\frac{1}{s^2}\right)\right)V_{01}^{0.5} = \frac{1}{s^4} + O\left(\frac{1}{s^5}\right) \tag{18}$$

$$\mathrm{V}_{01}^{0.5}(s) = \sum_{k=1}^{\infty}\frac{\lambda_k}{s^k}$$

$$\frac{d\mathrm{V}_{01}^{0.5}(s)}{ds} = \sum_{k=1}^{\infty}\frac{-k\lambda_k}{s^{k+1}}$$

By substitution in Eq.(18)

$$\sum_{k=1}^{\infty}\frac{-k\lambda_k}{s^{k+1}}+\frac{1}{s}\sum_{k=1}^{\infty}\frac{\lambda_k}{s^k}=\frac{1}{s^4}$$

$$\lambda_1=0, \lambda_2=0\ , \lambda_3=\lambda_4=\lambda_5=-1\,, \dots$$

$$\mathrm{V}_{01}^{0.5}(s)=-\sum_{k=3}^{\infty}\frac{1}{s^k}$$

Since $U_{01}^{0.5}=\frac{V_{01}^{0.5}+1}{s}$ then

$$U_{01}^{0.5}=\frac{1}{s}-\sum_{k=3}^{\infty}\frac{1}{s^{k+1}}$$

By taking $\mathrm{g}_{01}^{0.5}$-transform for both sides we get

$$u(x)=1-\frac{2}{3}x^{1.5}-\frac{2}{3}x^{2}-\frac{4}{15}x^{2.5}-\cdots$$

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