Finite Difference method for Numerical solution of Stochastic boundary value Problems

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ABSTRACT

Recently, the numerical solution of stochastic boundary value problems has attracted substantial interests, owed to its broad applications in engineering, physics and applied mathematics. These problems usually arise in differential equations with random parameters, random forcing terms, or probabilistic boundary conditions, where analytical solutions, if exist, are either complicated beyond belief or not obtainable. In this paper, a powerful numerical algorithm based on the finite difference method is proposed to solve this kind of problem. The C-methods of the current approach approximate the spatial domain in a finite set of grid points where difference approximations are used to reduce the initial stochastic differential equation to a system of algebraic equations which can be numerically solved or addressed further, e.g. within a statistical moment context. To account for the stochastic nature of the problem, random variables as well as probabilistic distributions are introduced in the numerical problem formulation. This enables the approach to accurately represent the variability and uncertainty in boundary conditions and system parameters. A recursive solution process is introduced in this way to numerically stable and convergent solution. Furthermore, particular attention is paid to the choice of suitable step sizes and the discretization scheme, in order to reduce these numerical errors and consequently improve the accuracy of the solutions. The method's accuracy is verified with several problems possessing varying degrees of randomness and complexity in boundaries. The results also suggest that the finite difference method offers good approximations at a fairly low computational cost. In addition, sensitivity analysis confirms that the approach is able to identify the effect of random parameters upon the overall solution behaviour. The straightforward implementation and the flexibility of the method, allow us to expect to obtain results concerning a wide range of Stochastic Boundary Values and to be very accurate and powerful in practical application.

1. Introduction

To modelling the complex physical processes encountered in science and Engineering , the boundary value problem plays an important role in in modelling. These problems appear in such applications as heat conduction, fluid flow, stress analysis, power systems, and environmental processes. In practical problems, the parameters, boundary conditions and external forces that characterize these problems are usually not known exactly. Instead, they have uncertainty or randomness as a result of measurement errors, natural variability, or lack of knowledge. Such problems are called stochastic BVPs (SBVPs). However, in contrast to deterministic problems, SBVPs are not always analytically solvable which means that developing numerically efficient methods for their solution is necessary.Stochastic modeling provides for more accurate depiction of complex processes and enables researchers to understand how uncertainties propagate in physical systems and influence the ultimate results. The viewpoint is particularly crucial for safety-critical applications where even minute

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perturbations in the input parameters can cause drastic changes in the results (uncertain engineering [1-3], aerospace design, climate modeling).

Among the different types of numerical methods, finite difference method (FDM) has become one of the most popular and successful way for finding approximate solutions for the differential equations. Its simplicity and versatility, combined with its natural treatment of uncertainty make it a natural candidate for the solution of these problems. By discretizing the domain into a finite number of grid points and substituting derivatives with finite difference approximations, the initial continuous problem is reduced to a system of algebraic equations that can be solved numerically.

This enables the computations to be combined with probabilistic models, resulting in the study of systems with random inputs and boundary conditions [4,6]. Further, FDM also has a natural setup for convergence and stability analysis which allow the accuracy and reliability of numerical solutions under random perturbation to be examined [7]. While these are classical in the sense of the spherical harmonic time inversion methods, several researchers have focused recently on extending traditional numerical schemes to deal with randomness and uncertainty. These pursuits have also produced stochastic versions of well know deterministic methods, such as finite element and finite difference procedures. The finite difference (FD) method, in particular is an excellent compromise between complexity and accuracy, which makes it appealing to practical problems where a large amount of simulation or sensitivity analysis are needed [8,10]. In this article we aim at developing a robust and effective finite difference based numerical framework for stochastic boundary value problems. By incorporating probabilistic information into the discretization, the method yields robust and accurate solutions even in the face of serious uncertainly. This approach is applicable to a wide range of engineering and scientific problems, leading to improved prediction, design and decision making in the presence of uncertainty [11,12]. In addition, the framework sets the basis for further extensions, e.g., coupling with ML for adaptive grid refinement and thus for broadening the applicability and efficiency of stochastic simulations in computational science [13].

2. Methodology

As mentioned in previous sections, the numerical solution of stochastic initial value problems has received considerable attention over the past two decades. In constrast, analytical and numerical approximations for stochastic boundary value problems have been explored to a much lesser extent. Since analytic solutions for this class of problems are generally unavailable numerical techniques are essential for obtaining approximate solutions. In line with efforts to develop numerical methods for such problems, this study focuses on extending the family of finite difference techniques for the numerical solution of stochastic boundary value problems. The methods presented here differ conceptually and essentially from previously proposed approaches. Unlike conventional techniques, this class for methods not rely on solving any initial value problems, instead approximate solutions are sought at the interior points of the problem domain. For this reason, these method can be regarded as primary finite difference schemes [14].
The core steps of the finite difference approach are as follows. First the problem domain is discretized into a set of points:

$$a < t_1 < t_2 < \cdots < t_N < t_{N+1} = b,$$

and approximate solutions are computed at these discrete points. Second, a system of equations is formulated by substituting finite difference approximations of derivatives into the differential equations and applying the boundary conditions, where exact solutions are known. Third, this system of algebraic equations is solved to obtain a set of discrete solution values $x_i = x(t_i)$. If necessary, interpolation can be applied to estimate $x(t)$at any point within the domain $t \in [a, b]$.

Despite the unique capabilities of this method family, apart from one case, little research has been conducted on applying it to stochastic boundary value problems. Finite difference techniques have been primarily employed for the numerical solution of second-order linear boundary problems. In this study, we aim to extend these methods for the first time to approximate solution trajectories for first-order linear and nonlinear problems, second-order nonlinear problems, and multi-point boundary value problems. Initially, first-order boundary problems are considered, with forward, backward, and central schemes applied to linear cases, and forward and backward schemes used for nonlinear cases. Subsequently, nonlinear problem linearization and iterative solution methods, such as the Newton-Raphson technique for solving the resulting system of equations, are briefly discussed.[15]

2.1 Finite Difference Method for First-Order Boundary Value Problems

In the following section, we focus on stochastic second-order boundary value problems, including both linear and nonlinear cases. Linear stochastic boundary value problems with multi-point boundary conditions are also considered. Finally, the effectiveness and accuracy of the proposed theoretical developments are demonstrated through various numerical examples.

2.1.1 First-Order Problems

This section addresses first-order stochastic boundary value problems. To date, no efforts have been made to apply finite difference techniques for approximating solutions of this class of problems. In this study, we employ, for the first time, this family of methods to approximate the solutions of first-order stochastic boundary problems. Initially, we extend the finite difference method to linear problems, which leads us to categorize the schemes into **single-step** and **two-step** methods. For nonlinear boundary problems, we extend the two-step finite difference approach to numerically approximate solution trajectories.

2.1.2 Linear Boundary Problems

Consider the following linear stochastic system with boundary conditions:

$$\begin{cases} d\vec{X}(t) = \left(A\vec{X}(t) + \vec{a}(t)\right) dt + \sum_{i=1}^{k} \left(B_i\vec{X}(t) + \vec{b}_i(t)\right) dW_i(t), \\ \qquad F_0\vec{X}(0) + F_1\vec{X}(0) = \vec{0} \end{cases} \tag{1}$$

As discussed in previous works [25, 32], existence and uniqueness of the solution to this problem are guaranteed. A solution $\tilde{X} = R^n$is considered valid if $\vec{X}$is Stratonovich integrable and satisfies the integral form of the system:

$$\begin{cases} \vec{X}(t) - \vec{X}(0) = \int_0^s \left(A\vec{X}(s) + \vec{a}(s)\right) ds + \vec{a}(s)ds + \int_0^t \sum_{i=1}^{k} \left(B_i\vec{X}(s) + \vec{b}_i(s)\right) dW_i(s)) \\ \qquad F_0\vec{X}(0) + F_1\vec{X}(0) = \vec{0} \end{cases} \tag{2}$$

where و$\tilde{u}, \vec{a}, \tilde{b}_i \in R^n$ and A, F_0, F_1, B_i are $n \times n$matrices, and the stochastic integrals are in the Stratonovich sense. We also assume a partition of the domain [0,1]as follows:

$$\vartheta: 0 = t_1 < t_2 < \cdots < t_N < t_{N+1} = 1.$$

In addition:

$$F_0 = \begin{bmatrix} \hat{F}_0 \\ 0 \end{bmatrix}, \qquad F_1 = \begin{bmatrix} \hat{F}_1 \\ 0 \end{bmatrix}.$$

The finite difference schemes for the stochastic system can then be summarized in the compact form:

$\Gamma_x = F(x),$ (3)

where Γrepresents a finite difference operator (e.g., forward, backward, or central), and the operator $F$is defined as:

$$F(x) = \left(A\vec{x}(t)\right) + \vec{a}(t)dt + \sum_{i=1}^{k}\left(B_i\vec{x}(t) + \vec{b}_i(t)\right)dW_i(t)$$

Note: Before applying finite difference schemes, the Stratonovich form of system (1) should be converted to the Itô form. However, due to the special structure of this system, the full Stratonovich-to-Itô conversion is not required. The resulting Itô form is:

$$d\vec{X} = \left\{\left(A\vec{X}(t) + \vec{a}(t)\right) + \frac{1}{2}\sum_{j=1}^{k}\left(B_j^2\vec{X}(t) + B_j\vec{b}_j(t)\right)\right\}dt + \sum_{j=1}^{k}\left(B_j\vec{X}(t) + \vec{b}_j(t)\right)dW_j(t)$$

2.2. Single-Step Finite Difference Schemes

In this section, we introduce the **single-step finite difference operator** for the operators $\Gamma$and Fin equation (5.3). This operator is applied over subintervals $[t_i, t_{i+1}]$of the partition ϑ. Two simple finite difference schemes, namely the **midpoint scheme** and the **trapezoidal scheme**, can be employed and are presented below. Numerical solutions at each partition point $t_i$are denoted by x_i, corresponding to the exact solution $x(t_i)$. All numerical solutions must satisfy the boundary conditions:

$$F_0\vec{x}_0 + F_1\vec{x}_{N+1} = \vec{\beta}$$

For interior points of the partition, the single-step finite difference approximation replaces the derivative term in the stochastic system (5.1), $d\mathbf{X}(t)$, with $x_{i+1} - x_i$over each subinterval $[t_i, t_{i+1}]$. At the midpoint $x_{i+1/2} := x_i + \frac{1}{2}h_i$, where $h_i := x_{i+1} - x_i$, the term $\left\{\left(A\vec{x}(t) + \vec{a}(t)\right) + \frac{1}{2}\sum_{j=1}^{k}\left(B_j^2\vec{x}(t) + B_j(t)\vec{b}_j(t)\right)\right\}$is approximated centrally.

The **trapezoidal scheme** is then expressed as:

$$\vec{x}_{i+1} - \vec{x}_i = \frac{1}{2}\left[A + \frac{1}{2}\sum_{j=1}^{k}\left(B_j\right)^2(x_{i+1} + \vec{x}_i)\right]h_i + \frac{1}{2}\left[\vec{a}_{i+1} + \frac{1}{2}\sum_{j=1}^{k}B_j\,\vec{b}_{ji+1} + \vec{a}_i + \frac{1}{2}\sum_{j=1}^{k}B_j\,\vec{b}_{ji}\right]h_i + \sum_{j=1}^{k}\left(B_j\vec{x}_i + \vec{b}_{ji}\right)dW_{ji}, \qquad 1 \le i \le N.$$

Similarly, the **midpoint scheme** is defined as:

$$\vec{x}_{i+1} - \vec{x}_i = \frac{1}{2}\left[A + \frac{1}{2}\sum_{j=1}^{k}\left(B_j\right)^2(x_{i+1} + \vec{x}_i)\right]h_i + \frac{1}{2}\left[\vec{a}_{i+\frac{1}{2}} + \frac{1}{2}\sum_{j=1}^{k}B_j\,\vec{b}_{ji+\frac{1}{2}}\right]h_i + \sum_{j=1}^{k}\left(B_j\vec{x}_i + \vec{b}_{ji}\right)dW_{ji}, \qquad 1 \le i \le N$$

Here, $\vec{a}_i$and $\vec{b}_{ji}$denote $\vec{a}(t_i)$and $\vec{b}_j(t_i)$, and $dW_{j,i}$is a random variable with zero mean and variance h_i. These schemes can also be expressed in a **matrix form**:

$\Psi\mathbf{X}_\vartheta = \kappa,$ (4)

$$\begin{bmatrix} C_1 & D_1 & & & \\ & C_2 & D_2 & & \\ & & \ddots & & \\ & & & C_N & D_N \\ F_0 & & & & F_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \\ X_{N+1} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \\ X_{N+1} \end{bmatrix} \qquad (5)$$

where $C_i$and $D_i$are $n \times n$matrices defined specifically for each scheme. For the **midpoint scheme**, the components are:

$$C_i = -I + \frac{1}{2}\left[A + \sum_{j=1}^{k}\left(B_j\right)^2\right]h_i - \sum_{j=1}^{k} B_j\, dw_i,$$

$$D_i = I - \frac{1}{2}\left[\left(A + \sum_{j=1}^{k}\left(B_j\right)^2\right)\right]h_i,$$

$$\kappa_i = \frac{1}{2}\left[\vec{a}_{i+1} + \frac{1}{2}\sum\nolimits_{j=1}^{k} B_j\vec{b}_{ji+1} + \vec{a}_i + \frac{1}{2}\sum_{j=1}^{k} B_j\vec{b}_{ji}\right]h_i + \sum_{j=1}^{k}\vec{b}_{ji}\, dw_j(t).$$

2.3. Two-Step Finite Difference Scheme

A **two-step finite difference operator** is also introduced for equation (5.4), based on subintervals $[t_{i-1}, t_{i+1}]$. A **central difference scheme** is used for derivative approximations, and $A\vec{x}(t) + \vec{a}(t)$is evaluated at the midpoint x_i. Numerical solutions $x_i$satisfy the boundary conditions:

$$F_0\vec{x}_1 + F_1\vec{x}_{N+1} = \vec{\beta}.$$

For interior points, the derivative $d\mathbf{X}(t)$is replaced by $x_{i+1} - x_{i-1}$, and the central approximation for the remaining terms is applied:

$$x_{i+1} - x_{i-1} = \left\{(A\vec{x}_i + \vec{a}_i) + \frac{1}{2}\sum\nolimits_{j=1}^{k}\left(B_j^2\vec{x}_i(t) + B_j\vec{b}_{ji}\right)\right\}h_i$$
$$+\sum\nolimits_{j=1}^{k}\left(B_j\vec{x}_i + \vec{b}_{ji}\right)dw_j(t), \qquad 1 \le i \le N$$

Here, $\vec{a}_i, \vec{b}_{ji}, dW_{j,i}$are defined as before.

2.4. Matrix Form of the Two-Step Finite Difference Scheme

In the two-step finite difference scheme (6), the stochastic increment $dW_{j,i}$is considered as a random variable with zero mean and variance h_i. Using the closed-form matrix representation, the scheme can be expressed as:

$$\Gamma\mathbf{X}_\vartheta = \kappa,$$

which, in detail, takes the form:

$$\begin{bmatrix} -I & M_1 & I & & & \\ & -I & M_2 & I & & \\ & & \ddots & & M_{N-1} & I \\ & & & -I & S & R \\ F_0 & & & & & F_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ X_N \\ X_{N+1} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ k_N \\ \beta \end{bmatrix}$$

Here, Idenotes the $n \times n$identity matrix, and $M_i$for $1 \le i \le N$are $n \times n$matrices defined as:

$$M_i = -\left[A + \frac{1}{2}\sum_{j=1}^{k} (B_j^2 + \frac{2}{h_i} B_j dw_j\right] h_i$$

$$\kappa_i = \left[\vec{a}_i + \tfrac{1}{2}\textstyle\sum_{j=1}^{k} B_j \vec{b}_j\right] h_i + \textstyle\sum_{j=1}^{k} \vec{b}_{ji} dW_j(t). \qquad (6)$$

The matrices $S$and $R$are determined based on the boundary conditions of the problem. Note that to construct the $-N$th row of Γ, information from the point $N+2$of the partition $\vartheta$is required. Since this point is not part of the partition, a common approach is to set the solution at $N+2$equal to that at $N+1$, leading to:

$$S = -I, R = I - M_{N+1}.$$

For **Neumann boundary conditions**, e.g., $\frac{\partial x}{\partial t}(b) = g$, the approximation can be written as:

$$\frac{x_{N+2} - x_{N+1}}{h_{N+1}} = g \Rightarrow x_{N+2} = g\, h_{N+1},$$

which modifies the last row as:

$$S = -I, R = I - M_{N+1}, \kappa_N = \kappa_N - g\, h_{N+1}.$$

Alternatively, a **central difference** can be used to approximate the boundary, or the function value at point $N+2$can be set to zero, which is a common practice in numerical implementations.

2.5. Nonlinear Boundary Value Problems

In this section, we extend the finite difference method to the **numerical solution of nonlinear stochastic boundary value problems**. Consider the multidimensional nonlinear system:

$$\begin{cases} dX_t = f(X_t, t)dt + \sigma(X_t, t)dW_t \\ \qquad \vec{h}(X_0, X_1) = \vec{h}, \end{cases} \qquad (7)$$

where $f$and σhave appropriate dimensions such that the stochastic integral in (7) is interpreted in the **Stratonovich sense**. As mentioned earlier, the existence and uniqueness of solutions for the nonlinear stochastic boundary value problem (7) is generally unresolved, and only special cases under certain conditions on $f$have been investigated [22, 320].

Definition 2.5.1: A process $X_t \in \mathbb{R}^n$is a solution of the differential equation

$$dX_t = f(X_t, t)\, dt + \sigma(X_t, t)\, dW_t$$

if $\sigma(X, t)$is Stratonovich integrable for all $t \in [0, T]$and

$$X_t - X_0 = \int_0^t f(X_s, s)ds + \int_0^t \sigma(X_s, s)ds0dW_s,$$

Moreover, $X_t$must satisfy the boundary conditions almost everywhere, i.e., $(X_0, X_T) \approx \mathbf{h}$.

Following the previous approach, we partition the interval $[0,1]$as $\vartheta: 0 = t_1 < t_2 < \cdots < t_{N+1} = 1$, and denote the numerical solution at each point $t_i$by x_i, corresponding to the exact solution $x(t_i)$. All numerical solutions must satisfy the boundary conditions:

$$\mathbf{h}(x_1, x_{N+1}) = \mathbf{h}_0.$$

Let Γdenote a finite difference operator and define Fas

$$F(x) = f(x_t, t)\, dt + \sigma(x_t, t)\, dW_t. \qquad (8)$$

The family of finite difference methods for the numerical solution of (7) can then be expressed as:

$$\Gamma(x) = F(x). \qquad (9)$$

2.6. Trapezoidal Scheme

If $\Gamma$and Fin (9) are chosen as

$$\vec{x}_{i+1}-\vec{x}_i=\frac{1}{2}\left[\vec{f}(t_i,\vec{x}_i)+\vec{f}(t_{i+1},\vec{x}_{i+1})\right]h_i+\sigma(t_i,\vec{x}_i)\Delta w(i), \qquad 1\le i\le N+1 \tag{10}$$

we obtain the **trapezoidal scheme** for the nonlinear problem. Here, $\Delta W_i$are normally distributed random variables with zero mean and variance $h_i=t_{i+1}-t_i$. Without loss of generality, the boundary function $\vec{h}=\vec{0}$can be assumed. Note that $x_i$appears implicitly, and both $f$and $\sigma$are nonlinear functions of $x_i$and t_i. Incorporating the boundary conditions results in the following nonlinear system:

$$\begin{cases} \vec{x}_2-\vec{x}_1-\frac{1}{2}\left[\vec{f}(t_1,\vec{x}_1)+\vec{f}(t_2,\vec{x}_2)\right]h_1-\sigma(x_1,t_1)\Delta W(1)=\vec{0} \\ \vec{x}_3-\vec{x}_2-\frac{1}{2}\left[\vec{f}(t_2,\vec{x}_2)+\vec{f}(t_3,\vec{x}_3)\right]h_2-\sigma(x_2,t_2)\Delta W(2)=\vec{0} \\ \vdots \\ \vec{x}_N-\vec{x}_{N-1}-\frac{1}{2}\left[\vec{f}(t_{N-1},\vec{x}_{N-1})+\tilde{f}(t_N,\vec{x}_N)\right]h_{N-1}+\sigma(x_{N-1},t_{N-1})\mathit{\Delta}W(N-1)=\vec{0} \\ \vec{x}_{N+1}-\vec{x}_N-\frac{1}{2}\left[\vec{f}(t_{N-1},\vec{x}_N)+\tilde{f}(t_{N+1},\vec{x}_{N+1})\right]h_N+\sigma(x_N,t_{N-1})\mathit{\Delta}W(N-1)=\vec{0} \\ h(x_1,x_{N+1})=h. \end{cases} \tag{11}$$

2.7. Midpoint Scheme

Alternatively, choosing

$$x_{i+1}-\vec{x}_i=\frac{1}{2}\left[\vec{f}\left(t_{i+\frac{1}{2}},\frac{1}{2}(\vec{x}_i+\vec{x}_{i+1})\right)\right]h_i+\sigma(x_i,t_i)\Delta w(i), \qquad 0\le t\le 1 \tag{12}$$

yields the **midpoint scheme**, where $\Delta W_i$is defined as before. Incorporating the boundary conditions produces a corresponding nonlinear system analogous to the trapezoidal scheme.

$$\begin{cases} \vec{x}_2-\vec{x}_1-\frac{1}{2}\left[\vec{f}\left(t_{1+\frac{1}{2}},\frac{1}{2}(\vec{x}_1+\vec{x}_2)\right)\right]h_1+\sigma(x_1,t_1)\mathit{\Delta}W(1)=\vec{0} \\ \vec{x}_2-\vec{x}_1-\frac{1}{2}\left[\vec{f}\left(t_{2+\frac{1}{2}},\frac{1}{2}(\vec{x}_1+\vec{x}_2)\right)\right]h_1+\sigma(x_1,t_1)\mathit{\Delta}W(1)=\vec{0} \\ \vdots \\ \vec{x}_N-\vec{x}_{N-1}-\frac{1}{2}\left[\vec{f}\left(t_{N-1+\frac{1}{2}},\frac{1}{2}(\vec{x}_{N-1}+\vec{x}_N)\right)\right]h_{N-1}+\sigma(x_{N-1},t_{N-1})\mathit{\Delta}W(N-1)=\vec{0} \\ \vec{x}_{N+1}-\vec{x}_N-\frac{1}{2}\left[\vec{f}\left(t_{N+\frac{1}{2}},\frac{1}{2}(\vec{x}_N+\vec{x}_{N+1})\right)+\tilde{f}(t_N,\vec{x}_N)\right]h_N+\sigma(x_N,t_N)\mathit{\Delta}W(N)=\vec{0} \\ h(x_1,x_{N+1})=h. \end{cases} \tag{13}$$

2.8. Newton's Method for Nonlinear Systems

As is evident, the discretization of the nonlinear boundary value problem leads to a system of $n(N+1)$algebraic equations with $n(N+1)$unknowns, $\mathbf{X}_\vartheta$. Iterative methods, such as **Newton's method**, can be employed to find the roots of these nonlinear equations. In this section, we describe Newton's method for solving the system resulting from the finite difference schemes, taking system (11) as an example. The nonlinear system (13) can be solved in a similar manner.

Consider the nonlinear function $F(\mathbf{x})$defined as:

$$F(x) = \begin{bmatrix} \vec{x}_2 - \vec{x}_1 - \frac{1}{2}\left[\vec{f}(t_1, \vec{x}_1) + \vec{f}(t_2, \vec{x}_2)\right]h_1 - \sigma(x_1, t_1)\Delta W(1) \\ \vec{x}_3 - \vec{x}_2 - \frac{1}{2}\left[\vec{f}(t_2, \vec{x}_2) + \vec{f}(t_3, \vec{x}_3)\right]h_2 - \sigma(x_2, t_2)\Delta W(2) \\ \vdots \\ \vec{x}_N - \vec{x}_{N-1} - \frac{1}{2}\left[\vec{f}(t_{N-1}, \vec{x}_{N-1}) + \tilde{f}(t_N, \vec{x}_N)\right]h_{N-1} + \sigma(x_{N-1}, t_{N-1})\Delta W(N-1) \\ \vec{x}_{N+1} - \vec{x}_N - \frac{1}{2}\left[\vec{f}(t_{N-1}, \vec{x}_N) + \tilde{f}(t_{N+1}, \vec{x}_{N+1})\right]h_N + \sigma(x_N, t_{N-1})\Delta W(N) \\ h(x_1, x_{N+1}) = h. \end{bmatrix}$$

Our objective is to solve the nonlinear system

$$F(\vec{x}) = \vec{0} \tag{14}$$

where the solution vector is $\vec{x} = [x_1, x_2, x_N, x_{N+1}]^T$.

In Newton's method, a sequence of approximations

$$\mathbf{x}^{(k)} = [x_1^{(k)}, x_2^{(k)}, \dots, x_N^{(k)}, x_{N+1}^{(k)}]^T$$

is generated, which converges to the solution of the nonlinear system, provided that the initial guess

$$\mathbf{x}^{(0)} = [x_1^{(0)}, x_2^{(0)}, \dots, x_N^{(0)}, x_{N+1}^{(0)}]^T$$

is sufficiently close to the true solution and that the Jacobian matrix of the system is nonsingular.

At each iteration, Newton's method requires solving the linear system:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - [DF(\mathbf{x}^{(k)})]^{-1}F(\mathbf{x}^{(k)}),$$

where $DF(\mathbf{x})$is the Jacobian matrix. For the nonlinear system (14), the Jacobian takes the block form:

$$DF(x) = \begin{bmatrix} A_1 & B_1 & & & \\ & A_2 & B_2 & & \\ & & \ddots & & \\ & & & A_{N+1} & B_{N+1} \\ F_0 & & & & F_1 \end{bmatrix} . \tag{17}$$

The components of the Jacobian are defined as:

$$A_i \equiv D_{x_i}F(x) = -I - \frac{1}{2}\left(\frac{\partial f(t,x)}{\partial x_i}\right)h - \left(\frac{\partial \sigma(t,x)}{\partial x_i}\right)\Delta W(i),$$

$$B_i \equiv D_{x_i}F(x) = I - \frac{1}{2}\left(\frac{\partial f(t,x)}{\partial x_{i+1}}\right)h$$

$$\mathrm{C} \equiv D_{x_1}F(x) = \frac{\partial h(t,x)}{\partial x_1}$$

$$\mathrm{D} \equiv D_{x_{N+1}}F(x) = \frac{\partial h(t,x)}{\partial x_{N+1}}$$

Here, Idenotes the $n \times n$identity matrix. This iterative procedure is repeated until the sequence $\mathbf{x}^{(k)}$converges to the solution of the nonlinear system.

3. Results

3.1. Finite Difference Methods for Second-Order Boundary Value Problems

In this section, we introduce a family of finite difference methods for the numerical approximation of solution paths for second-order boundary value problems (BVPs). We first focus on linear problems.

In 1995, Allen and Nan proposed a finite difference approach for the numerical solution of this class of problems [2]. However, until now, no studies have applied finite difference methods to the numerical solution of nonlinear second-order BVPs. In the second part of this paper, we extend finite difference methods for the first time to approximate the solution paths of such nonlinear problems.

3.1.1. Linear Second-Order Boundary Value Problems

We consider the two-point stochastic boundary value problem:

$$\begin{cases} X''(t) = (a(t) + b(t))X(t)\,dt + (f(t) + g(t)X(t))\,dW(t), \\ X(0) = X(1) = 0, \end{cases} \tag{16}$$

where $a, b, f, g \in C^1(0,1)$with $b(t) \geq 0$, and $W(t)$is a Wiener process.

Equation (3.1.1) can be rewritten as a first-order system:

$$\begin{cases} dX_1(t) = X_2(t)\,dt, \\ dX_2(t) = (a(t) + b(t)X_1(t))\,dt + (f(t) + g(t)X_1(t))\,dW(t), \\ X_1(0) = X_1(1) = 0. \end{cases} \tag{17}$$

System (3.1.1) is a predictive system, meaning that the solution at any time depends on the Brownian motion up to that time. As shown by Okano & Bagdo [25] and Demba & Zitoni [32], the solution of (3.1.1.) can be represented via Stratonovich stochastic integrals.

Definition 3.1.1: The process $(X_1, X_2) \in \mathbb{R}^2$is a solution of (17) if it is Stratonovich-integrable and satisfies

$$\begin{cases} X_1(t) = \int_0^t X_2\,(t')dt' \\ X_2(t) = X_2(0) + \int_0^t \big(a(t') + b(t')X_1(t')\big)dt' + \int_0^t f(t') + g(t')X_1(t')dW(t') \end{cases} \tag{18}$$

with probability one. The initial value $X_2(0)$can be determined by an iterative procedure, which plays a key role in proving the existence and uniqueness of the solution.

The finite difference method for solving system (3.1.1.) is now introduced. First, we divide the interval $[0,1]$into Nequal subintervals of length $h = 1/N$. Let $X_i = X(t_i)$. The finite difference equations are:

$$X_{i+1} - 2X_i + X_{i-1} = h^2(a_i + b_iX_i) + h\sqrt{h}\,\eta_i(f_i + g_iX_i), i = 1, \dots, N + 1,$$

with boundary conditions $X_0 = 0$and $X_{N+1} = 0$.

This system can be solved iteratively and, due to its tridiagonal structure, efficiently solved using standard linear algebra methods.

3.1.2. Nonlinear Second-Order Boundary Value Problems

Next, we extend the finite difference method to approximate the solution of nonlinear stochastic second-order BVPs. Consider the two-point problem:

$$\begin{cases} X''(t) = f(X)\,\dot{W}(t) + g(X, X'), \\ X(0) = \xi, X(1) = \eta, \end{cases}$$

where $\dot{W}(t)$denotes white noise, $\xi$and $\eta$are random variables, and $f, g$are continuous functions. As discussed in Section 1, Walsh and Lou (2002) studied such problems and provided conditions ensuring the existence and uniqueness of solutions [19].

$$L\bar{X} = \frac{d}{dt}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \qquad F\bar{X} = \frac{d}{dt}\begin{bmatrix} X_2 \\ a + bX_1 + (f + gX_1)\frac{dW}{dt} \end{bmatrix}$$

$$F_H\bar{X} = \frac{d}{dt}\begin{bmatrix} X_2 \\ a + bX_1 + (f + gX_1)\frac{dW}{dt} \end{bmatrix} \tag{19}$$

This section summarizes the proof for the existence and uniqueness of the solution to a Stochastic Boundary Value Problem (SBVP) with fixed boundary conditions (Dirichlet conditions), and introduces the initial steps of the Finite Difference Method (FDM) for second-order nonlinear problems.

Introduction: Existence and Uniqueness for Linear SBVPs

For the system of stochastic differential equations with initial conditions:

$$\begin{cases} L\bar{X} = F\bar{X} \\ \bar{X}_1(0) = 0, \alpha \in \mathbb{R} \\ \bar{X}_2(0) = \alpha \end{cases}$$

it is known that this problem possesses a unique continuous solution [15]. Assuming two solutions $\tilde{\bar{X}}$and $\hat{\bar{X}}$corresponding to initial conditions $\tilde{\alpha}$and $\hat{\alpha}$(where the Wiener process $W(t)$is the same):

$$\begin{cases} L\tilde{\bar{X}} = F(\tilde{\bar{X}}) \\ \tilde{\bar{X}}_1(0) = 0 \\ \tilde{\bar{X}}_2(0) = \tilde{\alpha} \end{cases} \text{and} \begin{cases} L\hat{\bar{X}} = F(\hat{\bar{X}}) \\ \hat{\bar{X}}_1(0) = 0 \\ \hat{\bar{X}}_2(0) = \hat{\alpha} \end{cases}$$

If $Z = \bar{X} - \hat{\bar{X}}$, then Zsatisfies the homogeneous problem:

$$\begin{cases} LZ = F_H Z \\ Z_1(0) = \tilde{\alpha} - \hat{\alpha} \end{cases}$$

Theorem 3.1.2: Existence and Uniqueness for Two-Point Boundary Conditions (Dirichlet)

The problem $L\bar{X} = F\bar{X}$with boundary conditions $X_1(1) = X_1(0) = 0$has a unique solution that is pathwise continuous.

Proof (Summary):

Assuming $\tilde{\alpha} = 0$and $\hat{\alpha} = \beta \neq 0$. A solution is constructed as a linear combination $\bar{X}(t) = \lambda\bar{\tilde{\bar{X}}}(t) + (1-\lambda)\hat{\bar{\hat{X}}}(t)$, where $\lambda = \frac{X_1(1)-\hat{X}_1(1)}{\bar{X}_1(1)-\hat{X}_1(1)}$. This combination satisfies the equation with initial conditions $\bar{X}_1(0) = 0$and $\bar{X}_2(0) = (1-\lambda)\beta = \sigma$, which proves existence. Uniqueness is established by showing that any other solution $\dot{\bar{X}}$must coincide with $\bar{X}(t)$due to the uniqueness of the Initial Value Problem (IVP).

Introduction of the Finite Difference Method (FDM) for the Linear System

Subsequently, the Finite Difference Method is introduced to solve system (16). First, the interval $[0,1]$is discretized into Nequally spaced subintervals, with a step size of $h = 1/N$. The difference equations take the form:

$$X_{i+1} - 2X_i + X_{i-1} = h^2(a_i + b_iX_i) + h\sqrt{h}\eta_i(f_i + g_iX_i), i = 1, \ldots, N+1 \tag{20}$$

subject to the boundary conditions (5.20). This system can be solved iteratively, and it is also solvable using methods for **tridiagonal systems**.

3.1.3. Second-Order Nonlinear Boundary Value Problems

In this section, the Finite Difference Method is extended to approximate the solution of second-order nonlinear stochastic boundary value problems. The two-point stochastic boundary value problem considered on the interval [0,1]is:

$$\begin{cases} X'' = f(X)\dot{W} + g(X, X') \\ X(0) = \xi \\ X(1) = \eta \end{cases} \tag{21}$$

where $\dot{W}$is White Noise, $\xi$and $\eta$are random variables, and $f, g$are continuous real-valued functions. As noted in the first section, Walsh and Lu studied this problem in 2002 [19]. They established the conditions under which this problem has a unique solution (which are assumed to hold here). Problem (21) is a concise representation of the following integral equation:

This problem can equivalently be written in integral form as:

$$\begin{cases} X_t = \xi + K_t + \int_0^t \int_0^s f(X_s) o dW_s\, ds + \int_0^t \int_0^s g(X_u, X'u) o du ds \\ X_1 = \eta \end{cases} \tag{22}$$

where $W_t$is a standard Brownian motion, $\circ\, dW_s$denotes the Stratonovich integral, and $K$is an unknown initial value. Setting $Y = X'$, the system becomes:

$$\begin{cases} X_t = \xi + \int_0^t Y_s ds \\ Y_t = K_t + \int_0^t f(X_s) o dw_s ds + \int_0^t g(X_s, Y_s) o ds \qquad 0 \le t \le 1 \\ X_1 = \eta \end{cases} \tag{23}$$

The finite difference method is applied by central differences for the first and second derivatives. Dividing the interval $[0,1]$into a partition $\vartheta: 0 = t_1 < t_2 < \cdots < t_{N+1} = 1$, the two-step central difference scheme is:

$$\frac{x_{i+1} - 2x_i + x_{i-1}}{h_i^2} = f(x_i)\Delta W_i + g(x_i, \frac{x_{i+1} - x_{i-1}}{2h_i}), \tag{24}$$

where $x_i \approx X(t_i)$and $\Delta W_i$is a normal random variable with mean zero and variance h_i.

Due to the nonlinearity of $f$and g, this leads to a nonlinear algebraic system. Including the boundary conditions, we have:

$$\begin{cases} \frac{x_3 - 2x_2 + x_1}{h_3^2} - f(x_2)\Delta W(2) - g\left(x_2, \frac{x_3 - x_1}{2h_2}\right) = \vec{0} \\ \frac{x_4 - 2x_3 + x_2}{h_3^2} - f(x_3)\Delta W(3) - g\left(x_3, \frac{x_4 - x_2}{2h_3}\right) = \vec{0} \\ \vdots \\ \frac{x_N - 2x_{N-1} + 2x_{N-2}}{h_{N-1}^2} - f(x_{N-1})\Delta W(N-1) - g\left(x_{N-1}, \frac{x_N - x_{N-2}}{2h_{N-1}}\right) = \vec{0}, \\ \frac{x_{N+1} - 2x_N + x_{N-1}}{h_N^2} - f(x_N)\Delta W(N-1) - g\left(x_N, \frac{x_{N+1} - x_{N-1}}{2h_N}\right) = \vec{0} \\ x_0 = \xi, \\ x_1 = \eta \end{cases} \tag{25}$$

As previously mentioned, standard numerical methods, such as Newton's method or Broyden's method, can be used to solve this nonlinear system.

Remark: Second-order boundary value problems can be transformed into first-order systems by introducing $X_1 = X, X_2 = X'$. For problem (21), this gives:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}' = \begin{bmatrix} 0 \\ f(X_1) \end{bmatrix} \dot{W} + \begin{bmatrix} X_2 \\ g(X_1, X_2) \end{bmatrix},$$

which is a first-order system that can be solved using the methods developed for first-order BVPs.

3.2. Linear Boundary Value Problems with Multi-Point Boundary Conditions

3.2.1. Problem Definition

In this section, we study problems of the following form:

$$L[X] = dW \tag{26}$$

where the differential operator $L$is defined as:

$$L = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0, D := \frac{d}{dt}$$

such that the coefficients $a_i$are continuous functions on the interval $[0,1]$. The problem (26) is also subject to the following multi-point boundary conditions:

$$\sum_{j=1}^{m} \alpha_{ij} X(t_j) = c_i, \tag{27}$$

where the points $t_j$are selected from the partition:

$$\pi: 0 \leq t_1 \leq \cdots \leq t_m \leq 1, m \leq n$$

Furthermore, we assume that the matrices α_{ij}in relation (27) are of full rank.

3.2.2. Reduction to a First-Order Stochastic System

The system of boundary value problems defined by (26) and (27) can be reduced to the following first-order stochastic system:

$$DY(t) + A(t)Y(t)dW = 0 \tag{28}$$

with the boundary constraints:

$$\sum_{j=1}^{m} \alpha_{ij} Y(t_j) = c_i, \quad 1 \leq i \leq n, \tag{29}$$

Here, $Y(t) = (Y_1(t), Y_2(t) \dots, Y_n(t))$is a vector, and its components are defined as $Y_i(t) = D^{n-i}X(t)$for $1 \leq i \leq n$.

3.2.3. Definition of the Matrix A(t)

The matrix $A(t)$is defined as:

$$B(t) = (W(t), 0, \dots, 0)$$

$$A(t) = \begin{bmatrix} a_{n-1}(t) & a_{n-2}(t) & \cdots & a_1(t) & a_0(t) \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{bmatrix} \tag{29}$$

3.2.4. Background and Related Work

Subsequently, we will develop difference schemes for the first-order system (28) with boundary conditions (29). Alabert and Ferrante [5] investigated this type of boundary value problem in 2003. They explored various properties of the solution to these problems, including Markov properties. It is notable that in [5], the term **"Functional Boundary Conditions"** was used for the boundary conditions (27).

3.2.5. Discretization of the Domain

To begin, we consider a partition of the interval [0,1]that includes the partition points π:

$$\vartheta: 0 = t_1 < t_2 < \cdots < t_N < t_{N+1}$$

For the system (28), we consider a single-step difference operator. This operator will be based on values corresponding to the subinterval $[t_i, t_{i+1}]$of the partition ϑ. A forward difference scheme can be used to approximate the derivative term. We denote the numerical solutions at each point of the partition $\vartheta$by x_i, which corresponds to the exact solution $X(t_i)$.

3.2.6. Numerical System and Boundary Conditions

In this context, we emphasize that all numerical solutions must satisfy the boundary conditions of the problem, namely:

$$\sum_{j=1}^{m} \alpha_{ij} x_j = c_i, 1 \leq i \leq n$$

Applying the above explanations to the boundary value problem (28) and (29) results in the following system of linear equations:

$$\begin{cases} \frac{x_{i+1}-x_{i-1}}{h_i} + A_i x_i = dW \\ \sum_{j=1}^{m} \alpha_{ij} u_j = c_i. \end{cases} \tag{30}$$

Another form of the above system, presented with more detail, is as follows:

$$\begin{cases} \frac{x_3-x_1}{h_2} + A_2 x_2 - \Delta W(2) = 0, \\ \frac{x_4-x_2}{h_3} + A_3 x_3 - \Delta W(3) = 0, \\ \vdots \\ \frac{x_{N+1}-x_{N-1}}{h_N} + A_N x_N - \Delta W(N) = 0, \\ \sum_{j=1}^{m} \alpha_{ij} u_j - c_i = 0. \end{cases} \tag{31}$$

3.2.7. The Matrix Closed Form

The system of linear equations (5.31) can be presented in a **closed matrix form** as:

$$\Gamma X_{\vartheta} = \kappa \tag{32}$$

Specifically:

$$\begin{bmatrix} M & I & & & & \\ & & M_2 & & & \\ & & \ddots & I & & \\ & & & & M_{N-1} & I \\ & & & & M_N & I \\ \cdots & & \cdots & \alpha_m & \cdots & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \vdots \\ x_N \\ x_{N+1} \end{bmatrix} = \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \\ \vdots \\ \kappa_N \\ \beta \end{bmatrix}$$

where $M_i$are $n \times n$matrices. Furthermore, the following relations hold for the components of the linear system (32):

$$M_i = h_i A_i - I, 1 \le i \le N+1$$

$$\kappa_i = \Delta W(i) h_i, 1 \le i \le N+1$$

As noted, system (5.32) is a linear system that can be easily solved using conventional numerical methods for linear systems.

4. Discussion

In this section, numerical results are presented to test and numerically verify the difference schemes introduced previously.

Example 4.1: Second-Order Two-Point Stochastic Boundary Value Problem

As the first example, we consider the following second-order two-point stochastic boundary value problem:

$$\begin{cases} X''(t) = c_1 dt + c_2 dt, \quad 0 \le t \le 1, \\ X(0) = 0, \\ X(1) = 0 \end{cases} \tag{33}$$

(Assuming the correct SDE form is $d^2X(t) = c_1\, dt + c_2\, dW(t)$)

Exact Solution:

The boundary value problem (5.33) has the following exact solution:

$$X(t) = c_1 \frac{t(1-t)}{2} + c_2(t-1)\int_0^t t' dW(t') + t\int_t^1 (t'-1) dW(t)$$

For this problem, $\{W(t), t \in [0,1]\}$is a one-dimensional Wiener process. In this example, $c_1 = c_2 = 1$is assumed.

Exact Expected Values:

The exact values for $E[X(t)]$and $E[X^2(t)]$are given by:

$$E[X(t)] = c_1 \frac{t(t-1)}{2}$$
$$E[X^2(t)] = \frac{(3c_1^2 + 4c_2^2)\, t^2(t-1)^2}{12}$$

4.2. Numerical Comparison:

Table 1.5 provides the numerical approximations for $E[X_1(1/2)]$and $E[X_1^2(1/2)]$using three different methods: the **Euler Shooting Method**, the **Heun Shooting Method**, and the **Finite Difference Method**. This comparison is provided to facilitate the assessment of the methods' performance. The problem is solved based on $10{,}000$independent realizations (trials).

Table 1: Approximations of $E[X_1(t)]$and $E[X_1^2(t)]$using the Euler Shooting, Heun Shooting, and Finite Difference methods.

Position t	$E[X]$**(Exact)**	$E[X^2]$**(Exact)**	$E[X]$**(Finite Difference)**	$E[X^2]$**(Finite Difference)**	$E[X]$**(Heun Shooting)**	$E[X^2]$**(Heun Shooting)**	$E[X]$**(Euler Shooting)**	$E[X^2]$**(Euler Shooting)**
0.0	-0.00000	0.0000	-0.0000	0.0000	-0.0000	0.0000	-0.00000	0.0000
0.2	-0.0800	0.0149	-0.0805	0.01497	-0.0800	0.01499	-0.0794	0.01465
0.4	-0.1200	0.0336	0.1201	0.03380	-0.1194	0.03338	-0.1190	0.03279
0.6	0.1200	0.0336	0.1193	0.03328	-0.1192	0.03346	-0.1190	0.03289
0.8	-0.0800	0.0149	0.0791	0.01472	-0.0793	0.01486	-0.0794	0.01465
1.0	-0.0000	0.0000	-0.0000	0.00000	-0.0000	0.00000	-0.0000	0.0000

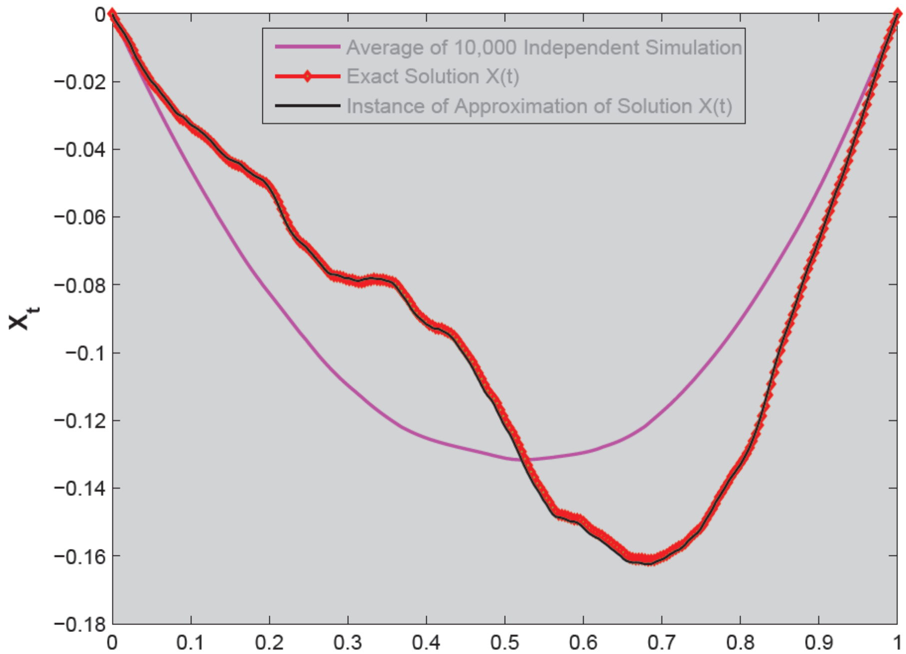

Figure 1: Illustrates the mean of 10,000numerical solutions obtained using the Finite Difference Method.

For this depiction, the number of partition points is assumed to be $N = 128$. In addition, one sample path of the numerical solution and one sample path of the exact solution are presented for visual comparison.

Example 4.2: Stochastic Boundary Value Problem with Integral Boundary Condition

As the second example, we consider the stochastic boundary value problem with the following integral boundary condition:

$$\begin{cases} dX(t) = dW(t), \qquad 0 \leq t \leq 1, \\ \qquad \int_0^1 X(t) = 0. \end{cases} \tag{34}$$

Exact Solution:

This boundary value problem has the following exact solution:

$$X(t) = -\int_0^1 W_t dt + W_t \qquad 0 \leq t \leq 1$$

where $\{W_t, 0 \leq t \leq 1\}$is a one-dimensional Wiener process.

Discretization of the Boundary Condition:

Initially, the integral in the boundary condition is approximated discretely at fixed points. In the implementations performed, five equal-spaced points in the interval [0,1]were used:

$$\int_0^1 X(t) = \sum_{i=1}^{5} h_i X(t_i).$$

Error Analysis:

Table 2.5 displays the numerical approximations for $E[\| X(t) - \bar{X}(t) \|]$and $E[\| X(t) - \bar{X}(t) \|^2]$, where the Finite Difference Method was used for approximation. Here, $\bar{X}(t)$is the approximate solution obtained from the Finite Difference Method, and $X(t)$is the exact solution of problem (5.34) at point t. The mean is calculated based on 10,000independent realizations.

Table 2: Approximations of $E[\| X(t) - \bar{X}(t) \|]$and $E[\| X(t) - \bar{X}(t) \|^2]$using the Finite Difference Method.

dt	$E[\mid X(t) - \bar{X}(t) \mid]$	$E[\mid X(t) - \bar{X}(t) \mid^2]$
0.327	0.2933	0.0327
0.0250	0.4652	0.0532
0.0886	0.7717	0.0886
0.0030	0.3191	0.1497

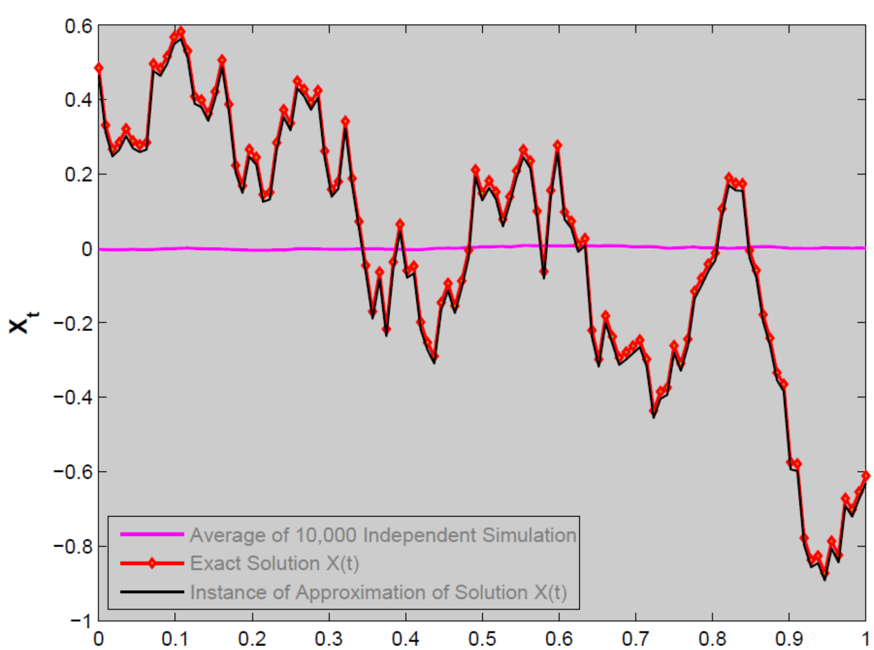

Figure 2: Illustrates the mean of 10,000numerical solution paths obtained using the Finite Difference Method.

Additionally, one path of the exact solution and one path of the numerical approximation are presented as examples.

5. Conclusions

In this paper, we introduced a family of Finite Difference Methods (FDM) for solving Stochastic Boundary Value Problems (SBVPs). Unlike the shooting methods mentioned previously in this paper, this family of difference schemes does not rely on solving an initial value problem to obtain the solution to the boundary value problem. Instead, it provides the solution directly at the designated grid points. Furthermore, compared to the aforementioned methods, the computations involved are simpler. Ultimately, after discretizing the problem domain and solving a resulting linear or nonlinear system of equations, we obtain an approximation of the boundary value problem's solution at every point of the partition. These factors constitute the main reasons why the use of this method has become widespread and popular among researchers. However, despite the demonstrated efficiency and accuracy of the methods discussed here, we have only utilized a fraction of this family's capabilities; specifically, we only presented the simple Finite Difference Method. The Dynamic Finite Difference Method (DFDM) could be the next step toward introducing more efficient schemes.

This is an approach where the necessary grid points for the finite difference method are determined dynamically based on the stability and behavioural characteristics of the solution paths for these stochastic problems.

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