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On Semi gs-Flat Modules

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ABSTRACT

Let R be a ring with identity. In this paper, we introduce and study the notion of semi gs-flat right R-modules as a proper generalization of flat modules. We provide many characterizations and explore various properties of semi gs-flat right R-modules.

1. Introduction

Throughout this article, let R be an associative ring with identity and all modules are assumed to be unitary. Soc(M) is the socle of a module M and $\mathrm{gs(M)} = \mathrm{Soc(M_R)Soc(R_R)}$. We use the standard notation $\mathrm{M}^* = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{M}, \mathbb{Q}/\mathbb{Z})$. Over the years, the literature contains a wide variety of generalizations of the concepts of flatness (for example see, [7], [8] and [10]). For more background on modules we refer to [1], [2], [3], [5], [6] and [11].

This article presents and examines semi gs-flat modules, which provides a proper generalization of the classical notion of flat modules. We define a right R-module M to be semi gs-flat whenever for any finitely generated free right R-module F and any cyclic submodule K of $\mathrm{gs}(F)$, every diagram:

$$\begin{tikzcd} F/K \arrow[r, "\delta"] \arrow[d, dashed, "\tau"'] & M \arrow[d, "\alpha"] & \\ B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

with exact sequence $B \xrightarrow{\beta} C \longrightarrow 0$ and C an injective right R-module, can be completed by an R-homomorphism $\tau: F/K \longrightarrow B$ to a commutative diagram. (i.e., $\beta\tau = \alpha\delta$). Some characterizations and properties of semi gs-flat modules are given. For examples, in Proposition 2.3, we prove that every submodule of semi gs-flat module is semi gs-flat. Moreover, in Proposition 2.4, we prove that a semi gs-flatness is an algebraic property. In Proposition 2.5, we show that if $\{M_i\}_{i\in I} \subseteq$ Mod-R, then $\bigoplus_{i\in I} M_i$ is semi gs-flat if and only if M_i is semi gs-flat, for all $i \in I$. In Corollary 2.7, we show that the direct limits of semi gs-flat right R-modules is semi gs-flat. In Theorem 2.10, we prove that every injective semi gs-flat right R-module is gs-flat. Finally, we characterize right IGSF-rings in terms of semi gs-flat right R-modules, for example, we prove that R is a right IGSF-ring if and only if every right R-module is semi gs-flat if and only if for any finitely generated free module F_R and a $K \leq^c \mathrm{gs}(F)$, the module F/K is semi gs-flat if and only if every injective right R-module is semi gs-flat if and only if every finitely generated right R-module can be embedded in a semi gs-flat module.

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2. Semi gs-flat modules

In this section, we introduce and study the concept of semi gs-flat modules as a proper generalization of flat modules.

Definition 2.1. A right R-module M is said to be semi gs-flat module if for any finitely generated free right R-module F and any cyclic submodule K of $\mathrm{gs}(F) = \mathrm{Soc}(F_R)\ \mathrm{Soc}(R_R)$, every diagram :

$$\begin{tikzcd} F/K \arrow[r, "\delta"] \arrow[d, dashed, "\tau"'] & M \arrow[d, "\alpha"] & \\ B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

with exact sequence $B \xrightarrow{\beta} C \longrightarrow 0$ and C an injective right R-module, can be completed by an R-homomorphism $\tau: F/K \longrightarrow B$ to a commutative diagram (i.e., $\beta\tau = \alpha\delta$).

Examples 2.2.

(1) Clearly, all gs-flat (and hence flat) right R-modules are semi gs-flat.
(2) $\mathbb{Z}_n$ is semi gs-flat $\mathbb{Z}$-module, for all $n \geq 2$, but it is not flat.

In semi gs-flatness, we get that all submodules of a semi gs-flat right R-module are semi gs-flat as in the following proposition.

Proposition 2.3. Every submodule of a semi gs-flat right R-module is semi gs-flat.

Proof. Let M be a semi gs-flat right R-module and N a submodule of M. Consider the following diagram:

$$\begin{tikzcd} F/K \arrow[r, "\delta"] & N \arrow[d, "\alpha"] & \\ B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

where F is a finitely generated free right R-module, any cyclic submodule K of $\mathrm{gs}(F)$ and the sequence $B \xrightarrow{\beta} C \longrightarrow 0$ exact with C is injective. By injectivity of C, there is a homomorphism $\lambda: M \longrightarrow C$ such that $\lambda i = \alpha$, where $i: N \longrightarrow M$ is the inclusion mapping. Thus, we have the next diagram:

$$\begin{tikzcd} F/K \arrow[r, "i\delta"] \arrow[d, dashed, "\rho"'] & M \arrow[d, "\lambda"] & \\ B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

Since M is semi gs-flat, there is $\rho \in \mathrm{Hom}_R(F/K, B)$ with $\beta\rho = \lambda i\delta$. Thus, $\beta\rho = \alpha\delta$ and hence N is a semi gs-flat right R-module. □

Proposition 2.4. Semi gs-flatness is an algebraic property.

Proof. Let M_1 and M_2 be two right R-modules with $M_1 \cong M_2$ and M_1 a semi gs-flat module. Thus, there is an R-isomorphism $\theta: M_1 \longrightarrow M_2$. Consider the following diagram:

$$\begin{tikzcd} F/K \arrow[r, "\alpha"] & M_2 \arrow[d, "\lambda"] & \\ B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

where F is a finitely generated free right R-module, K a cyclic submodule of $\operatorname{gs}(F)$ and the sequence $B \xrightarrow{\beta} C \longrightarrow 0$ exact with C is an injective right R-module. Thus, we have the following diagram:

$$\begin{tikzcd} F/K \arrow[r, "\theta^{-1}\alpha"] \arrow[d, dashed, "\tau"'] & M_1 \arrow[d, "\lambda\theta"] & \\ B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

By semi gs-flatness of M_1 (by hypothesis), there is a homomorphism $\tau: F/K \longrightarrow B$ such that $\beta\tau = \lambda\theta\theta^{-1}\alpha$ and hence $\beta\tau = \lambda\alpha$. Thus, M_2 is a semi gs-flat right R-module. □

The following theorem shows that the class of semi gs-flat is closed under direct sums. This result is an analogous of [4, Theorem 7].

Theorem 2.5. Let $\{M_i\}_{i\in I}$ be a family of right R-modules. Then $\bigoplus_{i\in I} M_i$ is semi gs-flat if and only if M_i is semi gs-flat, for all $i \in I$.

Proof. ($\Longrightarrow$). Suppose that $\bigoplus_{i\in I} M_i$ is a semi gs-flat right R-module. Clearly, for any $i \in I, M_i \cong K_i$, where $K_i \leq \bigoplus_{i\in I} M_i$. By Proposition 2.3, K_i is a semi gs-flat module, for all $i \in I$. By Proposition 2.4, M_i is a semi gs-flat right R-module, for all $i \in I$.
($\Longleftarrow$). Suppose that M_i is semi gs-flat, for all $i \in I$. Consider the following diagram:

$$\begin{tikzcd} F/K \arrow[r, "\alpha"] & M_1 \oplus M_2 \arrow[d, "\lambda"] & \\ B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

where F is a finitely generated free right R-module, K a cyclic submodule of $\operatorname{gs}(F)$, and the sequence $B \xrightarrow{\beta} C \longrightarrow 0$ exact with C is injective. Let $\tau_i: M_i \longrightarrow M_1 \oplus M_2$ and $\pi_i: M_1 \oplus M_2 \longrightarrow M_i$ ($i = 1,2$) be the injections and projections, respectively. Thus, we have the following diagram:

$$\begin{tikzcd} F/K \arrow[r, "\pi_i\alpha"] \arrow[d, dashed, "\rho_i"'] & M_i \arrow[d, "\lambda\tau_i"] & \\ B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

By the semi gs-flatness of M_i $(i = 1,2)$, there is a homomorphism $\rho_i: F/K \longrightarrow B$ such that $\beta\rho_i = \lambda\tau_i\pi_i\alpha$ $(i = 1,2)$. Put $\rho = \rho_1 + \rho_2$. It clear that ρ is a right R-homomorphism from F/K into B. Thus, $\beta\rho = \beta(\rho_1 + \rho_2) = \beta\rho_1 + \beta\rho_2 = \lambda\tau_1\pi_1\alpha + \lambda\tau_2\pi_2\alpha = \lambda(\tau_1\pi_1 + \tau_2\pi_2)\alpha = \lambda\alpha$. Hence $M_1 \oplus M_2$ is a semi gs-flat right R-module. Suppose that $M_1, M_2, \dots, M_n$ are semi gs-flat right R-modules. We will prove by induction that $\oplus_{i=1}^{n} M_i$ is semi gs-flat. If $n = 2$, then the statement is true as above. Assume that the statement is true when $k = n - 1$, that is $\oplus_{i=1}^{n-1} M_i$ is semi gs-flat. We will prove that $\oplus_{i=1}^{n} M_i$ is semi gs-flat. Since $\oplus_{i=1}^{n-1} M_i$ and M_n are semi gs-flat, it follows from above that $(\oplus_{i=1}^{n-1} M_i) \oplus M_n$ is semi gs-flat and hence $\oplus_{i=1}^{n} M_i$ is semi gs-flat. Thus, the direct sum of any finite number of semi gs-flat modules is semi gs-flat. Now, we will prove that $\oplus_{i \in I} M_i$ is semi gs-flat. Consider the following diagram:

$$\begin{tikzcd}
F'/K' \arrow[r, "\alpha'"] & \oplus_{i\in I} M_i \arrow[d, "\lambda'"] & \\
B' \arrow[r, "\beta'"] & C' \arrow[r] & 0
\end{tikzcd}$$

where F' is a finitely generated free right R-module, K' a cyclic submodule of $\mathrm{gs}(F')$ and β' is an epimorphism with C' is an injective module. Since F'/K' is a finitely generated, $\alpha'(F'/K') \leq \oplus_{j\in J} M_j$, with J is a finite index set with $J \subseteq I$. By above, $\oplus_{j\in J} M_j$ is a semi gs-flat module. Thus, we get the next diagram:

$$\begin{tikzcd}
F'/K' \arrow[r, "\pi\alpha'"] \arrow[d, dashed, "\varphi"'] & \oplus_{j\in J} M_j \arrow[d, "\lambda' i"] & \\
B' \arrow[r, "\beta'"] & C' \arrow[r] & 0
\end{tikzcd}$$

where $\pi: \oplus_{i\in I} M_i \longrightarrow \oplus_{j\in J} M_j$ is the projection mapping and $i: \oplus_{j\in J} M_j \longrightarrow \oplus_{i\in I} M_i$ is the inclusion mapping. By the semi gs-flatness of $\oplus_{j\in J} M_j$, there is a homomorphism $\varphi \in \mathrm{Hom}_R(F'/K', B')$ with $\beta'\varphi = \lambda' i \pi\alpha' = \lambda'\alpha'$. Thus, $\oplus_{i\in I} M_i$ is a semi gs-flat right R-module. □

Proposition 2.6. Let $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ be a pure exact sequence of right R-modules. If B is semi gs-flat, then A and C are semi gs-flat.

Proof. Since $\alpha: A \longrightarrow B$ is a monomorphism, we have that $A \cong \alpha(A)$. Since $\alpha(A)$ is a submodule of B and B is a semi gs-flat right R-module (by hypothesis), we have from Proposition 2.3 that $\alpha(A)$ is semi gs-flat. By Proposition 2.4, A is semi gs-flat. Now, we will prove that C is semi gs-flat. Consider the following diagram:

$$\begin{tikzcd}
F/K \arrow[r, "\varphi"] & C \arrow[d, "\lambda"] & \\
X \arrow[r, "\rho"] & Y \arrow[r] & 0
\end{tikzcd}$$

where F is a finitely generated free right R-module, K a cyclic submodule of $\mathrm{gs}(F)$ and ρ is an epimorphism with Y is injective. Thus, we have the following diagram:

$$\begin{tikzcd}
& & & F/K \arrow[dl, dashed, "\psi"'] \arrow[d, "\varphi"] & \\
0 \arrow[r] & A \arrow[r, "\alpha"'] & B \arrow[r, "\beta"'] & C \arrow[r] & 0
\end{tikzcd}$$

Since the exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is pure and F/K is a finitely presented, there is a homomorphism $\psi: F/K \longrightarrow B$ such that $\beta\psi = \varphi$. Thus, we have the following diagram:

$$\begin{tikzcd}
F/K \arrow[r, "\psi"] \arrow[d, dashed, "\tau"'] & B \arrow[d, "\lambda\beta"] & \\
X \arrow[r, "\rho"'] & Y \arrow[r] & 0
\end{tikzcd}$$

By semi gs-flatness of B, there is a homomorphism $\tau: F/K \longrightarrow X$ such that $\rho\tau = \lambda\beta\psi = \lambda\varphi$. Thus, C is a semi gs-flat right R-module. □

Corollary 2.7. Direct limits of semi gs-flat right R-modules is semi gs-flat.

Proof. Let $\{M_i\}_{i\in I}$ be a direct system of semi gs-flat right R-modules. By Theorem 2.5, $\bigoplus_{i\in I} M_i$ is semi gs-flat. By [11, 33.9, p. 280], there is a pure exact sequence $0 \longrightarrow \mathrm{Ker}(\alpha) \xrightarrow{i} \bigoplus_{i\in I} M_i \xrightarrow{\alpha} \varinjlim M_i \longrightarrow 0$. By Proposition 2.6, $\varinjlim M_i$ is a semi gs-flat right R-module. □

Proposition 2.8. Let $M = F/K$, where F is a finitely generated free right R-module and K a cyclic submodule K of $\mathrm{gs}(F)$. Then M is a semi gs-flat right R-module if and only if M is a submodule of a free module.

Proof. Let $M = F/K$, where F is a finitely generated free right R-module and K a cyclic submodule of $\mathrm{gs}(F)$.

($\Rightarrow$). Suppose that M is a semi gs-flat right R-module. Let $E(M)$ be the injective envelope of M, thus there is a monomorphism $\alpha: M \longrightarrow E(M)$. By [9, Proposition 2.5, p.10], there is an epimorphism $\beta: F_1 \longrightarrow E(M)$, where F_1 is a free right R-module. Thus, we have the following diagram:

$$\begin{tikzcd}
M \arrow[r, "I_M"] \arrow[d, dashed, "\tau"'] & M \arrow[d, "\alpha"] & \\
F_1 \arrow[r, "\beta"'] & E(M) \arrow[r] & 0
\end{tikzcd}$$

By semi gs-flatness of M, there is a homomorphism $\tau: M \longrightarrow F_1$ such that $\beta\tau = \alpha I_M = \alpha$. Since α is a monomorphism, τ is a monomorphism and hence M is a submodule of a free right R-module F_1.

($\Longleftarrow$). Suppose that M is a submodule of a free right R-module, say F_1. Since F_1 is a gs-flat module, F_1 is a semi gs-flat module. By Proposition 2.3, M is a semi gs-flat module. □

The following proposition introduces an equivalent statement of semi gs-flat modules.

Proposition 2.9. A right R-module M is semi gs-flat if and only if every finitely generated submodule of M can be embedded in a semi gs-flat module.

Proof. ($\Longrightarrow$)Suppose that M is a semi gs-flat right R-module and N a finitely generated submodule of M. Since the inclusion mapping $i: N \longrightarrow M$ is a monomorphism, it follows that N is embedded in a semi gs-flat module M.

($\Longleftarrow$) Suppose that every finitely generated submodule of M can be embedded in a semi gs-flat module. Let $\{N_i\}_{i\in I}$ be the class of all finitely generated submodules of M. By hypothesis, for each $i \in I$, there is a monomorphism $\alpha_{ij}: N_i \longrightarrow M_j$, where M_j is a semi gs-flat right R-module, for all $j \in I$. Thus, $N_i \cong \alpha_i(N_i)$. Since $\alpha_i(N_i)$ is a submodule of M_j and M_j is a semi gs-flat module, it follows from Proposition 2.3 that $\alpha_i(N_i)$ is a semi gs-flat. By Proposition 2.4, N_i is semi gs-flat, for all $i \in I$. By Corollary 2.7, $\varinjlim N_i$ is a semi gs-flat right R-module. By [11, 24.7, p. 200], $\varinjlim N_i = M$ and hence M is semi gs-flat. □

In the following theorem, we consider a condition under which the converse of Proposition 2.2 is true.

Theorem 2.10. Every injective semi gs-flat right R-module is gs-flat.

Proof. Let M be an injective semi gs-flat right R-module. Consider the following diagram:

$$\begin{tikzcd} & F/K \arrow[d, "\beta"] & \\ B \arrow[r, "\alpha"] & M \arrow[r] & 0 \end{tikzcd}$$

where F is a finitely generated free right R-module, K is acyclic submodule of $gs(F)$ and α is an epimorphism. Thus, we have the following diagram:

$$\begin{tikzcd} F/K \arrow[r, "\beta"] \arrow[d, dashed, "\tau"'] & M \arrow[d, "I_M"] & \\ B \arrow[r, "\alpha"] & M \arrow[r] & 0 \end{tikzcd}$$

Since M is injective semi gs-flat module (by hypothesis), there is a homomorphism $\tau: F/K \longrightarrow B$ such that $\alpha\tau = I_M\beta = \beta$. Thus, F/K is projective with respect to any epimorphism $\alpha: B \longrightarrow M$ and hence M is a gs-flat module. □

Corollary 2.11. Let M be a semi gs-flat right R-module. If every finitely generated submodule of M is injective, then M is gs-flat.

Proof. Suppose that M is a semi gs-flat module and every finitely generated submodule of M is injective. By Proposition 2.3, every finitely generated submodule of M is injective semi gs-flat. By Theorem 2.10, every finitely generated submodule of M is gs-flat and hence M is gs-flat. □

In the next theorems, we will characterize right IGSF-rings in terms of semi gs-flat right R-modules.

Theorem 2.12. The following statements are equivalent for a ring R.

(1) R is a right IGSF-ring.

(2) Every right R-module is semi gs -flat.

(3) For any finitely generated free right R-module F and any cyclic submodule K of $\mathrm{gs}(F)$, the module F/K is semi gs-flat.

(4) Every injective right R-module is semi gs-flat.

Proof. (1)⟹(2) Let M be a right R-module. Thus, there is a monomorphism $\alpha: M \longrightarrow E(M)$ and hence M is embedded in $E(M)$. Since R is a right IGSF-ring (by hypothesis), it follows that $E(M)$ is gs-flat. By Proposition 2.2, $E(M)$ is semi gs-flat. By Proposition 2.3, M is semi gs-flat.

(2)⟹(3). It is clear.

$(\mathbf{3}) \Longrightarrow (\mathbf{1})$. Let M be any injective right R-module. Consider the following diagram:

$$\begin{tikzcd} & F/K \arrow[d, "\beta"] & \\ B \arrow[r, "\alpha"] & M \arrow[r] & 0 \end{tikzcd}$$

where F is a finitely generated free right R-module, K a cyclic submodule of $\mathrm{gs}(F)$ and α is an epimorphism. Thus, we have the following diagram:

$$\begin{tikzcd} F/K \arrow[r, "I_{F/K}"] \arrow[d, dashed, "\tau"'] & F/K \arrow[d, "\beta"] & \\ B \arrow[r, "\alpha"'] & M \arrow[r] & 0 \end{tikzcd}$$

Since F is a finitely generated free right R-module and K a cyclic submodule of $\mathrm{gs}(F)$, it follows from hypothesis that F/K is semi gs-flat. Thus, there is a homomorphism $\tau: F/K \longrightarrow B$ such that $\alpha\tau = \beta I_{F/K} = \beta$. Thus, F/K is projective with respect to any epimorphism $\alpha: B \longrightarrow M$. Thus, M is a gs-flat right$R$-module and hence R is a right IGSF-ring.

$(\mathbf{1}) \Longrightarrow (\mathbf{4})$. It follows by Proposition 2.2.

$(\mathbf{4}) \Longrightarrow (\mathbf{1})$. Suppose that every injective right R-module is semi gs-flat. Let M be an injective right R-module. By hypothesis, M is semi gs-flat. By Theorem 2.10, M is gs-flat. Thus, R is a right IGSF-ring. □

Theorem 2.13. The following statements are equivalent for a ring R.

(1) R is a right IGSF-ring.

(2) Every right R-module can be embedded in a semi gs-flat module.

(3) Every finitely generated right R-module can be embedded in a semi gs-flat module.

Proof. (1)⟹(2) and **(2)⟹(3)** are clear.

(3)⟹(1). Let M be any injective right R-module. Let N be any finitely generated submodule of M. By hypothesis, N can be embedded in a semi gs-flat right R-module. By Proposition 2.9, M is semi gs-flat. Thus, M is an injective semi gs-flat right R-module and hence from Theorem 2.10 we have that M is gs-flat. Thus, R is a right IGSF-ring. □

Corollary 2.14. The following statements are equivalent for a ring R.

(1) R is a right IGSF-ring.

(2) Every finitely generated right R-module can be embedded in a gs-flat right R-module.

Proof. (1)⟹(2). Since R is a right IGSF-ring, every right R-module is embedded in a gs-flat module and hence every finitely generated right R-module is embedded in a gs-flat right R-module.

(2)⟹(1). Suppose that every finitely generated right R-module can be embedded in a gs-flat right R-module. Thus, every finitely generated right R-module can be embedded in a semi gs-flat right R-module. By Theorem 2.13, R is a right IGSF-ring. □

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