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Continuous Deformations in Nano Topological Spaces

Authors Names	**ABSTRACT**
Nadia Faiq Mohammed Publication date: 2 /9 /2026 ***Keywords:*** Nano space, nano continuous, nano homeomorphism, nano continuous deformation**.**	Continuous deformations theory is an exciting and very rapidly expanding research that arose from algebraic topology. Regarding this, the existing work is concerned with studying an important special case of continuous deformations, so-called nano continuous deformations, which are a new class of transformations designed for nano spaces. Building on these deformations, we present the concept of nano homotopy equivalence. It is a framework that classifies nano spaces according to the existence of nano continuous deformations between them. Consequently, the most significant results have been provided concerning these deformations.

1. Introduction

One of the crucial aspects that has arisen from algebraic topology is continuous deformation. It means the process of smoothly transforming one shape or object into another without any discontinuities. This concept is considered critically important to study essential attributes of spaces; for further details, see [1, 2]. On the other side, extensive topologists are continually creating new topological spaces. In 1968, Chang [3] presented the notion of fuzzy topological spaces. Furthermore, in 2011, Shabir and Naz [4] were the first to address soft topological spaces, which have since garnered the attention of the research team, as seen in references [5, 6]. The theoretical framework of nano topology was first addressed by Thivagar and Richard [7] in 2013, which will be considered in this work. It expands upon ordinary topological concepts by enhancing the notion of open sets. Those spaces frequently result in the study of extended rough sets and their applications in logic and information systems and are significantly used by the group of experts; see [8]- [13].

More significantly, in this research, the connection between continuous deformation and nano topological spaces makes new research prospects available, providing enhanced invariants and more techniques for utilising them, particularly in cases where classical topological tools are ineffectual. The study of nano continuous deformation enriches the classical theory, facilitating the investigation of new varieties of deformation and connectivity in the framework of generalised topological frameworks.

2. Preliminaries

Definition (2.1) [14]: Suppose that $(\mathcal{M}, \mathfrak{N})$ is a space of approximation, where $\mathcal{M}$ is the universe together with $\mathfrak{N}$ is a relation an of indiscernibility upon $\mathcal{M}$. Then the approximation of $\mathfrak{L}_{\mathfrak{N}}$ - lower as well $\mathfrak{U}_{\mathfrak{N}}$ upper of a $\mathfrak{M} \subseteq \mathcal{M}$ can be expressed as follows.

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$$\mathfrak{L}_{\mathfrak{N}}(\mathfrak{M}) = \bigcup_{m\in\mathfrak{M}}\{\mathfrak{N}(m): \mathfrak{N}(m) \subseteq \mathfrak{M}\}$$

$$\mathfrak{U}_{\mathfrak{N}}(\mathfrak{M}) = \cup_{m\in\mathfrak{M}} \{\mathfrak{N}(m): \mathfrak{N}(m) \cap \mathfrak{M} \neq \emptyset\}.$$

where $\mathfrak{N}(m)$ is the class of equivalence containing an element $m \in \mathfrak{M}$. The boundary area of $\mathfrak{M}$ is illustrated by $\mathfrak{B}_{\mathfrak{N}}(\mathfrak{M})$ and stated as $\mathfrak{B}_{\mathfrak{N}}(\mathfrak{M}) = \mathfrak{T}_{\mathfrak{N}}(\mathfrak{M}) - \mathfrak{U}_{\mathfrak{N}}(\mathfrak{S})$.

Definition (2.2) [15]: Assuming that $(\mathcal{M}, \mathfrak{N})$ is a space of approximation with $\mathfrak{M} \subseteq \mathcal{M}$. Then, the topology $\tau_{\mathfrak{N}}(\mathfrak{M}) = \{\mathcal{M}, \emptyset, \mathfrak{L}_{\mathfrak{N}}(\mathfrak{M}), \mathfrak{U}_{\mathfrak{N}}(\mathfrak{M}), \mathfrak{B}_{\mathfrak{N}}(\mathfrak{M})\}$ over $\mathcal{M}$ is considered as nano in context with $\mathfrak{M}$. The space $(\mathcal{M}, \tau_{\mathfrak{N}}(\mathfrak{M}))$ is the nano topology and the components from $\tau_{\mathfrak{N}}(\mathfrak{M})$ are known nano open.

Example (2.3) [15]: Presuming that $\mathcal{M} = \{u, v, w, r\}$ and $\mathfrak{M} = \{w, r\}$. Suppose $\mathcal{M}/_{\mathfrak{N}} = \{\{v\}, \{r\}, \{u, w\}\}$. Then $\tau_{\mathfrak{N}}(\mathfrak{M}) = \{\mathcal{M}, \emptyset, \{r\}, \{u, w\}, \{u, w, r\}\}$.

Definition (2.4) [15]: Suppose that $(\mathcal{M}, \tau_{\mathfrak{N}}(\mathfrak{M}))$ and $(\mathcal{N}, \tau_{\mathfrak{N}}(\mathfrak{N}))$ are two nano topological spaces. The function $\mathcal{K}: \mathcal{M} \to \mathcal{N}$ is stated nano-continuous whether the image of inverse for all nano open of $\mathcal{N}$ is a nano open of $\mathcal{M}$.

Definition (2.5) [15]: Let the spaces $(\mathcal{M}, \tau_{\mathfrak{N}}(\mathfrak{M}))$ and $(\mathcal{N}, \tau_{\mathfrak{N}}(\mathfrak{N}))$ be nano topology. The function $\mathcal{K}: \mathcal{M} \to \mathcal{N}$ is claimed nano homeomorphism whether its injective, surjective, nano continuous and $\mathcal{K}^{-1}$ nano continuous.

3. Main Results

In this section, the nano continuous deformations are explained along with the corresponding topological results that can be inferred from them.

Definition (3.1): Suppose $(\mathcal{M}, \tau_{\mathfrak{N}}(\mathfrak{M}))$ and $(\mathcal{N}, \tau_{\mathfrak{N}}(\mathfrak{N}))$ are nano topological spaces and the functions $\mathcal{K}, \mathcal{L}: \mathcal{M} \to \mathcal{N}$ are nano continuous. A nano continuous deformation between $\mathcal{K}$ and $\mathcal{L}$, denote by $\mathcal{K} \simeq_{nano} \mathcal{L}$, is a nano homotopy -that is to say, there exist a nano continuous

$$\mathcal{H}: \mathcal{M} \times I \to \mathcal{N}$$

Such that:

$$\mathcal{H}(u, 0) = \mathcal{K}(u) \text{ and } \mathcal{H}(u, 1) = \mathcal{L}(u), \qquad \text{for all } u \in \mathcal{M},$$

This means $\mathcal{K}$ can be nano continuously deformed into $\mathcal{L}$ through the function $\mathcal{H}: \mathcal{M} \times \mathrm{I} \to \mathcal{N}$ which actually being a family of functions $\{\mathcal{K}_t: \mathcal{M} \to \mathcal{N}\}$, for $t \in [0,1]$ where $\mathcal{K}_t = \mathcal{H}(u, \mathrm{t})$.

If $\mathcal{K}$ is a nano homotopic to a constant, then it is called nano null- homotopic.

Example (3.2): Consider $\mathcal{M} = \{u, v\}$ whenever $\mathcal{M}/_{\mathfrak{R}} = \{\{u\},\{v\}\}$ and $\mathfrak{M} = \{u\}$.Then, $\tau_{\mathfrak{R}}(\mathfrak{M}) = \{\mathcal{M}, \emptyset, \{u\}\}$. Assume that $\mathcal{N} = \{1,2,3\}$ along with $\mathcal{N}/_{\mathfrak{R}} = \{\{1\},\{2,3\}\}$ and $\mathfrak{N} = \{1\}$. So, $\tau_{\mathfrak{R}}(\mathfrak{N}) = \{\mathcal{N}, \emptyset, \{1\}\}$. Let $\mathcal{K}$ and $\mathcal{L}$ be nano continuous functions from $\mathcal{M}$ to $\mathcal{N}$ are given as

$$\mathcal{K}(u) = 1, \mathcal{K}(v) = 3,$$

and

$$\mathcal{L}(u) = 1, \mathcal{L}(v) = 2.$$

Let us construct $\mathcal{H}: \mathcal{M} \times \mathrm{I} \rightarrow \mathcal{N}$ as follwes:

$$\mathcal{H}(u, 0) = \mathcal{K}(u) \text{ and } \mathcal{H}(u, 1) = \mathcal{L}(u)$$

It has been noted that

$$\mathcal{H}(u, 0) = 1, \mathcal{H}(u, 1) = 1$$

$$\mathcal{H}(u, 0) = 3, \mathcal{H}(u, 1) = 2$$

Thus, $\mathcal{K}$ can be nano continuously deformed into $\mathcal{L}$ through the function $\mathcal{H}$.

Proposition (3.3): Considering maps $\mathcal{K}, \mathcal{K}^*: \mathcal{M} \rightarrow \mathcal{N}$ together with $\mathcal{L}, \mathcal{L}^*: \mathcal{N} \rightarrow \mathcal{D}$ are nano continuous. Assume composite maps $\mathcal{L} \circ \mathcal{K}$ and $\mathcal{L}^* \circ \mathcal{K}^*$. Then $\mathcal{L} \circ \mathcal{K} \simeq_{nano} \mathcal{L}^* \circ \mathcal{K}^*$ whether $\mathcal{K} \simeq_{nano} \mathcal{K}^*$ as well $\mathcal{L} \simeq_{nano} \mathcal{L}^*$,

Proof. Suppose $\mathcal{H}: \mathcal{M} \times \mathrm{I} \rightarrow \mathcal{N}$ is a nano homotopy between $\mathcal{K}$ and $\mathcal{K}^*$and suppose $\mathcal{G}: \mathcal{N} \times I \rightarrow \mathcal{D}$ be a nano homotopy between $\mathcal{L}$ and $\mathcal{L}^*$. Defining a map

$$\mathcal{E}: \mathcal{M} \times I \rightarrow \mathcal{D}$$

with

$$\mathcal{E}(u, t) = \mathcal{G}(\mathcal{H}(u, t), t).$$

Obviously, $\mathcal{E}$ is nano continuous. Furthermore

$$\mathcal{E}(u, 0) = \mathcal{G}(\mathcal{H}(u, 0), 0) = \mathcal{G}(\mathcal{K}(u), 0) = \mathcal{L}(\mathcal{K}(u))$$

and

$$\mathcal{E}(u, 1) = \mathcal{G}(\mathcal{H}(u, 1), 1) = \mathcal{G}((\mathcal{K}^*(u), 1) = \mathcal{L}^*(\mathcal{K}^*(u)).$$

As a result, $\mathcal{L} \circ \mathcal{K}$ is a nano continuously deformed into $\mathcal{L}^* \circ \mathcal{K}^*$ through the function $\mathcal{E}$.

Proposition (3.4): Suppose that $\mathcal{K}, \mathcal{L}: \mathcal{M} \rightarrow \mathcal{N}$ and $\phi: \mathcal{N} \rightarrow \mathcal{D}$ are nano continues functions so that $\mathcal{K}$ is nano continuously deformed into $\mathcal{L}$. Then $\phi \circ \mathcal{K}$ is nano continuously deformed into $\phi \circ \mathcal{L}$.

Proof. Assume that $\mathcal{H}: \mathcal{M} \times \mathrm{I} \rightarrow \mathcal{N}$ is a nano homotopy between $\mathcal{K}$ and $\mathcal{L}$. So, $\mathcal{H}$ is nano continuous and

$$\mathcal{H}(u, 0) = \mathcal{K}(u) \text{ and } \mathcal{H}(u, 1) = \mathcal{L}(u), \qquad \text{for all } u \in \mathcal{M}.$$

To show that $\phi \circ \mathcal{K}$ is nano continuously deformed into $\phi \circ \mathcal{L}$. Let us describe $\mathcal{G}: \mathcal{M} \times I \rightarrow \mathcal{D}$ such that

$$\mathcal{G}(u, t) = \phi\big(\mathcal{H}(u, t)\big) \qquad \forall u \in \mathcal{M}, t \in I$$

It is observed that $\mathcal{G}$ is nano continuous because of ϕ is nano continuous. Additionally, we have

$$\mathcal{G}(u, 0) = \phi\big(\mathcal{H}(u, 0)\big) = \phi\big(\mathcal{K}(u)\big) = \phi \circ \mathcal{K}(u)$$

and

$$\mathcal{G}(u, 1) = \phi\big(\mathcal{H}(u, 1)\big) = \phi\big(\mathcal{L}(u)\big) = \phi \circ \mathcal{L}(u)$$

Then $\phi \circ \mathcal{K} \simeq_{nano} \phi \circ \mathcal{L}$.
Now, take the assumption the two nano topological spaces $(\mathcal{M}, \tau_{\mathfrak{R}}(\mathfrak{M}))$ and $\big(\mathcal{N}, \tau_{\mathfrak{R}}(\mathfrak{N})\big)$, and consider the set of all nano continuous maps from $\mathcal{M}$ to $\mathcal{N}$, represented by $\mathrm{Map}(\mathcal{M}, \mathcal{N})$. Thus, an equivalency relation on $\mathrm{Map}(\mathcal{M}, \mathcal{N})$ is a nano homotopy.

Definition (3.5): Let $\mathcal{K}: \mathcal{M} \rightarrow \mathcal{N}$ be a nano continuous function. Then $\mathcal{K}$ is stated to be nano homotopy equivalence if there is a nano continuous map $\mathcal{L}: \mathcal{N} \rightarrow \mathcal{M}$ to enable

$$\mathcal{K} \circ \mathcal{L} \simeq_{nano} id_{\mathcal{N}} \text{ and } \mathcal{G} \circ \mathcal{L} \simeq_{nano} id_{\mathcal{M}}$$

whereas $id_{\mathcal{M}}$ and $id_{\mathcal{N}}$ are the identity maps on $\mathcal{M}$ and $\mathcal{N}$, respectively.

The map $\mathcal{L}$ in the above definition is said to be a nano homotopy inverse to $\mathcal{K}$. Moreover, it is obvious that every nano homeomorphism $\mathcal{K}: \mathcal{M} \rightarrow \mathcal{N}$ is a nano homotopy equivalence by considering $\mathcal{L} = \mathcal{K}^{-1}$.

It is well known that two spaces can be considered identical in nano topology when they are nano homeomorphic. Still, this concept is in general quite difficult to hold. Consequently, we will introduce the notion of nano homotopy equivalence to achieve a somewhat coarser classification.

Definition (3.6): Suppose $(\mathcal{M}, \tau_{\mathfrak{R}}(\mathfrak{M}))$ and $\big(\mathcal{N}, \tau_{\mathfrak{R}}(\mathfrak{N})\big)$ represent nano topological spaces. The two spaces are called nano homotopy equivalent (refers $\mathcal{M} \simeq_{nano} \mathcal{N}$) whenever there is a nano homotopy equivalence map $\mathcal{K}$ from $\mathcal{M}$ to $\mathcal{N}$.

Proposition (3.7): The relation of equivalency on nano topological spaces is classified as a nano homotopy equivalent.

Proof. Suppose that $(\mathcal{M}, \tau_{\mathfrak{N}}(\mathfrak{M}))$, $(\mathcal{N}, \tau_{\mathfrak{N}}(\mathfrak{N}))$ and $(\mathcal{O}, \tau_{\mathfrak{N}}(\mathfrak{O}))$ are nano topological spaces. To prove

a) Reflexivity ($\mathcal{M} \simeq_{nano} \mathcal{M}$). Let us define the identity map $id_{\mathcal{M}}: \mathcal{M} \to \mathcal{M}$ and simply it is a nano homeomorphism. Consequently, it is a nano homotopy equivalency.

b) Symmetric ($\mathcal{M} \simeq_{nano} \mathcal{N} \Rightarrow \mathcal{N} \simeq_{nano} \mathcal{M}$). Let $\mathcal{K}: \mathcal{M} \to \mathcal{N}$ be a nano homotopy equivalence with nano homotopy inverse $\mathcal{L}$. So, $\mathcal{L}: \mathcal{N} \to \mathcal{M}$ is a nano homotopy equivalence, with nano homotopy inverse $\mathcal{K}$.

c) Transitive ($\mathcal{M} \simeq_{nano} \mathcal{N}$ and $\mathcal{N} \simeq_{nano} \mathcal{O} \Rightarrow \mathcal{M} \simeq_{nano} \mathcal{O}$). Suppose $\mathcal{K}: \mathcal{M} \to \mathcal{N}$ is a nano homotopy equivalence with nano homotopy inverse $\mathcal{L}$, and $\mathcal{K}^*: \mathcal{N} \to \mathcal{O}$ is a nano homotopy equivalence, with nano homotopy inverse $\mathcal{L}^*$. By applying Proposition 3.3, we obtain $\mathcal{K}^* \circ \mathcal{K}: \mathcal{M} \to \mathcal{O}$ is a nano homotopy equivalence, with homotopy inverse $\mathcal{L} \circ \mathcal{L}^*$.

Proposition (3.8): Let $(\mathcal{M}, \tau_{\mathfrak{N}}(\mathfrak{M}))$ and $(\mathcal{N}, \tau_{\mathfrak{N}}(\mathfrak{N}))$ be nano homeomorphic spaces. Then they are nano homotopy equivalent.

Proof. Let us consider $(\mathcal{M}, \tau_{\mathfrak{N}}(\mathfrak{M}))$ is nano homeomorphic to $(\mathcal{N}, \tau_{\mathfrak{N}}(\mathfrak{N}))$. So, there exist nano homeomorphism

$$\mathcal{K}: \mathcal{M} \to \mathcal{N}$$

where $\mathcal{K}$ is bijective, nano continuous and $\mathcal{K}^{-1}$ nano continuous. Thus, we obtain

$$\mathcal{K}^{-1} \circ \mathcal{K} = id_{\mathcal{M}} \text{ and } \mathcal{K} \circ \mathcal{K}^{-1} = id_{\mathcal{N}}$$

Nevertheless, $id_{\mathcal{M}} \simeq_{nano} id_{\mathcal{M}}$ and $id_{\mathcal{N}} \simeq_{nano} id_{\mathcal{N}}$. So,

$$\mathcal{K}^{-1} \circ \mathcal{K} \simeq_{nano} id_{\mathcal{M}} \text{ and } \mathcal{K} \circ \mathcal{K}^{-1} \simeq_{nano} id_{\mathcal{N}}.$$

Consequently, $(\mathcal{M}, \tau_{\mathfrak{N}}(\mathfrak{M}))$ and $(\mathcal{N}, \tau_{\mathfrak{N}}(\mathfrak{N}))$ are nano homotopy equivalent.

4. Conclusions

Our interest in this work was to introduce a new kind of continuous deformation, which we refer to as nano continuous deformations. Alongside these deformations, we present the concept of nano homotopy equivalence, a framework that enables the classification of nano spaces into groups based on the presence of nano continuous deformations between them. Additionally, we provide several propositions that are associated with nano continuous deformations and nano homotopy equivalence to support these foundational ideas.

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