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G-Projective Modules

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Abstract. We introduce G-projective modules as a dualization of s-injective modules. Some new equivalent statements of t-semisimple modules are obtained by using the new concept.

1. Introduction

Throughout our paper every rings are associative with 1, and all modules are unitary right R-modules. By $J(M)$ (resp., $Z(M_R)$, $Z_2(M_R)$, $\mathrm{soc}(M_R)$, $S(M_R)$, $S = \mathrm{End}(M_R)$) we indicate the Jacobson radical (resp., right singular, right second singular, right socle, sum of all right nonsingular simple, endomorphism ring) of M, and we denote $J = J(R_R)$. Also we use ($A \leq^{max} M$, $A \leq^{\oplus} M$, $A \ll M$, $A \leq^{tes} M$, and $A \leq^{ess} M$) to express that A is (a maximal, a direct summand, a small, a t-essential, and an essential) submodule of M, respectively. For a right (resp., a left) ideal K, we denote $t_S(K) = \{\alpha \in S: \alpha K \subseteq Z_2(R_R)\}$ (resp., $t_R(K) = \{r \in R: Kr \subseteq Z_2(R_R)\}$.

In [1,2,3,4], we can recall the concepts as: " a module M is called t-semisimple (resp., t-extended) if $M = Z_2(M) \oplus M'$ where M' is a nonsingular semisimple (resp., a nonsingular extending) module; a module M is called s-N-injective if every homomorphism $f: K \longrightarrow M$ extends to N, and M is said to be strongly s-injective if it is s-N-injective for every $N \in$ Mod-R; a ring R is called right V-ring (resp., right SI-ring) if every simple (resp., singular) R-module is injective".

Sh. Asgari, A. Haghany and Y. Tolooei [1] introduced the concept t-semisimple module. A. R. Mehdi and A. S. Tayyah [8] gave some results on t-semisimple rings. In our paper, we obtain new equivalent statements of t-semisimple module by using new concept namely G-projective module. Also, we give examples to show that the G-projectivity and projectivity are distinct.

2. G-Projective Modules

Definition 2.1. Let M and N be R-modules. We call that the module M is G-N-projective if every R-epimorphism $g: N \longrightarrow A$ where A is the image of $N/Z_2(N)$ and every R-homomorphism $f: M \longrightarrow A$, there exists an R-homomorphism $h: M \longrightarrow N$ such that $gh = f$. We say M G-projective (respectively, strongly G-projective) if it is G-R-projective (respectively, G-N-projective $\forall N \in$ Mod-R).

Example 2.2.

(1) Every N-projective module is G-N-projective.
(2) Every $\mathbb{Z}$-module is G-$\mathbb{Z}_4$-projective, since $Z_2(\mathbb{Z}_4) = \mathbb{Z}_4$. But the $\mathbb{Z}$-module $\mathbb{Z}_4/<2>$ is not $\mathbb{Z}_4$-projective because $<2>$ is not direct summand of $\mathbb{Z}_4$.

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(3) The projective modules are strongly G-projective ,but this is not reversible. For instance, the ring $\mathbb{Z}_4$ is Z_2-torsion, thus $(\mathbb{Z}_4/<2>)_{\mathbb{Z}_4}$ is G-projective, and so $(\mathbb{Z}_4/<2>)_{\mathbb{Z}_4}$ is strongly G-projective by Proposition 2.3(6) below. But obviously $(\mathbb{Z}_4/<2>)_{\mathbb{Z}_4}$ is not projective.

(4) If a ring R is nonsingular or an R-module M is nonsingular, then M is projective iff M is strongly G-projective. For, since there is an R-epimorphism $g: R^{(I)} \longrightarrow M$ for some index set I, the map g splits because $Z_2(R^{(I)}) = 0$ whenever $Z_2(R_R) = 0$ or $g\left(Z_2(R^{(I)})\right) \subseteq Z_2(M) = 0$ and since M is strongly G-projective. Therefore $M \cong R^{(I)}/\ker(g) \cong A \subseteq^{\oplus} R^{(I)}$, and so M is projective.

(5) For all finitely generated R-module M. If R or M is nonsingular, then M is G-projective iff M is projective (by (4) and Proposition 2.3(6)).

(6) The $\mathbb{Z}$-module $\mathbb{Z}_4$ is not strongly G-projective, since $Z_2(\mathbb{Z}) = 0$ and by using (4) (in particular, $\mathbb{Z}_4$ is not G-projective by using (5)), but $\mathbb{Z}_4$ is G-$\mathbb{Z}_4$-projective by (2).

Proposition 2.3. The next statements hold:

(1) If N any R-module, then the class of all G-N-projective R-modules is closed under isomorphisms, direct sums and summands.

(2) If $N_1 \cong N_2$, then M is G-N_1-projective iff M is G-N_2-projective.

(3) If $A \hookrightarrow B$, then M is G-B/A-projective whenever M is G-B-projective.

(4) Let $\{N_i\}_{i=1}^{n}$ be the collection of R-modules. Then M is G-$\bigoplus_{i=1}^{n} N_i$-projective iff M is G-N_i-projective for all i.

(5) Let $\{N_i\}_{i\in I}$ be the collection of R-modules and M a finitely generated R-module. Then M is G-$\bigoplus_{i\in I} N_i$-projective iff M is G-N_i-projective for all $i \in I$.

(6) Every finitely generated G-projective R-module is strongly G-projective.

Proof. (1) and (2) understandably .

(3) Consider below diagram

$$\begin{array}{ccccccc} & & M & & & & \\ & & \downarrow f & & & & \\ B & \xrightarrow{\pi} & B/A & \xrightarrow{g} & K & \longrightarrow & 0 \end{array}$$

Where K is an image of $(B/A)/(Z_2(B/A))$. Define $\alpha: (B/A)/\pi(Z_2(B)) \longrightarrow (B/A)/Z_2(B/A)$ by $\alpha(x + \pi(Z_2(B)) = x + Z_2(B/A)$, so α is an R-epimorphism and $\alpha\pi' = \pi''$ where π' (respectively, π'') is the canonical from B/A to $(B/A)/\pi(Z_2(B))$ (respectively, B/A to $(B/A)/Z_2(B/A)$). Since K is an image $(B/A)/Z_2(B/A)$, we can define R -epimorpism $g': (B/A)/Z_2(B/A) \longrightarrow K$ by $g'(y + Z_2(B/A) = g(y)$ where $g = g'\pi''$, and so $g'\alpha\pi' = g'\pi'' = g$. Therefore, $g'\alpha\pi'\pi = g\pi$. Since $\pi'\pi(Z_2(B)) = 0$, K is an image of $B/Z_2(B)$. Now by hypothesis, there is $\beta: M \longrightarrow B$ with $\beta\pi g = f$, so f can be lifted to $\beta\pi$, and hence M is G-B/A-projective.

(4) Let $g: N_1 \oplus N_2 \longrightarrow (N_1 \oplus N_2)/K$ be the canonical map such that $Z_2(N_1 \oplus N_2) \subseteq K$. By using the similar argument of [5, 18.2(2)] and by (3), we can obtain this property.

(5) Using the similar argument of [5, 18.2(3)] and by (3).

(6) Let M is G-projective R-module. For all R-module N, there is an R-epimorphism $f: R^{(I)} \longrightarrow N$. By (5), M is G-$R^{(I)}$-projective. Thus (3) leads to M is G-N-projective, and hence M is strongly G-projective. ■

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From the concept of G-projectivity, several properties of t-semisimple modules will be given.

Theorem 2.4. The following statements are equivalent:

(1) Every R-module is G-A-projective.
(2) Every quotient of A is G-A-projective.
(3) A is t-semisimple R-module.

Proof. (2)⇒(3) Let $Z_2(A) \hookrightarrow B \hookrightarrow A$. Since A/B is G-A-projective, then the canonical map $\pi: A \longrightarrow A/B$ is split and hence $B \subseteq^{\oplus} A$. Therefore by [1, Theorem 2.3], A is t-semisimple.

(3)⇒(1) Since A is t-semisimple module, then $A = Z_2(A) \oplus A'$ where A' is a semisimple and nonsingular submodule. Now, let $g: A' \longrightarrow B$ any R-epimorphism and $f: M \longrightarrow B$ any R-homomorphism. Since $\ker(g) \subseteq^{\oplus} A'$, thus g splits, and there is an R-homomorphism $g': B \longrightarrow A'$ with $gg' = I_B$. Consider the R-homomorphism $h =: g'f: M \longrightarrow A'$. Thus $gh = gg'f = I_B f = f$. Therefore M is G-A'-projective. Obviously, M is G-$Z_2(A)$-projective, by Definition 2.1, so Proposition 2.3(4) implies that M is G-$(Z_2(A) \oplus A')$-projective, and hence M is G-A-projective. ■

Theorem 2.5. The following statements are equivalent:

(1) Every R-module is strongly G-projective.
(2) Every R-module is G-projective.
(3) Every simple R-module is G-projective.
(4) Every nonsingular R-module is quasi-continuous.
(5) R is a right t-semisimple ring.

Proof. (1)⇔(5) By Theorem 2.4 and [1, Theorem 3.2 (2)].

(2)⇔ (5) and (5) ⇒(3) By Theorem 2.4.

(3)⇒(5) Let $Z_2(R) \hookrightarrow I \subseteq^{max} R$. Since R/I is G-projective, then the canonical map $\pi: R \longrightarrow R/I$ is split and hence $I \subseteq^{\oplus} R$. So by [1, Theorem 3.8], R is t-semisimple.

(5)⇒(4) Since every injective R-module is quasi-continuous and by [1, Theorem 3.2(4)].

(4)⇒(5) Let K be a submodule of $R/Z_2(R_R)$, Since $R/Z_2(R_R)$ is nonsingular, then $K \oplus (R/Z_2(R_R))$ is quasi-continuous, thus by [6, Corollary 2.14], K is $R/Z_2(R_R)$-injective and hence $K \subseteq^{\oplus} (R/Z_2(R_R))$. This implies that $R/Z_2(R_R)$ is semisimple, so R is t-semisimple ring. ■

Proposition 2.6. A ring R is right t-semisimple iff the below conditions hold:

(1) $I = t_R(t_S(I))$, for every right ideal I contains $Z_2(R_R)$.
(2) If $A = a_1 R \oplus a_2 R$ where $a_2 R$ is nonsingular, then A is G-A-projective.

Proof. (⇒) By Theorem 2.5(1) and [1, Proposition 2.19].

(⇐) Let K be a nonsingular cyclic R-module, there exists epimprphism $f: R \longrightarrow K$. By hypothesis, $R \oplus K$ is G-$(R \oplus K)$-projective. Define $\alpha = f\pi_R: R \oplus K \longrightarrow K$ by $\alpha(a+b) = f(a)$ for all $a \in R, b \in K$ where π_R is the projection map . Since K is nonsingular; i.e, $Z_2(R_R) \subseteq \ker(f)$, and so $Z_2(R \oplus K) = Z_2(R_R) \oplus Z_2(K) \subseteq \ker(\alpha)$. Therefore there exists $\beta: R \oplus K \longrightarrow R \oplus K$ such that $\alpha\beta = \pi_K$ where π_K is the projection map, since $R \oplus K$ is G-$(R \oplus K)$-projective. Define $\delta: K \longrightarrow R$ by $\delta(b) = \pi_R \beta\eta(b)$ for all $b \in K$, where η is the injection map from K to $R \oplus K$. Therefore, $f\delta = f\pi_R\beta\eta = \alpha\beta\eta = \pi_K\eta = I_K$, where I_K is the identity map, and hence $\ker(f) \subseteq^{\oplus} R$. Thus $K \cong R/\ker(f) \cong B \subseteq^{\oplus} R$, and this leads to K is projective. Now, let N any t-closed in R. Thus [2,

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Proposition 2.6] leads to R/N is cyclic nonsingular R-module, so R/N is projective, that is $N \subseteq^{\oplus} R$. Therefore R is t-extended, so by using (1) and applying [1, Proposition 2.19] we conclude that R is right t-semisimple. ■

Recall that a ring R is said to be SP-ring (resp., mininjective) if $\mathrm{Soc}(R_R)$ is projective (resp., if $\mathrm{Soc}(R_R) \cap J = 0$ (see [7]).

Proposition 2.7. The next conditions are corresponding for any ring R:

(1) R is right t-semisimple universally mininjective.
(2) R is right t-semisimple SP-ring.
(3) R is right t-semisimple and $\mathrm{Soc}(R_R)$ is projective with respect to every exact sequence $A \xrightarrow{g} B \longrightarrow 0$ where the image of $A/Z_2(A)$ is proper in B.
(4) $R = Z_2(R_R) \oplus \mathrm{Soc}(R_R)$.
(5) $\mathrm{Soc}(R_R)$ is injective and $\mathrm{Soc}(R_R) \subseteq^{tess} R$.

Proof. (4)⇒(1)⇒(2)⇒(3) Understandable.

(3)⇒(2) By hypothesis and Theorem 2.5.

(2)⇒(4) Let K be a simple right ideal, then K is projective, and henceforth K is nonsingular. Therefore $\mathrm{Soc}(R_R) = S(R)$, and consequently $R = Z_2(R_R) \oplus \mathrm{Soc}(R_R)$.

(1)⇒(5) By [1, Proposition 1.1], $\mathrm{Soc}(R_R) \subseteq^{tess} R$. In other hand, the correspondence between (1) and (4) indicates that $\mathrm{Soc}(R_R)$ is nonsingular. Thus [1, Theorem 3.2] is leads to $\mathrm{Soc}(R_R)$ is injective.

(5)⇒(1) By supposition, $R = A \oplus \mathrm{Soc}(R_R)$, so $J = J(A)$. Thus $J \cap \mathrm{Soc}(R_R) = 0$ and this indicates that R is universally mininjective. Since $\mathrm{Soc}(R_R) \subseteq^{tess} R$, hence by [1, Proposition 1.1] A is Z_2-torsion, so $Z_2(R_R) = A \oplus \left(Z_2(R_R) \cap \mathrm{Soc}(R_R)\right)$. But $\mathrm{Soc}(R_R) = \left(Z_2(R_R) \cap \mathrm{Soc}(R_R)\right) \oplus B$ where B is injective. Thus

$R = A \oplus \mathrm{Soc}(R_R) = A \oplus \left(\left(Z_2(R_R) \cap \mathrm{Soc}(R_R)\right) \oplus B\right) = Z_2(R_R) \oplus B$ where B is injective. Hence R is t-semisimple by [1, Theorem 3.2]. ■

Proposition 2.8. The next conditions are corresponding:

(1) R is a right SI-ring, right V-ring and right noetherian.
(2) If every R-module is G-A-projective, then A is injective.
(3) Every t-semisimple R-module is injective.

Proof. (1)⇒(3) Let K be t-semisimple R-module, then $K = Z_2(K) \oplus S(K)$. By [3, Proposition 3.12], $S(K)$ is injective. However, $Z_2(K)$ is strongly s-injective by [4, Theorem 1(2), p. 29]. Therefore by [4, Proposition 3(4), p. 27], K is injective.

(3)⇒(1) By [3, Proposition 3.12], R is right V-ring right noetherian. Now, let K any R-module, $Z_2(K) \oplus S(K)$ is injective, and hence $Z_2(K)$ is injective. By [4, Proposition 3, p. 27], K is strongly s-injective, so [4, Theorem 1, p. 29] leads to the ring R is right SI-ring.

(2)⇔(3) By Theorem 2.4. ■

Proposition 2.9. If M is strongly G-projective R-module and $M/Z_2(M)$ has projective cover, then $M = P \oplus A$, where P is projective and A is Z_2-torsion.

Proof. If $Z_2(M) \neq M$, and consider the canonical map $\pi_1: M \longrightarrow M/Z_2(M)$ and the projective cover $\pi_2: K \longrightarrow M/Z_2(M)$ with $\ker(\pi_2) \ll K$ (see [5, Definition 19.4]). By hypothesis, we can find $h: M \longrightarrow K$ satisfying $\pi_1 = \pi_2 h$. Let $x \in K$,

$\pi_2(x) = m + Z_2(M) = \pi_1(m) = \pi_2 h(m)$, for some $m \in M$, so $\pi_2(x - h(m)) = 0$, that is $x - h(m) \in \ker(\pi_2)$, but $x = x - h(m) + h(m) \in \ker(\pi_2) + h(M)$, this means that $K = \ker(\pi_2) + h(M)$. Since $\ker(\pi_2) \ll K$, $K = h(M)$, so $M/\ker(h)$ is projective, and $\ker(h) \subseteq^{\oplus} M$. Since $\pi_1(\ker(h)) = \pi_2(h(\ker(h))) = 0$, so $\ker(h) \subseteq Z_2(M)$. Therefore $M = P \oplus \ker(h)$ with A is projective and $\ker(h)$ is Z_2-torsion.

Remarks 2.10.

(1) The reverse of above proposition is not correct, since the $\mathbb{Z}$-module $\mathbb{Z}_4 = 0 \oplus \mathbb{Z}_4$ with $Z_2(\mathbb{Z}_4) = \mathbb{Z}_4$, but $\mathbb{Z}_4$ is not strongly G-projective by Example 2.2(6).

(2) Consider M is strongly G-projective and $M/Z_2(M)$ has projective cover, then the following hold:

(i) If $Z_2(M) \ll M$, then M is projective.

(ii) $P \subseteq^{tess} M$ for some projective module P (by Proposition 2.9 and [1, Proposition 1.1]).

الخلاصة

نحن قدمنا المقاسات الأسقاطية من النمط G كمفهوم رديف للمقاسات الأغمارية من النمط s. بعض العبارات المكافئة الجديدة للمقاسات شبه البسيطة من النمط t استنتجت بواسطة استخدام هذا المفهوم الجديد.

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