

Semi-Hereditary Ring Related Other Rings Have Ideals is Principles and Projective

Authors Names	ABSTRACT
<i>Fatima Adnan Owaid^a</i> <i>Majid Mohammed Abed^{b,*}</i> Publication date: 30 / 6 /2026 Keywords: Regular ring, S-H-ring, Ore ring, Q-rings, Skew field.	In this article, we study many relationships between semi-hereditary ring (shortly S-H-ring) and other rings which have some properties of an ideal of the ring R. We prove that if R is semi-simple with Ore property, this leads the ring R is S-H-ring. On the other hand, Q-ring with some properties of L-F-group imply to S-H-ring property. Finally, some new results and more definitions with important examples have been presented.

1. Introduction

The scientist Shizuo Endo was one of the first scientists to presented a new study on a special type of ring named Principle-projective ring (shortly P.Pr. ring). Where in [1], he introduces general notice about the ring, a commutative ring is P.pr. if and only if the totally quotient ring (shortly Q-ring) N of $\mathcal{R}$ is a regular and the Q-ring A_x of $\mathcal{R}$, respect to every maximal ideal x of $\mathcal{R}$ is integral domain. In [2], Huh, Kim and Lee achieved generalization of p.pr. ring. A normal ring is a p.pr. ring iff it is right p.pr. ring. In [3], M.M. Abed, Kareem, F.F. presented a generalization about the types of ideals in algebra and studied it on an advanced type of ring. In [4], definition of p.pr. ring such that the ring is called P.Pr. if $\text{ann}(N) = \langle e \rangle$ such that $\emptyset \neq N \subset \mathcal{R}$, $e = e^2$.

2. Preliminaries

In this part, we present more concept which we use it in this paper later. Therefore, we should start with the following definition.

Definition 1. [5] We named $\mathcal{R}$ is a regular ring if for all $x \in \mathcal{R}, y \in \mathcal{R}$, so $xyx = x$.

Definition 2. [6] An $\mathcal{R}$ -module M named singular if $Z(M) = M$, also it is nonsingular when $Z(M) = 0$.

Definition 3. [7] An $\mathcal{R}$ -module M is named cyclic if it $M = \langle x \rangle, x \in M$.

Definition 4. [8] A module M is named hollow if each proper submodule $N \ni N \hookrightarrow M$, and $\ll M$.

Definition 5. [9] Any module M is named faithful if and only if $\forall r \in \mathcal{R} [rM = 0 \rightarrow r = 0]$.

Definition 6. [10] If $\emptyset \neq X \subseteq \mathcal{R}$, X is annihilator it to be the set of elements $m \in M \ni xm = 0 \forall x \in X$.

Definition 7. [11] Any ring $\mathcal{R}$ is semi simple if it is a direct sum of minimal simple ideal.

Definition 8. [12] A module M is named Artinian if it satisfies the descending chain condition (DCC) on submodule of M , and it is Noetherian if satisfies the ascending chain condition (ACC). On the other hand, M is simple if $M \neq 0$ and have only submodules $\{0\}$ and M .

3. The Main Results

In this section, we study and give some properties of the relation between p.pr. ring and S-H-ring. Now we define S-H-ring by the following:

Definition 1. [13] $\mathcal{R}$ is S-H-ring iff $\mathcal{R}/K$ is regular and $\mathcal{R}_m \mathcal{R}$ is a maximal ideal $m \in \mathcal{R}$ is a valuation ring.

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Theorem 2. Let $\mathcal{R}$ be an Ore and p.pr. rings. If $\mathcal{R}x$ is a right S-H-ring, $x \in X(\mathcal{R})$, $\mathcal{R}$ is S-H-ring.

Definition 3. Let $(I, +, \cdot)$ be an ideal I of the ring $(\mathcal{R}, +, \cdot)$. So $(\mathcal{R}I, +, \cdot)$ is a ring and it is named Q-ring.

Example 4.

1. $(\mathbb{Z}_8/I = (2), +, \cdot)$ is Q-ring.
2. $(\mathbb{Z}_{18}/I = (3), +, \cdot)$ is Q-ring.

Definition 5. [14] A ring $\mathcal{R}$ which has a classical right Q-ring is said to be a right Ore ring.

Remark 6. Any Q-ring is named Ore ring.

Definition 7. [15] A skew field is a ring $\mathcal{R}$ and x is abelian group. $\mathcal{R}$ has a multiplicative identity. Every $0 \neq x$ in $\mathcal{R}$ has a multiplicative inverse in $\mathcal{R}$. Multiplication in $\mathcal{R}$ is not necessarily commutative.

Example 8. The real number $\mathcal{R}$ and complex number C are field, hence also skew fields with commutative multiplication.

Lemma 9. For a ring $\mathcal{R}$, Skew-field G is L-F-group. If H is (f-subgroup) of $\mathcal{R}$ with 1, so $\mathcal{R}$ is p.pr. ring.

Proof. From Skew-field, we say $\mathcal{R}$ is a regular. But $\mathcal{R}$ is finitely subgroup with 1 in $\mathcal{R}$ this implies the group ring is p.pr. ring.

Proposition 10.

1. $\mathcal{R}$ be an Ore ring and it is Skew-field.
2. G be a L-F-group.
3. $\mathcal{R}x$ is a right S-H-ring, $x \in X(\mathcal{R})$.
4. H is f-subgroup of $\mathcal{R}$ with 1.

Thus $\mathcal{R}$ is right S-H-ring.

Proof. Since $\mathcal{R}$ is Skew-field, G is L-F-group with H is f-subgroup of $\mathcal{R}$ with 1, $\mathcal{R}$ is p.pr. ring (by Lemma 9), and we have $\mathcal{R}$ is Ore ring and $\mathcal{R}x$ is a right S-H-ring, $x \in X(\mathcal{R})$. So $\mathcal{R}$ is right S-H-ring (Theorem 2).

Definition 11. Any ring $\mathcal{R}$ is named semi-simple if it is a direct sum of minimal simple ideal.

Example 12. Any field F (like $\mathcal{R}$ or C) is also a semi simple ring.

Now, before presenting the following theorem, we need to clarify (group ring), where If N is a ring, G is a group, $\mathcal{R}$ equal $\mathcal{R}(G, N)$ denotes the group ring. Then a typical element of R is finite formal sum

$$r = \sum r(g)g, r(g) \in N, g \in G.$$

Theorem 13. Let $\mathcal{R}$ be a semi simple ring such that satisfies (Lemma 9). Thus $\mathcal{R}$ is p.pr. ring.

Proof. Where $\mathcal{R}$ is a semi simple ring, then $\mathcal{R}$ is a regular. Therefore from (Lemma 9), we have every group ring is p.pr. ring. Thus $\mathcal{R}$ is p.pr. ring.

Proposition 14. Assume that $\mathcal{R}$ is semi simple Ore ring and satisfies all condition in (Lemma 9), and $\mathcal{R}x$ is a right S-H-ring where $x \in X(\mathcal{R})$. Then a ring $\mathcal{R}$ is right S-H-ring.

Proof. We have $\mathcal{R}$ is semi simple and by (Theorem 13) $\mathcal{R}$ is p.pr. ring. But $\mathcal{R}$ has Ore property and $\mathcal{R}x$ is right S-H-ring where $x \in X(\mathcal{R})$, then by (Theorem 2) $\mathcal{R}$ is S-H-ring.

Definition 15. [12] A projective module P means $\exists Q \ni P \oplus Q$ is a free.

Example 16. $\mathcal{R} = \mathbb{Z}$ and $F = \mathbb{Z}^n$.

Theorem 17. For a ring $\mathcal{R}$, G is f-l-group. If H f-subgroup of G with 1 and cyclic modules are projective, $\mathcal{R}$ is p.pr. ring.

Proof. Sincerer cyclic modules are projective, therefore by Theorem a semi simple for $\mathcal{R}$ (Lemma 9), and we only short exact:

$$\{0\} \rightarrow I \rightarrow \mathcal{R} \rightarrow \frac{\mathcal{R}}{I} \rightarrow 0 \text{ is splits } \exists I \text{ is an ideal of } \mathcal{R}.$$

Also $\frac{\mathcal{R}}{I}$ is cyclic over $\mathcal{R}$, $\mathcal{R}$ is semi simple, $\mathcal{R}$ is Von-regular. But H is finitely-subgroup of G . So, $\mathcal{R}$ is p.pr. ring.

Proposition 18. Let

1. $\mathcal{R}$ be an Ore ring
2. G be F-L-group.
3. H is f-subgroup of G with 1.
4. Cyclic modules are projective.
5. $\mathcal{R}x$ is a right semi hereditary where $x \in X(\mathcal{R})$.

Then a ring $\mathcal{R}$ is right S-H-ring.

Proof. By previous theorem we get $\mathcal{R}$ is p.pr. ring. On the other hand, since $\mathcal{R}$ is Ore ring and $\mathcal{R}x$ is a right semi hereditary where $x \in X(\mathcal{R})$. It is clear that (by Theorem 2) $\mathcal{R}$ is S-H-ring.

Lemma 19. For a simple ring has (dcc) on Prime ideal is p.pr. ring.

Proof. Let $\mathcal{R}$ be a simple ring. We have 2-sided for $\text{ann}(M)$, $\text{ann}(M) = 0$. Also, M is faithful. Then $\mathcal{R}$ is faithful simple. Therefore $\mathcal{R}$ is primitive. This leads $\mathcal{R}$ is J-semi simple ($J(\mathcal{R})$ equal $\{0\}$). Thus $\mathcal{R}$ is a semi primitive (p.pr. ring).

Proposition 20. Let a ring $\mathcal{R}$ is simple Ore ring, where it is satisfies (dcc) on prime ideal. $\mathcal{R}x$ is a right semi hereditary where $x \in X(\mathcal{R})$. Then a ring $\mathcal{R}$ is right S-H-ring.

Proof. Since $\mathcal{R}$ is simple and satisfies (dcc), so (by lemma 19) $\mathcal{R}$ is p.pr. ring, and we have $\mathcal{R}$ is Ore ring, $\mathcal{R}x$ is a right semi hereditary where $x \in X(\mathcal{R})$. Then (by Theorem 2) we get $\mathcal{R}$ is right S-H-ring.

Theorem 21. Let a ring $\mathcal{R}$ be a commutative. If $\mathcal{R}$ is strictly generalized right p.pr. ring, so $\mathcal{R}$ is p.pr. ring.

Proposition 22. Let $\mathcal{R}$ be a commutative Ore ring, and it is strictly generalized right p.pr. ring, $\mathcal{R}x$ is a right semi hereditary where $x \in X(\mathcal{R})$. Then $\mathcal{R}$ is right S-H-ring.

Proof. Since $\mathcal{R}$ is commutative, and is a strictly generalized right p.pr. ring it is clear that $\mathcal{R}$ is p.pr. ring (by above Theorem). Also, we have $\mathcal{R}$ is Ore ring and $\mathcal{R}x$ is a right semi hereditary where $x \in X(\mathcal{R})$. Then (by Theorem 2) we get $\mathcal{R}$ is right S-H-ring.

Definition 23. [12] An element e in a ring $\mathcal{R}$ is idempotent if $e^2 = e$.

Lemma 24. [16] Let $\mathcal{R}$ be a P.Pr. ring, $f \in \mathcal{R}$, so $\text{ann}(f)$ is generated by $e^2 = e$ of $\mathcal{R}$.

Proposition 25. Let $\mathcal{R}$ is Q-ring and $f \in \mathcal{R}$. Hence $\text{ann}(f)$ is generated by $e^2 = e$ of $\mathcal{R}$. $\mathcal{R}x$ is S-H-ring, $x \in X(\mathcal{R})$. Thus $\mathcal{R}$ is right S-H-ring.

Proof. We have for each $f \in \mathcal{R}$, then $\text{ann}(f)$ is generated by $e^2 = e$ of $\mathcal{R}$. Thus $\mathcal{R}$ is p.pr. ring (by Lemma 24). and we have by previous remark any Q-ring is Ore ring. Therefore with $\mathcal{R}x$ is a right S-H-ring, $x \in X(\mathcal{R})$. Thus $\mathcal{R}$ is S-H-ring (by Theorem 2).

Theorem 26. [16] Suppose that $\mathcal{R}$ be a ring. So $\mathcal{R}$ is p.pr. ring if $\mathcal{R}[x]$ is a p.pr. ring.

Proposition 27. Let $\mathcal{R}$ be a Q-ring. Let $\mathcal{R}[x]$ is a p.pr. ring. If $\mathcal{R}x$ is S-H-ring, $x \in X(\mathcal{R})$. Hence $\mathcal{R}$ is S-

H-ring.

Proof. Since $\mathcal{R}[x]$ is a p.pr. ring, then $\mathcal{R}$ is p.pr. ring. And $\mathcal{R}$ is Ore ring because it is quotient. On the other hand, since $\mathcal{R}x$ is a right S-H-ring, $x \in X(\mathcal{R})$. Thus by (Theorem 2) $\mathcal{R}$ is S-H-ring.

Lemma 28. [16] For a non-zero ring $\mathcal{R}$. If $\mathcal{R}$ is an integral domain, then $\mathcal{R}$ is p.pr. ring with trivial idempotent.

Proposition 29. Let $\mathcal{R}$ be a non-zero integral domain Ore ring. If $\mathcal{R}x$ is S-H-ring, $x \in X(\mathcal{R})$. Hence $\mathcal{R}$ is S-H-ring.

Proof. We have (by Lemma above) If $\mathcal{R}$ is an integral domain for a non-zero ring $\mathcal{R}$, then $\mathcal{R}$ is p.pr. ring. Thus $\mathcal{R}$ is p.pr. ring. And $\mathcal{R}$ is Ore ring, $\mathcal{R}x$ is S-H-ring, $x \in X(\mathcal{R})$. Then (by Theorem 2) $\mathcal{R}$ is S-H-ring.

4. Conclusions

S-H-rings are very important in the ring theory. Some new results are obtained and new definitions are founded in this article. In (Th. 21), we showed that if $\mathcal{R}$ is a commutative and Ore property, this means $\mathcal{R}$ has an ideal with projective and principal properties. Also, in (Pro. 18), we proved that $\mathcal{R}$ is P.pr. ring when there exist some conditions such as finitely locally group has finitely locally subgroup and $\mathcal{R}$ is S-H-ring.

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