

Purely Critically Small Compressible Modules

Authors Names	ABSTRACT
Doaa Riyadh Abed Publication date: 15 / 9 /2026 Keywords: pure Module, small compressible Module.	Let D be unitary left Module above ring with identity R . Throughout this paper, we present purely small compressible and purely critically small compressible Modules. As well, we give remarks and examples, many properties of such Modules are investigated.

1. Introduction

Let D be unitary left Module above ring with identity R . A submodule W of R -Module D is small Submodule ($W \ll D$) if $W + U = D$ then $U = D$ for any Submodule U of D [1]. If an R -Module D contains embedded R -Modules in all of its non-zero small Submodules then D be small compressible. That is, if $\exists f : D \rightarrow W$ which is monomorphism whenever $0 \neq W \ll D$ [2]. A small compressible Module D be critically small compressible (c.s.c.) If unable to embed D in some proper factor factor D/U where U is non-zero small Submodule of D . If $U \cap IU = IU$ for each ideal I of R , then a Submodule U of R -Module D is pure[3]. Throughout this research, we give purely small compressible modules. Furthermore, purely small compressible Modules have a specific type known as purely critically small compressible Modules.

An R -Module D be purely small compressible (p.s.c) if for each non-zero pure small submodule B of D , there exists a monomorphism $A : D \rightarrow B$. Equally, D be p.s.c if $\exists$ a monomorphism $A : D \rightarrow B$ whenever $B \neq 0$ and pure small Submodule from D . A ring R be p.s.c if R as R -Module be p.s.c. Each small compressible Module be p.s.c. The opposite is not correct. For example, Z_4 as Z -Module be p.s.c which is not small compressible. An R -Module D be purely critically small compressible (p.c.s.c) if D is purely small compressible and D unable to be embedded in at all Modules of its quotient D/U where $U \ll D$ is proper pure submodule.

2. Examples and Remarks

- (1) Each c.s.c Module is p.c.s.c. In particular Z as Z -Module be p.c.s.c since it is c.s.c.
- (2) Z_{p^∞} as Z -Module is not p.c.s.c.
- (3) Q as Z -Module is not p.c.s.c.
- (4) Z_n as Z -Module is not p.c.s.c $\forall n > 1$.
- (5) Every von Neumann regular ring R be c.s.c iff R be p.c.s.c.

(6) Every regular Module D be p.c.s.c iff D be c.s.c.

Proposition 2: Each non-zero pure small Submodule of p.s.c Module be p.s.c.

Proof: Take D is p.s.c Module as well as R is not zero and it is a pure small Submodule from D . Assume U be pure small Submodule from R . Thus U is pure in D (4, Remarks 1.2.8,p.19). Thus $\exists$ a monomorphism $f: R \rightarrow U$, from this time if: $U \rightarrow R$ be monomorphism wherever $i: U \rightarrow R$ be inclusion homomorphism. Therefore R be p.s.c.

Proposition 3: Let D be p.c.s.c Module then each pure small Submodule of D that is not zero is p.c.s.c.

Proof: Assume $0 \neq U$ and is pure small Submodule from D . Thus as a result of prop.2, U is p.s.c. Assume V be pure Submodule of U . Thus by [5, 3.1.5(2) and (3)], U be pure in D also U/V be pure in D/V . Assume that $\exists$ a monomorphism, $\alpha: U \rightarrow U/V$. Since D be p.s.c there is a monomorphism, $f: D \rightarrow U$. At that moment the composition $D \xrightarrow{f} U \xrightarrow{\alpha} U/V \xrightarrow{i} D/V$ is a monomorphism. But D is included in D/V which contradicts p.c.s.c of D . Thus U is p.c.s.c.

Corollary4: A non-zero direct summand from p.c.s.c be p.c.s.c.

A homomorphism from pure small Submodule of D into D is defined as a pure small partial endomorphism of Module D .

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- (1) Let N be pure small Submodule of Module D then the inclusion homomorphism $i: N \rightarrow D$ is pure small partial endomorphism of D .
- (2) Each homomorphism from direct summand of R -Module D into D is pure small partial endomorphism of D .
- (3) Every partial endomorphism of regular (semisimple) Module D is pure small partial endomorphism of D .

Proposition 6: Let D be fully stable Module with each Submodule of the proper pure is pure. If D be p.c.s.c, thus each non-zero pure small partial endomorphism from D is monomorphism.

Proof: Assume pure small Submodule $0 \neq N$ of D and non-zero partial endomorphism $0 \neq f: N \rightarrow D$ thus $N/\ker f \cong f(N)$. Hence each of $\ker f$ and $f(N)$ be pure in D , but D be p.s.c thus there is a monomorphism, $g: D \rightarrow f(N)$. Then the composition $D \xrightarrow{g} f(N) \cong N/\ker f \xrightarrow[\text{incl.}]{i} D/\ker f$ is an embedding of D into $D/\ker f$ which is contradiction, but D is p.c.s.c. Thus $\ker f = 0$ implies f is monomorphism.

Proposition 7: Let D be p.s.c Module in which each pure Submodule of D has pure quotient. If each pure small partial endomorphism of D that is non-zero is a monomorphism, then D be p.c.s.c.

Proof: Assume D be not p.c.s.c. Therefore, $\exists$ small pure non-zero Submodule U of D also $f: D \rightarrow D/U$ which is monomorphism. Hence D is isomorphic to K/U . But K/U is pure Submodule in D/U and U is pure in D , Thus by [5,3.1.5(4)], K is pure in D . The composition $K \xrightarrow{\pi} K/U \xrightarrow[\text{iso.}]{\varphi} D$ is pure small

partial endomorphism of D . Therefore $\varphi\pi$ is a monomorphism implies $\ker(\varphi\pi) = \ker\pi = U$ which is a contradiction.

Proposition 8: Each p.c.s.c Module is indecomposable.

Proof: Let D be p.c.s.c Module. Suppose D is decomposable. Thus $D = U \oplus V$ where U and V are non-zero proper pure small Submodules of D . Since, $V \simeq D/U$. Let $\alpha : V \rightarrow D/U$ be an isomorphism, since D be p.s.c, thus $\exists$ a monomorphism $f : D \rightarrow V$ and therefore $\alpha f : D \rightarrow D/U$ is a monomorphism, which is a contradiction. The opposite is untrue. For instance, Q as a $\mathbb{Z}$ -Module is not p.c.s.c. but is indecomposable.

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