

Derivation of a New Numerical Method for Approximating Single Integrals Based on Boole and Simpson's Rules

Authors Names	ABSTRACT
^aMariam Ahmed Hamdi ^bSafaa M. Aljassas Publication date: 2 / 10 /2026 Keywords: Numerical integration , Single integrals , Romberg Accelerating , Newton-Coates formulas	This research aims to derive a new numerical method and its correction terms (error values) for approximating single integrals using Boole's and Simpson's formulas. This method addresses cases where the integrand is continuous, yet its derivative exhibits a singularity (or irregularity) at one of the endpoints of the integration interval. To accelerate convergence, the Romberg acceleration technique was employed. A numerical comparison was conducted between the proposed method and established numerical integration techniques. This comparison utilized a set of numerical examples to evaluate accuracy, the rate of convergence, the number of sub-intervals required to achieve a specific level of precision, and computational efficiency. The results demonstrate the proposed method's ability to achieve high levels of accuracy with fewer sub-intervals in certain cases, highlighting its numerical efficiency in approximating single integrals. The proposed method has been named the Boole–Simpson method and is designated by the symbol (BS).

1. Introduction

Numerical integration is a fundamental topic in numerical analysis, playing a crucial role in approximating integrals that cannot be computed analytically or are complex and impractical to handle using traditional analytical methods. Definite integrals appear extensively in numerous scientific and engineering applications; consequently, the development of numerical methods characterized by accuracy, efficiency, and rapid convergence has become a significant area of study within numerical analysis.

Among the well-known methods in numerical integration are the Newton–Cotes formulas, which serve as the foundation for several numerical integration rules most notably Boole's rule and Simpson's rule. While these rules are efficient in many scenarios, improving approximation accuracy and accelerating convergence particularly when dealing with integrands whose derivatives exhibit irregular behavior at specific points within the integration interval remains a vital area for the development of more efficient numerical methods.

Numerous academic studies have focused on constructing and developing numerical techniques aimed at obtaining approximate solutions to complex mathematical problems, with particular emphasis on analyzing error behavior and determining rates of convergence. Al-Sharifi [1], Hilal et al. [2], and Fox [3], alongside other researchers [5, 6, 7, 8], have made significant contributions to the field of numerical integration by proposing various numerical formulas and techniques and analyzing their properties regarding accuracy, stability, and error terms. Accordingly, this research focuses on deriving a new numerical method for approximating univariate integrals by combining Boole's and Simpson's rules to form a hybrid numerical formula termed the Boole–Simpson (BS) method. The study also involves deriving correction terms associated with the proposed method; these terms help improve approximation accuracy and address scenarios where the precision of traditional rules is compromised by the behavior of derivatives at specific points within the integration interval.

To enhance the efficiency of the proposed method and accelerate convergence, Romberg acceleration was integrated with the Boole –Simpson approach. A numerical study was then conducted to compare the results of the proposed method against those obtained using Boole's and Simpson's rules (with Romberg acceleration). This comparison evaluated approximation accuracy, result stability, convergence speed, and the number of sub-intervals required to achieve a specific level of precision. Ultimately, the study aims to assess the numerical efficiency of the proposed method and its capacity to improve numerical integration performance.

2. Derivation of the Boole-Simpson method (BS) and its accompanying correction terms

The general formulas for the two methods Boole And we Simpson's rules are respectively:

$$J(h) = \frac{2h}{45} \left[7f(x_0) + 32 \sum_{k=1,5,9,\dots}^{n-3} (f(x_k) + f(x_{k+2})) + 12 \sum_{k=2,6,10,\dots}^{n-2} f(x_k) + 14 \sum_{k=4,8,14,\dots} f(x_k) + 7f(x_n) \right] \\ - \frac{2}{945} h^6 (f_{4n}^{(5)} - f_0^{(5)}) + \frac{1}{900} h^8 (f_{4n}^{(7)} - f_0^{(7)}) - \dots \quad \dots(1)$$

$$J(h) = \frac{h}{3} \left[f(x_0) + f(x_{2n}) + 2 \sum_{i=1}^{\frac{n}{2}-1} f(x_0 + 2ih) + 4 \sum_{i=1}^{\frac{n}{2}} f(x_0 + (2i-1)h) \right] \\ - \frac{1}{180} h^4 (f_{2n}^{(3)} - f_0^{(3)}) + \frac{1}{1512} h^6 (f_{2n}^{(5)} - f_0^{(5)}) - \dots \quad \dots(2)$$

By adding the formulas above we get

$$J(h) = \frac{h}{90} \left[29 \left(f(x_0) + f(x_{4n}) \right) + 124 \sum_{k=0}^{n-1} \left(f(x_{4k+1}) + f(x_{4k+3}) \right) + 54 \sum_{k=1}^n f(x_{4k-2}) \right. \\ \left. + 58 \sum_{k=1}^{n-1} f(x_{4k+4}) \right] + E_{BS} \quad \dots(3)$$

We will call the suggest numerical method by(Boole-Simpson method) (BS)

And its correction terms, if the function is continuous, are

$$E_{BS}(h) = -\frac{1}{360} h^4 (f_{4n}^{(3)} - f_0^{(3)}) - \frac{11}{15120} h^6 (f_{4n}^{(5)} - f_0^{(5)}) + \frac{1}{1920} h^8 (f_{4n}^{(7)} - f_0^{(7)}) - \dots \quad \dots(4)$$

Where $v_i \in (x_{4n}, x_0)$ So if the integrator is a continuous function and its derivatives exist at every point of the integration interval $[x_0, x_{4n}]$,

Using the mean-value theorem in differentiation Mean-value theorem for derivatives for formula (4) we get:-

$$E_{BS} = -\frac{(x_{4n} - x_0)h^4}{360} \left(f^{(4)}(\vartheta_1) \right) - \frac{11(x_{4n} - x_0)h^6}{15120} \left(f^{(6)}(\vartheta_2) \right) + \frac{(x_{4n} - x_0)h^8}{1920} \left(f^{(8)}(\vartheta_3) \right) \\ \dots(5)$$

then we can write the error formulas as follows:

$$J - BS = A_{BS}h^4 + B_{BS}h^6 + C_{BS}h^8 + \dots \quad \dots(6)$$

Where $A_{BS}, B_{BS}, C_{BS}, \dots$ are Constants

3. Derivation of Correction Terms for Continuous Integrands with Singular Derivatives of Numerical Method Boole-Simpson (BS)

In this section, we present three theorems for deriving correction terms for integrands that are continuous but have singular derivatives at either the lower term or upper term of integration, or at both endpoints.

3.1: Theorem: Let the function $f(x)$ be continuous and differentiable at each point of the interval except at the point $x = x_0$, the approximate value of the integral $J = \int_{x_0}^{x_{4n}} f(x) dx$ can be evaluated using **Boole-Simpson method (BS)** and the error formula is

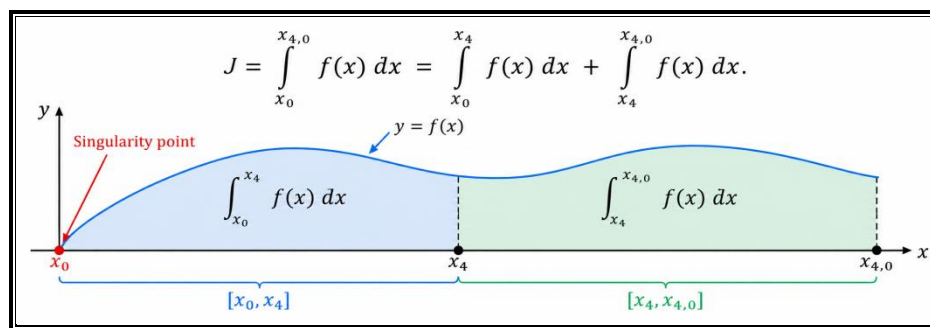
$$E_{BS}(h) = \left[-\frac{1}{360} h^4 D^3 + \frac{1}{360} h^5 D^4 - \frac{1}{630} h^7 D^6 + \dots \right] f_1 + A_{BS} h^4 + B_{BS} h^6 + C_{BS} h^8 + \dots$$

Where $A_{BS}, B_{BS}, \dots$ are constants that depend on the derivatives of the function.

Proof: Suppose that the function $f(x)$ is defined at each point of the integration interval $[x_0, x_{4n}]$ and it is not improper and derivatives of the function are not defined at the point (x_0) . This means that the Taylor's series of functions exists at each point of the integration region except at (x_0) .

We can write the integral J by :

$$J = \int_{x_0}^{x_{4n}} f(x) dx = \int_{x_0}^{x_4} f(x) dx + \int_{x_4}^{x_{4n}} f(x) dx$$



Fig(1) Partitioning the integration interval to derive correction terms for upper point singularities of BS

Formula

Substituting BS rule for the second integral, we obtain:

$$\int_{x_4}^{x_{4n}} f(x) dx = \frac{h}{90} \left[29(f(x_4) + f(x_{4n})) + 124 \sum_{i=1}^{n-1} (f(x_{4i+1}) + f(x_{4i+3})) + 54 \sum_{i=2}^n f(x_{4i-2}) + 58 \sum_{i=2}^{n-1} f(x_{4i}) \right]$$

$$E_{BS}(h) = -\frac{1}{360} h^4 (f_{4n}^{(3)} - f_4^{(3)}) - \frac{11}{15120} h^6 (f_{4n}^{(5)} - f_4^{(5)}) + \frac{1}{1920} h^8 (f_{4n}^{(7)} - f_4^{(7)}) - \dots + R \quad \dots(7)$$

E_{BS} are the correction terms, and R the error is the cutoff from the correction terms after several terms.

$$h = \frac{x_{4n} - x_4}{4n}$$

For the first integral on the integration interval $[x_0, x_4]$, use Taylor's series for $f(x)$ about x_4 :

$$f(x) = f_4 + (x - x_4)f_4' + \frac{(x - x_4)^2}{2!} f_4'' + \frac{(x - x_4)^3}{3!} f_4^{(3)} + \frac{(x - x_4)^4}{4!} f_4^{(4)} + \frac{(x - x_4)^5}{5!} f_4^{(5)} + \frac{(x - x_4)^6}{6!} f_4^{(6)} + \dots \quad \dots(8)$$

Assuming that all derivatives of exist at the point $x = x_4$

By integrate equation (8) on (x_0, x_4) we get:

$$\int_{x_0}^{x_4} f(x) dx = 4hf_4 - 8hf_4' + \frac{32h^3}{3} f_4'' - \frac{32h^4}{3} f_4^{(3)} + \frac{128h^5}{15} f_4^{(4)} - \frac{256h^6}{45} f_4^{(5)} + \frac{1024h^7}{315} f_4^{(6)} - \dots \quad (9)$$

Substituting the points $x_0, x_0 + h, x_0 + 2h$ and $x_0 + 3h$ into formula (8) we obtain:

$$f(x_0) = f_4 - 4hf_4' + \frac{16h^2}{2!} f_4'' - \frac{64h^3}{3!} f_4''' + \frac{256h^4}{4!} f_4^{(4)} - \frac{1024h^5}{5!} f_4^{(5)} + \frac{4096h^6}{6!} f_4^{(6)} - \dots \quad \dots(10)$$

$$f(x_1) = f_4 - 3hf_4' + \frac{9h^2}{2!} f_4'' - \frac{27h^3}{3!} f_4^{(3)} + \frac{81h^4}{4!} f_4^{(4)} - \frac{243h^5}{5!} f_4^{(5)} + \frac{729h^6}{6!} f_4^{(6)} - \dots \quad \dots(11)$$

$$f(x_0 + 2h) = f_4 - 2hf_4' + \frac{4h^2}{2!} f_4'' - \frac{8h^3}{3!} f_4^{(3)} + \frac{16h^4}{4!} f_4^{(4)} - \frac{32h^5}{5!} f_4^{(5)} + \frac{64h^6}{6!} f_4^{(6)} - \dots \quad \dots(12)$$

$$f(x_0 + 3h) = f_4 - hf_4' + \frac{h^2}{2!} f_4'' - \frac{h^3}{3!} f_4^{(3)} + \frac{h^4}{4!} f_4^{(4)} - \frac{h^5}{5!} f_4^{(5)} + \frac{h^6}{6!} f_4^{(6)} - \dots \quad \dots(13)$$

By combining the formulas for (9), (10), (11), (12) and (13) we obtain:-

$$\int_{x_0}^{x_4} f(x) dx = \frac{h}{90} (29f_0 + 124f_1 + 54f_2 + 124f_3 + 29f_4) + \frac{h^3}{180} f_4'' - \frac{10h^4}{552} f_4^{(3)} + \frac{64h^5}{1920} f_4^{(4)} + \dots \quad \dots(14)$$

And by adding formula (14) to formula (7), we conclude that:

$$\int_{x_0}^{x_{4n}} f(x) dx = \frac{h}{90} \left[29(f(x_0) + f(x_{4n})) + 124 \sum_{k=0}^{n-1} (f(x_{4k+1}) + f(x_{4k+3})) + 54 \sum_{k=1}^n f(x_{4k-2}) + 58 \sum_{k=1}^{n-1} f(x_{4k+4}) \right] + \left[-\frac{1}{360} h^2 D + \frac{1}{900} h^3 D^2 - \frac{34}{1920} h^4 D^3 + \dots \right] f_4 + A_{BS} h^4 + B_{BS} h^6 + \dots$$

Since $f(x_4) = Ef(x_3) = f(x_0 + 3h)$, $E = e^{hD}$ then

$$\int_{x_0}^{x_{4n}} f(x) dx = \frac{h}{90} \left[29(f(x_0) + f(x_{4n})) + 124 \sum_{k=0}^{n-1} (f(x_{4k+1}) + f(x_{4k+3})) + 54 \sum_{k=1}^n f(x_{4k-2}) + 58 \sum_{k=1}^{n-1} f(x_{4k+4}) \right] + \left[\frac{1}{360} h^4 D^3 + \frac{1}{360} h^5 D^4 - \frac{1}{630} h^7 D^6 + \dots \right] f_1 + A_{BS} h^4 + B_{BS} h^6 + \dots \quad \dots(15)$$

Since $A_{BS}, B_{BS}, \dots$ constants depend on the derivatives of the function at the point $x = x_{4n}$

3.2: Theorem: Let the function $f(x)$ be continuous and differentiable at each point of the interval except at the point $x = x_{4n}$, the approximate value of the integral $J = \int_{x_0}^{x_{4n}} f(x) dx$ can be evaluated using BS method and the error formula is

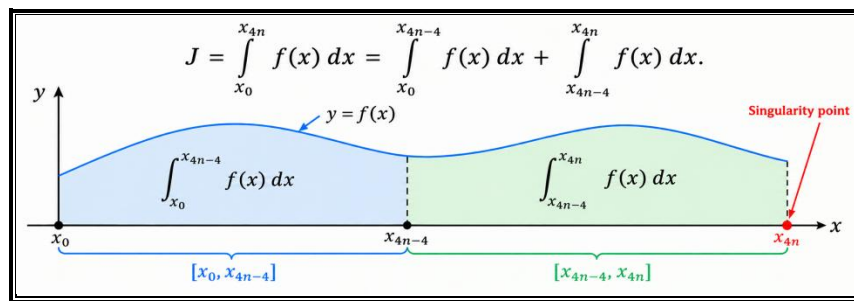
$$E_{BS}(h) = \left[\frac{1}{360} h^4 D^3 + \frac{1}{360} h^5 D^4 - \frac{1}{630} h^7 D^6 + \dots \right] f_{4n} + A'_{BS} h^4 + B'_{BS} h^6 + \dots$$

Where $A'_{BS}, B'_{BS}, \dots$ are constants that depend on the derivatives of the function.

Proof: Suppose that the function $f(x)$ is defined at each point of the integration interval $[x_0, x_{4n}]$ and it is not improper and derivatives of the function are not defined at the point (x_{2n}) . This means that the Taylor's series of functions exists at each point of the integration region except at (x_{2n}) .

We can write the integral J by :

$$J = \int_{x_0}^{x_{4n}} f(x) dx = \int_{x_0}^{x_{4n-4}} f(x) dx + \int_{x_{4n-4}}^{x_{4n}} f(x) dx$$



Fig(2) Partitioning the integration interval to derive correction terms for endpoint singularities of BS Formula

Substituting BS rule for the first integral, we obtain:

$$\int_{x_0}^{x_{4n-4}} f(x) dx = \frac{h}{90} \left[29(f(x_0) + f(x_{4n-4})) + 124 \sum_{i=1}^{n-5} (f(x_{4i+1}) + f(x_{4i+3})) + 54 \sum_{i=2}^{n-4} f(x_{4i-2}) + 58 \sum_{i=2}^{n-5} f(x_{4i}) \right] + \left[\frac{1}{360} h^4 (f_{4n-4}^{(3)} - f_0^{(3)}) - \frac{11}{15120} h^6 (f_{4n-4}^{(5)} - f_0^{(5)}) + \frac{1}{1920} h^8 (f_{4n-4}^{(7)} - f_0^{(7)}) - \dots + R \right] \quad \dots(16)$$

The function is continuous in the interval of integration $[x_0, x_{4n-4}]$.

E_{SY} are the correction terms, and R the error is the cutoff from the correction limits after several terms.

$h = \frac{x_{4n-4} - x_0}{4n}$, For the first integral on the integration interval $[x_{4n-4}, x_{4n}]$, use Taylor's series for $f(x)$ about x_{4n-4} :

$$f(x) = f_{4n-4} + (x - x_{4n-4})f'_{4n-4} + \frac{(x - x_{4n-4})^2}{2!}f''_{4n-4} + \frac{(x - x_{4n-4})^3}{3!}f^{(3)}_{4n-4} + \frac{(x - x_{4n-4})^4}{4!}f^{(4)}_{4n-4} + \frac{(x - x_{4n-4})^5}{5!}f^{(5)}_{4n-4} + \frac{(x - x_{4n-4})^6}{6!}f^{(6)}_{4n-4} + \dots \quad (17)$$

Assuming that all derivatives of exist at the point $x = x_{4n-4}$, By integrate equation (17) on (x_{4n-4}, x_{4n}) we get:

$$\int_{x_{4n-4}}^{x_{4n}} f(x) dx = 4hf_{4n-4} + 8h^2f'_{4n-4} + \frac{32h^3}{3}f''_{4n-4} + \frac{32h^4}{3}f^{(3)}_{4n-4} + \frac{128h^5}{15}f^{(4)}_{4n-4} - \frac{256h^6}{45}f^{(5)}_{4n-4} + \frac{512h^7}{315}f^{(6)}_{4n-4} - \dots \quad (18)$$

Substituting the points x_{4n} , x_{4n-1} , x_{4n-2} and x_{4n-3} into formula (17) we obtain:

$$f(x_{4n}) = f_{4n-4} + 4hf'_{4n-4} + 8h^2f''_{4n-4} + \frac{32h^3}{3}f^{(3)}_{4n-4} + \frac{32h^4}{3}f^{(4)}_{4n-4} + \frac{128h^5}{15}f^{(5)}_{4n-4} + \frac{256h^6}{45}f^{(6)}_{4n-4} + \dots \quad (19)$$

$$f(x_{4n-1}) = f_{4n-4} + 3hf'_{4n-4} + \frac{9h^2}{2!}f''_{4n-4} + \frac{9h^3}{2!}f^{(3)}_{4n-4} + \frac{27h^4}{8}f^{(4)}_{4n-4} + \frac{81h^5}{40}f^{(5)}_{4n-4} + \frac{81h^6}{80}f^{(6)}_{4n-4} + \dots \quad (20)$$

$$f(x_{4n-2}) = f_{4n-4} + 2hf'_{4n-4} + 2h^2f''_{4n-4} + \frac{4h^3}{3}f^{(3)}_{4n-4} + \frac{2h^4}{3}f^{(4)}_{4n-4} + \frac{4h^5}{15}f^{(5)}_{4n-4} + \frac{4h^6}{45}f^{(6)}_{4n-4} + \dots \quad (21)$$

$$f(x_{4n-3}) = f_{4n-4} + hf'_{4n-4} + \frac{h^2}{2!}f''_{4n-4} + \frac{h^3}{3!}f'''_{4n-4} + \frac{h^4}{4!}f^{(4)}_{4n-4} + \frac{h^5}{5!}f^{(5)}_{4n-4} + \frac{h^5}{6!}f^{(5)}_{4n-4} \dots \quad (22)$$

By combining the formulas for (22), (21), (20), (19) and (18) we obtain:-

$$\int_{x_{4n-4}}^{x_{4n}} f(x) dx = \frac{h}{90}(29f_{4n-4} + 124f_{4n-3} + 54f_{4n-2} + 124f_{4n-1} + 29f_{4n}) + \frac{h^3}{180}f''_{4n-4} + \frac{10h^4}{552}f^{(3)}_{4n-4} + \frac{64h^5}{1920}f^{(4)}_{4n-4} + \dots \quad (23)$$

And by adding formula (23) to formula (16), we conclude that:

$$\int_{x_0}^{x_{4n}} f(x) dx = \frac{h}{90} \left[29(f(x_0) + f(x_{4n})) + 124 \sum_{k=0}^{n-1} (f(x_{4k+1}) + f(x_{4k+3})) + 54 \sum_{k=1}^n f(x_{4k-2}) + 58 \sum_{k=1}^{n-1} f(x_{4k+4}) \right] + \left[\frac{1}{360}h^2D + \frac{1}{900}h^3D^2 + \frac{34}{1920}h^4D^3 + \dots \right] f_{4n-4} + A'_{BS}h^4 + B'_{BS}h^6 + \dots$$

Since $E^{-1}f_{4n} = f_{4n-1} = (x_{4n} - h)$, $E^{-1} = e^{-hD}$ are

$$\int_{x_0}^{x_{4n}} f(x) dx = \frac{h}{90} \left[29 \left(f(x_0) + f(x_{4n}) \right) + 124 \sum_{k=0}^{n-1} \left(f(x_{4k+1}) + f(x_{4k+3}) \right) + 54 \sum_{k=1}^n f(x_{4k-2}) \right. \\ \left. + 58 \sum_{k=1}^{n-1} f(x_{4k+4}) \right] + \left[\frac{1}{360} h^4 D^3 + \frac{1}{360} h^5 D^4 + \frac{1}{630} h^7 D^6 + \dots \right] f_{4n-1} + A'_{BS} h^4 + B'_{BS} h^6 + \dots \quad \dots(24)$$

Since $A'_{BS}, B'_{BS}, \dots$ constants depend on the derivatives of the function at the point $x = x_0$

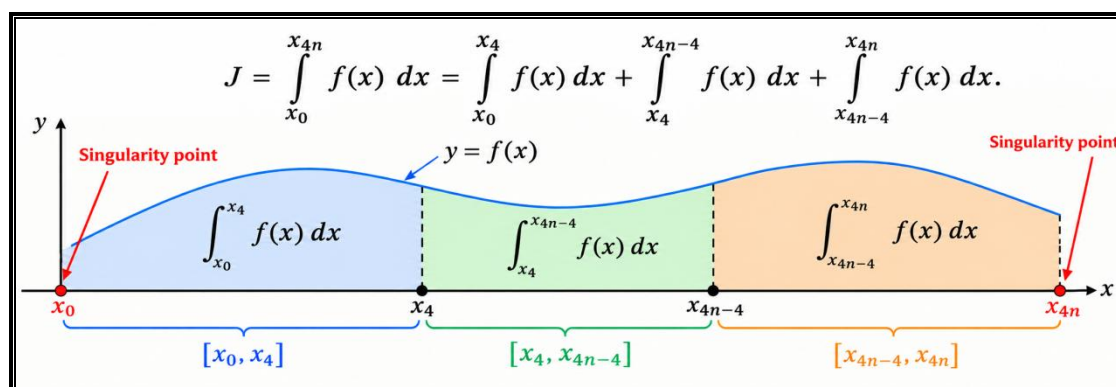
3.3: Theorem : Let the function $f(x)$ be continuous and differentiable at each point of the interval except at the two point $x = x_0$ and $x = x_{4n}$ then the error formula is

$$E_{BS}(h)(f_0 + f_{4n}) = \left[-\frac{1}{360} h^4 D^3 + \frac{1}{360} h^5 D^4 - \frac{1}{630} h^7 D^6 + \dots \right] f_1 + \left[\frac{1}{360} h^4 D^3 + \frac{1}{360} h^5 D^4 + \frac{1}{630} h^7 D^6 + \dots \right] f_{4n-1}$$

Proof: Suppose that the function $f(x)$ is defined at each point of the integration interval $[x_0, x_{4n}]$ and it is not improper and derivatives of the function are not defined at the point (x_{4n}) . This means that the Taylor's series of functions exists at each point of the integration region except at x_0 and x_{4n}

We can write the integral J by :

$$J = \int_{x_0}^{x_{4n}} f(x) dx = \int_{x_0}^{x_4} f(x) dx + \int_{x_4}^{x_{4n-4}} f(x) dx + \int_{x_{4n-4}}^{x_{4n}} f(x) dx \quad \dots(25)$$



Fig(3) Partition of the Integration Interval for Deriving Correction Terms in the Case of Endpoints Singularities in the BS Formula

Substituting BS rule for the second integral, we obtain:

$$\int_{x_4}^{x_{4n-4}} f(x) dx = \frac{h}{90} \left[29(f(x_4) + f(x_{4n-4})) + 124 \sum_{i=1}^{n-5} (f(x_{4i+1}) + f(x_{4i+3})) + 54 \sum_{i=2}^{n-4} f(x_{4i-2}) + 58 \sum_{i=2}^{n-5} f(x_{4i}) \right] + E_{BS} + R$$

$$E_{BS}(h) = -\frac{1}{360} h^4 (f_{4n-4}^{(3)} - f_4^{(3)}) - \frac{11}{15120} h^6 (f_{4n-4}^{(5)} - f_4^{(5)}) + \frac{1}{1920} h^8 (f_{4n-4}^{(7)} - f_4^{(7)}) - \dots \quad \dots(1-57)$$

The function is continuous in the interval of integration $[x_2, x_{2n-2}]$. E_{BS} are the correction terms, and R the error is the cutoff from the correction terms after several terms. $h = \frac{x_{4n-4} - x_4}{4n}$. Similar to the steps in proofs of Theorems 1 and 2, we can evaluate the first and the last integrals in equation (15) respectively. and then add the result to equation (24) to obtain

$$\int_{x_0}^{x_{4n}} f(x) dx = \frac{h}{90} \left[29(f(x_0) + f(x_{4n})) + 124 \sum_{k=0}^{n-1} (f(x_{4k+1}) + f(x_{4k+3})) + 54 \sum_{k=1}^n f(x_{4k-2}) + 58 \sum_{k=1}^{n-1} f(x_{4k+4}) \right] + \left[-\frac{1}{360} h^2 D + \frac{1}{900} h^3 D^2 - \frac{34}{1920} h^4 D^3 + \dots \right] f_1 + \left[\frac{1}{360} h^2 D + \frac{1}{900} h^3 D^2 + \frac{34}{1920} h^4 D^3 + \dots \right] f_{4n-1}$$

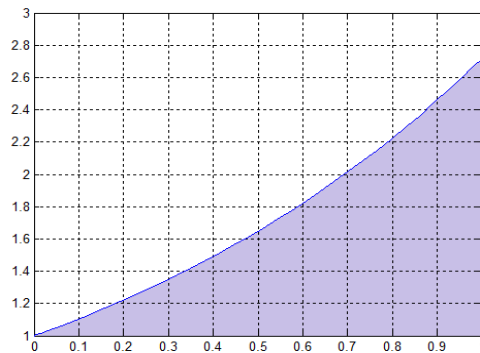
4. Integrals with improper integrands in one or both ends of the integration

Suppose that $J = \int_{x_0}^{x_{4n}} f(x) dx$, $f(x)$ is continuous on the integration region $[x_0, x_{4n}]$ but it is not defined at the points (x_0) and (x_{4n}) or on one of them. Thus we cannot apply any of the above three theorems, so to evaluate the improper integral, we will ignore the value of the function on impropriety point [9].

5. Examples

1.5: The integral $I = \int_0^1 e^x dx$ that its function had been shown in Figure4, has analytic value 1.71828182845905 which approximates to fourteen decimal digits. In this example, the integrand is continuous in integration interval the suitable correction terms are

$$E_{BS}(h) = A_{BS} h^4 + B_{BS} h^6 + C_{BS} h^8 + \dots \quad \text{where } A_{BS}, B_{BS}, C_{BS}, \dots \text{ are constants.}$$

Fig:(4): Geometric shape for e^x

When applying the Boole–Simpson method combined with Romberg acceleration an approximate integral value with a precision of fourteen decimal place was obtained using $n=32$ sub-intervals. In contrast, Boole’s rule when combined with Romberg acceleration, achieved a precision of thirteen decimal places at the same value of $n=32$. Meanwhile, Simpson’s rule required $n=64$ sub-intervals to reach the same level of accuracy achieved by Boole’s rule.

These results demonstrate that the Boole–Simpson method with Romberg acceleration achieved the highest level of numerical precision among the three methods at $n=32$ reaching fourteen decimal places whereas Boole’s rule achieved thirteen decimal places with the same number of sub-intervals. Simpson’s rule, however, required doubling the number of sub-intervals to $n=64$ to match the precision level achieved by Boole’s rule at $n=32$. This highlights the effectiveness of the proposed method in enhancing numerical approximation accuracy while using fewer sub-intervals.

Table (1) shows the approximate value of the integral $I = \int_0^1 e^x dx = 1.71828182845905$ using the RBS, RB, RS methods.			
n	RBS	RB	RS
4	1.71828268792476	1.85914091422952	1.71777653196690
8	1.71828182879453	1.71828268792476	1.71828099651212
16	1.71828182845908	1.71828182879453	1.71828182812625
32	1.71828182845905	1.71828182845904	1.71828182845901
64			1.71828182845904

5.2: The integral $\int_0^1 \sqrt{x^3} dx$ shown in Figure (5) has an irregular derivative in the lower term ($x = 0$) of the integral, and the irregularity is radical. Applying Theorem 1 the appropriate correction terms for the Boole -Simpson method (BS), Boole and Simpson's rules , respectively are:

$$E_{BS}(h) = a h^{2.5} + A_{BS} h^4 + B_{BS} h^6 + \dots$$

$$E_B(h) = b h^{2.5} + A_B h^6 + B_B h^8 + \dots$$

$$E_S(h) = c h^{2.5} + A_S h^4 + B_S h^6 + \dots$$

when $a, b, c, A_{BS}, B_{BS}, \dots, A_S, B_S, \dots, A_B, B_B, \dots$, are contents

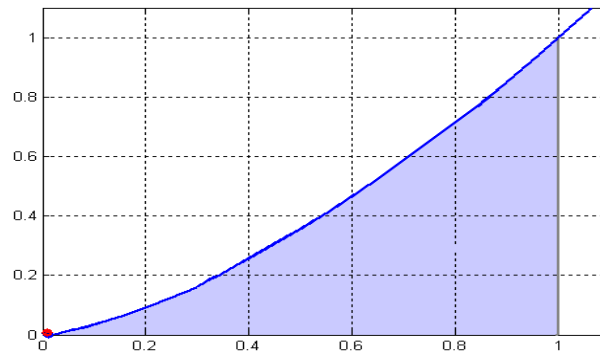


Fig:(5): Geometric shape for $\int_0^1 \sqrt{x^3} dx$

Table (2) shows the approximate value of the integral $\int_0^1 \sqrt{x^3} dx = 0.4$ using the RBS, RB, RS methods.

n	RBS	RB	RS
4	0.40236892706218	0.40001596765721	0.39849490502780
8	0.40001596765721	0.4000000016387	0.39998621098921
16	0.40000009743428	0.40000000067913	0.39999990650504
32	0.40000000045435	0.40000000000234	0.39999999955137
64	0.40000000000000	0.40000000000000	0.39999999999892
128			0.40000000000000

When applying the BS method with Romberg acceleration, the correct value was obtained with a precision of fourteen decimal places at $n=64$. In contrast, Boole's rule with Romberg acceleration achieved the same level of precision with the same number of intervals, whereas Simpson's rule with Romberg acceleration achieved the same precision only at $n=128$. These results demonstrate the superiority of the RBS method over the other methods in terms of numerical accuracy and convergence speed; it achieves the highest level of precision using fewer subdivisions, reflecting greater numerical efficiency and a superior ability to estimate the integral's value.

5.3 : The integral $\int_0^1 \frac{1}{\sqrt[4]{1-x}} dx$ shown in Figure (6) has an irregular in upper term ($x=1$) of the integral, and the irregularity is relative. Applying Theorem 2 the appropriate correction terms for the Boole -Simpson method (BS), Boole and Simpson's rules , respectively are:

$$E_{BS}(h) = a_1 h^{3/4} + A_{BS} h^4 + B_{BS} h^6 + \dots, E_B(h) = b_1 h^{3/4} + A_B h^6 + B_B h^8 + \dots$$

$$E_S(h) = c_1 h^{3/4} + A_S h^4 + B_S h^6 + \dots$$

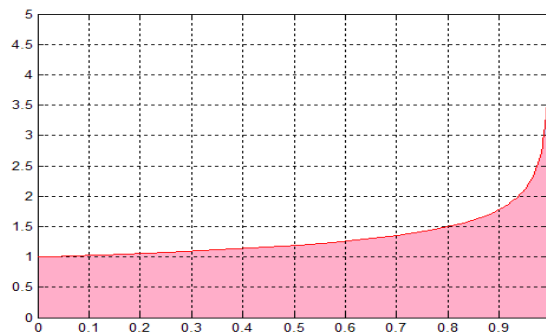


Fig:(6): Geometric shape for $\frac{1}{\sqrt[4]{1-x}}$

Table (3) shows the approximate value of the integral $\int_0^1 \frac{1}{\sqrt[4]{1-x}} dx$ using the RBS, RB, RS methods.			
n	RBS	RB	RS
4	1.33333516947891	1.33334080290574	1.333523287717824
8	1.3333333214579	1.33333276328903	1.33332126992864
16	1.33333333333221	1.33333333341734	1.33333332760538
32	1.33333333333345	1.3333333332476	1.33333333266379
64	1.33333333333333	1.33333333333263	1.33333333335314
128		1.33333333333333	1.33333333333335

When the BS method was applied with Romberg acceleration, the correct value was obtained with an accuracy of up to fourteen decimal places at $n=64$. In contrast, Boole rule achieved the same level of accuracy at $n=128$, while Simpson's rule achieved only thirteen decimal places at $n=128$.

These results demonstrate the superiority of the BS method over the other methods in terms of numerical accuracy and convergence speed. It achieves the highest level of accuracy using fewer division operations, reflecting higher numerical efficiency and a better ability to approximate the integral value.

5.4 : The integral $\int_0^1 \sqrt{x-x^2} dx$ shown in Figure (1.10) has an irregular derivative in the lower and upper term ($x=0$) and ($x=1$) of the integral, and the irregularity is radical. Applying Theorem 3 the

appropriate correction terms for the Boole -Simpson method (BS), Boole and Simpson's rules , respectively are

$$E_{BS}(h) = a_1 h^{1.5} + a_2 h^{2.5} + a_3 h^{3.5} + a_4 h^{4.5} + \dots$$

$$E_B(h) = b_1 h^{2.5} + b_2 h^{3.5} + b_3 h^{4.5} + b_4 h^{5.5} + \dots$$

$$E_S(h) = c_1 h^{1.5} + c_2 h^{2.5} + c_3 h^{3.5} + c_4 h^{4.5} + \dots$$

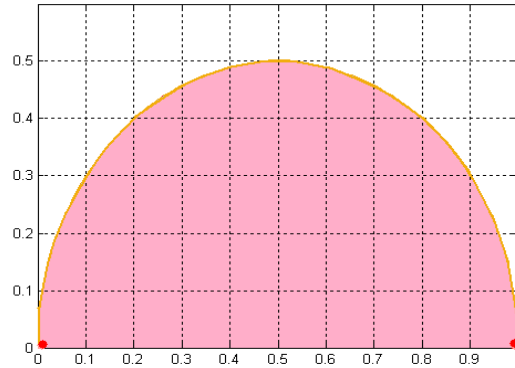


Fig:(1.10): Geometric shape for $\sqrt{x - x^2}$

Table (4) shows the approximate value of the integral $\int_0^1 \sqrt{x - x^2} dx$ using the RBS, RB, RS methods.			
n	RBS	RB	RS
4	0.392699235331923	0.39269907955512	0.396376132573495
8	0.392699087999736	0.39269908041540	0.39279043612371
16	0.392699081697757	0.39269908089400	0.392698964956792
32	0.392699081698735	0.39269908119121	0.392699044297214
64	0.392699081698724	0.392699081698735	0.392699081698690
128		0.392699081698724	0.392699081698731

When the BS method was applied with Romberg acceleration, the correct value was obtained with an accuracy of fourteen decimal places at $n=64$. In contrast, Boole rule achieved the same level of accuracy at $n=128$, while the Simpson's rules only achieved twelve decimal places at the same value of $n=128$. These results demonstrate that the BS method outperforms the other methods in terms of numerical accuracy and convergence speed, achieving the highest level of accuracy using fewer divisions, reflecting higher numerical efficiency and a better ability to approximate the integral value.

6. Conclusions

Numerical results demonstrated that the Boole-Simpson method with Romberg acceleration (RBS) achieved a high level of accuracy and efficiency in approximating the values of single integrals for the cases considered whether the integrand was continuous over the integration interval or exhibited irregularities or singularities at one of the interval's endpoints. Furthermore, the results presented in the tables indicate that applying the proposed method improved the rate of convergence and yielded stable approximations close to the reference values of the integrals under study.

Based on these numerical results, the Boole-Simpson method with Romberg acceleration demonstrated a strong capability to handle the cases studied and achieve high levels of accuracy using a suitable number of sub-intervals. Numerical comparisons with the Boole and Simpson rules each combined with Romberg acceleration—showed that the proposed method achieved the required level of accuracy in the studied examples using fewer sub-intervals in certain cases. This highlights its numerical efficiency in enhancing the convergence rate and reducing the number of subdivisions needed to reach the target level of accuracy.

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