

Results on Light Functions via h-open Sets

Authors Names	ABSTRACT
Hussam Numan Solag Publication date: 2 / 10 /2026 Keywords: h-light, h-totally disconnected, h-connected, h-Hausdorff, h-open	In this article we propose new kind of light function by using h-open sets specifically h-light function and bring in the relationship between them. Through the same set, we also formulated new terminologies known as h-totally disconnected spaces and functions as well h^* -totally disconnected and h^{**} -totally disconnected functions. Various facts, results, and examples are included to support the findings of this study.

1. Introduction

Light functions have been investigated by several researchers, including J. J. Charatonik and K. Omijaniowski. In our paper, we introduce and study another class of light functions, referred to as h-light open functions. Related contributions to the theory of light functions [2] have also been made by researchers such as M. Władysław [1], M K Fort, and G Sh. Mohammed, among others. Our work extends these studies by considering the properties and behavior of h-light open functions.

The following basic definitions are required for our study. A subset A of space X if $A \subseteq \text{Int}(AUH)$ where H subset of X is nonempty, open, and proper. The collection of each h-open sets in X is implied by T^h and the complement of every h-open set is an h-closed set [4-5]. Every open set is h-open. Suppose topological space (X, T) and let A is a subset of X , a point $\mu \in A$ is named h-interior point of the set A if $\mu \in H \subseteq A$ there exist $H \in T^h$ containing μ . where T_h the family which consisting of h-open sets. A space X is defined to be h-Hausdorff if whenever $x, y \in X$ are distinct point, there are h-open sets H_x, K_y which contain x and y , respectively, and satisfy $H_x \cap K_y = \emptyset$. A function $f: X \rightarrow Y$ is referred as h-open (h^* -open and h^{**} -open) where $f(K)$ is h-open in Y [3], for every open (h-open) set K in X . If X and Y are spaces, and f is a function from X into Y that is bijective, continuous, and h-closed (h-open), then f is considered an h-homeomorphism. A topological space X is termed totally disconnected space provided that, for any two distinct points $a, b \in X$ there exist a disconnection $A \cup B$ of X and $b \in B$ and $a \in A$. Suppose a function $f: X \rightarrow Y$ be surjective function. Hence f is termed a totally disconnected function, if and only if the image $f(H)$ is a totally disconnected subset of Y corresponding to every totally disconnected set H in X .

2. Results and Discussion

Definition 2.1: A subset of a space X is h-open whenever, for each nonempty proper open set U in X such that $A \subseteq \text{Int}(UUA)$. The complement of an h-open set is termed h-closed set. [6-7].

Remark 2.2: For all open (closed) set is necessarily h-open (h-closed), however, the converse is not valid in general.

Example 2.3: In indiscrete space (R, T) , The rational number set Q is h -open. but not open set.

Definition 2.4: A topological space X with two nonempty h -open sets A & B in X , then the union of $A \cup B$ is called h -disconnection in X whenever $A \cup B = X$ and $A \cap B = \emptyset$

Example 2.5: in example 2.3 the set Q and Q^c are h -disconnection of indiscrete space (R, T) .

Definition 2.6: A topology space X is h -totally disconnected, whenever, given any two distinct points $q, p \in X$ there exist an h -disconnected HUK of X such that $p \in H$ & $q \in K$.

Example 2.7: The discrete space (R, T) is h -totally space.

Remark 2.8: Each totally disconnected space is h -totally disconnected (shortly we shall use h -totally. disc. Space) however the converse not true.

Example 2.9: The indiscrete space (R, T) is h -totally. disc. space but not totally disc.

Definition 2.10: A surjective function $F: X \rightarrow Y$ is termed h -light function if for each $y \in Y$, $F^{-1}(y)$ is h -totally disc. set.

Example 2.11: Every light function is an h -light function however, the converse may not true

Remark 2.12: Any function F from R with indiscrete topology onto R with any topology is h -totally is h -light function and not light function.

Definition 2.13: A space X is called h -Hausdorff, for every distinct points x, y there exist two disjoint h -open set U and V such that it contains x and y respectively.

Remark 2.14: For every Hausdorff space is h -Hausdorff however, the opposite is not hold in general.

Example 2.15: the indiscrete space (R, T) is h -Hausdorff space but not Hausdorff.

Proposition 2.16: Every h -totally disc. (totally disc.) space is h -Hausdorff

Proof: Let X be h -totally disc. space and x_1 & x_2 be two distinct points belong to X which is h -totally disc. space, that is, there is h -disconnection $A \cup B$ in X with $x_1 \in A$ & $x_2 \in B$ such that $A \cup B = X$ and $A \cap B = \emptyset$. Infer X is h -Hausdorff.

Remark 2.17: The converse of the preceding proposition does not hold.

Example 2.18: The space R , equipped with the usual topology, is h -Hausdorff. but not totally disc. space.

Giving two distinct points $x, y \in R$, satisfying $x < y$, there exist disjoint open intervals around them, for example

$$U = \left(x - \frac{y-x}{3}, x + \frac{y-x}{3}\right), \quad V = \left(y - \frac{y-x}{3}, y + \frac{y-x}{3}\right) \text{ thus } R \text{ is } h\text{-Hausdorff.}$$

But not totally disc. it is well known that a space is totally disc. if it's only connected subsets, are singletons. But $[1,2] \subset \mathbb{R}$ is connected, and contains infinitely many points, therefore $\mathbb{R}$ is h-Hausdorff but not totally disc.

Proposition 2.19: Consider the topological space X . If X is h-totally disconnected, it follows that, for every $x \in X$, the h-connected component $C_X(x)$ of is the singleton $\{x\}$.

i.e. X is h-totally disconnected $\Leftrightarrow C_X(x) = \{x\} \forall x \in X$

Proof: let X is h-totally disconnected, so $C_X(x) = \{x\}$. Assume X is totally disconnected.

By definition, every h-connected subset of X is a singleton. Giving that $C_X(x)$ is h-connected subset of X , we have $C_X(x) = \{y\}$ with some $y \in X$.

But $x \in C_X(x)$ by the definition of the component containing x . There for $x = y$ thus giving that $C_X(x) = \{x\}$, and since x was arbitrary and $x \in X$

Now to prove the other side if $C_X(x) = \{x\}$ with each $x \in X$, thus X is h-totally disc.

We need to prove that X is h-totally disc.

Suppose, contrary to the claim, that X fails to be h-totally disconnected. Then there exists a connected subset $A \subseteq X$ containing at least two distinct points. Choose $x, y \in A$ and $x \neq y$.

Since A is h-connected and contains x , and $C_X(x)$ is the largest subset which is h-connected containing x , we must get $A \subseteq C_X(x)$. Therefor $y \in C_X(x)$ but our assumption says $C_X(x) = \{x\}$. Thus $y \in \{x\}$, which implies $y = x$ which is contrary to $y \neq x$. Therefore, there cannot exist a h-connected subset containing two distinct points. Hence every h-connected subset of is a singleton, so

X is h-totally disc.

Definition 2.20: A space X is termed h-disc. giving that its union of disjoint non empty h-open set so the space X is h-connected whenever it's not h-disconnected.

Remark 2.21: Every disconnected space is h-disconnected space but the opposite may not be true.

Example 2.22: The indiscrete topology space $(\mathbb{R}, T)$ is h-disconnected space but not disconnected.

Proposition 2.23: Every h-totally disconnected space with at least two points is h- disconnected

Proof: Suppose is h-totally disconnected and. If were h-connected, then itself would be a h-connected component. But in a h-totally disconnected space every connected component is a singleton. Hence would have to be a singleton, contradicting. Therefore, is h-disconnected.

Definition 2.24: A function f from space X into space Y is called h -open whenever every open set U in X , we get $f(U)$ is h -open in Y .

Remark 2.25: Each open function is h -open however the converse statement may fail.

Example 2.26: The identity function T_R from R with cofinite topology into R with indiscrete topology is h -open but not open function.

Proposition 2.27: For a bijective h -open function $F:X \rightarrow Y$, so Y is h -totally disc. space in case of X is totally disc. space.

Proof: Let e_1 & e_2 belong to Y with $e_1 \neq e_2$ and by bijectivity, there is m_1 & m_2 in X with $m_1 \neq m_2$ and $F(m_1) = e_1$, $F(m_2) = e_2$. But X is totally disc., so there are two nonempty disjoint open set L and H such that $L \cup H = X$ with $m_1 \in L$ and $m_2 \in H$. Since F is an h -open function, then $F(L)$ & $F(H)$ are h -open sets and $e_1 = F(m_1) \in F(L)$ & $e_2 = F(m_2) \in F(H)$. But $F(L) \cap F(H) = F(L \cap H) = F(\emptyset) = \emptyset$ (since F injective). Also $F(L) \cup F(H) = F(L \cup H) = F(X) = Y$ (with F is a surjective). Infer Y is h -totally disc.

Corollary 2.28: If F is homeomorphism, then the above proposition valid.

Definition 2.29: A function $f:X \rightarrow Y$ is called h^* -open whenever for each h -open set U in X , $f(U)$ has an open image in Y .

Proposition 2.30: every bijective h^* -open function $F:X \rightarrow Y$ if X h -totally disc. then Y is totally disc. also, It's h -totally disc.

Proof: by above **Definition 2.29** and by the same contact the proof.

Definition 2.31: A function $f:X \rightarrow Y$ is defined as inversely h -totally disc. whenever, giving a totally disc. set G in X , then $F(G)$ is totally disc.

Definition 2.32: A function f from space X into space Y is called h^* -totally disc. whenever $G \subseteq X$ is h -totally disc, then the image $f(G)$ is totally disc.

Definition 2.33: A function f from space X into space Y is named h^{**} -totally disc. whenever $G \subseteq X$ is h -totally disc., then the image $f(G)$ is h -totally disc. in Y

Proposition 2.34: Consider the surjective functions $\alpha:X \rightarrow Z$ and $\beta: Z \rightarrow Y$. we have:

- 1- let ℓ is h -light function and α be h^{**} -totally disc. function it follows that β is an h -light function.
- 2- let β be injective function and ℓ is an h -light function we have ℓ is an h -light function

Proof:

- 1- Assume $y \in Y$, then $\ell^{-1}(y)$ be an h -totally disc. set within X . But α satisfies the h^{**} -totally disc., so $\alpha(\ell^{-1}(y))$ inherits h -totally disc. in Z . Since $\alpha((\beta \circ \alpha)^{-1}(y)) = \alpha(\alpha^{-1} \circ \beta^{-1}(y)) = \alpha(\alpha^{-1}(\beta^{-1}(y))) = \beta^{-1}(y)$. So, β is an h -light function.
- 2- Assume $z \in Z$, then $\beta(z) \in Y$. But ℓ is h -light function then $\ell^{-1}(\beta(z))$ is h -totally disc. set in X since $\ell^{-1}(\beta(z)) = (\beta \circ \alpha)^{-1}(\beta(z)) = \alpha^{-1} \circ \beta^{-1}(\beta(z)) = \alpha^{-1}(\beta^{-1}(\beta(z))) = \alpha^{-1}(z)$ infer α is an h -light function.

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