

Optimization-Homotopy-Taylor Series According to Laplace Technique for generalized Non-linear Pantograph Delay Differential Equations of Higher Order to Obtain Analytic and Closed Solutions

Authors Names	ABSTRACT
Ahmed Hanoon Abud^a Publication date: 2 /10 /2026 Keywords Approximate solution, Delay differential equations, Optimization-Homotopy, Taylor series, Laplace transform.	In this study, optimization-homotopy technique which is a powerful tool to find the analytic solution for non-linear generalized delay Pantograph differential equations of higher order with the concept of homotopy approach is used. Taylor series with Laplace technique is applied to approximate solution. The main advantage for this technique is to get more an accurate solution with more extended of the convergence region for whole time interval therefore, this technique is a capable of provide us a continuous representation of approximate solution, which gives better information and behavior of solution for over time interval for the problem under consideration. It is reducing more complex calculations which need more times. Exact solution is conducted by expand the iteration of classical homotopy results using Taylor series. Selected examples to investigate the efficiency, capability, accurately and easily for the proposed technique.

1. Introduction

Closed form is very hard to find, therefore numerically and analytic solutions is very important her¹⁻⁴ (Abdullah S, Safdar ;1-11,2024).Laplace homotopy analysis method to nonlinear fractional models for evolution equations and heat-typed is applied⁵.

Also, interested, the researchers applied different idea for this form⁵⁻⁷, Analytical with numerical Investigation for the Inhomogeneous Pantograph is discussed⁸. Application of Pantograph equation with Caputo derivative is studied⁹. Fractional Pantograph differential equation is solved by Galerkin method¹⁰. Nonlinear Pantograph type diffusion PDEs, exact solutions is discussed¹¹.

.Analytical study of delay Pantograph differential equation to find exact solution¹². A multi-pantograph equation's approximate solution with variable coefficients¹³⁻¹⁴.

$$s(\delta) = \alpha s(\delta) + \sum_{i=1}^3 l_i(\delta) s(g_i \delta) + w(\delta), \quad \delta \geq 0, \quad (1)$$

$s(0) = \gamma$, $\alpha, \gamma \in \mathbb{R}$, with $l_i(s)$ and $w(\delta)$ analytically function ; $0 < g_i < 1$. Also, numerically with analytic solution property of (1) with $w(\delta) = 0$, and $l_i(\delta) = l_i$, studied¹⁴. Adomian-Homotopy for solving non-linear of theses type of equation is presented¹⁵. The generalized pantograph equation is solved numerically using the Taylor method.¹⁶

$$s^{(p)}(\delta) = \sum_{v=0}^V \sum_{k=0}^{p-1} Q_{vk}(\delta) s^{(k)}(\sigma_v \delta + \alpha_v) + w(\delta), \quad (2)$$

with stationary conditions:

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$$\sum_{k=0}^{p-1} a_{ik} s^{(k)}(0) = \varsigma_i, \quad i = 0, 1, \dots, p-1, \quad (3)$$

$Q_{vk}(\delta)$ with $w(\delta)$ analytically function $\sigma_v, \alpha_v, \varsigma_i$ and a_{ik} are complex constant or real. Homotopy with Adomian approach for solving matrix delay differential equation is studied¹⁷. In my study, considered the two type 1 and 2 for this equation.

Type 1.

$$\begin{aligned} s'(\delta) &= \alpha s(\delta) + w(\delta, s(\delta), s(\sigma_1(\delta)), s(\sigma_2(\delta)), \dots, s(\sigma_3(\delta))), \\ s(0) &= s_0. \end{aligned} \quad (4)$$

Type 2.

$$\begin{aligned} s^{(p)}(\delta) &= w(\delta, s(\delta), s(\sigma_1(\delta)), s(\sigma_2(\delta)), \dots, s(\sigma_3(\delta))) \\ \sum_{k=0}^{p-1} a_{ik} s^{(k)}(0) &= \varsigma_i, \quad i = 0, 1, \dots, p-1. \end{aligned} \quad (5)$$

where w and $\sigma_i, \quad i = 1, 2, \dots, 3$, analytical function; a_{ik}, ς_i and α constant.

For this study, approximate and closed solutions for generalized non-linear Pantograph delay differential equations of higher order by optimization-homotopy technique associated with Taylor series according to Laplace transform which make calculations more easier with reduce time and get more accurate solutions and extend for all convergences regions to all time intervals is conducted. These results are increasing whenever more iterate n -times. Present method a capable for providing a continuously representations of approximately solutions, it give best informations and behavior solution to over time intervals for problem under consideration. Exact solution is conducted via Taylor technique to approximate solution. Finally, to know accurate results under delay influence for this technique.

2. Optimization-Homotopy-Taylor Series According to Laplace Technique

Now, we should to review a homotopy approach¹ as:

$$\begin{cases} s'(\delta) = \alpha s(\delta) + w(\delta, s(\delta), s(\sigma_1(\delta)), s(\sigma_2(\delta)), \dots, s(\sigma_3(\delta))), \\ s(0) = s_0. \end{cases} \quad (6)$$

By apply homotopy:

$$\mathcal{L}[\phi(t, p)] = \frac{\partial \phi(t, p)}{\partial t}$$

Non-linear defined:

$$N[\phi(t, p)] = \frac{\partial \phi(t, p)}{\partial t} - \alpha \phi - \phi \quad (7)$$

Then, zero order given:

$$(1-p)\mathcal{L}[\phi(t, p) - s(0)] = p\hbar H(t)N[\phi(t, p)] \quad (8)$$

With $H(t) = 1$ and $\hbar = -1$,

$$(1 - p)\mathcal{L}[\phi(t, p)] - s(0)] + pN[\phi(t, p)] = 0 \quad (9)$$

Moreover; $\phi(0, p) = s_0$

$p = 0$ and $p = 1$, in Eq. (14), as:

$$\mathcal{L}[\phi(t, 0) - s(0)] = 0 \quad (10)$$

$\phi(t, 0) = s(0)$; With:

$$N[\phi(t, 1)] = 0 \quad (11)$$

Then, we have

$$\phi(t, p) = \phi(t, 0) + \sum_{\mathfrak{M}=1}^{\infty} \frac{1}{\mathfrak{M}!} \left. \frac{\partial^{\mathfrak{M}} \phi(t, p)}{\partial p^{\mathfrak{M}}} \right|_{p=0} p^{\mathfrak{M}}$$

from Eq. (10), we have:

$$\phi(t, p) = s(0) + \sum_{\mathfrak{M}=1}^{\infty} s_{\mathfrak{M}}(t) p^{\mathfrak{M}} \quad (12)$$

$$s_{\mathfrak{M}}(t) = \frac{1}{\mathfrak{M}!} \left. \frac{\partial^{\mathfrak{M}} \phi(t, p)}{\partial p^{\mathfrak{M}}} \right|_{p=0}$$

Due to Eq. (11), the matrix series (12) becomes the following if the auxiliary linear operator, the initial condition $s(0)$, the auxiliary parameter h , and the auxiliary function $H(t)$ are all appropriately selected so that the series (12) converges at $p = 1$:

$$s(t) = s(0) + \sum_{\mathfrak{M}=1}^{\infty} s_{\mathfrak{M}}(t)$$

Now, vector is defined: $\vec{s}_n(t) = \{s_0(t), s_1(t), \dots, s_n(t)\}$.

Then, the $\mathfrak{M}$ th-order form is:

$$\mathcal{L}[s_{\mathfrak{M}}(t) - \chi_{\mathfrak{M}} s_{\mathfrak{M}-1}(t)] = \hbar H(t) R_{\mathfrak{M}}(\vec{s}_{\mathfrak{M}-1}) \quad (13)$$

$$\text{And; } R_{\mathfrak{M}}(s_{\mathfrak{M}-1}) = \frac{1}{(\mathfrak{M}-1)!} \left. \frac{\partial^{\mathfrak{M}-1} N[\phi(t, p)]}{\partial p^{\mathfrak{M}-1}} \right|_{p=0}, \quad \chi_{\mathfrak{M}} = \begin{cases} 0, & \mathfrak{M} \leq 1, \\ 1, & \mathfrak{M} > 1. \end{cases}$$

$$R_{\mathfrak{M}} s_{\mathfrak{M}-1} = \frac{\partial s_{\mathfrak{M}-1}}{\partial t} - \alpha s_{\mathfrak{M}-1} - \sum_{i=0}^{\mathfrak{M}-1} R_{\mathfrak{M}}(s_{\mathfrak{M}-1}) + (1 - \chi_{\mathfrak{M}})$$

From above, the $\mathfrak{M}$ th-order form of Eq. (13), $\mathfrak{M} \geq 1$ will be:

$$s_{\mathfrak{M}}(t) = \chi_{\mathfrak{M}} s_{\mathfrak{M}-1} + \hbar \int_0^t (\dot{s}_{\mathfrak{M}-1}(\delta) - s_{\mathfrak{M}-1}(\delta) - \sum_{i=0}^{\mathfrak{M}-1} R_{\mathfrak{M}}(s_{\mathfrak{M}-1}) + (1 - \chi_{\mathfrak{M}})) d\delta$$

$$= \chi_{\mathfrak{M}} s_{\mathfrak{M}-1} + \hbar [s_{\mathfrak{M}-1}(t) - (1 - \chi_{\mathfrak{M}}) s_{\mathfrak{M}-1}(0)] + \hbar \int_0^t (s_{\mathfrak{M}-1}(\delta) - \sum_{i=0}^{\mathfrak{M}-1} R_{\mathfrak{M}}(s_{\mathfrak{M}-1}) + (1 - \chi_{\mathfrak{M}})) d\delta \quad (14)$$

Then $s_{\mathfrak{M}}(t)$, $\mathfrak{M} \geq 1$; easily to find by Mathematica or Maple. It get

$$s(t) = \sum_{k=0}^M s_k(t). \quad (15)$$

Now, Taylor-Laplace is applied to Eq. (15), as follow:

$$\text{Taylor}[\text{Laplace}[s(t) = \sum_{k=0}^M s_k(t)]], \quad (16)$$

After that, optimization technique to Eq. (16) is conducted, and the accuracy is increasing when we more n-times iteration. More accurately with more extended of the convergence region over time interval is obtained. Finally and importantly, exact solution (closed form) is conducted by Taylor series through expand a classical approximation HAM. Proposed technique give more an accurate solution with easily and reduce a time with a complex calculation with capable for providing a continuously representations of approximately solutions, it give best information and behavior of solution for over time interval for the problem under consideration as we see in the following examples.

3. Computational Experiment

Currently, we implement the proposed technique which discussed on different types of non-linear quadratic differential equations of higher order. Result shown this technique is very simply, more an accurate and effectiveness.

Example 1

Consider the non-linear Pantograph delay differential equation of third order¹¹

$$\begin{cases} \{ s'''(t) = 2s^2(\frac{t}{2}) - 1, \\ s(0) = 0, s'(0) = 1, s''(0) = 0. \end{cases} \quad 0 \leq t \leq 15, \quad (17)$$

with the exact solution $s(t) = \sin t$. Now, as in Eq. (16), apply optimization-homotopy technique with Taylor series according to Laplace transform on Eq. (17), which explained above, as:

$$s_m = \chi_m s_{m-1} + \hbar \int_0^t \int_0^t \int_0^t \left(s_{m-1}'''(r) - 2 \left[\sum_{i=0}^{m-1} s_i s_{m-1-i} \left(\frac{r}{2} \right) \right] + (1 - \chi_m)(1) \right) dr dr dr$$

by Mathematica software, very simply to get our results as it shown in Table 1. and Figure 1. below.

Table 1. Numerical results of example 1.

t	Proposed technique	Exact Solution	Absolute Error
0.	0	0	0
1.	0.841470984807897	0.841470984807897	0
2.	0.909297426825682	0.909297426825682	0
3.	0.141120008059867	0.141120008059867	1.2326×10^{-32}
4.5	- 0.977530117665097	- 0.977530117665097	1.97215×10^{-31}
5.5	-0.75540325570385	-0.75540325570392	4.29066×10^{-29}
8.	0.989358246623424	0.989358246623382	1.77987×10^{-27}
10.	- 0.544021110889424	- -0.54402111088937	2.93535×10^{-27}
11.	- 0.999990206551384	- 0.999990206550704	4.63473×10^{-25}
12. 5	- 0.066321897348399 2	- 0.066321897351200 7	7.84838×10^{-24}
13. 5	0.803784426564902	0.803784426551621	1.76377×10^{-22}
15.	0.650287840109716	0.650287840157117	2.2468×10^{-21}

Now for the exact solution, expand the iterations of the homotopy approximation using Taylor series, as:

Taylor series (s_m) = $t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + \frac{1}{362880}t^9$ which is the same exact solution

Taylor series (exact) = $t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + \frac{1}{362880}t^9$.

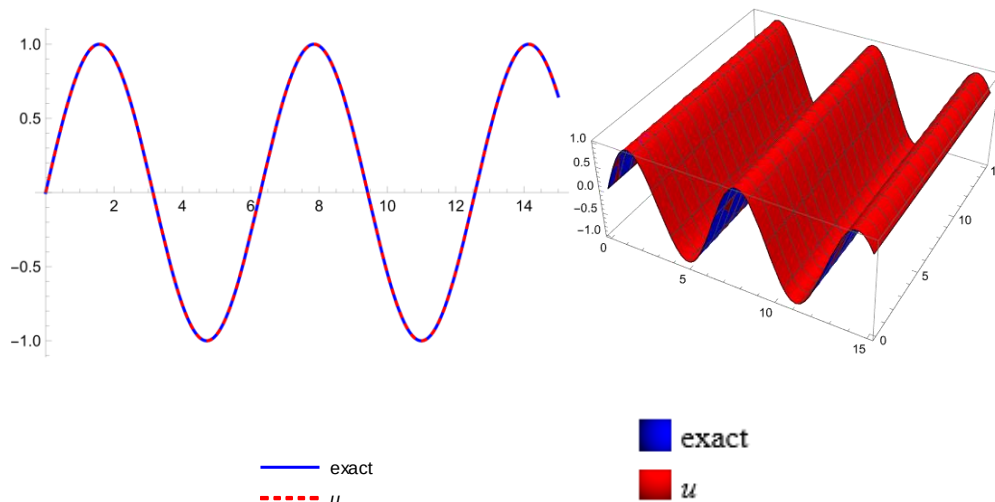


Fig. 1 exact and approximate solution

Example 2

Consider the non-linear pantograph delay differential equation of second order¹¹

$$\begin{cases} s''(t) = \frac{3}{4}s(t) + s(\frac{t}{2}) - t^2 + 2, \\ s(0) = 0, s'(0) = 1, \end{cases} \quad 0 \leq t \leq 15, \quad (18)$$

Closed form $s(t) = t^2$. Now, as in Eq. (16), apply optimization-homotopy technique with Taylor series according to Laplace transform on Eq. (18), which explained above, as:

$$s_m = \chi_m s_{m-1} + \hbar \int_0^t \int_0^t \left(s_{m-1}''(r) - \frac{3}{4}s(r) - s\left(\frac{r}{2}\right) + (1 - \chi_m)(t^2 - 2) \right) dr dr$$

by Mathematica software, very simply to get our results as it shown in Table 2 and Figure 2 below.

Table 2. Numerical results of example 2.

t	Proposed technique	Exact Solution	Absolute Error
0.	0	0	0
1.	0.25	0.25	0

2.	4.	4.	0
3.	9.	9.	0
4.5	20.25	20.25	0
5.5	30.25	30.25	0
8.	64.	64.	0
10.	99.999999	100.	7.29034×10^{-26}
11.	120.99999	121.	3.10015×10^{-22}
12.	156.249999	156.25	2.38148×10^{-17}
13.	182.249999	182.25	2.08035×10^{-14}
15.	224.9999	225.	2.21205×10^{-10}

Now, for the exact solution, expand the iterations of the homotopy approximation using Taylor series, as: Tylor series $s(t) = t^2$ which is the same a closed form. Taylor series (exact)= t^2 .

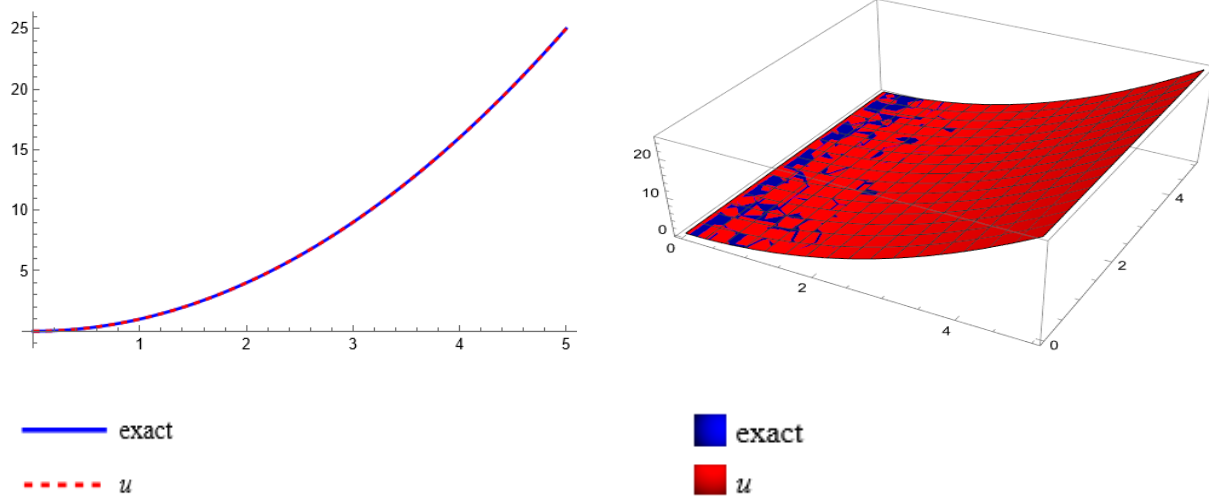


Fig. 2 exact and approximate solution

Example 3

Consider the non-linear Pantograph delay differential equation of first order¹¹

$$\begin{cases} s'(t) = -2s^2\left(\frac{t}{2}\right) + 1, \\ s(0) = 0. \end{cases} \quad 0 \leq t \leq 15, \quad (18)$$

Closed form $s(t) = \sin t$. Now, as in Eq. (16), apply optimization-homotopy technique with Taylor series according to Laplace transform on Eq. (18), which explained above and by Mathematica software, very simply to get our results as it shown in Table 3 and Figure 3 below.

t	Proposed technique	Exact Solution	Absolute Error
0.	0	0	0
1.	0.841470984807897	0.841470984807897	0
2.	0.909297426825682	0.909297426825682	0
3.	0.141120008059867	0.141120008059867	6.93335×10^{-33}
4.5	- 0.977530117665097	- 0.977530117665097	1.10934×10^{-31}
5.5	- 0.705540325570385	- 0.705540325570392	4.58649×10^{-29}
8.	0.989358246623405	0.989358246623382	5.23064×10^{-28}
10.	- 0.544021110889415	- -0.54402111088937	2.0619×10^{-27}
11.	-0.99999020655054	- 0.999990206550704	2.68894×10^{-26}
12.5	- 0.066321897349442	- 0.066321897351200 7	3.09289×10^{-24}
13.5	0.803784426563327	0.803784426551621	1.37032×10^{-22}
15.	0.650287840113028	0.650287840157117	1.94382×10^{-21}

Numerical results of example 3.

Now, for the exact solution, expand the iterations of the homotopy approximation using Taylor series,

as: Tylor series $(s_m) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362880}$, which is the same exact solution

Taylor series (exact) $= t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362880}$.

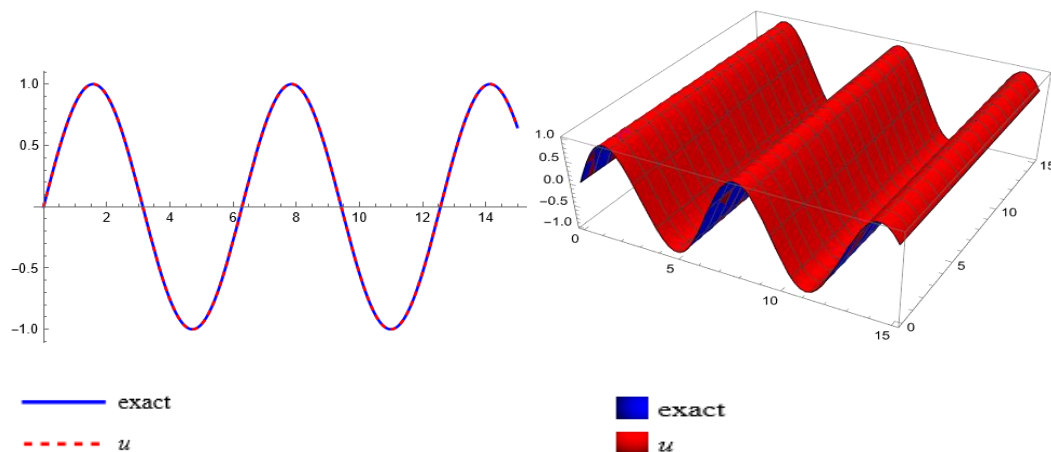


Fig. 3 exact and approximate solution

4. Conclusions

Optimization-Homotopy-Taylor technique according to Taylor series Laplace transform has been successfully implemented for solving non-linear generalized delay pantograph differential equations with higher order. Some examples are presented to investigate the reliability of this approach. We observe, from the above examples that the result for approach is great despite having only a few iteration. Results indicated that This approach can converge to precise results with the fewest iterations accurately solutions with a wider range of the convergence region over time interval as well as reduce a time of calculations and complex computation and we can improve all results with more iteration. This technique is a capable of provide us a continuous representation of approximate solution, which gives better information and behavior of solution for over time interval for the problem under consideration. Importantly, Exact solution is conducted by expand the iteration of classical HAM results using Taylor series.

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