

Structural and Computational Analysis of Weak Hausdorff-Type Separation in Alpha-Topologized Graphs

Authors Names	ABSTRACT
Balqees K. Mahmoud ^a <i>Publication date: 3 /10 /2026</i> <i>Keywords: alpha-open sets; computational topology; graph incidence topology; Hausdorff separation; II-induced topology; pendant vertices; weak separation</i>	Weak Hausdorff-type conditions on graph-associated topologies are usually expressed as qualitative yes/no properties, even though finite graph models allow the unavoidable overlap of neighborhoods to be measured exactly. This paper develops a structural and computational framework for alpha-topologized graphs. For the incidence topology generated by edge-endpoint sets, three quantitative invariants are studied: the alpha-overlap order, the mean alpha-overlap, and the zero-overlap separation ratio. A bounded alpha-weak Hausdorff hierarchy alpha-WH(k) is introduced to avoid the collapse of the ordinary finite-intersection condition on finite spaces. The incidence model is shown to be alpha-stable, and its complete separation structure is characterized by pendant vertices. The total overlap is expressed by a pendant separation load, from which sharp extremal results and closed formulas for standard graph families are obtained. To ensure that the alpha operator plays a genuinely nontrivial role, a second model based on the II-induced topology is analyzed. In that model the alpha-refinement changes the topology for 463 of the 994 connected non-isomorphic graphs of orders three through seven, and overlap orders as large as seven occur. Exhaustive Graph Atlas calculations compare the two models, while a reproducible benchmark on random labeled trees demonstrates the computational advantage of the linear-time pendant-load algorithm over a quadratic pairwise scan. The results provide a quantitative bridge between weak topological separation and measurable graph structure, while clearly separating alpha-stable and non-alpha-stable graph topologies.

1. Introduction

Separation axioms describe how effectively a topology distinguishes different points by neighborhoods. In topological structures induced by graphs, this question has a direct combinatorial interpretation because neighborhood overlap can reflect adjacency, incidence, degree, and local support structure. The alpha-open sets introduced by Njåstad [1] provide a classical generalization of openness, while graph-associated topologies have supplied several discrete models in which topological properties can be translated into graph-theoretic conditions [4, 5, 9, 10, 11, 12, 13].

Previous work by Mahmoud and Yousif considered cut-points and separations in alpha-connected spaces [6], compatibility and edge structures in alpha-topological spaces [7], and feebly Hausdorff spaces in alpha-topological spaces using graphs [8]. In the latter paper, alpha-Hausdorff, alpha-almost Hausdorff, alpha-weakly Hausdorff, and alpha-feebly Hausdorff conditions were related through qualitative implications and graph conditions such as finite and unique adjacency. That framework motivates a quantitative question that was not addressed there: when separation fails, how much neighborhood overlap is unavoidable, and which graph structures are responsible for it?

The question is particularly important for finite graphs. If weak Hausdorff separation is defined only by requiring two neighborhoods to have finite intersection, then every finite topological space satisfies that finiteness condition automatically. A useful computational hierarchy must therefore retain the size of the intersection rather than only its finiteness. This paper introduces a bounded condition alpha-WH(k), together with normalized overlap indices that measure worst-case, average, and complete pairwise separation.

A second issue concerns the role of alpha-openness itself. In the incidence topology used for the main structural theorems, the alpha-refinement turns out to coincide with the original incidence topology. This alpha-stability is mathematically useful because it yields exact formulas, but it also makes it necessary to test the proposed indices on a graph topology where alpha-refinement is genuinely nontrivial. For that purpose, the paper also studies the II-induced topology introduced from the independent and incidence topologies [12]. Recent graph-topological work, including separation axioms for II-induced spaces [12] and P-topological graph spaces [13], shows that distinct graph

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constructions can have substantially different separation behavior.

The main contributions are: (i) a precise bounded weak Hausdorff hierarchy based on minimal alpha-neighborhood overlap; (ii) exact incidence-topology characterizations in terms of leaves and support vertices; (iii) a sharp extremal theorem identifying the star as the unique connected graph with minimum zero-overlap ratio and leaf-free graphs as the maximizers; (iv) a non-alpha-stable II-induced comparison in which overlap orders greater than one occur; (v) exhaustive calculations over all 994 connected non-isomorphic graphs of orders three through seven; and (vi) a reproducible large-graph benchmark showing the scaling advantage of the closed structural formula.

2. Preliminaries and Graph Topologies

Throughout the paper, $G=(V,E)$ is a finite simple graph. Unless explicitly stated otherwise, the graph is connected and has no isolated vertices. Its order and size are $n=|V|$ and $m=|E|$. The degree of v is $d(v)$. A vertex of degree one is called a leaf (or pendant vertex); its unique neighbor is its support vertex. Standard topological terminology concerning interiors, closures, and neighborhoods follows Willard [2], while standard graph terminology follows [3, 16].

Definition 2.1. For an edge $e=uv$, let $I(e)=\{u,v\}$. The incidence topology $\tau I(G)$ on V is generated by the subbasis $\{I(e): e \in E\}$ [4, 11].

Definition 2.2. Let $N(v)$ be the open graph neighborhood of v . Define $D(v)=V \setminus N(v)$, the set consisting of v together with the vertices not adjacent to v . The independent topology $\tau D(G)$ is generated by the subbasis $\{D(v): v \in V\}$; this is the independent-topology convention used in the II-induced construction [12].

Definition 2.3. The II-induced topology is $\tau II(G)=\tau D(G) \cap \tau I(G)$, i.e. the family of sets that are open in both the independent and incidence topologies [12].

Definition 2.4. For a topological space (X,τ) , a set $A \subseteq X$ is alpha-open if $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$ [1]. The family of alpha-open sets is denoted τ^α ; Njåstad showed that this family forms a topology [1].

Definition 2.5 (minimal alpha-neighborhood). Let (X,τ) be a finite topological space. Since τ^α is a topology [1], the standard minimal-neighborhood construction for finite topological spaces [17] applies to (X,τ^α) . For $x \in X$, define $M_\alpha(x) = \bigcap \{U \in \tau^\alpha: x \in U\}$. Because X is finite, only finitely many alpha-open neighborhoods occur, so this intersection is itself alpha-open.

Convention 2.6 (almost-Hausdorff terminology). To remove an ambiguity that can arise from the phrase “exactly one point,” this paper uses alpha-almost Hausdorff to mean that every two distinct points admit alpha-open neighborhoods whose intersection has cardinality at most one. Equivalently, this is alpha-WH(1) below. This convention preserves the monotone implication alpha-Hausdorff $\Rightarrow$ alpha-almost Hausdorff and is the only almost-Hausdorff convention used in the computational results.

3. Quantitative Weak Hausdorff-Type Separation

Definition 3.1. For distinct $x,y \in X$ define the pairwise alpha-overlap

$$ca(x,y) = |M_\alpha(x) \cap M_\alpha(y)|.$$

For an integer $k \geq 0$, the space satisfies bounded alpha-weak Hausdorff separation of order k , abbreviated alpha-WH(k), if for every distinct x and y there exist alpha-open neighborhoods U of x and W of y with $|U \cap W| \leq k$.

Theorem 3.2. For distinct x and y , $ca(x,y)$ is the minimum possible value of $|U \cap W|$ over all alpha-open neighborhoods U of x and W of y .

Proof. Every alpha-open neighborhood of x contains $M_\alpha(x)$, and every alpha-open neighborhood of y contains $M_\alpha(y)$. Hence $M_\alpha(x) \cap M_\alpha(y) \subseteq U \cap W$. Choosing $U=M_\alpha(x)$ and $W=M_\alpha(y)$ attains this lower bound. $\square$

Definition 3.3. The alpha-overlap order is

$$\kappa_\alpha(G) = \max \{ca(x,y): x,y \in V, x \neq y\}.$$

Thus $\kappa_\alpha(G)$ is the smallest k for which alpha-WH(k) holds. In particular, alpha-Hausdorff is equivalent

to $\kappa=0$, and the alpha-almost Hausdorff convention of this paper is equivalent to $\kappa \leq 1$.

Definition 3.4. Let $N=C(n,2)$. The mean alpha-overlap and zero-overlap separation ratio are

$$\bar{\omega}_\alpha(G) = (1/N) \sum_{x < y} c_\alpha(x,y), \quad Z_\alpha(G) = (1/N) |\{ \{x,y\} : c_\alpha(x,y)=0 \}|.$$

The number $Z_\alpha(G)$ lies in $[0,1]$. It equals one when every pair of distinct vertices admits disjoint minimal alpha-neighborhoods, while values near zero indicate pervasive unavoidable overlap.

Proposition 3.5 (matrix form). Let B_α be the zero-one matrix whose (i,j) -entry is 1 exactly when $v_j \in M_\alpha(v_i)$. Then $C_\alpha = B_\alpha B_\alpha^T$ has off-diagonal entry (i,j) equal to $c_\alpha(v_i, v_j)$.

Proof. The scalar product of rows i and j counts exactly the points belonging to both minimal alpha-neighborhoods. $\square$

4. Exact Structure in the Incidence Alpha-Topology

Lemma 4.1. In the incidence topology of a simple graph without isolated vertices, the minimal open neighborhood $U(v)$ is $\{v\}$ when $d(v) \geq 2$, and $\{v, s(v)\}$ when $d(v)=1$.

Proof. The subbasic sets containing v are the endpoint sets of the edges incident with v . If at least two distinct edges are incident with v , simplicity forces their common endpoint to be v . If exactly one edge is incident with v , its endpoint set is $\{v, s(v)\}$. $\square$

Theorem 4.2 (alpha-stability). For every finite simple graph without isolated vertices, $\tau I(G)^\alpha = \tau I(G)$.

Proof. Every incidence-open set is alpha-open. Conversely, let A be alpha-open and $v \in A$. If $d(v) \geq 2$, Lemma 4.1 gives $U(v) = \{v\} \subseteq A$. Suppose $d(v)=1$ and $s=s(v)$. If $s \notin A$, then $U(v) = \{v, s\}$ is not contained in A , so $v \notin \text{Int}(A)$. Since $s \notin A$, also $s \notin \text{Int}(A)$, and therefore $U(v) \cap \text{Int}(A) = \emptyset$. In a finite topology, $v \in \text{Cl}(B)$ exactly when $U(v) \cap B \neq \emptyset$, so $v \notin \text{Cl}(\text{Int}(A))$, hence $v \notin \text{Int}(\text{Cl}(\text{Int}(A)))$, contradicting alpha-openness. Thus $s \in A$ and $U(v) \subseteq A$. Every point of A has its minimal incidence neighborhood inside A , so A is incidence-open. $\square$

Theorem 4.2 makes the incidence model exactly computable: no enumeration of additional alpha-open sets is needed.

Theorem 4.3 (pairwise characterization). Let G be connected with $n \geq 3$. For distinct vertices x and y , $c_\alpha(x,y)=1$ exactly when either (i) one is a leaf and the other is its support vertex, or (ii) x and y are distinct leaves with the same support. In every other case $c_\alpha(x,y)=0$.

Proof. By Lemma 4.1 and Theorem 4.2, every non-leaf has minimal alpha-neighborhood $\{v\}$, while a leaf l has $\{l, s(l)\}$. Two singleton neighborhoods are disjoint. A leaf neighborhood meets a singleton neighborhood precisely at its support, and two leaf neighborhoods can meet only at a common support. $\square$

Corollary 4.4. For connected G with $n \geq 3$, $\kappa(G)=0$ exactly when G has no leaves; otherwise $\kappa(G)=1$. Hence the incidence alpha-topology is alpha-Hausdorff iff $\delta(G) \geq 2$, while every such connected graph is alpha-WH(1).

Definition 4.5. Let $L(G)$ be the set of leaves and $S(G)$ their support vertices. For $s \in S(G)$, let $\ell(s)$ be the number of leaves adjacent to s . Define the pendant separation load

$$\Pi(G) = |L(G)| + \sum_{s \in S(G)} C(\ell(s), 2).$$

Theorem 4.6. For every connected simple graph of order $n \geq 3$,

$$\sum_{x < y} c_\alpha(x,y) = \Pi(G), \quad \bar{\omega}_\alpha(G) = \Pi(G)/C(n,2), \quad Z_\alpha(G) = 1 - \Pi(G)/C(n,2).$$

Proof. Each leaf contributes one positive-overlap pair with its support. For a fixed support s , every pair among its $\ell(s)$ leaves contributes one additional positive-overlap pair. Theorem 4.3 shows that there are no other positive-overlap pairs and that every positive overlap equals one. $\square$

Theorem 4.7 (sharp extremal theorem). Among connected simple graphs on $n \geq 3$ vertices,

$$0 \leq \Pi(G) \leq C(n,2), \quad 0 \leq Z_\alpha(G) \leq 1.$$

Moreover, $Z_\alpha(G)=1$ iff G has no leaves (equivalently $\delta(G) \geq 2$), and $Z_\alpha(G)=0$ iff G is the star $K_{1,n-1}$.

Proof. Let $L=|L(G)|$. By Theorem 4.6 and concentration of the leaf multiplicities,

$\Pi(G) \leq L + C(L, 2) = C(L+1, 2) \leq C(n, 2)$. Equality at the upper end requires $L = n-1$ and all leaves to share the unique non-leaf support, which is exactly $K_{1, n-1}$. The lower equality $\Pi = 0$ is equivalent to $L = 0$. Substitution into the formula for Z gives the stated characterizations. $\square$

Theorem 4.8 (fixed-leaf bounds). If G has L leaves, then

$$L \leq \Pi(G) \leq L + C(L, 2).$$

The lower equality occurs when each support has at most one leaf, and the upper equality when all leaves share one support. Thus, for fixed n and L , concentrating leaves lowers $Z\alpha$, whereas dispersing them raises $Z\alpha$ whenever both arrangements are feasible.

Proposition 4.9 (pendant update). Attach a new leaf to a non-leaf support s having r leaf neighbors. The attachment creates $r+1$ new positive-overlap pairs, so Π increases by $r+1$.

$$\Pi(G+) = \Pi(G) + r + 1, \quad Z\alpha(G+) = 1 - [\Pi(G) + r + 1] / C(n+1, 2).$$

Corollary 4.10 (support-concentration principle). Move a leaf from a support with r leaves to one with q leaves, keeping both supports non-leaves. Then $\Delta\Pi = q - r + 1$. Hence $Z\alpha$ increases when the original support has at least two more leaves than the destination.

4.1 Linear-time incidence computation

The closed formula makes it unnecessary to construct $\tau I(G)$, enumerate alpha-open sets, or examine all $C(n, 2)$ vertex pairs. A single degree scan identifies leaves and support multiplicities.

Algorithm 1. Input a connected simple graph $G = (V, E)$, $n \geq 3$. (1) compute all degrees; (2) for every leaf l , increment the counter $\ell(s(l))$; (3) compute $\Pi = |L| + \sum C(\ell(s), 2)$; (4) set $\kappa = 0$ if $|L| = 0$ and $\kappa = 1$ otherwise; (5) return $\bar{\omega} = \Pi / C(n, 2)$ and $Z = 1 - \Pi / C(n, 2)$.

Proposition 4.11. With adjacency-list input, Algorithm 1 runs in $O(n+m)$ time and $O(n)$ auxiliary memory.

5. Exact Formulas for Standard Incidence Families

Proposition 5.1 (paths). For P_3 , $\Pi = 3$ and $Z = 0$. For $n \geq 4$, $\Pi(P_n) = 2$, $\kappa(P_n) = 1$, and $Z(P_n) = 1 - 4/[n(n-1)]$.

Proposition 5.2 (stars). For $K_{1, m}$, $m \geq 2$, $\Pi = C(m+1, 2)$, $\kappa = 1$, $\bar{\omega} = 1$, and $Z = 0$.

Proposition 5.3 (leaf-free families). For cycles C_n ($n \geq 3$), complete graphs K_n ($n \geq 3$), and complete bipartite graphs $K_{p, q}$ with $p, q \geq 2$, $\Pi = 0$, $\kappa = 0$, and $Z = 1$.

Proposition 5.4 (double stars). For the double star $D_{p, q}$ obtained by joining two supports and attaching p and q leaves to them,

$$\begin{aligned} \Pi(D_{p, q}) &= p + q + C(p, 2) + C(q, 2), \\ Z\alpha(D_{p, q}) &= 1 - [p + q + C(p, 2) + C(q, 2)] / C(p + q + 2, 2). \end{aligned}$$

Thus $D_{3, 1}$ and $D_{2, 2}$ both have six vertices and four leaves, but their separation ratios are $8/15 = 0.5333$ and $9/15 = 0.6000$. This illustrates that leaf concentration, not only leaf count, controls weak separation.

Table 1. Exact incidence-alpha separation indices for representative graph families.

Graph	n	Leaves	$\Pi(G)$	κ	Mean overlap	$Z\alpha$
P_3	3	2	3	1	1.0000	0.0000
P_4	4	2	2	1	0.3333	0.6667
P_6	6	2	2	1	0.1333	0.8667
P_{10}	10	2	2	1	0.0444	0.9556
$K_{1, 4}$	5	4	10	1	1.0000	0.0000
$K_{1, 6}$	7	6	21	1	1.0000	0.0000
$D_{2, 2}$	6	4	6	1	0.4000	0.6000
$D_{3, 1}$	6	4	7	1	0.4667	0.5333
C_6	6	0	0	0	0.0000	1.0000
K_5	5	0	0	0	0.0000	1.0000
$K_{2, 3}$	5	0	0	0	0.0000	1.0000

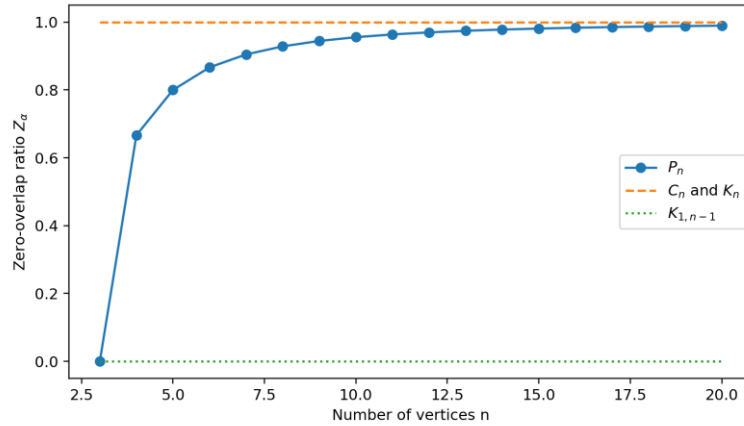


Figure 1. Exact incidence-alpha zero-overlap ratio for standard graph families, $3 \leq n \leq 20$.

5.1 Worked matrix example: P4

Take P4 in the vertex order v_1, v_2, v_3, v_4 . Then $M(v_1) = \{v_1, v_2\}$, $M(v_2) = \{v_2\}$, $M(v_3) = \{v_3\}$, and $M(v_4) = \{v_3, v_4\}$, so

$$B\alpha = [[1, 1, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 1, 1]],$$

$$C\alpha = B\alpha B\alpha^T = [[2, 1, 0, 0], [1, 1, 0, 0], [0, 0, 1, 1], [0, 0, 1, 2]].$$

The entries above the main diagonal sum to 2, hence $\Pi(P4) = 2$. Their maximum is 1, and $Z(P4) = 1 - 2/C(4, 2) = 2/3$.

6. Beyond Alpha-Stability: II-Induced Alpha-Topology

The incidence formulas are exact, but Theorem 4.2 shows that alpha-refinement does not change τI . To see how the overlap indices behave when refinement is nontrivial, we now use $\tau II = \tau D \cap \tau I$ [12], where κ may exceed one.

Example 6.1 (strict alpha-refinement on P3). Let P3 have vertices $\{1, 2, 3\}$ and edges 12 and 23. Then

$$\tau D = \{\emptyset, \{2\}, \{1, 3\}, V\}, \quad \tau I = \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, V\},$$

$$\tau II = \{\emptyset, \{2\}, V\}, \quad \text{but } \tau II^\alpha = \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, V\}.$$

Hence $\tau II^\alpha \neq \tau II$. The minimal alpha-neighborhoods are $\{1, 2\}$, $\{2\}$, and $\{2, 3\}$; every distinct pair has overlap one, so $\kappa = 1$ and $Z = 0$.

Example 6.2 (a genuinely higher bounded level). For P4, the II-induced topology is indiscrete: $\tau II = \{\emptyset, V\}$. Its alpha-refinement is also indiscrete, so every minimal alpha-neighborhood is V . Therefore $c\alpha(x, y) = 4$ for every distinct pair and $\kappa(P4) = 4$. Thus P4 satisfies alpha-WH(4) but not alpha-WH(k) for $k < 4$ in this model.

Examples 6.1 and 6.2 show why the parameter k is useful: unlike the incidence topology, II-induced alpha-topologies can populate several levels of the bounded hierarchy rather than only $k=0$ and $k=1$.

7. Exhaustive Graph Atlas Experiment

7.1 Data and verification procedure

Calculations used NetworkX version 3.6.1 and the graph_atlas_g function to access the Read-Wilson Graph Atlas [14, 15]. Rather than sample, we retained every connected non-isomorphic graph with $3 \leq n \leq 7$, giving 994 graphs. For each graph, τI and τD were built from the subbases in Section 2 and intersected to obtain τII . We then applied $A \sqcap \text{Int}(\text{Cl}(\text{Int}(A)))$ directly, obtained the minimal alpha-neighborhoods, and calculated $c\alpha$, κ , $\bar{\omega}\alpha$, and $Z\alpha$. Incidence results were also checked with the pendant-load formula; both methods agreed for all 994 graphs.

For the II-induced spaces, we also recorded whether τII^α differed from τII , showing how often alpha-

refinement actually changed the topology.

Table 2. Graph Atlas results for the incidence and II-induced alpha-topologies.

n	Graphs	Inc. $\kappa=0$ %	Inc. mean Z	II $\kappa=0$ %	II $\kappa \leq 1$ %	II mean Z	II α -change %	II max κ
3	2	50.00	0.5000	50.00	100.00	0.5000	50.00	1
4	6	50.00	0.7500	16.67	50.00	0.4167	33.33	4
5	21	52.38	0.8762	14.29	66.67	0.4762	38.10	5
6	112	54.46	0.9345	8.93	66.07	0.5256	42.86	6
7	853	59.44	0.9654	6.33	68.70	0.6035	47.36	7

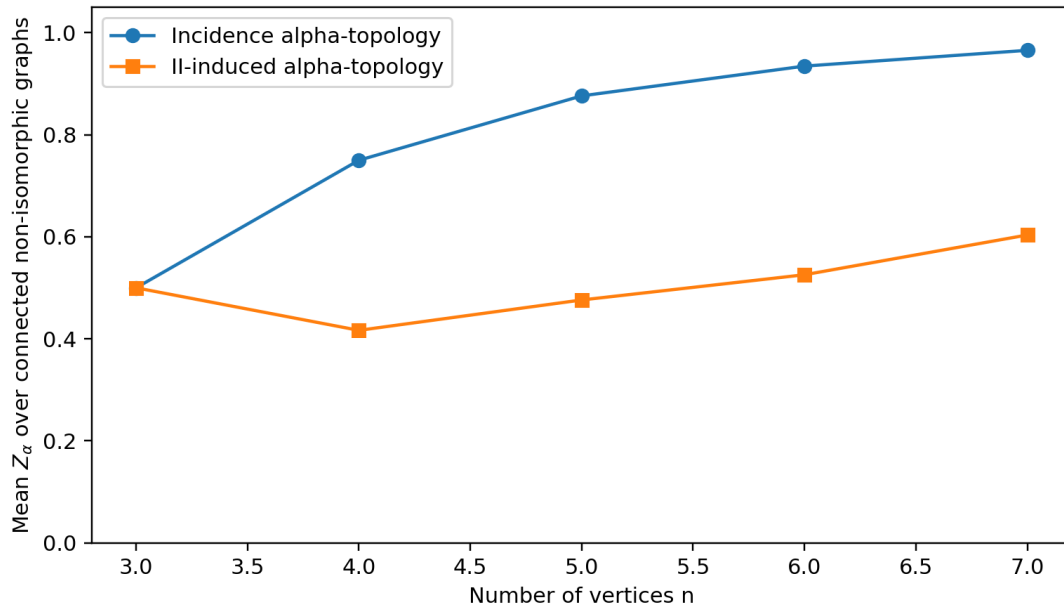


Figure 2. Average zero-overlap ratio Z_α for the incidence and II-induced models on connected non-isomorphic graphs with $3 \leq n \leq 7$.

7.2 Interpretation of the numerical results

The two constructions behave differently on the same atlas sample. In the incidence model, alpha-refinement never changes the topology and κ never exceeds one. In the II-induced model, τII^α differs from τII for 463 of 994 graphs (46.58%), and κ reaches 7. Of these spaces, 69 are alpha-Hausdorff and 679 satisfy alpha-WH(1), so the II-induced model occupies more levels of the bounded hierarchy. Mean Z_α also separates the two models. For incidence it rises from 0.5000 at order three to 0.9654 at order seven, compared with 0.5000 and 0.6035 for the II-induced model. Incidence overlap is confined to leaves and supports, whereas $\tau D \cap \tau I$ can produce larger minimal neighborhoods and therefore broader unavoidable overlap.

8. Large-Graph Computational Scaling

To compare computational cost, we timed two incidence implementations on seeded random labeled trees with $n=100, 250, 500, 1000, 2000$. One used the pendant-load formula; the other scanned all $C(n, 2)$ vertex pairs. Both had to return the same Π , κ , $\bar{\omega}$, and Z values before timing. Table 3 gives median wall-clock times from the same Python environment. Absolute timings depend on implementation, but the growth follows $O(n+m)$ versus $O(n^2)$.

Table 3. Illustrative runtime benchmark on seeded random labeled trees.

n	m	Leaves	Pendant-load ms	Pairwise ms	Speedup
100	99	41	0.058	1.020	17.7×
250	249	99	0.054	4.015	74.3×
500	499	187	0.114	16.436	144.6×
1000	999	366	0.217	69.603	320.0×
2000	1999	736	0.444	273.982	617.4×

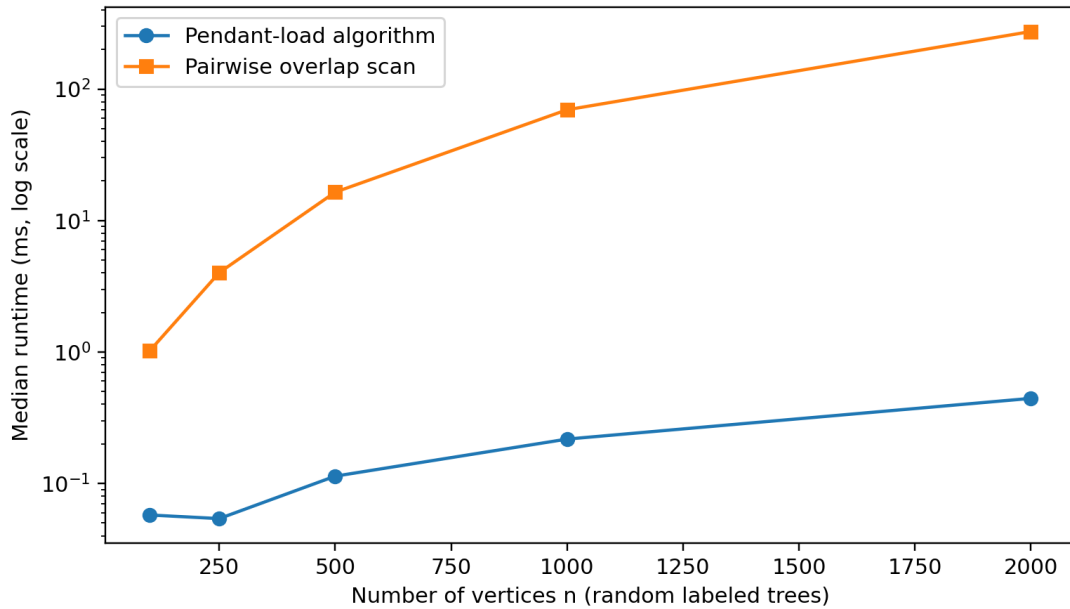


Figure 3. Median runtime scaling of the incidence pendant-load algorithm versus a direct pairwise overlap scan.

At $n=2000$, the structural formula required about 0.44 ms in the present implementation, compared with about 274 ms for the pairwise scan. The exact timing factor should not be interpreted as universal hardware performance, but the observed growth agrees with the theoretical linear-versus-quadratic complexity analysis.

9. Discussion

The present framework strengthens earlier qualitative weak-Hausdorff work in three directions. First, it replaces a binary label by an exact overlap scale. Second, in the incidence model it identifies the complete obstruction structure: leaves cause every positive overlap, and the way leaves are concentrated among supports determines the total separation loss. Third, the II-induced comparison shows that the alpha operator is not intrinsically redundant in graph topology; nearly half of the Graph Atlas spaces tested undergo a strict alpha-refinement, and higher bounded weak-Hausdorff levels occur.

The extremal theorem gives a particularly transparent interpretation of $Z\alpha$. For connected incidence graphs of fixed order, leaf-free graphs achieve perfect pairwise separation $Z=1$, while the star uniquely achieves $Z=0$. Between these extremes, $\Pi(G)$ quantifies the exact penalty caused by leaf-support incidences and shared supports. The fixed-leaf bounds additionally show that two graphs with the same order and the same number of leaves may have different separation quality solely because of leaf concentration.

For the II-induced model, k retains information that a simple “finite intersection” condition loses on a finite space. It records the smallest unavoidable overlap bound. The path examples show the distinction: P3 has strict alpha-refinement with $\kappa=1$, whereas P4 remains indiscrete and requires $k=4$. Thus one graph family can occupy different levels of the bounded hierarchy.

These conclusions are limited to the finite simple-graph setting and the topologies studied here. The incidence formulas rely on incidence-specific proofs and should not be assumed for independent, II-induced, P-topological, directed, weighted, or infinite locally finite settings. Such extensions need separate arguments, which keeps the exact incidence theory distinct from the exploratory II-induced computations.

10. Conclusions

A structural and computational theory of weak Hausdorff-type separation has been developed for alpha-topologized graphs. The bounded alpha-WH(k) condition records the smallest uniform bound on unavoidable neighborhood intersection, while κ , mean overlap, and Z summarize worst-case, average, and complete pairwise separation. In the incidence topology, alpha-refinement is stable and the full separation structure is governed by pendant vertices. The pendant separation load gives exact formulas, sharp extremal characterizations, and a linear-time algorithm. The star $K_{1,n-1}$ is the unique connected graph with $Z=0$, whereas every connected leaf-free graph has $Z=1$.

The II-induced comparison resolves the main limitation of the alpha-stable incidence model. Alpha-refinement is strict for 463 of the 994 connected non-isomorphic graphs of orders three through seven, and overlap orders larger than one occur, reaching seven in the tested range. Together, the incidence and II-induced results show that the proposed indices can describe both exact structural regimes and genuinely nontrivial alpha-refinements. Future work may develop closed characterizations for II-induced and P-topological spaces, study stability under vertex and edge deletion, and extend the overlap framework to directed, weighted, and infinite locally finite graphs.

Reproducibility Note

The finite enumeration was performed using the `graph_atlas_g` function in NetworkX version 3.6.1 to access the Graph Atlas. The connected graphs of orders 3–7 were filtered by order and connectedness, with no random sampling. The larger-tree benchmark used fixed pseudorandom seeds and excluded graph-generation time. For the incidence model, the matrix and pendant-load computations were required to agree exactly. For the II-induced model, the independent and incidence topologies were generated from their stated subbases, intersected to form τII , and then alpha-refined directly from the defining interior-closure-interior condition.

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