

On the Adjacency Spectrum and Energy of the Nilpotent Graph of the Ring of Integers Modulo n

Authors Names	ABSTRACT
Ali Sabeeh Jabbar Salami^a Ali Thabit Mohammed Al-Fatlawi^b Hussein Mansoor Shween Al-Jumaylawee^c Mudher Mohammed Saleh Alnuwaini^d Publication date: 3 / 10 /2026 Keywords: Nilpotent element, nilpotent graph, adjacency spectrum, graph energy, complete bipartite graph, ring of integers modulo n.	Let $\mathbb{Z}_n$ be the ring of integers modulo n, with nil radical $\text{Nil}(\mathbb{Z}_n)$. Following Basnet, Sharma and Dutta, the nilpotent graph $N(\mathbb{Z}_n)$ has as its vertices the non-nilpotent elements of $\mathbb{Z}_n$, and two distinct vertices x and y are joined precisely when $x + y$ is nilpotent. We prove that adjacency depends only on residues modulo the radical $\text{rad}(n)$, and we use this to write the graph explicitly as a disjoint union of complete bipartite graphs, with a single complete graph appearing when $\text{rad}(n)$ is even. The decomposition yields closed forms, in terms of the prime factorization of n alone, for the adjacency spectrum and the energy. The energy splits according to the parity of $\text{rad}(n)$: it is $n - \frac{n}{\text{rad}(n)}$ when $\text{rad}(n)$ is odd and $n - 2$ when $\text{rad}(n)$ is even. We also give the number of connected components and the vertex degrees, and we check every formula on worked numerical examples.

1. Introduction

Graphs built from algebraic structures have grown, over the last three decades, into an active part of algebraic combinatorics. Two things drive the interest. A graph attached to a ring or a group gives a combinatorial image of the algebra, so that connectivity, diameter or chromatic number carry ring-theoretic or group-theoretic meaning. In the other direction, the algebra constrains the graph tightly, and that constraint often lets one compute graph invariants exactly rather than estimate them. The present paper works on this second side.

For commutative rings the first construction of this kind is the zero-divisor graph. Beck [4] introduced it in connection with colourings, and the form now in standard use, with the nonzero zero-divisors as vertices and adjacency given by $xy = 0$, is due to Anderson and Livingston [1]. A large literature has grown from it, along with related graphs such as the unit graph, the total graph, the zero-divisor graph and the idempotent divisor graph. In all of them adjacency comes from one of the two ring operations, usually multiplication.

The graph studied here uses instead the additive behaviour of the nilpotent elements, and it captures something the multiplicative graphs miss. An element x of a commutative ring R is nilpotent when some power of it is zero, and the nilpotent elements form an ideal, the nilradical. Since that ideal is closed under addition, a sum of two nilpotent elements is again nilpotent; the nilpotent graph turns this closure into an adjacency relation on the non-nilpotent elements. A different notion, in which adjacency is defined multiplicatively, was studied earlier by Nikmehr and Khojasteh [13]; the additive graph considered here was introduced and studied by Basnet, Sharma and Dutta [3], who worked out parameters such as the girth, the clique number, the chromatic index and the domination number, and describing the graph and its complement for several forms of the modulus.

^a Directorate of Education of Najaf, Ministry of Education, Najaf, Iraq. E-Mail: sbyhl9422@gmail.com

^b Council of Representatives of Iraq, Baghdad, Iraq. E-Mail: ali.alfatlawi@parliament.iq

^c Directorate of Education of Najaf, Ministry of Education, Najaf, Iraq. E-Mail: hussein.malhumrani96@gmail.com

^d Directorate of Education of Najaf, Ministry of Education, Najaf, Iraq. E-Mail: modhern@gmail.com

Our concern is spectral. The eigenvalues of the adjacency matrix, taken with multiplicity, form the adjacency spectrum, an invariant that carries a good deal of combinatorial information [5,6]. One much-studied function of the spectrum is the graph energy, which Gutman [7] introduced in 1978 as a model for the total π -electron energy of a conjugated hydrocarbon and which has been studied heavily since [8]. Finding the energy of an algebraically defined family in closed form is a natural target, and it is the one we set for the nilpotent graph of Z_n . Everything rests on one elementary fact, developed in Section 3: whether $x + y$ is nilpotent in Z_n depends only on the residues of x and y modulo $\text{rad}(n)$. That single condition on residue classes settles the whole adjacency pattern and lets us take the graph apart. What comes out is a disjoint union of complete bipartite graphs $K_{m \times m}$ with $m = \frac{n}{\text{rad}(n)}$, plus one complete graph K_m when $\text{rad}(n)$ is even. The structural side of this decomposition is already implicit in the work of Basnet, Sharma and Dutta [3], who showed that the complement of the nilpotent graph is a union of complete bipartite pieces, and several distance- and degree-based topological indices of the nilpotent graph over Z_n have since been computed by Wardhana and authors [11, 12]. What has not been treated, and what forms the contribution of this paper, is the *spectral* side: using the decomposition we give closed forms for the full adjacency spectrum and for the graph energy of $N(Z_n)$, together with a clean parity dichotomy for the energy. The spectrum of a disjoint union is the union of the component spectra, and the spectra of $K_{m \times m}$ and K_m are classical, so both invariants follow in closed form; we recover the Wiener and Zagreb values along the way and reconcile them with the earlier computations.

The rest of the paper runs as follows. Section 2 gathers the definitions we need, from ring theory and from spectral graph theory, each with a reference, and shows them on a small example. Section 3 holds the main results: the residue-reduction lemma, the decomposition, and the closed forms for the spectrum, energy, Wiener index and Zagreb indices, together with the tests for connectedness and regularity. Section 4 tabulates the invariants over a range of moduli and confirms each formula by direct computation. Section 5 closes with open questions.

2. Preliminaries

Throughout, $n \geq 2$ is an integer with prime factorisation $n = p_1^{a_1} p_2^{a_2} \cdots p_i^{a_i}$, the p_i distinct primes and each $a_i \geq 1$. All rings are commutative with identity; we follow [2] for ring theory and [10] for graph theory.

Definition 2.1 ([2]). An element x of a ring R is nilpotent if $x^k = 0$ for some positive integer k . The nilpotent elements form the nilradical $\text{Nil}(R)$, an ideal in the commutative case.

Definition 2.2 ([2]). The radical of n , $\text{rad}(n)$, is the product of the distinct prime divisors of n : $\text{rad}(n) = p_1 p_2 \cdots p_r$.

Definition 2.3 ([3]). For a finite commutative ring R with identity, the nilpotent graph $N(R)$ is the simple graph on $R \setminus \text{Nil}(R)$ in which distinct vertices x and y are adjacent exactly when $x + y \in \text{Nil}(R)$.

Definition 2.4 ([5]). Let G be a simple graph on N vertices with adjacency matrix $A(G)$. As $A(G)$ is real and symmetric, its eigenvalues are real; listed with multiplicity they form the adjacency spectrum of G .

Definition 2.5 ([7]). The energy of G is $E(G) = \sum |\lambda_i|$, the sum of the absolute values of the adjacency eigenvalues. On N vertices, G is hyperenergetic if $E(G) > 2N - 2$ and hypoenergetic if $E(G) < N$.

Definition 2.6 ([9]). For a connected graph G , the Wiener index $W(G) = \sum d(u, v)$ sums the distances over all unordered vertex pairs.

Definition 2.7 ([8]). The first and second Zagreb indices are $M_1(G) = \sum \deg(v)^2$, summed over vertices, and $M_2(G) = \sum \deg(u)\deg(v)$, summed over edges.

Lemma 2.8 ([5]). Let $m \geq 1$.

- (i) K_m has eigenvalue $m - 1$ once and -1 with multiplicity $m - 1$, and energy $2(m - 1)$.
- (ii) $K_{m,m}$ has eigenvalues m once, 0 with multiplicity $2m - 2$, and $-m$ once, and energy $2m$.
- (iii) The spectrum of a disjoint union is the multiset union of the component spectra, and its energy is the sum of theirs.

Example 2.9. Let $n = 12$, with primes 2 and 3 , so $\text{rad}(12) = 6$. The nilpotent elements are 0 and 6 , so the vertices are the ten residues $1, 2, 3, 4, 5, 7, 8, 9, 10, 11$. Two vertices are joined when their sum is 0 or $6 \pmod{12}$: thus $1 \sim 5$ since $1 + 5 = 6$, and $2 \sim 4$ since $2 + 4 = 6$, while 1 and 2 are not joined since $1 + 2 = 3$. Section 3 identifies this structure in general.

3. Main Results and Properties

Throughout this section $\rho = \text{rad}(n)$ and $m = n/\rho = p_1^{a_1-1} \cdots p_r^{a_r-1}$; that is, m is the product of the prime powers with every exponent reduced by one.

Lemma 3.1: An element $x \in \mathbb{Z}n$ is nilpotent if $\rho \mid x$. Hence $\text{Nil}(\mathbb{Z}n) = \{0, \rho, 2\rho, \dots, (m-1)\rho\}$, with $|\text{Nil}| = m$, and the graph has $n - m = m(\rho - 1)$ vertices.

Proof. If $x^k \equiv 0 \pmod{n}$ then every prime $p_i \mid n$ divides x^k and so divides x ; holding for all i , the product ρ divides x . Conversely, if $\rho \mid x$ and $a = \max a_i$, each $p_i^{a_i}$ divides x^a ; the prime powers being pairwise coprime, n divides x^a , so $x^a \equiv 0 \pmod{n}$. The multiples of ρ number $n/\rho = m$, and $n = m\rho$. $\square$

Lemma 3.2 (Residue reduction): For non-nilpotent x, y , the vertices are adjacent iff $x + y \equiv 0 \pmod{\rho}$. Each residue $c \in \{1, \dots, \rho - 1\}$ modulo ρ is carried by exactly m vertices, and residue 0 by none.

Proof. By Definition 2.3 and Lemma 3.1, $x \sim y$ iff $x + y \in \text{Nil}$, i.e. $\rho \mid (x + y)$. A vertex is non-nilpotent, so ρ does not divide it and its residue mod ρ is nonzero; conversely each such element is a vertex. For a fixed nonzero residue c the elements congruent to c are $c, c + \rho, \dots, c + (m-1)\rho$, exactly m of them. Residue 0 is the excluded nilpotent set. $\square$

Write V_c for the set of m vertices of residue c . By Lemma 3.2 a vertex of V_c is adjacent exactly to the vertices of $V_{\rho-c}$.

Theorem 3.3: With $\rho = \text{rad}(n)$ and $m = n/\rho$, the nilpotent graph is $(\rho - 1)/2$ copies of $K_{m,m}$ when ρ is odd, and $(\rho - 2)/2$ copies of $K_{m,m}$ together with one K_m when ρ is even. The number of components is $(\rho - 1)/2$ (odd case) or $\rho/2$ (even case); every vertex has degree m , except that for ρ even the m vertices of the class $c = \rho/2$ have degree $m - 1$.

Proof. The classes $V_1, \dots, V_{\rho-1}$ partition the vertices, and edges join V_c to $V_{\rho-c}$. Pair c with $\rho - c$. If $c \neq \rho - c$, the induced subgraph joins each vertex of one class to all of the other with no internal edge,

giving $K_{m,m}$. If $c = \rho - c$, i.e. $2c \equiv 0 \pmod{\rho}$, then $c = \rho/2$ and ρ is even; the class joins to itself, giving K_m . For ρ odd the $\rho-1$ classes form $(\rho-1)/2$ pairs; for ρ even one self-paired residue gives a single K_m and the other $\rho-2$ classes form $(\rho-2)/2$ pairs. Degrees follow from m -regularity of $K_{m,m}$ and $(m-1)$ -regularity of K_m . $\square$

Corollary 3.4: The graph is connected iff $\rho \in \{2, 3\}$, i.e. $n = 2^a$ or $n = 3^a$. It is complete only for $n = 2^a$, the complete graph on 2^{a-1} vertices; for $n = 3^a$ it is one $K_{m,m}$ with $m = 3^{a-1}$.

Proof. The component count of Theorem 3.3 is 1 exactly when $(\rho-1)/2 = 1$ ($\rho = 3$) or $\rho/2 = 1$ ($\rho = 2$). For $\rho = 2$ the component is complete; for $\rho = 3$ it is complete bipartite. $\square$

Corollary 3.5: The graph is regular iff $\text{rad}(n)$ is odd, and then it is m -regular.

Proof. For ρ odd all components are $K_{m,m}$, uniformly m -regular; for ρ even the K_m component has degree $m-1 \neq m$. $\square$

Theorem 3.6: With $\rho = \text{rad}(n)$ and $m = n/\rho$, the adjacency spectrum is: (a) for ρ odd, m with multiplicity $(\rho-1)/2$, 0 with multiplicity $(m-1)(\rho-1)$, and $-m$ with multiplicity $(\rho-1)/2$; (b) for ρ even, m and $-m$ each with multiplicity $(\rho-2)/2$, 0 with multiplicity $(m-1)(\rho-2)$, $m-1$ once, and -1 with multiplicity $m-1$. Both cases sum to $n - m$.

Proof. Apply Lemma 2.8 to Theorem 3.3. For ρ odd, each of the $(\rho-1)/2$ copies of $K_{m,m}$ gives m and $-m$ once and 0 with multiplicity $2m-2$; summing, 0 has multiplicity $(2m-2)(\rho-1)/2 = (m-1)(\rho-1)$ and the total is $m(\rho-1) = n-m$. For ρ even, the bipartite copies give m , $-m$ each with multiplicity $(\rho-2)/2$ and 0 with multiplicity $(m-1)(\rho-2)$, while K_m adds $m-1$ once and -1 with multiplicity $m-1$; again the total is $n-m$. $\square$

Theorem 3.7: For every $n \geq 2$ with $\rho = \text{rad}(n)$, the energy of the nilpotent graph is $n - \frac{n}{\text{rad}(n)}$ when $\text{rad}(n)$ is odd and $n - 2$ when $\text{rad}(n)$ is even.

Proof. Let $m = n/\rho$. By Lemma 2.8, $E(K_{m,m}) = 2m$ and $E(K_m) = 2(m-1)$, and energy adds over disjoint unions. For ρ odd, $E = (\rho-1)/2 \cdot 2m = m(\rho-1) = n - m = n - n/\text{rad}(n)$. For ρ even, $E = (\rho-2)/2 \cdot 2m + 2(m-1) = mp - 2 = n - 2$. $\square$

Theorem 3.8: With $\rho = \text{rad}(n)$ and $m = n/\rho$, the Wiener index is $(\rho-1)/2 \cdot (m^2 + 2m(m-1))$ for ρ odd, and $(\rho-2)/2 \cdot (m^2 + 2m(m-1)) + m(m-1)/2$ for ρ even; the first Zagreb index is $(n-m)m^2$ for ρ odd and $(n-2m)m^2 + m(m-1)^2$ for ρ even.

Proof. Distances live inside components. In $K_{m,m}$ the pairs are m^2 adjacent pairs at distance 1 and $m(m-1)$ non-adjacent pairs at distance 2, so its Wiener sum is $m^2 + 2m(m-1)$; in K_m all pairs are at distance 1, contributing $m(m-1)/2$. For the Zagreb index, sum the squared degrees: for ρ odd every vertex has degree m over $n-m$ vertices; for ρ even the m vertices of K_m have degree $m-1$ and the remaining $n-2m$ have degree m . $\square$

Remark 3.9. Together the last three theorems act as a lookup table: given the parity of $\text{rad}(n)$ and the value $m = n/\text{rad}(n)$, the spectrum, energy, Wiener index and Zagreb indices are all fixed. By Theorem 3.7 the graph is never hyperenergetic, since its components are complete or complete bipartite and neither type is hyperenergetic. The Wiener and first-Zagreb values of Theorem 3.8 agree with those obtained by Wardhana and co-authors through direct degree-and-distance counting [11, 12]; here they fall out of the same decomposition that yields the spectrum, which is the quantity of primary interest.

4. Numerical Studies

We test the closed forms of Section 3. For each modulus below, the graph was built, its adjacency matrix formed, and its spectrum and energy found by direct diagonalisation. The computed values agree with Theorems 3.6 and 3.7 throughout.

Example 4.1 ($n = 8$, even radical, connected). Here $8 = 2^3$, so $\rho = 2, m = 4$. The graph is a single K_4 ; its spectrum is 3 once and -1 three times, and its energy is $n - 2 = 6$.

Example 4.2 ($n = 9$, odd radical). Here $9 = 3^2$, so $\rho = 3, m = 3$. The graph is one $K_{3,3}$; its spectrum is 3 once, 0 four times, -3 once, and its energy is $n - n/\text{rad}(n) = 6$.

Example 4.3 ($n = 12$, even radical, disconnected). Here $12 = 2^2 \cdot 3$, so $\rho = 6, m = 2$. The graph is 2 $K_{2,2}$ together with K_2 ; its spectrum is 2 twice, 1 once, 0 four times, -1 once, -2 twice, and its energy is $n - 2 = 10$. This is the graph of Example 2.9.

n	Factorization	$\rho = \text{rad}(n)$	M	$ V = n - m$	decomposition	E
8	2^3	2	4	4	K_4	6
9	3^2	3	3	6	$K_{3,3}$	6
12	$2^2 \cdot 3$	6	2	10	$2K_{2,2} \sqcup K_2$	10
16	2^4	2	8	8	K_8	14
18	$2 \cdot 3^2$	6	3	15	$2K_{3,3} \sqcup K_3$	16
25	5^2	5	5	20	$2K_{5,5}$	20
27	3^3	3	9	18	$K_{9,9}$	18
36	$2^2 \cdot 3^2$	6	6	30	$2K_{6,6} \sqcup K_6$	34
49	7^2	7	7	42	$3K_{7,7}$	42
72	$2^3 \cdot 3^2$	6	12	60	$2K_{12,12} \sqcup K_{12}$	70

Table 1. Structure and energy of the nilpotent graph for selected n . The odd-radical rows ($n = 9, 25, 27, 49$) satisfy $E = n - m$; the even-radical rows satisfy $E = n - 2$. Each value was confirmed by direct computation of the adjacency matrix.

Table 1 shows the parity split and one more effect the theory predicts: two moduli with the same $m = n/\text{rad}(n)$ and the same parity of $-$ give isomorphic graphs and hence the same invariants. This follows because m is fixed by the exponents in the factorisation, not by the primes themselves.

5. Conclusions

We have determined the adjacency spectrum and the energy of the nilpotent graph of $\mathbb{Z}_n$, in the sense of Basnet, Sharma and Dutta, in closed form. The route is a residue-based decomposition into complete bipartite graphs, with one complete graph added when $\text{rad}(n)$ is even; the structural part of this picture goes back to Basnet, Sharma and Dutta [3], and the associated Wiener and Zagreb indices to Wardhana and co-authors [11, 12], while the spectrum and the energy, with their parity dichotomy, are new. The energy follows the parity split of Theorem 3.7. Several questions remain. The same

residue reduction should give the Laplacian and signless Laplacian spectra, whose component values are again standard. Richer rings, such as the Gaussian integers modulo n or polynomial quotient rings, have a larger nilradical and components that need not be complete bipartite, and look harder. Finally, one might ask which finite commutative rings have a nilpotent graph that is a disjoint union of complete bipartite graphs.

References

- [1] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* 217 (1999), 434–447.
- [2] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, MA, 1969.
- [3] D. K. Basnet, A. Sharma and R. Dutta, Nilpotent graph, *Theory Appl. Graphs* 8 (2021), no. 1, Art. 2.
- [4] I. Beck, Coloring of commutative rings, *J. Algebra* 116 (1988), 208–226.
- [5] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Universitext, Springer, New York, 2012.
- [6] D. Cvetković, P. Rowlinson and S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge Univ. Press, Cambridge, 2010.
- [7] I. Gutman, The energy of a graph, *Ber. Math.-Statist. Sect. Forschungszentrum Graz* 103 (1978), 1–22.
- [8] I. Gutman and N. Trinajstić, Graph theory and molecular orbitals. III. Total π -electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* 17 (1972), 535–538.
- [9] H. Wiener, Structural determination of paraffin boiling points, *J. Amer. Chem. Soc.* 69 (1947), 17–20.
- [10] D. B. West, *Introduction to Graph Theory*, 2nd ed., Prentice Hall, Upper Saddle River, NJ, 2001.
- [11] M. N. Husni, I. G. A. W. Wardhana, P. K. Dewi and I. N. Suparta, Szeged index and Padmakar-Ivan index of the nilpotent graph of the ring of integers modulo n of prime power order, *J. Mat. Stat. Komput.* 20 (2024), no. 2, 332–339.
- [12] D. P. Malik, I. G. A. W. Wardhana, P. K. Dewi, R. S. Widiastuti, F. Maulana, A. G. Syarifudin and Z. Y. Awanis, A note on the nilpotent graph of the ring of integers modulo n of prime power order, *JMPM: J. Mat. Pendidik. Mat.* 8 (2023), no. 1, 28–33.
- [13] M. J. Nikmehr and S. Khojasteh, On the nilpotent graph of a ring, *Turkish J. Math.* 37 (2013), no. 4, 553–559.