Journal of Iraqi Al-Khwarizmi (JIKh) Volume:10 Issue:2 Year: 2026 pages: 473-486

Forward Differences of Entire Solutions of Second-Order Complex Differential Equations

Haneen Abass Saleh

Publication date: 9 /10 /2026

Keywords: *second order complex differential equation, entire solution, coefficient function, forward difference, periodic coefficient, Floquet multiplier, zeros, Wronskian, power series, argument principle*

ABSTRACT

In this paper the effect of the translation of the coefficient by a fixed non-zero amount on the forward differences of entire solutions of second order complex differential equations with entire coefficients is examined. The basic coefficient-difference identity implies that the absence of shift periodicity is the same forcing as provided by translation. If the coefficient induced by a nontrivial forward difference, the full local principal part at each zero is determined and an exact criterion is given for when the induced meromorphic coefficient is an entire function. Coefficient periodicity and preservation of the original homogeneous equation is then described by a shift-Wronskian evolution identity. A nonconstant polynomial exponent is proven to be linear, and for exponential coefficients, the shift preservation condition is simplified to a discrete resonance condition. For the periodic case, classical Floquet translation results in a second-order difference relation, a parabolic-trace consequence and a zero-transfer classification of solutions and higher forward differences. Examples of constant-coefficient, exponential-Bessel, Airy and quadratic-exponential are used to illustrate the theory. Numerical examples of zero-counting, monodromy computation, and power-series residuals are only provided for the purpose of performing reproductions and checks of the exact analytic results.

1. Introduction

The second-order linear differential equations in the complex plane are linked to complex analysis, oscillation theory, value distribution and special-function theory. We take into account the central model.

$$f''(z)+A(z)f(z)=0, \qquad (1)$$

where A is entire. The coefficient can be determined by the growth and zero distribution of the global solutions of the analytic continuation of every solution obtained from finite initial data, since each one is extended analytically to all of the plane. The classical work of Bank and Laine has laid the foundations of the whole coefficient and oscillation of solutions [3] and more recent work has continued in this direction using real zero configurations, Bank-Laine functions, completely regular growth, and deficient values or asymptotic values [5-12]. More recent examples include the classification of second order equations that have a basis of solutions with only real zeroes [5], constructions of zero-distributions for finite order Bank-Laine functions [6] and a result on asymptotic values and deficiencies of functions satisfying y" + A(z)y = 0 [7] for 2026.

The paper by Alhily and Saleh [1] is directly inspiring for the present study. It explores the impact of the coefficient function on the behaviour of solutions of a second order complex differential equation; it explicitly deals with the class H(z) = e^p(z) of coefficient functions; it introduces power series representations and applications. They have also considered growth and boundedness in their previous work [2] via the coefficient function. We keep this coefficient-centred perspective but use another operator on the solution: translation and forward difference,

Department of Mathematics, College of Basic Education, University of Diyala, Diyala, Iraq. Email: basicmathte6@uodiyala.edu.iq

T_c f(z)=f(z+c), Δ_c f=(T_c-I)f. (2)

The most important question is not just what is the behavior of the forward difference, but how does the shift geometry of A control the equation, the zeros, and the higher differences of f? The central structural identity is

L_A (Δ_c f)=-Δ_c A(z)f(z+c)(3)

It shows the coefficient mismatch A(z + c) − A(z) as the exact forcing term that is created when translated.

The second aspect of the literature that is relevant to the problem is the differences of entire and meromorphic functions. Halburd and Korhonen introduced a Nevanlinna theory for the difference operator [13] and [14] and Bergweiler and Langley discussed the zeros of forward differences [15], while Heittokangas et al. connected value sharing of shifts to periodicity [16]. This line is still open for further research, with recent work on partial shared-value conditions and higher-order difference operators [23,24]. The results indicate that translating is not only a discrete stand in for differentiation, but also has global information about a function that is already independent.

A third classical element is added with periodic coefficients: the Floquet theory. Translation by one period keeps the solution space of (1) two dimensional and the period-translation matrix has determinant one due to the fact the Wronskian is constant. It is standard Floquet structure and is not new here but used in the way convenient for entire complex solutions, in the classical style of Magnus and Winkler [21] and Eastham [22].

The contribution and scope of the paper is therefore made deliberately apart from the known theory. There is no need for the Floquet matrix relation to be classical, and the identity (3) is immediate. The manuscript-specific deductions are the local pole classification of the induced coefficient B_c=-g″/g, including an explicit residue formula at simple zeros; the shift-Wronskian evolution law that makes the coefficient mismatch obvious at the level of linear independence; zero consequences at coefficient-mismatch points; and unified treatment of periodic shifts, exact examples, and reproducible computation. These results are accurate and applicable, but here no more than an extension of the present theory, not a replacement for Bank-Laine methods or difference-Nevanlinna methods.

They are organized in this way. The coefficient-shift identities, the induced-coefficient criterion, higher differences and the power-series recurrence are developed in Section 2. The periodic coefficients, the e^p family, Floquet translation and zeros are covered in Section 3. In Section 4 exact worked examples are provided. Section 5 offers computational verification and reproducibility diagnostics. Limitations and clear new directions are provided in Section 6.2.
Coefficient Shifts and Induced Coefficients

Let $L_A = D^2 + A(z)$, so that (1) is $L_A f = 0$.

2.1. Basic coefficient-difference identity

Lemma 2.1 (coefficient-difference identity). Let A be entire, let $f \not\equiv 0$ solve (1), and let $c \neq 0$. If

$$g(z) = \Delta_c f(z) = f(z + c) - f(z),$$

then

$$\boxed{L_A g(z) = [A(z) - A(z + c)] f(z + c).} \quad (4)$$

Proof. Since $f''(z) = -A(z)f(z)$,

$$g''(z) = f''(z+c) - f''(z) = -A(z+c)f(z+c) + A(z)f(z).$$

Adding $A(z)g(z) = A(z)f(z+c) - A(z)f(z)$ yields (4). ▫

Thus the coefficient difference

$$\Delta_c A(z) = A(z+c) - A(z) \tag{5}$$

is exactly the forcing mechanism produced when the shift is measured with the original differential operator.

2.2. Local pole structure of the induced coefficient

Assume $g = \Delta_c f \not\equiv 0$. Away from the zeros of g, define

$$B_c(z) = -\frac{g''(z)}{g(z)}. \tag{6}$$

Using $g = f(z+c) - f(z)$ and the differential equation,

$$B_c(z) = \frac{A(z+c)f(z+c) - A(z)f(z)}{g(z)} = A(z+c) + [A(z+c) - A(z)]\frac{f(z)}{g(z)}.$$

Thus g satisfies

$$g''(z) + B_c(z)g(z) = 0 \tag{7}$$

wherever the quotient is defined. The next result gives the exact local singularity carried by a zero of g.

Theorem 2.2 (local pole classification). Let ζ be a zero of g of multiplicity $m \geq 1$, and write

$$g(z) = (z-\zeta)^m u(z), \qquad u(\zeta) \neq 0.$$

Then the principal part of $B_c = -g''/g$ at ζ is

$$-\frac{m(m-1)}{(z-\zeta)^2} - \frac{2m}{z-\zeta}\frac{u'(\zeta)}{u(\zeta)}.$$

Consequently:

1. if $m \geq 2$, then B_c has an unavoidable double pole at ζ;
2. if $m = 1$, then B_c has at most a simple pole and

$$\operatorname{Res}(B_c;\zeta) = -\frac{g''(\zeta)}{g'(\zeta)} = -\frac{[A(\zeta) - A(\zeta+c)]f(\zeta+c)}{g'(\zeta)};$$

3. the meromorphic function B_c extends to an entire function on $\mathbb{C}$ if and only if every zero ζ of g is simple and

$$[A(\zeta) - A(\zeta+c)]f(\zeta+c) = 0.$$

Proof. From $g = (z-\zeta)^m u$, direct differentiation gives

$$\frac{g''}{g} = \frac{m(m-1)}{(z-\zeta)^2} + \frac{2m}{z-\zeta}\frac{u'}{u} + \frac{u''}{u}.$$

The stated principal part follows immediately. If $m \geq 2$, its quadratic coefficient is nonzero, so no cancellation can remove the double pole. If $m = 1$, then $g'(\zeta) = u(\zeta)$ and $g''(\zeta) = 2u'(\zeta)$, which yields the residue formula. Finally, at a zero of g, identity (4) reduces to

$$g''(\zeta) = [A(\zeta) - A(\zeta+c)]f(\zeta+c).$$

Hence a simple zero is removable precisely when the displayed forcing term vanishes. Applying this condition at every zero proves the global criterion. ▫

Corollary 2.3 (zeros at coefficient-mismatch points). Assume B_c is entire. If $\zeta \in Z(\Delta_c f)$ and

$$A(\zeta) \neq A(\zeta + c),$$

then

$$f(\zeta) = f(\zeta + c) = 0. \tag{8}$$

Proof. Theorem 2.2 forces $f(\zeta + c) = 0$, and $\Delta_c f(\zeta) = 0$ then gives $f(\zeta) = 0$. ▫

2.3. Shift-Wronskian evolution

For a nontrivial solution f, define

$$W_c(f)(z) = W(f, T_c f)(z) = f(z)f'(z + c) - f'(z)f(z + c).$$

Unlike the ordinary Wronskian of two solutions of the *same* equation, this quantity need not be constant when the coefficient is not translation invariant.

Proposition 2.4 (Wronskian mismatch identity). For every entire coefficient A, every nontrivial solution f, and every $c \neq 0$,

$$\boxed{W_c'(f)(z) = [A(z) - A(z + c)]f(z)f(z + c).}$$

In particular, $W_c(f)$ is constant if and only if A is c-periodic.

Proof. Differentiating the shift Wronskian and using $f'' = -Af$ gives

$$W_c' = f(z)f''(z + c) - f''(z)f(z + c) = [A(z) - A(z + c)]f(z)f(z + c).$$

If A is periodic, the right side vanishes. Conversely, if $W_c' \equiv 0$, then the product

$$[A(z) - A(z + c)]f(z)f(z + c)$$

is identically zero. Since neither f nor $f(\cdot + c)$ is identically zero, their product is nonzero on a nonempty open set; the identity theorem therefore forces $A(z + c) = A(z)$ identically. ▫

2.4. Higher forward differences

For $n \geq 1$,

$$\Delta_c^n f(z) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(z + kc). \tag{9}$$

Proposition 2.5. Under the hypotheses of Lemma 2.1,

$$\boxed{L_A(\Delta_c^n f) = A(z)\Delta_c^n f - \Delta_c^n(Af).} \tag{10}$$

Equivalently, expansion of the two forward differences gives

$$L_A(\Delta_c^n f)(z) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} [A(z) - A(z + kc)]f(z + kc).$$

Proof. Differentiation commutes with translation, so $(\Delta_c^n f)'' = \Delta_c^n(f'') = -\Delta_c^n(Af)$. Adding $A(z)\Delta_c^n f$ gives (10), and the expanded expression follows from (9). ▫

2.5. Power-series recurrence

The power-series viewpoint is directly connected with the motivating work [1]. Let

$$A(z) = \sum_{m=0}^{\infty} \alpha_m z^m, \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{11}$$

Substitution into (1) gives

$$(n+2)(n+1)a_{n+2} + \sum_{m=0}^{n} \alpha_m \, a_{n-m} = 0,$$

hence

$$\boxed{a_{n+2} = -\frac{1}{(n+2)(n+1)} \sum_{m=0}^{n} \alpha_m \, a_{n-m}.} \tag{12}$$

The values $a_0 = f(0)$ and $a_1 = f'(0)$ determine the unique entire solution.

3. Periodicity, Floquet Translation, and Zero Transfer

3.1. Preservation of the original homogeneous equation

Proposition 3.1 (periodicity characterization). Let A be entire, let $f \not\equiv 0$ solve (1), and let $c \neq 0$. The following are equivalent:

1. $\Delta_c f$ solves $y'' + A(z)y = 0$;
2. $A(z+c) = A(z)$ for every $z \in \mathbb{C}$;
3. the shift Wronskian $W_c(f)$ is constant.

In particular, the existence of even one nontrivial solution whose first difference solves the same equation, or whose shift Wronskian is constant, forces A to be c-periodic.

Proof. The equivalence of 1 and 2 follows from (4): if $A(z+c) = A(z)$, the forcing term vanishes; conversely, if $L_A(\Delta_c f) = 0$, then

$$[A(z) - A(z+c)]f(z+c) \equiv 0,$$

and the identity theorem gives periodicity. The equivalence of 2 and 3 is Proposition 2.4. ▫

Corollary 3.2. If A is c-periodic, then every $\Delta_c^n f$ is a solution of (1).

Proof. In (10), $A(z+kc) = A(z)$ for all integers k. ▫

3.2. The coefficient family $A(z) = e^{p(z)}$

The class $e^{p(z)}$ is central in the motivating source [1] and in its earlier companion study [2].

Theorem 3.3 (exponential-coefficient resonance). Let

$$A(z) = e^{p(z)},$$

where p is entire, and let $c \neq 0$. The following are equivalent:

1. $A(z+c) = A(z)$ for all z;
2. $p(z+c) - p(z) = 2\pi i k$ for one fixed integer k;
3. there exists a nontrivial solution f of $f'' + e^{p(z)} f = 0$ such that $\Delta_c f$ solves the same equation;
4. for every solution f, $\Delta_c f$ solves the same equation.

If p is a polynomial and nonconstant, then necessarily

$$p(z) = az + b, \qquad ac = 2\pi i k, \qquad k \neq 0. \tag{13}$$

Proof. The equivalence of 1, 3, and 4 follows from Proposition 3.1. If 1 holds, then

$$e^{p(z+c)-p(z)} = 1.$$

Thus $q(z) = p(z + c) - p(z)$ takes values in the discrete set $2\pi i\mathbb{Z}$. Because q is entire and $\mathbb{C}$ is connected, q is constant, proving 2. The converse is immediate. If p is a polynomial of degree $d \geq 2$, then $p(z + c) - p(z)$ has degree $d - 1$, so it cannot be constant. Hence $d \leq 1$. In the nonconstant case $p(z) = az + b$ with $a \neq 0$, and because $c \neq 0$, the resonance constant is nonzero, so $k \neq 0$. ▫

3.3. Floquet Translation

Assume throughout this subsection that A is c-periodic. Let $\mathcal{S}$ be the two-dimensional solution space of (1). Translation defines a linear map

$$T_c: \mathcal{S} \to \mathcal{S}, \qquad (T_c f)(z) = f(z + c). \tag{14}$$

This is the standard period-translation operator of Floquet theory [21,22].

Proposition 3.4 (period-translation relation). Relative to any basis of $\mathcal{S}$, the operator T_c is represented by a matrix $M_c \in SL(2, \mathbb{C})$. If

$$\tau_c = \mathrm{tr} M_c,$$

then every solution of (1) satisfies

$$\boxed{f(z + 2c) - \tau_c f(z + c) + f(z) = 0.} \tag{15}$$

Equivalently,

$$\boxed{\Delta_c^2 f + (2 - \tau_c)\Delta_c f + (2 - \tau_c) f = 0.} \tag{16}$$

Proof. Let f_1, f_2 be a basis. Periodicity makes $f_1(\cdot + c), f_2(\cdot + c)$ solutions, so translation is represented by a constant matrix M_c. Since equation (1) has no first-derivative term, $W(f_1, f_2)$ is constant. Hence

$$W\big(f_1(\cdot + c), f_2(\cdot + c)\big) = W(f_1, f_2),$$

which gives $\det M_c = 1$. The Cayley-Hamilton identity

$$M_c^2 - \tau_c M_c + I = 0$$

yields (15). Substituting $T_c = I + \Delta_c$ gives (16). ▫

Corollary 3.5 (Floquet multiplier formula). If

$$f(z + c) = \mu f(z), \qquad \mu \neq 0, \tag{17}$$

then for every $n \geq 1$,

$$\boxed{\Delta_c^n f = (\mu - 1)^n f.} \tag{18}$$

Corollary 3.6 (parabolic trace). If $\tau_c = 2$, then

$$\Delta_c^2 f \equiv 0 \tag{19}$$

for every solution f, and $\Delta_c f$ is c-periodic. If, in addition, $M_c \neq I$, then the space of c-periodic solutions is one-dimensional; a generic solution is not periodic, but its first difference is a nonzero periodic solution.

Proof. Equation (16) gives $\Delta_c^2 f = 0$. Then

$$T_c(\Delta_c f) - \Delta_c f = \Delta_c^2 f = 0,$$

so $\Delta_c f$ is periodic. If $M_c \neq I$ and $\mathrm{tr}M_c = 2 = 2\sqrt{\det M_c}$, then $M_c - I$ is a nonzero nilpotent 2×2 matrix and its kernel is one-dimensional. ▫

3.4. Zero-transfer classification

For a solution f, define the shift Wronskian

$$W_c(f) = W(f, T_c f). \tag{20}$$

Explicitly, $W_c(f) = f(z)f'(z+c) - f'(z)f(z+c)$. When A is c-periodic, both functions are solutions of the same equation, hence $W_c(f)$ is constant.

Theorem 3.7 (zero-transfer classification). Let A be c-periodic and $f \not\equiv 0$ solve (1).

1. If $W_c(f) \neq 0$, then
$$Z(f) \cap Z(\Delta_c f) = \varnothing. \tag{21}$$
2. If $W_c(f) = 0$, then $f(z+c) = \mu f(z)$ for some $\mu \neq 0$. If $\mu \neq 1$, then for every $n \geq 1$,
$$Z(\Delta_c^n f) = Z(f) \tag{22}$$
with the same multiplicities. If $\mu = 1$, then $\Delta_c f \equiv 0$.

Proof. If $W_c(f) \neq 0$ and ζ were a common zero of f and $\Delta_c f$, then $f(\zeta + c) = f(\zeta) = 0$, so $W_c(f)(\zeta) = 0$, a contradiction. If $W_c(f) = 0$, the two solutions f and $T_c f$ are linearly dependent, so (17) holds. Formula (18) gives the remaining assertions, including equality of multiplicities. ▫

Corollary 3.8 (exact zero-count transfer for an eigen-solution). Under the second alternative of Theorem 3.7, if $\mu \neq 1$, then for every $n \geq 1$ the zero divisors of f and $\Delta_c^n f$ coincide. Hence, for every $r > 0$,

$$n\left(r, \frac{1}{\Delta_c^n f}\right) = n\left(r, \frac{1}{f}\right), \qquad N\left(r, \frac{1}{\Delta_c^n f}\right) = N\left(r, \frac{1}{f}\right),$$

and, whenever the exponent of convergence is defined in the usual sense,

$$\lambda(\Delta_c^n f) = \lambda(f).$$

Thus the Floquet multiplier controls not only zero-set equality but the full zero-counting data of every higher forward difference.

4. Exact Worked Examples

The four examples below illustrate complementary regimes: constant periodic behavior, a periodic exponential-Bessel model with nontrivial Floquet structure, a nonperiodic Airy model, and a quadratic-exponential coefficient treated by power series.

4.1. Constant coefficient: $f'' + f = 0$

Take

$$A(z) = 1, \qquad f'' + f = 0. \tag{23}$$

A basis is $\cos z, \sin z$. For every $c \in \mathbb{C}$, translation is represented by

$$M_c = \begin{pmatrix} \cos c & \sin c \\ -\sin c & \cos c \end{pmatrix}. \tag{24}$$

Thus $\det M_c = 1$ and $\tau_c = 2\cos c$, and every solution satisfies

$$f(z+2c)+f(z)=2\cos(c)f(z+c). \tag{25}$$

For $f(z)=\sin z$ and $c=\pi/2$,

$$\Delta_{\pi/2}f(z)=\cos z-\sin z.$$

The zeros of f are $k\pi$, whereas the zeros of $\Delta_{\pi/2}f$ are $\pi/4+k\pi$; the zero sets are disjoint, as predicted by Theorem 3.7. For $c=\pi$, $f(z+\pi)=-f(z)$, so $\Delta_{\pi}f=-2f$ and the zeros coincide with multiplicity. For $c=2\pi$, $\Delta_{2\pi}f\equiv 0$. The three alternatives of Theorem 3.7 therefore occur already in the constant-coefficient equation.

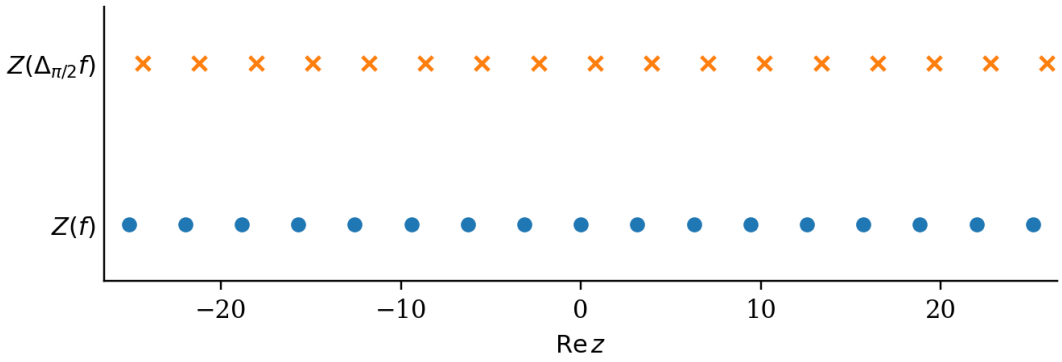


Figure 1. *Exact zero sets for the constant-coefficient model with shift π/2.*

4.2. Exponential-Bessel Model

Consider

$$f''(z)+e^{2z}f(z)=0. \tag{26}$$

With $t=e^{z}$ and $f(z)=F(t)$,

$$f'(z)=tF'(t), \qquad f''(z)=tF'(t)+t^{2}F''(t),$$

so (26) becomes

$$t^{2}F''(t)+tF'(t)+t^{2}F(t)=0, \tag{27}$$

Bessel's equation of order zero. Hence

$$f_1(z)=J_0(e^{z}) \tag{28}$$

is an entire solution.

The coefficient has period $c=\pi i$, since $e^{2(z+\pi i)}=e^{2z}$. Because J_0 is even,

$$f_1(z+\pi i)=J_0(-e^{z})=J_0(e^{z})=f_1(z),$$

so f_1 is a periodic eigen-solution with multiplier 1.

The nontrivial Jordan structure can in fact be proved exactly, rather than inferred numerically. The classical order-zero Bessel expansion [25] may be written locally as

$$\frac{\pi}{2}Y_0(t)=(\log(t/2)+\gamma)J_0(t)+\Phi(t),$$

where Φ is an even entire function. Lifting the logarithm through $t=e^{z}$ defines an entire second solution

$$f_2(z)=(z-\log 2+\gamma)J_0(e^{z})+\Phi(e^{z}).$$

Since both J_0 and Φ are even,

$$f_2(z+\pi i) = f_2(z) + \pi i\, f_1(z).$$

Thus, in the basis (f_1, f_2), period translation is represented exactly by the nonidentity Jordan matrix

$$\begin{pmatrix} 1 & \pi i \\ 0 & 1 \end{pmatrix}.$$

In particular, both Floquet multipliers equal 1, $\tau_{\pi i} = 2$, and Corollary 3.6 gives

$$\boxed{\Delta_{\pi i}^2 f \equiv 0.} \tag{29}$$

This identity holds for every solution of (26), while the subspace of πi-periodic solutions is one-dimensional. Section 5.2 independently checks the same Jordan invariants using a canonically normalized fundamental matrix.

4.3. Nonperiodic Airy Model

Consider

$$f''(z) + zf(z) = 0. \tag{30}$$

The function

$$f(z) = \mathrm{Ai}(-z)$$

is an entire solution because $\mathrm{Ai}''(w) = w\mathrm{Ai}(w)$. For any $c \neq 0$, $A(z) = z$ is not c-periodic. Identity (4) gives the exact inhomogeneous equation

$$\boxed{(\Delta_c f)'' + z\Delta_c f = -c\, f(z+c).} \tag{31}$$

No random-point experiment is needed to establish (31); it is an exact consequence of the Airy differential equation and Lemma 2.1. This example makes clear that the coefficient difference itself, rather than numerical approximation, is the relevant obstruction.

4.4. Quadratic-Exponential Power-Series Model

Take

$$A(z) = e^{z^2}, \qquad f(0) = 1, \qquad f'(0) = 0. \tag{32}$$

Here $e^{z^2} = 1 + z^2 + z^4/2! + \cdots$. Using (12), the solution begins

$$f(z) = 1 - \frac{z^2}{2} - \frac{z^4}{24} + \frac{z^6}{720} + \frac{89z^8}{40320}. \tag{33}$$

The omitted remainder is $O(z^{10})$. The next two nonzero coefficients are $a_{10} = 95/145152$ and $a_{12} = 11531/95800320$. Since e^{z^2} has no nonzero translational period, Proposition 3.1 implies that for every $c \neq 0$, no nontrivial $\Delta_c f$ can satisfy the same homogeneous equation (1). The series recurrence nevertheless provides a direct and reproducible way to approximate the entire solution on compact sets.

5. Numerical Verification

The computations in this section are diagnostics and consistency checks; they do not replace the analytic proofs. Floating-point argument-principle counts are not described as interval-certified results.

5.1. Argument-Principle Zero Count

For a function F with no zeros on $|z| = R$,

$$n_R(F) = \frac{1}{2\pi i}\int_{|z|=R} \frac{F\prime(z)}{F(z)}\, dz \tag{34}$$

counts the zeros in the disk with multiplicity. Numerical global-zero methods based on this principle are well established [17,18]. For $f(z) = \sin z$ and $g = \Delta_{\pi/2} f$, the winding counts agree with the exact zero sets:

Table 1. Floating-point argument-principle cross-check for the constant-coefficient model.

Radius R	$n_R(f)$	$n_R(g)$	**sampled** $\min_{\|z\|=R}\|f(z)\|$	**sampled** $\min_{\|z\|=R}\|g(z)\|$
5	3	3	0.9589	0.6753
10	7	6	0.5440	0.2951
15	9	10	0.6503	0.1094
20	13	13	0.9129	0.5049

The exact formulas determine the true counts here; the numerical procedure serves only as an implementation check.

5.2. Bessel Monodromy Matrix

To test the nontrivial Floquet statement in Section 4.2, a canonical fundamental matrix $Y(z)$ was normalized by

$$Y(0) = I$$

and numerically continued along the segment $z = it$, $0 \le t \le \pi$, using a high-order adaptive Runge-Kutta method. The resulting entries of the period-translation matrix were:

Table 2. Computed entries of the period-translation matrix for the exponential-Bessel model.

Matrix entry	**Computed value**
$(M_{\pi i})_{11}$	$1 + 1.057854954546i$
$(M_{\pi i})_{12}$	$1.839488890947i$
$(M_{\pi i})_{21}$	$-0.608352195202i$
$(M_{\pi i})_{22}$	$1 - 1.057854954546i$

The real parts of the two off-diagonal entries were below 10^{-15} in magnitude. The structural diagnostics were:

Table 3. Numerical monodromy diagnostics for the exponential-Bessel coefficient.

Diagnostic	**Computed value**
$\|\det M_{\pi i} - 1\|$	1.44×10^{-15}
$\|\mathrm{tr} M_{\pi i} - 2\|$	5.86×10^{-15}
$\| M_{\pi i} - I \|_2$	2.447841086149
$\| (M_{\pi i} - I)^2 \|_\infty$	1.77×10^{-14}

Thus the canonical numerical matrix is consistent with the exact Jordan form derived in Section 4.2: $\det M = 1$, $\mathrm{tr} M = 2$, $(M - I)^2 = 0$, while $M \neq I$. The computation is therefore an independent change-of-basis check of the special-function derivation, not the source of the Floquet conclusion.

5.3. Power-Series Residual Convergence

Let f_N be the truncation at degree N of the solution (33). On $|z| = 1$, the maximum differential-equation residual decreases as follows:

Table 4. Residual decay for the truncated power-series solution.

Truncation degree N	$\max_{\lvert z\rvert=1} \lvert f_N''(z) + e^{z^2} f_N(z)\rvert$
8	7.91×10^{-2}
12	2.55×10^{-3}
16	2.27×10^{-4}
20	1.57×10^{-5}
24	2.56×10^{-7}
30	3.92×10^{-9}

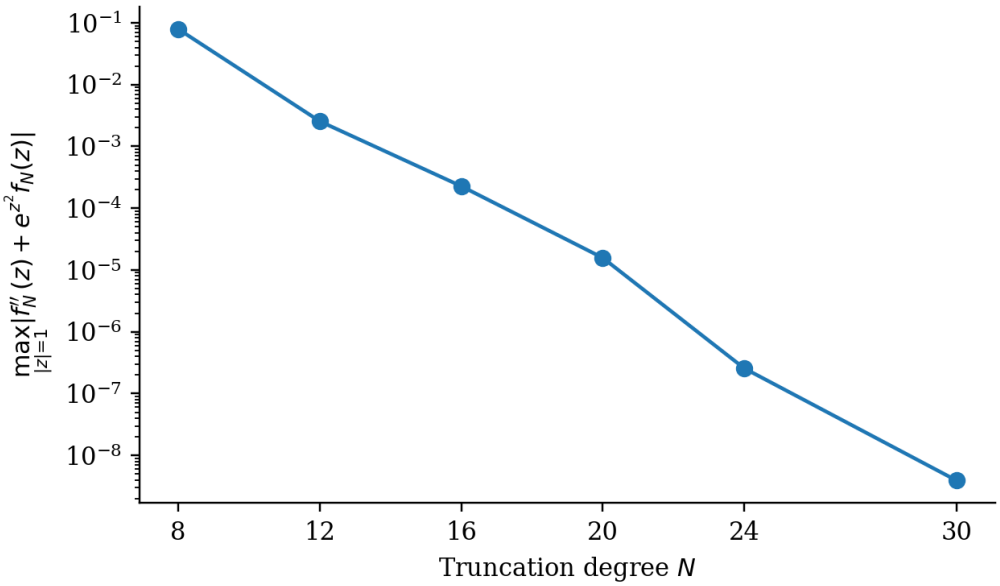


Figure 2. *Residual decay for the truncated power-series solution in the quadratic-exponential model.*

The decay is consistent with convergence of the Taylor solution on compact disks. For publication-grade validated numerics, the residual and zero-count computations should be repeated with interval arithmetic or another certified enclosure method.

6. Conclusions and Further Directions

6.1. Main Conclusions

The following are the main findings of the analysis.

The coefficient-shift identity concentrates the coefficient mismatch as the precise forcing that occurs due to translation, and the coefficient periodicity is the boundary between the two ranges of coefficient inheritance and forced behavior.

2. At the zeros of an induced coefficient associated with a nontrivial forward difference, if several zeros are present, then the induced coefficient has multiple zeros, producing double poles, and if just one zero is present, then the induced coefficient has removable-singularity principal part: Theorem 2.2.

3. The shift-Wronskian offers a complete diagnostic of translation invariance: If it is constant, so is the coefficient, and if it is periodic, so will be the structure of the shifted solutions.

4. When the coefficient has the form of an exponential function, shift preservation is the same as a discrete resonance condition; on the other hand, if the exponent is not a constant function, Theorem 3.3 implies that it is linear.

Floquet translation in the periodic setting yields the same second-order difference relation and the zero-transfer alternatives to the solutions and higher forward differences; the cases of constant and exponential-Bessel are explicitly realized.

6.2. Consider the limitations and further directions associated with this research.

Most of the main identities are related to uniqueness, translation and classical Floquet theory, etc. and the results are exact and structurally useful. It is therefore desirable to consider them as a localized structural addition rather than as alternatives to deeper oscillation or Nevanlinna methods.

The following directions are left open in the framework developed here:

The induced coefficients are beyond periodic, 1. Characterize the pairs of coefficient and solution where all the local principal parts in Theorem 2.2 are zero throughout the entire region.

2. Density (zero) and Floquet data. Discuss Floquet multipliers and with the exponent of convergence of zeros of eigen-solutions and their forward differences for periodic entire coefficients of finite order.

3. Exponential-sum coefficients. Using the results of modern representation [8-10] and the multiplier formula (18) attain explicit zero-distribution statements for the higher forward differences of finite-order solutions.

4. Quantitative value sharing. In the spirit of recent difference-operator uniqueness results [23,24] replace the exact alternatives in Theorem 3.7 with Nevanlinna counting inequalities involving common zeros or common small functions.

5. Certified computation. Substitute sample boundary clearances and floating-point monodromies by interval certified argument-principle counts and validated ODE continuation.

These directions combine the theme of the coefficient and transition from exact inheritance identities to quantitative information on growth and zero distribution.

Appendix A contains a collection of procedures for use with the Python programming language.

Below are some small exercises showing some of the numerical methods of Section 5. Thes e compute the winding-number zero count, and the integral around the complex period path for the Bessel model, respectively.**import** numpy **as** np
from scipy.integrate **import** solve_ivp

```
def winding_count(F, R, samples=8192):
    theta = np.linspace(0.0, 2*np.pi, samples, endpoint=False)
    z = R*np.exp(1j*theta)
    values = F(z)
    closed = np.concatenate([values, values[:1]])
    angle = np.unwrap(np.angle(closed))
    count = int(round((angle[-1] - angle[0])/(2*np.pi)))
    clearance = float(np.min(np.abs(values)))
    return count, clearance


def bessel_monodromy():
    def rhs(t, Y):
        z = 1j*t
        y1, v1, y2, v2 = Y
        A = np.exp(2*z)
        return [1j*v1, -1j*A*y1, 1j*v2, -1j*A*y2]

    sol = solve_ivp(
        rhs, [0, np.pi], [1+0j, 0+0j, 0+0j, 1+0j],
        method='DOP853', rtol=1e-13, atol=1e-15
    )
    y1, v1, y2, v2 = sol.y[:, -1]
    return np.array([[y1, y2], [v1, v2]])
```

The winding count is a floating-point diagnostic unless supplemented by rigorous enclosure of the continuous boundary. The monodromy computation is likewise a numerical consistency check, not a proof of the Floquet relations.

References

[1] S. S. Alhily and H. A. Saleh, "The effect of the Coefficient Function on the Solution Behavior for the Second-Order Complex Differential Equation," *Iraqi Journal of Science*, 63(4) (2022), 1653-1661. DOI: 10.24996/ijs.2022.63.4.23.

[2] H. A. Saleh and S. S. Alhily, "Growth and Bounded Solution of Second-Order of Complex Differential Equations Through of Coefficient Function," *IOP Conference Series: Materials Science and Engineering*, 871 (2020), 012057. DOI: 10.1088/1757-899X/871/1/012057.

[3] S. B. Bank and I. Laine, "On the oscillation theory of $f'' + Af = 0$ where A is entire," *Transactions of the American Mathematical Society*, 273(1) (1982), 351-363. DOI: 10.2307/1999210.

[4] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, de Gruyter Studies in Mathematics 15, Walter de Gruyter, Berlin, 1993.

[5] W. Bergweiler, A. Eremenko, and L. Rempe, "Second order linear differential equations with a basis of solutions having only real zeros," *Journal d'Analyse Mathematique*, 152 (2024), 53-108. DOI: 10.1007/s11854-023-0294-z.

[6] Y. Zhang, "Zero Distribution of Finite Order Bank-Laine Functions," *Journal of Geometric Analysis*, 34 (2024). DOI: 10.1007/s12220-024-01679-6.

[7] J. K. Langley, "Asymptotic Values, Deficiencies and Bank-Laine Functions," *Computational Methods and Function Theory*, 26 (2026), 481-510. DOI: 10.1007/s40315-025-00593-4.

[8] X.-Y. Li, J. Wang, and Z.-T. Wen, "Representation of finite order solutions to linear differential equations with exponential sum coefficients," *Journal of Differential Equations*, 423 (2025), 118-132. DOI: 10.1016/j.jde.2024.12.035.

[9] J. Heittokangas, K. Ishizaki, I. Laine, and K. Tohge, "Exponential polynomials in the oscillation theory," *Journal of Differential Equations*, 272 (2021), 911-937. DOI: 10.1016/j.jde.2020.10.015.

[10] J. Heittokangas, K. Ishizaki, K. Tohge, and Z.-T. Wen, "Value distribution of exponential polynomials and their role in the theories of complex differential equations and oscillation theory," *Bulletin of the London Mathematical Society*, 55(1) (2023), 1-77. DOI: 10.1112/blms.12719.

[11] J. Long, "Growth of Solutions of Second Order Complex Linear Differential Equations with Entire Coefficients," *Filomat*, 32(1) (2018), 275-284. DOI: 10.2298/FIL1801275L.

[12] G. Zhang and L. Yang, "Infinite growth of solutions of second order complex differential equations with entire coefficient having dynamical property," *Applied Mathematics Letters*, 112 (2021), 106708. DOI: 10.1016/j.aml.2020.106708.

[13] R. G. Halburd and R. J. Korhonen, "Difference analogue of the lemma on the logarithmic derivative with applications to difference equations," *Journal of Mathematical Analysis and Applications*, 314 (2006), 477-487. DOI: 10.1016/j.jmaa.2005.04.010.

[14] R. G. Halburd and R. J. Korhonen, "Nevanlinna theory for the difference operator," *Annales Academiae Scientiarum Fennicae Mathematica*, 31(2) (2006), 463-478.

[15] W. Bergweiler and J. K. Langley, "Zeros of differences of meromorphic functions," *Mathematical Proceedings of the Cambridge Philosophical Society*, 142(1) (2007), 133-147. DOI: 10.1017/S0305004106009777.

[16] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo, and J. Zhang, "Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity," *Journal of Mathematical Analysis and Applications*, 355(1) (2009), 352-363. DOI: 10.1016/j.jmaa.2009.01.053.

[17] M. Dellnitz, O. Schütze, and Q. Zheng, "Locating all the zeros of an analytic function in one complex variable," *Journal of Computational and Applied Mathematics*, 138(2) (2002), 325-333. DOI: 10.1016/S0377-0427(01)00371-5.

[18] H. Chen, "On locating the zeros and poles of a meromorphic function," *Journal of Computational and Applied Mathematics*, 402 (2022), 113796. DOI: 10.1016/j.cam.2021.113796.

[19] J. Heittokangas, K. Ishizaki, I. Laine, and K. Tohge, "Complex oscillation and nonoscillation results," *Transactions of the American Mathematical Society*, 372(9) (2019), 6161-6182.

[20] J. Heittokangas, K. Ishizaki, K. Tohge, and Z.-T. Wen, "Zero distribution and division results for exponential polynomials," *Israel Journal of Mathematics*, 227(1) (2018), 397-421. DOI: 10.1007/s11856-018-1738-5.

[21] W. Magnus and S. Winkler, *Hill's Equation*, Interscience Tracts in Pure and Applied Mathematics 20, Interscience Publishers, New York, 1966.

[22] M. S. P. Eastham, *The Spectral Theory of Periodic Differential Equations*, Scottish Academic Press, Edinburgh, 1973.

[23] F. Lü and Z. Yang, "Meromorphic Functions Partially Share Three Values with Their Difference Operators," *Chinese Annals of Mathematics, Series B*, 47(2) (2026), 421-428. DOI: 10.1007/s11401-025-0060-x.

[24] J.-X. Zhang and J.-F. Chen, "Uniqueness Theorems Related to Higher-Order Difference Operators of Entire Functions," *Journal of Contemporary Mathematical Analysis*, 61(2) (2026), 150-163. DOI: 10.3103/S1068362325600461.

[25] NIST Digital Library of Mathematical Functions, Section 10.8, "Power Series: Bessel and Hankel Functions," especially Eq. 10.8.2 for Y_0, National Institute of Standards and Technology, accessed September 4, 2026.