

Some Generalization of Hypergraphs Through Hypergroup

Authors Names	ABSTRACT
<i>Sarah Nahed Abdul Abbas</i> <i>Publication date: 10/10/2026</i> <i>Keywords: graph, hypergraph, generalized hypergraph, gdegree of vertex, continuous hypergraph, hypergroups</i>	In this paper, we initiate the notion generalized hypergraph which is an extension of hypergraph and graph, and study some classes of them which are induced by hypergroups. In particular, we give some formula for degree of vertices in this situation.

1. Introduction

During the last decades, graph theory has been widely studied and has found many applications in various branches of mathematics and other sciences. Therefore, it has always been of interest. See [3, 5, 7, 8]. However, some structures have not been applicable to it, and hypergraphs have been defined, studied and used as a complex generalization of graphs. In short, in a hypergraph, an edge can be related to more than two vertices. For more details, see the references [1, 2, 11, 13, 14]. In this article, we want to express a generalization of hypergraphs. In this generalization, the edges related to vertices can even have elements other than vertices. To examine a class of this new structure, we have gone to locally compact hypergroups to create a generalized hypergraph by taking advantage of the convolution product between measures. After stating some definitions, we study the degree of each vertex and under some conditions state a formula for it.

2. Main Results

Recall that if V is a non-empty set and E is a nonempty subset of the power set of V , then the pair (V, E) is called a hypergraph. This hypergraph will be a graph whenever for every $e \in E$ we have $|e| \leq 2$, where $|e|$ denotes the number of elements of the set e .

In this section, first we present the notion generalized hypergraph, which is an extension of hypergraph, and next, we will study this new concept.

Definition 2.1. Assume that Ω is a non-empty set, and fix some $\emptyset \neq V \subseteq \Omega$. Let E be a non-empty collection of the non-void subsets of Ω . Then, the pair $H := (V, E)$ is called a generalized hypergraph, or simply a generalized hypergraph.

Note that in the above definition we do not limit ourself to finite sets. Also, we mention that any hypergraph is a generalized hypergraph. In fact, while in the above definition any element of E is a subset of V , then H will be a hypergraph. For example setting $\Omega := \mathbb{N}, V := \{2, 4, 5, 7, 10\}$ and $E := \{\{1, 2, 4\}, \{4, 10\}, \{2, 4, 10\}, \{3\}\}$, the pair (V, E) is a generalized hypergraph which is not a hypergraph. In the following definition, we define some classes of generalized hypergraphs which are induced by a locally compact hypergroup. Hypergroups were introduced in the papers [10, 9, 12], and in the book [4] one can study regarding them. Roughly speaking, a hypergroup has enough structure for that its measure

space to be a convolution algebra, and also the Fourier analysis can be applied for them. For more details we refer to [4]. Throughout, we consider a general hypergroup K equipped with $*$ as the convolution product, and the mapping $x \rightarrow x^-$ as involution. Also, $\text{supp}(\mu)$ denotes the support of a regular measure μ , and ϵ_a is the point-mass measure at $a \in K$.

Definition 2.2. A locally compact Hausdorff space K equipped with $x \rightarrow x^-$ from K to K , and $(\mu, \nu) \mapsto \mu * \nu$ from $M(K) \times M(K)$ to $M(K)$ is called a hypergroup if

- (1) $M(K)$ with $+$ and $*$ is a unital algebra with ϵ_e as the unit, where $e \in K$ is a fixed element,
- (2) for every $x, y \in K$, $e \in \text{supp}(\epsilon_x * \epsilon_y)$ iff $y = x^-$,
- (3) for every $x, y \in K$, $\text{supp}(\epsilon_x * \epsilon_y)$ is compact and $\epsilon_x * \epsilon_y$ is a probability measure,
- (4) the map $(x, y) \mapsto \text{supp}(\epsilon_x * \epsilon_y)$ is continuous,
- (5) for every $\mu, \nu \in M(K)$, $(\mu * \nu)^- = \nu^- * \mu^-$.

A hypergroup K is called Hermitian if its involution is the identity mapping i.e. $x^- = x$ for all $x \in K$. Also, for every $A, B \subseteq K$, $A * B$ denotes the union of all $\text{supp}(\epsilon_a * \epsilon_b)$, where $a \in A$ and $b \in B$. For simply, put $x * B := \{x\} * B$ for all $x \in K$.

Definition 2.3. Assume that K is a locally compact hypergroup, and $\emptyset \neq A, B \subseteq K$. Assume that $\emptyset \neq V \subseteq K$, and let

$$E_{A,B,V} := \{\text{supp}(\epsilon_x * \epsilon_y) : x \in A, y \in B\}.$$

Then, $H := (V, E_{A,B,V})$ is called a continuous generalized hypergraph induced by K . Also, we denote $E_{A,V} := E_{A,A,V}$ and $E_K := E_{K,K}$.

Remark 2.4. Note that in the case that for every $(a, b) \in A \times B$ we have $\text{supp}(\epsilon_a * \epsilon_b) \subseteq V$, the pair $(V, E_{A,B,V})$ will be a hypergraph. In particular, if H is a subhypergroup of K (i.e. $H^- * H \subseteq H$), then for every $A, B \subseteq H$, the pair $(H, E_{A,B,H})$ is a hypergraph.

Next, always K denotes a locally compact hypergroup.

Definition 2.5. Assume that $H := (V, E)$ is a generalized hypergraph, and $a \in V$. Then, the number of all elements $e \in E$ with $a \in e$ is called the g -degree of a , and is denoted by $gd_H(a)$.

Let $A, B \subseteq K$. Then, we say that the pair (A, B) has DC (i.e. disjoint convo) property whenever for every distinct $a_1, a_2 \in A$ and $b_1, b_2 \in B$ we have

$$\text{supp}(\epsilon_{a_1} * \epsilon_{b_1}) \neq \text{supp}(\epsilon_{a_2} * \epsilon_{b_2})$$

In particular, whenever

$$\text{supp}(\epsilon_{a_1} * \epsilon_{b_1}) \cap \text{supp}(\epsilon_{a_2} * \epsilon_{b_2}) = \emptyset$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$ with $(a_1, b_1) \neq (a_2, b_2)$, then we say that (A, B) has PDC property.

Theorem 2.6. Assume that $A, B \subseteq K$ are finite and (A, B) has the DC property. Let there be $r \in \mathbb{N}$ and $s \in \mathbb{N}_0$ with

$$|\text{supp}(\epsilon_a * \epsilon_b)| \geq r \text{ and } |\text{supp}(\epsilon_a * \epsilon_b) \cap \text{supp}(\epsilon_c * \epsilon_d)| \leq s$$

for all $a, c \in A$ and $b, d \in B$ with $(a, b) \neq (c, d)$. Then

$$\frac{|A||B|r^2}{r + (|A||B| - 1)s} \leq |A * B|.$$

In particular, if (A, B) has PDC property, then $|A * B| \geq |A||B|r$.
Proof. Denote

$$\mathcal{J} := \{(x, a, b) \in (A * B) \times A \times B : x \in \text{supp}(\epsilon_a * \epsilon_b)\},$$

and for every $x \in A * B$ we define

$$\theta(x) := |\{(a, b) \in A \times B : x \in \text{supp}(\epsilon_a * \epsilon_b)\}|.$$

Note that

$$\sum_{x \in A * B} \theta(x) = |\mathcal{J}| = \sum_{(a,b) \in A \times B} |\text{supp}(\epsilon_a * \epsilon_b)| \geq |A||B|r. \#(2.1)$$

This implies that

$$\sum_{x \in A * B} \theta(x) \geq |A||B|r.$$

On the other hand, thanks to the hypothesis we have

$$\begin{aligned} 2 \sum_{x \in A * B} \binom{\theta(x)}{2} &= \sum_{\substack{(a,b), (c,d) \in A \times B \\ (a,b) \neq (c,d)}} |\text{supp}(\epsilon_a * \epsilon_b) \cap \text{supp}(\epsilon_c * \epsilon_d)| \\ &\leq 2s \binom{|A||B|}{2}, \end{aligned}$$

so,

$$\sum_{x \in A * B} \binom{\theta(x)}{2} \leq s \binom{|A||B|}{2}.$$

Now, since

$$\theta(x)^2 = \theta(x) + 2 \binom{\theta(x)}{2}$$

one can conclude that

$$\begin{aligned}\sum_{x \in A * B} \theta(x)^2 &= \sum_{x \in A * B} \theta(x) + 2 \sum_{x \in A * B} \binom{\theta(x)}{2} \\ &\leq \sum_{x \in A * B} \theta(x) + 2s \binom{|A||B|}{2} \\ &= |J| + s|A||B|(|A||B| - 1).\end{aligned}$$

By the inequality

$$\left(\sum_{x \in A * B} \theta(x) \right)^2 \leq |A * B| \sum_{x \in A * B} \theta(x)^2.$$

and the first equality in (2.1) we conclude that

$$\sum_{x \in A * B} \theta(x)^2 \leq |J| + s|A||B|(|A||B| - 1).$$

Therefore

$$|A * B| \geq \frac{|A||B|r^2}{r + (|A||B| - 1)s} \quad \square$$

Example 2.7. For every $x, y \in [0, \infty)$ define

$$\epsilon_x * \epsilon_y := \frac{1}{2} \epsilon_{|x-y|} + \frac{1}{2} \epsilon_{x+y}.$$

Then, by [4], $K := [0, \infty)$ is a Hermitian hypergroup equipped with the Euclidean topology. For every $x, y, a, b \in [0, \infty)$ with $x > y > 0$ and $a > b > 0$ we have

$$\text{supp}(\epsilon_x * \epsilon_y) = \text{supp}(\epsilon_a * \epsilon_b)$$

if and only if $\{x - y, x + y\} = \{a - b, a + b\}$. This holds if and only if $x - y = a - b$ and $x + y = a + b$, and this is equivalent with $x = a$ and $y = b$. On the other words, in general, for every distinct $a, b \in (0, \infty)$ and distinct $x, y \in (0, \infty)$ we have

$$\text{supp}(\epsilon_x * \epsilon_y) = \text{supp}(\epsilon_a * \epsilon_b)$$

if and only if $\min\{x, y\} = \min\{a, b\}$ and $\max\{x, y\} = \max\{a, b\}$. This implies that if $A, B \subseteq (0, \infty)$ with $A \cap B = \emptyset$, then the pair (A, B) has DC property.

Example 2.8. Assume that $K := \mathbb{N} \cup \{0\}$, so thanks to [9], K is a discrete Hermitian hypergroup with 0 as the identity element and for every $a, b \in \mathbb{N}$,

$$\epsilon_a * \epsilon_b := \epsilon_{\max\{a, b\}}$$

if $a \neq b$, and

$$\epsilon_a * \epsilon_a := \frac{1}{23^{a-1}} \epsilon_0 + \sum_{k=1}^{a-1} \frac{1}{3^{a-k}} \epsilon_k + \frac{1}{2} \epsilon_a.$$

This implies that $\text{supp}(\epsilon_a * \epsilon_b) = \max\{a, b\}$ if $a \neq b$, and $\text{supp}(\epsilon_a * \epsilon_a) = \{0, 1, 2, \dots, a\}$. Let $x, y \in K$ be distinct, and also $a, b \in K$ be distinct. In this case, by some simple calculation one can see that

$$\text{supp}(\epsilon_x * \epsilon_a) = \text{supp}(\epsilon_y * \epsilon_b)$$

if and only "both x, a and y, b are distinct and $\max\{a, x\} = \max\{b, y\}$ ", or $x = a = b = y$.

Notation 2.9. Let $C, D \subseteq K$. Then, we denote

$$D^- := \{x^- : x \in D\},$$

and

$$C \odot D^- := \bigcup \{\text{supp}(\epsilon_x * \epsilon_{y^-}) : x \in C, y \in D, x \neq y\}.$$

Remark 2.10. By [10], for every $A, B, C \subseteq K$ we have $A \cap (B * C) \neq \emptyset$ if and only if $B \cap (A * C^-) \neq \emptyset$. In the next proof, we will use this fact.

Theorem 2.11. Assume that K is a locally compact hypergroup, $\emptyset \neq A, B \subseteq K$ such that

$$B \cap ((A^- \odot A) * B) = A \cap (A * (B \odot B^-)) = \emptyset, A * B \subseteq V, \#(2.2)$$

and let $H := (V, E_{A,B,V})$ be the continuous generalized hypergraph induced by K . Then, for every $a \in V$ we have

$$gd_H(a) = \text{card}((a * B^-) \cap A) \cdot \text{card}(B).$$

Proof. It would be enough to prove that $gd_H(a) = \text{card}\Omega$, where

$$\Omega := \{(x, y) \in A \times B : x \in (a * B^-) \cap A\}.$$

For this, first note that by [10], for every $x \in A$ and $y \in B$ we have $a \in x * y$ if and only if $x \in a * y^-$, and this holds if and only if $(x, y) \in \Omega$. Now, we prove that for every distinct (x_1, y_1) and (x_2, y_2) in Ω , we have $x_1 * y_1 \neq x_2 * y_2$. For this, let $(x_1, y_1) \neq (x_2, y_2)$.

Case 1: $y_1 \neq y_2$. By the assumption we have $x_2 \notin A * (B \odot B^-)$. This implies that $x_2 \notin x_1 * y_1 * y_2^-$, and so, thanks to [10], we have $(x_1 * y_1) \cap (x_2 * y_2) = \emptyset$.

Case 2: $x_1 \neq x_2$. By the assumption, $y_2 \notin x_2^- * x_1 * y_1$, and thus $(x_2 * y_2) \cap (x_1 * y_1) = \emptyset$.

Therefore, the proof is complete. $\square$

Example 2.12. Assume that K is the hypergroup as in Example 2.8. Let $A := \{5\}$ and $B := \{5, 7, 9, 11\}$. Then, we have

$$B \odot B = \{7,9,11\}, A * (B \odot B) = \{7,9,11\}$$

and so, $A \cap (A * (B \odot B)) = \emptyset$. Trivially, $B \cap ((A^- \odot A) * B) = \emptyset$ because A is singleton. Put

$$V = \{0,1,2,3,4,5,6,7,9,11,15,20\}.$$

Then, $A * B = \{0,1,2,3,4,5,7,9,11\} \subseteq V$. By the above theorem, setting $H := (V, E_{A,B,V})$ we have

$$gd_H(6) = \text{card}((6 * B^-) \cap A) \times \text{card}(B) = \text{card}(\{6,7,9,11\} \cap \{5\}) \times 4 = 0,$$

and

$$\begin{aligned} gd_H(7) &= \text{card}((7 * B^-) \cap A) \times \text{card}(B) \\ &= \text{card}(\{1,2,3,4,5,6,7,9,11\} \cap \{5\}) \times 4 = 4. \end{aligned}$$

Example 2.13. Let K be the hypergroup as in Example 2.7. Setting $A = \{7,8\}$ and $B = (2,3)$ we have

$$\begin{aligned} B \odot B^- &= (0,1) \cup (2,6), A * (B \odot B^-) = (7,8) \cup (8,9) \cup (9,14), \\ \text{so, } A \cap (A * (B \odot B^-)) &= \emptyset. \text{ Also,} \\ A^- \odot A &= \{1,15\}, (A^- \odot A) * B = (1,2) \cup (3,4) \cup (12,13) \cup (17,18), \end{aligned}$$

hence, $B \cap ((A^- \odot A) * B) = \emptyset$.

This new introduced structure (generalized hypergraph) can be basic for new studies same as researches regarding graphs and hypergraphs, specially related to a locally compact hypergroup.

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