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Subordination Preserving of Convolution of Functions Associated with Differential Operator

Authors Names
Ahmed Salim Mohammed[1]
Zaniab Aodeh A. Mohmmed[2]
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ABSTRACT

The present submission includes some subordination preserving and superordination results derived from a new differential transformation operator for convolution of Hurwitz- Lerch zeta function with a meromorphic function defined within the punctured unit disk $\mathcal{U}^* = \{s: s \in \mathbb{C}: 0 < |s| < 1\}$. Furthermore, those results were used to obtain the sandwich theorem.

1. Introduction

For s belonging in the unit disk in the complex plane that defined as $\mathcal{U} := \{s: s \in \mathbb{C}: |s| < 1\}$ the symbol $\mathcal{H}(\mathcal{U})$ refers to the collection of functions , known as the form

$$\mathcal{H}(\mathcal{U}) = \{\mathfrak{F}: \mathcal{U} \to \mathbb{C}: \mathfrak{F} \text{ holomorphic in } \mathcal{U}\},$$

and

$\mathcal{H}(\mathcal{U}) \supset \mathcal{H}[\varepsilon, i] := \{\mathfrak{F} \in \mathcal{H}(\mathcal{U}): \mathfrak{F}(s) = \varepsilon + \varepsilon_i s^i + \varepsilon_{i+1} s^{i+1} + \cdots, s \in \mathcal{U}\}$. It is worth that the special collections and where obtained through special cases of ε and i , where $\mathcal{H}_0 = \mathcal{H}[0,1]$ and $\mathcal{H}_1 = \mathcal{H}[1,1]$ when $\varepsilon = 0$, $i = 1$ and $\varepsilon = i = 1$ respectively .

Given two holomorphic functions, $\mathfrak{f}(s)$ and $\mathfrak{S}(s)$ in $\mathcal{U}$. Suppose $\mathcal{B}(s)$ be Schwarz function belongs to $\mathcal{U}$ with $\mathcal{B}(0) = 0$, then we define that $\mathfrak{f}(s)$ is subordinate to $\mathfrak{S}(s)$ or $\mathfrak{S}(s)$ is superordinate to $\mathfrak{f}(s)$, and $|\mathcal{B}(s)| < 1$ in which $\mathfrak{f}(s) = \mathfrak{S}(\mathcal{B}(s))(for\ all\ s \in \mathcal{U})$. In such case we write $\mathfrak{f}(s) < \mathfrak{S}(s)$ or $\mathfrak{S}(s) < \mathfrak{f}(s)$.

To be more precise, in situations when a function $\mathfrak{f}(s)$ univalent in $\mathcal{U}$, so the the function $\mathfrak{f}(s)$ is subordinate to the function $\mathfrak{S}(s)$ iff $\mathfrak{f}(0) = \mathfrak{S}(0)$ and $\mathfrak{f}(\mathcal{U}) \subset \mathfrak{S}(\mathcal{U})$.

Let Σ denot the collection of all meromorphic functions that defined by

$$k(s) = s^{-1} + \sum_{i=0}^{\infty} \mathcal{E}_i s^i \ (s \in \mathcal{U}^* = \mathcal{U} - \{0\}) \ . \qquad (1.1)$$

Note that the collection of all functions that are meromorphic starlike of order β $(0 \le \beta < 1)$, meromorphic convex of order $\beta (0 \le \beta < 1)$ are defined as ::

$$\Sigma^* := \{k: k \in \sum: \mathcal{R}e\left(\frac{-sk'(s)}{k(s)}\right) > \beta \ , where\ k(s) \neq 0\},$$

$$\Sigma_k := \{k: k \in \sum: \mathcal{R}e\left(-\left(1 + \frac{sk''(s)}{k'(s)}\right)\right) > \beta \ , where\ k'(s) \neq 0\},$$

respectively (see [2], [3] and [11]). Hurwitz- Lerch zeta is one of the important special functions that has received considerable attention in researcher's studies , this function is sometimes abbreviated as (HL-Z) (see [8]) and is known by the form

$$\zeta(s, \lambda, \xi) = \frac{1}{\xi^{\lambda}} + \sum_{i=1}^{\infty} \frac{s^i}{(i+\xi)^{\lambda}} \ ,$$

where $\xi \in \mathbb{C} - \mathbb{Z}_0^{-1}$, $\mathbb{Z}_0^{-1} := \{0\} \cup \{-1, -2, \ldots.\}$, $\lambda \in \mathbb{C}$, $s \in \mathcal{U} = \mathcal{U}^* \cup \{0\}$, $\mathcal{R}e(\lambda) > 1$, for $s \in \partial\mathcal{U}$.

[1,2]*Department of Mathematics, College of Education, University of AL-Qadisiyah, Diwaniya, Iraq*
Email: ahmed.salem198814@gmail.com, Email: zainab.aodeh@qu.edu.iq

Ghanim et al. [5] define a new function depend on HL-Z function Denoted by $g(\lambda,\xi)$ and known by the form

$$g(\lambda,\xi)=\frac{1}{s}+\sum_{i=0}^{\infty}\left(\frac{1+\xi}{(1+i)+\xi}\right)^{\lambda}s^{i}\ . \qquad (1.2)$$

So for $k\in\Sigma$, we define $\mathfrak{D}_{\xi}^{\lambda_m}$ as a new differential operator such that

$$\mathfrak{D}_{\xi}^{\lambda_m}=(\mathfrak{D}^{m}*g(\lambda,\xi)*k)(s)$$
$$=\frac{1}{s}+\sum_{i=0}^{\infty}\left(\frac{1+\xi}{(1+i)+\xi}\right)^{\lambda}(2+i)^{m}\xi_i s^{i}\quad ,$$
$$\mathfrak{D}_{\xi}^{\lambda_1}=\frac{(s^2(g(\lambda,\xi)*k(s)))'}{s}$$
$$=\frac{1}{s}+\sum_{i=0}^{\infty}\left(\frac{1+\xi}{1+i+\xi}\right)^{\lambda}(i+2)\xi_i s^{i}\ ,$$
$$\mathfrak{D}_{\xi}^{\lambda_2}=\mathfrak{D}_{\xi}^{\lambda_1}(\mathcal{D}g_{\xi}^{\lambda_1})\ \cong\frac{(s^2\mathfrak{D}_{\xi}^{\lambda_1})'}{s}$$
$$=\frac{1}{s}+\sum_{i=0}^{\infty}\left(\frac{1+\xi}{1+i+\xi}\right)^{\lambda}(2+i)^{2}\xi_i s^{i}\ ,$$
$$\vdots$$
$$\mathfrak{D}_{\xi}^{\lambda_m}=\frac{1}{s}+\sum_{i=0}^{\infty}\left(\frac{1+\xi}{1+i+\xi}\right)^{\lambda}(2+i)^{m}\xi_i s^{i}\ , \qquad (1.3)$$

where $m\in\mathbb{N}$, $\lambda\in\mathbb{C}$ such that $\mathscr{R}e(\lambda)>1$ when $\xi\in\mathbb{C}-\mathbb{Z}_0^{-1}$, $\mathbb{Z}_0^{-1}:=\{0\}\cup\{-1,-2,\dots.\}$ and $s\in\mathcal{U}^{*}:=\mathcal{U}-\{0\}$. Thus for $k\in\Sigma$, we observe that

$$\left(\mathfrak{D}_{\xi}^{\lambda_{m+1}}k\right)(s)=s\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}k\right)(s)\right)'+2\left(\mathfrak{D}_{\xi}^{\lambda_m}k\right)(s). \qquad (1.4)$$

Now , the new collection of meromorphic functions that represented by $\sum_{\xi}^{\lambda_m}$ defined as :

$$\sum_{\xi}^{\lambda_m}:=$$
$$\left\{k:k\in\Sigma\,,sk(s)\neq 0\ \text{and}\ s\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}k\right)(s)\right)\neq 0\ for\ s\in\mathcal{U}\,,\ m\in\mathbb{N},\lambda\in\mathbb{C},\text{such that}\ \mathscr{R}e(\lambda)>1\right\}\ \text{and}$$

$\mathfrak{D}_{\xi}^{\lambda_m}$ refer to the differential operator that defined by (1.3) . The collection that indicated by the symbol Σ is contain all the meromorphic functions of the form (1.1) . Many studies have included the topic of preserving subordination and superordination of holomorphic univalent(or multivalent) functions (see [15], [17]), meromorphic multivalent functions (see [4] , [20]) and univalent which are meromorphic functions (see [1], [6], [7], [9], [10] and [12] − [14]) in punctured disk involving an operators . Here we mention some important basic definitions relevant to the research topic.

Definition 1.1 [19] . Let $\Psi:\mathbb{C}^2\longrightarrow\mathbb{C}$ and $\mathscr{F}(s)$ be univalent in $\mathcal{U}$. If φ is homomorphic in $\mathcal{U}$ and satisfies $\Psi\big(\varphi(s),s\varphi'(s)\big)\prec\mathscr{F}(s)\ \ (s\in\mathcal{U})$. (1.5)

Than we say that the differential subordination in (1.5) has a solution represented by φ. The univalent function $\pi(s)$ is a dominant of the solutions of (1.5) , in other words , $\pi(s)$ is a dominant if $\varphi(s)\prec\pi(s)$ for every φ fulfill the differential subordination in (1.5) , a best Dominant is a dominant function $\tilde{\pi}(s)$ such that $\tilde{\pi}(s)\prec\pi(s)$ for all subordinants of $\pi(s)$ in (1.5) .

Definition 1.2 [16] . For a univalent functions φ, $\mathbb{N}\big(\varphi(s),s\varphi'(s)\big)$ and a holomorphic function $\mathscr{F}(s)$ in $\mathcal{U}$ and satisfy $\mathscr{F}(s)\prec\mathbb{N}\big(\varphi(s),s\varphi'(s)\big)(s\in\mathcal{U})$, (1.6)

where $\mathbb{N}:\mathbb{C}^2\longrightarrow\mathbb{C}$, we say the differential superordination in (1.6) has the solution represented by φ , a subordinate of the solutions of (1.6) is an holomorphic function that represented by π, to be more clear , π(s) is a subordinate if $\pi(s)\prec\varphi(s)$, for every $\varphi(s)$ verifies the differential superordinatin in

(1.6) ,a function $\tilde{\pi}(s)$ called the best subordinate for all subordinates $\pi(s)$ of (1.6) if it is univalent and satisfies the subordination condition $\pi(s) \prec \tilde{\pi}(s)$.

Miller and Mocanu [16] defined the set of functions $\mathfrak{Q}$, which refers to the collection of functions $k(s)$ that are holomorphic and one to one on D, where $D = \bar{\mathcal{U}} - \mathfrak{E}(k(s))$ and

$$\mathfrak{E}\big(k(s)\big) = \{\xi : \xi \in \partial\mathcal{U} \ \ and \ \lim_{s\to\xi} k(s) = \infty\},$$

and are such that $k'(\xi) \neq 0$ $(\xi \in \partial\mathcal{U} - \mathfrak{E}(k(s))$.

In addition to the definitions mentioned above, and to obtain new results based on previous findings, we mention some of those results.

Lemma 1.1 [2] . A function $\mathscr{L}(s, \tilde{r})$ of the form $\mathscr{L}(s, \tilde{r}) = \xi_1(\tilde{r})s + \xi_2(\tilde{r})s^2 + \cdots (\xi_1(\tilde{r}) \neq 0, \tilde{r} \geq 0)$ that maps $\mathcal{U} \times [0,1)$ onto the complex plane which has the property $\lim_{r\to\infty}|\xi_1(\tilde{r})| = \infty$, is a subordinate chain iff $\mathscr{Re}\left(\frac{\frac{s\partial h(s,\tilde{r})}{\partial s}}{\frac{\partial h(s,\tilde{r})}{\partial \tilde{r}}}\right) > 0$ $(s \in \mathcal{U}, \tilde{r} \geq 0)$.

Lemma 1.2 [17] . For a function $\mathcal{S}$ satisfies the condition $\mathscr{Re}(\mathcal{S}(id\,, \tilde{r}) \leq 0$ $, \forall\, d \in \mathbb{R}\ \&\ \forall \tilde{r}$ satisfies the inequality$\tilde{r} \leq \frac{-i(1+d^2)}{2}, i \in \mathbb{N}$,where $\mathcal{S}: \mathbb{C}^2 \longrightarrow \mathbb{C}$.

If the function $\varphi(s) = 1 + c_i s^i + c_{i+1}s^{i+1} + \cdots$ is holomorphic in $\mathcal{U}$ and $\mathscr{Re}\left(\mathcal{S}\big(\varphi(s), s\varphi'(s)\big)\right) > 0$ $(s \in \mathcal{U})$, then $\mathscr{Re}\big(\varphi(s)\big) > 0$, $\forall\, s \in \mathcal{U}$.

Lemma 1.3 [18]**.** For every $s \in \mathcal{U}$,suppose that $v_1(v_1 \neq 0), v_2 \in \mathbb{C}$, and $\mathscr{F}(s)$ belonging to $\mathscr{H}(\mathcal{U})$ with $\mathscr{F}(s) = d \ at\ s = 0$. If $\mathscr{Re}\big(v_1\mathscr{F}(s) + v_2\big) > 0$ $(s \in \mathcal{U})$, then the differential equation

$$\pi(s) + \frac{s\pi'(s)}{v_1\pi(s)+v_2} = \mathscr{F}(s) \ \ (s \in \mathcal{U}\,, \pi(0) = d)\,,$$

has a holomorphic solution in $\mathcal{U}$ and satisfies $\mathscr{Re}(v_1\pi(s) + v_2) > 0$.

Lemma 1.4 [19] **.** Assume that for $\varphi \in \mathfrak{Q}$ with $\varphi(0) = \varepsilon$ and a function $\pi(s) = \varepsilon + \varepsilon_i s^i + \varepsilon_{i+1}s^{i+1} + \cdots$ be a holomorphic in $\mathcal{U}$ such that $\pi(s) \neq \varepsilon$, If $\pi(s) \prec \varphi(s)$, then there exist points $\xi_0 \in \partial\mathcal{U} \setminus \mathfrak{E}(k(s))$ and $s_0 = t_0 e^{i\rho} \in \mathcal{U}$, for each

$$\pi\big(u_{t_0}\big) \subset \varphi(u), \pi(s_0) = \varphi(\xi_0) \text{ and } s_0\pi'(s_0) = j\xi_0\varphi'(\xi_0)\ (j > i, i = 1,2, \ldots.)\,.$$

Lemma 1.5 [16]. Assume that $\pi \in \mathscr{H}[\varepsilon, 1]$ and $\mathcal{T}: \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a function such that $\mathcal{T}\big(\pi(s), s\pi'(s)\big) = \mathscr{F}(s)(s \in \mathcal{U})$. If $\mathscr{L}(s, \tilde{r}) = \mathcal{T}(\pi(s), s\tilde{r}\pi'(s))$ is a subordination chain and $\tilde{\pi} \in \mathscr{H}[\varepsilon, 1] \cap \mathfrak{Q}$, then

$$\mathscr{F}(s) \prec \mathcal{T}(\tilde{\pi}(s), s\tilde{\pi}'(s))\,,$$

implies $\pi(s) \prec \tilde{\pi}(s)$. Furthermore , if a function $\pi \in \mathfrak{Q}$ is a univalent solution of the equation $\mathcal{T}\big(\pi(s), s\pi'(s)\big) = \mathscr{F}(s)$, then π is the best subordinant .

The sandwich – type theorem is investigate in the present submission , we discussed several preserving subordination and superordination properties for convolution of meromorphic function with HL-Z function involving differential operator $\mathfrak{D}_{\xi}^{\lambda_m}$.

2. Fundamental Results Involving the Differential Operator $\mathfrak{D}_{\xi}^{\lambda_m}$.

The following theorem imply an important result about the subordination of convex meromorphic function associated with the differential operator defined by (1.4).

Theorem 2.1. Let $v = -\left(\frac{(1+b)-\Im}{2b\Im}\right) \geq -\frac{1}{2}(b \geq 1, \Im > 0)$, (2.1)

suppose that $\mathfrak{T} \in \Sigma$, such that

$$\mathcal{T}(s) = (1-b)\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}(s)\right)^{\Im} + \frac{b\mathfrak{D}_{\xi}^{\lambda_{m+1}}\mathfrak{T}(s)}{\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}(s)}\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}(s)\right)^{\Im}. \qquad (2.2)$$

If $\mathscr{Re}\left\{1+\frac{s\mathcal{T}''(s)}{\mathcal{T}'(s)}\right\} > v\,,(s\in\mathcal{U})$, (2.3)

and $k\in\Sigma$, then the subordination condition

$$(1-b)\left(s\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}k\right)(s)\right)\right)^{\Im}+b\left(\frac{\left(\mathfrak{D}_{\xi}^{\lambda_{m+1}}k\right)(s)}{\left(\mathfrak{D}_{\xi}^{\lambda_m}k\right)(s)}\left(s\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}k\right)(s)\right)\right)^{\Im}\right)\prec\mathcal{T}(s). \quad (2.4)$$

Implies $\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}k\right)(s)\right)^{\Im}\prec\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}\right)(s)\right)^{\Im}$. (2.5)

Moreover , $\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}\right)(s)\right)^{\Im}$ represent a best dominant of the subordination condition on (2.5).

Proof. Suppose that for $s\in\mathcal{U}$,the following holomorphic functions $\mathcal{B}_1(s)$ and $\mathcal{B}_2(s)$ in $\mathcal{U}$ are defined as

$$\mathcal{B}_1(s)=\left(s\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}k\right)(s)\right)\right)^{\Im}\text{ and }\mathcal{B}_2(s)=\left(s\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}\right)(s)\right)\right)^{\Im}. \quad (2.6)$$

Suppose that , without privation of generality , on $\bar{\mathcal{U}}$ there is a holomorphic as well as univalent function $\mathcal{B}_2(s)$ with property $\mathcal{B}_2'(\xi)\neq 0\ (|\xi|=1)$. If not , so with respect to $\mathfrak{y}\in(0,1)$, we have new functions $\mathcal{B}_1(s)=\mathcal{B}_1(\mathfrak{y}s)$ and $\mathcal{B}_2(s)=\mathcal{B}_2(\mathfrak{y}s)$ have the desired properties on $\bar{\mathcal{U}}$, using these functions is useful in proving the results of this theorem . Therefore , by letting $\mathfrak{y}\to 1$, we will obtain the results. So, we can prove that $\mathscr{Re}\left(\mathcal{Q}(s)\right)>0\ (s\in\mathcal{U})$, if

$$\mathcal{Q}(s)=1+\frac{s\mathcal{B}_2{}''(s)}{\mathcal{B}_2{}'(s)}\ (s\in\mathcal{U})\,. \quad (2.7)$$

To show that the real part of $\mathcal{Q}(s)$ lies in the right – hand half of the complex plane, we begin with the derivative of $\mathcal{B}_2(s)$, which will be in the form

$$\mathcal{B}_2{}'(s)=\Im\left(s\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}\right)(s)\right)\right)^{\Im}\left[\frac{s\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}\right)(s)+\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}\right)(s)}{s\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}\right)(s)\right)}\right],$$

that is ,

$$s\mathcal{B}_2'(s)=\Im\,\mathcal{B}_2(s)\left[\frac{\left(\mathfrak{D}_{\xi}^{\lambda_{m+1}}\mathfrak{T}\right)(s)}{\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}\right)(s)}-1\right],$$

It goes without saying that, we have

$$\frac{bs\mathcal{B}_2'(s)}{\Im}+b\,\mathcal{B}_2(s)=b\,\mathcal{B}_2(s)\frac{\left(\mathfrak{D}_{\xi}^{\lambda_{m+1}}\mathfrak{T}\right)(s)}{\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}\right)(s)},$$

after several calculations, we get

$$\frac{bs\mathcal{B}_2'(s)}{\Im}+\mathcal{B}_2(s)=b\,\mathcal{B}_2(s)\left(b\frac{\left(\mathfrak{D}_{\xi}^{\lambda_{m+1}}\mathfrak{T}\right)(s)}{\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}\right)(s)}\right)+(1-b)\,\mathcal{B}_2(s)\,,$$

in other expression

$$\frac{bs\,\mathcal{B}_2'(s)}{\Im}+\mathcal{B}_2(s)=\mathcal{T}(s)\,. \quad (2.8)$$

By performing simple calculations after taking the derivative for both sides of (2.8), we have

$$\mathcal{T}'(s)=\left(1+\frac{b}{\Im}\right)\mathcal{B}_2'(s)+\frac{b}{\Im}s\mathcal{B}_2''(s)\,. \quad (2.9)$$

Since $\mathcal{Q}(s)=1+\frac{s\mathcal{B}_2''(s)}{\mathcal{B}_2'(s)}$, we conclude that

$$\mathcal{T}'(s)=\left(1+\frac{b}{\Im}\right)\mathcal{B}_2'(s)+\frac{b}{\Im}\left(\mathcal{Q}(s)\mathcal{B}_2'(s)-\mathcal{B}_2'(s)\right)$$
$$=\mathcal{B}_2'(s)+\frac{b}{\Im}\mathcal{Q}(s)\mathcal{B}_2'(s)\,.$$

That is

$$\mathcal{T}'(s) = \left(1 + \frac{b}{\Im}\mathcal{Q}(s)\right)\mathcal{B}_2'(s) \ . \qquad (2.10)$$

After simple computation by taking the derivative to both sides of (2.10) with respect to s, we get

$\frac{s\mathcal{T}''(s)}{\mathcal{T}'(s)} = \frac{s\mathcal{B}_2''(s)}{\mathcal{B}_2'(s)} + \frac{b}{\Im + b\,\mathcal{Q}(s)} s\mathcal{Q}'(s)$, that is

$$1 + \frac{s\mathcal{T}''(s)}{\mathcal{T}'(s)} = 1 + \frac{s\mathcal{B}_2''(s)}{\mathcal{B}_2'(s)} + \frac{b}{\Im + b\,\mathcal{Q}(s)} s\mathcal{Q}'(s)$$

$$= \mathcal{Q}(s) + \left(\frac{1}{\frac{\Im}{b} + \mathcal{Q}(s)}\right) s\mathcal{Q}'(s)$$

$$= \mathcal{F}(s) \ \ (s \in \mathcal{U}). \qquad (2.11)$$

Since by (2.2) and (2.11) that

$\mathcal{F}(s) = \mathcal{Q}(s) + \left(\frac{1}{\frac{\Im}{b} + \mathcal{Q}(s)}\right) s\mathcal{Q}'(s)$.

That is

$$\mathcal{F}(s) + \frac{\Im}{b} = \mathcal{Q}(s) + \left(\frac{s\mathcal{Q}'(s)}{\frac{\Im}{b} + \mathcal{Q}(s)}\right) + \frac{\Im}{b}$$

$$= \frac{\left(\mathcal{Q}(s) + \frac{\Im}{b}\right)^2 + s\mathcal{Q}'(s)}{\frac{\Im}{b} + \mathcal{Q}(s)}$$

$$= \left(\mathcal{Q}(s) + \frac{\Im}{b}\right) + \frac{s\mathcal{Q}'(s)}{\frac{\Im}{b} + \mathcal{Q}(s)} .$$

Therefor

$$\mathcal{R}e\left(\mathcal{F}(s) + \frac{\Im}{b}\right) > 0 \ \ (s \in \mathcal{U}) . \qquad (2.12)$$

Moreover, from lemma 1.3 , we conclude that the function $\mathcal{Q}(s)$ belonging to $\mathcal{H}(\mathcal{U})$ is the solution to the differential equation in (2.11) such that $\mathcal{Q}(s) = \mathcal{F}(s) = 1 \ at \ s = 0$,

let $\mathcal{S}(\omega_1, \omega_2) = \omega_1 + \frac{\omega_2}{\omega_1 + \frac{\Im}{b}} + v$,

where v is given by (2.1), form (2.11) and (2.12) , we get for $s \in \mathcal{U}$

$$\mathcal{R}e\{\mathcal{S}(\mathcal{Q}(s), s\mathcal{Q}'(s))\} > 0 \quad .$$

To demonstrate the validity of the condition

$$\mathcal{R}e\{\mathcal{S}(id\ , \tilde{r})\} \leq 0 \ \ (d \in \mathbb{R}\ , \tilde{r} \leq -\tfrac{1+d^2}{2}) . \qquad (2.13)$$

This will be as follows :

$$\mathcal{R}e\{\mathcal{S}(id\ , \tilde{r})\} = \mathcal{R}e\left\{id + \frac{r}{\frac{\Im}{b} + id} + v\right\}$$

$$= \frac{\frac{\tilde{r}\Im}{b}}{\left(\frac{\Im}{b}\right)^2 + d^2} + v$$

$$\leq \frac{\frac{\tilde{r}\,\Im}{b} + v\left(\left(\frac{\Im}{b}\right)^2 + d^2\right)}{\left(\frac{\Im}{b}\right)^2 + d^2}$$

$$= \frac{-\frac{\Im}{b}\left(\frac{1+d^2}{2}\right) + v\left(\frac{\Im}{b}\right)^2 + d^2}{\left(\frac{\Im}{b}\right)^2 + d^2}$$

$$= \frac{-\left\{\left[\frac{\Im}{b} - 2v\right]d^2 - 2v\left(\frac{\Im}{b}\right)^2 + \frac{\Im}{b}\right\}}{2\left(\left(\frac{\Im}{b}\right)^2 + d^2\right)}$$

$$= \frac{-\Psi(d,b,v)}{2(d^2+\left(\frac{\Im}{b}\right)^2)},$$

where $\Psi(d,b,v) = \left[\frac{\Im}{b} - 2v\right] d^2 - 2v\left(\frac{\Im}{b}\right)^2 + \frac{\Im}{b}.$ (2.14)

For v given by (2.4) and $\Psi(d,b,v)$ is a positive , since $\Im > 0$ and $b \geq 1$, $d \in \mathbb{R}$, this imply that $\Psi(d,b,v) > 0$ and this ensures that the mathematical relationship is a chive in (2.13) , an application of lemma 1.2 , imply that

for $s \in \mathcal{U}$, we get $\mathscr{R}e(\mathcal{Q}(s)) > 0$.

Now we will prove that $\mathcal{B}_1(s) \prec \mathcal{B}_2(s)$, when $\mathcal{B}_2(s)$ is convex according to the definition on $\mathcal{Q}(s)$.

To proceed with the argument, we will assume that the function $\mathscr{L}(s,\tilde{r})$ is defined in the from

$$\mathscr{L}(s,\tilde{r}) = \mathcal{B}_2(s) + \frac{(1+\tilde{r})s\mathcal{B}_2'(s)}{\frac{\Im}{b}} \quad (0 \leq \tilde{r} < \infty, s \in \mathcal{U}). \tag{2.15}$$

Differentiate both sided of (2.15) w.r.to s, we obtain

$\frac{\partial \mathscr{L}(s,\tilde{r})}{\partial s}|_{s=0} = (1 + \frac{(1+\tilde{r})}{\frac{\Im}{b}})\mathcal{B}_2'(0) \neq 0$.

Differentiate both sided of (2.15) w.r.to $\widetilde{r}$, we get

$\frac{\partial \mathscr{L}(s,\tilde{r})}{\partial \tilde{r}} = \frac{s\mathcal{B}_2'(s)}{\frac{\Im}{b}}$.

Hence

$$\frac{s\frac{\partial \mathscr{L}(s,\tilde{r})}{\partial s}}{\frac{\partial \mathscr{L}(s,\tilde{r})}{\partial \tilde{r}}} = \frac{\Im}{b} + (1+\tilde{r})\left(1 + \frac{s\mathcal{B}_2''(s)}{\mathcal{B}_2'(s)}\right)$$

$$= \frac{\Im}{b} + (1+\tilde{r})\ \mathcal{Q}(s).$$

So $\mathscr{R}e\left(\frac{s\frac{\partial \mathscr{L}(s,\tilde{r})}{\partial s}}{\frac{\partial \mathscr{L}(s,\tilde{r})}{\partial \tilde{r}}}\right) = \mathscr{R}e\left(\frac{\Im}{b} + (1+\tilde{r})\ \mathcal{Q}(s)\right)$ $(0 \leq \tilde{r} < \infty, s \in \mathcal{U})$.

We deduct that $\mathscr{L}(s,\tilde{r})$ is a subordination chain by applying lemma 1.1, so from this , we have

$\mathcal{T}(s) = \mathcal{B}_2(s) + \frac{s\mathcal{B}_2'(s)}{\frac{\Im}{b}} = \mathscr{L}(s,0)$ and since $\mathscr{L}(s,0) \prec \mathscr{L}(s,\tilde{r})$ for $0 \leq \tilde{r} < \infty$, therefore

$$\mathscr{L}(\xi,\tilde{r}) \notin \mathscr{L}(u,0) = \mathcal{T}(\mathcal{U}) \quad (0 \leq \tilde{r} < \infty, \xi \in \partial u). \tag{2.16}$$

If $\mathcal{B}_1(s) \nprec \mathcal{B}_2(s)$, lemma 1.4 imply that $\exists \xi_0 \in \partial\mathcal{U}$ and $\exists s_0 \in \mathcal{U}$ with property $\mathcal{B}_1(s_0) = \mathcal{B}_2(\xi_0)$, such that

$$s_0\mathcal{B}_1'(s_0) = (1+\tilde{r})\ \xi_0\mathcal{B}_2(\xi_0). \tag{2.17}$$

Therefore, according to (1.6) and (2.16), we get

$$\mathscr{L}(\xi_0,\tilde{r}) = \mathcal{B}_2(\xi_0) + \frac{(1+\tilde{r})s_0\mathcal{B}_2'(\xi_0)}{\frac{\Im}{b}}$$

$$= \mathcal{B}_1(s_0) + \frac{s_0\mathcal{B}_1'(s_0)}{\frac{\Im}{b}}$$

$$= \left(s_0\left(\mathfrak{D}_\xi^{\lambda_m}k\right)(s_0)\right)^{\Im} + s_0\Im\left(s_0\left(\mathfrak{D}_\xi^{\lambda_m}k\right)(s_0)\right)^{\Im-1}\left[\frac{s_0\left(\left(\mathfrak{D}_\xi^{\lambda_m}k\right)(s_0)\right)' + \left(\left(\mathfrak{D}_\xi^{\lambda_m}k\right)(s_0)\right)}{\frac{\Im}{b}}\right]$$

$$= \left(s_0\left(\left(\mathfrak{D}_\xi^{\lambda_m}k\right)(s_0)\right)\right)^{\Im} + \frac{\Im\left(s_0\left(\left(\mathfrak{D}_\xi^{\lambda_m}k\right)(s_0)\right)\right)^{\Im}}{\frac{\Im}{b}}\left[\frac{\left(\mathfrak{D}_\xi^{\lambda_{m+1}}k\right)(s_0)}{\left(\mathfrak{D}_\xi^{\lambda_m}k\right)(s_0)} - 1\right]$$

$$= (1-b)\left(s_0\left(\left(\mathfrak{D}_\xi^{\lambda_m}k\right)(s_0)\right)\right)^{\Im} + b\left(\frac{\left(\mathfrak{D}_\xi^{\lambda_{m+1}}k\right)(s_0)}{\left(\mathfrak{D}_\xi^{\lambda_m}k\right)(s_0)}\right)\left(s_0\left(\left(\mathfrak{D}_\xi^{\lambda_m}k\right)(s_0)\right)\right)^{\Im} \in \mathcal{T}(\mathcal{U}).$$

This disagree the condition of subordination to the theorem.
Thus , we have $\mathcal{B}_1(s)$ is a subordinate to $\mathcal{B}_2(s)$.
As a result, in the case of $\mathcal{B}_1(s) = \mathcal{B}_2(s)$, the best dominant function will be $\mathcal{B}_2(s)$. Thus we obtain the desired result .
Next theorem proves the existence of the best dominant by discussing the result of super-ordination of function defined by convolution coupled with a differential operator.

Theorem 2.2. For the functions $\mathfrak{T}_j(s)$ $(j = 1,2)$ belonging to $\sum$, and $\left(\left(\mathfrak{D}_{\xi}^{\lambda_m} k\right)(s)\right)^{\mathfrak{J}}$ belonging to $\mathfrak{Q}$, let $\mathcal{T}(s)$ be a function of the form

$$\mathcal{T}(s) = (1-b)\left(s\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}_2\right)(s)\right)^{\mathfrak{J}} + b\left(\frac{\left(\mathfrak{D}_{\xi}^{\lambda_{m+1}}\mathfrak{T}\right)(s)}{\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}\right)(s)}\right)\left(s\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}_2\right)(s)\right)\right)^{\mathfrak{J}},$$

satisfy the condition

$$\mathscr{R}e\left\{1 + \frac{s\mathcal{T}''(s)}{\mathcal{T}'(s)}\right\} > v\,, \qquad (2.18)$$

if $\tilde{\mathcal{T}}(s)$ is univalent in $\mathcal{U}$, where $\tilde{\mathcal{T}}(s) = (1-b)\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}_1\right)(s)\right)^{\mathfrak{J}} + b\left(\frac{\left(\mathfrak{D}_{\xi}^{\lambda_{m+1}}k\right)(s)}{\left(\mathfrak{D}_{\xi}^{\lambda_m}k\right)(s)}\right)\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}_1\right)(s)\right)^{\mathfrak{J}}$

and $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$ such that $\mathscr{R}e(\lambda) > 0$, $\delta \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, v is given by (2.1). Then

$$\mathcal{T}(s) \prec \tilde{\mathcal{T}}(s)\,,$$

ensure that

$$\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}_2\right)(s)\right)^{\mathfrak{J}}(s) \prec \left(\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}_1\right)(s)\right)^{\mathfrak{J}}.$$

Thus the function $\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}_2\right)(s)\right)^{\mathfrak{J}}$ is a best dominant .

Proof. Let the functions $\mathcal{B}_1(s)$, $\mathcal{B}_2(s)$ and $\mathcal{Q}(s)$ are given as in equations (2.6) and (2.7), Utilizing the same approach employed in the proof of Theorem 2.1, we obtain the following result:

$\mathcal{R}e\{\mathcal{Q}(s)\} > 0 \quad (s \in \mathcal{U})$.

To find the required proof, which is the best dominant ,we note that this is obtained by the validity of the result $\mathcal{B}_2(s) \prec \mathcal{B}_1(s)$, and following the same proof method the function for theorem 2.1 , we conclude from the fact that $\mathscr{L}(s, \tilde{r})$ is defined by the form in (2.15) and the fact that $\mathcal{B}_1(s)$ is convex, it becomes clear that $\mathscr{L}(s, \tilde{r})$ is a subordination chain of those justifications , then , using Lemma 1.5, we conclude that $\mathcal{B}_2(s) \prec \mathcal{B}_1(s)$. Further explanation, as a final result obtained from applying lemma 1.3 and lemma 1.5, $\mathcal{B}_2(s)$ is a solution that represents a univalent function of the differential equation

$$\mathcal{T}(s) = \mathcal{B}_2(s) + \frac{bs\mathcal{B}_2{}'(s)}{\mathfrak{J}} = \psi\left(\mathcal{B}_2(s), s\,\mathcal{B}_2{}'(s)\right).$$

Then $\mathcal{B}_2(s)$ is the best subordinant .
The following result is an composite of theorems (1.1) and (1.2), which known as sandwich theorem .

Theorem 2.3. Suppose that the functions $k(s)$, $\mathfrak{T}_j(s)$ $(j = 1,2)$ and $\left(\left(\mathfrak{D}_{\xi}^{\lambda_m} k\right)(s)\right)^{\mathfrak{J}}$ belonging to $\mathfrak{Q}$, for $\mathcal{T}_j(s)$ of the form

$$\mathcal{T}_j(s) = (1-b)\left(s\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}_j(s)\right)^{\mathfrak{J}} + b\left(\frac{\left(\mathfrak{D}_{\xi}^{\lambda_{m+1}}\mathfrak{T}_j\right)(s)}{\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}_j\right)(s)}\right)\left(s\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathfrak{T}_j\right)(s)\right)\right)^{\mathfrak{J}},$$

satisfy the condition

$$\mathscr{R}e\left\{1+\frac{s\mathcal{T}_j''(s)}{\mathcal{T}_j'(s)}\right\}>v\,, \tag{2.19}$$

if $\mathcal{T}(s)=(1-b)\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}k\right)(s)\right)^{\Im}+b\left(\frac{\left(\mathfrak{D}_{\xi}^{\lambda_{m+1}}k\right)(s)}{\left(\mathfrak{D}_{\xi}^{\lambda_m}k\right)(s)}\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}k\right)(s)\right)^{\Im}\right.$ is univalent in $\mathcal{U}$, where $m\in\mathbb{N}$, $\lambda\in\mathbb{C}$ such that $\mathscr{R}e(\lambda)>0$, $\delta\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$, $s\in\mathcal{U}$ and v is given by (2.1). Then

$$\mathcal{T}_1(s)\prec\mathcal{T}(s)\prec\mathcal{T}_2(s)\,,$$

implies that

$$\widetilde{\mathcal{T}_1}(s)\prec\mathcal{B}_1(s)\prec\widetilde{\mathcal{T}_2}(s)\,,$$

where $\widetilde{\mathcal{T}_1}(s)=\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathcal{T}_1\right)(s)\right)^{\Im}$ refer to the best subordinant

$\widetilde{\mathcal{T}_2}(s)=\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}\mathcal{T}_2\right)(s)\right)^{\Im}$ refer to the best dominant and $\mathcal{B}_1(s)=\left(\left(\mathfrak{D}_{\xi}^{\lambda_m}k\right)(s)\right)^{\Im}$.

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